Towards an extension of Taubes' Gromov invariant to Calabi–Yau 3-folds

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(Based on joint work with Mohan Swaminathan)

Overview

1. Statement of results
   - Background knowledge
   - Wendl's solution to the generic super-rigidity conjecture
   - Main Theorem
2. Ingredients of the Proof
   - Outline of Proof
   - Necessary condition for bifurcations
   - Sufficient condition for bifurcations
   - Linear wall-crossing correction
3. Further directions
   - Comparison to BPS
   - Other questions
4. Bonus: proofs

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Gromov–Witten theory

- Let (X, ω) be a symplectic Calabi–Yau 3-fold, i.e. (X, ω) is a 6-dimensional closed symplectic manifold with $c_1(TX, \omega) = 0$.
- Denote by $\mathcal{J}(X, \omega)$ the space of almost complex structures on X compatible with ω.
- Given $J \in \mathcal{J}(X, \omega)$, $g \geq 0$ and $A \in H_2(X, \mathbb{Z})$, the moduli space $\overline{\mathcal{M}}_g(X, J, A)$ has virtual dimension 0.
  $\rightsquigarrow$ Gromov–Witten invariant $\mathrm{GW}_{A,g} \in \mathbb{Q}$, independent of J.
- Because of multiple covers and ghost components (which may have non-trivial automorphisms) these invariants are not $\mathbb{Z}$-valued.

Taubes' Gromov invariant

- For a closed symplectic 4-manifold, Taubes defined the so-called Gromov invariant Gr as a suitable $\mathbb{Z}$-weighted count of **embedded** J-holomorphic curves.
- Taubes later also proved the famous identity SW = Gr.
- The definition relies on intersection theory in dimension 4, which is not available in higher dimensions.

Question

How to extend Taubes' construction to symplecitc Calabi–Yau 3-folds to define a $\mathbb{Z}$-valued invariant by counting embedded J-holomorphic curves?

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Simple/multiply-covered dichotomy

- Any non-constant J-holomorphic map $f' : \Sigma' \to X$ with Σ' smooth can be factored uniquely as

$$\Sigma' \xrightarrow{\varphi} \Sigma \xrightarrow{f} X$$

where φ is a holomorphic (branched) cover and $f : \Sigma \to X$ is a **simple** J-holomorphic map.

Fact

For a symplectic Calabi–Yau 3*-fold* (X, ω)*, away from a codimension* 2 *subset of* $\mathcal{J}(X, \omega)$*, all* **simple** *holomorphic curves are* **embedded** *and have* **pairwise disjoint images***.*

- Restrict attention to J as in the above fact for the remainder.
- Then we can assume $\Sigma \xrightarrow{f} X$ is an embedded J-holomorphic curve and $f' : \Sigma' \to X$ can be an arbitrary non-constant **stable** map.

Super-rigidity

Definition

$J \in \mathcal{J}(X, \omega)$ is called **super-rigid** if, for all non-constant J-holomorphic stable maps

$$\Sigma' \xrightarrow{\varphi} \Sigma \subset X,$$

we have $\ker(\varphi^* D^N_{\Sigma,J}) = 0$, where

$$D^N_{\Sigma,J} : \Omega^0(\Sigma, N_\Sigma) \to \Omega^{0,1}(\Sigma, N_\Sigma)$$

is the **normal deformation operator** of the embedded J-curve $\Sigma \subset X$.

- If J is super-rigid, then given any sequence of embedded J-curves $\Sigma_n \subset X$ (of bounded genus and ω-area), we can find a subsequence converging to an **embedded** J-curve $\Sigma \subset X$.
- There are only **finitely many** embedded J-curves with fixed genus and homology class if J is super-rigid.

Generic super-rigidity

Conjecture (Bryan–Pandharipande '01)

Super-rigid almost complex structures in $\mathcal{J}(X, \omega)$ form a Baire subset.

- This conjecture has recently been resolved.

Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $\mathcal{J}(X, \omega)$ where super-rigidity fails has codimension ≥ 1.

