Rouquier dimension, quantitative intersection of skeleta and Orlov's conjecture

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(Based on joint work with Laurent Côté)
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Question

Given a generic compactly supported Hamiltonian diffeomorphism $\phi : X \to X$, what is the size of $|\mathfrak{c}_X \cap \phi(\mathfrak{c}_X)|$?

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- The skeleton $\mathfrak{c}_X$ could be rather singular in general, seems hard to develop a Floer theory for such objects.

Quantitative intersection of skeleta

Theorem (B–Côté)

Suppose $2c_1(X) = 0$ *and* $SH^*(X;\mathbb{C})$ *is isomorphic to the affine coordinate ring of a smooth n-dimensional affine variety over* $\mathbb{C}$*. Assume there exists an object* $L_0 \in \mathcal{W}(X)$ *such that the degree* 0 *part of the closed open string map*

$$\mathcal{CO}^0 : SH^0(X) \to HW^*(L_0, L_0)$$

is an isomorphism. Then

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If $(X, \lambda) = (T^*G, \lambda)$ where G is a simply-connected compact Lie group (note that λ is just Weinstein homotopic to pdq), then

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- An object L_0 as above is called a *homological section* (Pomerleano).

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- New cases covered: all log Calabi–Yau surfaces with the choice of complex structures from Hacking–Keating.
- Singular surfaces arising as mirrors of Milnor fibers à la Lekili–Ueda.

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- The generation time of G is the minimum of all such n.
- The *Rouquier dimension* Rdim $\mathcal{T}$ is the minimum of generation times ranging over all split-generators of $\mathcal{T}$.
- For a pre-triangulated k-linear A_∞ category $\mathcal{C}$ (e.g. the wrapped Fukaya category $\mathcal{W}(X)$), define Rdim $\mathcal{C}$:= Rdim $H^0(\text{Perf } \mathcal{C})$.

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- (bounded from above by *the diagonal dimension* c.f. Elagin–Lunts) If the diagonal bimodule $\Delta_{\mathcal{C}}$ admits an l-step resolution in terms of Yoneda bimodules, then Rdim $\mathcal{C} \leq l - 1$.

Basic properties of Rdim

- (Elagin): let Γ be a quiver, i.e. a directed graph which is connected, finite and admitting no loops or cycles. Denote by $k[\Gamma]$ the path algebra of Γ. Then Rdim $(H^0(\text{Perf } k[\Gamma])) = 0$ if Γ is of Dynkin ADE type, otherwise Rdim $(H^0(\text{Perf } k[\Gamma])) = 1$.

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- (Elagin–Lunts) If the A_∞ category $\mathcal{C}$ has an object L such that the Yoneda right module $\mathcal{Y}_L^r$ is proper and $\hom(L, L) = \Lambda^\bullet V$ as A_∞ algebras where V is an r-dimensional graded vector space supported in degree 1, then Rdim $\mathcal{C} \geq r$.

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- The L as above is called a *point-like object*. For derived categories of coherent sheaves, they arise from skyscraper sheaves; for Fukaya categories, they come from exact Lagrangian torus.

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Assume (X, λ) has properly embedded cocore Lagrangian discs (this is a generic condition). Then given a generic compactly supported Hamiltonian diffeomorphism $\phi : X \to X$, we have Rdim $\mathcal{W}(X) + 1 \leq |\mathfrak{c}_X \cap \phi(\mathfrak{c}_X)|$.

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Concerning minimal number of critical points of Weinstein Lefschetz fibrations, Rdim $\mathcal{W}(X) \leq Lef_w(X, \lambda) - 1$.
If (X, λ) has arboreal Lagrangian skeleton and admits a "good" arboreal sectorial cover and $\dim_{\mathbb{R}} X = 2n \leq 6$, then Rdim $\mathcal{W}(X) \leq n$.

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Proposition

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□

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- (Lekili–Ueda) Homological mirrors of (A-side) affine Calabi–Yau surfaces from Milnor fibers, could be singular and stacky.
- The proof is based on bounding Rdim from above by n using the local-to-global descent formula and symplectic flexibility, completely different from known approaches. The lower bound Rdim $\geq n$ follows from the existence of point-like object in D^bCoh from sky-scraper sheaves.

Theorem (B–Côté, lower bound)

Suppose there exists a split-generator $K \in \mathcal{W}(X)$ such that $\hom^\bullet(K, K)$ is a Noetherian module over a sub-ring R of the symplectic cohomology $SH^\bullet(X)$, where R acts on $\hom^\bullet(K, K)$ via the closed-open string map $\mathcal{CO}$. Then

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- Combined with the upper bounds, we see $|\mathfrak{c}_X \cap \phi(\mathfrak{c}_X)| \geq n + 1$.

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- Any simply-connected compact Lie group G has the rational homotopy type of product of odd dimension spheres. We can compute that $\dim_R H_\bullet(\Omega_p G) = \mathrm{rank} G$ for some $R \subset H_{\bullet+n}(\mathcal{L}G)$.

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- It is shown by Abbondandolo–Schwarz that $\mathcal{I}$ coincides with $\mathcal{CO}$ under the isomorphisms $H_{\bullet+n}(\mathcal{L}M) \simeq SH^\bullet(T^*M)$ and $H_\bullet(\Omega_p M) \simeq HW^\bullet(F_p, F_p)$.
- Any simply-connected compact Lie group G has the rational homotopy type of product of odd dimension spheres. We can compute that $\dim_R H_\bullet(\Omega_p G) = \mathrm{rank} G$ for some $R \subset H_{\bullet+n}(\mathcal{L}G)$.
- Therefore $|\mathfrak{c}_X \cap \phi(\mathfrak{c}_X)| \geq \mathrm{Rdim}\, \mathcal{W}(T^*G) + 1 \geq \mathrm{rank} G + 1$.

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- Ganatra–Pardon–Shende, Chantraine–Dimitroglou Rizell–Ghiggni–Golovko: the object $\Delta \subset \mathcal{W}(\overline{X} \times X)$ is quasi-isomorphic to an iterated cone of product of cocore Lagrangians of length $|\mathfrak{c}_X \cap \phi(\mathfrak{c}_X)|$.

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- Descent formula of GPS: $\mathcal{W}(T^*M)$ is recovered as a homotopy colimit of $T^*(\cap_{i \in I} U_i)$.
- We can use this local-to-global principle to obtain the inequality Rdim $\mathcal{W}(T^*M) \leq n$.

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- The wrapped Fukaya categories of $X_{\mathfrak{T}}$ is quasi-isomorphic to Perf $k[\Gamma]$ where Γ is the underlying quiver of $\mathfrak{T}$. In particular, Rdim $\mathcal{W}(X_{\mathfrak{T}}) = 0$ or 1 depending on whether Γ is of ADE type or not.

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Our current result restricts to $n \leq 2$ but it could be generalized in principle.

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Question

How to define a notion of Lusternick–Schnirelmann category of Weinstein manifolds based on arborealization? How is this related to the cone length of the underlying topological spaces of arboreal skeleta?

Thanks for your attention!