BILINEAR ESTIMATES ON CURVED SPACE-TIMES

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Abstract. To settle the L^2 bonded curvature conjecture for the Einstein-vaccum equations one needs to prove bilinear type estimates for solutions of the homogeneous wave equation on a fixed background with H^2 local regularity. In this paper we introduce a notion of primitive parametrix for the homogeneous wave equation for which we can prove, under very broad assumptions, the desired bilinear estimates.

1. introduction

In this paper we address the issue of proving bilinear estimates for solutions of the wave equation,

$$\square_{\mathbf{g}}\phi = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha\big(\mathbf{g}^{\alpha\beta}\partial_\beta\phi\big) = 0, \tag{1}$$

in an asymptotically flat curved spacetime $(\mathcal{M}, \mathbf{g})$ with limited regularity. We assume that $\mathcal{M}$ is endowed with a time function t whose level hypersurfaces define a spacelike foliation Σ_t. We are interested in solving the initial value problem for (1),

$$\phi[0] := \big(\phi(0), \partial_t\phi(0)\big) = (\phi_0, \phi_1) \tag{2}$$

for given functions ϕ_0, ϕ_1 on Σ_0. We shall assume the existence of coordinates x^1, x^2, x^3 on Σ_0 relative to which

$$\|\phi[0]\|_{H^s} := \|\phi_0\|_{H^s(\Sigma_0)} + \|\phi_1\|_{H^{s-1}(\Sigma_0)} < \infty \tag{3}$$

for some $s \le 2$.

To motivate our results consider the flat case when $(\mathcal{M}, \mathbf{g})$ is the four dimensional Minkowski space, t its canonical time function and $\square$ the flat D'Alembertian. We recall the following result, see [Kl-Ma],

Theorem 1.1. *Consider ϕ, ψ solutions of the flat wave equation*

$$\square\phi = \square\psi = 0$$

and Q one of the following null forms,

$$Q_0(\phi,\psi) = \partial^\alpha\phi\cdot\partial_\alpha\psi, \qquad Q_{\alpha\beta}(\phi,\psi) = \partial_\alpha\phi\cdot\partial_\beta\psi - \partial_\beta\phi\cdot\partial_\alpha\psi, \quad \forall\alpha\neq\beta.$$

Then,

$$\|Q(\phi,\psi)\|_{L^2(\mathbb{R}^{3+1})} \lesssim \|\phi[0]\|_{H^2(\mathbb{R}^3)}\cdot\|\psi[0]\|_{H^1(\mathbb{R}^3)} \tag{4}$$

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In what follows we shall give an equivalent formulation of the estimate (4) in terms of the energy fluxes acros a family of null hypersurfaces.

To every unit vector $\omega \in \mathbb{S}^2$ we consider the family of functions

$$u_\omega = t - x \cdot \omega \tag{5}$$

Observe that each $u = u_\omega$ is an optical function, i.e.,

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u\, \partial_\beta u = 0. \tag{6}$$

The level hypersurfaces ${}^{(\omega)}\mathcal{H}$ are null hyperplanes. Let

$$ {}^{(\omega)}N = \omega \cdot \nabla = \omega^1\partial_1 + \omega^2\partial_2 + \omega^3\partial_3$$

be the vectorfield tangent to Σ_t perpendicular to the direction ω. Observe that ${}^{(\omega)}N(u_\omega) = -1$ and

$$ {}^{(\omega)}L = \partial_t + \omega \cdot \nabla = \partial_t + {}^{(\omega)}N \tag{7}$$

is the null generator vectorfield of ${}^{(\omega)}\mathcal{H}$. Consider also the null conjugate vectorfield,

$$ {}^{(\omega)}\underline{L} = \partial_t - \omega \cdot \nabla = \partial_t - {}^{(\omega)}N. \tag{8}$$

Clearly ${}^{(\omega)}L$, ${}^{(\omega)}\underline{L}$ form a null pair, i.e. $< {}^{(\omega)}L, {}^{(\omega)}\underline{L} >= -2$. Denote by ${}^{(\omega)}S_{t,u}$ the 2 dimensional surfaces of intersection between ${}^{(\omega)}\mathcal{H}_u$ and Σ_t and by ${}^{(\omega)}\nabla$ the induced covariant derivative on ${}^{(\omega)}S_{t,u}$. On each point on ${}^{(\omega)}S_{t,u}$ we can choose an orthonormal frame $({}^{(\omega)}e_a)_{a=1,2}$. Together with ${}^{(\omega)}L = {}^{(\omega)}e_4$ and ${}^{(\omega)}\underline{L} = {}^{(\omega)}e_3$ they form a null frame ${}^{(\omega)}e_1$, ${}^{(\omega)}e_2$, ${}^{(\omega)}e_3$, ${}^{(\omega)}e_4$ at the particular point.

