

Brief introduction to the nonlinear stability of Kerr

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October 28, 2022

Abstract

This is a brief introduction to the sequence of works [63], [39], [61], [62] and [81] which establish the nonlinear stability of Kerr black holes with small angular momentum. We are delighted to dedicate this article to Demetrios Christodoulou for whom we both have great admiration. The first author would also like to thank Demetrios for the magic moments of friendship, discussions and collaboration he enjoyed together with him.

1 Kerr stability conjecture

1.1 Kerr spacetime

Let $(\mathcal{K}(a, m), \mathbf{g}_{a,m})$ denote the family of Kerr spacetimes depending on the parameters m (mass) and a (with $J = am$ angular momentum). In Boyer-Lindquist coordinates the Kerr metric is given by

$$\mathbf{g}_{a,m} = -\frac{q^2 \Delta}{\Sigma^2} (dt)^2 + \frac{\Sigma^2 (\sin \theta)^2}{|q|^2} \left(d\phi - \frac{2amr}{\Sigma^2} dt \right)^2 + \frac{|q|^2}{\Delta} (dr)^2 + |q|^2 (d\theta)^2, \quad (1.1)$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2mr, & q = r + ia \cos \theta, \\ \Sigma^2 = (r^2 + a^2)|q|^2 + 2mra^2(\sin \theta)^2 = (r^2 + a^2)^2 - a^2(\sin \theta)^2 \Delta. \end{cases} \quad (1.2)$$

The asymptotically flat¹ metrics $\mathbf{g}_{a,m}$ verify the Einstein vacuum equations

$$\text{Ric}(\mathbf{g}) = 0, \quad (1.3)$$

¹That is they approach the Minkowski metric for large r .

are stationary and axially symmetric², possess well-defined event horizons $r = r_+$ (the largest root of $\Delta(r) = 0$), domain of outer communication $r > r_+$ and smooth future null infinity \mathcal{I}^+ where $r = +\infty$. The metric can be extended smoothly inside the black hole region, see Figure 1. The boundary $r = r_-$ (the smallest root of $\Delta(r) = 0$) inside the black hole region is a Cauchy horizon across which predictability fails³.

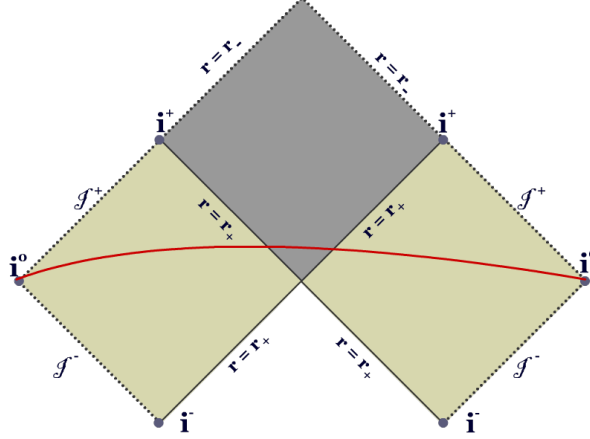


Figure 1: Penrose diagram of Kerr for $0 < |a| < m$. The surface $r = r_+$, the larger root of $\Delta = 0$, is the event horizon of the black hole, $r > r_+$ the domain of outer communication, \mathcal{I}^+ is the future null infinity, corresponding to $r = +\infty$.

Here are some of the most important properties of $\mathcal{K}(a, m)$:

- $\mathcal{K}(a, m)$ possesses a canonical family of null pairs, called *principal null pairs*, of the form $(\lambda e_4, \lambda^{-1} e_3)$, with $\lambda > 0$ an arbitrary scalar function, and

$$e_4 = \frac{r^2 + a^2}{|q|^2} \partial_t + \frac{\Delta}{|q|^2} \partial_r + \frac{a}{|q|^2} \partial_\phi, \quad e_3 = \frac{r^2 + a^2}{\Delta} \partial_t - \partial_r + \frac{a}{\Delta} \partial_\phi. \quad (1.4)$$

- The horizontal structure, perpendicular to e_3, e_4 , denoted \mathcal{H} , is spanned by the vectors

$$e_1 = \frac{1}{|q|} \partial_\theta, \quad e_2 = \frac{a \sin \theta}{|q|} \partial_t + \frac{1}{|q| \sin \theta} \partial_\phi. \quad (1.5)$$

The distribution generated by \mathcal{H} is non-integrable for $a \neq 0$.

²That is $\mathcal{K}(a, m)$ possess two Killing vectorfields: the stationary vectorfield $\mathbf{T} = \partial_t$, which is time-like in the asymptotic region, away from the horizon, and the axial symmetric Killing field $\mathbf{Z} = \partial_\phi$.

³Infinitely many smooth extensions are possible beyond the boundary.

- The horizontal structure (e_3, e_4, \mathcal{H}) has the remarkable property that all components of the Riemann curvature tensor \mathbf{R} , decomposed relative to them, vanish with the exception of those which can be deduced from $\mathbf{R}(e_a, e_3, e_b, e_4)$.
- $\mathcal{K}(a, m)$ possesses the Killing vectorfields \mathbf{T}, \mathbf{Z} which, in BL coordinates, are given by $\mathbf{T} = \partial_t, \mathbf{Z} = \partial_\phi$.
- In addition to the symmetries generated by \mathbf{T}, \mathbf{Z} , $\mathcal{K}(a, m)$ possesses also a non-trivial Killing tensor⁴, i.e. a symmetric 2-tensor $\mathbf{C}_{\alpha\beta}$ verifying the property $\mathbf{D}_{(\gamma}\mathbf{C}_{\alpha\beta)} = 0$. The tensor carries the name of its discoverer B. Carter, see [15], who made use of it to show that the geodesic flow in Kerr is integrable. Its presence, in addition to \mathbf{T} and \mathbf{Z} , as a higher order symmetry, is at the heart of what Chandrasekhar, see [17], called the most striking feature of Kerr, “the separability of all the standard equations of mathematical physics in Kerr geometry”.
- The Carter tensor can be used to define the Carter operator

$$\mathcal{C} = \mathbf{D}_\alpha(\mathbf{C}^{\alpha\beta}\mathbf{D}_\beta), \quad (1.6)$$

a second order operator which commutes with $\square_{a,m}$. This property plays a crucial role in the proof of our stability result, Theorem 1.1, more precisely in Part II of [39].

1.2 Kerr stability conjecture

The discovery of black holes, first as explicit solutions of EVE and later as possible explanations of astrophysical phenomena⁵, has not only revolutionized our understanding of the universe, it also gave mathematicians a monumental task: to test the physical reality of these solutions. This may seem nonsensical since physics tests the reality of its objects by experiments and observations and, as such, needs mathematics to formulate the theory and make quantitative predictions, not to test it. The problem, in this case, is that black holes are by definition non-observable and thus no direct experiments are possible.

⁴Given by the expression $\mathbf{C} = -a^2 \cos^2 \theta \mathbf{g} + O$, $O = |q|^2 (e_1 \otimes e_1 + e_2 \otimes e_2)$.

⁵According to Chandrasekhar “Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent that they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described exactly by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single two parameter family of solutions for their description, they are the simplest as well.”

Astrophysicists ascertain the presence of such objects through indirect observations⁶ and numerical experiments, but both are limited in scope to the range of possible observations or the specific initial conditions in which numerical simulations are conducted. One can rigorously check that the Kerr solutions have vanishing Ricci curvature, that is, their mathematical reality is undeniable. But to be real in a physical sense, they have to satisfy certain properties which, as it turns out, can be neatly formulated in unambiguous mathematical language. Chief among them⁷ is the problem of stability, that is, to show that if the precise initial data corresponding to Kerr are perturbed a bit, the basic features of the corresponding solutions do not change much⁸.

Conjecture (Stability of Kerr conjecture). *Vacuum, asymptotically flat, initial data sets, sufficiently close to $Kerr(a, m)$, $|a|/m < 1$, initial data, have maximal developments with complete future null infinity and with domain of outer communication⁹ which approaches (globally) a nearby Kerr solution.*

1.3 Resolution of the conjecture for slowly rotating black holes

1.3.1 Statement of the main result

The goal of this article is to give a short introduction to our recent result in which we settle the conjecture in the case of slowly rotating Kerr black holes.

Theorem 1.1. *The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a $Kerr(a_0, m_0)$ initial data set, for sufficiently small $|a_0|/m_0$, has a complete future null infinity \mathcal{I}^+ and converges in its causal past $\mathcal{J}^{-1}(\mathcal{I}^+)$ to another nearby Kerr spacetime $Kerr(a_f, m_f)$ with parameters (a_f, m_f) close to the initial ones (a_0, m_0) .*

⁶The physical reality of these objects was recently put to test by LIGO which is supposed to have detected the gravitational waves generated in the final stage of the in-spiraling of two black holes. Rainer Weiss, Barry C. Barish and Kip S. Thorne received the 2017 Nobel prize for their “decisive contributions” in this respect. The 2020 Nobel prize in Physics was awarded to R. Genzel and A. Ghez for providing observational evidence for the presence of super massive black holes in the center of our galaxy, and to R. Penrose for his theoretical foundational work: his concept of a trapped surface and the proof of his famous singularity theorem.

⁷Other such properties concern the rigidity of the Kerr family, see [46] for a current survey, or the dynamical formations of black holes from regular configurations, see the [23], [59] and the introduction to [3] for an up to date account of more recent results.

⁸If the Kerr family would be unstable under perturbations, black holes would be nothing more than mathematical artifacts.

⁹This presupposes the existence of an event horizon. Note that the existence of such an event horizon can only be established a posteriori, upon the completion of the proof of the conjecture.

