ON THE LINEAR STABILITY OF BLACK HOLES

The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unravelled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate.
[S. Chandrasekhar]

1. Problem of evolution

2. Kerr spacetimes

3. Stability of Minkowski space

4. Linear stability of Kerr

5. Main results

PROBLEM OF EVOLUTION

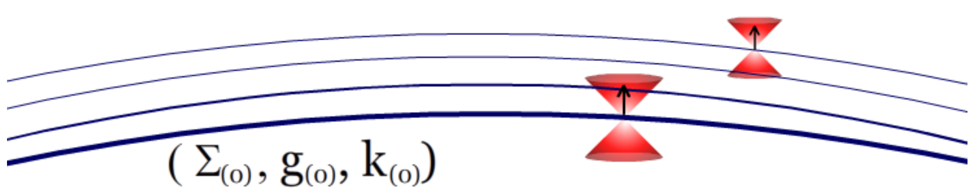
$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}(\Phi)$$

Data $(\Sigma_{(0)}, g_{(0)}, k_{(0)}, \Phi_{(0)})+$ constraints

Asympt. flatness (AF)

Developments i : Data $\longrightarrow (\mathcal{M}, g, \Phi)$

Vacuum $\text{Ric}(g) = 0.$



MGFHD Maximal Global Future Hyperbolic Development

EXPLICIT SOLUTIONS

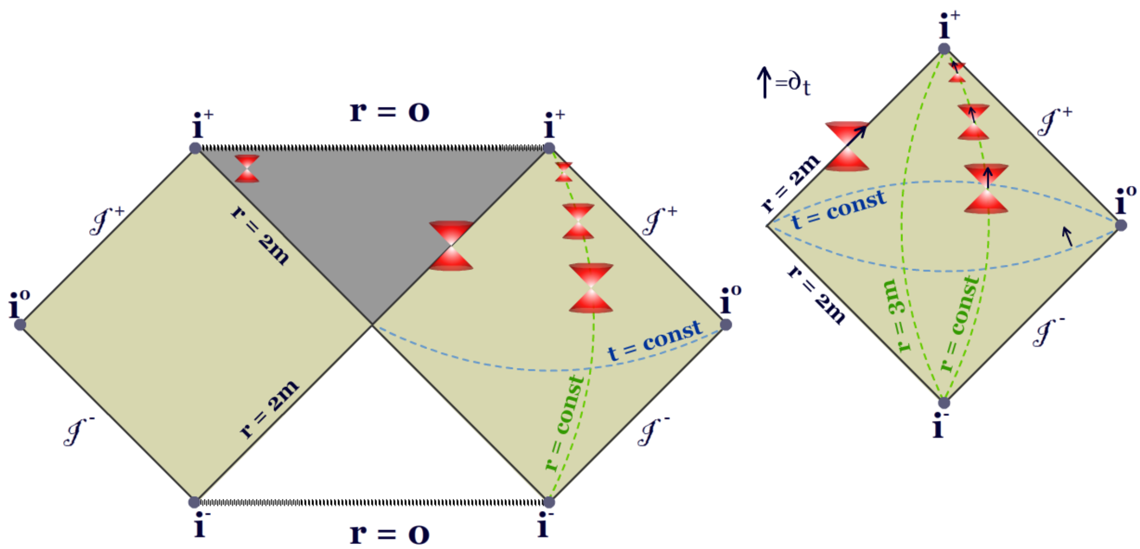
- Minkowski $\mathbb{R}^{1+3} = \mathcal{K}(0,0)$
- Schwarzschild $\mathcal{K}(0,m)$
- Kerr $\mathcal{K}(a,m)$, $0 \leq a < m$.

QUESTIONS

1. Are there other AF, stationary, solutions ?
2. Are Kerr spacetimes stable ?

SCHWARZSCHILD SPACETIME

$$-(1-\frac{2m}{r})dt^2+(1-\frac{2m}{r})^{-1}dr^2+r^2d\sigma^2_{\mathbb{S}^2}$$



- Event horizon $r = 2m$,
- Black and white holes $r < 2m$
- Exterior domains $r > 2m$.
- Photon sphere $r = 3m$.