- Wendl actually proves a more precise statement which determines the codimensions of the various strata of this subset (corresponding to the Galois group of the covers involved and their representations).
- Doan–Walpuski arXiv:2006.01352 have provided a more general perspective on this result and clarified the conditions that make Wendl's proof work for Cauchy–Riemann operators.

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Main result

Theorem 1 (B.–Swaminathan, 2021)

Let (X, ω) be a CY 3-fold. Fix a primitive class $A \in H_2(X, \mathbb{Z})$ and an integer $h \geq 0$. For a super-rigid $J \in \mathcal{J}(X, \omega)$, define the *virtual count of embedded genus h curves of class* $2A$ to be the **integer**

$$\mathrm{Gr}_{2A,h}(X, \omega, J) = \sum_{C' : 2A, h} \mathrm{sgn}(C') + \sum_{g \leq h} \sum_{C : A, g} \mathrm{sgn}(C) \cdot \mathrm{w}_{2,h}(D^N_{C,J})$$

where the first sum is over embedded J-curves C' of genus h and class $2A$, the second sum is over all genera $0 \leq g \leq h$ and J-curves C of genus g and class A and $\mathrm{w}_{2,h}$ are suitably defined integer weights.
Then, $\mathrm{Gr}_{2A,h}(X, \omega, J)$ is independent of the choice of super-rigid J and

$$\mathrm{Gr}_{2A,h}(X, \omega) := \mathrm{Gr}_{2A,h}(X, \omega, J) \tag{1}$$

defines a symplectic invariant of (X, ω).

- Doan–Walpuski arXiv:1910.12338 prove that the count of embedded J-curves of *primitive* homology classes defines an invariant.
- Our result is the first non-trivial extension of Taubes' Gr to CY 3-folds when multiple covers are present and all genera are allowed.
- In fact, $\mathrm{Gr}_{2A,h}$ is a symplectic deformation invariant of (X, ω). This follows from the fact that all the results about super-rigidity continue to hold when we replace ω-compatible almost complex structures by ω-tame almost complex structures.

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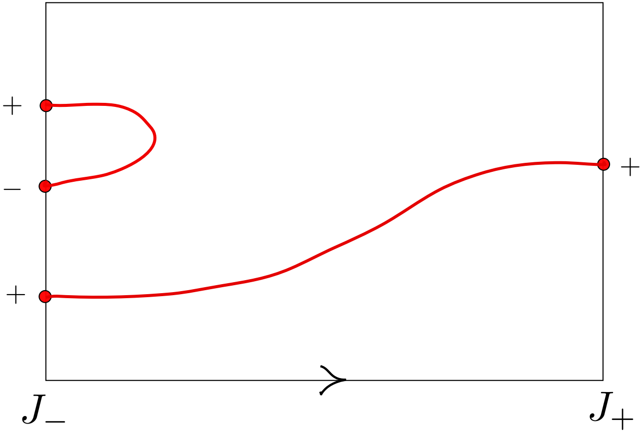
Outline of proof

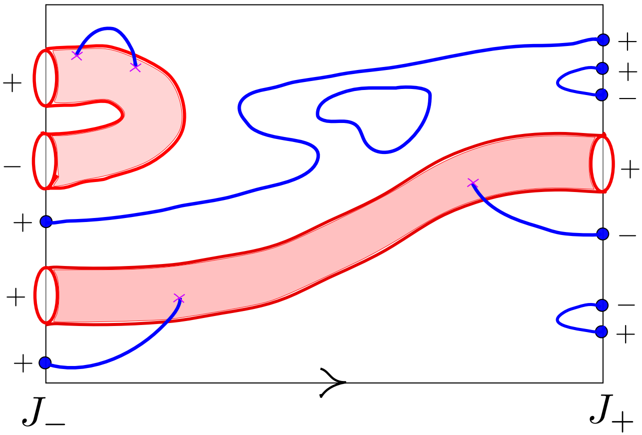
- Super-rigid $J \Rightarrow$ finitely many embedded J-curves in any fixed homology class and genus. Therefore, the sums defining $\mathrm{Gr}_{2A,h}(X, \omega, J)$ are finite.
- To show symplectic invariance, we take a *generic* path (J_t) joining two given super-rigid almost complex structures J_- and J_+. In view of Wendl's theorem, we just need to explicitly understand the bifurcations that occur at the discrete set of times t for which J_t fails to be super-rigid.
- The "correction terms" $\mathrm{w}_{2,h}(D^N_{C,J})$ are designed to cancel out the bifurcations. The construction is inspired by gauge theory.