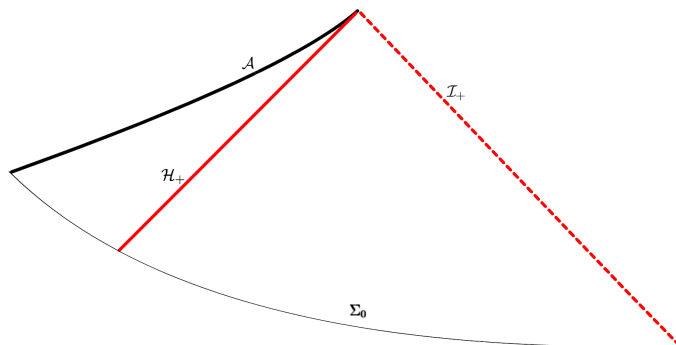


Figure 2: The Penrose diagram of the final space-time in Theorem 1.1 with initial hypersurface Σ_0 , future space-like boundary \mathcal{A} , and \mathcal{I}^+ the complete future null infinity. The hypersurface \mathcal{H}_+ is the future event horizon of the final Kerr.

The precise version of the result, all the main features of the architecture of its proof, as well as detailed proofs for most of the main steps are to be found in [63]. The full proof relies also on our joint work [39] with E. Giorgi, our papers [61], [62] on GCM spheres, and the extension [81] to GCM hypersurfaces by D. Shen.

1.3.2 Brief comments on the proof

We will discuss the main ideas of the proof in more details in section 4. It pays however to give already a graphic sense of the main building blocks of our approach, which we call *general covariant modulated*(GCM), admissible spacetimes.

The main features of these spacetimes $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ are as follows:

- The capstone of the entire construction is the sphere S_* , on the future boundary Σ_* of ${}^{(ext)}\mathcal{M}$, which verifies a set of specific extrinsic and intrinsic conditions denoted by the acronym GCM.
- The spacelike hypersurface Σ_* , initialized at S_* , verifies a set of additional GCM conditions.
- Once Σ_* is specified the whole GCM admissible spacetime \mathcal{M} is determined by a more conventional construction, based on geometric transport type equations¹⁰.

¹⁰More precisely ${}^{(ext)}\mathcal{M}$ can be determined from Σ_* by a specified outgoing foliation terminating in the timelike boundary \mathcal{T} , ${}^{(int)}\mathcal{M}$ is determined from \mathcal{T} by a specified incoming one, and ${}^{(top)}\mathcal{M}$ is a complement of ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ which makes \mathcal{M} a causal domain.

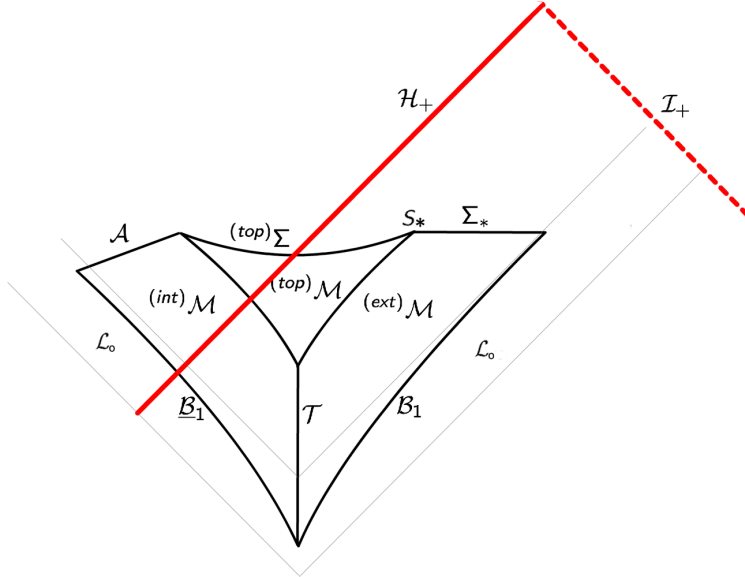


Figure 3: The Penrose diagram of a finite GCM admissible space-time $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$. The future boundary Σ_* initiates at the GCM sphere S_* . The past boundary of \mathcal{M} , $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$, is included in the initial layer \mathcal{L}_0 , in which the spacetime is assumed given.

- The construction, which also allows us to specify adapted null frames¹¹, is made possible by the covariance properties of the Einstein vacuum equations.
- The past boundary $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$ of \mathcal{M} , which is itself to be constructed, is included in the initial layer \mathcal{L}_0 in which the spacetime is assumed to be known¹², i.e. a small vacuum perturbation of a Kerr solution.

The proof of Theorem 1.1 is centered around a limiting argument for a continuous family of such spacetimes \mathcal{M} together with a set of bootstrap assumptions (BA) for the connection and curvature coefficients, relative to the adapted frames. Assuming that a given finite, GCM admissible, spacetime \mathcal{M} saturates BA we reach a contradiction as follows:

- First improve BA for some of the components of the curvature tensor with respect to the frame. These verify equations (called Teukolsky equations) which decouple, up to terms quadratic in the perturbation, and are treated by wave equations methods.

¹¹In our work we prefer to talk about horizontal structures, see the brief discussion in section 4.3.2. Another important novelty in the proof of Theorem 1.1 is that it relies on non-integrable horizontal structures, see section 4.3.2.

¹²The passage from the initial data specified on Σ_0 to the initial layer spacetime \mathcal{L}_0 , can be justified by arguments similar to those of [57] [58], based on the methods introduced in [25].

- Use the information provided by these curvature coefficients together with the gauge choice on \mathcal{M} , induced by the GCM condition on Σ_* , to improve BA for all other Ricci and curvature components.
- Use these improved estimates to extend \mathcal{M} to a strictly larger spacetime \mathcal{M}' and then construct a new GCM sphere S'_* , a new boundary Σ'_* which initiates on S'_* , and a new GCM admissible spacetime \mathcal{M}' , with Σ'_* as boundary, strictly larger than \mathcal{M} .

Remark 1.2. *The critical new feature of this argument is the fact that the new GCM sphere S'_* has to be constructed as a co-dimension 2 sphere in \mathcal{M}' with no reference to the initial conditions¹³. This construction appears first in [60] in a polarized situation. The general construction appears in the GCM papers [61], [62]. The construction of Σ'_* from S'_* appears first in [60] in the polarized case. The general construction used in our work is due to D. Shen [81].*

2 Linear and nonlinear stability

2.1 Notions of nonlinear stability

Consider a stationary solution ϕ_0 of a nonlinear evolution equation

$$\mathcal{N}[\phi] = 0. \tag{2.1}$$

There are two distinct notions of stability, *orbital stability*, according to which small perturbations of ϕ_0 lead to solutions ϕ which remain close for all time, and *asymptotic stability (AS)* according to which the perturbed solutions converge as $t \rightarrow \infty$ to ϕ_0 . In the case where ϕ_0 is non trivial, there is a third notion, which we call *asymptotic orbital stability (AOS)*, to describe the fact that the perturbed solutions may converge to a different stationary solution. This happens if ϕ_0 belongs to a multi parameter smooth family of stationary solutions, or by applying a gauge transform to ϕ_0 which keeps the equation invariant¹⁴.

For quasilinear equations¹⁵, such as EVE, a proof of stability means necessarily AS or AOS stability. Both require a detailed understanding of the *decay properties* of the linearized

¹³See a more detailed discussion in section 4.3.1.

¹⁴In the case of Kerr, both cases are present as we shall see below.

¹⁵Orbital stability can be established directly (i.e. without establishing the stronger version) only in rare occasions, such as for hamiltonian equations with weak nonlinearities.

equation, i.e.

$$\mathcal{L}[\phi_0]\psi = 0, \tag{2.2}$$

with $\mathcal{L}[\phi_0]$ the Fréchet derivative $\mathcal{N}'[\phi_0]$. This is, essentially, a linear hyperbolic system with variable coefficients which, typically, presents instabilities¹⁶.

In the exceptional situation, when stability can ultimately be established, one can tie all the instability modes to the following properties of the nonlinear equation:

- M1. If ϕ_λ is a family of stationary solutions, near ϕ_0 , verifying $\mathcal{N}[\phi_\lambda] = 0$. Then $\psi_0 = (\frac{d}{d\lambda}\phi_\lambda)_{\lambda=0}$ verifies $\mathcal{N}'[\phi_0]\psi_0 = 0$, i.e. ψ_0 is a nontrivial, stationary, bound state of the linearized equations (2.2).
- M2. If Φ_λ is a smooth family of diffeomorphisms of the background manifold, $\Phi_0 = I$, such that $\mathcal{N}[\Phi_\lambda^*(\phi_0)] = 0$. Then $\Psi_0 = (\frac{d}{d\lambda}\Phi_\lambda^*(\phi_0))_{\lambda=0}$ verifies $\mathcal{N}'[\phi_0]\Psi_0 = 0$, i.e. Ψ_0 is also a stationary bound state of the linearized equation (2.2).

These linear instabilities are responsible for the fact that a small perturbation of the fixed stationary solution ϕ_0 may not converge to ϕ_0 but to another nearby stationary solution¹⁷.

To prove the asymptotic convergence of ϕ to a final state ϕ_f , different from ϕ_0 , we need to establish sufficiently strong rates of decay¹⁸ for $\phi - \phi_f$. Rates of decay however are strongly *coordinate dependent*, i.e. dependent on the choice of the diffeomorphism (or gauge) Φ in which decay is measured. Thus, to prove a nonlinear stability, result we need to know both the final state ϕ_f and the coordinate system Φ_f in which sufficient decay, and thus convergence to ϕ_f , can be established. The difficulty here is that neither ϕ_f nor Φ_f can be determined a-priori (from the initial perturbation), they have to emerge dynamically in the process of convergence. Moreover, in all examples involving nonlinear wave equations in 1 + 3 dimensions, the nonlinear terms have to also cooperate, that is an appropriate version of the so called null condition has to be verified.

To summarize, given a nonlinear system $\mathcal{N}[\phi] = 0$ which possesses both a smooth family of stationary solutions ϕ_λ and a smooth family of invariant diffeomorphisms a proof of the nonlinear stability of ϕ_0 requires the following ingredients:

¹⁶In unstable situations (2.2) may have exponentially growing solutions, see for example [33].

