Equivariant Cerf theory and perturbative SU(n) Casson invariants

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Table of Contents

Casson Invariant

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- Roughly speaking, the Casson invariant is defined by "counting" irreducible SU(2) representations of $\pi_1(Y)$.
- Here, an SU(2) representation is called *irreducible*, if the commutator of its image is equal to the center of SU(2) (namely $\{\pm 1\}$). Otherwise, it is called *reducible*.

Casson Invariant

Proposition (Casson)

Suppose $\lambda(Y) \neq 0$, then $\pi_1(Y)$ has an irreducible SU(2)–representation.

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Corollary (Casson)

Suppose the Rokhlin invariant of Y is non-zero, then $\pi_1(Y)$ has an irreducible SU(2)–representation.

Table of Contents

The construction of Casson invariant

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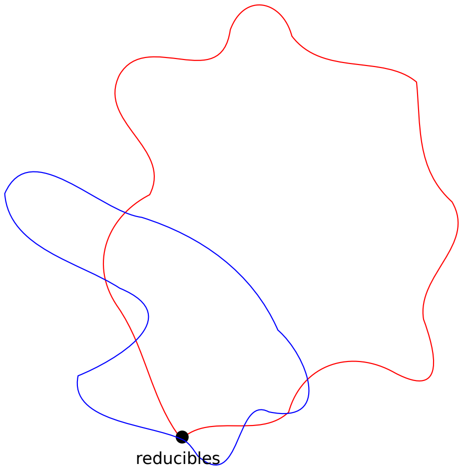
- The SU(2) character variety (i.e. the space of SU(2) representations up to conjugations) of $\pi_1(H_i)$ can be viewed as subsets of of the character variety of $\pi_1(F)$.
- The character variety of $\pi_1(Y)$ is equal to the intersection of the character varieties of $\pi_1(H_1)$ and $\pi_1(H_2)$ in the character variety of $\pi_1(F)$.

The construction of Casson invariant



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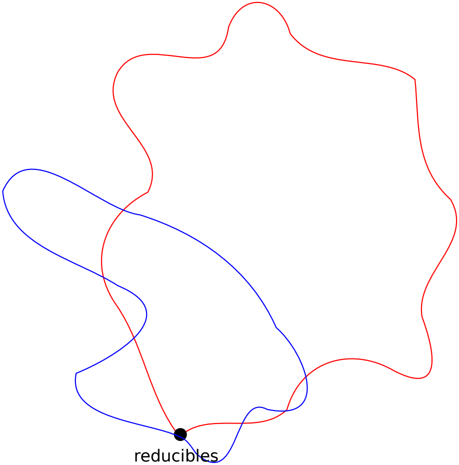

- It turns out that the intersection number is always even. The Casson invariant $\lambda(Y)$ is defined to be $1/2$ times the intersection number.

Table of Contents

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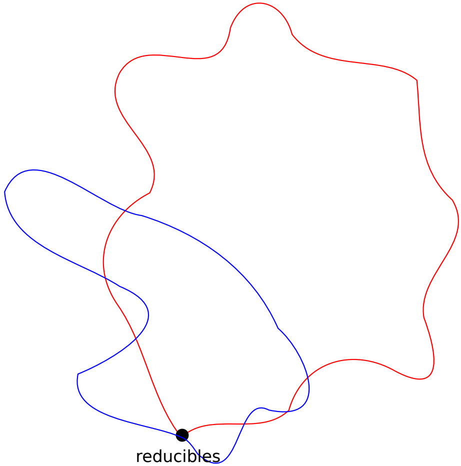

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- Boyer-Nicas (1990), Walker (1990),
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- Curtis (1994) [$SO(3), U(2), Spin(4), SO(4)$].

Main result

Construction

We can generalize the definition of the Casson invariant to $\mathrm{SU}(n)$ using gauge theory.

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We can generalize the definition of the Casson invariant to SU(n) using gauge theory.

This construction generalizes a previous work of Boden-Herald (1998) on gauge-theoretic SU(3) Casson invariants.

Table of Contents

The Chern-Simons functional

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- Since $\pi_1(\mathrm{SU}(2)) = \pi_2(\mathrm{SU}(2)) = 0$, every SU(2) bundle over Y is trivial.
- The conjugation classes of SU(2) representations of $\pi_1(Y)$ are in one-to-one correspondence with the isomorphism classes of flat SU(2) connections on the trivial SU(2)–bundle.