Given a function f we denote,

$$|{}^{(\omega)}\overline{\nabla} f|^2 = |{}^{(\omega)}\nabla f|^2 + |{}^{(\omega)}Lf|^2 = \sum_{a=1,2} |{}^{(\omega)}e_a(f)|^2 + |{}^{(\omega)}L(f)|^2 \tag{9}$$

Remark 1.2. The null function conjugate to $u_\omega = t - x\cdot\omega$ is $\underline{u}_\omega = t + x\cdot\omega$. Observe that $\underline{u}_\omega = u_{-\omega}$. Also remark that ${}^{(\omega)}\underline{L}$ is the null generator of the null folition generated by $\underline{u}_\omega$. Together u_ω, $\underline{u}_\omega$ define a canonical double null foliation.

According to the standard energy inequality for the flat wave equation $\Box$ we have, for each $\omega \in \mathbb{S}^2$,

$$\sup_u \int_{{}^{(\omega)}\mathcal{H}_u} |{}^{(\omega)}\overline{\nabla}\phi|^2 = E[\phi] \tag{10}$$

where $E[\phi]$ denotes the total energy of ϕ i.e.,

$$E[\phi] = \int_{\Sigma_0} \big(|\partial_t\phi|^2 + |\nabla\phi|^2\big)dx \tag{11}$$

We are now ready to reformulate (4)

$$\|Q(\phi,\psi)\|_{L^2(\mathbb{R}^{3+1})} \lesssim \|\phi[0]\|_{H^2} \cdot \big(\sup_{u,\omega}\int_{{}^{(\omega)}\mathcal{H}_u} |{}^{(\omega)}\overline{\nabla}\psi|^2\big)^{\frac12} \tag{12}$$

This is the form of the standard bilinear estimates in Minkowski space which we shall try to generalize to nonflat backgrounds.

In what follows we sketch the proof of (12) for the null forms Q_{ij}. The other null forms can be treated in the same manner. As well known the general solution of the initial value problem

$$\Box\phi = 0, \qquad \phi[0] = (\varphi_0, \varphi_1)$$

can be written in the form,

$$\begin{aligned}
\phi(t,x) &= \phi^+ + \phi^- = \int e^{i(t|\xi|+x\cdot\xi)}\widehat{f_+}(\xi)d\xi + \int e^{i(t|\xi|-x\cdot\xi)}\widehat{f_-}(\xi)d\xi\\
\widehat{f^+}(\xi) &= \frac{1}{2}\big(\widehat{\varphi_0}(\xi) + \frac{1}{|\xi|}\widehat{\varphi_1}(\xi)\big)\\
\widehat{f^-}(\xi) &= \frac{1}{2}\big(\widehat{\varphi_0}(\xi) - \frac{1}{|\xi|}\widehat{\varphi_1}(\xi)\big)
\end{aligned}$$

Introducing spherical coordinates $\xi = \lambda\omega$, $\lambda = |\xi|$, $\omega \in \mathbb{S}^2$ we can rewrite,

$$\begin{aligned}
\phi^-(t,x) &= \int e^{i\lambda(t-x\cdot\omega)}\widehat{f_+}(\lambda\omega)\lambda^2 d\lambda d\omega = \int e^{i\lambda u_\omega(t,x)}\widehat{f_+}(\lambda\omega)\lambda^2 d\lambda d\omega\\
\phi^+(t,x) &= \int e^{i\lambda(t+x\cdot\omega)}\widehat{f_+}(\lambda\omega)\lambda^2 d\lambda d\omega = \int e^{i\lambda u_{-\omega}(t,x)}\widehat{f_-}(\lambda\omega)\lambda^2 d\lambda d\omega
\end{aligned}$$

Observe now that the proof (12) for $Q(\phi,\psi)$ reduces to the following:

Proposition 1.3. *Let*

$$\Phi_f(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u_\omega(t,x)}\lambda^2\hat{f}(\lambda\omega)d\lambda d\omega \tag{13}$$

be a special, complex valued, solution of the free wave equation. Then,

$$Q(\phi^{(a)},\psi) \lesssim \|f\|_{H^2}\cdot\big(\sup_{u,\omega}\int_{{}^{(\omega)}\mathcal{H}_u} |{}^{(\omega)}\overline{\nabla}\psi|^2\big)^{\frac{1}{2}} \tag{14}$$

Proof : We easily calculate,

$$\begin{aligned}
Q_{ij}(\phi_a,\psi) &= \int_{\mathbb{S}^2}\int_{\mathbb{R}_+} Q_{ij}(\,u_\omega,\psi)e^{i\lambda\,u_\omega}\lambda^3\hat{f}(\lambda\,\omega)\,d\lambda\,d\omega\\
&= \int_{\mathbb{S}^2} Q_{ij}(\,u_\omega,\psi)\big(\int_{\mathbb{R}_+} e^{i\lambda\,u_\omega}\lambda^3\hat{f}(\lambda\,\omega)\,d\lambda\big)d\omega
\end{aligned} \tag{15}$$