¹⁷The methodology of tracking this asymptotic final state, in general different from ϕ_0 , is usually referred to as modulation. See for example [70],[72] for how modulation theory can be used to deal with some examples of scalar nonlinear dispersive equations.

¹⁸To control the nonlinear terms of the equation.

- The only non-decaying modes of the linearized equation $\mathcal{L}(\phi_0)\psi = 0$ are those due to the items M1–M2 above. In particular there are no exponentially growing modes.
- A dynamical construction of both the final state ϕ_f and the final gauge Φ_f in which convergence to the final state takes place.
- The nonlinear terms in the equation

$$\mathcal{L}[\phi_f]\psi = N(\psi)$$

obtained by expanding the equation $\mathcal{N}[\phi]$ near ϕ_f , in the gauge given by the diffeomorphism Φ_f , has to verify an appropriate version of the null condition.

2.2 The case of the Kerr family

The issue of the stability of the Kerr family has been at the center of attention of GR physics and mathematical relativity for more than half a century, ever since their discovery by Kerr in [51]. In this case we have to deal not only with a 2-parameter family of solutions, corresponding to the parameters (a, m) , but also with the entire group of diffeomorphisms¹⁹ of \mathcal{M} . In what follows we try to discuss the main difficulties of the problem. In doing that it helps to compare these to those arising in the simplest case when $a = m = 0$, i.e. stability of Minkowski.

2.3 Stability of Minkowski space

Until very recently the only space-time for which full nonlinear stability had been established was the Minkowski space, see [25]. The proof is based on some important PDE advances of late last century:

- (i) Robust approach, based on the vectorfield method, to derive quantitative decay based on generalized energy estimates and commutation with (approximate) Killing and conformal Killing vectorfields.
- (ii) The *null condition* identifying the deep mechanism for nonlinear stability, i.e. the specific structure of the nonlinear terms which enables stability despite the low decay of the perturbations.

¹⁹Indeed, according to the covariant properties of the Einstein vacuum equations we cannot distinguish between \mathbf{g} and $\Phi^*\mathbf{g}$, for any diffeomorphism Φ of \mathcal{M} .

- (iii) Elaborate bootstrap argument according to which one makes educated assumptions about the behavior of solutions to nonlinear wave equations and then proceeds, by a long sequence of a-priori estimates, to show that they are in fact satisfied. This amounts to a *conceptual linearization*, i.e. a method by which the equations become, essentially, linear²⁰ without actually linearizing them.

The main innovation in the proof in [25] is the choice of an appropriate gauge condition, readjusted dynamically through the convergence process, by a continuity argument, which allows one to separate the curvature estimates, treated by hyperbolic methods, from the estimates for the connection coefficients. The key point is that these latter verify transport or elliptic equations in which the curvature terms appear as sources. Thus both the curvature components and connection coefficients can be controlled by a bootstrap argument. The gauge condition is based on the constructive choice of a maximal time function t and two outgoing optical functions ${}^{(int)}u$ ²¹ and ${}^{(ext)}u$ ²² covering the interior and exterior parts of the spacetime.

Another novelty of [25] was the reliance on null frames adapted to the S -foliations induced by the level surfaces of t and u . These define integrable horizontal structures (in the language of part I of [39]), by contrast with the non-integrable ones used in the proof of Theorem 1.1 and discussed in section 4.3.2. The functions t, u and this integrable horizontal structure can be used to define approximate Killing vectorfields used in estimating the curvature.

2.4 Main difficulties

There are a few major obstacles in passing from the stability of Minkowski to that of Kerr:

1. The first one was already discussed in section 2.1 in the general context of the stability of a stationary solution ϕ_0 . In the case when ϕ_0 is trivial there are no nontrivial bound states for the linearized problem and thus we expect that the final state does actually coincide with ϕ_0 . This is precisely the case for the special member

²⁰Note that in the context of EVE, and other quasilinear hyperbolic systems, this differs substantially from the usual notion of linearization around a fixed background.

²¹The interior optical function is initialized on a timelike geodesic from the initial hypersurface.

²²The exterior optical function ${}^{(ext)}u$ is initialized on the last slice $t = t_*$, by the construction of a foliation (inverse lapse foliation) initialized at space-like infinity. It is thus readjusted dynamically as $t_* \rightarrow \infty$.

of the Kerr family $a = m = 0$, i.e. the Minkowski space²³ (\mathbb{R}^{1+3}, m) . On the other hand, in perturbations of Kerr, general covariance affects the entire construction of the spacetime. In the proof of Theorem 1.1 the crucial concept of a GCM admissible spacetime is meant to deal with both finding the final parameters and the gauge in which convergence to the final state takes place.

2. A fundamental insight in the stability of the Minkowski space was that the Bianchi identities decouple at first order from the null structure equations which allows one to control curvature first, as a Maxwell type system (see [24]), and then proceed with the rest of the solution. This cannot work for perturbations of Kerr due to the fact that some of the null components²⁴ of the curvature tensor are non-trivial in Kerr.
3. Even if one succeeds in tackling the above mentioned issues, there are still major obstacles in understanding the decay properties of the solution. Indeed, when one considers the simplest, relevant, linear equation on a fixed Kerr background, i.e. the scalar wave equation $\square_{a,m}\psi = 0$, one encounters serious difficulties to prove decay. Below is a very short description of these:

- *The problem of trapped null geodesics.* This concerns the existence of null geodesics²⁵ neither crossing the event horizon nor escaping to null infinity, along which solutions can concentrate for arbitrary long times. This leads to degenerate energy-Morawetz estimates which require a very delicate analysis.
- *The trapping properties of the horizon.* The horizon itself is ruled by null geodesics, which do not communicate with null infinity and can thus concentrate energy. This problem was solved by understanding the so called red-shift effect associated to the event horizon, which counteracts this type of trapping.
- *The problem of superradiance.* This is the failure of the stationary Killing field $\mathbf{T} = \partial_t$ to be everywhere timelike in the domain of outer communication²⁶, and thus, of the associated conserved energy to be positive. Note that this problem is absent in Schwarzschild and, in general, for axially symmetric solutions of EVE. In both cases however there still is a degeneracy along the horizon.

²³Note however that even though the linearized system around Minkowski does not contain instabilities, the proof of the nonlinear stability of the Minkowski space in [25] takes into account (in a fundamental way!) general covariance. Indeed the presence of the ADM mass affects the causal structure of the far, asymptotic, region of the perturbed space-time.

²⁴With respect to the *principal null directions of Kerr*, i.e a distinguished null pair which diagonalizes the full curvature tensor, the middle component $P = \rho + i^* \bar{\rho}$ is nontrivial.

²⁵In the Schwarzschild case, these geodesics are associated with the celebrated photon sphere $r = 3m$.

²⁶The stationary Killing vectorfield \mathbf{T} is timelike only outside of the so-called ergoregion.

- *Superposition problem.* This is the problem of combining the estimates in the near region, close to the horizon, (including the ergoregion and trapping) with estimates in the asymptotic region, where the spacetime is close to Minkowski.
4. Though, as seen above, the analysis of the scalar wave equation in Kerr presents formidable difficulties, it is itself just a vastly simplified model problem. A more realistic equation is the so called spin 2 wave equation, or Teukolsky equation, which presents many new challenges²⁷.
 5. The full linearized system, whatever its formulation, presents many additional difficulties due to its complex tensorial structure and the huge gauge covariance of the equations²⁸. The crucial breakthrough in this regard is the observation, due to Teukolsky [89], that the extreme components of the linearized curvature tensor are both gauge invariant (see below in section 2.5) and verify decoupled spin 2 equations, that is the Teukolsky equations mentioned above.
 6. A crucial simplification of the linear theory, by comparison to the full nonlinear case, is that one can separate the treatment of the gauge invariant extreme curvature components from all the other gauge invariant quantities. In the nonlinear case this separation is no longer true, all quantities need to be treated simultaneously. Moreover, methods based on separation of variables, developed to treat scalar and spin 2 wave equations in Kerr, are incompatible with the nonlinear setting which requires, instead, robust methods to derive decay.

2.5 Linear stability

Linear stability for the vacuum equations is formulated in the following way. Given the Einstein tensor $\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}g_{\alpha\beta}$ and a stationary solution \mathbf{g}_0 , i.e. a fixed Kerr metric, one has to solve the system of equations

$$\mathbf{G}'(\mathbf{g}_0) \delta\mathbf{g} = 0. \quad (2.3)$$

The covariant properties of the Einstein equations, i.e. the equivalence between a solution \mathbf{g} and $\Phi^*(\mathbf{g})$, leads us to identify $\delta\mathbf{g}$ with $\delta\mathbf{g} + \mathcal{L}_X(\mathbf{g}_0)$ for arbitrary vectorfields X in \mathcal{M} , i.e.

$$\delta\mathbf{g} \equiv \delta\mathbf{g} + \mathcal{L}_X(\mathbf{g}_0). \quad (2.4)$$

²⁷Unlike the scalar wave equation $\square_{a,m}\psi = 0$, which is conservative, the Teukolsky equation is not, and we thus lack the most basic ingredient in controlling the solutions of the equation, i.e. energy estimates.

²⁸As mentioned earlier, rates of decay are heavily dependent on a proper choice of gauge, thus affecting the issue of convergence.

We can now attempt to formulate a version of linear stability for (2.3), loosely related to the nonlinear stability of Kerr conjecture, as follows.

Definition 2.1. *By linear stability of the Kerr metric \mathbf{g}_0 we understand a result which achieves the following:*

Given an appropriate initial data for a perturbation $\delta\mathbf{g}$, find a vectorfield X such that, after projecting away the bound states generated by the parameters a, m according to M1–M2 in section 2.1, all solutions of the form $\delta\mathbf{g} + \mathcal{L}_X\mathbf{g}_0$ to (2.3), decay, relative to an appropriate null frame of $\mathcal{K}(a_0, m_0)$, sufficiently fast in time.