The Chern-Simons functional

- Let $\mathcal{A}$ be the affine space of all connections on the trivial SU(2)–bundle.

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- Let $\mathcal{G}$ be the gauge group.
- Define the Chern-Simons functional on $\mathcal{A}$ as

$$\mathrm{CS}(\theta + a) := \int_Y Tr\Big(da \wedge a + \frac{2}{3} a \wedge a \wedge a\Big),$$

where θ is the trivial connection.

The Chern-Simons functional

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- The isomorphism classes of flat connections are given by the $\mathcal{G}$–orbits of the critical points of the Chern-Simons functional.

Table of Contents

Taubes' theorem

One can perturb the Chern-Simons functional such that all critical points are cut out transversely (i.e. non-degenerate).

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Theorem (Taubes, 1990)

The Casson invariant of Y equals half of the (signed) counting of the irreducible critical points of the perturbed Chern-Simons functional.

Taubes' theorem

One can perturb the Chern-Simons functional such that all critical points are cut out transversely (i.e. non-degenerate).

Theorem (Taubes, 1990)

The Casson invariant of Y equals half of the (signed) counting of the irreducible critical points of the perturbed Chern-Simons functional.

In particular, the (signed) counting of the irreducible critical points of the perturbed Chern-Simons functional is independent of the perturbation.

Finite dimensional analogue

- Suppose M is an n–dimensional closed manifold, let f be a Morse function, let n_k be the number of critical points of f with index k, then

$$\sum_{k=0}^{n}(-1)^k n_k$$

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Finite dimensional analogue

- Suppose M is an n–dimensional closed manifold, let f be a Morse function, let n_k be the number of critical points of f with index k, then

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  is independent of the function f.
- Cerf's theorem: a generic path from two Morse functions has only finitely many birth-death singularities.

Table of Contents

Morse functions with group actions

Suppose M is endowed with a G–action.

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Definition

A smooth G–invariant function on M is called G–Morse, if at each critical point p of M, the kernel of $\mathrm{Hess}_p f$ is equal to the tangent space of the orbit of p.

Morse function with G-actions

The finite dimensional analogue of counting critical orbits of the Chern-Simons functional:

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Find a way to count critical orbits of G–Morse functions on M, such that it is independent of the Morse function.

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- Require that the counting equals $(-1)^k$ for irreducible (i.e. the stabilizer being trivial) critical points with index k.
- Cerf's theorem no longer holds for the set of irreducible critical orbits, therefore one must also count reducible points.

Bifurcation

- Let $M = \mathbb{R}$, $G = \{\pm 1\}$. Suppose G acts on M by multiplication.

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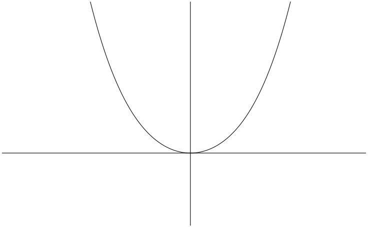
- Consider the family $f_t(x) = x^2 - tx^4$, for $t \in [-1, 1]$.

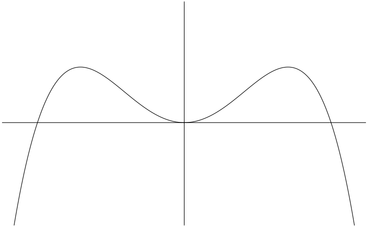
Bifurcation

When $t < 0$, there is a unique irreducible critical point.

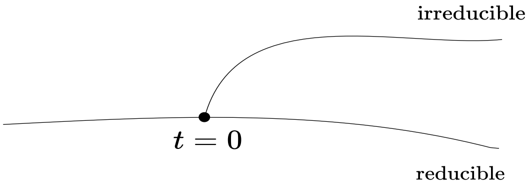


Bifurcation

When $t > 0$, there are two critical orbits.



Moduli space of critical points



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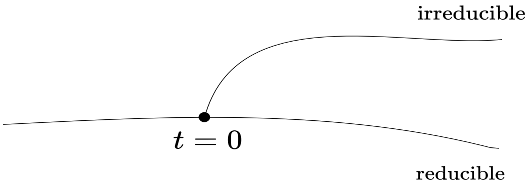
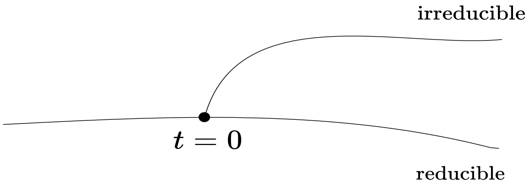


- More generally, let $M = \mathbb{R}^n$, $G = \{\pm 1\}$. Suppose G acts on M by multiplication on x_1, and $f_t(x) = \pm x_1^2 \pm x_2^2 \cdots \pm x_n^2 - t x_1^4$.

