PROBLEMS IN PDE III. GENERAL IDEAS

- Fundamental solutions.
- A-priori estimates
- Boot-strap and continuity arguments
- Method of generalized solutions
- Microlocal analysis and paradifferential calculus

EXPLICIT REPRESENTATIONS. FUNDAMENTAL SLUTIONS.

HEAT EQ. $-\partial_t u + \Delta u = 0, \quad u(0,x) = u_0(x)$

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} E_d(t, x-y) u_0(y) dy \\ E_d(t,x) &= (4\pi t)^{-d/2} H(t) e^{-|x|^2/4t} \end{aligned}$$

LAPLACE EQ. $\Delta\phi = 0$

$$\begin{aligned} \phi &= \int_{\mathbb{R}^d} K_d(x-y) f(y) dy \\ K_d(x) &= \big((2-d)\omega_d\big)^{-1} |x|^{2-d} \\ K_2(x) &= (2\pi)^{-1} \log |x| \end{aligned}$$

WAVE EQ.

$$\Box\phi = 0, \qquad \phi(0) = f_0, \quad \partial_t\phi(0) = f_1,$$

$$\begin{aligned} u(t,x) &= \partial_t\Big((4\pi t)^{-1} \int_{|x-y|=t} f_0(y) da(y)\Big) \\ &+ (4\pi t)^{-1} \int_{|x-y|=t} f_1(y) da(y) \end{aligned}$$

EXPLICIT REPRESENTATIONS. FOURIER TRANSFORM.

HEAT EQ. $-\partial_t u + \Delta u = 0, \quad u(0,x) = u_0(x)$

$$\widehat{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(t,x)dx, \qquad \widehat{u}(0,\xi) = \widehat{u}_0(\xi)$$

$$\begin{aligned} \partial_t \widehat{u}(t,\xi) &= -\xi^2 \widehat{u}(t,\xi), \\ \widehat{u}(t,\xi) &= \widehat{u_0}(\xi) e^{-t|\xi|^2}. \end{aligned}$$

$$u(t,x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi$$

WAVE EQ.

$$\Box \phi = 0, \qquad \phi(0) = f_0, \quad \partial_t \phi(0) = f_1$$

$$\begin{aligned} u(t,x) &= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\xi} \cos(t|\xi|) \widehat{f}_0(\xi) d\xi \\ &+ (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\xi} \frac{\sin(t|\xi|)}{|\xi|} \widehat{f}_0(\xi) d\xi \end{aligned}$$

COMPARISON

A PRIORI ESTIMATES, ELLIPTIC.

1. L^2 -ESTIMATES

- $\|\partial^2 u\|^2_{L^2(\mathbb{R}^d)} = \|\Delta u\|^2_{L^2(\mathbb{R}^d)}$
- $\|\partial u\|^2_{L^2(\mathbb{R}^2)} = \|\mathrm{d}iv\, u\|^2_{L^2(\mathbb{R}^2)} + \|\mathrm{c}url\, u\|^2_{L^2(\mathbb{R}^2)}$
- $\int_S |\partial^2 u|^2_g da_g + \int_S K|\partial u(x)|^2_g da_g = \int_S |\Delta_S u|^2 da_g$

2. MAXIMUM PRINCIPLE $\quad u \in C(\overline{D}) \cap C^2(D)$

- $\Delta u = 0$
- $a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u = 0,\ c \leq 0.$

$$\max_{x \in \overline{D}} |u(x)| = \max_{x \in \partial D} |u(x)|$$

3. HARNAK INEQUALITY

4. CALDERON-ZYGMUND ESTIMATES

$$\sum_{i,j=1}^{d} \|\partial_i \partial_j u\|_{L^p(\mathbb{R}^d)} \le C_p \|\Delta u\|_{L^p(\mathbb{R}^d)}, \ 1 < p < \infty.$$

5. SCHAUDER ESTIMATES

$$\sum_{i,j=1}^{d} \|\partial_i \partial_j u\|_{C^{0,\alpha}(\mathbb{R}^d)} \le C_\alpha \|\Delta u\|_{C^{0,\alpha}(\mathbb{R}^d)}$$

WHERE,

$$\|f\|_{C^{0,\alpha}(D)} = \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

FREEZING COEFFICIENTS Extend Schauder estimates to second order elliptic operators,

$$Lu(x) \ = \ a^{ij}(x)\partial_i \partial_j u(x) + b^i(x)\partial_i u(x) + c(x)u(x)$$

$$\|\partial^2 u\|_{C^{0,\alpha}} + \|\partial u\|_{C^{0,\alpha}} \le C\Big(\|f\|_{C^{0,\alpha}} + \|u\|_{C^0}\Big)$$

A PRIORI ESTIMATES, HYPERBOLIC.