Stable maps of class A and genus $\leq h$ (preview)



- Embedded curves of class A are shown in **red**.

Stable maps of class $2A$ and genus h (preview)



- Double covers of embedded curves of class A are shown in **red**.
- Embedded curves of class $2A$ are shown in **blue**.
- Bifurcations are shown in **magenta**.

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Necessary condition for bifurcations: statement

Theorem 2 (B.–Swaminathan, 2021)

Suppose V is a smooth manifold and let $\mathcal{F} : V \to \mathcal{J}(X, \omega)$ be a smooth family. Let $x_\nu \to x$ be a convergent sequence in V, with $J_\nu := \mathcal{F}(x_\nu)$ for $\nu \geq 0$ and $J := \mathcal{F}(x)$. Let $h \geq 0$ be an integer and suppose that we have a sequence $(J_\nu, \varphi_\nu : \Sigma'_\nu \to X)$ of **simple** J_ν-curves of genus h converging, in the Gromov topology, to a stable map

$$(J, C' \xrightarrow{\varphi} C \subset X) \tag{2}$$

with C being a smooth **embedded** J-curve of genus $g \leq h$ and $\varphi : C' \to C$ being an element of $\overline{\mathcal{M}}_h(C, k)$ for some integer $k \geq 1$. Then, exactly one of the following must be true.

- $\varphi : C' \to C$ is an isomorphism ("no bifurcation occurs").
- The natural pullback map $\varphi^* : \ker D^N_{C,J} \to \ker \varphi^* D^N_{C,J}$ is injective but not surjective ("a multi-valued normal deformation exists.")

Necessary condition for bifurcations: comments

- Theorem 2 is quite general. It works for *any* almost complex manifold.
- Results of this kind date back to Taubes' 1996 paper, and appeared subsequently in the work of Ionel–Parker, Zinger, Wendl. These precursors are all established based on rescaling arguments.
- The rescaling argument requires existence of a "core" curve and certain transversality assumptions (e.g. $\ker D^N_{C,J} = 0$ in Wendl's proof). Our result does not need any of these assumptions.
- In the situation of Theorem 1, V is an interval, it allows us to rule out degenerations into nodal stable maps (possibly with ghost components).

Necessary condition for bifurcations: comments

- Our result also implies that a sequence of embedded curves of a fixed genus lying in a primitive homology class can't limit to an embedded curve with some ghost component(s) attached to it.
- Combined with Ionel–Parker's Kuranishi model for birth-death bifurcations, we can recover Doan–Walpuski's work on counting embedded curves of primitive homology class without appealing to any delicate gluing analysis.

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Sufficient condition for bifurcations: statement

The next result provides a partial converse to Theorem 2 by giving a precise description of what happens to the number of embedded curves when a bifurcation occurs.

Theorem 3 (B.–Swaminathan, 2021)

Let (J_t) be a *generic* path in $\mathcal{J}(X, \omega)$. Assume that for $t = 0$ there exists an embedded rigid J_0-curve $C \subset X$ along with a d-fold genus h branched multiple cover $\varphi : \Sigma \to C$ along which a non-trivial multi-valued normal deformation exists. If φ determines an **elementary wall type**, then

1. $\mathrm{Aut}(\varphi) \subset \mathbb{Z}/2\mathbb{Z}$ and,
2. the change in the signed count of embedded curves of genus h and class $d[C]$ near φ, when going from $t < 0$ to $t > 0$, is given by $\pm 2/|\mathrm{Aut}(\varphi)|$.

