Regularity of the Future Event Horizon in Perturbations of Kerr

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Abstract

The goal of the paper is to show that the event horizons of the spacetimes constructed in [16], see also [19], in the proof of the nonlinear stability of slowly rotating Kerr spacetimes $\mathcal{K}(a_0, m_0)$, are necessarily smooth null hypersurfaces. The results remain valid for the entire range of $|a_0|/m_0$ for which stability can be established. We show in fact, see Theorem 1.7, that they depend on a much weaker set of smallness assumptions than those derived in [16].

Contents

1 Introduction

It is known that in dynamical situations the event horizon[1] (EV) is typically non-smooth,[2] even non-differentiable on a dense set, see [8], [7]. We show that, in contrast to the general situation, the future event horizons of the spacetimes constructed in [16], see also [19] are necessarily smooth null hypersurfaces. Recall that, according to the main result of [16], the future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a $Kerr(a_0, m_0)$ initial data set, for sufficiently small $|a_0|/m_0$, has a complete future null infinity $\mathcal{I}^+$ and converges in its causal past $\mathcal{J}^{-1}(\mathcal{I}^+)$ to another nearby Kerr spacetime $Kerr(a_f, m_f)$ with parameters (a_f, m_f), close to (a_0, m_0), and possesses a future event horizon $\mathcal{H}^+$. The goal of the present paper is to show that $\mathcal{H}^+$ is in fact smooth. We also show that the result remains true for the entire range of $|a_0|/m_0$ for which stability can be established.

Theorem 1.1 (Regularity)**.** *The event horizon $\mathcal{H}^+$ of the perturbed Kerr spacetime $\mathcal{K}(a, m)$ constructed in [16] is regular.*[3] *Moreover, as already shown in [16], in the coordinates used there, $\mathcal{H}^+$ lies near $r = r_+ := m_f + \sqrt{m_f^2 - a_f^2}$, where m_f, a_f are the final mass and angular momentum. The result holds*

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[1] Recall that an embedded achronal hypersurface $\mathcal{H}$ in a Lorentzian manifold $\mathcal{M}$ is a future horizon if it is ruled by future null geodesics, i.e. every point $p \in \mathcal{H}$ belongs to a future, inextendible, null geodesic $\Gamma \subset \mathcal{H}$, called a generator (Note that Γ is allowed to have past end points in $\mathcal{H}$ but no future end points) of $\mathcal{H}$. The primary example is that of the future event horizon $\mathcal{H}^+ := \partial\mathcal{J}^-(\mathcal{I}^+)$ of an asymptotically flat Lorentzian manifold with complete future null infinity $\mathcal{I}^+$, see for example [14], page 312 or [24].

[2] This is expected to be the case for black hole mergers, see [11] and the references within. The authors of [11] argue that in realistic physical situations the EV is smooth at late times and take this, see Section 2.2, as an assumption in their analysis.

[3] In the future of the initial Cauchy hypersurface. See Definition 2.19 for the precise notion of regularity used here. We note that [10] also contains a proof of regularity of EV for the special case of the ultimately Schwarzschildian spacetimes constructed in that work. See also [5].

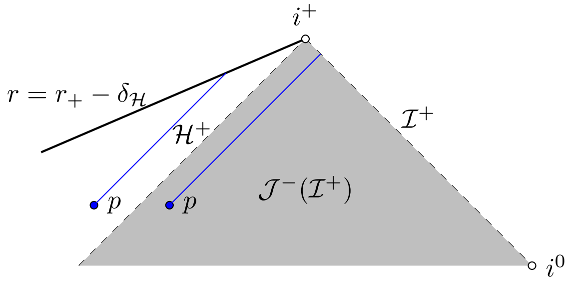

Figure 1: The event horizon

in fact for the full sub-extremal case $|a_f| < m_f$ *provided that the main estimates*[4] *derived in [16] remain valid.*

We also prove the following uniqueness result.