KERR SPACETIMES

$\mathcal{K}_{m,a}, \quad 0 \le a < m.$

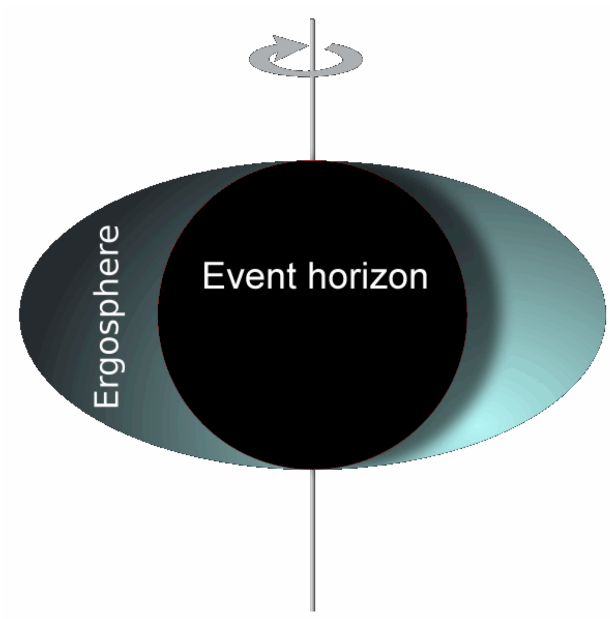
$$-\frac{\rho^2\Delta^2}{\Sigma^2}dt^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2\theta}{\rho^2}(d\phi - \frac{2amr}{\Sigma^2}dt)^2$$

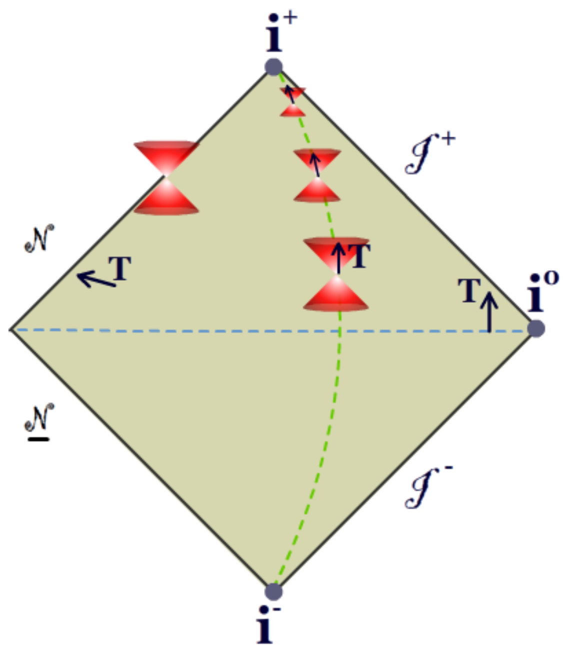
$$\begin{aligned} \Delta &= r^2 + a^2 - 2mr, \qquad a < m \\ \rho^2 &= r^2 + a^2 \cos^2\theta \\ \Sigma^2 &= (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta \end{aligned}$$

- **Horizon** $\quad r = r_+ := m + \sqrt{m^2 - a^2}.$

- **Black Hole** $\quad r < r_+$

KERR SPACETIME

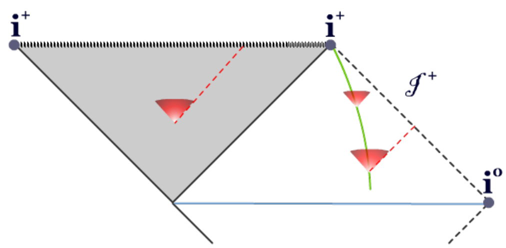




$$r_+ = m + \sqrt{m^2 - a^2}, \;\; r_e = m + \sqrt{m^2 - a^2 \cos^2\theta}$$

- **Exterior domain** $\quad r \geq r_+$
- **Ergo-region** $\quad r_+ \leq r \leq r_e$
- **Photon-region** $\quad r\Delta = m(r^2 - a^2)$

COSMIC CENSORSHIP AND STABILITY OF KERR



Null geodesics in and outside black holes.

