Bifurcation of embedded curves and Gopakumar–Vafa invariants

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(Based on joint work with Mohan Swaminathan)

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Overview

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   - Gromov–Witten invariants
   - The Gopakumar–Vafa formula and BPS invariants
2. Background-II
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   - Wendl's Theorem
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Gromov–Witten invariants

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  $\rightsquigarrow$ Gromov–Witten invariant $\mathrm{GW}_{A,g} \in \mathbb{Q}$, independent of J.
- Because of maps with non-trivial automorphism groups, these invariants are not $\mathbb{Z}$-valued and don't directly enumerate curves.

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Question

How to define integer-valued counts of curves in CY 3-folds?

GV formula and BPS invariants

Conjecture (Gopakumar–Vafa '98)

There exist numbers $BPS_{A,h}$ for all $h \geq 0$ and $A \in H_2(X, \mathbb{Z})$ satisfying the following identity

$$\sum_{A \neq 0, g \geq 0} GW_{A,g} t^{2g-2} q^A = \sum_{A \neq 0, h \geq 0} BPS_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \left(\frac{kt}{2} \right) \right)^{2h-2} q^{kA}$$

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such that

- *(integrality) $BPS_{A,h} \in \mathbb{Z}$;*
- *(finiteness) for fixed A, $BPS_{A,h} = 0$ for $h \gg 1$.*

Remark: the numbers $\mathrm{GW}_{A,g}$ and $\mathrm{BPS}_{A,h}$ uniquely determine each other via the above formula.

GV formula and BPS invariants

Theorem (Ionel–Parker, 2018, Integrality)

There exist integers $BPS_{A,h}$ for $h \geq 0$ and $A \in H_2(X, \mathbb{Z})$ satisfying the Gopakumar–Vafa formula.

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For any $A \in H_2(X, \mathbb{Z})$, we have $BPS_{A,h} = 0$ for $h \gg 1$.

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For any $A \in H_2(X, \mathbb{Z})$*, we have* $BPS_{A,h} = 0$ *for* $h \gg 1$*.*

- Both proofs use the symplectic geometry.
- Problem: the proofs do not illustrate the geometric meaning of $\mathrm{BPS}_{A,h}$.

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- Their approaches were made rigorous by Maulik–Toda.
- Informally speaking, $\mathrm{BPS}_{A,h}$ is conjectured as a count of sheaves, via the perverse sheaf of vanishing cycles.
- Issues: hard to check deformation invariance; hard to verify the GV formula except very special examples.

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A reminder of notation: denote by $J(X, \omega)$ the space of almost complex structures on X which are compatible with ω.

The super-rigidity conjecture

Fact

Away from a codimension 2 subset of $\mathcal{J}(X, \omega)$, all **simple** *holomorphic curves are* **embedded** *and have* **pairwise disjoint images**.

- Restrict attention to J as in the above fact.

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Fact

Away from a codimension 2 subset of $\mathcal{J}(X, \omega)$, all **simple** *holomorphic curves are* **embedded** *and have* **pairwise disjoint images**.

- Restrict attention to J as in the above fact.
- Any non-constant J-holomorphic stable map $f' : \Sigma' \to X$ can then be factored uniquely as

$$\Sigma' \xrightarrow{\varphi} \Sigma \xrightarrow{f} X$$

where Σ is a smooth closed Riemann surface, f is a J-holomorphic embeddeing and φ is holomorphic.

The super-rigidity conjecture

Definition (Super-rigidity)

$J \in \mathcal{J}(X, \omega)$ is called **super-rigid** if, for all stable J-holomorphic maps

$$\Sigma' \xrightarrow{\varphi} \Sigma \subset X$$

we have $\ker(\varphi^* D^N_{\Sigma,J}) = 0$, where $D^N_{\Sigma,J}$ is the **normal Cauchy–Riemann operator** of the embedded J-curve $\Sigma \subset X$.

- Morally speaking, if J is super-rigid, any deformation of a J-holomorphic map $u : \Sigma \to X$ comes from deforming the branched cover part of the factorization $\Sigma \to \tilde{\Sigma} \to X$.
- In particular, any embedded curve is *rigid*.

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Proposition

Suppose J is super-rigid. Given $A \in H_2(X, \mathbb{Z})$ and $g \geq 0$, there are just finitely many embedded J-holomorphic curves of genus g and class A.

