HAWKING'S LOCAL RIGIDITY THEOREM WITHOUT ANALYTICITY

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Abstract. We prove the existence of a Hawking Killing vector-field in a full neighborhood of a local, regular, bifurcate, non-expanding horizon embedded in a smooth vacuum Einstein manifold. The result extends a previous result of Friedrich, Rácz and Wald, see [13, Proposition B.1], which was limited to the domain of dependence of the bifurcate horizon. So far, the existence of a Killing vector-field in a full neighborhood has been proved only under the restrictive assumption of analyticity of the space-time. Using this result we provide the first unconditional proof that a stationary black hole solution must posess an additional, rotational Killing field in an open neighborhood of the event horizon. This work is accompanied by a second paper, where we prove a uniqueness result for smooth stationary black hole solutions which are close (in a very precise, geometric sense) to the Kerr family of solutions, for arbitrary $0 < a < m$.

Contents

1. Introduction

It is widely expected[1] that the domains of outer communications of regular, stationary, four dimensional, vacuum black hole solutions are isometrically diffeomorphic to those of the Kerr black holes. Due to gravitational radiation, general, asymptotically flat, dynamic solutions of the Einstein-vacuum equations ought to settle down, asymptotically, into a stationary regime. Thus the conjecture, if true, would characterize all possible asymptotic

[1]See reviews by B. Carter [7] and P. Chusciel [9] for a history and review of the current status of the conjecture.

states of the general vacuum evolution. A similar scenario is supposed to hold true in the presence of matter.

So far the conjecture is known to be true[2] if, besides reasonable geometric and physical conditions, one assumes that the space-time metric in the domain of outer communications is *real analytic*. This last assumption is particularly restrictive, since there is apriori no reason that general stationary solutions of the Einstein field equations should be analytic in the ergoregion, i.e. the region where the stationary Killing vector-field becomes space-like. The strategy of the well known proof starts with the observation, due to Hawking, that the event horizon of a general stationary metric is non-expanding and the stationary Killing field must be tangent to it. Specializing to the future event horizon Hawking [14] (see also [18]) proved the existence of a non-vanishing vector-field $\mathbf{K}$ tangent to the null generators of the horizon and Killing to any order along it. Under the assumption of real analyticity of the space-time metric one can prove, by a Cauchy-Kowalewski type argument (see [14] and the rigorous argument in [10]), that the Hawking Killing vector-field $\mathbf{K}$ can be extended to a neighborhood of the entire domain of outer communications. Therefore, it follows that the spacetime $(\mathbf{M}, \mathbf{g})$ is not just stationary but also axi-symmetric, at least in the case when the stationary vector-field does not coincide with $\mathbf{K}$ (i.e. in the case of rotating horizon). To derive uniqueness, one then appeals to the theorem of Carter and Robinson[3] which shows that the exterior region of a regular, stationary, axi-symmetric vacuum black hole must be isometrically diffeomorphic to a Kerr exterior of mass M and angular momentum $a < M$. The proof of this result originally obtained by Carter [5] and Robinson [23], has been strengthened and extended by many authors, notably Mazur [22], Bunting [4], Weinstein [27]; the most recent and complete account, which fills in various gaps in the previous literature is the recent paper of Chrusciel and Costa [12].

In this paper we present the first unconditional proof of the existence of the Hawking vector-field in a small neighborhood of the event horizon, with no recourse to analyticity. As we explain in the next subsection, our main new idea, is to use unique continuation arguments, based on Carleman estimates, for solutions of the vacuum Einstein equations in a neighborhood of the bifurcation sphere. Based on such arguments we are able to extend the Hawking vector-field, from the causal future and past of the bifurcation sphere, to a small neighborhood of it in the domain of outer communications. We also state (in Theorem 1.3 below, which is proven in [2]) an important application to the black hole uniqueness problem. Though, for the moment, the result of [2] is only perturbative (it applies only to stationary black holes that are "close" to one of the Kerr family of solutions, in a very precise, geometric, sense) we strongly believe that the unconditional local results and methods of this paper will provide a vital ingredient in any future attempt to prove the full black hole uniqueness conjecture (with no closeness condition).

[2]By combining results of Hawking [14], Carter [5], and Robinson [23], see also the recent work of Chrusciel-Costa [12].

[3]This second step does not require analyticity.

1.1. **Statement of main results.** Let $(\mathbf{M}, \mathbf{g})$ to be a smooth[4] vacuum Einstein space-time. Let S be an embedded spacelike 2-sphere in $\mathbf{M}$ and let $\mathcal{N}, \underline{\mathcal{N}}$ be the null boundaries of the causal set of S, i.e. the union of the causal future and past of S. We fix $\mathbf{O}$ to be a small neighborhood of S such that both $\mathcal{N}, \underline{\mathcal{N}}$ are regular, achronal, null hypersurfaces in $\mathbf{O}$ spanned by null geodesic generators orthogonal to S. We say that the triplet $(S, \mathcal{N}, \underline{\mathcal{N}})$ forms a local, regular, bifurcate, non-expanding horizon in $\mathbf{O}$ if both $\mathcal{N}, \underline{\mathcal{N}}$ are non-expanding null hypersurfaces (see definition 2.1) in $\mathbf{O}$. Our main result is the following:

Theorem 1.1. *Given a local, regular, bifurcate*[5] $(S, \mathcal{N}, \underline{\mathcal{N}})$ *in a smooth, vacuum Einstein space-time* $(\mathbf{O}, \mathbf{g})$*, there exists an open neighborhood* $\mathbf{O}' \subseteq \mathbf{O}$ *of* S *and a non-trivial Killing vector-field* $\mathbf{K}$ *in* $\mathbf{O}'$*, which is tangent to the null generators of* $\mathcal{N}$ *and* $\underline{\mathcal{N}}$*. In other words, every local, regular, bifurcate, non-expanding horizon is a Killing bifurcate horizon.*

It is already known, see [13], that such a Killing vector-field exists in a small neighborhood of S intersected with the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. The extension of $\mathbf{K}$ to a full neighborhood of S has been known to hold only under the restrictive additional assumption of analyticity of the space-time (see [14], [18], [13]). The novelty of our theorem is the existence of Hawking's Killing vector-field $\mathbf{K}$ in a full neighborhood of the 2-sphere S, without making any analyticity assumption. The assumption that the non-expanding horizon in Theorem 1.1 is *bifurcate* is essential for the proof; this assumption is consistent with the application to our global result.

In applications to our global result we also need the following:

Theorem 1.2. *Assume that* $(S, \mathcal{N}, \underline{\mathcal{N}})$ *is a local, regular, bifurcate, non-expanding horizon in a vacuum Einstein space-time* $(\mathbf{O}, \mathbf{g})$ *which possesses a Killing vector-field* $\mathbf{T}$ *tangent to* $\mathcal{N} \cup \underline{\mathcal{N}}$ *and not identically vanishing on* S*. Then, there exists an open neighborhood* $\mathbf{O}' \subseteq \mathbf{O}$ *of* S *and a non-trivial rotational Killing vector-field* $\mathbf{Z}$ *in* $\mathbf{O}'$ *which commutes with* $\mathbf{T}$*. Moreover we can construct a regular double null foliation*[6] $(u, \underline{u})$ *in* $\mathbf{O}' \subseteq \mathbf{O}$*, with* $u = 0$ *on* $\mathcal{N}$ *and* $\underline{u} = 0$ *on* $\underline{\mathcal{N}}$*, such that* $\mathbf{Z}$ *leaves invariant the* 2*-surfaces* $S_{u,\underline{u}}$ *on which* $u, \underline{u}$ *are constant.*

A related version of the first part of our result was known only in the special case when the space-time is analytic[7]. In view of theorem 1.1, there exists a Hawking vector-field $\mathbf{K}$, in a full neighborhood of S. One can easily show that it must commute with $\mathbf{T}$. We then show that there exist constants λ_0 and $t_0 > 0$ such that

$$\mathbf{Z} = \mathbf{T} + \lambda_0 \mathbf{K} \tag{1.1}$$

[4] $\mathbf{M}$ is assumed to be a connected, oriented, C^∞ 4-dimensional manifold without boundary.

[5] Hawking's original rigidity theorem relies instead on a non-degeneracy assumption. We note however that the two assumptions are in fact related, see [24].

[6] That is $\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha u \mathbf{D}_\beta u = \mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha \underline{u} \mathbf{D}_\beta \underline{u} = 0$ and $\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha u \mathbf{D}_\beta \underline{u}$ nowhere vanishing.

[7] Hawking's rigidity theorem, see [14], asserts that, under some global causality, asymptotic flatness and connectivity assumptions, a stationary, rotating, non-degenerate, real analytic spacetime must be axially symmetric.

is a rotation with period t_0. The main constants λ_0 and t_0 can be determined on the bifurcation sphere S. While this first part of theorem 1.2 can, in principle, be recovered from theorem 1.1 using existing results in the literature, the second, more precise version of the result seems new to us. This precise version, as well as all the other complementary results proved in section 4, is important in our main application below.

We remark that the two theorems presented below are general, local, results which we expect to play a crucial role in a future, general, classification of stationary[8], smooth, vacuum, black holes. For the moment, however, the only global result we can prove is the following perturbative one.

Theorem 1.3. *Any regular stationary black-hole solution of the vacuum Einstein equations, which is a perturbation of a Kerr solution $\mathcal{K}(a,m)$ with $0 < a < m$ is in fact the Kerr solution.*

A precise version of the result[9], together with its proof, can be found in [2]. The perturbation condition is expressed geometrically by assuming that the Mars-Simon tensor $\mathcal{S}$ of the stationary space-time (see [21] and [16]) is sufficiently small. The proof of theorem 1.3 uses theorems 1.1 and 1.2 as a first step. We start by defining the Killing fields $\mathbf{K}, \mathbf{Z}$ in a neighborhood of S and then extend them to the entire space-time by using the level sets of a canonically defined function y. We show that these level sets are conditionally pseudo-convex, as in [16], as long as the the Mars-Simon tensor $\mathcal{S}$ is sufficiently small. Once $\mathbf{K}$ and $\mathbf{Z}$ are extended to the entire space-time, the proof then follows by appealing to the well known results of Carter [5] and Robinson [23], see also the more complete account of [12]. One could remark here that the methods of the present paper would suffice to derive this theorem if one were to restrict attention to stationary black hole solutions which are perturbations of a Kerr solution $\mathcal{K}(a,m)$ with $a << m$, since in that case our construction provides a rotational Killing vector field in the entire ergoregion, which can then be extended to the entire domain of outer communications by analyticity. However, the stronger methods that we introduce in [2] cover the full range of perturbations of all Kerr solutions and give an explicit bound on the closeness in terms of the natural Mars-Simon tensor.

1.2. **Main Ideas.** We recall that a Killing vector-field $\mathbf{K}$ in a vacuum Einstein space-time must verify the covariant wave equation

$$\square_{\mathbf{g}} \mathbf{K} = 0. \tag{1.2}$$

The main idea in [13] was to construct $\mathbf{K}$ as a solution to (1.2) with appropriate, characteristic, boundary conditions on $\mathcal{N} \cup \underline{\mathcal{N}}$. As known, the characteristic initial value problem

[8] As known the existence of the Hawking vector-field plays a fundamental role in the classification theory of stationary black holes (see [14] or [12] and references therein for a more complete treatment of the problem).

[9] For pedagogical reasons we thought it preferable to publish the general, very satisfactory, local results of theorems 1.1 -1.2 separately from the result of theorem 1.3, which we expect to improve in the future.

is well posed in the domain of dependence of $\mathcal{N}\cup\underline{\mathcal{N}}$ but ill posed in its complement. To avoid this fundamental difficulty we rely instead on a completely different strategy[10]. The main idea, which allows us to avoid using (1.2) or some other system of PDE's in the ill posed region, is to first construct $\mathbf{K}$ in the domain of dependence of $\mathcal{N}\cup\underline{\mathcal{N}}$ as a solution to (1.2), extend $\mathbf{K}$ by Lie dragging along the null geodesics transversal to $\mathcal{N}$, consider its associated flow Ψ_t, and show that, for small $|t|$, the pull back metric $\Psi_t^*\mathbf{g}$ must coincide with $\mathbf{g}$, in view of the fact they they are both solutions of the Einstein vacuum equations and coincide on $\mathcal{N}\cup\underline{\mathcal{N}}$. To implement this idea we need to prove a uniqueness result for two Einstein vacuum metrics $\mathbf{g}$, $\mathbf{g}'$ which coincide on $\mathcal{N}\cup\underline{\mathcal{N}}$. The precise result is given in our crucial theorem[11] 3.3.

At the heart of the problem is the issue of fixing a gauge. Given the equations $Ric(\mathbf{g}) = 0, Ric(\mathbf{g}')=0$, and the Carleman estimates in [16] for the wave operator, an obvious such gauge choice would be the wave gauge $\square_{\mathbf{g}}x^\alpha = 0$; this choice would lead to a system of wave equations for the components of the two metrics $\mathbf{g}$, $\mathbf{g}'$ in the given coordinate system. Unfortunately such a coordinate system would have to be constructed starting with data on $\mathcal{N}\cup\underline{\mathcal{N}}$ which requires one to solve the same ill posed problem.