Remark 2.2. *The definition above is necessarily vague. What is the meaning of sufficiently fast? In fact various components of the metric $\delta\mathbf{g}$, relative to the canonical null frame of $\mathcal{K}(a, m)$, are expected to decay at different polynomial rates, some of which are not even integrable. Unlike in the nonlinear context, where one needs precise rates of decay for components of the curvature tensor and Ricci coefficients, as well as their derivatives, to be able to control the nonlinear terms, in linear theory any type of nontrivial control of solutions may be regarded as satisfactory²⁹. Thus linear stability, as formulated above, can only be regarded as a vastly simplified model problem. Nevertheless the study of linear stability of the Kerr family has turned out to be useful in various ways, as we shall see below.*

Historically, the following versions of linear stability have been considered.

- (a) *Metric Perturbations.* At the level of the metric itself, i.e. as above in (2.3).
- (b) *Curvature Perturbations.* Via the Newman-Penrose (NP) formalism, based on null frames.

The strategy followed in both cases³⁰ is:

- Find components of the metric (in case (a)) or curvature tensor (in case (b)), invariant with respect to linearized gauge transformations 2.4 which verify simple, decoupled, wave equations. The main insight of this type was the discovery, by Teukolsky [89], in the context of (b) above, that the extreme components of the linearized curvature tensor verify both these properties.

²⁹This is in fact what happens in the literature on linear theory. Thus, for example, in their well known linear stability result around Schwarzschild [30], the authors derive satisfactory results (compatible with what is needed in nonlinear theory) for components of the curvature tensor, and some Ricci coefficients, but not all.

³⁰In the article we refer mainly to the curvature perturbation approach.

- Analyze these components by showing one of the following:
 - There are no exponentially growing modes. This is known as mode stability.
 - Boundedness for all time.
 - Decay (sufficiently fast) in time.
- Find a linearized gauge condition, i.e. a vectorfield X , such that all remaining (gauge dependent) components (at the metric or curvature level) inherit the property mentioned above: no exponentially growing modes, boundedness, or decay in time. In the physics literature this is known as the problem of reconstruction.

2.5.1 Mode stability

All results on the linear stability of Kerr in the physics literature during the 10-15 years after Roy Kerr’s 1963 discovery, often called the “Golden Age of Black Hole Physics”, are based on mode decompositions. One makes use of the separability³¹ of the linearized equations, more precisely the Teukolsky equations, on a fixed Kerr background, to derive simple ODEs for the corresponding modes. One can then show, by ingenious methods, that these modes cannot exhibit exponential growth. The most complete result of this type is due to Bernard Whiting [93].

The obvious limitation of these results are as follows:

- They are far from even establishing the boundedness of general solutions to the Teukolsky equations, let alone to establish quantitative decay for the general solutions.
- Results based on mode decompositions depend strongly on the specific symmetries of Kerr which cannot be adapted to perturbations of Kerr.

Robust methods to deal with both issues have been developed in the mathematical community, based on the vectorfield method which we discuss below.

2.5.2 Classical vectorfield method

The vectorfield method, as an analytic tool to derive decay, was first developed in connection with the wave equation in Minkowski space. As well known, solutions of the wave

³¹See discussion in section 1.1.

equation $\square\psi = 0$ in the Minkowski space \mathbb{R}^{n+1} both conserve energy and decay uniformly in time like $t^{-\frac{n-1}{2}}$. While conservation of energy can be established by a simple integration by parts, and is thus robust to perturbations of the Minkowski metric, decay was first derived either using the Kirchoff formula or by Fourier methods, which are manifestly not robust. An integrated version of local energy decay, based on an inspired integration by parts argument, was first derived by C. Morawetz [74], [75]. The first derivation of decay based on the commutations properties of \square with Killing and conformal Killing vectorfields of Minkowski space together with energy conservation appear in [52] and [55]. The same method also provides precise information about the decay properties of derivatives of solutions with respect to the standard null frame of Minkowski space, an important motivating factor in the discovery of the null condition [53], [22] and [54].

The crucial feature of the methodology initiated by these papers, to which we refer as the classical vectorfield method, is that it can be easily adapted to perturbations of the Minkowski space. As such the method has had numerous applications to nonlinear wave equations and played an important role in the proof of the nonlinear stability of Minkowski space, as discussed in section 2.3. It has also been applied to later versions of the stability of Minkowski in [57]-[58], [67], [9], [66], [44], [43], [40], and extensions of it to Einstein equation coupled with various matter fields in [10], [34], [68], [11], [92], [65], [48].

2.5.3 New vectorfield method

To derive decay estimates for solutions of wave equations on a Kerr background one has to substantially refine the classical vectorfield method. The new vectorfield method is an extension of the classical method which compensates for the lack of enough Killing and conformal Killing vectorfields in Kerr by introducing, new, cleverly designed, vectorfields whose deformation tensors have coercive properties in different regions of spacetime, not necessarily causal. The method has emerged in the last 20 years in connection to the study of boundedness and decay for the scalar wave equation in Schwarzschild and Kerr, see section 3.2 for more details.

2.6 Model problems

To solve the stability of Kerr conjecture one has to deal simultaneously with all the difficulties mentioned above. This is, of course, beyond the abilities of mere humans. Instead the problem was tackled in a sequence of steps based on a variety of simplified model problems, in increasing order of difficulty. To start with we can classify model problems based on the following criteria:

1. Whether the result refers to Schwarzschild i.e. $a = 0$, slowly rotating Kerr i.e. $|a| \ll m$ or full non-extremal Kerr $|a| < m$.
2. Whether the result refers to linear or nonlinear stability.
3. Whether the result, in linear theory, refers to scalar wave equation, i.e. spin 0, Teukolsky equation, i.e. spin 2, or the full linearized system.
4. Whether the stability result, in linear theory, is a mode stability result, a boundedness result or one that establishes quantitative decay.

3 Short survey of model problems

We give below a short outline of the main developments concerning linear and nonlinear model problems for the Kerr stability problem, paying special attention to those which had a measurable influence on our work.

3.1 Mode stability results

These are mode stability results, using the method of separation of variables, obtained in the physics community roughly during the period 1963-1990. They rely on what Chandrasekhar called the most striking feature of Kerr i.e. “*the separability of all the standard equations of mathematical physics in Kerr geometry*”.

1. *Regge-Wheeler (1957)*. Even before the discovery of the Kerr solution physicists were interested in the mode stability of Schwarzschild space, i.e. $\mathcal{K}(0, m)$. The first important result goes back to T. Regge and J.A Wheeler [78], in which they analyzed linear, metric perturbations, of the Schwarzschild metric. They showed that in a suitable gauge, equation (2.3) decouples into even-parity and odd-parity perturbations, corresponding to axial and polar perturbations. The most important discovery in that paper is that of the master Regge-Wheeler equation, a wave equation with a favorable potential, verified by an invariant scalar component ϕ of the metric, i.e.

$$\square_m \phi = V \phi, \quad V = \frac{4}{r^2} \left(1 - \frac{2m}{r} \right). \quad (3.1)$$

where \square_m denotes the wave operator of the Schwarzschild metric of mass m . The R-W study was completed by Vishveshwara [90] and Zerilli [94]. A gauge-invariant formulation of *metric perturbations* was then given by Moncrief [73].

2. *Teukolsky (1973)*. The curvature perturbation approach, near Schwarzschild, based on the Newman-Penrose (NP) formalism was first undertaken by Bardeen-Press [8]. This approach was later extended to the Kerr family by Teukolsky [89], see also [77], who made the important discovery that the extreme curvature components, relative to a principal null frame, are gauge invariant and satisfy decoupled, separable, wave equations. The equations, bearing the name of Teukolsky, are roughly of the form

$$\square_{a,m}\psi = \mathcal{L}[\psi] \tag{3.2}$$

where $\mathcal{L}[\psi]$ is a first order linear operator in ψ .

3. *Chandrasekhar (1975)*. In [16] Chandrasekhar initiated a transformation theory relating the two approaches. He exhibited a transformation which connects the Teukolsky equations to a Regge-Wheeler type equation. In the particular case of Schwarzschild the transformation takes the Teukolsky equation to the Regge-Wheeler equation in (3.1). The Chandrasekhar transformation was further elucidated and extended by R. Wald [91] and recently by Aksteiner and al [1].

Though originally it was meant only to unify the Regge-Wheeler approach with that of Teukolsky, the Chandrasekhar transformation, and various extensions of it, turn out to play an important role in the field.

4. *Whiting (1989)*. As mentioned before the full mode stability, i.e. lack of exponentially growing modes, for the Teukolsky equation on Kerr is due to Whiting [93] (see also [83], in the case of the scalar wave equation, and [5] [88] for stronger quantitative versions).
5. *Reconstruction*. Once we know that the Teukolsky variables, i.e. the extreme components of the curvature tensor verify mode stability, i.e. there are no exponentially growing modes, it still remains to deal with the problem of reconstruction, i.e. to find a gauge relative to which all other components of the curvature and Ricci coefficients enjoy the same property. We refer the reader to Wald [91] and the references within for a treatment of this issue in the physics literature.

3.2 Quantitative decay for the scalar wave equation

As mentioned in section 2.5.1, mode stability is far from establishing even the boundedness of solutions. To achieve that³² and, more importantly, to derive realistic decay estimates, one needs an entirely different approach based on what we called the “new vectorfield method” in section 2.5.3. The method has emerged in connection to the study of boundedness and decay for the scalar wave equation in $\mathcal{K}(a, m)$,

$$\square_{a,m}\phi = 0. \tag{3.3}$$

The starting and most demanding part of the new method, which appeared first in [12], is the derivation of a global, combined, *Energy-Morawetz* estimate which degenerates in the trapping region. Once an Energy-Morawetz estimate is established one can commute with the Killing vectorfields of the background manifold, and the so called red shift vectorfield introduced in [26], to derive uniform bounds for solutions. The most efficient way to also get decay, and solve the *superposition problem* (see section 2.4), originating in [27], is based on the presence of family of *r^p-weighted*, quasi-conformal vectorfields defined in the non-causal, far *r* region of spacetime³³.