Theorem 1.2 (Uniqueness)**.** *Under the same assumptions, any point* p *that is not on* $\mathcal{H}^+$ *either lies in the past of null infinity, or satisfies the property that, any future null geodesic emanating from it hits the spacelike hypersurface* $\{r = r_+ - \delta_{\mathcal{H}}\}$ *in the black hole region* $\mathcal{M}\backslash\mathcal{J}^-(\mathcal{I}^+)$ *(*$\delta_{\mathcal{H}}$ *is a small positive constant). In particular, any future complete null hypersurface, different from* $\mathcal{H}^+$*, cannot stay in the region* $\{|r - r_+| \leq \delta_{\mathcal{H}}\}$ *where* $\mathcal{H}^+$ *is located, see Figure 1.*

The global nonlinear stability of Kerr spacetime is a major topic of interest in general relativity. An affirmative answer for the case of small angular momentum has been established in a series of work [17, 18, 22, 16, 12]. The restriction of small angular momentum only comes from [12] and all other works apply in the full subextremal range. Important futher conclusions and applications have been established afterwards, including An–He–Shen [3] on the angular momentum memory effect along future null infinity, Klainerman–Shen–Wan [15] on canonical future null infinity and elimination of supertranslation ambiguities, and An–He [1, 2] on global dynamics of apparent horizon, formation of black holes and new proof of Penrose inequality. In the current work, we study the regularity of the event horizon and its uniqueness property as a null hypersurface.

1.1 A sufficient set of assumptions and a general theorem

The proof of Theorems 1,2 depends in fact on a much weaker set of assumptions than those derived in the Kerr stability result [16]. These are encapsulated in the following definitions.

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[4] It is important to note that our results depend only on uniform bounds with respect to the perturbation parameter and not on decay with respect to the advanced time v, see Theorem 1.7. Such estimates, however, cannot be proved in the absence of a stability proof.

Definition 1.3. *We say a spacetime* $(\mathcal{M}, \mathbf{g})$ *with complete future null infinity is an* $O_N(\epsilon)$*-interior perturbation of the subextremal Kerr spacetime* $\mathcal{K}(a,m)$ $(|a| < m)$*, if we have* $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$*, where*

- ${}^{(ext)}\mathcal{M}$ *is in the past of future null infinity* $\mathcal{I}^+$*.*
- *The event horizon, i.e. the boundary of* $\mathcal{J}^-(\mathcal{I}^+)$*, is included in* ${}^{(int)}\mathcal{M}$*.*
- *The hypersurface* $\{r = r_0\}$ *with* $r_0 \gg 1$ *is a subset of* ${}^{(int)}\mathcal{M} \cap {}^{(ext)}\mathcal{M}$*.*
- ${}^{(int)}\mathcal{M}$ *admits an ingoing PG structure*[5] $\big\{(e_3, e_4, H); (r, v, \theta, \varphi)\big\}$*, with the linearized Ricci and curvature coefficients* $\check{\Gamma} = \Gamma - \Gamma_{a,m}$ *verifying the smallness estimates*

$$|(\nabla_3, \nabla_4, \nabla)^{\leq N}\check{\Gamma}| \leq \epsilon. \tag{1.1}$$

  Here Γ *and* $\Gamma_{a,m}$ *denote the set of all Ricci and null curvature coefficients of* ${}^{(int)}\mathcal{M}$*, respectively* $\mathcal{K}(a,m)$ *with respect to* (r, θ)*, see Definition 2.4.*

Remark 1.4. *While we do not give a precise definition of the future null infinity, it is natural to assume that all points in* ${}^{(ext)}\mathcal{M}$ *have large* r*-values, for some reasonable extension*[6] *of* r *from* ${}^{(int)}\mathcal{M}$ *to* ${}^{(ext)}\mathcal{M}$*.*

Remark 1.5. *Note that we make no quantitative smallness assumptions in* ${}^{(ext)}\mathcal{M}$ *and no decay assumptions in* ${}^{(int)}\mathcal{M}$*.*

Upon this consideration, for points in ${}^{(int)}\mathcal{M}$, we also adopt the following definition:

Definition 1.6. *We say that a point* $p \in {}^{(int)}\mathcal{M}$ *lies in the past of null infinity, denoted by* $p \in \mathcal{J}^-(\mathcal{I}^+)$*, if* $p \in \mathcal{J}^-({}^{(ext)}\mathcal{M})$*.*

According to Definition 1.3 we assume that the event horizon, i.e. the boundary of $\mathcal{J}^-(\mathcal{I}^+)$, is included in ${}^{(int)}\mathcal{M}$. In fact, as in [16], we can assume that ${}^{(int)}\mathcal{M}$ extends in the black hole region to a spacelike hypersurface $\{r = r_+ - \delta_{\mathcal{H}}\}$ for some small constant $\delta_{\mathcal{H}}$ (but much larger than the perturbation parameter ϵ), where $r_+ = m + \sqrt{m^2 - a^2}$.