We circumvent this difficulty by starting from the schematic identities,

$$\square_{\mathbf{g}}\mathbf{R} = \mathbf{R}*\mathbf{R}, \qquad \square_{\mathbf{g}'}\mathbf{R}' = \mathbf{R}'*\mathbf{R}'$$

with $\mathbf{R}*\mathbf{R}$, $\mathbf{R}'*\mathbf{R}'$ quadratic expressions in the curvatures $\mathbf{R},\mathbf{R}'$ of the Einstein vacuum metrics $\mathbf{g},\mathbf{g}'$. Subtracting the two equations we derive,

$$\square_{\mathbf{g}}(\mathbf{R}-\mathbf{R}') + \big(\square_{\mathbf{g}} - \square_{\mathbf{g}'}\big)\mathbf{R}' = (\mathbf{R}-\mathbf{R}')*(\mathbf{R}+\mathbf{R}').$$

We would like to rely on the uniqueness properties of covariant wave equations, as in [16], [17], but this is not possible due to the presence of the term $\big(\square_{\mathbf{g}} - \square_{\mathbf{g}'}\big)\mathbf{R}'$ which forces us to consider equations for $\mathbf{g}-\mathbf{g}'$ (and its first and second derivatives) expressed relative to an appropriate choice of a gauge condition. Instead of relying solely on coordinate conditions[12] we complement these by a pair of geometrically constructed frames v, v' (using parallel transport with respect to $\mathbf{g}$ and $\mathbf{g}'$ along certain null geodesics which emanate from $\mathcal{N}$ and are common to both metrics $\mathbf{g},\mathbf{g}'$) and derive ODE's for their difference $dv = v'-v$, as well as the difference $d\Gamma = \Gamma'-\Gamma$ between their connection coefficients, with source terms in $dR = \mathbf{R}'-\mathbf{R}$. In this way we derive a system of wave equations in dR coupled with ODE's in dv, $d\Gamma$ and their partial derivatives ∂dv, $\partial d\Gamma$ with respect to our fixed coordinate system. Since ODE's are clearly well posed it is natural to expect that the uniqueness results for covariant wave equations derived in [16], [17] can

[10] Such strategy is known, see for example [13, Remark B.1], as an alternative to the use of the wave equation (1.2), in the domain of dependence of $\mathcal{N}\cup\underline{\mathcal{N}}$. We show here that the strategy can also be used in the ill posed region of the complement of the domain of dependence

[11] A first version of this result was proved by one of the authors in [1]. The version we give here is more precise and the proof much simpler.

[12] Note that the original construction in [1] relies only on a system of coordinates which introduces many additional difficulties.

be extended to such coupled system; indeed we deduce that $dR = 0$, and hence $d\Gamma = 0$ so $dv = 0$ in a full neighborhood of S, which implies that the metrics $\mathbf{g}, \mathbf{g}'$ agree there. The precise result is stated and proved in Lemma 3.4.

In section 2 we construct a canonical null frame which will be used throughout the paper. We use the non-expanding condition to derive the main null structure equations along $\mathcal{N}$ and $\underline{\mathcal{N}}$, and construct the Hawking vector-field $\mathbf{K}$ in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ (see Proposition B.1 in [13]). In section 3, we show how to extend $\mathbf{K}$ to a full neighborhood of S. We also show that the extension must be locally time-like in the complement of the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$, see Proposition 3.5. In section 4 we prove Theorem 1.2. We first show that if $\mathbf{T}$ is another smooth Killing vector-field, tangent to $\mathcal{N} \cup \underline{\mathcal{N}}$, then it must commute with $\mathbf{K}$ in a full neighborhood of S. We then construct a rotational Killing vector-field $\mathbf{Z}$ as a linear combination of $\mathbf{T}$ and $\mathbf{K}$. As explained above we also show that $\mathbf{Z}$ leaves both u and $\underline{u}$ invariant. We also show that if σ_μ is the Ernst 1-form associated with $\mathbf{T}$ then $\mathbf{Z}^\mu \sigma_\mu = 0$. These additional results, in the presence of the (stationary) Killing vector-field $\mathbf{T}$, are important in the proof of theorem 1.3 in [2].

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2. Preliminaries

We restrict our attention to an open neighborhood $\mathbf{O}$ of S in which $\mathcal{N}, \underline{\mathcal{N}}$ are regular, achronal, null hypersurfaces, spanned by null geodesic generators orthogonal to S. During the proof of our main theorem and their consequences we will keep restricting our attention to smaller and smaller neighborhoods of S; for simplicity of notation we keep denoting such neighborhoods of S by $\mathbf{O}$.

We define two optical functions $u, \underline{u}$ in a neighborhood of S as follows. We first fix a smooth future-directed null pair $(L, \underline{L})$ along S, satisfying

$$\mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = 0, \quad \mathbf{g}(L, \underline{L}) = -1, \tag{2.1}$$

such that L is tangent to $\mathcal{N}$ and $\underline{L}$ is tangent to $\underline{\mathcal{N}}$. In a small neighborhood of S, we extend L (resp. $\underline{L}$) along the null geodesic generators of $\mathcal{N}$ (resp. $\underline{\mathcal{N}}$) by parallel transport, i.e. $\mathbf{D}_L L = 0$ (resp. $\mathbf{D}_{\underline{L}} \underline{L} = 0$). We define the function $\underline{u}$ (resp. u) along $\mathcal{N}$ (resp. $\underline{\mathcal{N}}$) by setting $u = \underline{u} = 0$ on S and solving $L(\underline{u}) = 1$ (resp. $\underline{L}(u) = 1$). Let $S_{\underline{u}}$ (resp. $\underline{S}_u$) be the level surfaces of $\underline{u}$ (resp. u) along $\mathcal{N}$ (resp. $\underline{\mathcal{N}}$). We define $\underline{L}$ at every point of $\mathcal{N}$ (resp. L at every point of $\underline{\mathcal{N}}$) as the unique, future directed null vector-field orthogonal to the surface $S_{\underline{u}}$ (resp. $\underline{S}_u$) passing through that point and such that $\mathbf{g}(L, \underline{L}) = -1$. We now define the null hypersurface $\underline{\mathcal{N}}_{\underline{u}}$ to be the congruence of null geodesics initiating on $S_{\underline{u}} \subset \mathcal{N}$ in the direction of $\underline{L}$. Similarly we define $\mathcal{N}_u$ to be the congruence of null geodesics initiating on $\underline{S}_u \subset \underline{\mathcal{N}}$ in the direction of L. Both congruences are well defined in a sufficiently small neighborhood of S in $\mathbf{O}$, which (according to our convention) we continue to call $\mathbf{O}$. The null hypersurfaces $\underline{\mathcal{N}}_{\underline{u}}$ (resp. $\mathcal{N}_u$) are the level sets of a function $\underline{u}$ (resp u) vanishing on $\underline{\mathcal{N}}$ (resp. $\mathcal{N}$). By construction

$$L = -\mathbf{g}^{\mu\nu}\partial_\mu u \partial_\nu, \qquad \underline{L} = -\mathbf{g}^{\mu\nu}\partial_\mu \underline{u}\partial_\nu. \tag{2.2}$$

In particular, the functions $u, \underline{u}$ are both null optical functions, i.e.

$$\mathbf{g}^{\mu\nu}\partial_\mu u \partial_\nu u = \mathbf{g}(L, L) = 0 \quad \text{and} \quad \mathbf{g}^{\mu\nu}\partial_\mu \underline{u}\,\partial_\nu \underline{u} = \mathbf{g}(\,\underline{L},\,\underline{L}) = 0. \tag{2.3}$$

We define,

$$\Omega = \mathbf{g}^{\mu\nu}\partial_\mu u\,\partial_\nu \underline{u} = \mathbf{g}(L,\,\underline{L}).$$

By construction $\Omega = -1$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$, but Ω is not necessarily equal to -1 in $\mathbf{O}$. Choosing $\mathbf{O}$ small enough, we may assume however that $\Omega \in [-3/2, -1/2]$ in $\mathbf{O}$.

To summarize, we can find two smooth optical functions $u, \underline{u} : \mathbf{O} \to \mathbb{R}$ such that,

$$\mathcal{N} \cap \mathbf{O} = \{p \in \mathbf{O} : u(p) = 0\}, \qquad \underline{\mathcal{N}} \cap \mathbf{O} = \{p \in \mathbf{O} : \underline{u}(p) = 0\}. \tag{2.4}$$

and,

$$\Omega \in [-3/2, -1/2] \quad \text{in } \mathbf{O}. \tag{2.5}$$

Moreover, by construction (with L, $\underline{L}$ defined by (2.2)) we have,

$$L(\underline{u}) = 1 \text{ on } \mathcal{N}, \qquad \underline{L}(u) = 1 \text{ on } \underline{\mathcal{N}}.$$

Using the null pair $\underline{L}, L$ introduced in (2.1), (2.2) we fix an associated null frame $e_1, e_2, e_3 = \underline{L}, e_4 = L$ such that $\mathbf{g}(e_a, e_a) = 1$, $\mathbf{g}(e_1, e_2) = \mathbf{g}(e_4, e_a) = \mathbf{g}(e_3, e_a) = 0$, $a = 1, 2$. At every point p in in $\mathbf{O}$, e_1, e_2 form an orthonormal frame along the 2-surface $S_{u,\underline{u}}$ passing through p. We denote by ∇ the induced covariant derivative operator on $S_{u,\underline{u}}$. Given a horizontal vector-field X, i.e. X tangent to the 2-surfaces $S_{u,\underline{u}}$ at every point in $\mathbf{O}$, we denote by $\nabla_3 X$, $\nabla_4 X$ the projections of $\mathbf{D}_{e_3}$ and $\mathbf{D}_{e_4}$ to $S_{u,\underline{u}}$. Recall the definition of the null second fundamental forms

$$\chi_{ab} = \mathbf{g}(\mathbf{D}_{e_a} L, e_b), \qquad \underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_{e_a}\underline{L}, e_b)$$

and the torsion

$$\zeta_a = \mathbf{g}(\mathbf{D}_{e_a} L,\, \underline{L}).$$

Definition 2.1. *We say that $\mathcal{N}$ is non-expanding if $\mathrm{tr}\,\chi = 0$ on $\mathcal{N}$. Similarly $\underline{\mathcal{N}}$ is non-expanding if $\mathrm{tr}\,\underline{\chi} = 0$ on $\underline{\mathcal{N}}$. The bifurcate horizon $(S, \mathcal{N}, \underline{\mathcal{N}})$ is called non-expanding if both $\mathcal{N}, \underline{\mathcal{N}}$ are non-expanding.*

The assumption that the surfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ are non-expanding implies, according to the Raychaudhuri equation,

$$\chi = 0 \text{ on } \mathcal{N} \cap \mathbf{O} \quad \text{and} \quad \underline{\chi} = 0 \text{ on } \underline{\mathcal{N}} \cap \mathbf{O}. \tag{2.6}$$

In addition, since the vectors e_1, e_2 are tangent to $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$ and $\mathbf{g}(L,\,\underline{L}) = -1$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$, we have $\zeta_a = -\mathbf{g}(D_{e_a}\underline{L}, L)$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$. Finally, it is known that the following components of the curvature tensor $\mathbf{R}$ vanish on $\mathcal{N}$ and $\underline{\mathcal{N}}$,

$$\mathbf{R}_{4a4b} = \mathbf{R}_{434b} = 0 \text{ on } \mathcal{N} \quad \text{and} \quad \mathbf{R}_{3a3b} = \mathbf{R}_{343b} = 0 \text{ on } \underline{\mathcal{N}}, \quad a, b = 1, 2. \tag{2.7}$$

Let, see [8], [20], $\alpha_{ab} = \mathbf{R}_{4a4b}$, $\beta_a = \mathbf{R}_{a434}$, $\rho = \mathbf{R}_{3434}$, $\sigma = {}^*\mathbf{R}_{3434}$, $\underline{\beta}_a = \mathbf{R}_{a334}$ and $\underline{\alpha}_{ab} = \mathbf{R}_{a3b3}$ denote the null components of $\mathbf{R}$. Thus, in view of (2.7) the only non-vanishing null components of $\mathbf{R}$ on S are ρ and σ. Since $[e_a, e_4](\underline{u}) = 0$ on $\mathcal{N} \cap \mathbf{O}$, it follows that $\mathbf{g}([e_a, e_4], e_3) = 0$ on $\mathcal{N} \cap \mathbf{O}$. Using $\mathbf{D}_L L = 0$, (2.6), and the definitions, we derive, on $\mathcal{N} \cap \mathbf{O}$,

$$\begin{aligned} &\mathbf{D}_{e_4} e_4 = 0, \quad \mathbf{D}_{e_a} e_4 = -\zeta_a e_4, \quad \mathbf{D}_{e_4} e_3 = -\sum_{a=1}^{2} \zeta_b e_b, \quad \mathbf{D}_{e_4} e_a = \nabla_{e_4} e_a - \zeta_a e_4, \\ &\mathbf{D}_{e_a} e_3 = \sum_{b=1}^{2} \underline{\chi}_{ab} e_b + \zeta_a e_3, \quad \mathbf{D}_{e_a} e_b = \nabla_{e_a} e_b + \underline{\chi}_{ab} e_4. \end{aligned} \tag{2.8}$$