To calculate $Q_{ij}(\,u_\omega,\psi)$ we expand as follows

$$\partial_i = \omega_i\ {}^{(\omega)}N + \sum_{a=1,2} X_{ai}\ {}^{(\omega)}e_a, \qquad X_{ai} =< \partial_i,\ {}^{(\omega)}e_a >$$

Observe that $\sum_i \omega^i X_{ai} = 0$ and $\delta_{ij} =< \partial_i, \partial_j >= \omega_i\,\omega_j + \sum_{a=1,2} X_{ai}X_{aj}$. Also, since

$${}^{(\omega)}e_a = \sum_i < \partial_i,\ {}^{(\omega)}e_a > \partial_i = \sum_i X_{ai}\,\partial_i$$

and $<\ {}^{(\omega)}e_a,\ {}^{(\omega)}e_b >= \delta_{ab}$ we deduce,

$$\sum_{i=1,2,3} X_{ai}X_{bi} = \delta_{ab}, \quad \sum_{a=1,2} X_{ai}X_{aj} = \delta_{ij} - \omega_i\,\omega_j, \quad \sum_i \omega^i X_{ai} = 0 \tag{16}$$

Since ${}^{(\omega)}N(u_\omega) = -1$ and ${}^{(\omega)}e(u_\omega) = 0$, $\partial_i u_\omega = -\omega_i$

$$
\begin{aligned}
Q_{ij}(u_\omega, \psi) &= \partial_i u_\omega \, \partial_j \psi - \partial_j u_\omega \, \partial_i \psi \\
&= -\omega_i(\omega_j N + \sum_{a=1,2} X_{aj} \, {}^{(\omega)}e_a)\psi + \omega_j(\omega_i N + \sum_{a=1,2} X_{ai} \, {}^{(\omega)}e_a)\psi \\
&= -\sum_{a=1,2} (\omega_i X_{aj} - \omega_j X_{ai}) \, {}^{(\omega)}e_a \psi
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|Q(\phi, \psi)|^2 &= \sum_{i,j=1,2,3} |Q_{ij}(u_\omega, \psi)| = 2 \sum_{a,b=1,2} \sum_i X_{ai} X_{bi} \, {}^{(\omega)}e_a \psi \, {}^{(\omega)}e_b \psi \\
&= \delta_{ab} \, {}^{(\omega)}e_a \psi \, {}^{(\omega)}e_b \psi = |{}^{(\omega)}\overline{\nabla}\psi|^2
\end{aligned}
$$

Thus, returning to (15),

$$
\begin{aligned}
|Q(\phi_a, \psi)| &\leq \int_{\mathbb{S}^2} |Q(u_\omega, \psi)| \, |\int_{\mathbb{R}_+} e^{i\lambda u_\omega} \lambda^3 \hat{f}(\lambda \omega) \, d\lambda| \, d\omega \\
&\leq \int_{\mathbb{S}^2} |{}^{(\omega)}\overline{\nabla}\psi| \, |\int_{\mathbb{R}_+} e^{i\lambda u_\omega} \lambda^3 \hat{f}(\lambda \omega) \, d\lambda| \, d\omega \\
&= \int_{\mathbb{S}^2} |{}^{(\omega)}\overline{\nabla}\psi| \, |J(u_\omega, \omega)| d\omega
\end{aligned}
$$

where,

$$
J(u, \omega) = \int_{\mathbb{R}_+} e^{i\lambda u} \lambda^3 \hat{f}(\lambda \omega) \, d\lambda \tag{17}
$$

depends only on u for every fixed ω. Now, by the Minkowski inequality,

$$
\|Q(\phi, \psi)\|_{L^2(\mathbb{R}^{3+1})} \leq \int_{\mathbb{S}^2} \| \, |{}^{(\omega)}\overline{\nabla}\psi| \, J(u_\omega, \omega) \, \|_{L^2(\mathbb{R}^{3+1})} \tag{18}
$$

For each fixed ω, we express the volume integral $dtdx$ in $\mathbb{R}^{3+1}$ with respect to the variables u_ω, t and the volume element dA_ω on ${}^{(\omega)}S_{t,u}$,

$$
dtdx = dt \, du \, dA_\omega.
$$

Observe that $dtdA_\omega$ is precisely the volume element on the null hypersurface ${}^{(\omega)}\mathcal{H}_u$. Thus, applying the Plancherel formula to the function $J(u_\omega, \omega)$ defined by (17),