Weak Cosmic Censorship. *Generic asymptotically flat initial data have MFGHD with complete future null infinity.*

Global stability of Kerr. *Small perturbations of Kerr initial data have MFGHD with a complete future null infinity which, within its DOC, behaves asymptotically like (another) Kerr.*

STABILITY OF MINKOWSKI SPACE

Theorem. *[Chr-Kl] Any asymptotically flat initial data set which is sufficiently close to the trivial one has a regular MFGHD.*

Main ideas

1. Cannot prove stability without *robust* decay.

2. To prove robust decay one needs approximate symmetries

3. To construct approximate symmetries one needs control of causal geometry

4. To control causal geometry one needs precise decay (*peeling*) for the curvature tensor.

VECTORFIELD METHOD

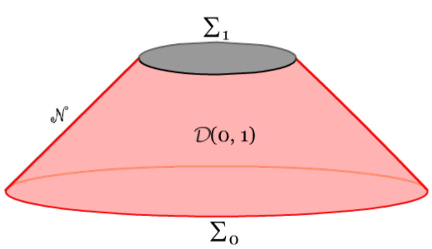
I. Generalized energy method
II. Commuting vectorfieds method

$$\Box_g \phi = 0, \qquad L(\phi) = g^{\mu\nu} D_\mu \phi D_\nu \phi$$
$$Q_{\alpha\beta} = D_\alpha \phi D_\beta \phi - \frac{1}{2} g_{\alpha\beta} L(\phi)$$

- Q is symmetric

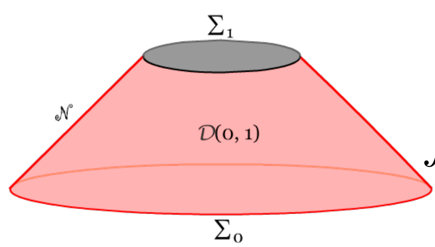
- Q is divergenceless

- $Q(X,Y) > 0$ if X, Y timelike, f- oriented



$$Q(T,T) \geq c|D\phi|^2$$
$$Q(T,L) \geq c|L\phi|^2 + |\nabla\!\!\!/ \phi|^2$$

GENERALIZED ENERGY



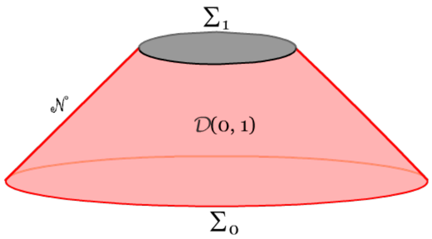
Main Lemma

$$\int_{\mathcal{N}} Q_w(X, L) + \int_{\Sigma_1} Q_w(X, T) = \int_{\Sigma_0} Q_w(X, T) - \int_{\mathcal{D}(0,1)} \text{Err}$$

Here X vectorfield, w scalar

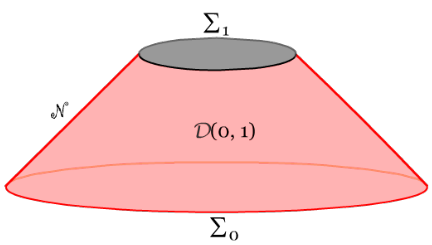
$$\begin{aligned} Q(X, Y) &= X(\phi)Y(\phi) - \frac{1}{2}g(X, Y)L(\phi) \\ Q_w(X, Y) &= Q(X, Y) + \frac{1}{2}w\,\phi\,Y(\phi) - \frac{1}{4}Y(w)\phi^2 \\ \text{Err}(w, X) &= \frac{1}{2}\Big(Q \cdot \mathcal{L}_X g + w\,L(\phi)\Big) - \frac{1}{4}\Box(w)\,\phi^2 \end{aligned}$$

Example 1. $\mathcal{L}_X g = 0, \quad g(X, X) < 0, \quad w = 0.$



$$\int_{\mathcal{N}} Q(X, L) + \int_{\Sigma_1} Q(X, T) = \int_{\Sigma_0} Q(X, T)$$

Example 2. $\mathcal{L}_X g = \Omega g, \; g(X, X) < 0, \; w = \Omega \frac{d-1}{2}.$



$$\int_{\mathcal{N}} Q_w(X, L) + \int_{\Sigma_1} Q_w(X, T) = \int_{\Sigma_0} Q_w(X, T)$$

Example 3. $\mathrm{Err}(w, X) \geq 0$

II. Commuting vectorfields. $\pi = \mathcal{L}_X g$

$$\begin{aligned} \Box_g(\mathcal{L}_X \phi) &= \mathcal{L}_X(\Box_g \phi) - \pi^{\alpha\beta} D_\alpha D_\beta \phi \\ &- \Big(2D^\beta \pi_{\alpha\beta} - D_\alpha(\mathrm{tr}\pi)\Big) D^\alpha \phi \end{aligned}$$