Lemma 2.2. *The null structure equations along $\mathcal{N}$ (see[13] Proposition 3.1.3 in [20]) reduce to*

$$\nabla_4 \zeta = 0, \quad curl\, \zeta = \sigma, \quad L(tr\, \underline{\chi}) + div\, \zeta - |\zeta|^2 = \rho. \tag{2.9}$$

Also, if X is an horizontal vector,

$$[\nabla_4, \nabla_a] X_b = 0.$$

As a consequence we also have

$$\nabla_4 (div\, \zeta) = 0. \tag{2.10}$$

Proof of Lemma 2.2. Indeed,

$$\mathbf{g}(\mathbf{D}_4 \mathbf{D}_a \underline{L}, e_4) - \mathbf{g}(\mathbf{D}_a \mathbf{D}_4 \underline{L}, e_4) = \mathbf{R}(e_a, e_4, e_3, e_4) = \beta_a$$

and, using (2.8), $\mathbf{g}(\mathbf{D}_a \mathbf{D}_4 \underline{L}, e_4) = \underline{L}_{4;4a} = 0$, $\mathbf{g}(\mathbf{D}_4 \mathbf{D}_a \underline{L}, e_4) = \underline{L}_{4;a4} = -\nabla_4 \zeta_a$. Hence, since β vanishes along $\mathcal{N}$, we deduce $\nabla_4 \zeta = 0$. Also,

$$\mathbf{g}(\mathbf{D}_4 \mathbf{D}_b \underline{L}, e_a) - \mathbf{g}(\mathbf{D}_b \mathbf{D}_4 \underline{L}, e_a) \;\; = \;\; \mathbf{R}(e_a, e_3, e_4, e_b) = \frac{1}{2} \gamma_{ab}\, \rho - \frac{1}{2} \in_{ab} \sigma$$

and, $\mathbf{g}(\mathbf{D}_4 \mathbf{D}_b \underline{L}, e_a) = \underline{L}_{a;b4} = \nabla_4 \underline{\chi}_{ab} - 2\zeta_a \zeta_b$, $g(\mathbf{D}_b \mathbf{D}_4 \underline{L}, e_a) = \underline{L}_{a;4b} = -\nabla_b \zeta_a - \zeta_a \zeta_b$. Hence,

$$\nabla_4 \underline{\chi}_{ab} - \zeta_a \zeta_b + \partial_b \zeta_a = \frac{1}{2} \rho\, \gamma_{ab} - \frac{1}{2} \sigma \in_{ab}.$$

Taking the symmetric part we derive, $\nabla_4 \mathrm{tr}\, \underline{\chi} - |\zeta|^2 + \mathrm{d}iv\, \zeta = \rho$ while taking the antisymmetric part yields, $curl\, \zeta = \sigma$ as desired. To check the commutation formula we

[13] The discrepancy with the corresponding formula is due to the different normalization for $\underline{L}$, i.e. $\mathbf{g}(L, \underline{L}) = -1$ instead of $\mathbf{g}(L, \underline{L}) = -2$.

write,

$$\begin{aligned}
\mathbf{D}_4\mathbf{D}_a X_b &= e_4(\mathbf{D}_a X_b) - \mathbf{D}_{\mathbf{D}_4 e_a} X_b - \mathbf{D}_a X_{\mathbf{D}_4 e_b}\\
&= e_4(\nabla_b X_a) - \mathbf{D}_{\nabla_4 e_a} X_b + \zeta_a \mathbf{D}_4 X_b - \mathbf{D}_a X_{\nabla_4 e_a} + \zeta_b \mathbf{D}_a X_4\\
&= \nabla_4\nabla_a X_b + \zeta_a \nabla_4 X_b\\
\mathbf{D}_a\mathbf{D}_4 X_b &= e_a(\mathbf{D}_4 X_b) - \mathbf{D}_{\mathbf{D}_a e_4} X_b - \mathbf{D}_4 X_{\mathbf{D}_a e_b}\\
&= e_a(\mathbf{D}_4 X_b) - \mathbf{D}_{\nabla_a e_4} X_b + \zeta_a \mathbf{D}_4 X_b - \mathbf{D}_4 X_{\nabla_a e_b}\\
&= \nabla_a\nabla_4 X_b + \zeta_a \nabla_4 X_b
\end{aligned}$$

Therefore,

$$[\mathbf{D}_4, \mathbf{D}_a] X_b = [\nabla_4, \nabla_a] X_b.$$

On the other hand, $[\mathbf{D}_4, \mathbf{D}_a] X_b = \mathbf{R}_{a4cb} X^c = 0$ in view of the vanishing of β and the Einstein equations. □

We define the following four regions I^{++}, I^{--}, I^{+-} and I^{-+}:

$$\begin{aligned}
I^{++} &= \{p \in \mathbf{O} : u(p) \geq 0 \text{ and } \underline{u}(p) \geq 0\}, \quad I^{--} = \{p \in \mathbf{O} : u(p) \leq 0 \text{ and } \underline{u}(p) \leq 0\},\\
I^{+-} &= \{p \in \mathbf{O} : u(p) \geq 0 \text{ and } \underline{u}(p) \leq 0\}, \quad I^{-+} = \{p \in \mathbf{O} : u(p) \leq 0 \text{ and } \underline{u}(p) \geq 0\}.
\end{aligned} \tag{2.11}$$

Clearly I^{++}, I^{--} coincide with the causal and future and past sets of S in $\mathbf{O}$. We construct first the Killing vector-field $\mathbf{K}$ in the causal region $I^{++} \cup I^{--}$.

Proposition 2.3. *Under the assumptions of Theorem 1.1, there is a small neighborhood* $\mathbf{O}$ *of* S *and a smooth Killing vector-field* $\mathbf{K}$ *in* $\mathbf{O} \cap \big(I^{++} \cup I^{--}\big)$ *such that*

$$\mathbf{K} = \underline{u} L - u\, \underline{L} \quad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}. \tag{2.12}$$

Moreover, in the region $\mathbf{O} \cap (I^{++} \cup I^{--})$ *where* $\mathbf{K}$ *is defined,* $[\,\underline{L}, \mathbf{K}] = -\,\underline{L}$.

This proposition, which depends on the main assumption that the surfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ are non-expanding, is well known, see [13, Proposition B.1]. For the sake of completeness, we provide its proof in Appendix B.

3. Extension of the Hawking vector-field to a full neighborhood

In the previous section we have defined our Hawking vector-field $\mathbf{K}$ in a neighborhood $\mathbf{O}$ of S intersected with $I^{++} \cup I^{--}$. To extend $\mathbf{K}$ in the exterior region $I^{+-} \cup I^{-+}$ we cannot rely on solving the wave equation (B.1); the characteristic initial value problem is ill posed in that region. We need to rely instead on a completely different strategy, sketched in the introduction. We extend $\mathbf{K}$ by Lie dragging it relative to $\underline{L}$ and show that, for small $|t|$, $\Psi_t^* \mathbf{g}$ must coincide with $\mathbf{g}$, where $\Psi_t = \Psi_{t,\mathbf{K}}$ is the flow generated by $\mathbf{K}$. We show that both metrics coincide on $\mathcal{N} \cup \underline{\mathcal{N}}$ and, since they both verify the vacuum Einstein equations, we prove that the must coincide in a full neighborhood of S.

To implement this strategy we first define the vector-field K' by setting $K' = \underline{u} L$ on $\mathcal{N} \cap \mathbf{O}$ and solving the ordinary differential equation $[\,\underline{L}, K'] = -\,\underline{L}$. The vector-field K'

is well-defined and smooth in a small neighborhood of S (since $\underline{L} \neq 0$ on S) and coincides with $\mathbf{K}$ in $I^{++} \cup I^{--}$ in $\mathbf{O}$. Thus $\mathbf{K} := K'$ defines the desired extension. This proves the following.

Lemma 3.1. *There exists a smooth extension of the vector-field* $\mathbf{K}$ *(defined in Proposition 2.3) to an open neighborhood* $\mathbf{O}$ *of* S *such that*

$$[\underline{L}, \mathbf{K}] = -\underline{L} \quad \text{in } \mathbf{O}. \tag{3.1}$$

It remains to prove that $\mathbf{K}$ is indeed our desired Killing vector-field. For $|t|$ sufficiently small, we define, in a small neighborhood of S, the map $\Psi_t = \Psi_{t,\mathbf{K}}$ obtained by flowing a parameter distance t along the integral curves of $\mathbf{K}$. Let

$$\mathbf{g}^t = \Psi_t^*(\mathbf{g}).$$

The Lorentz metrics $\mathbf{g}^t$ are well-defined in a small neighborhood of S, for $|t|$ sufficiently small. To show that $\mathbf{K}$ is Killing we need to show that in fact $\mathbf{g}^t = \mathbf{g}$. Since $\mathbf{K}$ is tangent to $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$ and is Killing in $I^{++} \cup I^{--}$, we infer that $\mathbf{g}^t = \mathbf{g}$ in a small neighborhood of S intersected with $I^{++} \cup I^{--}$. In view of the definition of $\mathbf{K}$ (see (3.1)),

$$\frac{d}{dt}\Psi_t^* \underline{L} = \lim_{h\to 0} \frac{\Psi_{t-h}^* \underline{L} - \Psi_t^* \underline{L}}{-h} = -\Psi_t^* \big(\lim_{h\to 0} \frac{\Psi_{-h}^* \underline{L} - \Psi_0^* \underline{L}}{h} \big) = -\Psi_t^* (\mathcal{L}_{\mathbf{K}} \underline{L}) = -\Psi_t^* \underline{L}.$$

We infer that,

$$\Psi_t^* \underline{L} = e^{-t} \underline{L}.$$

Now, given arbitrary vector-fields X, Y, we have $\mathbf{D}^t_{X^t} Y^t = \Psi_t^*(\mathbf{D}_X Y)$ where $\mathbf{D}^t$ denotes the covariant derivative induced by the metric $\mathbf{g}^t = \Psi_t^* g$ and $X^t = \Psi_t^* X$, $Y^t = \Psi_t^* Y$. In, particular $0 = \mathbf{D}^t_{\underline{L}^t} \underline{L}^t = e^{-2t} \mathbf{D}^t_{\underline{L}} \underline{L}$. This proves the following.

Lemma 3.2. *Assume* $\mathbf{K}$ *is a smooth vector-field verifying* (3.1) *and* $\mathbf{D}^t$ *the covariant derivative induced by the metric* $\mathbf{g}^t = \Psi_t^* g$. *Then,*

$$\mathbf{D}^t{}_{\underline{L}} \underline{L} = 0 \quad \text{in a small neighborhood of } S.$$

To summarize we have a family of metrics $\mathbf{g}^t$ which verify the Einstein vacuum equations $\mathbf{Ric}(\mathbf{g}^t) = 0$, $\mathbf{g}^t = \mathbf{g}$ in a small neighborhood of S intersected with $I^{++} \cup I^{--}$, and such that $\mathbf{D}^t{}_{\underline{L}} \underline{L} = 0$. Without loss of generality we may assume that both relations hold in $\mathbf{O}$. Thus Theorem 1.1 is an immediate consequence of the following:

Theorem 3.3. *Assume* $\mathbf{g}'$ *is a smooth Lorentz metric on* $\mathbf{O}$, *such that* $(\mathbf{O}, \mathbf{g}')$ *is a smooth Einstein vacuum space-time. Assume that*

$$\mathbf{g}' = \mathbf{g} \quad \text{in} \quad (I^{++} \cup I^{--}) \cap \mathbf{O} \quad \text{and} \quad \mathbf{D}'_{\underline{L}} \underline{L} = 0 \text{ in } \mathbf{O},$$

where $\mathbf{D}'$ *denotes the covariant derivative induced by the metric* $\mathbf{g}'$. *Then* $\mathbf{g}' = \mathbf{g}$ *in a small neighborhood* $\mathbf{O}' \subset \mathbf{O}$ *of* S.

Proof of Theorem 3.3. It suffices to prove the proposition in a neighborhood $\mathbf{O}(x_0)$ of a point x_0 in S in which we can introduce a fixed coordinate system x^α. Without loss of generality we may assume that

$$\mathbf{g}_{ij}(x_0) = \operatorname{diag}(-1,1,1,1), \qquad \sup_{x\in\mathbf{O}(x_0)} \sum_{j=0}^{6} |\partial^j \mathbf{g}(x)| \le A, \tag{3.2}$$

with $|\partial^j \mathbf{g}|$ denoting the sum of the absolute values of all partial derivatives of order j for all components of $\mathbf{g}$ in the given coordinate system. We may also assume, for the optical functions $u, \underline{u}$ introduced in section 2,

$$\sup_{x\in\mathbf{O}(x_0)} (|\partial^j u(x)| + |\partial^j \underline{u}(x)|) \le C_1 = C_1(A) \quad \text{ for } j = 0,\dots,4. \tag{3.3}$$

In the rest of the proof we will keep restricting to smaller and smaller neighborhoods of x_0; for simplicity of notation we keep denoting such neighborhoods by $\mathbf{O}(x_0)$.