Theorem 1.7 (Regularity and Uniqueness)**.** *Assume that* $\epsilon > 0$ *is a small constant satisfying* $\epsilon \leq \delta_*^{\frac{3}{2}}$*, where* $\delta_* := 1 - |a|/m$*. Define* $\delta = \epsilon\delta_*^{-\frac{1}{2}}$ *and consider a positive interger* N *satisfying* $N\delta \ll 1$*. Then the event horizon of an* $O_N(\epsilon)$*-interior perturbation of a subextremal Kerr spacetime* $\mathcal{K}(a,m)$ *is a* C^N *null hypersurface. Moreover, it is the only (future inextendible) null hypersurface that lies in the region* $\{|r - r_+| \leq \delta\}$*, where* $r_+ := m + \sqrt{m^2 - a^2}$*.*

Remark 1.8. *The spacetime constructed in [16] verifies the conditions in the theorem, as it satisfies* $\delta_* \sim 1$ *and* $N\epsilon \ll 1$*. Stronger assumptions on the spacetime, which are not ensured by the result of [16], could imply that* $\mathcal{H}^+$ *is* C^∞*; See Remark 4.14.*

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[5] An ingoing Principal Geodesic (PG) structure, see Section 2.4 for details, consists of a geodesic foliation given by the coordinates (r, v, θ, φ) with a null geodesic vector field e_3 satisfying $e_3(r) = 1$, $e_3(v) = e_3(\theta) = e_3(\varphi) = 0$, and the horizontal part of the null frame $\{e_3, e_4, e_a\}_{a=1,2}$ satisfying $e_a(r) = 0$. Note that the exact Kerr spacetime admits a canonical ingoing PG structure, see Section 2.3.2.

[6] Such an extension is automatic in the exterior part of the spacetime constructed in [16].

The spacetime constructed in [16] satisfies the Definition 1.3 (in particular, the quantitative estimates obtained in [16] are stronger as there are also decay estimates with respect to v), and hence Theorem 1.1 and Theorem 1.2 are corollaries of the result in the special case when $|a|/m \ll 1$.

1.2 Main features of the proof

We describe below the main features of the paper.

1. The strategy of the proof for the regularity theorem is described in Section 3.2.
2. Section 2 contains a detailed description of the spacetimes we consider, based on the stability results of [16], [12]. As mentioned, our results hold true for the entire subextremal range of $|a|/m$ for which stability can be established and, moreover, they only rely on uniform bounds and not on the more precise decay estimates derived in [16], [12]. Most of our analysis is restricted to the interior region ${}^{(int)}\mathcal{M}$, more precisely in a small neighborhood $\mathcal{R}_\delta := \{|r - r_+| \leq \delta\}$ ($\delta < \min\{\delta_{\mathcal{H}}/2, 1 - \frac{|a|}{m}\}$, $0 < \delta_{\mathcal{H}} \ll 1$) of EV, see (2.22). As in [16], [12], the perturbation estimates in Section 2.4 are described relative to a non-integrable frame, yet in the proof we need to invoke various integrable frames, such as that described in Section 2.5. The general change of frame formulas are described in Section 2.2 and the main estimates for the integrable frame are done in Section 2.6.
3. Section 3.1 contains two important criteria for the regularity of null cones over spheres as well as a description of the main steps in the proof of the regularity theorem, mentioned above.
4. Section 4 contains a detailed proof of the steps described in Section 3.2. The main quantitative estimates are obtained in Proposition 4.2 and Proposition 4.4, which makes use of the sign of ω near the horizon, see Lemma 2.9. Note that when $|a|$ is not small, especially when $|a|$ is close to m, a null structure near the horizon is used, as the effect of ω becomes small and general transformation coefficients, bounded by the size of $|a|$, are no longer small.
5. Section 5 contains the proof of the uniqueness theorem. The main quantitative ingredient, described in Proposition 5.2, makes again use of the sign of ω near $\mathcal{H}^+$.
6. The arguments made in the proof of our regularity and uniqueness results depend only on the assumptions made in Definition 1.3.