The first Energy-Morawetz type results for scalar wave equation (3.3) in Schwarzschild, i.e. $a = 0$, are due to Soffer-Blue [12], [13] and Blue-Sterbenz [14], based on a modified version of the classical Morawetz integral energy decay estimate. Further developments appear in the works of Dafermos-Rodnianski, see [26], [27], and Marzuola-Metcalf-Tataru-Tohaneanu [71]. The vectorfield method can also be extended to derive decay for axially symmetric solutions in Kerr, see [47] and³⁴ [85], but it is known to fail for general solutions in Kerr, see Alinhac [2].

In the absence of axial symmetry the derivation of an Energy-Morawetz estimate in $\mathcal{K}(a, m)$ for $|a/m| \ll 1$ requires a more refined analysis involving both the vectorfield method and either micro-local methods or mode decompositions. The first full quantitative decay³⁵ result, based on micro-local analysis techniques, is due to Tataru-Tohaneanu [87]. The derivation of such an estimate in the full sub-extremal case $|a| < m$ is even more

³²The first realistic boundedness result for solutions of the scalar wave equation in Schwarzschild appears in [50] based on a clever use of the energy method which takes into account the degeneracy of \mathbf{T} at the horizon.

³³These replace the scaling and inverted time translation vectorfields used in [52] or their corresponding deformations used in [25]. A recent improvement of the method allowing one to derive higher order decay can be found in [7].

³⁴In his Princeton PhD thesis Stogin establishes a Morawetz estimate even for the full subextremal case $|a| < m$.

³⁵See also [28] for the first proof of boundedness of solutions, based on mode decompositions.

subtle and was achieved by Dafermos-Rodnianski-Shlapentokh-Rothman [29] by combining the vectorfield method with a full separation of variables approach. A purely physical space proof of the Energy-Morawetz estimate for small $|a/m|$, which avoids both micro-local analysis and mode decompositions, was pioneered by Andersson-Blue in [4]. Their method, which extends the classical vectorfield method to include second order operators (in this case the Carter operator, see section 1.1), has the usual advantages of the classical vectorfield method, i.e it is robust with respect to perturbations. It is for this reasons that we rely on it in the proof of Theorem 1.1, more precisely in part II of [39].

3.3 Linear stability of Schwarzschild

A first quantitative proof of the linear stability of Schwarzschild spacetime was established³⁶ by Dafermos-Holzegel-Rodnianski (DHR) in [30]. Notable in their analysis is the treatment of the Teukolsky equation in a fixed Schwarzschild background. While the Teukolsky equation is separable, and amenable to mode analysis, it is not variational and thus cannot be treated directly by energy type estimates. As mentioned earlier in section 3.1, Chandrasekhar was able to relate the Teukolsky equation to the Regge-Wheeler (RW) equation, which is both variational and coercive (the potential V has a favorable sign). In [30] the authors rely on a physical space version of the Chandrasekhar transformation. Once decay estimates for the RW equation have been established, based on the technology developed earlier for the scalar wave equation in Schwarzschild, the authors recover the expected boundedness and decay for solutions to the original Teukolsky equation.

The remaining work in [30] is to derive similar control for the other curvature components and the linearized Ricci coefficients associated to the double null foliation. This last step requires carefully chosen gauge conditions, which the authors make within the framework of a double null foliation, initialized both on the initial hypersurface and the background Schwarzschild horizon³⁷. This gauge fixing from initial data leads to sub-optimal decay estimates for some of the metric coefficients³⁸ and is thus inapplicable to the nonlinear case. This deficiency was fixed in the PhD thesis of E. Giorgi, in the context of the linear stability of Reissner-Nordström, see [38], by relying on a linearized version of the GCM construction in [60].

³⁶A somewhat weaker version of linear stability of Schwarzschild was subsequently proved in [45] by using the original, direct, Regge-Wheeler, Zerilli approach combined with the vectorfield method and adapted gauge choices. See also [49] for an alternate approach of linear stability of Schwarzschild using wave coordinates.

³⁷The authors use a a scalar condition for the linearized lapse along the event horizon (part of what the authors call future normalized gauge), itself initialized from initial data, see (212) and (214) in [30].

³⁸See (250)–(252) and (254) in [30].

3.4 Linear stability of Kerr for small angular momentum

The first breakthrough result on the linear stability of Kerr, for $|a|/m \ll 1$, is due to Ma [69], see also [31]. Both results are based on a generalization of the Chandrasekhar transformation to Kerr which takes the Teukolsky equations, verified by the extreme curvature components, to generalized versions of the Regge-Wheeler (gRW) equation. Relying on separation of variables and vectorfield techniques, similar to those developed for the scalar wave equation in slowly rotating Kerr, the authors derive Energy-Morawetz and r^p estimates for the solutions of the gRW equations. These results were recently partially³⁹ extended to the full subextremal range in [84].

The first stability results for the full linearized Einstein vacuum equations near $\mathcal{K}(a, m)$, for $|a|/m \ll 1$, appeared in [6] and [41]. The first paper, based on the GHP formalism⁴⁰, see [37], builds on the results of [69] while the second paper is based on an adapted version of the metric formalism and builds on the seminal work of the authors on Kerr-de Sitter [42]. Though the ultimate relevance of these papers to nonlinear stability remains open, they are both remarkable results in so far as they deal with difficulties that looked insurmountable even ten years ago.

3.5 Nonlinear model problems

3.5.1 Nonlinear stability of Kerr-de Sitter

There is another important, simplified, nonlinear model problem which has drawn attention in recent years, due mainly to the striking achievement of Hintz and Vasy [42]. This is the problem of stability of Kerr-de Sitter concerning the Einstein vacuum equation with a strictly positive cosmological constant

$$\mathbf{R}_{\alpha\beta} + \Lambda \mathbf{g}_{\alpha\beta} = 0, \quad \Lambda > 0. \quad (3.4)$$

In their work, which relies in part on the important mode stability result of Kodama and Ishibashi [64], Hintz and Vasy were able to prove the nonlinear stability of the stationary part of Kerr-de Sitter with small angular momentum, the first nonlinear stability result of any nontrivial stationary solutions for the Einstein equations⁴¹. It is important to note

³⁹The analysis of [84] is still limited to modes. The authors have however announced a full proof of the result.

⁴⁰An adapted spinorial version of the NP formalism.

⁴¹This is also the first general nonlinear stability result in GR establishing asymptotic stability towards

that, despite the fact that, formally, the Einstein vacuum equation (1.3) is the limit⁴² of (3.4) as $\Lambda \rightarrow 0$, the global behavior of the corresponding solutions is radically different⁴³.

The main simplification in the case of stationary solutions of (3.4) is that the expected decay rates of perturbations near Kerr-de Sitter is exponential, while in the case $\Lambda = 0$ the decay is lower degree polynomial⁴⁴, with various components of tensorial quantities decaying at different rates, and the slowest decaying rate⁴⁵ being no better than t^{-1} . The Hintz-Vasy result was recently revisited in the work of A. Fang [35] [36] where he bridges the gap between the spectral methods of [42] and the vectorfield methods.

3.5.2 Nonlinear stability of Schwarzschild

The first nonlinear stability result of the Schwarzschild space was established in [60]. In its simplest version, the result states the following.

Theorem 3.1 (Klainerman-Szeftel [60]). *The future globally hyperbolic development of an axially symmetric, polarized, asymptotically flat initial data set, sufficiently close (in a specified topology) to a Schwarzschild initial data set of mass $m_0 > 0$, has a complete future null infinity \mathcal{I}^+ and converges in its causal past $\mathcal{J}^{-1}(\mathcal{I}^+)$ to another nearby Schwarzschild solution of mass m_f close to m_0 .*

The restriction to axial polarized perturbations is the simplest assumption which insures that the final state is itself Schwarzschild and thus avoids the additional complications of the Kerr stability problem. We refer the reader to the introduction in [60] for a full discussion of the result.

The proof is based on a construction based on GCM admissible spacetimes similar to that briefly discussed in section 1.3 in the context of slowly rotating Kerr. There are however several important simplifications to be noted:

a family of solutions, i.e. full quantitative convergence to a final state close, but different from the initial one.

⁴²To pass to the limit requires one to understand all global in time solutions of (3.4) with $\Lambda = 1$, not only those which are small perturbations of Kerr-de Sitter, treated by [42].

⁴³Major differences between formally close equations occur in many other contexts. For example, the incompressible Euler equations are formally the limit of the Navier-Stokes equations as the viscosity tends to zero. Yet, at fixed viscosity, the global properties of the Navier-Stokes equations are radically different from that of the Euler equations.

⁴⁴While there is exponential decay in the stationary part treated in [42], note that lower degree polynomial decay is expected in connection to the stability of the complementary causal region (called cosmological or expanding) of the full Kerr-de Sitter space, see e.g. [80].

⁴⁵Responsible for carrying gravitational waves at large distances so that they are detectable.

- The assumption of polarization makes the constructions of the GCM spheres S_* and spacelike hypersurface Σ_* significantly simpler, see Chapter 9 in [60], by comparison to the general case treated in [61], [62] and [81].
- The spacetime has only two components $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ and the null horizontal structures, defined on each component, are integrable.
- As in the case of the scalar wave equation on Schwarzschild space the main spin-2 Teukolsky wave equations can be treated by a vectorfield approach. This is no longer true in Kerr and even less so in perturbations of Kerr.