Consider now our old null frame $\widetilde{v}_{(1)} = e_1, \widetilde{v}_{(2)} = e_2, \widetilde{v}_{(3)} = L, \widetilde{v}_{(4)} = \underline{L}$ on $\mathcal{N} \cap \mathbf{O}(x_0)$ and define the vector-fields $v_{(1)},\ v_{(2)},\ v_{(3)},\ v_{(4)} = \underline{L}$ and $v'_{(1)},\ v'_{(2)},\ v'_{(3)},\ v'_{(4)} = \underline{L}$ by parallel transport along $\underline{L}$:

$$\begin{aligned} &\mathbf{D}_{\underline{L}} v_{(a)} = 0 \text{ and } v_{(a)} = \widetilde{v}_a \text{ on } \mathcal{N}\cap\mathbf{O}(x_0);\\ &\mathbf{D}'_{\underline{L}} v'_{(a)} = 0 \text{ and } v'_{(a)} = \widetilde{v}_a \text{ on } \mathcal{N}\cap\mathbf{O}(x_0). \end{aligned}$$

The vector-fields $v_{(a)}$ and $v'_{(a)}$ are well-defined and smooth in $\mathbf{O}(x_0)$. Let $\mathbf{g}_{(a)(b)} = \mathbf{g}(v_{(a)}, v_{(b)})$, $\mathbf{g}'_{(a)(b)} = \mathbf{g}'(v'_{(a)}, v'_{(b)})$. The identities $\mathbf{D}_{\underline{L}} v_{(a)} = \mathbf{D}'_{\underline{L}} v'_{(a)} = 0$ show that $\underline{L}(\mathbf{g}_{(a)(b)}) = \underline{L}(\mathbf{g}'_{(a)(b)}) = 0$. Since $\mathbf{g}_{(a)(b)} = \mathbf{g}'_{(a)(b)}$ along $\mathcal{N}$ it follows that

$$\mathbf{g}_{(a)(b)} = \mathbf{g}'_{(a)(b)} := h_{(a)(b)} \text{ and } \underline{L}(h_{(a)(b)}) = 0 \text{ in } \mathbf{O}(x_0). \tag{3.4}$$

For $a, b, c = 1, \dots 4$ let

$$\begin{aligned} &\Gamma_{(a)(b)(c)} = \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}} v_{(b)}), \qquad \Gamma'_{(a)(b)(c)} = \mathbf{g}'(v'_{(a)}, \mathbf{D}'_{v'_{(c)}} v'_{(b)}),\\ &(d\Gamma)_{(a)(b)(c)} = \Gamma'_{(a)(b)(c)} - \Gamma_{(a)(b)(c)}. \end{aligned}$$

For $a, b, c, d = 1, \dots, 4$ let

$$\begin{aligned} &\mathbf{R}_{(a)(b)(c)(d)} = \mathbf{R}(v_{(a)}, v_{(b)}, v_{(c)}, v_{(d)}), \qquad \mathbf{R}'_{(a)(b)(c)(d)} = \mathbf{R}'(v'_{(a)}, v'_{(b)}, v'_{(c)}, v'_{(d)}),\\ &(dR)_{(a)(b)(c)(d)} = \mathbf{R}'_{(a)(b)(c)(d)} - \mathbf{R}_{(a)(b)(c)(d)}. \end{aligned}$$

Clearly, $\Gamma_{(a)(b)(4)} = \Gamma'_{(a)(b)(4)} = 0$. Note that all quantities $\mathbf{g}_{(a)(b)}, \Gamma_{(a)(b)(c)},\ \mathbf{R}_{(a)(b)(c)(d)}$ are now scalar-valued functions over $\mathbf{O}(x_0)$. They arise by evaluating the corresponding tensors against the vectors $v_{(a)}, v_{(b)}, v_{(c)}, v_{(d)}$. We use now the definition of the Riemann curvature tensor to find a system of equations for $\underline{L}[(d\Gamma)_{(a)(b)(c)}]$. We have

$$\begin{aligned}\mathbf{R}_{(a)(b)(c)(d)} &= \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}}(\mathbf{D}_{v_{(d)}}v_{(b)}) - \mathbf{D}_{v_{(d)}}(\mathbf{D}_{v_{(c)}}v_{(b)}) - \mathbf{D}_{[v_{(c)},v_{(d)}]}v_{(b)})\\ &= \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}}(\mathbf{g}^{(m)(n)}\Gamma_{(m)(b)(d)}v_{(n)})) - \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(d)}}(\mathbf{g}^{(m)(n)}\Gamma_{(m)(b)(c)}v_{(n)}))\\ &+ \mathbf{g}^{(m)(n)}\Gamma_{(a)(b)(n)}(\Gamma_{(m)(c)(d)} - \Gamma_{(m)(d)(c)})\\ &= v_{(c)}(\Gamma_{(a)(b)(d)}) - v_{(d)}(\Gamma_{(a)(b)(c)}) + \mathbf{g}^{(m)(n)}\Gamma_{(a)(b)(n)}(\Gamma_{(m)(c)(d)} - \Gamma_{(m)(d)(c)})\\ &+ \mathbf{g}_{(a)(n)}[\Gamma_{(m)(b)(d)}v_{(c)}(\mathbf{g}^{(m)(n)}) - \Gamma_{(m)(b)(c)}v_{(d)}(\mathbf{g}^{(m)(n)})]\\ &+ \mathbf{g}^{(m)(n)}(\Gamma_{(m)(b)(d)}\Gamma_{(a)(n)(c)} - \Gamma_{(m)(b)(c)}\Gamma_{(a)(n)(d)}).\end{aligned}$$

We set $d = 4$ and use $\Gamma_{(a)(b)(4)} = v_{(4)}(\mathbf{g}^{(a)(b)}) = 0$ and $\mathbf{g}^{(a)(b)} = h^{(a)(b)}$; the result is

$$\underline{L}(\Gamma_{(a)(b)(c)}) = -h^{(m)(n)}\Gamma_{(a)(b)(n)}\Gamma_{(m)(4)(c)} - \mathbf{R}_{(a)(b)(c)(4)}.$$

Similarly,

$$\underline{L}(\Gamma'_{(a)(b)(c)}) = -h^{(m)(n)}\Gamma'_{(a)(b)(n)}\Gamma'_{(m)(4)(c)} - \mathbf{R}'_{(a)(b)(c)(4)}.$$

We subtract these two identities to derive

$$\underline{L}[(d\Gamma)_{(a)(b)(c)})] = {}^{(1)}F^{(d)(e)(f)}_{(a)(b)(c)}(d\Gamma)_{(d)(e)(f)} - (dR)_{(a)(b)(c)(4)} \tag{3.5}$$

for some smooth function ${}^{(1)}F$. This can be written schematically in the form

$$\underline{L}(d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(dR). \tag{3.6}$$

We will use such schematic equations for simplicity of notation[14].

For $a, b, c = 1, \dots, 4$ and $\alpha = 0, \dots, 3$ we define

$$\begin{aligned}(\partial d\Gamma)_{\alpha(a)(b)(c)} &= \partial_\alpha[(d\Gamma)_{(a)(b)(c)}];\\ (\partial dR)_{\alpha(a)(b)(c)(d)} &= \partial_\alpha[(dR)_{(a)(b)(c)(d)}],\end{aligned}$$

where ∂_α are the coordinate vector-fields relative to our local coordinates in $\mathbf{O}(x_0)$. By differentiating (3.6),

$$\underline{L}(\partial d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \tag{3.7}$$

Assume now that

$$\begin{aligned}&v_{(a)} = v^\alpha_{(a)}\partial_\alpha, \qquad v'_{(a)} = v'^\alpha_{(a)}\partial_\alpha,\\ &v'_{(a)} - v_{(a)} = (dv)^\alpha_{(a)}\partial_\alpha, \quad (dv)^\alpha_{(a)} = v'^\alpha_{(a)} - v^\alpha_{(a)},\end{aligned}$$

are the representations of the vectors $v_{(a)}$, $v'_{(a)}$, and $v'_{(a)} - v_{(a)}$ in our coordinate frame $\{\partial_\alpha\}_{\alpha=0,\dots,3}$. Since $[v_{(4)}, v_{(b)}] = -\mathbf{D}_{v_{(b)}}v_{(4)} = -\Gamma^{(c)}{}_{(4)(b)}v_{(c)}$, we have

$$v^\alpha_{(4)}\partial_\alpha(v^\beta_{(b)}) - v^\alpha_{(b)}\partial_\alpha(v^\beta_{(4)}) = -\Gamma_{(a)(4)(b)}v^\beta_{(c)}\mathbf{g}^{(a)(c)}.$$

Similarly,

$$v^\alpha_{(4)}\partial_\alpha(v'^\beta_{(b)}) - v'^\alpha_{(b)}\partial_\alpha(v^\beta_{(4)}) = -\Gamma'_{(a)(4)(b)}v'^\beta_{(c)}\mathbf{g}'^{(a)(c)}.$$

[14] In general, given $B = (B_1, \dots B_L) : \mathbf{O}(x_0) \to \mathbb{R}^L$ we let $\mathcal{M}_\infty(B) : \mathbf{O}(x_0) \to \mathbb{R}^{L'}$ denote vector-valued functions of the form $\mathcal{M}_\infty(B)_{l'} = \sum_{l=1}^L A^l_{l'}B_l$, where the coefficients $A^l_{l'}$ are smooth on $\mathbf{O}(x_0)$.

We subtract these two identities to conclude that, schematically,

$$\underline{L}(dv) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(dv). \tag{3.8}$$

As before, we define

$$(\partial dv)_{\alpha(b)}^{\beta} = \partial_\alpha[(dv)_{(b)}^{\beta}].$$

By differentiating (3.8) we have

$$\underline{L}(\partial dv) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv). \tag{3.9}$$

Finally, we derive a wave equation for dR. We start from the identity

$$(\square_{\mathbf{g}}\mathbf{R})_{(a)(b)(c)(d)} - (\square_{\mathbf{g}'}\mathbf{R}')_{(a)(b)(c)(d)} = \mathcal{M}_\infty(dR),$$

which follows from the standard wave equations satisfied by $\mathbf{R}$ and $\mathbf{R}'$ and the fact that $\mathbf{g}^{(m)(n)} = \mathbf{g}'^{(m)(n)} = h^{(m)(n)}$. We also have

$$\begin{aligned}&\mathbf{D}_{(m)}\mathbf{R}_{(a)(b)(c)(d)} - \mathbf{D}'_{(m)}\mathbf{R}'_{(a)(b)(c)(d)}\\ &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(d\mathbf{R}) + \mathcal{M}_\infty(\partial d\mathbf{R}).\end{aligned}$$

It follows from the last two equations that

$$\begin{aligned}&\mathbf{g}^{(m)(n)}v_{(n)}(v_{(m)}(\mathbf{R}_{(a)(b)(c)(d)})) - \mathbf{g}'^{(m)(n)}v'_{(n)}(v'_{(m)}(\mathbf{R}'_{(a)(b)(c)(d)}))\\ &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR).\end{aligned}$$

Since $\mathbf{g}^{(m)(n)} = \mathbf{g}'^{(m)(n)}$ it follows that

$$\begin{aligned}&\mathbf{g}^{(m)(n)}v_{(n)}(v_{(m)}((dR)_{(a)(b)(c)(d)}))\\ &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR).\end{aligned}$$

Thus

$$\square_{\mathbf{g}}(dR) = \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \tag{3.10}$$

This is our main wave equation.

We collect now equations (3.6), (3.7), (3.8), (3.9), and (3.10):

$$\begin{aligned}&\underline{L}(d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(dR);\\ &\underline{L}(\partial d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR);\\ &\underline{L}(dv) = \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(d\Gamma);\\ &\underline{L}(\partial dv) = \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma);\\ &\square_{\mathbf{g}}(dR) = \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR).\end{aligned} \tag{3.11}$$

This is our main system of equations. Since $\mathbf{g} = \mathbf{g}'$ in $I^{++} \cup I^{--}$, it follows easily that the functions $d\Gamma$, $\partial d\Gamma$, dv, ∂dv and dR vanish also in $I^{++} \cup I^{--}$. Therefore, the proposition follows from Lemma 3.4 below. □

Lemma 3.4. *Assume $G_i, H_j : \mathbf{O}(x_0) \to \mathbb{R}$ are smooth functions, $i = 1, \dots, I$, $j = 1, \dots, J$. Let $G = (G_1, \dots, G_I)$, $H = (H_1, \dots, H_J)$, $\partial G = (\partial_0 G_1, \dots, \partial_4 G_I)$ and assume that in $\mathbf{O}(x_0)$,*

$$\begin{cases} \square_{\mathbf{g}} G = \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H); \\ \underline{L}(H) = \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H). \end{cases} \tag{3.12}$$

Assume that $G = 0$ and $H = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}(x_0)$. Then, there exists a small neighborhood $\mathbf{O}'(x_0) \subset \mathbf{O}(x_0)$ of x_0 such that $G = 0$ and $H = 0$ in $\big(I^{+-} \cup I^{-+}\big) \cap \mathbf{O}'(x_0)$.

Unique continuation theorems of this type in the case $H = 0$ were proved by two of the authors in [16] and [17], using Carleman estimates. It is not hard to adapt the proofs, using similar Carleman estimates, to the general case; we provide all the details in appendix A. This completes the proof of Theorem 1.1.

We show now that the Killing vector-field $\mathbf{K}$ is timelike, in a quantitative sense, in a small neighborhood of S in the complement of $I^{++} \cup I^{--}$.