2 Preliminaries

2.1 Horizontal structures

We work within the framework of general horizontal structures $\big(e_3, e_4, H = \{e_3, e_4\}^\perp\big)$, where e_3 and e_4 are null vector fields with $g(e_3, e_4) = -2$, introduced in [12]. Given an arbitrary orthonormal basis

$\{e_1, e_2\}$ of H we consider the null frame $\{e_3, e_4, e_a\}$ $(a = 1, 2)$ and the associated Ricci and curvature coefficients[7]

$$\chi_{ab} = g(D_a e_4, e_b), \quad \underline{\chi}_{ab} = g(D_a e_3, e_b), \quad \eta_a = \frac{1}{2} g(D_3 e_4, e_a), \quad \underline{\eta}_a = \frac{1}{2} g(D_4 e_3, e_a), \quad \zeta_a = \frac{1}{2} g(D_a e_4, e_3),$$

$$\omega = \frac{1}{4} g(D_4 e_4, e_3), \quad \underline{\omega} = \frac{1}{4} (D_3 e_3, e_4), \quad \xi_a = \frac{1}{2} g(D_4 e_4, e_a), \quad \underline{\xi} = \frac{1}{2} g(D_3 e_3, e_a),$$

$$\alpha_{ab} = R_{a4b4}, \quad \beta_a = \frac{1}{2} R_{a434}, \quad \rho = \frac{1}{4} R_{3434}, \quad {}^*\rho = \frac{1}{4} {}^*R_{3434}, \quad \underline{\beta}_a = \frac{1}{2} R_{a334}, \quad \underline{\alpha}_{ab} = R_{a3b3}.$$

Here D denotes the spacetime covariant derivative operator. We further decompose

$$\chi_{ab} = \widehat{\chi}_{ab} + \frac{1}{2} \mathrm{tr}\, \chi\, \delta_{ab} + \frac{1}{2} {}^{(a)}\mathrm{tr}\, \chi \in_{ab}, \qquad \underline{\chi}_{ab} = \widehat{\underline{\chi}}_{ab} + \frac{1}{2} \mathrm{tr}\, \underline{\chi}\, \delta_{ab} + \frac{1}{2} {}^{(a)}\mathrm{tr}\, \underline{\chi} \in_{ab},$$

where the trace and anti-trace are defined by

$$\mathrm{tr}\, \chi := \delta^{ab} \chi_{ab}, \quad \mathrm{tr}\, \underline{\chi} := \delta^{ab} \underline{\chi}_{ab}, \quad {}^{(a)}\mathrm{tr}\, \chi := \in^{ab} \chi_{ab}, \quad {}^{(a)}\mathrm{tr}\, \underline{\chi} := \in^{ab} \underline{\chi}_{ab}.$$

and the horizontal volume form $\in_{12} = - \in_{21} = \frac{1}{2} \in (e_1, e_2, e_3, e_4) = 1$. Recall that the horizontal structure is integrable if and only if the asymmetric traces ${}^{(a)}\mathrm{tr}\, \chi$, ${}^{(a)}\mathrm{tr}\, \underline{\chi}$ vanish identically.