Moduli space of critical points



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- The index of the reducible critical point changes by 1 at $t = 0$.

Weighted counting on reducibles

- Count the reducible orbits by a linear function of its index to cancel the bifurcation.

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- Count the reducible orbits by a linear function of its index to cancel the bifurcation.
- The idea probably dates back to earlier projects of Mrowka-Ruberman-Saveliev and Kronheimer-Mrowka.
- Works if there is only one type of reducible orbit, which holds for SU(3) connections over an integer homology sphere.

SU(3) Casson invariant

Theorem (Boden-Herald, 1998)

A generic 1-parameter family of perturbations of the Chern-Simons functional only has finitely many singularities, that are either birth-death singularities or equivalent to the bifurcations shown above.

As a consequence,

Theorem (Boden-Herald, 1998)

The following formula defines an SU(3) Casson invariant

$$\sum_{[A]\in\mathcal{M}_h^*}(-1)^{Sf(\theta,A)} - \frac{1}{2}\sum_{[A]\in\mathcal{M}_h^r}(-1)^{Sf(\theta,A)}(Sf_{\mathfrak{h}^\perp}(\theta,A) - 4CS(\hat{A}) + c)$$

for any constant c.

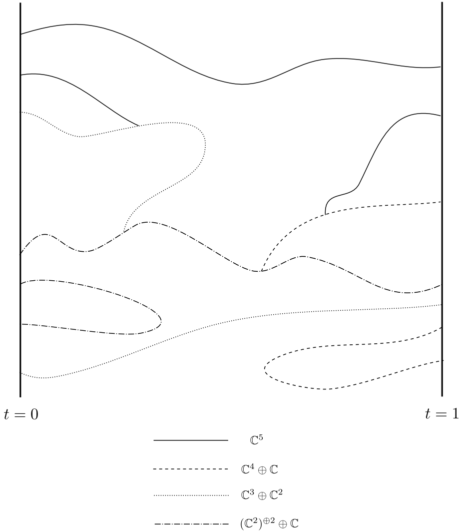
SU(n) Casson invariant

- More than one strata of reducible connections.

SU(n) Casson invariant

- More than one strata of reducible connections.
- More complicated bifurcations.

Possible bifurcations for SU(5)



SU(n) Casson invariant

Theorem (Bai-Z., 2020)

Every 1-parameter family of perturbations of the SU(n) Chern-Simons functional can be perturbed so that it only has finitely many bifurcations, and each bifurcation is given by a standard form.

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Theorem (Bai-Z., 2020)

There exists an SU(n) Casson invariant of the form

$$\sum_{[A]\in\mathcal{M}_h^*} (-1)^{Sf(\theta,A)} + \sum_{[A]\in\mathcal{M}_h^r} w(A),$$

where $w(A)$ is a weight function depending on the spectral flow and the Chern-Simons functional.

An explicit formula when $n = 4$

$$\sum_{[A]\in\mathcal{M}_\pi(4,1)} (-1)^{\dim Sf_{(4,1)}(A)}$$

$$+ \sum_{[A]\in\mathcal{M}_\pi((1,1),(3,1))} (-1)^{\dim Sf_{(1,1),(3,1)}(A)-1} \cdot \frac{1}{2} \operatorname{ind}_{(1,1),(3,1)^\perp}(A)$$

$$+ \sum_{[A]\in\mathcal{M}_\pi((2,1),(2,1))} (-1)^{\dim Sf_{(2,1),(2,1)}(A)-1} \cdot \frac{1}{2} \operatorname{ind}_{(2,1),(2,1)^\perp}(A)$$

$$+\frac{1}{2} \sum_{[A]\in\mathcal{M}_\pi(2,2)} (-1)^{\dim Sf_{(2,2)}(A)} \cdot \frac{1}{3} \operatorname{ind}_{(2,2)^\perp}(A) \cdot \Big(\frac{1}{3} \operatorname{ind}_{(2,2)^\perp}(A) + 1\Big)$$

$$+\frac{1}{2} \sum_{[A]\in\mathcal{M}_\pi((1,2),(2,1))} (-1)^{\dim Sf_{((1,2),(2,1))}(A)-1} \cdot \frac{1}{4} \operatorname{ind}^{(2)}_{((1,2),(2,1))^\perp}(A)$$

$$\cdot \Big(\frac{1}{4} \operatorname{ind}^{(2)}_{((1,2),(2,1))^\perp}(A) + 1\Big).$$

