PROBLEMS IN PDE I

- **Main linear equations**: Laplace, Wave, Klein-Gordon, Heat and Schrödinger equations. Cauchy-Riemann and Maxwell systems of equations.

- **Examples of nonlinear equations**: Minimal surfaces, Burger, nonlinear Klein-Gordon. Incompressible Euler and Navier Stokes equations. Compressible Euler equations.

- **Simple conservation laws and coercive estimates**.

- **Lagrangean formalism**. Variational principle and conservation laws via Noether's theorem

MAIN LINEAR EQUATIONS

- (LAPLACE EQUATION)

$$\Delta u = 0,$$

- (HEAT EQUATION)

$$-\partial_t u + k\Delta u = 0,$$

- (WAVE EQUATION)

$$-\frac{1}{c^2}\,\partial_t^2 u + \Delta u = 0,$$

- (SCHRÖDINGER EQUATION)

$$i\partial_t u + k\Delta u = 0,$$

- (KLEIN-GORDON EQUATION)

$$-\frac{1}{c^2}\,\partial_t^2 u + \Delta u - \frac{mc^2}{\hbar^2} u = 0.$$

OTHER EQUATIONS

- (MINIMAL SURFACE EQUATION)

$$\partial_x\left(\frac{\partial_x u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right)+\partial_y\left(\frac{\partial_y u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right)=0.$$

- (NON-LINEAR WAVE EQ.)

$$-\partial_t^2 u+\Delta u+V'(u)=0. \tag{1}$$

- (NONLINEAR SHRÖDINGER)

$$i\partial_t u+\Delta u+|u|^k u=0. \tag{2}$$

- (BURGER EQUATION)

$$\partial_t+u\partial_x u=0.$$

- (KDV EQUATION)

$$\partial_t u+uu_x+u_{xxx}=0.$$

SYMMETRIES

- SCALING
- ROTATIONS, GALILEAN, LORENTZ TRS. EUCLIDEAN AND MINKOWSKI METRICS
- CONFORMAL TRS.

BASIC PROPERTIES

- LOCALITY
- LINEARITY

SYSTEM OF EQUATIONS

- (CAUCHY RIEMANN)

$$\partial_1 u_2 - \partial_2 u_1 = 0, \qquad \partial_1 u_1 + \partial_2 u_2 = 0$$

,

- (DIRAC EQUATION)

$$\begin{aligned} \mathcal{D}u - mu &= 0 \\ \mathcal{D} &= i\gamma^\alpha \partial_\alpha u \\ \gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha &= -2g^{\alpha\beta} \\ \mathcal{D}^2 &= \Box I \end{aligned}$$

FACT. There exists 4×4 matrices verifying the above relations.

MAXWELL VACUUM EQUATIONS

ELECTROMAGNETIC FIELD

$$F = F_{\alpha\beta} dx^{\alpha} dx^{\beta}, \qquad F = dA$$

MAXWELL EQTS.

$$\partial_{[\gamma} F_{\alpha\beta]} = 0, \qquad \partial^{\beta} F_{\alpha\beta} = 0$$

DUAL PAIR

$$\partial_{[\gamma} \, {}^*F_{\alpha\beta]} = 0, \qquad \partial^{\beta} \, {}^*F_{\alpha\beta} = 0$$

HODGE DUAL $\quad {}^*F_{\alpha\beta} = \frac{1}{2} \in_{\alpha\beta\gamma\delta} F^{\gamma\delta}.$

ELECTRIC-MAGNETIC DECOMP.

$$E_i = F_{0i}, \quad H_i = {}^*F_{0i}$$

$$\begin{aligned} \partial_t E &- \mathrm{c}url\, H = 0, \quad \mathrm{d}iv\, E = 0 \\ \partial_t H &+ \mathrm{c}url\, E = 0, \quad \mathrm{d}iv\, H = 0. \end{aligned}$$

GAUGE FIELDS

MATRIX LIE GROUP G=SO(n)

LIE ALGEBRA $\mathcal{G} = so(n)$

LIE BRACKET $[A, B] = AB - BA$

JACOBI IDENTITY

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.$$

KILLING SCALAR FORM $< A, B >= -\mathrm{Tr}(AB^T)$.