For a spacetime vector field X, we define its projection onto the horizontal structure H by

$${}^{(h)}X := X + \frac{1}{2} g(X, e_3) e_4 + \frac{1}{2} g(X, e_4) e_3.$$

A k-covariant tensor field U is called horizontal, if

$$U(X_1, \cdots, X_k) = U({}^{(h)}X_1, \cdots, {}^{(h)}X_k).$$

The horizontal covariant derivative operator ∇ is defined by

$$\nabla_X Y := {}^{(h)}(D_X Y) = D_X Y - \frac{1}{2} \underline{\chi}(X, Y) e_4 - \frac{1}{2} \chi(X, Y) e_3$$

for two horizontal vector fields X, Y. Similarly, one can define $\nabla_3 X$ and $\nabla_4 X$ as the projections of $D_3 X$ and $D_4 X$. Then the horizontal covariant derivative can be generalized for tensors in the standard way

$$\nabla_Z U(X_1, \cdots, X_k) = Z(U(X_1, \cdots X_k)) - U(\nabla_Z X_1, \cdots, X_k) - \cdots - U(X_1, \cdots, \nabla_Z X_k),$$

and similarly for $\nabla_3 U$ and $\nabla_4 U$.

The left dual of a horizontal 1-form ψ and a horizontal covariant 2-tensor U are defined by

$${}^*\psi_a := \in_{ab} \psi_b, \quad ({}^*U)_{ab} := \in_{ac} U_{cb}.$$

For two horizontal 1-forms ψ, ϕ, we also define

$$\psi \cdot \phi := \delta^{ab} \psi_a \phi_b, \quad \psi \wedge \phi := \in^{ab} \psi_a \phi_b, \quad (\psi \widehat{\otimes} \phi)_{ab} = \psi_a \phi_b + \psi_b \phi_a - \delta_{ab} \psi \cdot \phi.$$

In particular $|\psi| := (\psi \cdot \psi)^{\frac{1}{2}}$ with the straightforward generalization to general horizontal covariant tensors.

Similarly we define the derivative operators

$$\mathrm{div}\, \psi := \delta^{ab} \nabla_a \psi_b, \quad \mathrm{curl}\, \psi := \in^{ab} \nabla_a \psi_b, \quad (\nabla \widehat{\otimes} \psi)_{ab} := \nabla_a \psi_b + \nabla_b \psi_a - \delta_{ab}\, \mathrm{div}\, \psi.$$

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[7] Here ${}^*R$ is defined by ${}^*R_{\alpha\beta\mu\nu} = \frac{1}{2} \in_{\mu\nu}{}^{\rho\sigma} R_{\alpha\beta\rho\sigma}$, with $\in$ the volume form on $\mathcal{M}$.

2.2 Null frame transformations

A general frame transformation $(f, \underline{f}, \lambda)$ between two null horizontal structures (e_3, e_4, H) and (e_3', e_4', H') can be written in the form, see Section 2.2 in [16],

$$
\begin{aligned}
e_4' &= \lambda\left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\right),\\
e_a' &= e_a + \frac{1}{2}\underline{f}_a f^b e_b + \frac{1}{2}\underline{f}_a e_4 + \left(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{f}_a\right)e_3,\\
e_3' &= \lambda^{-1}\left(\left(1 + \frac{1}{2}f\cdot\underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2\right)e_3 + \left(\underline{f}_b + \frac{1}{4}|\underline{f}|^2 f^b\right)e_b + \frac{1}{4}|\underline{f}|^2 e_4\right).
\end{aligned}
\tag{2.1}
$$

The Ricci coefficients transform according to Proposition 2.2.3 in [16]. In our work we will consider frame transformations of the form[8]

$$
e_4' = \lambda\left(e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3\right), \quad e_a' = e_a + \frac{1}{2}f_a e_3, \quad e_3' = \lambda^{-1}e_3. \tag{2.2}
$$

Definition 2.1. *If $\psi = \psi_{a_1\ldots a_k}$ is an H-horizontal tensor, we define the associated H'-horizontal tensor by the formula*

$$
\tilde{\psi}_{a_1\ldots a_k} = \tilde{\psi}(e_{a_1}', \ldots, e_{a_k}') = \psi(e_{a_1}, \ldots, e_{a_k}). \tag{2.3}
$$

When taking derivatives with respect to the primed frame, we define for an H-horizontal k-tensor ψ

$$
\nabla_b'\psi_{a_1\cdots a_k} := \nabla_b'\tilde{\psi}_{a_1\cdots a_k} = e_b'(\tilde{\psi}_{a_1\cdots a_k}) - \tilde{\psi}(D_{e_b'}e_{a_1}', \cdots, e_{a_k}') - \cdots - \tilde{\psi}(e_{a_1}', \cdots, D_{e_b'}e_{a_k}').
$$

