PROBLEMS IN PDE II

- FIRST ORDER EQTS.
- CAUCHY PROBLEM. CHARACTERISTIC SURFACES
- CAUCHY-KOWALEWSKY
- ILL POSED AND WELL POSED PROBLEMS
- REGULARITY OR BREAK-DOWN. SCALING

FIRST CLASSIFICATION

- ORDER
- LINEAR AND NONLINEAR
- SEMILINEAR, QUASILINEAR, FULLY NONLINEAR
- HOMOGENEITY

LINEAR: Laplace, Wave, Heat, Schrödinger, Maxwell

SEMILINEAR: Yang-Mills

QUASILINEAR: Minimal surfaces, Compressible and incompressible Euler, Einstein field equations, MHD.

FULLY NONLINEAR: Monge -Ampere

FIRST ORDER EQTS.

LINEAR: $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\quad a : \mathbb{R}^d \longrightarrow \mathbb{R}$

$$a^i(x)\partial_i u(x) = f(x).$$

CHARACTERISTIC SYSTEM

$$\frac{dx^i}{ds} = a^i(x(s)), \qquad i = 1, \dots d$$

METHOD OF CHARACTERSITICS

$$\frac{d}{ds}u\Big(x(s)\Big) = f(x(s))$$

PROPAGATION PROPERTIES

QUASILINEAR: $a^1, a^2 \dots a^d, f : \mathbb{R}^d \longrightarrow \mathbb{R}$

$$a^i(x, u(x))\partial_i u(x) = f(x, u(x))$$

BURGER EQUATION

$$\partial_t u + u\partial_x u = 0, \qquad u(0,x) = u_0(x)$$

CHARACTERISTIC EQUATION

$$\frac{dx}{ds} = u(s, x(s)), \quad x(0) = x_0$$

CHARACTERISTIC METHOD

$$\frac{d}{ds} u(s, x(s)) = 0$$

HENCE $\qquad u(s, x(s)) = u_0(x_0).$

HOW DO YOU CONTINUE ?

$$u(s, x_0 + s u_0(x_0)) = u_0(x_0)$$

BREAK-DOWN OF SOLUTIONS

INITIAL VALUE PR. FOR ODE'S

$$A(x, u(x))\partial_x u = F(x, u(x)), \quad u(x_0) = u_0$$

RECURSIVELY,

$$u = u_0 + (x - x_0)u_1 + \frac{1}{2}(x - x_0)^2 u_2 + \ldots$$

NONCHARACTERISTIC CONDITION

$$\det A(x_0, u_0) \neq 0.$$

CAUCHY- KOWALEWSKY If noncharacteristic condition is verified there exists a unique solution defined in a vicinity of x_0.

FUNDAMENTAL THEOREM OF ODE Need only noncharacteristic condition and A, F locally Lipschitz

PICARD ITERATION

$$\begin{aligned} \partial_x u_{(n)}(x) &= A^{-1} F(x, u_{(n-1)}(x)), \\ u_{(n-1)}(x_0) &= u_0 \end{aligned}$$

INITIAL VALUE PR. FOR PDE'S

$$A^i(x,u(x))\partial_i u(x) = F(x,u(x)); \quad u|_{\mathcal{H}} = u_0$$

DEFINITION. *I. V. P , is non-characteristic at x_0 of $\mathcal{H}$, if we can determine all other higher partial derivatives of u at x_0, uniquely, in terms of the data.*

CHARACTERISTIC SURFACES Characteristic at every point

SCALAR CASE $u = u(x),\ x = (x^1, x^2)$,

$$\sum_{i=1}^{2} a^i(x,u(x))\partial_i u(x) = f(x,u(x)),$$

CHARACTERISTIC SYSTEM

$$\frac{dx^i}{ds} \ = \ a^i(x^1(s), x^2(s)), \quad i = 1,2$$

characteristic curves $\Longleftrightarrow$ characteristic

$$A^i(x,u(x))\partial_i u(x) \;=\; F(x,u(x)),\ u|_{\mathcal{H}} = u_0$$

$\mathcal{H}$ regular hypersurface with normal N

NON-CHARACTERISTIC CONDITION:

$$\sum_{i=1}^{n} a^i(x_0,u_0(x_0))N_i(x_0) \neq 0$$

SECOND ORDER EQUATIONS

$$\sum_{i,j=1}^{d} a^{ij}(x)\partial_i\partial_j u = f(x,u(x),\partial u(x))$$

NON-CHARACTERISTIC CONDITION:

$$\sum_{i,j=1}^{d} a^{ij}(x_0)n_i(x_0)n_j(x_0) \neq 0.$$

PROPOSITION *Assume ellipticity condition*

$$a^{ij}(x)\xi_i\xi_j > 0, \quad \forall\,\xi \in \mathbb{R}^d, \quad \forall x \in \mathbb{R}^d$$

Then no surface in $\mathbb{R}^d$ *can be characteristic.*

EXAMPLE. Laplace and minimal surface equations.

EXAMPLE. $$\Box u = f$$

PROPOSITION. *All hypersurf.* $\psi(t,x) = 0$,

$$(\partial_t \psi)^2 = \sum_{i=1}^{d} (\partial_i \psi)^2,$$

are characteristic.

EXAMPLES.

$$\begin{aligned} \psi_+(t,x) &= (t - t_0) + |x - x_0| \\ \psi_-(t,x) &= (t - t_0) - |x - x_0| \end{aligned}$$

GENERAL WAVE EQUATIONS

$$a^{00}(t,x)\partial_t^2 u - \sum_{i,j} a^{ij}(t,x)\partial_i \partial_j u = 0,$$

CHARACTERISTICS

$$a^{00}(\partial_t \psi)^2 = \sum_{i,j} a^{ij} \partial_i \psi \partial_j \psi.$$

CAUCHY-KOWALEWSKY

THEOREM[Cauchy-Kowalevsky] *If the coefficients, the hypersurface $\mathcal{H}$ and initial conditions are real analytic and if $\mathcal{H}$ is non-characteristic at x_0, there exists locally, near x_0, a unique real analytic solution.*

ILL POSED PROBLEMS

DEFINITION. *A given problem for a PDE is said to be well posed if both existence and uniqueness of solutions can be established for arbitrary data which belong to a specified large space of functions, which includes the class of smooth functions. Moreover the solutions must depend continuously on the data*

STANDARD CLASSIFICATION

ELLIPTIC *A linear, or quasi-linear, $N \times N$ system with no characteristic hyper-surfaces is called elliptic.*
The well posed problems are *boundary value problems*.

HEURISTIC PRINCIPLE: *Classical solutions of elliptic equations with smooth (or real analytic) coefficients in a regular domain D are smooth (or real analytic), in the interior of D, independent of how smooth are the boundary conditions.*

HYPERBOLIC Have a " full set " of characteristic surfaces along which singularities propagate.

PARABOLIC

DISPERSIVE