$$< A, [B, C] >= - < [A, B], C >$$

CONNECTION 1-FORM $A = A_\alpha dx^\alpha$,

$$A_\alpha : \mathbb{R}^{1+3} \longrightarrow \mathcal{G}$$

COVARIANT DERIVATIVE

Given $\psi : \mathbb{R}^{1+3} \longrightarrow \mathcal{G}$,

$$\mathbf{D}_\mu^{(A)}\psi = \mathbf{D}_\mu\psi + [A_\mu, \psi]$$

GAUGE TRANSFORMATIONS

$$\tilde{\psi} \;=\; U^{-1}\psi\, U, \qquad \tilde{A}_\alpha = U^{-1}A_\alpha\, U + \left(\partial_\alpha U^{-1}\right) U$$

with $U : \mathbb{R}^{1+3} \longrightarrow G$

PROPOSITION

$$\mathbf{D}_\mu^{(\tilde{A})}\tilde{\psi} \;=\; U^{-1}\left(\mathbf{D}_\mu^{(A)}\psi\right) U = \mathbf{D}^{\widetilde{A}}\psi$$

Proof.

$$\partial_\alpha\left(U^{-1}\psi U\right) =$$
$$= \left(\partial_\alpha U^{-1}\right)\psi U + U^{-1}\left(\partial_\alpha\psi\right) U + U^{-1}\psi\left(\partial_\alpha U\right)$$
$$= U^{-1}\left(-\left(\partial_\alpha U\right) U^{-1}\psi + \partial_\alpha\psi + \psi\left(\partial_\alpha U\right) U^{-1}\right) U$$
$$= U^{-1}\left(\partial_\alpha\psi + [\psi, \left(\partial_\alpha U\right) U^{-1}]\right) U$$

CONTINUE ! $\square$

PROPOSITION. $\mathbf{D}_\alpha \mathbf{D}_\beta \psi - \mathbf{D}_\beta \mathbf{D}_\alpha \psi = [F_{\alpha\beta}, \psi]$ with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$. Moreover F is invariant under gauge transformations.

$$\widetilde{F^{(\tilde{A})}} \left(\equiv U^{-1} F^{(\tilde{A})} U \right) = F^{(A)}$$

Proof.

$$\begin{aligned}
\mathbf{D}_\alpha \mathbf{D}_\beta \psi &= \partial_\alpha \left(\mathbf{D}_\beta \psi \right) + [A_\alpha, \mathbf{D}_\beta \psi] \\
&= \partial_\alpha \left(\partial_\beta \psi + [A_\beta, \psi] \right) \\
&+ \left[A_\alpha, \partial_\beta \psi + [A_\beta, \psi] \right] \\
&= \partial_\alpha \partial_\beta \psi + [\partial_\alpha A_\beta, \psi] + [A_\beta, \partial_\alpha \psi] \\
&+ [A_\alpha, \partial_\beta \psi] + [A_\alpha, [A_\beta, \psi]]
\end{aligned}$$

$$\left(\mathbf{D}_\alpha \mathbf{D}_\beta - \mathbf{D}_\beta \mathbf{D}_\alpha \right) \psi = [\partial_\alpha A_\beta - \partial_\beta A_\alpha, \psi] + \underbrace{\left[A_\alpha, [A_\beta, \psi] \right] - \left[A_\beta, [A_\alpha, \psi] \right]}_{[[A_\alpha, A_\beta], \psi]} = [F_{\alpha\beta}, \psi]$$

□

BIANCHI IDENTITIES

$$\mathbf{D}_\alpha F_{\beta\gamma} + \mathbf{D}_\gamma F_{\alpha\beta} + \mathbf{D}_\beta F_{\gamma\alpha} = 0$$

YANG-MILLS EQUATIONS

$$\mathbf{D}_{[\alpha}F_{\beta\gamma]} = 0, \qquad \mathbf{D}^{\beta}F_{\alpha\beta} = 0$$

DUAL PAIR

$$\mathbf{D}_{[\alpha}\,{}^{*}F_{\beta\gamma]} = 0, \qquad \mathbf{D}^{\beta}\,{}^{*}F_{\alpha\beta} = 0$$

$${}^{*}F_{\alpha\beta} = \frac{1}{2}\in_{\alpha\beta\gamma\delta}F^{\gamma\delta}, \qquad {}^{*}({}^{*}F) = -F$$

E.M. DECOMP. $E_i = F_{0i}, \quad H_i = {}^{*}F_{0i}.$

GAUGE CONDITIONS

- Lorentz $\partial^{\mu}A_{\mu} = 0.$
- Coulomb $\partial^{i}A_{i} = 0.$
- Temporal $A_0 = 0.$

OTHER IMPORTANT SYSTEMS

NAVIER-STOKES EQTS.