We also define $\nabla_3'\psi$ and $\nabla_4'\psi$ similarly. By abuse of language, we will often refer to $\tilde{\psi}$ as simply ψ.[9]

Lemma 2.2. *Consider a null frame $\{e_3, e_4, e_a\}$ for which $\underline{\xi} = 0$, i.e. e_3 is proportional to a geodesic vectorfield. Then,*

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[8] Note that the general transformation can be obtained by composing this with a second transformation which reverses the roles of $e_3 - e_4$, i.e. with $f = 0$ and $\underline{f}$ nontrivial.

[9] In [16] and [12] $\tilde{\psi}$ was systematically denoted by ψ.

1. *Under the transformation* (2.2), *we have*[10]

$$\begin{aligned}
\lambda^{-1}\chi'_{ab} &= \chi_{ab} + \nabla'_a f_b + f_a\eta_b + f_a\zeta_b - \frac{1}{4}|f|^2\underline{\chi}_{ab} - \underline{\omega} f_a f_b,\\
\lambda^{-1}\omega' &= \omega - \frac{1}{2}\lambda^{-1}e'_4(\log\lambda) + \frac{1}{2}f^a(\zeta - \underline{\eta})_a - \frac{1}{4}|f|^2\underline{\omega} - \frac{1}{4}f^a f^b\underline{\chi}_{ab},\\
\lambda^{-2}\xi'_a &= \xi_a + \frac{1}{2}\lambda^{-1}\nabla'_4 f_a + \frac{1}{2}f_b\chi_{ba} + \omega f_a + \frac{1}{4}|f|^2\eta_a - \frac{1}{4}|f|^2\underline{\eta}_a + \frac{1}{2}f_a f_b\zeta_b - \frac{1}{8}|f|^2 f_b\underline{\chi}_{ba},\\
\zeta'_a &= \zeta_a - e'_a(\log\lambda) - \underline{\omega} f_a - \frac{1}{2}f_b\underline{\chi}_{ab},\\
\underline{\eta}'_a &= \underline{\eta}_a + \frac{1}{2}f_b\underline{\chi}_{ba},\\
\lambda\,\underline{\chi}'_{ab} &= \underline{\chi}_{ab},\\
\lambda^{-1}\beta'_a &= \beta_a + \frac{3}{2}(f\rho + {}^*f\,{}^*\rho) + \frac{1}{4}|f|^2\underline{\beta}_a - f_a f_b\underline{\beta}_b - f_c\,{}^*f_a\,{}^*\underline{\beta}_c - \frac{1}{2}|f|^2 f_c\underline{\alpha}_{ac} + f^a f^b f^c\underline{\alpha}_{bc}.
\end{aligned} \tag{2.4}$$

2. *Given an H-horizontal tensor* ψ *we have, schematically,*

$$\nabla'_{a_1}\cdots\nabla'_{a_i}\psi = \nabla_{e'_{a_1}}\cdots\nabla_{e'_{a_i}}\psi + f\cdot\underline{\chi}\cdot(\nabla_{e'_a})^{\le i-1}\psi + \nabla'^{\le i-1}(f\cdot\underline{\chi}\cdot\psi). \tag{2.5}$$

   where the differentiation ∇' *is defined in Definition 2.1.*

3. *Similar formulas hold true*[11] *for frame transformations of the form*

$$e'_4 = e_4, \quad e'_a = e_a + \frac{1}{2}\underline{f}_a e_4, \quad e'_3 = e_3 + \underline{f}^a e_a + \frac{1}{4}|\underline{f}|^2 e_4,$$

   for which $\xi = 0$.