Proposition 3.5. *Let $\mathbf{K}$ be the Killing vector-field, constructed above, in a neighborhood $\mathbf{O}$ of S. Then there is a neighborhood $\mathbf{O}' \subset \mathbf{O}$ of S such that*

$$\mathbf{g}(\mathbf{K}, \mathbf{K}) \le u\underline{u} \quad \text{in } (I^{+-} \cup I^{-+}) \cap \mathbf{O}'. \tag{3.13}$$

In particular, the vector-field $\mathbf{K}$ is timelike in the set $\mathbf{O}' \setminus (I^{++} \cup I^{--})$.

Proof of Proposition 3.5. Since $\mathbf{K}$ is a Killing vector-field in $\mathbf{O}$, we have

$$\square_{\mathbf{g}}(\mathbf{K}^\beta \mathbf{K}_\beta) = 2\mathbf{D}^\alpha(\mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta) = 2\mathbf{D}^\alpha \mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta = -4 \quad \text{on } S. \tag{3.14}$$

Indeed, $\square_{\mathbf{g}} \mathbf{K} = 0$ and it follows from (B.3) that $2\mathbf{D}^\alpha \mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta = 4\mathbf{D}^3 \mathbf{K}^4 \mathbf{D}_3 \mathbf{K}_4 = -4$ on S. Since $\mathbf{K}_\beta \mathbf{K}^\beta = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$ (see (2.12)), we have $\mathbf{K}_\beta \mathbf{K}^\beta = u\underline{u}f$ on $\mathbf{O}$ for some smooth function $f : \mathbf{O} \to \mathbb{R}$. Using (3.14) on S and the fact that $u = \underline{u} = 0$ on S, we derive

$$-4 = \mathbf{D}^\alpha \mathbf{D}_\alpha(u\underline{u}f) = 2f\mathbf{D}^\alpha u \mathbf{D}_\alpha \underline{u} = -2f\, \underline{L}(u) L(\underline{u}) = -2f.$$

Thus $f = 2$ on S, and the bound (3.13) follows for a sufficiently small $\mathbf{O}'$. $\square$

4. Further results in the presence of a symmetry

The goal of this section is to prove Theorem 1.2. So far we have constructed a smooth Killing vector-field $\mathbf{K}$ defined in an open set $\mathbf{O}$ such that $\mathbf{K} = \underline{u}L - u\underline{L}$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$.

Assume in this section that the space-time $(\mathbf{O}, \mathbf{g})$ admits a smooth Killing vector-field $\mathbf{T}$, which is tangent to the null hypersurfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$. We do not assume in this section that $\mathbf{T}$ does not vanish identically on S; however, the rotational Killing vector-field $\mathbf{Z} = \mathbf{T} + \lambda_0 \mathbf{K}$ constructed in Proposition 4.2 is non-trivial if and only if $\mathbf{T}$ does not vanish identically on S.

We recall several definitions (see [16, Section 4] for a longer discussion and proofs of some identities). In $\mathbf{O}$ we define the 2-form $F_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{T}_\beta$ and the complex valued 2-form,

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + i\,{}^*F_{\alpha\beta} = F_{\alpha\beta} + (i/2)\in_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}, \tag{4.1}$$

where $\in$ denotes the volume form in O. Let $\mathcal{F}^2 = \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$. We define also the Ernst 1-form

$$\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu} = \mathbf{D}_\mu(-\mathbf{T}^\alpha \mathbf{T}_\alpha) - i \in_{\mu\beta\gamma\delta} \mathbf{T}^\beta \mathbf{D}^\gamma \mathbf{T}^\delta. \tag{4.2}$$

It is easy to check that, in $\mathbf{O}$

$$\begin{cases} \mathbf{D}_\mu \sigma_\nu - \mathbf{D}_\nu \sigma_\mu = 0; \\ \mathbf{D}^\mu \sigma_\mu = -\mathcal{F}^2; \\ \sigma_\mu \sigma^\mu = \mathbf{g}(\mathbf{T},\mathbf{T})\mathcal{F}^2. \end{cases} \tag{4.3}$$

Proposition 4.1. *There is an open set $\mathbf{O}' \subseteq \mathbf{O}$, $S \subseteq \mathbf{O}'$ such that*

$$[\mathbf{T},\mathbf{K}] = 0 \ \textit{in}\ \mathbf{O}'. \tag{4.4}$$

In addition, if $\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu}$ is the Ernst 1-form associated to $\mathbf{T}$ (see (4.2)), then

$$\mathbf{K}^\mu \sigma_\mu = 0 \ \textit{in}\ \mathbf{O}'. \tag{4.5}$$

Proof of Proposition 4.1. We show first that

$$[\mathbf{T},\mathbf{K}] = 0 \qquad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}. \tag{4.6}$$

By symmetry, it suffices to check that $[\mathbf{T},\mathbf{K}] = 0$ on $\mathcal{N} \cap \mathbf{O}$. We first observe that $[\mathbf{T}, L]$ is proportional to L. Indeed, since the null second fundamental form of $\mathcal{N}$ is symmetric and $\mathbf{T}$ is both Killing and tangent to $\mathcal{N}$, we have for every $X \in T(\mathcal{N})$,

$$\begin{aligned} \mathbf{g}([\mathbf{T}, L], X) &= \mathbf{g}(\mathbf{D_T} L, X) - \mathbf{g}(\mathbf{D}_L \mathbf{T}, X) = \mathbf{g}(\mathbf{D_T} L, X) + \mathbf{g}(\mathbf{D}_X \mathbf{T}, L) \\ &= \mathbf{g}(\mathbf{D_T} L, X) - \mathbf{g}(\mathbf{T}, \mathbf{D}_X L) = \chi(\mathbf{T}, X) - \chi(X, \mathbf{T}) = 0. \end{aligned}$$

Consequently $[\mathbf{T}, L]$ must be proportional to L, i.e. $[\mathbf{T}, L] = fL$. Since $\mathbf{D}_L L = 0$ and $\mathbf{T}$ commutes with covariant derivatives we derive,

$$\begin{aligned} 0 &= \mathcal{L}_\mathbf{T}(\mathbf{D}_L L) = \mathbf{D}_{\mathcal{L}_T L} L + \mathbf{D}_L(\mathcal{L}_\mathbf{T} L) \\ &= \mathbf{D}_{fL} L + \mathbf{D}_L(fL) = L(f)L. \end{aligned}$$

Therefore

$$[\mathbf{T}, L] = fL \qquad \text{and} \qquad L(f) = 0 \qquad \text{on } \mathcal{N} \cap \mathbf{O}. \tag{4.7}$$

On the other hand, in view of the definition of $\underline{u}$ we have $\mathbf{T}(L(\underline{u})) - L(\mathbf{T}(\underline{u})) = fL(\underline{u})$. Hence,

$$L(f\underline{u} + \mathbf{T}(\underline{u})) = 0.$$

Since $\mathbf{T}$ is tangent to S and $\underline{u} = 0$ on S, we deduce that $f\underline{u} + \mathbf{T}(\underline{u})$ vanishes on S, thus

$$\mathbf{T}\underline{u} + f\underline{u} = 0, \qquad \text{on } \mathcal{N} \cap \mathbf{O}.$$

Now, $[\mathbf{T}, \underline{u}L] = \mathbf{T}(\underline{u})L + \underline{u}[\mathbf{T}, L] = \big(\mathbf{T}(\underline{u}) + f\underline{u}\big)L = 0$. The identity (4.6) follows since $\mathbf{K} = \underline{u}L$ on $\mathcal{N} \cap \mathbf{O}$.

Let $V = [\mathbf{T},\mathbf{K}] = \mathcal{L}_\mathbf{T}\mathbf{K}$ on $\mathbf{O}$. Since $\square_\mathbf{g}\mathbf{K} = 0$ and $\mathbf{T}$ is Killing, we derive, after commuting covariant and Lie derivatives,

$$0 = \mathcal{L}_\mathbf{T}(\square_\mathbf{g}\mathbf{K}) = \square_\mathbf{g}(\mathcal{L}_\mathbf{T}\mathbf{K}) = \square_\mathbf{g} V.$$

Since V vanishes on $(\mathcal{N}\cup\underline{\mathcal{N}})\cap\mathbf{O}$, it follows that V vanishes in $(I^{++}\cup I^{--})\cap\mathbf{O}'$, for some smaller neighborhood $\mathbf{O}'$ of S. (due to the well-posedness of the characteristic initial-value problem); it also follows that V vanishes in $(I^{+-}\cup I^{-+})\cap\mathbf{O}'$ using Lemma 3.4 with $H=0$. This completes the proof of (4.4).

We prove now the identity (4.5). Since $\mathbf{K}$ and $\mathbf{T}$ commute we observe that $\mathcal{L}_{\mathbf{K}}\mathcal{F}=0$ in $\mathbf{O}$. Moreover, using the first identity in (4.3),

$$\mathbf{D}_\alpha(\mathbf{K}^\mu\sigma_\mu)=\mathbf{D}_\alpha\mathbf{K}^\mu\sigma_\mu+\mathbf{K}^\mu\mathbf{D}_\mu\sigma_\alpha=\mathcal{L}_{\mathbf{K}}(\sigma_\alpha)=2\mathcal{L}_{\mathbf{K}}(\mathbf{T}^\beta\mathcal{F}_{\beta\alpha})=0.$$

The identity (4.5) follows since $\mathbf{K}$ vanishes on S. □

To complete the proof of Theorem 1.2 it suffices to prove the following:

Proposition 4.2. *There is a constant $\lambda_0\in\mathbb{R}$ and an open neighborhood $\mathbf{O}'\subseteq\mathbf{O}$ of S such that the vector-field*

$$\mathbf{Z}=\mathbf{T}+\lambda_0\mathbf{K}$$

has periodic orbits in $\mathbf{O}'$. In other words, there is $t_0>0$ such that $\Psi_{t_0,\mathbf{Z}}=\mathrm{Id}$ in $\mathbf{O}'$.

Moreover there exists a choice of a null pair $(L,\,\underline{L})$ such that, in $\mathbf{O}'$,

$$\mathbf{g}(\mathbf{Z},L)=\mathbf{g}(\mathbf{Z},\,\underline{L})=0,\quad [\mathbf{Z},L]=[\mathbf{Z},\,\underline{L}]=0, \tag{4.8}$$

and

$$\mathbf{g}(\mathbf{T},L)=-\lambda_0 u,\quad \mathbf{g}(\mathbf{T},\,\underline{L})=\lambda_0\underline{u},\quad [\mathbf{T},L]=\lambda_0 L,\quad [\mathbf{T},\,\underline{L}]=-\lambda_0\,\underline{L}. \tag{4.9}$$

Observe that the main constants λ_0 and t_0 can be determined on the bifurcation sphere S. We show below that Proposition 4.2 follows from the following lemma.

Lemma 4.3. *There is a constant $t_0>0$ such that $\Psi_{t_0,\mathbf{T}}=\mathrm{Id}$ in S. In addition, there is a constant $\lambda_0\in\mathbb{R}$ and a choice of the null pair $(L,\underline{L})$ along S (satisfying (2.1)) such that*

$$[\mathbf{T},L]=\lambda_0 L\quad \textit{and}\quad [\mathbf{T},\underline{L}]=-\lambda_0\underline{L}\quad \textit{on } S. \tag{4.10}$$

Proof of Proposition 4.2. Assuming Lemma 4.3, the main identities we need to prove are

$$\mathbf{g}(\mathbf{T},\,\underline{L})=\lambda_0\underline{u}\quad\text{along }\mathcal{N},\qquad \mathbf{g}(\mathbf{T},L)=-\lambda_0 u\quad\text{along }\underline{\mathcal{N}}. \tag{4.11}$$

Indeed, assume that the first identity in (4.11) holds. Since

$$\underline{L}(\mathbf{g}(\mathbf{T},\,\underline{L})-\lambda_0\underline{u})=\mathbf{g}(\mathbf{D}_{\underline{L}}\mathbf{T},\,\underline{L})+\mathbf{g}(\mathbf{T},\mathbf{D}_{\underline{L}}\,\underline{L})-\lambda_0\,\underline{L}(\underline{u})=0,$$

it follows that $\mathbf{g}(\mathbf{T},\,\underline{L})=\lambda_0\underline{u}$ in $\mathbf{O}''$, for some open set $\mathbf{O}''$, $S\subseteq\mathbf{O}''$. Thus, in $\mathbf{O}''$,

$$\mathcal{L}_{\mathbf{T}}\,\underline{L}_\alpha=-\mathcal{L}_{\mathbf{T}}(\mathbf{D}_\alpha\underline{u})=-\mathbf{D}_\alpha(\mathbf{T}^\beta\mathbf{D}_\beta\underline{u})=\mathbf{D}_\alpha(\lambda_0\underline{u})=\lambda_0\mathbf{D}_\alpha\underline{u}=-\lambda_0\,\underline{L}_\alpha,$$

therefore $[\mathbf{T},\,\underline{L}]=-\lambda_0\,\underline{L}$ in $\mathbf{O}''$. Moreover, the first identity in (4.11) and the fact that $\mathbf{K}=\underline{u}L$ along $\mathcal{N}$ show that

$$\mathbf{g}(\mathbf{Z},\,\underline{L})=\mathbf{g}(\mathbf{T},\,\underline{L})+\lambda_0\mathbf{g}(\mathbf{K},\,\underline{L})=\lambda_0\underline{u}-\lambda_0\underline{u}=0\qquad\text{along }\mathcal{N}.$$

The same arguments as before show that $\mathbf{g}(\mathbf{Z},\,\underline{L})=0$ and $[\mathbf{Z},\,\underline{L}]=0$ in $\mathbf{O}''$. To summarize, the first identity in (4.11) shows that

$$\mathbf{g}(\mathbf{T},\,\underline{L})=\lambda_0\underline{u},\quad [\mathbf{T},\,\underline{L}]=-\lambda_0\,\underline{L},\quad \mathbf{g}(\mathbf{Z},\,\underline{L})=0,\quad [\mathbf{Z},\,\underline{L}]=0$$

in an open set $\mathbf{O}''$ containing S. The other four identities in (4.8) and (4.9) follow in the same way from the second identity in (4.11). Finally, the identity $\Psi_{t_0,\mathbf{Z}} = \mathrm{Id}$ holds in an open set containing S because it holds on S (the first part of Lemma 4.3) and we have established the commutation identities $[\mathbf{Z}, L] = [\mathbf{Z}, \underline{L}] = 0$.