EULER EQTS.

COMPRESSIBLE EULER EQTS.

MAGNETO-HYDRODYNAMICS.

BOLTZMAN EQTS.

EINSTEIN FIELD EQTS.

PDE IN GEOMETRY

1.) HODGE THEORY

2.) UNIFORMIZATION THEOREM *S is a 2-d, compact, Riemann surface with metric g, Gauss curvature $K = K(g)$ and Euler characteristic $\chi(S)$. There exists a conformal transformation of the metric g, i.e. $\tilde{g} = e^{2u} g$, for a smooth function u, such that the Gauss curvature $\tilde{K}$ of the new metric $\tilde{g}$ is identical equal to 1, 0 or -1 according to whether $\chi(S) > 0$, $\chi(S) = 0$ or $\chi(S) < 0$.*

Proof. In the case $\chi(S) = 2$ we are led to the semilinear elliptic equation,

$$\Delta_S u + e^{2u} = K$$

$\square$

3.) POINCARE CONJECTURE, GEOMETRIZATION.

VARIATIONAL PRINCIPLE

- SPACE OR SPACE-TIME
- COLLECTION OF FIELDS
- LAGRANGEAN DENSITY
- ACTION INTEGRAL

$$\mathcal{S} = \mathcal{S}[\psi, g : \mathcal{U}] = \int_{\mathcal{U}} L[\psi] dv_g$$

- COMPACT VARIATIONS $\quad \psi = \psi_{(s)}$

ACTION PRINCIPLE: *Acceptable solutions are stationary.*

$$\frac{d}{ds}\mathcal{S}[\psi_{(s)}, g; \mathcal{U}]\Big|_{s=0} = 0.$$

SCALAR FIELD EQUATION

FIELD $\phi : \mathbb{R}^{1+3} \longrightarrow \mathbb{R}$ or $\mathbb{C}$

LAGRANGEAN

$$L[\phi] = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

VARIATION

$$\phi_{(S)} = \phi + s\dot{\phi} + O(s^2)$$

$$\begin{aligned} \frac{d}{ds}\mathcal{S}(s)\Big|_{s=0} &= \int_{\mathcal{U}} [-g^{\mu\nu}\partial_\mu \dot{\phi} \partial_\nu \phi - V'(\phi)\dot{\phi}]\sqrt{-g}dx \\ &= \int_{\mathcal{U}} \dot{\phi}[\Box_g \phi - V'(\phi)]dv_g] \end{aligned}$$

where $\Box_g$ is the D'Alembertian,

$$\Box_g \phi = \frac{1}{\sqrt{-g}} \partial_\mu \Big(g^{\mu\nu} \sqrt{-g}\, \partial_\nu \phi \Big)$$

ACTION PRINCIPLE

$$\Box_g \phi - V'(\phi) = 0,$$

MAXWELL EQUATIONS

FIELD $F = F_{\alpha\beta} dx^\alpha dx^\beta$, dF=0.

POTENTIAL $F = dA, \quad A = A_\alpha dx^\alpha$

LAGRANGEAN $L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$

VARIATION $F_{(s)} = dA_{(s)} = dA + s d\dot{A} + O(s^2)$

$$\dot{F}_{\mu\nu} = \partial_\mu \dot{A}_\nu - \partial_\nu \dot{A}_\mu$$

$$-\frac{d}{ds}\mathcal{S}(s)\Big|_{s=0} = \frac{1}{2}\int_{\mathbf{M}} \dot{F}_{\mu\nu} F^{\mu\nu} dv_g$$

$$= \frac{1}{2}\int_{\mathcal{U}} (\partial_\mu \dot{A}_\nu - \partial_\nu \dot{A}_\mu) F^{\mu\nu} dv_g = \int_{\mathcal{U}} \partial_\mu \dot{A}_\nu F^{\mu\nu} dv_g$$

$$= -\int_{\mathcal{U}} \dot{A}_\nu \left(\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right) \right) dv_g$$