Table of Contents

Local models of bifurcations

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- If G acts linearly on $\mathbb{R}^n$, and if the action of G is transitive on the unit sphere, then $f_t(x) = \|x\|^2 - t\|x\|^4$ is G–Morse when $t \neq 0$.

Local models of bifurcations

If G acts linearly on $\mathbb{R}^n$, but the action of G is not necessarily transitive on the unit sphere, then we can obtain a local bifurcation model by further perturbing $f_t(x) = \|x\|^2 - t\|x\|^4$ when $t > 0$.

Local models of bifurcations

- After perturbation, the function f_t is G–Morse for $t \neq 0$. When $t < 0$, it has one critical orbit at 0; when $t > 0$, it has one critical orbit at 0 and a finite set of critical orbits on the sphere with radius $\sqrt{\frac{t}{2}}$ corresponding to the critical orbits of a G–Morse function on the unit sphere.

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- We can also extend f_t to a function F_t on $\mathbb{R}^n \oplus \mathbb{R}^m$, where $F_t(x, y) = F(x) \pm \|y\|^2$.

Local models of bifurcations

If the action of G on M is not linear, the at each point there is a local slice. The previous construction gives a local bifurcation model on the slice, which extends to a neighborhood of the orbit.

Precise statement of the main result

Theorem (finite-dimensional case)

Every 1-parameter family between two G–Morse functions can be C^0–perturbed to a smooth family, such that there are only finitely many degeneracies, and every degeneracy is either a birth-death singularity, or a bifurcation as described above.

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Theorem (finite-dimensional case)

Every 1-parameter family between two G–Morse functions can be C^0–perturbed to a smooth family, such that there are only finitely many degeneracies, and every degeneracy is either a birth-death singularity, or a bifurcation as described above.

Remark

The classical Cerf theorem states that the desired property can be achieved by a generic C^∞–perturbation. However, our result only holds for C^0–perturbations. In fact, the function $f_t(z) := t|z|^2 + \mathrm{Re}(z^3)$ on $\mathbb{C} \cong \mathbb{R}^2$ cannot be perturbed in C^2 so that all the bifurcations are birth-death or as given above.

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The same statement holds for 1-parameter families of perturbed Chern-Simons functionals.

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The same statement holds for 1-parameter families of perturbed Chern-Simons functionals.

For perturbed Chern-Simons functionals, the index is replaced by the spectral flow.

Table of Contents

Equivariant transversality

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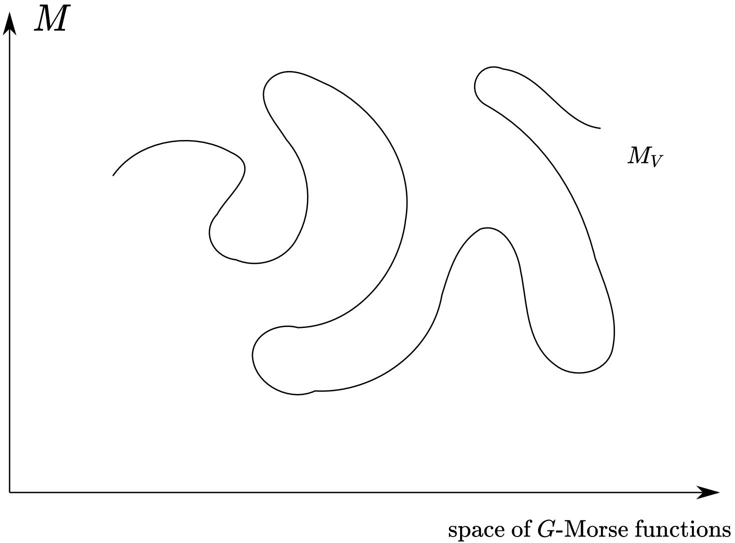
- Let $\mathcal{M}$ be the space of (f, p), where f is a G–Morse function, p is a point on M.
- In the infinite-dimensional case, f is a perturbed Chern-Simons functional, p is a connection.
- Consider the space $\mathcal{M}_V$ of (f, p), such that $\ker \mathrm{Hess}_p f$ is isomorphic to a given representation V of $\mathrm{Stab}\,(p)$.