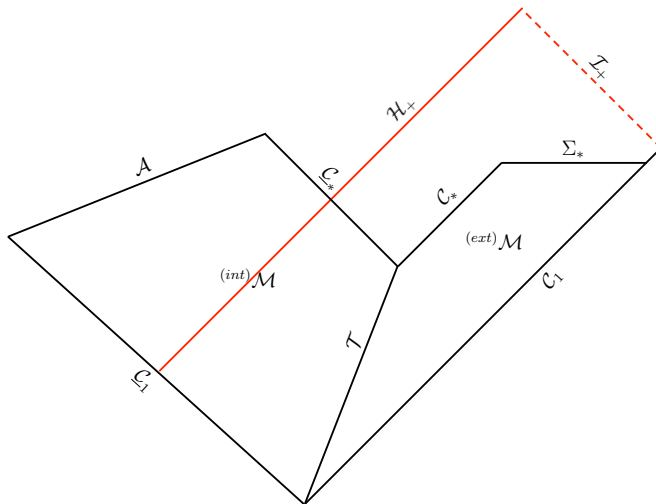


Figure 4: The GCM admissible space-time \mathcal{M} . By comparison to Figure 3, \mathcal{M} does not have ${}^{(top)}\mathcal{M}$, the past boundaries $\mathcal{C}_0 \cup \underline{\mathcal{C}}_0$ and future boundary $\underline{\mathcal{C}} \cup \mathcal{C}_*$ are null and the horizontal structures (induced by geodesic foliations) are integrable. As in Theorem 1.1, the crucial GCM sphere S_* is defined and constructed with no reference to the initial data.

Recently Dafermos-Holzegel-Rodnianski-Taylor [32] have extended⁴⁶ the result of [60] by properly preparing a co-dimension 3 subset of the initial data such that the final state is still Schwarzschild. Like in [60], the starting point of [32] is to anchor the entire construction on a far away⁴⁷ GCM type sphere S_* , in the sense of [61] [62], with no direct reference to the initial data. It also uses the same definition of the angular momentum as in (7.19) of [62]. Finally, the spacetime in [32] is separated in an exterior region ${}^{(ext)}\mathcal{M}$ and an interior region ${}^{(int)}\mathcal{M}$, with the ingoing foliation of ${}^{(int)}\mathcal{M}$, initialized based on the information induced by ${}^{(ext)}\mathcal{M}$, as in [60]. We note, however, that [32] does not use the

⁴⁶The novelty of [32], compared to [60], is the well preparation of the initial data, based on an additional three dimensional modulation. Note however that [32] requires substantially stronger asymptotic conditions for the initial data compared to [60].

⁴⁷That is $r \gg u$, similar to the dominant in r condition (3.3.4) of [60].

geodesic foliation of [60], but instead both $^{(int)}\mathcal{M}$ and $^{(ext)}\mathcal{M}$ are foliated by double null foliations, and thus, the process of estimating the gauge dependent variables is somewhat different.

4 Main ideas in the proof of Theorem 1.1

4.1 The bootstrap region

As mentioned in section 1.3.2, the proof of Theorem 1.1 is centered around a limiting argument for a family of carefully constructed finite *generally covariant modulated (GCM) admissible spacetimes* $\mathcal{M} = ^{(ext)}\mathcal{M} \cup ^{(top)}\mathcal{M} \cup ^{(int)}\mathcal{M}$. As can be seen in Figure 5 below, the future boundary of the spacetime is given by $\mathcal{A} \cup ^{(top)}\Sigma \cup \Sigma_*$ where Σ_* is a spacelike, generally covariant modulated (GCM) hypersurface, that is a hypersurface verifying a set of crucial, well-specified, geometric conditions, essential to our proof of convergence to a final state.

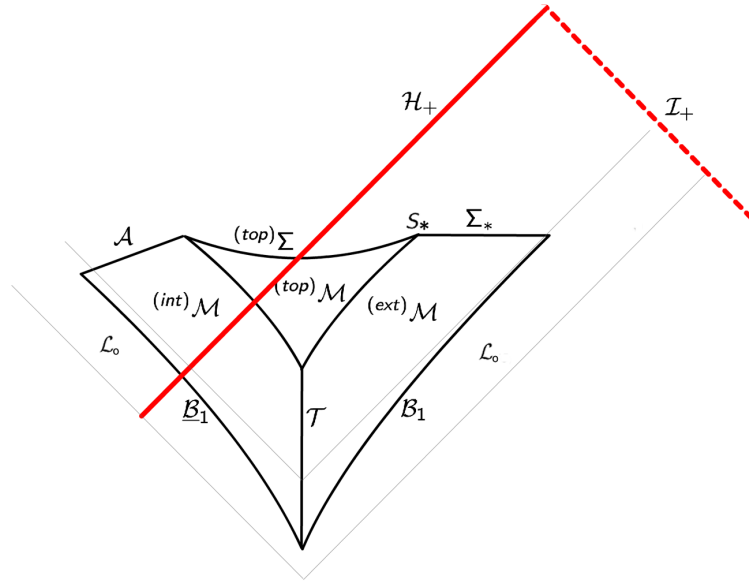


Figure 5: The Penrose diagram of a finite GCM admissible space-time $\mathcal{M} = ^{(ext)}\mathcal{M} \cup ^{(top)}\mathcal{M} \cup ^{(int)}\mathcal{M}$. The spacetime is prescribed in the initial layer \mathcal{L}_0 and has $\mathcal{A} \cup ^{(top)}\Sigma \cup \Sigma_*$ as future boundary, with Σ_* a spacelike “generally covariant modulated (GCM)” hypersurface. Its past boundary, $\underline{\mathcal{B}}_1 \cup \underline{\mathcal{B}}_1$, is itself part of the construction. $^{(ext)}\mathcal{M}$ is initialized by the GCM hypersurface Σ_* while $^{(int)}\mathcal{M}$ is initialized on \mathcal{T} by the foliation induced by $^{(ext)}\mathcal{M}$. The main innovation is the GCM sphere S_* , defined and constructed with no reference to the initial data prescribed in the initial data layer \mathcal{L}_0 .

The capstone as well as the most original part of the entire construction is the sphere S_* , the future boundary of Σ_* , which verifies a set of rigid, extrinsic and intrinsic, conditions. Once Σ_* is specified the whole GCM admissible spacetime \mathcal{M} is determined by a more conventional construction, based on geometric transport type equations. More precisely ${}^{(ext)}\mathcal{M}$ can be determined from Σ_* by a specified outgoing foliation terminating in the timelike boundary \mathcal{T} , ${}^{(int)}\mathcal{M}$ is determined from \mathcal{T} by a specified incoming one, and ${}^{(top)}\mathcal{M}$ is a complement of ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ which makes \mathcal{M} a causal domain⁴⁸. The past boundary $\mathcal{B}_1 \cup \underline{\mathcal{B}}_1$ of \mathcal{M} , which is itself to be constructed, is included in the initial layer \mathcal{L}_0 in which the spacetime is assumed to be known, i.e. a small perturbation of a Kerr solution. The passage from the initial data specified on Σ_0 to the initial layer spacetime \mathcal{L}_0 is justified by D. Shen in [82] by arguments similar to those of [57]-[58], based on the mathematical methods and techniques introduced in [25].

Each of the spacetime regions ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$, ${}^{(top)}\mathcal{M}$ come equipped with specific geometric structure including specific choices of null frames and functions such as r, u, \underline{u} . These are first defined on Σ_* and then transported to ${}^{(ext)}\mathcal{M}$, ${}^{(int)}\mathcal{M}$, ${}^{(top)}\mathcal{M}$.

4.2 Main intermediary results

The proof of Theorem 1.1 is divided in nine separate steps, Theorems M0–M8. These steps are briefly described below, see section 3.7 in [63] for the precise statements:

1. *Theorem M0 (Control of the initial data in the bootstrap gauge)*. The smallness of the initial perturbation is given in the frame of the initial data layer \mathcal{L}_0 . Theorem M0 transfers this control to the bootstrap gauge in the initial data layer.
2. *Theorems M1–M2 (Decay estimates for α (Theorem M1) and $\underline{\alpha}$ (Theorem M2))*. This is achieved using Teukolsky equations and a Chandrasekhar type transform in perturbations of Kerr.
3. *Theorems M3–M5 (Decay estimates for all curvature, connection and metric components)*. This is done making use of the GCM conditions on Σ_* as well as the control of α and $\underline{\alpha}$ established in Theorems M1 and M2. The proof proceeds in the following order:
 - Theorem M3 provides the crucial decay estimates on Σ_* ,
 - Theorem M4 provides the decay estimates on ${}^{(ext)}\mathcal{M}$,

⁴⁸This is required because of the fact that, in our construction, the future boundary of ${}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$ is not causal. By contrast, in [60], $\mathcal{M} = {}^{(ext)}\mathcal{M} \cup {}^{(int)}\mathcal{M}$.

- Theorem M5 provides the decay estimates on $^{(int)}\mathcal{M}$ and $^{(top)}\mathcal{M}$.
4. *Theorems M6 (Existence of a bootstrap spacetime)*. This theorem shows that there exists a GCM admissible spacetime satisfying the bootstrap assumptions, hence initializing the bootstrap procedure.
 5. *Theorems M7 (Extension of the bootstrap region)*. This theorem shows the existence of a slightly larger GCM admissible spacetime satisfying estimates improving the bootstrap assumptions on decay.
 6. *Theorem M8 (Control of the top derivatives estimates)*. This is based on an induction argument relative to the number of derivatives, energy-Morawetz estimates and the Maxwell like character of the Bianchi identities.

The paper [63] provides the proof of Theorem M0, Theorems M3 to M7, and half of Theorem M8 (on the control of Ricci coefficients and metric components). The proof of Theorems M1 and M2, and of the other half of Theorem M8 (on the control of curvature components), based on nonlinear wave equations techniques, are provided in [39]. The construction of GCM spheres in [61] [62], and of GCM hypersurfaces in [81] are used in the proof of Theorems M6 and M7 to construct respectively the terminal GCM sphere S_* and the last slice hypersurface Σ_* from S_* .

4.3 Main new ideas of the proof

Here is a short description of the main new ideas in the proof of Theorem 1.1 and how they compare with ideas used in other nonlinear results.