ACTION PRINCIPLE

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right) = 0$$

YANG-MILLS EQUATIONS

FIELD $F = dA + [A, A].$

LAGRANGEAN $L = -\frac{1}{4} < F_{\mu\nu}, F^{\mu\nu} >_{\mathcal{G}}$

VARIATION $A_{(s)} = A + s\dot{A} + O(s^2)$

$$\dot{F}_{\mu\nu} \;=\; \partial_\mu \dot{A}_\nu - \partial_\nu \dot{A}_\mu + [\dot{A}_\mu, A_\nu] + [A_\mu, \dot{A}_\nu]$$

$$\frac{d}{ds}\mathbf{S}(s)\Big|_{s=0} = -\frac{1}{2}\int_{\mathcal{U}} < \dot{F}_{\alpha\beta}, F^{\alpha\beta} >_{\mathcal{G}} dv_m$$
$$= -\int_{\mathcal{U}} < \partial_\alpha \dot{A}_\beta, F^{\alpha\beta} > + < [A_\alpha, \dot{A}_\beta], F^{\alpha\beta} > dv_m$$
$$= \int_{\mathcal{U}} < \dot{A}_\beta, \partial_\alpha F^{\alpha\beta} > + < \dot{A}_\beta, \left[A_\alpha, F^{\alpha\beta}\right] > dv_m$$

ACTION PRINCIPLE

$$D_\nu^{(A)} F^{\mu\nu} = 0$$

ENERGY-MOMENTUM TENSOR,

THEOREM. For each variational system of equations we can define a symmetric 2− tensor $T_{\mu\nu}$ which verifies the *local conservation laws*

$$D^\nu T_{\mu\nu} \;=\; 0$$

POSITIVITY For an acceptable physical theory T must be positive, i.e. for any time-like, future oriented vector-fields $X^\mu\partial_\mu$, $Y = Y^\nu\partial_\nu$,

$$T(X,Y) := T_{\mu\nu}X^\mu Y^\nu > 0.$$

NOETHER THEOREM To any continuous symmetry of the physical space (i.e. Minkowki) there corresponds a conservation law.

Proof. Set $P_\mu = T_{\mu\nu}X^\nu$

$$\begin{aligned} D^\mu P_\mu &= T_{\mu\nu}D^\mu X^\nu = \frac{1}{2}T_{\mu\nu}\,{}^{(X)}\pi^{\mu\nu} \\ {}^{(X)}\pi_{\mu\nu} &= D_\mu X_\nu + D_\nu X_\mu \end{aligned}$$

□

KILLING VECTOR-FIELDS IN MINKOWSKI SPACE,

DEFINITION. A vector-field X is called Killing if ${}^{(X)}\pi = 0$. It is called conformal Killing if ${}^{(X)}\pi$ is proportional to the metric.

1. Generators of translations :
$$T_{(\mu)} = \frac{\partial}{\partial x^\mu}$$
2. Generators of the Lorentz rotations
$$L_{(\mu\nu)} = x_\mu \partial_\nu - x_\nu \partial_\mu$$
3. Generators of the scaling transformations:
$$S = x^\mu \partial_\mu$$
4. Generators of the inverted translations
$$K_{(\mu)} = 2x_\mu \, x^\rho \frac{\partial}{\partial x^\rho} - (x^\rho x_\rho) \frac{\partial}{\partial x^\mu}$$

EXAMPLES,

1. Scalar field equation,

$$T_{\alpha\beta} = \frac{1}{2}\left(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2V(\phi))\right)$$

2. Maxwell equations,

$$T_{\alpha\beta} = F_{\alpha}{}^{\mu}F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}(F_{\mu\nu}F^{\mu\nu})$$

3. Yang-Mills equations,

$$T_{\alpha\beta} = < F_{\alpha}{}^{\mu}, F_{\beta\mu} >_{\mathcal{G}} - \frac{1}{4}g_{\alpha\beta}(< F_{\mu\nu}, F^{\mu\nu} >_{\mathcal{G}})$$

PROPOSITION. In all the above examples $T_{\alpha\beta}$ verifies the positive energy condition.

Proof. Let $L, \underline{L}$ be the two future directed null vectors corresponding to the two complementary null directions, normalized by,

$$< L, \underline{L} >= -2,$$

such that X, Y are linear combinations with positive coefficients of $L, \underline{L}$. Proposition will follow from,

$$T(L, L) \geq 0, \quad T(\underline{L}, \underline{L}) \geq 0, \quad T(L, \underline{L}) \geq 0.$$

NULL FRAME:

$E_{(d+1)} = L, \quad E_{(d)} = \underline{L}, \quad E_{(1)}, \ldots, E_{(d-1)}.$

$$\begin{aligned} m(E_{(i)}, E_{(d)}) &= m(E_{(i)}, E_{(d+1)}) = 0, \\ m(E_{(i)}, E_{(j)}) &= \delta_{ij}, \quad i, j = 1, \ldots, d-1. \end{aligned}$$

$\square$