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Sketch of proof.

Argue by contradiction: given a sequence of distinct J-holomorphic curves Σ_n of genus g and class A, Gromov compactness shows they converge to a J-holomorphic stable map with image contained in some embedded curve Σ_∞. One can "rescale" this sequence along the normal bundle of Σ_∞ to construct a non-trivial element in $\ker(\varphi^* D^N_{\Sigma_\infty, J})$ for some $\Sigma' \xrightarrow{\varphi} \Sigma$. This violates the super-rigidity of J. □

The super-rigidity conjecture

- If J is super-rigid, then given any sequence of embedded J_n-curves $\Sigma_n \subset X$ (of bounded genus and area), with $J_n \to J$, we can find a subsequence converging to an embedded J-curve $\Sigma \subset X$.

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- This allows us to separate embedded curves from multiple covers!

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- It has a long history of false proofs.
- This conjecture was settled by Wendl in 2019.

Wendl's Theorem

Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $\mathcal{J}(X, \omega)$ where super-rigidity fails has codimension ≥ 1. In particular, a generic J is super-rigid.

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- The actual result determines the codimensions of the various strata of this subset (corresponding to the Galois group of the covers involved and their representations).
- This suggests one could try to define $\mathbb{Z}$-valued counts of embedded curves using super-rigid J.

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- Even if there are finitely many embedded curves, a naive count may not be deformation invariant: bifurcations could happen when varying $J \in J(X, \omega)$.
- The comparison with GW invariants requires extra work.

Results

- Our recent paper (arXiv:2106.01206) addresses parts of these issues.
- We study the bifurcations in the space of embedded curves which occur when we cross one of the "walls" from Wendl's theorem.
- We study how the (Euler numbers of) obstruction bundles change under some simple bifurcations. This leads to the comparison with GW invariants.

Bifurcations (I)

Theorem A (B–Swaminathan., 2021)

Let $\{J_t\}_{t\in[-1,1]}$ be a generic path in $\mathcal{J}(X,\omega)$. Assume that there exists an embedded rigid J_0-curve $\Sigma \subset X$ along with a d-fold genus h branched multiple cover $\varphi : \Sigma' \to \Sigma$ which has non-trivial normal deformations. If this cover determines an **elementary wall type**, then $\mathrm{Aut}(\varphi) \subset \mathbb{Z}/2\mathbb{Z}$ and the change in the signed count of embedded curves of genus h and class $d[\Sigma]$ near φ is given by $\pm 2/|\mathrm{Aut}(\varphi)|$.

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- The technical condition of "elementary wall type" is satisfied by a large class of branched covers. For example, this includes all d-fold covers $\Sigma' \to \Sigma$ with generalized automorphism group S_d.

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- For the proof, we study the local structure near $(J_0, \varphi : \Sigma' \to \Sigma \subset X)$ of the moduli space

$$\overline{\mathcal{M}}_h(X, \{J_t\}, dA)$$

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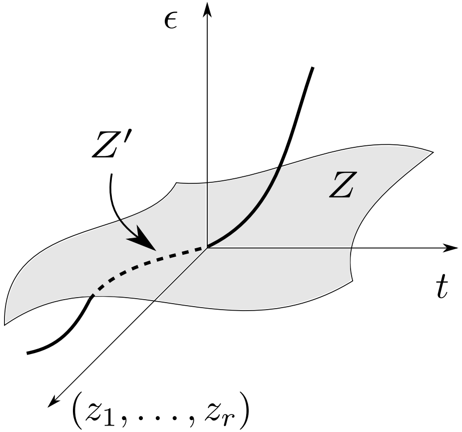
where $A = [\Sigma] \in H_2(X, \mathbb{Z})$.
- We obtain a local Kuranishi model by applying the implicit function theorem. We then analyze the first few terms in the Taylor expansion of the Kuranishi map to complete the proof.

Schematic picture of the local Kuranishi model



$(z_1, \dots, z_r)$ are coordinates on $T_\varphi \mathcal{M}_h(\Sigma, d)$, ϵ is a coordinate $\ker(\varphi^* D^N_{\Sigma,J})$ and Z, Z' are the local irreducible components of the moduli space.