It remains to prove the identities (4.11). By symmetry, it suffices to prove the first identity,

$$\mathbf{g}(\mathbf{T}, \underline{L}) = \lambda_0 \underline{u} \quad \text{along } \mathcal{N}. \tag{4.12}$$

Since $\mathbf{T}$ is tangent to $\mathcal{N}$, we have $\mathbf{g}(\mathbf{T}, \mathbf{D}_L \underline{L}) = -\mathbf{g}(\mathbf{D}_\mathbf{T} L, \underline{L})$ along $\mathcal{N}$ (using (2.8)). It follows from (4.7) and (4.10) that $[\mathbf{T}, L] = \lambda_0 L$ along $\mathcal{N}$. Thus, along $\mathcal{N}$,

$$L(\mathbf{g}(\mathbf{T}, \underline{L})) = \mathbf{g}(\mathbf{D}_L \mathbf{T}, \underline{L}) + \mathbf{g}(\mathbf{T}, \mathbf{D}_L \underline{L}) = \mathbf{g}(\mathbf{D}_L \mathbf{T}, \underline{L}) - \mathbf{g}(\mathbf{D}_\mathbf{T} L, \underline{L}) = \lambda_0.$$

The identity (4.12) follows since $\mathbf{g}(\mathbf{T}, \underline{L}) = 0$ on S. This completes the proof of the proposition. $\square$

Proof of Lemma 4.3. The existence of the period t_0 is a standard fact concerning Killing vector-fields on the sphere[15]. In particular all nontrivial orbits of S are compact and diffeomorphic to $\mathbb{S}^1$. To prove (4.10), in view of (4.7) it suffices to prove that there is $\lambda_0 \in \mathbb{R}$ and a choice of the null pair $(L, \underline{L})$ on S such that

$$\mathbf{g}([\mathbf{T}, L], \underline{L}) = -\lambda_0, \qquad \mathbf{g}([\mathbf{T}, \underline{L}], L) = \lambda_0 \qquad \text{on } S.$$

Both identities are equivalent to

$$\mathbf{T}^\alpha \underline{L}^\beta \mathbf{D}_\alpha L_\beta - L^\alpha \underline{L}^\beta \mathbf{D}_\alpha \mathbf{T}_\beta = -\lambda_0,$$

which is equivalent to

$$\lambda_0 = F_{43} - \mathbf{g}(\zeta, \mathbf{T}).$$

We thus have to show that there exist a choice of the null pair $e_4 = L, e_3 = \underline{L}$ along S such that the scalar function below is constant along S,

$$H := F_{43} - \mathbf{g}(\zeta, \mathbf{T}). \tag{4.13}$$

Under a scaling transformation $e_4' = f e_4, e_3' = f^{-1} e_3$ the torsion ζ changes according to the formula,

$$\zeta' = \zeta - \nabla \log f.$$

Therefore, in the new frame,

$$H' = F_{4'3'} - \mathbf{g}(\zeta', \mathbf{T}) = F_{43} - \mathbf{g}(\zeta, \mathbf{T}) + \mathbf{T}(\log f) = H + \mathbf{T}(\log f).$$

Consequently, we are led to look for a function f such that $H + \mathbf{T}(\log f)$ is a constant. Taking $\hat{H}$ to be the average of H along the integral curves of $\mathbf{T}$ and solving the equation

$$\mathbf{T}(\log f) \;\; = \;\; -H + \hat{H}, \tag{4.14}$$

it only remains to prove that $\hat{H}$ is constant along S.

[15] If $\mathbf{T} \equiv 0$ on S then any value of $t_0 > 0$ is suitable. In this case, the conclusion of Proposition 4.2 is that $\mathbf{T} + \lambda_0 \mathbf{K} \equiv 0$ in $\mathbf{O}'$ for some $\lambda_0 \in \mathbb{R}$.

Since $\mathbf{T}$ is Killing we must have,

$$\mathbf{D}_\alpha \mathbf{D}_\beta T_\gamma = T^\lambda \mathbf{R}_{\lambda\alpha\beta\gamma}. \tag{4.15}$$

Using (4.15) and the formulas (2.8) on S we derive,

$$\mathbf{T}^\lambda \mathbf{R}_{\lambda a 43} = \mathbf{D}_a \mathbf{D}_4 \mathbf{T}_3 = e_a(\mathbf{D}_4 \mathbf{T}_3) = e_a(F_{43}).$$

Thus, since $\mathbf{T}$ is tangent to S and $\mathbf{T}^b \mathbf{R}_{ba43} = \mathbf{T}^b \mathbf{R}_{ab34} = \mathbf{T}^b \in_{ab} \sigma$ (recall that $\sigma = {}^*\mathbf{R}_{3434}$, $\mathbf{R}_{ab34} = \in_{ab} \sigma$)

$$e_a(F_{43}) = \mathbf{T}^b \mathbf{R}_{ba43} = \mathbf{T}^b \in_{ab} \sigma. \tag{4.16}$$

In particular, the function H defined in (4.13) is constant on S if $\mathbf{T} \equiv 0$ on S. Thus we may assume in the rest of the proof that the set $\Lambda = \{p \in S : \mathbf{T}_p = 0\}$ is finite.

On the other hand, writing $\nabla_a \zeta_b - \nabla_b \zeta_a = \in_{ab} curl\, \zeta$,

$$\begin{aligned} e_a \mathbf{g}(\zeta, \mathbf{T}) &= \nabla_a \zeta_b \mathbf{T}^b + \zeta_b \nabla_a \mathbf{T}_b = (\nabla_a \zeta_b - \nabla_b \zeta_a)\mathbf{T}^b + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a \\ &= \in_{ab} curl\, \zeta \mathbf{T}^b + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a. \end{aligned}$$

Recall that the torsion ζ verifies the equation,

$$curl\, \zeta = \sigma. \tag{4.17}$$

Therefore,

$$e_a \mathbf{g}(\zeta, \mathbf{T}) = \in_{ab} \mathbf{T}^b \sigma + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a. \tag{4.18}$$

Since $H = F_{43} - \zeta \cdot \mathbf{T}$ we deduce,

$$e_a(H) = -\zeta^b \nabla_a \mathbf{T}_b - \nabla_{\mathbf{T}} \zeta_a. \tag{4.19}$$

Consider the orthonormal frame e_1, e_2 on $S \setminus \Lambda$,

$$e_1 = X^{-1}\mathbf{T}, \qquad X^2 = \mathbf{g}(\mathbf{T}, \mathbf{T}).$$

Since $e_1(X) = 0$ and $e_1 = X^{-1}\mathbf{T}$, we have

$$\nabla_{\mathbf{T}} e_2 = -F_{12} e_1.$$

We claim that, with respect to this local frame,

$$\nabla_2(H) = -\mathbf{T}(\zeta_2). \tag{4.20}$$

Indeed,

$$\begin{aligned} \nabla_2(H) &= -\zeta^1 \nabla_2 \mathbf{T}_1 - \zeta^2 \nabla_2 \mathbf{T}_2 - \mathbf{g}(\nabla_{\mathbf{T}} \zeta, e_2) \\ &= -\zeta^1 F_{21} - \mathbf{T}\mathbf{g}(\zeta, e_2) + \mathbf{g}(\zeta, \nabla_{\mathbf{T}} e_2) \\ &= -\mathbf{T}\mathbf{g}(\zeta, e_2) - \zeta^1 F_{21} - \zeta^1 F_{12} \\ &= -\mathbf{T}(\zeta_2). \end{aligned}$$

We now fix a a non-trivial orbit γ_0 of $\mathbf{T}$ in $S \setminus \Lambda$. Consider the geodesics initiating on γ_0 and perpendicular to it and ϕ the corresponding affine parameter. More precisely we choose a vector V on γ_0 such that $\mathbf{g}(V, V) = 1$ and extend it by parallel transport along the geodesics perpendicular to γ_0. Then choose ϕ such that $V(\phi) = 1$ and $\phi = 0$ on γ_0.

This defines a system of coordinates t, ϕ in a neighborhood U of γ_0, such that $\partial_t = T$, $\nabla_{\partial_\phi}\partial_\phi = 0$ in U and $\mathbf{g}(\partial_t, \partial_\phi) = 0$, $\mathbf{g}(\partial_\phi, \partial_\phi) = 1$ on Γ_0. Since ∂_t is Killing we must have $X^2 = -\mathbf{g}(\partial_t, \partial_t)$ and $\mathbf{g}(\partial_\phi, \partial_\phi)$ independent of t. Moreover,

$$\partial_\phi \mathbf{g}(\partial_t, \partial_\phi) \;=\; \mathbf{g}(\nabla_{\partial_\phi}\partial_t, \partial_\phi) + \mathbf{g}(\partial_t, \nabla_{\partial_\phi}\partial_\phi) = \mathbf{g}(\nabla_{\partial_t}\partial_\phi, \partial_\phi) = \frac{1}{2}\partial_t \mathbf{g}(\partial_\phi, \partial_\phi) = 0.$$

Hence, since $\mathbf{g}(\partial_t, \partial_\phi) = 0$ on Γ_0 we infer that $\mathbf{g}(\partial_t, \partial_\phi) = 0$ in U. Similarly,

$$\partial_\phi \mathbf{g}(\partial_\phi, \partial_\phi) = 2\mathbf{g}(\nabla_{\partial_\phi}\partial_\phi, \partial_\phi) = 0$$

and therefore, $\mathbf{g}(\partial_\phi, \partial_\phi) = 1$ in U. Thus, in U, the metric $\mathbf{g}$ takes the form,

$$d\phi^2 + X^2(\phi)dt^2. \tag{4.21}$$

Therefore, with $\mathbf{T} = \partial_t$, $e_2 = \partial_\phi$, we deduce from (4.20), everywhere in U,

$$\partial_\phi H = -\partial_t \mathbf{g}(\zeta, \partial_\phi). \tag{4.22}$$

Thus, integrating in t and in view of the fact that the orbits of ∂_t are closed, we infer that $\hat{H}$ is constant along S, as desired. □

Appendix A. Proof of Lemma 3.4

We will use a Carleman estimate proved by two of the authors in [16, Section 3], which we recall below. Let $\mathbf{O}(x_0)$ a coordinate neighborhood of a point $x_0 \in S$ and coordinates x^α as in (3.2). We denote by $B_r = B_r(x_0)$, the set of points $p \in \mathbf{O}(x_0)$ whose coordinates $x = x^\alpha$ verify $|x - x_0| \le r$, relative to the standard euclidean norm in $\mathbf{O}(x_0)$. Consider two vector-fields $V = V^\alpha \partial_\alpha, W = W^\alpha \partial_\alpha$ on $\mathbf{O}(x_0)$ which verify, that,

$$\sup_{x \in \mathbf{O}(x_0)} \sum_{j=0}^{4} (|\partial^j V(x)| + |\partial^j W(x)|) \le A, \tag{A.1}$$

where A is a large constant (as in (3.2)), and $|\partial^j V(x)|$ denotes the sum of the absolute values of all partial derivatives of order j of all components of V in our given coordinate system. When $j = 1$ we write simply $|\partial V(x)|$.