$$
\begin{aligned}
\| \, |{}^{(\omega)}\overline{\nabla}\psi| \, J(u_\omega, \omega) \, \|^2_{L^2(\mathbb{R}^{3+1}_+)} &= \int_0^\infty \int_{-\infty}^{+\infty} \int_{{}^{(\omega)}S_{t,u}} |{}^{(\omega)}\overline{\nabla}\psi|^2 \, |J(u, \omega)|^2 dt du dA_\omega \\
&= \int_{-\infty}^{+\infty} |J(u, \omega)|^2 \left(\int_{{}^{(\omega)}\mathcal{H}_u} |{}^{(\omega)}\overline{\nabla}\psi|^2 \right) du \\
&\lesssim \sup_{u,\omega} \int_{{}^{(\omega)}\mathcal{H}_u} |{}^{(\omega)}\overline{\nabla}\psi|^2 \cdot \int_{-\infty}^{+\infty} |J(u, \omega)|^2 du \\
&= \sup_{u,\omega} \int_{{}^{(\omega)}\mathcal{H}_u} |{}^{(\omega)}\overline{\nabla}\psi|^2 \int_0^\infty \lambda^6 |\hat{f}(\lambda\omega)|^2 d\lambda
\end{aligned}
$$

Thus, returning to (18),

$$\begin{aligned}\|Q(\phi,\psi)\|_{L^2({\mathbb{R}_+}^{3+1})} &\le \sup_{u,\omega}\Big(\int_{^{(\omega)}\mathcal{H}_u} |\,^{(\omega)}\overline{\nabla}\psi|^2\Big)^{\frac12}\Big(\int_{\mathbb{S}^2}\int_0^\infty \lambda^6|\hat{f}(\lambda\omega)|^2 d\lambda\Big)^{\frac12}\\ &\lesssim \|\nabla^2 f\|_{L^2(\mathbb{R}^3)}\cdot\sup_{u,\omega}\Big(\int_{^{(\omega)}\mathcal{H}_u} |\,^{(\omega)}\overline{\nabla}\psi|^2\Big)^{\frac12}\end{aligned}$$

as desired. ■

Remark 1.4. The representation

$$\Phi_f = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u_\omega}\widehat{f}(\lambda\omega)\,d\lambda d\omega$$

corresponds to the decomposition of a solution of the homogeneous wave equation into a superposition of traveling waves

$$^{(\omega)}\Phi_f := \int_0^\infty e^{i\lambda u_\omega}\widehat{f}(\lambda\omega)\,d\lambda.$$

This decomposition was used, along similar lines, by D. Tataru in the proof of bilinear product estimates in Minkowski space in his pioneering work on the wave maps equation [Ta]. The goal of this work is to point out that expressing solutions of the wave equation as a superposition of travelling waves does not only provide a quick proof of the bilinear estimate (14) in flat space but, more to the point, it provides the right framework for generalization to the wave equation on a curved spacetime background with limited regularity.

2. Geometric set-up.

In order to formulate the precise version of the bilinear estimate on a curved background we first need to introduce the relevant geometric objects.

1). *Space-like foliation:* We foliate space-time $(\mathcal{M},\mathbf{g})$ by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. We denote by T the unit future oriented normal to Σ_t and define the lapse function n of the foliation Σ_t according to the formula

$$n^{-1} = \nabla_T t. \tag{19}$$

The second fundamental form k_{ij} of the Σ_t foliation is given by

$$k_{ij} = -\frac{1}{2}\mathcal{L}_T\mathbf{g}_{ij} \tag{20}$$

2). *Optical functions and null hyperplanes:* We assume given a family of null hypersurfaces $(\,u_\omega)$ with $\omega\in\mathbb{S}^2$,

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\, u_\omega\;\partial_\beta\, u_\omega = 0. \tag{21}$$

defineed in a canonical way such that u_ω become the flat null hypersurfaces $u_\omega - t - x\cdot\omega$ in the flat case. This can be achieved, for example, by assuming that on Σ_0 the level surfaces of $u_\omega(0,x)$ are minimal planes asymptotic to $x\cdot\omega$. More precisely, relative to coordinates x^i on Σ_0 for which the metric is asymptotically flat, i.e. $g_{ij} - (1+2M/r)\delta_{ij} = O(r^{-1})$ as $r \to 0$, we have

$$u_\omega(0,x) - x\cdot\omega = O(1), \quad \text{as } r \to \infty$$

The level hypersufaces $u_\omega = u$ of the optical function u_ω are null hyperpsurfaces associated with the direction ω which we denote by ${}^{(\omega)}\mathcal{H}_u$. The null geodesic generator of ${}^{(\omega)}\mathcal{H}_u$ is given by,

$$^{(\omega)}L = -\mathbf{g}^{\alpha\beta}\partial_a\, u_\omega \partial_\beta. \tag{22}$$

3.) *Induced t-foliation of ${}^{(\omega)}\mathcal{H}_u$:* The 2-surfaces ${}^{(\omega)}S_{t,u} = \Sigma_t \cap {}^{(\omega)}\mathcal{H}_u$ form a foliation of the hull hypersurfaces ${}^{(\omega)}\mathcal{H}_u$. The null lapse ${}^{(\omega)}a$ of this foliation of is given by

$$^{(\omega)}a^{-1} = - < {}^{(\omega)}L, T >= T(u_\omega). \tag{23}$$

We denote by ${}^{(\omega)}\gamma$ the restriction of the metric $\mathbf{g}$ to ${}^{(\omega)}S_{t,u}$ and by ${}^{(\omega)}\nabla$ its induced covariant derivative.