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- The almost complex structure J defines a Cauchy–Riemann operator $D^N_{\Sigma,J}$ on $N_X\Sigma$ which is the deformation operator.
- Many objects relevant to Gromov–Witten theory could be discussed locally along $N_X\Sigma$.
- In particular, the tangent-obstruction theory of the moduli spaces of stable maps comes from $D^N_{\Sigma,J}$.

Obstruction bundles

Definition

A Cauchy–Riemann operator D on N is said to be **super-rigid**, if $\ker(\varphi^* D) = 0$ for all (possibly branched) holomorphic covers $\varphi : \Sigma' \to \Sigma$. For super-rigid D and integers $d \geq 2$ and $h \geq 0$, we define the (canonically oriented) **cokernel bundle** $\mathcal{N}^{(d,h)}_{\Sigma,D} \to \overline{\mathcal{M}}_h(\Sigma, d)$ by

$$[\varphi : \Sigma' \to \Sigma] \mapsto \operatorname{coker}(\varphi^* D).$$

Since, $\operatorname{vdim} \overline{\mathcal{M}}_h(\Sigma, d) = \operatorname{rank} \mathcal{N}^{(d,h)}_{\Sigma,D}$, this bundle has a well-defined **virtual Euler number** $e_{d,h}(D) \in \mathbb{Q}$.

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The number $e_{d,h}(D)$ arises from local contribution from Σ to the Gromov–Witten invariant $\mathrm{GW}_{d[\Sigma],h}$.

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Theorem B (B–Swaminathan, 2021)

Let $\mathcal{D} = \{D_t\}_{t \in [-1,1]}$ be a generic 1-parameter family of Cauchy–Riemann operators on N. Assume that $([\varphi : \Sigma' \to \Sigma], t) \mapsto \mathrm{coker}(\varphi^* D_t)$ gives a vector bundle of the expected rank on the space

$$\overline{\mathcal{M}}_h(\Sigma, d) \times [-1, 1] \setminus \Delta \times \{0\}$$

where $\Delta \subset \mathcal{M}_h(\Sigma, d)$ is a finite set where super-rigidity fails for D_0. Then,

$$e_{d,h}(D_+) - e_{d,h}(D_-) = \sum_{p \in \Delta} \frac{2 \cdot \mathrm{sgn}(\mathcal{D}, p)}{|\mathrm{Aut}(p)|}$$

with $\mathrm{sgn}(\mathcal{D}, p) \in \{-1, +1\}$ determined by the behavior of $\mathcal{D}$ near p.

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This is the wall-crossing formula for local Gromov–Witten invariants.

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- The argument is based on a standard cobordism-type discussion, combined with a local finite dimensional reduction to a model case.
- It's possible to proof the assertion using other frameworks, such as Ruan–Tian's perturbative definition of GW invariants.

An application

Theorem C (B–Swaminathan, 2021)

Given any primitive homology class $A \in H_2(X, \mathbb{Z})$, the number $\mathrm{BPS}_{2A,g}(X) \in \mathbb{Z}$ is a weighted count of embedded J-holomorphic genus g curves (of classes $2A$ and A) when J is super-rigid for $g = 0, 1$.

- Our construction showcases the fact that the GW invariants are not enumerative in general: curves with ghost components contribute nontrivially.

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- Our construction showcases the fact that the GW invariants are not enumerative in general: curves with ghost components contribute nontrivially.
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- In reality, the genus 1 BPS number $\mathrm{BPS}_{2A,1}(X)$ is a weighted sum of virtual counts of embedded genus 1 curves, reflecting the "ghost contribution" of genus 0 curves.
- We need to use Zinger's reduced GW invariants of genus 1 to make the comparison. There is no existing construction of reduced invariants for $g \geq 2$, which serves an obstruction of generalizing our result.

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- Symplectic invariance of this definition follows from Theorem A.
- The verification of the GV formula uses Theorem B and the standard computation of $e_{2,0}(\bar{\partial})$ for $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

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We hope to address this in future work. For curves with representing more general classes in $H_2(X, \mathbb{Z})$, a concrete expression of the weighted sum formula also requires extra work.

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One can put our bifurcation analysis into the context of wall-crossings of calibrated submanifolds. See the recent speculation of Donaldson on special Lagrangian submanifolds.

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A first step to answer this question seems to require a compactification of moduli space of curves different from the stable map compactification.

Thank you!