SYMMETRIES AND DECAY IN MINKOWSKI SPACE $\mathbb{R}^{d+1}$

Theorem. There exists an expression $\mathcal{Q}[\phi](t)$, constructed by the *vectorfield method*, such that $\mathcal{Q}[\phi](t) = \mathcal{Q}[\phi](0)$ if $\Box\phi = 0$ and, with $u = t - |x|, \underline{u} = t + |x|$,

$$|\phi(t,x)| \le c \frac{1}{(1+\underline{u})^{\frac{n-1}{2}}(1+|u|)^{\frac{1}{2}}} \sup_{t\ge 0} \mathcal{Q}[\phi](t)$$

- Generators of translations : $\mathbb{T}_\mu = \frac{\partial}{\partial x^\mu}$.
- Generators of rotations $\mathbb{L}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$.
- Generator of scaling: $\mathbb{S} = x^\mu \partial_\mu$.
- Generators of inverted translations $\mathbb{K}_\mu = 2x_\mu x^\rho \frac{\partial}{\partial x^\rho} - (x^\rho x_\rho)\frac{\partial}{\partial x^\mu}$.

LINEAR STABILITY OF KERR $\mathcal{K}(a, m)$

Can the vectorfield method still be applied ?

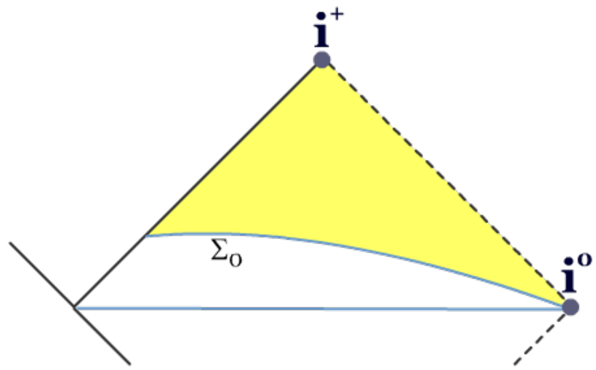
- Only two linearly independent Killing vectorfields, $\mathbb{T}$ and $\mathbb{Z}$

- $\mathbb{T}$ becomes space-like in the ergo-region. Even for $a = 0$, $\mathbb{T}$ becomes null on the horizon. Thus $Q(\mathbb{T}, T)$ is degenerate for any t-like T.

- Trapped null geodesics

MAIN RESULTS

Theorem 1. $(\mathcal{M}, g)$ smooth, stationary , A.F. Any solution of $\Box_g \phi = 0$, with reasonable data on Σ_0 is bounded in the colored region.

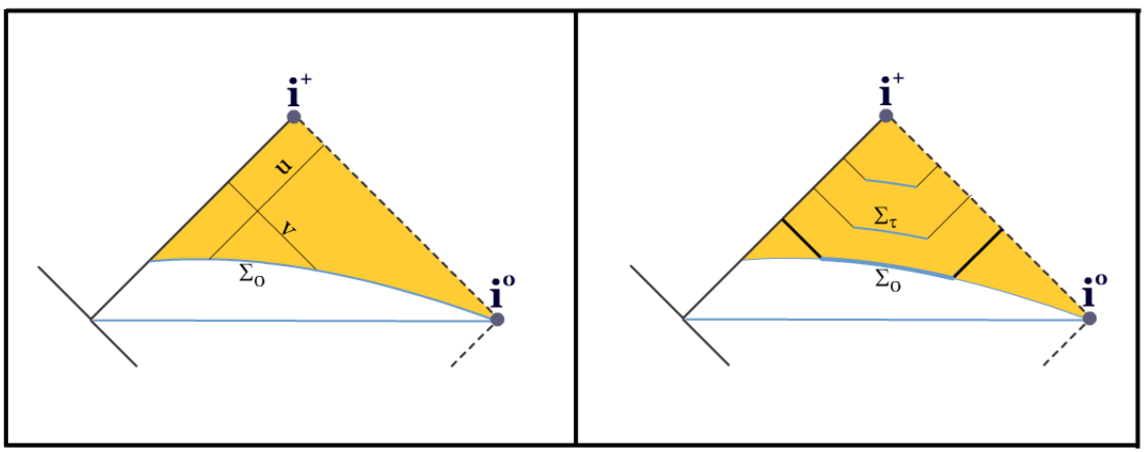


Theorem 2.[Decay in Schwarzschild] Consider $\Box_g \phi = 0$ in $\mathcal{K}(0, m)$ with data on Σ_0. Then, with $u = t - r^*$, $\underline{u} = t + r^*$,