4.) *Null pair:* Recall the definition of the null geodesic vectorfield, see (22), ${}^{(\omega)}L = -\mathbf{g}^{\alpha\beta}\partial_\alpha u_\omega\, \partial_\beta$. Clearly, ${}^{(\omega)}L(u_\omega) = 0$. Moreover, since $\langle\, {}^{(\omega)}L, T\,\rangle = -\nabla_T\, u_\omega = -\,{}^{(\omega)}a^{-1}$, it follows that

$$^{(\omega)}L = {}^{(\omega)}a^{-1}(T + {}^{(\omega)}N),$$

where ${}^{(\omega)}N$ is the outward unit normal to ${}^{(\omega)}S_{t,u}$ in Σ_t.

At any point $P \in {}^{(\omega)}S_{t,u} \subset {}^{(\omega)}\mathcal{H}_u$ we denote by $\underline{L}_\omega$ the null vector conjugate to L relative to the t-foliation, i.e.

$$\langle L_\omega, \underline{L}_\omega\rangle = -2\,, \qquad \langle \underline{L}_\omega, X\rangle = 0 \quad \text{for all } X \in T_p(S_{t,u_\omega}).$$

It follows by a simple calculation that

$$^{(\omega)}\underline{L} = {}^{(\omega)}a(T - {}^{(\omega)}N) = {}^{(\omega)}a(\,{}^{(\omega)}a\,{}^{(\omega)}L + 2T). \tag{24}$$

Thus,

$$\langle\, {}^{(\omega)}L,\, {}^{(\omega)}\underline{L}\rangle = -\,{}^{(\omega)}\underline{L}(u_\omega) = -2\,{}^{(\omega)}aT(\,u_\omega) = -2 \tag{25}$$

Therefore $L, \underline{L}$ form a null pair, which we call the canonical null pair associated to the t, u_ω-foliation. An arbitrary orthonormal frame on ${}^{(\omega)}S_{t,u}$ will be denoted by $(\,{}^{(\omega)}e_a)_{a=1,2}$. Clearly,

$$\langle\, {}^{(\omega)}e_a,\, {}^{(\omega)}L\rangle = \langle\, {}^{(\omega)}e_a,\, {}^{(\omega)}\underline{L}\rangle = 0, \quad \langle\, {}^{(\omega)}e_a,\, {}^{(\omega)}e_b\rangle = \delta_{ab}.$$

Together with the null pair ${}^{(\omega)}e_4 = {}^{(\omega)}L$ and ${}^{(\omega)}e_3 = {}^{(\omega)}\underline{L}$ we obtain a null frame, ${}^{(\omega)}e_1$, ${}^{(\omega)}e_1$, ${}^{(\omega)}e_3$, ${}^{(\omega)}e_4$.

Given a function f we denote,

$$|\,{}^{(\omega)}\overline{\nabla} f|^2 = |\,{}^{(\omega)}\nabla f|^2 + |\,{}^{(\omega)}Lf|^2 = \sum_{a=1,2} |\,{}^{(\omega)}e_a(f)|^2 + |\,{}^{(\omega)}L(f)|^2 \tag{26}$$

5.) *Null connection coeficients:* The null second fundamental forms ${}^{(\omega)}\chi$, ${}^{(\omega)}\underline{\chi}$ of the foliation S_{t,u_ω} are given by

$$ {}^{(\omega)}\chi_{ab} = \langle \nabla_{{}^{(\omega)}e_a} {}^{(\omega)}L\,,\ {}^{(\omega)}e_b\rangle, \qquad \underline{\chi}_{ab} = \langle \nabla_{{}^{(\omega)}e_a} {}^{(\omega)}\underline{L}\,,\ {}^{(\omega)}e_b\rangle \tag{27} $$

The torsion is given by,

$$ {}^{(\omega)}\zeta_a = \frac{1}{2}\langle \nabla_{{}^{(\omega)}e_a} {}^{(\omega)}L\,,\ {}^{(\omega)}\underline{L}\rangle \tag{28} $$

We also denote ${}^{(\omega)}\mathrm{tr}\chi = \delta^{ab}\,{}^{(\omega)}\chi_{ab}$ and ${}^{(\omega)}\hat{\chi} = {}^{(\omega)}\hat{\chi} - \frac{1}{2}\,{}^{(\omega)}\mathrm{tr}\chi\delta$.