1. BASIC ENERGY ESTIMATES

2. HIGHER ENERGY ESTIMATES

3. SOBOLEV INEQUALITIES AND UNIFORM BOUNDS

4. VECTORFIELD METHOD AND DECAY

BOOT-STRAP AND CONTINUITY ARGUMENTS

Theorem. $\partial_t^2 u = -V'(u)$, $V : \mathbb{R} \to \mathbb{R}$ *smooth,* $V(0) = V'(0)$ *and* $V''(0) > 0$, *such as* $V(u) = \frac{1}{2}c^2u^2 - u^3$, *for some* $1 \geq c > 0$.
Then, for all $u(0) = u_0$, $\partial_t u(0) = u_1$ *sufficiently small, there exists a unique global solution of the equation, which remains close the origin, i.e.* $|u(t)| + |\partial_t u(t)|$ *stays small for all* $t \geq 0$.

$E(t) = E(0) \leq \delta, \qquad E(t) = \frac{1}{2}\Big((\partial_t u(t))^2 + c^2 u(t)^2\Big) -$

BOOTSTRAP ASSUMPTION

$$A(T): \quad E'(T) := \sup_{t \in [0,T]} (\partial_t u(t))^2 + u(t)^2 \leq \epsilon^2,$$

Let $T_m \leq T_*$ be the largest time for which A(T) holds. By the continuity of u and $\partial_t u$, if T_m is finite, we must have,

$$E'(T_m) = \epsilon^2$$

IN FACT. If ϵ, δ small $\Rightarrow E'(T_m) \leq \frac{1}{2}\epsilon^2$

$$-\Delta u + u^3 = f(x), \qquad u|_{\partial D} = 0$$

SIMPLE PROBLEM:

$$-\Delta u = f(x), \qquad u|_{\partial D} = 0$$

INTRODUCE:

$$-\Delta u + tu^3 = f(x), \qquad u|_{\partial D} = 0$$

SHOW, Set $J \subset [0, 1]$ of values of t for which the problem can be solved in some functional space X, is both open and closed.

OPEN Implicit function theorem and Schauder estimates. $f \in C^{\alpha}(\bar{D})$, $u \in X = C^{2,\alpha}(\bar{D})$

CLOSED Need an a-priori estimate. Maximum principle.

GENERALIZED SOLUTIONS.

DIRICHLET PRINCIPLE: Harmonic functions u in $D \subset \mathbb{R}^d$ with $u|\partial D = f$ are minimizers of,

$$\|v\|_{Dr}^2 = \frac{1}{2}\int_D |\nabla v|^2 = \frac{1}{2}\sum_{i=1}^{d}\int_D |\partial_i v|^2,$$

in an appropriate functional space X.
RIEMANN: Use Dirichlet Principle to solve,

$$\Delta u = 0, \qquad u|_{\partial D} = u_0.$$

WEAK SOLUTION For any $\phi \in C^\infty(\overline{D})$

$$\sum_i \int_D \partial_i u \partial_i \phi = 0,$$

INHOMOG. PR. $\Delta u = f, u|_{\partial D} = 0$

$$I(v) = \|v\|_{Dr}^2 - \int_D v(x) f(x) dx$$

SOBOLEV SPACE $H_0^1(D)$

$$Lu = \partial_i(a^{ij}(x)\partial_j u) + b^i(x)\partial_i u(x) + c(x)u(x) = 0$$

WEAK SOLUTIONS For any $\phi \in C_0^1(D)$,

$$\int_D a^{ij}(x)\partial_j u(x)\partial_i \phi(x) = 0$$

Theorem[Di-Giorgi-Nash] *Assume that* a^{ij}, b, c *are measurable, bounded almost everywhere (a.e.) in* D *and that* a^{ij} *verify the ellipticity condition,*

$$a^{ij}(x)\xi_i\xi_j > c|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d, \quad \text{and a.e.} \quad x \in D$$

Then, every generalized solution $u \in H^1(D)$ *of the equation* $Lu = 0$*must be continuous in* D*, and in fact Hölder continuous for some exponent* $\delta > 0$*, i.e.* $u \in C^{0,\delta}(D)$*.*