Definition A.1. *A family of weights $h_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}_+$, $\epsilon \in (0, \epsilon_1)$, $\epsilon_1 \le A^{-1}$, will be called V-conditional pseudo-convex if for any $\epsilon \in (0, \epsilon_1)$*

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x \in B_{\epsilon^{10}}} \sum_{j=1}^{4} \epsilon^j |\partial^j h_\epsilon(x)| \le \epsilon/\epsilon_1, \quad |V(h_\epsilon)(x_0)| \le \epsilon^{10}, \tag{A.2}$$

$$\mathbf{D}^\alpha h_\epsilon(x_0)\mathbf{D}^\beta h_\epsilon(x_0)(\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) \ge \epsilon_1^2, \tag{A.3}$$

and there is $\mu \in [-\epsilon_1^{-1}, \epsilon_1^{-1}]$ such that for all vectors $X = X^\alpha \partial_\alpha$ at x_0

$$\begin{aligned}&\epsilon_1^2[(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2]\\ &\le X^\alpha X^\beta(\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2}(|X^\alpha V_\alpha(x_0)|^2 + |X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2).\end{aligned} \tag{A.4}$$

A function $e_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}$ will be called a negligible perturbation if

$$\sup_{x\in B_{\epsilon^{10}}} |\partial^j e_\epsilon(x)| \le \epsilon^{10} \qquad \textit{for } j = 0, \dots, 4. \tag{A.5}$$

Our main Carleman estimate, see [16, Section 3], is the following:

Lemma A.2. *Assume $\epsilon_1 \le A^{-1}$, $\{h_\epsilon\}_{\epsilon\in(0,\epsilon_1)}$ is a V-conditional pseudo-convex family, and e_ϵ is a negligible perturbation for any $\epsilon \in (0, \epsilon_1]$. Then there is $\epsilon \in (0, \epsilon_1)$ sufficiently small (depending only on ϵ_1) and $\widetilde{C}_\epsilon$ sufficiently large such that for any $\lambda \ge \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}})$*

$$\lambda\|e^{-\lambda f_\epsilon}\phi\|_{L^2} + \|e^{-\lambda f_\epsilon}|\partial\phi|\,\|_{L^2} \le \widetilde{C}_\epsilon \lambda^{-1/2}\|e^{-\lambda f_\epsilon}\,\square_{\mathbf{g}}\phi\|_{L^2} + \epsilon^{-6}\|e^{-\lambda f_\epsilon} V(\phi)\|_{L^2}, \tag{A.6}$$

where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$.

We will only use this Carleman estimate with $V = 0$. In this case the pseudo-convexity condition in Definition A.1 is a special case of Hörmander's pseudo-convexity condition [15, Chapter 28]. We also need a Carleman estimate to exploit the ODE's in (3.12).

Lemma A.3. *Assume $\epsilon \le A^{-1}$ is sufficiently small, e_ϵ is a negligible perturbation, and $h_\epsilon : B_{\epsilon^{10}} \to \mathbf{R}_+$ satisfies*

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x\in B_{\epsilon^{10}}} \sum_{j=1}^{2} \epsilon^j |\partial^j h_\epsilon(x)| \le 1, \quad |W(h_\epsilon)(x_0)| \ge 1. \tag{A.7}$$

Then there is $\widetilde{C}_\epsilon$ sufficiently large such that for any $\lambda \ge \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}})$

$$\|e^{-\lambda f_\epsilon}\phi\|_{L^2} \le 4\lambda^{-1}\|e^{-\lambda f_\epsilon} W(\phi)\|_{L^2}, \tag{A.8}$$

where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$.

Proof of Lemma A.3. Clearly, we may assume that ϕ is real-valued and let $\psi = e^{-\lambda f_\epsilon}\phi \in C_0^\infty(B_{\epsilon^{10}})$. We have to prove that

$$\|\psi\|_{L^2} \le 4\|\lambda^{-1}W(\psi) + W(f_\epsilon)\psi\|_{L^2}. \tag{A.9}$$

By integration by parts,

$$\begin{aligned}
&\int_{B_{\epsilon^{10}}} [\lambda^{-1}W(\psi) + W(f_\epsilon)\psi] \cdot W(f_\epsilon)\psi \, d\mu \\
&= \int_{B_{\epsilon^{10}}} [W(f_\epsilon)\psi]^2 \, d\mu - (2\lambda)^{-1} \int_{B_{\epsilon^{10}}} \psi^2 \cdot \mathbf{D}_\alpha(W(f_\epsilon)W^\alpha) d\mu.
\end{aligned}$$

In view of (A.7) and the assumption (A.1)

$$|W(f_\epsilon)| \ge 1 \quad \text{and} \quad |\mathbf{D}_\alpha(W(f_\epsilon)W^\alpha)| \le \widetilde{C}_\epsilon \quad \text{in} \quad B_{\epsilon^{10}},$$

provided that ϵ is sufficiently small. Thus, for λ sufficiently large,

$$\int_{B_{\epsilon^{10}}} [\lambda^{-1}W(\psi) + W(f_\epsilon)\psi] \cdot W(f_\epsilon)\psi \, d\mu \ge \frac{1}{2}\int_{B_{\epsilon^{10}}} [W(f_\epsilon)\psi]^2 \, d\mu,$$

and the bound (A.9) follows. $\square$

Proof of Lemma 3.4. It suffices to prove that $G = 0$ and $H = 0$ in $I^{+-}_{\widetilde{c}}$, for some $\widetilde{c}$ sufficiently small. We fix $x_0 \in S$ and set

$$h_\epsilon = \epsilon^{-1}(u+\epsilon)(-\underline{u}+\epsilon) \quad \text{and} \quad e_\epsilon = \epsilon^{10}N^{x_0}, \tag{A.10}$$

where $u, \underline{u}$ are the optical functions defined in section 2 and $N^{x_0}(x) = |x - x_0|^2 = \sum_{\alpha=0,1,2,3}|x^\alpha - x_0^\alpha|^2$, the square of the standard euclidean norm.

It is clear that e_ϵ is a negligible perturbation, in the sense of (A.5), for ϵ sufficiently small. Also, it is clear that h_ϵ verifies the condition (A.7), for ϵ sufficiently small and $W = 2\underline{L}$.

We show now that there is $\epsilon_1 = \epsilon_1(A)$ sufficiently small such that the family of weights $\{h_\epsilon\}_{\epsilon\in(0,\epsilon_1)}$ is 0-conditional pseudo-convex, in the sense of Definition A.1. Condition (A.2) is clearly satisfied, in view of the definition and (3.3). To verify conditions (A.3) and (A.4), we compute, in the frame e_1, e_2, e_3, e_4 defined in section 2,

$$e_1(h_\epsilon) = e_2(h_\epsilon) = 0, \quad e_3(h_\epsilon) = -\Omega(1-\epsilon^{-1}\underline{u}), \quad e_4(h_\epsilon) = \Omega(1+\epsilon^{-1}u) \tag{A.11}$$

in $B_{\epsilon^{10}}(x_0)$, and

$$\begin{aligned}&(\mathbf{D}^2h_\epsilon)_{ab} = O(1), \quad (\mathbf{D}^2h_\epsilon)_{3a} = O(1), \quad (\mathbf{D}^2h_\epsilon)_{4a} = O(1), \quad a,b = 1,2,\\ &(\mathbf{D}^2h_\epsilon)_{33} = O(1), \quad (\mathbf{D}^2h_\epsilon)_{44} = O(1), \quad (\mathbf{D}^2h_\epsilon)_{34} = -\Omega^2\epsilon^{-1} + O(1)\end{aligned} \tag{A.12}$$

in $B_{\epsilon^{10}}(x_0)$, where $O(1)$ denotes various functions on $B_{\epsilon^{10}}(x_0)$ with absolute value bounded by constants that depends only on A. Thus

$$\mathbf{D}^\alpha h_\epsilon(x_0)\mathbf{D}^\beta h_\epsilon(x_0)(\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon\mathbf{D}_\alpha\mathbf{D}_\beta h_\epsilon)(x_0) = 2 + \epsilon O(1).$$

This proves (A.3) if ϵ_1 is sufficiently small. Similarly, if $X = X^\alpha e_\alpha$ then, with $\mu = \epsilon_1^{-1/2}$ we compute

$$\begin{aligned}&X^\alpha X^\beta(\mu\mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha\mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2}|X^\alpha\mathbf{D}_\alpha h_\epsilon(x_0)|^2\\ &= \mu((X^1)^2 + (X^2)^2) + 2(\epsilon^{-1}-\mu)X^3X^4 + \epsilon^{-2}(X^3-X^4)^2 + O(1)\sum_{\alpha=1}^4(X^\alpha)^2\\ &\geq (\mu/2)((X^1)^2 + (X^2)^2) + (\epsilon^{-1}/2)((X^3)^2 + (X^4)^2),\end{aligned}$$

provided that ϵ_1 is sufficiently small. This completes the proof of (A.4).

It follows from the Carleman estimates in Lemmas A.2 and A.3 that there is $\epsilon = \epsilon(A) \in (0, c)$ (where c is the constant in Lemma 3.4) and a constant $\widetilde{C} = \widetilde{C}(A) \geq 1$ such that

$$\begin{aligned}&\lambda\|e^{-\lambda f_\epsilon}\phi\|_{L^2} + \|e^{-\lambda f_\epsilon}|\partial\phi|\,\|_{L^2} \leq \widetilde{C}\lambda^{-1/2}\|e^{-\lambda f_\epsilon}\,\square_{\mathbf{g}}\phi\|_{L^2};\\ &\|e^{-\lambda f_\epsilon}\phi\|_{L^2} \leq \widetilde{C}\lambda^{-1}\|e^{-\lambda f_\epsilon}\underline{L}(\phi)\|_{L^2},\end{aligned} \tag{A.13}$$

for any $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$ and any $\lambda \geq \widetilde{C}$, where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$. Let $\eta : \mathbb{R} \to [0,1]$ denote a smooth function supported in $[1/2,\infty)$ and equal to 1 in $[3/4,\infty)$. For $\delta \in (0,1]$,

$i = 1, \dots, I$, $j = 1, \dots J$ we define,

$$\begin{aligned} G_i^{\delta,\epsilon} &= G_i \cdot \mathbf{1}_{I_c^{+-}} \cdot \eta(-u\underline{u}/\delta) \cdot \big(1 - \eta(N^{x_0}/\epsilon^{20})\big) = G_i \cdot \widetilde{\eta}_{\delta,\epsilon} \\ H_j^{\delta,\epsilon} &= H_j \cdot \mathbf{1}_{I_c^{+-}} \cdot \eta(-u\underline{u}/\delta) \cdot \big(1 - \eta(N^{x_0}/\epsilon^{20})\big) = H_j \cdot \widetilde{\eta}_{\delta,\epsilon}. \end{aligned} \tag{A.14}$$

Clearly, $G_i^{\delta,\epsilon}, H_j^{\delta,\epsilon} \in C_0^\infty(B_{\epsilon^{10}}(x_0) \cap \mathbf{E})$. We would like to apply the inequalities in (A.13) to the functions $G_i^{\delta,\epsilon}, H_j^{\delta,\epsilon}$, and then let $\delta \to 0$ and $\lambda \to \infty$ (in this order).

Using the definition (A.14), we have

$$\begin{aligned} &\square_{\mathbf{g}} G_i^{\delta,\epsilon} = \widetilde{\eta}_{\delta,\epsilon} \cdot \square_{\mathbf{g}} G_i + 2\mathbf{D}_\alpha G_i \cdot \mathbf{D}^\alpha \widetilde{\eta}_{\delta,\epsilon} + G_i \cdot \square_{\mathbf{g}} \widetilde{\eta}_{\delta,\epsilon}; \\ &\underline{L}(H_j^{\delta,\epsilon}) = \widetilde{\eta}_{\delta,\epsilon} \cdot \underline{L}(H_j) + H_j \cdot \underline{L}(\widetilde{\eta}_{\delta,\epsilon}). \end{aligned}$$

Using the Carleman inequalities (A.13), for any $i = 1, \dots, I$, $j = 1, \dots, J$ we have

$$\begin{aligned} &\lambda \cdot \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} G_i\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} |\partial^1 G_i|\,\|_{L^2} \le \widetilde{C} \lambda^{-1/2} \cdot \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} \square_{\mathbf{g}} G_i\|_{L^2} \\ &+ \widetilde{C} \Big[\|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha G_i \mathbf{D}^\alpha \widetilde{\eta}_{\delta,\epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot G_i(|\square_{\mathbf{g}} \widetilde{\eta}_{\delta,\epsilon}| + |\partial^1 \widetilde{\eta}_{\delta,\epsilon}|)\|_{L^2}\Big] \end{aligned} \tag{A.15}$$

and

$$\|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} H_j\|_{L^2} \le \widetilde{C} \lambda^{-1} \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} \underline{L}(H_j)\|_{L^2} + \widetilde{C} \lambda^{-1} \|e^{-\lambda f_\epsilon} \cdot H_j \underline{L}(\widetilde{\eta}_{\delta,\epsilon})\|_{L^2}, \tag{A.16}$$

for any $\lambda \ge \widetilde{C}$. Using the main identities (3.12), in $B_{\epsilon^{10}}(x_0)$ we estimate pointwise

$$\begin{aligned} |\square_{\mathbf{g}} G_i| &\le M \sum_{l=1}^{I} \big(|\partial^1 G_l| + |G_l|\big) + M \sum_{m=1}^{J} |H_j|, \\ |\underline{L}(H_j)| &\le M \sum_{l=1}^{I} \big(|\partial^1 G_l| + |G_l|\big) + M \sum_{m=1}^{J} |H_j|, \end{aligned} \tag{A.17}$$

for some large constant M. We add inequalities (A.15) and (A.16) over i, j. The key observation is that, in view of (A.17), the first terms in the right-hand sides of (A.15) and (A.16) can be absorbed into the left-hand sides for λ sufficiently large. Thus, for any λ sufficiently large and $\delta \in (0, 1]$,

$$\begin{aligned} &\lambda \sum_{i=1}^{I} \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} G_i\|_{L^2} + \sum_{j=1}^{J} \|e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon} H_j\|_{L^2} \le \widetilde{C} \lambda^{-1} \sum_{j=1}^{J} \|e^{-\lambda f_\epsilon} \cdot H_j |\partial \widetilde{\eta}_{\delta,\epsilon}|\|_{L^2} \\ &+ \widetilde{C} \sum_{i=1}^{I} \Big[\|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha G_i \mathbf{D}^\alpha \widetilde{\eta}_{\delta,\epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot G_i(|\square_{\mathbf{g}} \widetilde{\eta}_{\delta,\epsilon}| + |\partial \widetilde{\eta}_{\delta,\epsilon}|)\|_{L^2}\Big]. \end{aligned} \tag{A.18}$$