3. Bilinear estimates for primitive parametrices

To prove bilnear estimates analogous to those of proposition 1.3 we need to make certain assumptions for our spacetime $(\mathcal{M}, \mathbf{g})$.

Assumption 1. We assume that there exists a global system of coordinates t, x^1, x^2, x^3 on $(\mathcal{M}, \mathbf{g})$ relative to which the metric takes the form,

$$ ds^2 = -n^2 dt^2 + g_{ij} dx^i dx^j. $$

We assume that the lapse n verifies, $c < n(t,x) \le c^{-1}$ for some fixed $c > 0$, uniformly in (t,x). Moreover we shall assume that the induced metric g is euclidean on Σ_0. This last assumption is not compatible with the application we have in mind, i.e. to vacuum Einstein equations, but there are obvious ways to modify the definition below such that it applies to a nonflat initial hypersurface Σ_0. Observe that under this assumption we can pick the initial data for u_ω on Σ_0 such that

$$ u_\omega(0,x) = x\cdot\omega. $$

Assumption 2. For every $\omega \in \mathbb{S}^2$ our spacetime can be foliated, in a regular fashion, by a family of null hypersurfaces ${}^{(\omega)}\mathcal{H}_u$, $-\infty < u < +\infty$. The null lapse ${}^{(\omega)}a$ verifies,

$$ c < {}^{(\omega)}a(t,x) < c^{-1} $$

for some fixed $c > 0$, uniformly in ω, (t,x).

We now consider the following operator,

$$ \Phi_f(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u_\omega(t,x)}\lambda^2 \hat{f}(\lambda\omega) d\lambda d\omega \tag{29} $$

where $\hat{f}$ denotes the standard Fourier transform of f relative to the euclidean coordinates x^i of Σ_0. We shall call this a primitive parametrix, see remark below, solution of the wave equation. To justify the definition we note the following:

Lemma 3.1. *We have the following identity:*

$$ \square_{\mathbf{g}}\Phi_f = \int_{\mathbb{S}^2}\int_{\mathbb{R}_+} {}^{(\omega)}tr\chi(t,x) e^{i\lambda\, u_\omega(t,x)}\lambda^3 \hat{f}(\lambda\,\omega)\, d\lambda\, d\omega \tag{30} $$

Proof : The proof of (30) follows from the definition of the optical function $\mathbf{g}^{\alpha\beta}\partial_\alpha u_\omega \partial_\beta u_\omega = 0$ and the simple calculation $\Box_{\mathbf{g}} u_\omega = {}^{(\omega)}\mathrm{tr}\chi$. ■

Observe that in flat space ${}^{(\omega)}\mathrm{tr}\chi \equiv 0$ and thus Φ_f is an exact solution of the homogeneous wave equation.

Remark 3.2. Formula (29) is the first term in the standard geometric optics approximation. Typically one improves the approximation by solving a sequence of transport equations, see [Ba-Ch] for an application of this construction to quasilinear wave equations. Unfortunately each additional transport equation requires more differentiability for the background metric. Our choice here, to take the most primitive approximation involving only the solution u_ω of the eikonal equation (21), is somewhat related to that made by Smith-Tataru in [Sm-Tat]. However they use a wave packet construction which depends heavily on coordinates. The coordinate dependence manifests itself in the dependence of the accuracy of approximation on the Cristoffel symbols Γ of the metric $\mathbf{g}$, i.e., if w_λ is a wave packet at frequency λ then

$$\Box_{\mathbf{g}} w_\lambda = \Gamma \cdot \partial w_\lambda + ...$$

To show that w_λ is an acceptable approximation requires the Strichartz estimate

$$\|\Gamma\|_{L^2_t L^\infty_x} \lesssim \|\mathbf{g}\|_{L^\infty_t H^s_x}, \qquad s > 2$$

which severely limits the validity of this approximation to the metrics with regularity H^s for $s > 2$. The parametrix introduced in (29) does not depend on a particular choice of coordinates. Moreover, we have shown in [Kl-Rodn1] that the error coefficients ${}^{(\omega)}\mathrm{tr}\chi \in L^\infty_{t,x}$ in (30) for an Einstein metric with the bounded curvature flux, which corresponds to H^2 regularity.

Another crucial difference is that angular and physical space localizations that are part of the wave packet construction require a dyadic frequency localization of the parametrix. In other words, the wave packet construction can only approximate a single dyadic piece $P_\lambda\phi$ of the solution ϕ of a homogeneous wave equation. Here P_λ is a Fourier space projection on the region $\{\xi : \ \lambda \le |\xi| \le 2\lambda\}$ and the wave packet has dimensions $1 \times \lambda^{-1} \times (\lambda^{-\frac{1}{2}})^{n-1}$. This would introduce additional difficulties in the proof of the bilinear estimate for the true solutions of the wave equation due to the lack of orthogonality with respect to λ.