$$|\phi| \le \frac{C}{\underline{u}}, \qquad |r\phi| \le \frac{C_R}{|u|^{1/2}}, \;\; r \ge R > 2m$$

Theorem[Decay in Kerr, $0 \leq a << m$] Any solution of $\Box\phi = 0$ decays uniformly at the rates,

$$|r^{1/2}\phi| \leq C\tau^{-1+\delta}, \qquad |r\phi| \leq C\tau^{-\frac{1-\delta}{2}}$$



Decay for $\mathcal{K}(0, m)$ with respect to $u = t - r^*$ and $v = \underline{u} = t + r^*$.

Decay for $\mathcal{K}(a, m)$ with respect to a $\mathbb{T}$-equivariant foliation Σ_τ.

MAIN IDEAS

1. Red shift vectorfield

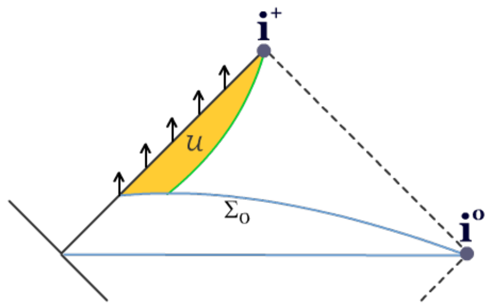
2. Modified Morawetz vectorfield

3. Decompose into super-radiant and sub-radiant frequencies

4. Patching of non-causal vectorfields

5. New mechanism for decay

RED SHIFT VECTORFIELD



Proposition[Dafermos-Rodnianski] Event horizon $\mathcal{N}$ of a regular, asymptotically flat stationary spacetime admits a $\mathbb{T}$- invariant neighborhood $\mathcal{U}$ of $\mathcal{N}$ and a strictly time-like, smooth, vector-field $\mathbb{H}$ on $\mathcal{U}$, both invariant with respect to the $\mathbb{T}$ flow ϕ_τ, $\tau \geq 0$, such that, for a constant $c > 0$,

$$^{(\mathbb{H})}\pi \cdot Q \geq c\, Q(\mathbb{H}, \mathbb{H})$$

Moreover, given any $\Lambda > 0$, we can choose $\mathbb{H}$ such that, all along $\mathcal{N}$,

$$^{(\mathbb{H})}\pi \cdot \mathbf{Q} \geq c\, e_3(\phi)^2 + \Lambda\Big((e_4(\phi)^2 + |\nabla\!\!\!/ \phi|^2\Big)$$

MODIFIED MORAWETZ VECTORFIELD

Idea: Find $X = f\partial_{r^*}$, $w = w(f)$,

$$\mathrm{Err}(\phi; w, X) \geq 0, \quad \text{at } r = 3m$$

$$r^* := r + 2m\log(r-2m) - 3m - 2m\log m.$$

- $f = 1, \quad w = \frac{\mu}{r}, \quad \mu = 1 - \frac{2m}{r}$:

$$\frac{1}{2} Q_{(w)} \cdot {}^{(X)}\pi = \frac{r-3m}{r^2} |\nabla\!\!\!/ \phi|^2$$