We let now $\delta \to 0$ and $\lambda \to \infty$, as in [16, Section 6], to conclude that $\mathbf{1}_{B_{\epsilon^{40}}(x_0) \cap I^{+-}} G_i = 0$ and $\mathbf{1}_{B_{\epsilon^{40}}(x_0) \cap I^{+-}} H_j = 0$. The main ingredient needed for this limiting procedure is the inequality

$$\inf_{B_{\epsilon^{40}}(x_0) \cap I_c^{+-}} e^{-\lambda f_\epsilon} \ge e^{\lambda/\widetilde{C}} \sup_{\{x \in B_{\epsilon^{10}}(x_0) \cap I_c^{+-} : N^{x_0} \ge \epsilon^{20}/2\}} e^{-\lambda f_\epsilon},$$

which follows easily from the definition (A.10). The lemma follows. $\square$

Appendix B. Proof of Proposition 2.3

Following [13] we construct the smooth vector-field $\mathbf{K}$ as the solution to the characteristic initial-value problem,

$$\square_{\mathbf{g}}\mathbf{K} = 0, \qquad \mathbf{K} = \underline{u}L - u\,\underline{L} \quad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}. \tag{B.1}$$

As well known, see [25], the characteristic initial value problem for wave equations of type (B.2) is well posed. Thus the vector-field $\mathbf{K}$ is well-defined and smooth in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ in $\mathbf{O}$. Let $\pi_{\alpha\beta} = {}^{(\mathbf{K})}\pi_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{K}_\beta + \mathbf{D}_\beta \mathbf{K}_\alpha$. We have to prove that $\pi = 0$ in a neighborhood of S intersected to $I^{++} \cup I^{--}$. It follows from (B.1), using the Bianchi identities and the Einstein vacuum equations, that π verifies the covariant wave equation,

$$\square_{\mathbf{g}}\pi_{\alpha\beta} = 2\mathbf{R}^{\mu}{}_{\alpha\beta}{}^{\nu}\pi_{\mu\nu}. \tag{B.2}$$

In view of the standard uniqueness result for characteristic initial value problems, see [25], the statement of the proposition reduces to showing that $\pi = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathbf{O}$. By symmetry, it suffices to prove that $\pi = 0$ on $\mathcal{N} \cap \mathbf{O}$. The proof relies on our main hypothesis, that the surfaces $\mathcal{N}$ and $\underline{\mathcal{N}}$ are non-expanding.

Since $\mathbf{K} = \underline{u}L$ on $\mathcal{N} \cap \mathbf{O}$ is tangent to the null generators of $\mathcal{N}$, it follows that

$$\mathbf{D}_4\mathbf{K}_3 = -1, \quad \mathbf{D}_4\mathbf{K}_4 = \mathbf{D}_a\mathbf{K}_4 = \mathbf{D}_4\mathbf{K}_a = \mathbf{D}_a\mathbf{K}_b = 0, \quad a, b = 1, 2. \tag{B.3}$$

Thus, on $\mathcal{N} \cap \mathbf{O}$

$$\pi_{44} = \pi_{a4} = \pi_{ab} = 0 \quad a, b = 1, 2. \tag{B.4}$$

To prove that the remaining components of π vanish we use the wave equation $\square_{\mathbf{g}}\mathbf{K} = 0$, which gives

$$\mathbf{D}_3\mathbf{D}_4\mathbf{K}_\mu + \mathbf{D}_4\mathbf{D}_3\mathbf{K}_\mu = \sum_{a=1}^{2} \mathbf{D}_a\mathbf{D}_a\mathbf{K}_\mu \quad \text{on } \mathcal{N} \cap \mathbf{O}.$$

Since $\mathbf{D}_3\mathbf{D}_4\mathbf{K}_\mu - \mathbf{D}_4\mathbf{D}_3\mathbf{K}_\mu = \mathbf{R}_{34\mu\nu}\mathbf{K}^\nu$ on $\mathcal{N} \cap \mathbf{O}$ (using (2.7)), we derive

$$2\mathbf{D}_4\mathbf{D}_3\mathbf{K}_\mu = \sum_{a=1}^{2} \mathbf{D}_a\mathbf{D}_a\mathbf{K}_\mu - \mathbf{R}_{34\mu\nu}\mathbf{K}^\nu,\ \mu = 1, 2, 3, 4, \text{ on } \mathcal{N} \cap \mathbf{O}. \tag{B.5}$$

We set first $\mu = 4$. It follows from (B.3) that $\mathbf{D}_4\mathbf{D}_3\mathbf{K}_4 = 0$. In addition, $\mathbf{D}_3\mathbf{K}_4 = 1$ on S (the analogue of the first identity in (B.3) along the hypersurface $\underline{\mathcal{N}}$). Using (2.8) and (B.3), $\mathbf{D}_4\mathbf{D}_3\mathbf{K}_4 = L(\mathbf{D}_3\mathbf{K}_4)$. Thus $\mathbf{D}_3\mathbf{K}_4 = 1$ on $\mathcal{N}$, which implies

$$\pi_{34} = 0 \quad \text{on } \mathcal{N}. \tag{B.6}$$

We use now the equation (B.5) with $\mu = a \in \{1, 2\}$ to calculate $P_a := \pi_{a3}$ along $\mathcal{N}$. It follows from (B.3) and (2.7) that $\mathbf{D}_a\mathbf{D}_b\mathbf{K}_c = 0$, $a, b, c = 1, 2$, and $\mathbf{R}_{34a\nu}\mathbf{K}^\nu = 0$ on $\mathcal{N}$. A simple computation shows that $\mathbf{D}_a\mathbf{K}_3 = \underline{u}\zeta_a$, thus $P_a = \mathbf{D}_3\mathbf{K}_a + \underline{u}\zeta_a$. Thus, using (2.8), $\mathbf{D}_3\mathbf{K}_4 = 1$, and $\mathbf{D}_b\mathbf{K}_c = 0$ on $\mathcal{N}$, we derive

$$\begin{aligned}0 = \mathbf{K}_{b;34} &= e_4(\mathbf{K}_{b;3}) - \mathbf{K}_{\mathbf{D}_{e_4}e_b;e_3} - \mathbf{K}_{e_b;\mathbf{D}_{e_4}e_3} = e_4(P_b - \underline{u}\zeta_b) - \mathbf{K}_{\nabla_4 e_b;e_3} + \zeta_b K_{e_4;e_3}\\ &= \nabla_4(P_b - \underline{u}\zeta_b) + \zeta_b = \nabla_4 P_b - \underline{u}\nabla_4\zeta_b.\end{aligned}$$

Thus

$$\nabla_4 P_a = \underline{u}\nabla_4\zeta_a \quad \text{ on } \mathcal{N}.$$

On the other hand, along $\mathcal{N}$, ζ verifies the transport equation,

$$\nabla_4\zeta_a = -\mathbf{R}_{a434} = 0.$$

Therefore, along $\mathcal{N}$,

$$\nabla_4 P_a = 0.$$

Since $P_a = \pi_{a3} = 0$ on S it follows that

$$\pi_{a3} = 0 \quad \text{ on } \mathcal{N}. \tag{B.7}$$

Similarly, denoting $Q = \pi_{33} = 2\mathbf{D}_3\mathbf{K}_3$, we have, according to (B.5) with $\mu = 3$,

$$\mathbf{D}_4\mathbf{D}_3\mathbf{K}_3 = \frac{1}{2}\big(\sum_{a=1}^{2}\mathbf{D}_a\mathbf{D}_a\mathbf{K}_3 - \rho\underline{u}\big), \qquad \rho = \mathbf{R}_{3434}. \tag{B.8}$$

Now, since we already now that π_{3b} vanishes on $\mathcal{N}$,

$$\mathbf{K}_{3;34} = e_4(\mathbf{K}_{3;3}) - \mathbf{K}_{\mathbf{D}_{e_4}e_3;e_3} - \mathbf{K}_{e_3;\mathbf{D}_{e_4}e_3} = \frac{1}{2}e_4(Q) + \sum_{b=1}^{2}\zeta_b\pi_{3b} = \frac{1}{2}e_4(Q). \tag{B.9}$$

On the other hand, using (2.8), $\mathbf{K}_{3;4} = -1$, $\mathbf{K}_{a;b} = 0$, and $\mathbf{K}_{3;a} = \underline{u}\zeta_a$,

$$\begin{aligned}\mathbf{K}_{3;ab} &= e_b(\mathbf{K}_{3;a}) - \mathbf{K}_{e_3;\mathbf{D}_{e_b}e_a} - \mathbf{K}_{\mathbf{D}_{e_b}e_3;e_a}\\ &= \partial_b(\underline{u}\zeta_a) - \underline{\chi}_{ba}\mathbf{K}_{3;4} - \zeta_b\mathbf{K}_{3;a}\\ &= \partial_b(\underline{u}\zeta_a) + \underline{\chi}_{ba} - \underline{u}\zeta_a\zeta_b,\end{aligned}$$

thus

$$\sum_{a=1}^{2}\mathbf{D}_a\mathbf{D}_a\mathbf{K}_3 \quad = \underline{u}(\mathrm{d}iv\,\zeta - |\zeta|^2) + \mathrm{tr}\,\underline{\chi}. \tag{B.10}$$

Therefore, equation (B.8) takes the form

$$L(Q) \quad = \quad \mathrm{tr}\,\underline{\chi} + \underline{u}(\mathrm{d}iv\,\zeta - |\zeta|^2 - \rho). \tag{B.11}$$

On the other hand we have the following structure equation on $\mathcal{N}$,

$$L(\mathrm{tr}\,\underline{\chi}) + \mathrm{d}iv\,\zeta - |\zeta|^2 - \rho = 0. \tag{B.12}$$

Thus, differentiating (B.11) with respect to L and applying (B.12) we derive,

$$\begin{aligned} L(L(Q)) &= L(\mathrm{tr}\,\underline{\chi}) + (\mathrm{d}iv\,\zeta - |\zeta|^2 - \rho) + \underline{u}L(\mathrm{d}iv\,\zeta - |\zeta|^2 - \rho) \\ &= -\mathrm{d}iv\,\zeta + |\zeta|^2 + \rho + (\mathrm{d}iv\,\zeta - |\zeta|^2 - \rho) + \underline{u}L(\mathrm{d}iv\,\zeta - |\zeta|^2 - \rho). \end{aligned}$$

Using null structure equations, it is not hard to check that

$$L(\mathrm{d}iv\,\zeta) = L(|\zeta|^2) = L(\rho) = 0 \quad \text{along } \mathcal{N}. \tag{B.13}$$

Indeed, the last identity follows from (2.7) and [20, Proposition 3.2.4]. The identity $L(|\zeta|^2) = 0$ follows from the transport equation $\nabla_4 \zeta_a = 0$. Therefore,

$$L(L(Q)) \;=\; 0 \qquad \text{along} \quad \mathcal{N}.$$

Since $L(Q) = 0$ on S (using again(B.11) restricted to S where both $\mathrm{t}r\,\underline{\chi}$ and $\underline{u}$ vanish), we infer that $L(Q) = 0$ along $\mathcal{N}$. Since $Q = 0$ on S we conclude that $Q = 0$ along $\mathcal{N}$ as desired. Thus $\pi_{33} = 0$, as desired.

To prove the second part of the proposition, $[\,\underline{L}, \mathbf{K}] = -\,\underline{L}$ in a neighborhood of S in $I^{++} \cup I^{--}$, we notice first that $\mathbf{g}(\mathbf{K},\,\underline{L}) = -\underline{u}$ on $\mathcal{N}$ since $\mathbf{K} = \underline{u}L - u\,\underline{L}$ on $\mathcal{N}$. Since

$$\underline{L}(\mathbf{g}(\mathbf{K},\,\underline{L}) + \underline{u}) = \mathbf{g}(\mathbf{D}_{\underline{L}}\mathbf{K},\,\underline{L}) + \mathbf{g}(\mathbf{K}, \mathbf{D}_{\underline{L}}\,\underline{L}) = 0,$$

it follows that $\mathbf{g}(\mathbf{K},\,\underline{L}) = -\underline{u}$ in a neighborhood of S in $I^{++} \cup I^{--}$. Thus, in this neighborhood, $\mathbf{K}(\underline{u}) = \underline{u}$. Therefore, since $\mathbf{K}$ is Killing,

$$\mathcal{L}_{\mathbf{K}}\,\underline{L}_\alpha = -\mathcal{L}_{\mathbf{K}}(\mathbf{D}_\alpha \underline{u}) = -\mathbf{D}_\alpha(\mathbf{K}^\beta \mathbf{D}_\beta \underline{u}) = -\mathbf{D}_\alpha \underline{u} = \,\underline{L}_\alpha,$$

as desired.

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