The main result of this paper is the following,

Theorem 3.3. *Assume that the spacetime* $(\mathcal{M}, \mathbf{g})$ *verifies* **Assumptions 1,2** . *Let* $Q_{ij}(\phi,\psi)$ *denote the null forms, relative to our given coordinates* $Q_{ij}(\phi,\psi) = \nabla_i\phi\nabla_j\psi - \nabla_j\phi\nabla_i\psi$. *Then, for any smooth* ψ,

$$\|Q(\Phi_f,\psi)\|_{L^2_t L^2_x} \lesssim \|f\|_{H^2(\Sigma_0)} \cdot \sup_{u,\omega} \| {}^{(\omega)}\overline{\nabla}\psi\|_{L^2({}^{(\omega)}\mathcal{H}_u)}. \tag{31}$$

where $L^2_t L^2_x$ *denotes the spacetime norm in a slab* $0 \le t \le T$,

$$\|f\|^2_{L^2_t L^2_x} = \int_0^T \Big(\int_{\Sigma_t} |f(t,x)|^2 n \sqrt{\det g}\, dx \Big) dt$$

Proof : The proof is exactly as in Minkowski space. We compute

$$Q_{ij}(\Phi_f,\psi)=\int_{\mathbb{S}^2}\int_{\mathbb{R}_+}Q_{ij}(u_\omega,\psi)e^{i\lambda u_\omega}\lambda^3\hat{f}(\lambda\,\omega)\,d\lambda\,d\omega$$

Therefore, defining $|Q|=g^{ik}g^{jl}Q_{ij}Q_{kl}$ where g_{ij} denotes the induced metric on Σ_t and g^{ij} its inverse,

$$|Q(\Phi_f,\psi)|\lesssim\int_{\mathbb{S}^2}|\,Q(u_\omega,\psi)\,|\int_{\mathbb{R}_+}e^{i\lambda u_\omega}\lambda^3\hat{f}(\lambda\,\omega)\,d\lambda\,|d\omega$$

Moreover, we can express the coordinate derivatives ∂_i relative to the orthonormal frame ${}^{(\omega)}N$, ${}^{(\omega)}e$ on Σ_t as follows,

$$\partial_i={}^{(\omega)}N_i\,{}^{(\omega)}N+\sum_{a=1,2}X_{ai}\,{}^{(\omega)}e_a,\qquad {}^{(\omega)}N_i=\langle{}^{(\omega)}N,\partial_i\rangle,\qquad X_{ai}=<\partial_i,{}^{(\omega)}e_a>$$

Since $\langle{}^{(\omega)}e_a,{}^{(\omega)}e_b\rangle=\delta_{ab}$ we easily infer that $X_{ai}X_{bj}g^{ij}=\delta_{ab}$. Also, since $\langle{}^{(\omega)}N\,,\,{}^{(\omega)}N\rangle=1$, $\langle{}^{(\omega)}N\,,\,{}^{(\omega)}e_a\rangle=0$ we easily check that $g^{ij}\,{}^{(\omega)}N_i\cdot X_{ai}=0$. Hence,

$$X_{ai}X_{bj}g^{ij}=\delta_{ab},\qquad g^{ij}\,{}^{(\omega)}N_i\cdot X_{ai}=0. \tag{32}$$

Now, using that

$$\nabla_{{}^{(\omega)}e_a}u_\omega=0,\quad \nabla_{{}^{(\omega)}N}u_\omega=-a^{-1}$$

we have $\partial_i\,u_\omega=-a^{-1}\,{}^{(\omega)}N_i$. Hence,

$$\begin{aligned}Q_{ij}(\,u_\omega,\psi)&=\partial_i\,u_\omega\partial_j\psi-\partial_j\,u_\omega\partial_i\psi\\&=-\,{}^{(\omega)}a^{-1}\big[\,{}^{(\omega)}N_i\big({}^{(\omega)}N_jN(\psi)+X_{aj}\,{}^{(\omega)}e_a(\psi)\big)-{}^{(\omega)}N_j\big({}^{(\omega)}N_iN(\psi)+X_{ai}\,{}^{(\omega)}e_a(\psi)\big)\big]\\&={}^{(\omega)}a^{-1}\big({}^{(\omega)}N_jX_{ai}-{}^{(\omega)}N_iX_{aj}\big)\end{aligned}$$

Using (32) we easily infer that,

$$|Q(u_\omega,\psi)|={}^{(\omega)}a^{-1}|\,{}^{(\omega)}\overline{\nabla}\psi|.$$