- $X = f(r^*)\partial_{r^*}$, $w = f' + \frac{2\mu}{r}$

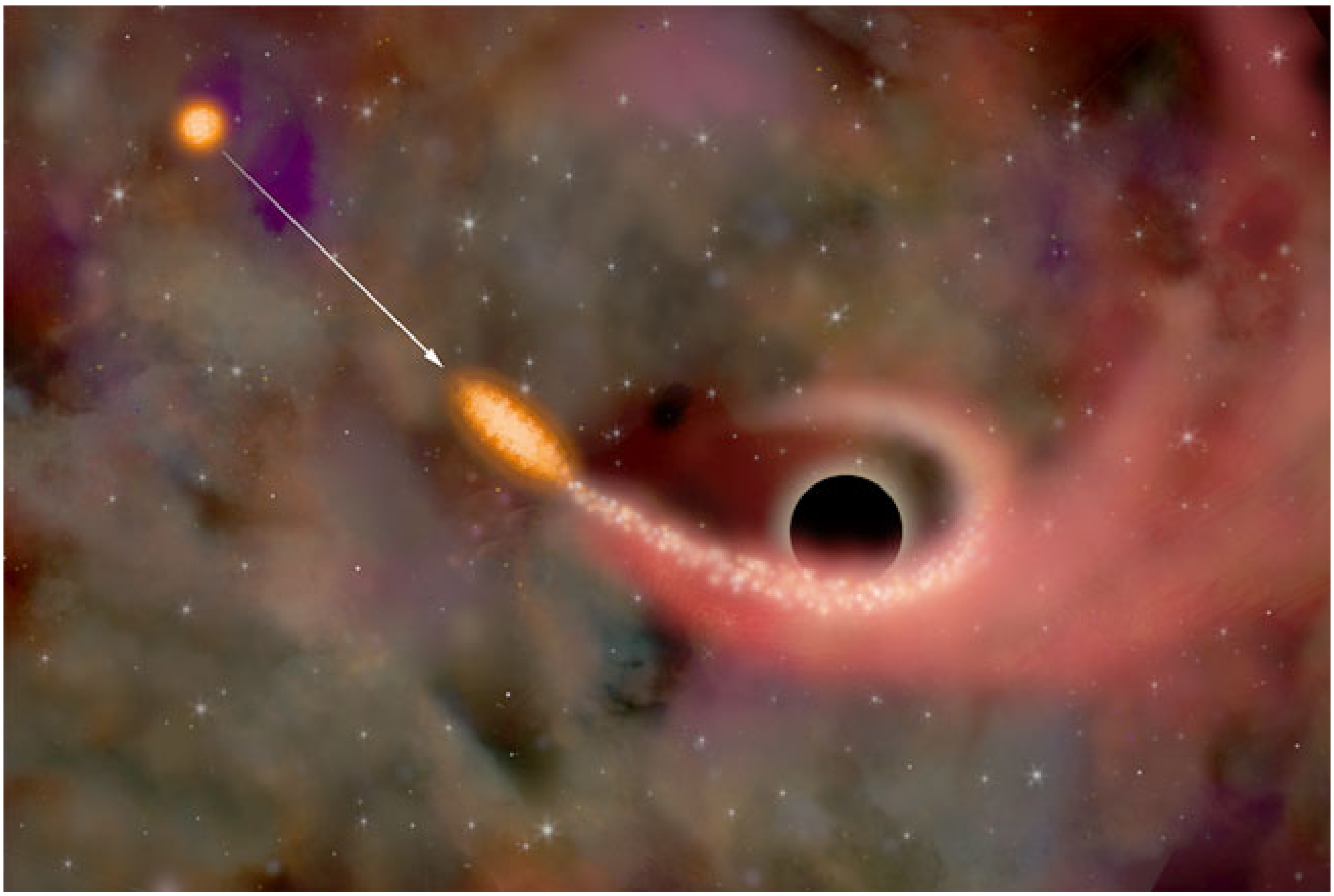
$$\mathrm{Err}(w, X) = f\frac{r-3m}{r^2}|\nabla\!\!\!/ \phi|^2 + f'\mu^{-1}(\partial_{r^*}\phi)^2 - \frac{1}{4}\Delta(w)\phi^2$$

Want: $f' \geq 0$, $\;f\,\frac{r-3m}{r^2} \geq 0$, $\;\Delta w \leq 0$.