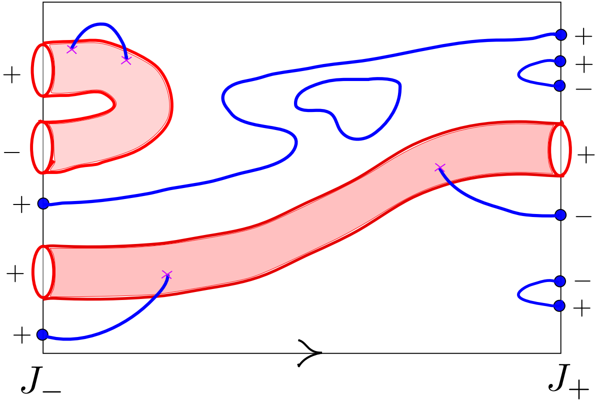
Sufficient condition for bifurcations: comments

- The technical notion of *elementary wall type* covers a large class of branched covers. For example, this includes all d-fold covers $\Sigma \xrightarrow{\varphi} C$ for which Σ is smooth, φ has the expected number of distinct branch points and the Galois group of φ is S_d.
- Theorem 3 comes out of an explicit analysis of local Kuranishi models along paths which are sufficiently *generic*.

1. Firstly, γ should be transverse to the codimension 1 walls in $\mathcal{J}(X, \omega)$. *This follows from Wendl's work.* Condition (W)
2. Secondly, γ should miss a certain codimension 1 subset of the wall (the *degeneracy locus*). This needs some work and is the place where the *elementary* nature of φ is used. *(Our contribution)* Condition (BS)

Summary of bifurcation analysis

Theorems 2 and 3 together show that the following picture, introduced earlier, is indeed a faithful depiction of the moduli space of stable maps of class $2A$ and genus h along a generic path γ.



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Linear wall-crossing invariant

- As the previous picture shows, embedded curves (of class $2A$ and genus h) may appear or disappear when bifurcations occur. We need to account for this phenomenon by using a suitably defined *correction term* to get a symplectic invariant.
- For this purpose, we define the algebraic invariant $\mathrm{w}_{2,h}(\cdot) \in \mathbb{Z}$ for the normal deformation operator of any embedded pseudo-holomorphic curve of genus $g \leq h$ in class A. It can be viewed as a kind of *equivariant spectral flow*. Showing that it is well-defined takes some work and the proof uses Theorem 2 (as well as a result on generic embeddedness of normal multi-valued deformations).

Linear wall-crossing invariant

- Such constructions probably first originated in gauge theory (e.g. in Walker '92, Boden–Herald '98, Mrowka–Ruberman–Saveliev '11).
- Putting the definition of the *linear wall-crossing invariant* $\mathrm{w}_{2,h}$ together with the bifurcation analysis (Theorems 2 and 3) yields the proof of invariance in Theorem 1.

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BPS invariants and GV formula

Conjecture (Gopakumar–Vafa '98)

Let X be a symplectic CY 3-fold. Then, there exist numbers $BPS_{A,h} \in \mathbb{Z}$ for all $h \geq 0$ and $A \in H_2(X, \mathbb{Z})$ satisfying the following identity

$$\sum_{A \neq 0, g \geq 0} GW_{A,g} t^{2g-2} q^A = \sum_{A \neq 0, h \geq 0} BPS_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \left(\frac{kt}{2} \right) \right)^{2h-2} q^{kA}.$$

Moreover, for any $A \in H_2(X, \mathbb{Z})$, we have $BPS_{A,h} = 0$ for $h \gg 0$.

Theorem (Ionel–Parker, 2018)

Gopakumar–Vafa's integrality conjecture holds.

Theorem (Doan–Ionel–Walpuski arXiv:2103.08221)

Gopakumar–Vafa's finiteness conjecture holds.

Comparison to BPS

- We can't expect $\mathrm{Gr}_{2A,h}(X, \omega)$ to exactly equal the BPS invariants for $h > 0$. Instead, we suspect that the following is true.