We then derive

$$|Q(\Phi_f,\psi)|\quad\lesssim\quad\int_{\mathbb{S}^2}|Q(u_\omega,\psi)|\;|\int_{\mathbb{R}_+}e^{i\lambda u_\omega}\lambda^3\hat{f}(\lambda\,\omega)|=\int_{\mathbb{S}^2}|\,{}^{(\omega)}a^{-1}\,{}^{(\omega)}\overline{\nabla}\psi|\,|J(\,u_\omega,\omega)$$

where,

$$J(\,u_\omega,\omega)=\int_0^\infty e^{i\lambda\,u_\omega}\lambda^3\hat{f}(\lambda\,\omega)\,d\lambda$$

By the Minkowski inequality,

$$\|Q(\Phi_f,\psi)\|_{L^2_tL^2_x}\lesssim\int_{\mathbb{S}^2}d\omega\|\,{}^{(\omega)}a^{-1}\cdot\,{}^{(\omega)}\overline{\nabla}\psi\,\cdot J(\,u_\omega,\omega)\|_{L^2_tL^2_x}$$

We now express the volume element $n\sqrt{\det g}dtdx$ relative to the foliation ${}^{(\omega)}S_{t,u}$ i.e $n\sqrt{\det g}dtdx={}^{(\omega)}a\,n\,dt\,du\,dA_\omega$ with dA_ω denoting the area element on ${}^{(\omega)}S_{t,u}$.

Thus, using our assumptions on ${}^{(\omega)}a, n$,

$$
\begin{aligned}
\| {}^{(\omega)}a^{-1}\, {}^{(\omega)}\overline{\nabla}\psi\, J(u_\omega,\omega)\|^2_{L^2_tL^2_x} &= \int_0^T dt\int_{-\infty}^{+\infty} du\int_{{}^{(\omega)}S_{t,u}} |{}^{(\omega)}\overline{\nabla}\psi|^2\cdot|J(u,\omega)|^2\, {}^{(\omega)}a^{-1}n\;dA_\omega\\
&\lesssim \int_{-\infty}^{+\infty}|J(u,\omega)|^2du\int_0^T dt\int_{{}^{(\omega)}S_{t,u}}|{}^{(\omega)}\overline{\nabla}\psi|^2\cdot\,dA_\omega\\
&\lesssim \int_{-\infty}^{+\infty}|J(u,\omega)|^2du\int_{{}^{(\omega)}\mathcal{H}_u}|{}^{(\omega)}\overline{\nabla}\psi|^2\\
&\lesssim \sup_{u,\omega}\int_{{}^{(\omega)}\mathcal{H}_u}|{}^{(\omega)}\overline{\nabla}\psi|^2\int_{-\infty}^{+\infty}|J(u,\omega)|^2du
\end{aligned}
$$

By Plancherel, $\int_{-\infty}^{+\infty}|J(u,\omega)|^2du=\int_0^\infty\lambda^6|\hat f(\lambda\omega)|^2d\lambda$. Hence,

$$
\begin{aligned}
\|Q(\Phi_f,\psi)\|^2_{L^2_tL^2_x} &\lesssim \sup_{u,\omega}\int_{{}^{(\omega)}\mathcal{H}_u}|{}^{(\omega)}\overline{\nabla}\psi|^2\int_{\mathbb{S}^2}d\omega\int_0^\infty\lambda^6|\hat f(\lambda\omega)|^2d\lambda\\
&\le \|f\|^2_{H^2}\cdot\sup_{u,\omega}\int_{{}^{(\omega)}\mathcal{H}_u}|{}^{(\omega)}\overline{\nabla}\psi|^2
\end{aligned}
$$

as desired. ■

4. Further comments

The result of theorem 3.3 is just a first step in proving bilinear estimates for exact solutions $\square_{\mathbf{g}}\phi=\square_{\mathbf{g}}\psi=0$ in Einstein vacuum spacetime $(\mathcal{M},\mathbf{g})$ with L^2 bounded curvature. The main remaining task is to show that the primitive parametrix Φ_f of (29) can be used to define a good notion of approximate solution to the homogeneous wave equation. To do this we need to show that the $L^2_tH^1_x$ norm of the error term,

$$\int_{\mathbb{S}^2}\int_{\mathbb{R}_+}{}^{(\omega)}\mathrm{tr}\chi(t,x)e^{i\lambda\, u_\omega(t,x)}\lambda^3\hat f(\lambda\,\omega)\,d\lambda\,d\omega$$

in (30) is small in a suitable sense. To verify this we need al lot more information about both Σ_t and ${}^{(\omega)}\mathcal{H}_u$ foliations. In particular it is quite clear that nothing can bee done unless, at least, ${}^{(\omega)}\mathrm{tr}\chi$ is uniformly bounded. To show that, however, is highly nontrivial under our limited regularity assumptions. In the sequence of paper [Kl-Rodn1],[Kl-Rodn2], [Kl-Rodn3] we have developed methods to prove such results, indeed have proved one for the particular case of a geodesic foliation on a fixed null hypersurface verifying only a small curvature flux condition, compatible with H^2 regularity. We have developed those methods with the expectation that they can applied precisely in this context.

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