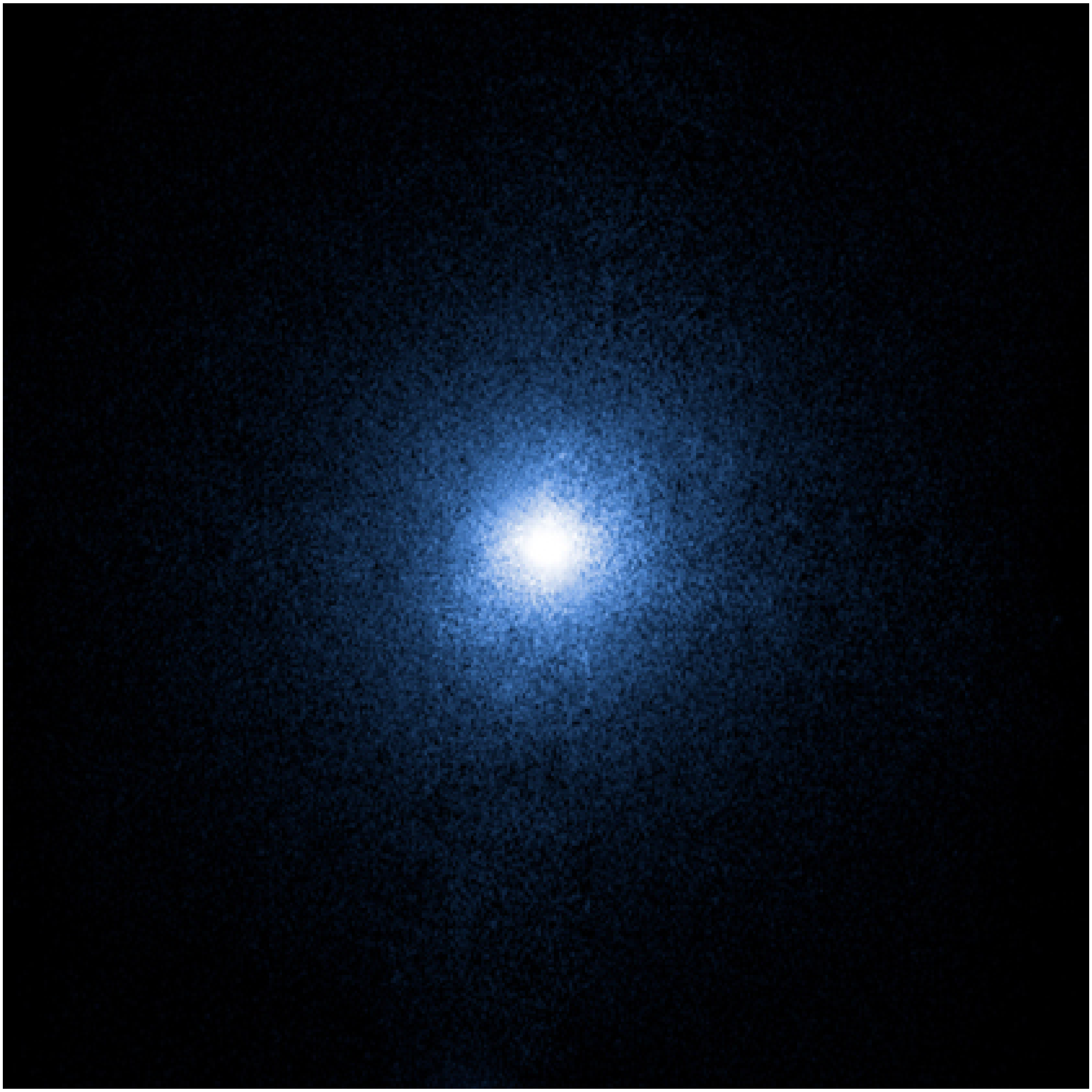
Can be done near $r = 3m$.

BOUNDEDNESS THEORM

- $\mathbb{T}$ is everywhere time-like.
- $\mathbb{T}$ becomes null on the horizon.
- $\mathbb{T}$ becomes space-like near the horizon