Conjecture

$\mathrm{Gr}_{2A,h}(X, \omega)$ coincides with the *virtual count of embedded clusters*, denoted $e_{2A,h}(X, \omega)$, defined by Ionel–Parker '18.

- We show the conjecture is true for $h = 0$.
- This is proved using a more general result about the change of the (virtual) Euler number of an obstruction bundle along a generic path of $\mathbb{R}$-linear Cauchy–Riemann operators. We use the virtual fundamental class (VFC) package developed by Pardon '16.

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Further directions

- We are still working *towards* the full extension of Taubes' Gromov invariant with one obvious obstacle: our sufficient condition for bifurcations (Theorem 3) only deals with *elementary wall types*. We hope to address this limitation in future work.
- Motivated by the present work, we may pose the following question.

Question

Is it possible to give an algebro-geometric construction of this analogue of Taubes' Gromov invariant for CY 3-folds?

Since one doesn't expect generic super-rigidity to hold in the algebro-geometric context, answering this question probably requires a novel compactification of the moduli space of embedded curves.

Bonus: proofs

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Necessary condition for bifurcations: proof idea

- The proof is by a careful examination of the infinitesimal deformations and obstructions of the two moduli spaces

$$\overline{\mathcal{M}}_h(\mathcal{M}_g^{\text{emb}}(X, \gamma, [C]), d) \subset \overline{\mathcal{M}}_h(X, \gamma, d[C])$$

at the point $(t, \Sigma \xrightarrow{\varphi} C \subset X)$. Here, the latter is the usual stable map moduli space along γ while the former consists of those stable maps which factor through an embedded genus g curve in class $[C]$.

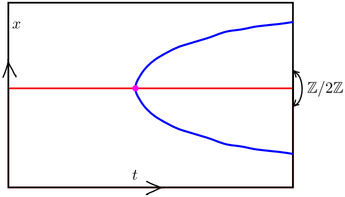
- The key assertion to prove is the following. The fibre of the normal bundle of the above inclusion, after suitably "thickening" both moduli spaces, at the point $(t, \Sigma \xrightarrow{\varphi} C \subset X)$ is canonically isomorphic to the vector space

$$\frac{\ker \varphi^* D^N_{C,J}}{\ker D^N_{C,J}}.$$

Sufficient condition for bifurcations: proof idea

Focus on Taubes' setting of a 2:1 cover $T' \xrightarrow{\varphi} T$ of a torus by a torus.

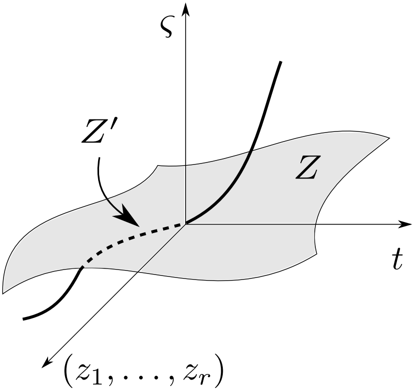
- One has to argue that the moduli space locally looks like this:



Here, $x \in \mathbb{R}$ is a coordinate on $\ker \varphi^* D^N_{T,J_0}$, double covers close to φ are shown in **red** and embedded curves are shown in **blue** and picture represents the locus $x(t - x^2) = 0$ modulo the non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on x.

- More precisely, one argues that the Kuranishi map is given by $f(t, x) = x(\alpha t + \beta x^2 + \cdots)$ with $\alpha \neq 0$ corresponding to genericity Condition (W) and $\beta \neq 0$ corresponding to genericity Condition (BS).

Schematic picture of the local Kuranishi model



In the picture, $(z_1, \ldots, z_r) \in \mathbb{C}^r$ are coordinates on $T_\varphi \mathcal{M}_h(C, d)$, $\varsigma \in \mathbb{R}$ is a coordinate on $\ker(\varphi^* D^N_{C,J})$ and Z, Z' are the local irreducible components of the moduli space near $(0, \Sigma \xrightarrow{\varphi} C \subset X)$.

Thanks for your attention!