Equivariant transversality



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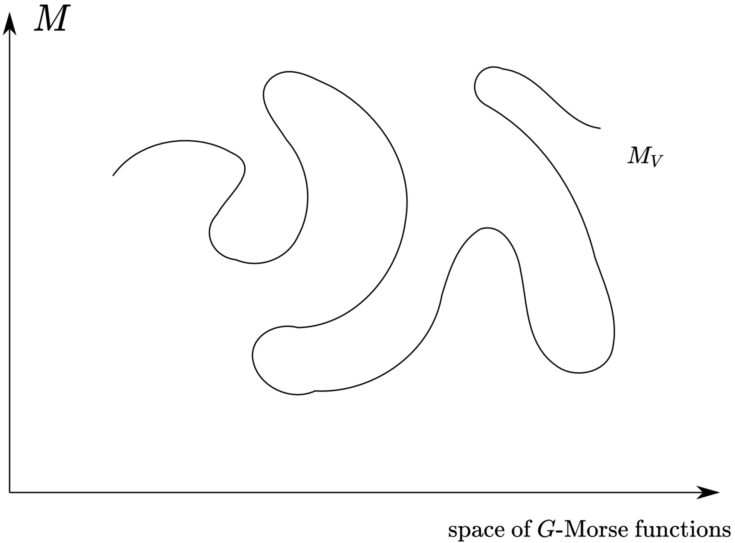

- $\mathcal{M} = \sqcup \mathcal{M}_V$.

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Equivariant transversality

- The space $\mathcal{M}_V \subset \mathcal{M}$ is regular.
- The projection of $\mathcal{M}_V$ to the first component (i.e. the affine space of the G–Morse functions) has negative index unless $V = 0$.
- The index is -1 if and only if V is irreducible.

Equivariant transversality

- As a consequence, a generic path of G–Morse functions intersects the images of $\mathcal{M}_V$ for $V \neq 0$ at only countably many degeneracies, and is disjoint from the image of $\mathcal{M}_V$ if V is reducible.

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- Therefore, a generic path only consists of countably many degeneracies, and $\ker \mathrm{Hess}_p f$ is an irreducible representation of $\mathrm{Stab}\,(p)$ at every degeneracy.

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- As a consequence, a generic path of G–Morse functions intersects the images of $\mathcal{M}_V$ for $V \neq 0$ at only countably many degeneracies, and is disjoint from the image of $\mathcal{M}_V$ if V is reducible.
- Therefore, a generic path only consists of countably many degeneracies, and $\ker \mathrm{Hess}_p f$ is an irreducible representation of $\mathrm{Stab}\,(p)$ at every degeneracy.
- We can further improve the argument to show that every degenerate point is isolated and they form a closed set, therefore there are only finitely many degeneracies.

Table of Contents

Kuranishi's reduction argument

- If $F : \mathcal{M}_1 \to \mathcal{M}_2$ is a smooth Fredholm map between (possibly finite-dimensional) Banach manifolds, suppose $F(x) = y$, then locally $F^{-1}(y)$ is modeled by the zero set of a map from $\ker df(x)$ to $\operatorname{coker} df(x)$.

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- Apply Kuranishi reduction to the gradients of the G–Morse function or the perturbed Chern-Simons functional.
- Reduces to the finite dimensional case where M is given by an irreducible representation of G.

Induction

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- If the G–action on M is trivial, then it follows from the classical Cerf theory.
- By the previous arguments, we may assume that M is an irreducible representation of G, therefore the Hessians of f_0 and f_1 at the origin are scalar multiples of the identity.

Induction

- If the Hessians of f_0 and f_1 have the same sign at the origin, we can modify the family such that there is no degeneracy at the origin, therefore the the result follows from the induction hypothesis on $M - B(\epsilon)$.

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- If the Hessians of f_0 and f_1 have the same sign at the origin, we can modify the family such that there is no degeneracy at the origin, therefore the the result follows from the induction hypothesis on $M - B(\epsilon)$.
- If the Hessians of f_0 and f_1 have opposite signs at the origin, reduce to the previous case by concatenating the family with a standard bifurcation.

Table of Contents

Further questions

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- The topology of equivariant diffeomorphism groups.

Thanks!