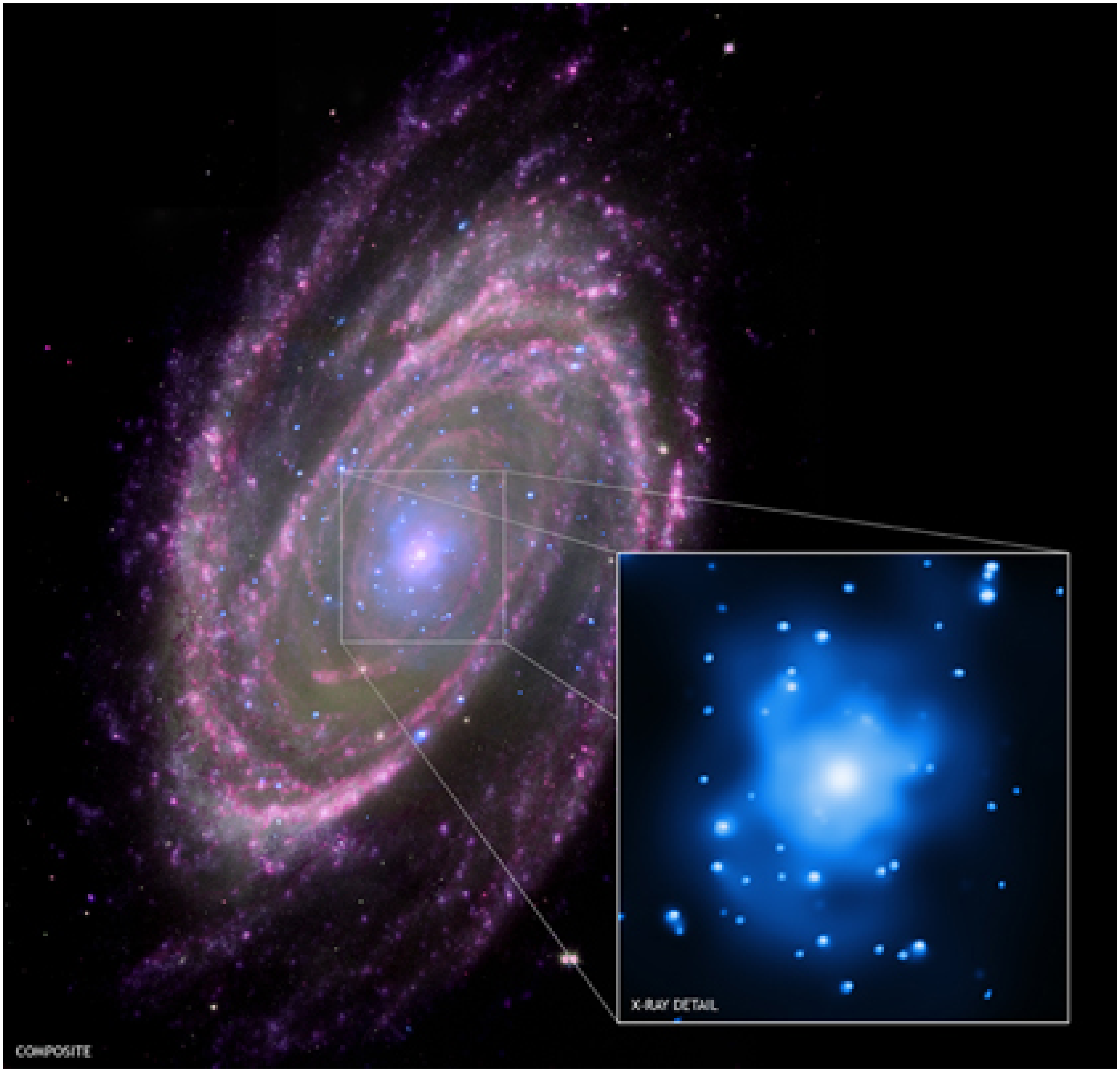
X-Rays Indicate Star Ripped Up by Black Hole
Illustration Credit: M. Weiss, CXC, NASA



Credit: NASA/CXC/SAO riginally discovered in 1964, Cygnus X-1 has been observed intensely since

In the 1970s, X-ray and optical observations led to the conclusion that Cygnus X-1 contained a black hole, the first one identified

Because it is only 6,000 light years from Earth, Cygnus X-1 is a very bright and therefore a good target for astronomers to study he Cygnus X-1 system consists of a black hole with a mass about 10 times that of the Sun in a close orbit with a blue supergiant star with a mass of about 20 Suns. Gas flowing away from the supergiant in a fast stellar wind is focused by the black hole, and some of this gas forms a disk that spirals into the black hole. The gravitational energy release by this infalling gas powers the X-ray emission from Cygnus X-1.



Credit: X-ray: NASA/CXC/Wisconsin/D.Pooley

This composite NASA image of the spiral galaxy M81, located about 12 million light years away, includes X-ray data from the Chandra X-ray Observatory (blue), optical data from the Hubble Space Telescope (green), infrared data from the Spitzer Space Telescope (pink) and ultraviolet data from GALEX (purple). The inset shows a close-up of the Chandra image. At the center of M81 is a supermassive black hole that is about 70 million times more massive than the Sun.