4.3.1 GCM admissible spacetimes

- As mentioned already the crucial concept in the proof of Theorem 1.1 is that of a GCM admissible spacetime, whose construction is anchored by the GCM sphere S_* in Figure 5. GCM spheres⁴⁹, are codimension 2 compact surfaces, unrelated to the initial conditions, on which specific geometric quantities take Schwarzschildian values (made possible by taking into account the full general covariance of the Einstein vacuum equations). In addition to these extrinsic conditions the sphere S_* is endowed with a choice of “effective⁵⁰ isothermal coordinates”, (θ, φ) verifying the following properties:

⁴⁹See the discussion in the introductions to [61], [62].

⁵⁰This is meant to insure the rigidity of the uniformization map, see [62].

- The metric on S_* takes the form $g = e^{2\phi} r^2 ((d\theta)^2 + \sin^2 \theta (d\varphi)^2)$.
 - The integrals on S_* of the $\ell = 1$ modes⁵¹ $J^{(0)} := \cos \theta$, $J^{(-)} := \sin \theta \sin \varphi$ and $J^{(+)} := \sin \theta \cos \varphi$ vanish identically.
- Given the GCM sphere S_* and the effective isothermal coordinates (θ, φ) on it, our GCM procedure allows us, in particular, to define the mass m , the angular momentum a and a virtual axis of rotation which converge, in the limit, to the final parameters a_f, m_f and the axis of rotation of the final Kerr⁵². We refer the reader to section 7.2 in [62] for our intrinsic definition of a and of the virtual axis of symmetry on a GCM sphere.
 - The boundary Σ_* , called a GCM hypersurface, is initialized at S_* and verifies additional conditions. In the polarized setting the first such construction appears [60]. The general case needed for our theorem is treated in [81].
 - The concepts of GCM spheres has appeared first in [60] in the context of polarized symmetry. The construction of GCM spheres, without any symmetries, in realistic perturbations of Kerr, is treated in [61], [62]⁵³.
 - The main novelty of the GCM approach is that it relies on gauge conditions initialized at a far away co-dimension 2 sphere S_* , with no direct reference to the initial conditions. Previously known geometric constructions, such as in [25], [57] and [56], were based on codimension-1 foliations constructed on spacelike or null hypersurfaces and initialized on the initial hypersurface⁵⁴. Gauge conditions initialized from the future with no direct reference to the initial conditions, which was initiated in [60], have since been used in other works, see [38] [40] [32].
 - The GCM construction introduces the following new important conceptual difficulty. The foliation on Σ_* , induced from the far away sphere S_* , needs to be connected, somehow, to the initial conditions (i.e. the initial layer \mathcal{L}_0 in Figure 5). This is achieved in both [60] and [63] by transporting⁵⁵ the sphere S_* to a sphere S_1 in the the initial layer and compare it, using the rigidity properties of the GCM conditions, to a sphere of the initial data layer. This induces a new foliation of the initial layer which differs substantially from the original one, due to a shift of the center of mass

⁵¹This is a natural generalization of $\ell = 1$ spherical harmonics.

⁵²Previous definitions of the angular momentum in General Relativity were given in [79], [19], [20], see also [86] for a comprehensive discussion of the subject.

⁵³See also chapter 16 of [32] in the particular case of perturbations of Schwarzschild, where the same concept appears instead under the name “teleological”.

⁵⁴The first such construction appears in the proof of the nonlinear stability of the Minkowski space [25] where the “inverse lapse foliation” was constructed on the “last slice”, initialized at spacelike infinity i^0 . Similar constructions, where the last slice is null rather than spacelike, appear in [57] and [56].

⁵⁵That is, we transport the $\ell = 1$ modes of some quantities from S_* to S_1 , see section 8.3.1 in [63].

frame of the final black hole, known in the physics literature as a gravitational wave recoil⁵⁶.

4.3.2 Non integrability of the horizontal structure

As mentioned in section 1.1, the canonical horizontal structure induced by the principal null directions (e_3, e_4) in (1.4) of Kerr are non integrable. The lack of integrability is dealt with by the Newman-Penrose (NP) formalism by general null frames (e_3, e_4, e_1, e_2) , with e_1, e_2 a specified basis⁵⁷ of the horizontal structure induced by the null pair (e_3, e_4) . It thus reduces all calculations to equations involving the Christoffel symbols of the frame, as scalar quantities. This un-geometric feature of the formalism makes it difficult to use it in the nonlinear setting of the Kerr stability problem. Indeed complex calculations depend on higher derivatives of all connection coefficients of the NP frame rather than only those which are geometrically significant. This seriously affects and complicates the structure of non-linear corrections and makes it difficult to avoid artificial gauge type singularities⁵⁸. This difficulty is avoided in [25] by working with a tensorial approach adapted to S -foliations, i.e. $\{e_3, e_4\}^\perp$ coincides, at every point, with the tangent space to S .

In our work we extend, with minimal changes, the tensorial approach introduced in [25] to general non-integrable foliations. The idea is very simple: we define Ricci coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \underline{\zeta}, \xi, \underline{\xi}, \omega, \underline{\omega}$ exactly as in [25], relative to an arbitrary basis of vectors $(e_a)_{a=1,2}$ of $\mathcal{H} := \{e_3, e_4\}^\perp$. In particular, the null fundamental forms χ and $\underline{\chi}$, are given by

$$\underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_a e_3, e_b), \quad \chi_{ab} = \mathbf{g}(\mathbf{D}_a e_4, e_b).$$

Due to the lack of integrability of \mathcal{H} , the null fundamental forms χ and $\underline{\chi}$ are no longer symmetric. They can be both decomposed as follows

$$\chi_{ab} = \frac{1}{2} \text{tr} \chi \delta_{ab} + \frac{1}{2} \epsilon_{ab} \text{}^{(a)}\text{tr} \chi + \widehat{\chi}_{ab}, \quad \underline{\chi}_{ab} = \frac{1}{2} \text{tr} \underline{\chi} \delta_{ab} + \frac{1}{2} \epsilon_{ab} \text{}^{(a)}\text{tr} \underline{\chi} + \widehat{\underline{\chi}}_{ab},$$

where the new scalars $\text{}^{(a)}\text{tr} \chi, \text{}^{(a)}\text{tr} \underline{\chi}$ measure the lack of integrability of the horizontal structure. The null curvature components are also defined as in [25],

$$\alpha_{ab} = \mathbf{R}_{a4b4}, \quad \beta_a = \frac{1}{2} \mathbf{R}_{a434}, \quad \underline{\beta}_a = \frac{1}{2} \mathbf{R}_{a334}, \quad \underline{\alpha}_{ab} = \mathbf{R}_{a3b3}, \quad \rho = \frac{1}{4} \mathbf{R}_{3434}, \quad \text{}^* \rho = \frac{1}{4} \text{}^* \mathbf{R}_{3434}.$$

⁵⁶We refer the reader to section 8.3 in [63] for the details.

⁵⁷Or rather the complexified vectors $m = e_1 + ie_2$ and $\bar{m} = e_1 - ie_2$.

⁵⁸There are no smooth, global choices of a basis (e_1, e_2) . The choice (1.5) in Kerr, for example, is singular at $\theta = 0, \pi$.

The null structure and null Bianchi equations can then be derived as in the integrable case, see chapter 7 in [25]. The only new features are the presence of the scalars ${}^{(a)}\text{tr}\chi$, ${}^{(a)}\text{tr}\underline{\chi}$ in the equations. Finally we note that the equations acquire additional simplicity if we pass to complex notations⁵⁹,

$$\begin{aligned} A &:= \alpha + i^* \alpha, & B &:= \beta + i^* \beta, & P &:= \rho + i^* \rho, & \underline{B} &:= \underline{\beta} + i^* \underline{\beta}, & \underline{A} &:= \underline{\alpha} + i^* \underline{\alpha}, \\ X &:= \chi + i^* \chi, & \underline{X} &:= \underline{\chi} + i^* \underline{\chi}, & H &:= \eta + i^* \eta, & \underline{H} &:= \underline{\eta} + i^* \underline{\eta}, & Z &:= \zeta + i^* \zeta. \end{aligned} \quad (4.1)$$

4.3.3 Frame transformations and choice of frames

Given an arbitrary perturbation of Kerr, there is no a-priori reason to prefer an horizontal structure to any other one obtained from the first by another perturbation of the same size. It is thus essential that we consider all possible frame transformations from one horizontal structure (e_4, e_3, \mathcal{H}) to another one $(e'_4, e'_3, \mathcal{H}')$ together with the transformation formulas $\Gamma \rightarrow \Gamma'$, $R \rightarrow R'$ they generate. The most general transformation formulas between two null frames is given in Lemma 3.1 of [61]. It depends on two horizontal 1-forms f, \underline{f} and a real scalar function λ and is given by

$$\begin{aligned} e'_4 &= \lambda \left(e_4 + f^b e_b + \frac{1}{4} |f|^2 e_3 \right), \\ e'_a &= \left(\delta_a^b + \frac{1}{2} \underline{f}_a f^b \right) e_b + \frac{1}{2} \underline{f}_a e_4 + \left(\frac{1}{2} f_a + \frac{1}{8} |f|^2 \underline{f}_a \right) e_3, \\ e'_3 &= \lambda^{-1} \left(\left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16} |f|^2 |\underline{f}|^2 \right) e_3 + \left(\underline{f}^b + \frac{1}{4} |\underline{f}|^2 f^b \right) e_b + \frac{1}{4} |\underline{f}|^2 e_4 \right). \end{aligned} \quad (4.2)$$

The very important transformation formulas $\Gamma \rightarrow \Gamma'$, $R \rightarrow R'$ are given in Proposition 3.3 of [61].

The case of $\mathcal{K}(a, m)$, $a \neq 0$ presents an interesting new feature which can be described as follows:

- To capture the simplicity induced by the principle null directions in Kerr it is natural to work with non-integrable frames. We do in fact define all our main quantities relative to frames for which all quantities which vanish in Kerr are of the size of the perturbation.
- A crucial aspect of all important results in GR, based on integrable S -foliations, is that one can rely on elliptic Hodge theory on each 2-surface S . This is no longer

⁵⁹The dual here is taken with respect to the antisymmetric horizontal 2-tensor ϵ_{ab} .

possible in context where our main quantities and the basic equations they verify are defined relative to non integrable frames. In our work we deal with this problem by passing back and forth, whenever needed, from the main non-integrable frame to a well chosen adapted integrable frame, according to the transformation formulas mentioned above.

4.3.4 Renormalization procedure and the canonical complex 1-form \mathfrak{J}

We first notice that our main complex quantities introduced in (4.1) take a particularly simple form in the principal null frame (1.4) of Kerr:

$$\begin{aligned} A = \underline{A} = B = \underline{B} = 0, \quad P &= -\frac{2m}{q^3}, \\ \widehat{X} = \underline{\widehat{X}} = 0, \quad \text{tr}X &= \frac{2}{q} \frac{\Delta}{|q|^2}, \quad \text{tr}\underline{X} = -\frac{2}{\bar{q}}, \\ Z &= \frac{aq}{|q|^2} \mathfrak{J}, \quad H = \frac{aq}{|q|^2} \mathfrak{J}, \quad \underline{H} = -\frac{a\bar{q}}{|q|^2} \mathfrak{J}, \end{aligned} \tag{4.3}$$

where $q = r + ia \cos \theta$, and where the regular⁶⁰ complex 1-form \mathfrak{J} is given by

$$\mathfrak{J}_1 = \frac{i \sin \theta}{|q|}, \quad \mathfrak{J}_2 = \frac{\sin \theta}{|q|}, \tag{4.4}$$

see sections 2.4.2 and 2.4.3 in [63]. In particular, the following holds for the complexified horizontal tensors of (4.1) in the principal null frame (1.4) of Kerr:

- the complex scalars P , $\text{tr}X$ and $\text{tr}\underline{X}$ are functions of r and $\cos \theta$,
- the non vanishing complex 1-forms H , \underline{H} and Z consist of functions of r and $\cos \theta$ multiplied by \mathfrak{J} ,
- the traceless symmetric complex 2-tensors A , \underline{A} , \widehat{X} and $\underline{\widehat{X}}$ vanish identically.

Based on that observation, for a given horizontal structure perturbing the one of Kerr, we can define a renormalization procedure by which, once we have⁶¹ suitable constants (a, m) , suitable scalar functions (r, θ) , and a suitable complex 1-form \mathfrak{J} , and after subtracting the

⁶⁰Note that \mathfrak{J} is regular including at $\theta = 0, \pi$.

⁶¹The constants m and a are computed on our GCM sphere S_* , see section 4.3.1. r , θ and \mathfrak{J} are chosen on S_* , transported to Σ_* and then to \mathcal{M} . The horizontal structure is also defined first on Σ_* and then transported to \mathcal{M} .

corresponding values in Kerr computed from $(a, m, r, \theta, \mathfrak{J})$ for all the Ricci and curvature coefficients, we obtain quantities which are first order in the perturbation.

More precisely, once (a, m) , (r, θ) and \mathfrak{J} have been chosen, we renormalize the quantities in (4.1) that do not vanish in Kerr as follows⁶²:

$$\begin{aligned} \check{P} &:= P + \frac{2m}{q^3}, & \check{\text{tr}X} &:= \text{tr}X - \frac{2}{q} \frac{\Delta}{|q|^2}, & \check{\text{tr}\underline{X}} &:= \text{tr}\underline{X} + \frac{2}{\bar{q}}, \\ \check{Z} &:= Z - \frac{aq}{|q|^2} \mathfrak{J}, & \check{H} &:= H - \frac{aq}{|q|^2} \mathfrak{J}, & \check{\underline{H}} &:= \underline{H} + \frac{a\bar{q}}{|q|^2} \mathfrak{J}. \end{aligned} \tag{4.5}$$

4.3.5 Principal Geodesic and Principal Temporal structures

In addition to the GCM gauge conditions on Σ_* , we need to construct a gauge on \mathcal{M} which relates the non integrable horizontal structure to the scalars (r, θ) and the complex 1-form \mathfrak{J} . Two such gauges were introduced in [63]:

- Principal Geodesic (PG) structure, which is a generalization of the geodesic foliation to non-integrable horizontal structures,
- Principal Temporal (PT) structure, which favors transport equations along a null direction.

The PG structure is well suited for decay estimates, but fails to be well posed. Indeed, due to the lack of integrability of the horizontal structure, we cannot control the null structure equations⁶³ without a loss of derivative. The PT structure, on the other hand, is designed so that the loss of derivatives in the null structure equations, in the incoming or outgoing direction, is completely avoided. Note however that the PT structure is not well suited to the derivation of decay estimates on $^{(ext)}\mathcal{M}$ where r can take arbitrary large values. In [63] we work with both gauge conditions, depending on the goal we want to achieve, and rely on the transformation formulas (4.2) to pass from one to the other.

In the outgoing normalization both the outgoing PG and PT structures consist of a choice (e_3, e_4, \mathcal{H}) , with e_4 null geodesic, together with a scalar functions r, θ and a complex 1-form \mathfrak{J} such that $e_4(r) = 1$, $e_4(\theta) = 0$, $\nabla_4(q\mathfrak{J}) = 0$ where $q = r + ia \cos \theta$. In addition:

1. In a PG structure the gradient of r , given by $N = \mathbf{g}^{\alpha\beta} \partial_\beta r \partial_\alpha$, is perpendicular to \mathcal{H} ,

⁶²The renormalization is written here in the case of a null pair (e_3, e_4) with an ingoing normalization.

⁶³In integrable situation, like in the case of S -foliations, the Hodge systems on the leaves of the S -foliation allows us to avoid the loss.

2. In a PT structure $\underline{H} = -\frac{a\bar{q}}{|q|^2}\check{\mathfrak{J}}$, i.e. $\check{\underline{H}} = 0$ in view of (4.5).

A similar definition of incoming PG and PT structures is obtained by interchanging the roles of e_3, e_4 . Note that both structures still need to be initialized. The outgoing PG and PT structures of ${}^{(ext)}\mathcal{M}$ are both initialized on Σ_* from the GCM frame of Σ_* , while the ingoing PT structures of ${}^{(int)}\mathcal{M}$ and ${}^{(top)}\mathcal{M}$ are initialized on the the timelike hypersurface \mathcal{T} , see Figure 5, using the data induced by the outgoing structures.

4.3.6 Control of the extreme curvature components A, \underline{A}

It was already observed by Teukolsky that, in linear theory, the extreme components of the curvature are both gauge invariant and verify decoupled wave equations⁶⁴. In our nonlinear context this translates to the statement that the horizontal 2-tensors A, \underline{A} , defined relative to an $O(\epsilon)$ perturbation of the principal frame of Kerr, are $O(\epsilon^2)$ -invariant, relative to $O(\epsilon)$ frame transformations⁶⁵, and verify tensorial wave equations of the form

$$\dot{\square}_2 A + L[A] = \text{Err}(\check{\Gamma}, \check{R}), \quad \dot{\square}_2 \underline{A} + \underline{L}[\underline{A}] = \underline{\text{Err}}(\check{\Gamma}, \check{R}). \quad (4.6)$$

Here $\dot{\square}_2$ denotes the wave operator on horizontal symmetric traceless 2-tensors, L and \underline{L} are linear first order operators and $\check{\Gamma}, \check{R}$ denote the linearized Ricci and curvature coefficients. The error terms $\text{Err}(\check{\Gamma}, \check{R}), \underline{\text{Err}}(\check{\Gamma}, \check{R})$ are nonlinear expressions in $\check{\Gamma}, \check{R}$.

In linear theory, i.e. when \mathbf{g} is the Kerr metric and the error terms are not present, these equations have been treated by [30] in Schwarzschild⁶⁶ and by [69] and [31] in slowly rotating⁶⁷ Kerr, i.e. $|a|/m \ll 1$. More precisely both results derive realistic decay estimates for A, \underline{A} . The methods are however not robust. Indeed, a crucial ingredient in the proof, the Energy-Morawetz estimates, is based on separation of variables. The control of A and \underline{A} in perturbations of Kerr in [39] contains the following new features:

- *Derivation of the gRW equation.* The derivation of the generalized Regge-Wheeler equations in Kerr, in [69] and [31], is done starting with the complex, scalar, Teukolsky equations, derived via the NP, or GHP formalism, by applying a Chandrasekhar type transformation. In part I of [39] we extend their derivation, using our non-integrable horizontal formalism, to perturbations of Kerr. By contrast with [69], [31], we derive gRW equations for the horizontal symmetric traceless 2-tensors⁶⁸

⁶⁴See discussion in section 2.5.

⁶⁵This means that $f, \check{f}, \lambda - 1$ are $O(\epsilon)$ in the transformation formulas (4.2).

⁶⁶See discussion in section 3.3.

⁶⁷See discussion in section 3.4.

⁶⁸Derived from A, \underline{A} , see Definition 5.2.2 and 5.3.3 in [39].

$\mathfrak{q}, \underline{\mathfrak{q}}$, rather than for complex scalars. The main difficulty here is to make sure that the non-linear error terms verify a favorable structure.

- *Nonlinear error terms.* The control of the nonlinear terms and their associated null structure was already understood in perturbations of Schwarzschild in [60] and is extended to perturbations of Kerr in [39].
- *Energy-Morawetz.* To derive energy-morawetz estimates for A, \underline{A} in Part II of [39] we vastly extend the pioneering idea of Andersson and Blue [4], based on commutations with \mathbf{T}, \mathbf{Z} and the second order Carter operator \mathcal{C} , developed in the context of the scalar wave equation in slowly rotating Kerr, to treat our tensorial Teukolsky and gRW equations in perturbations of Kerr.

4.3.7 Comment on the full sub-extremal range

Though the full sub-extremal range $|a| < m$ remains open we remark that a large part of our work does not require the smallness of $|a|/m$. This is the case for [61] [62] [81] and [63]. In fact the smallness assumption is only needed in [39], mostly in the derivation of the main energy-morawetz estimates in parts II and III.

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