Proof. The first part of the lemma is a special case of Proposition 2.2.3 in [16]. To prove the second part we proceed as follows:

$$\begin{aligned}
\nabla'_b\psi_{a_1\cdots a_k} &= \nabla'_b\tilde{\psi}_{a_1\cdots a_k} = e'_b(\tilde{\psi}_{a_1\cdots a_k}) - \tilde{\psi}(D_{e'_b}e'_{a_1},\cdots,e'_{a_k}) - \cdots - \tilde{\psi}(e'_{a_1},\cdots,D_{e'_b}e'_{a_k})\\
&= e'_b(\psi_{a_1\cdots a_k}) - \tilde{\psi}(D_{e'_b}e'_{a_1},\cdots,e'_{a_k}) - \cdots - \tilde{\psi}(e'_{a_1},\cdots,D_{e'_b}e'_{a_k})\\
&= (D_{e'_b}\psi)(e_{a_1},\cdots,e_{a_k}) + \psi(D_{e'_b}e_{a_1},\cdots,e_{a_k}) + \cdots + \psi(e_{a_1},\cdots,D_{e'_b}e_{a_k})\\
&\quad - \tilde{\psi}(D_{e'_b}e'_{a_1},\cdots,e'_{a_k}) - \cdots - \tilde{\psi}(e'_{a_1},\cdots,D_{e'_b}e'_{a_k})\\
&= \nabla_{e'_b}\psi_{a_1\cdots a_k} + \sum_{i=1}^{k}\Big(g(D_{e'_b}e_{a_i},e_c) - g(D_{e'_b}e_{a'_i},e'_c)\Big)\psi_{a_1\cdots c\cdots a_k}\\
&= \nabla_{e'_b}\psi_{a_1\cdots a_k} + \sum_{i=1}^{k}\left(-\frac{1}{2}f_{a_i}\underline{\chi}_{bc}\psi_{a_1\cdots c\cdots a_k} + \frac{1}{2}f_c\underline{\chi}_{ba_i}\psi_{a_1\cdots c\cdots a_k}\right).
\end{aligned} \tag{2.6}$$

Note that all terms on the last line, including $\nabla_{e'_b}\psi_{a_1\cdots a_k} = (\nabla_b + \frac{1}{2}f_b\nabla_3)\psi_{a_1\cdots a_k}$, are H-horizontal $(k+1)$-tensors. This serves as a general formula when we apply ∇' to an H-horizontal tensor ψ.

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[10] Note that $\nabla' f$ and $\nabla'_4 f$ in the transformation formulas above are defined according to the definition (2.3).

[11] By the usual $e_3 - e_4$ duality.

To write $\nabla'_a \nabla'_b \psi$ using $\nabla_{e'_a} \nabla_{e'_b} \psi$, we repeat the calculation in (2.6) with ψ replaced by the last line of (2.6), and obtain schematically

$$\nabla'_a \nabla'_b \psi = \nabla_{e'_a} \nabla_{e'_b} \psi + f \cdot \underline{\chi} \cdot \nabla_{e'_c} \psi + \nabla'(f \cdot \underline{\chi} \cdot \psi).$$

Repeating this we obtain, inductively,

$$\nabla'_{a_1} \cdots \nabla'_{a_i} \psi = \nabla_{e'_{a_1}} \cdots \nabla_{e'_{a_i}} \psi + f \cdot \underline{\chi} \cdot (\nabla_{e'_a})^{\leq i-1} \psi + \nabla'^{\leq i-1} (f \cdot \underline{\chi} \cdot \psi)$$

as stated. $\square$

Remark 2.3. *When* $\lambda = 1$ *in* (2.2)*, one can also derive similarly as* (2.6) *the relation*

$$\nabla'_4 \psi_{a_1 \cdots a_k} = \nabla_{e'_4} \psi_{a_1 \cdots a_k} + \sum_{i=1}^{k} \left(-\frac{1}{2} f_{a_i} \left(\underline{\eta}_c + f_b \underline{\chi}_{bc} \right) \psi_{a_1 \cdots c \cdots a_k} + \frac{1}{2} f_c \left(\underline{\eta}_{a_i} + f_b \underline{\chi}_{b a_i} \right) \psi_{a_1 \cdots c \cdots a_k} \right),$$

and (2.5) *can be generalized to*

$$(\nabla'_4, \nabla')^i \psi = (\nabla_{e'_4}, \nabla_{e'_a})^i \psi + f \cdot (\underline{\eta}, f \cdot \underline{\chi}) \cdot (\nabla_{e'_4}, \nabla_{e'_a})^{\leq i-1} \psi + (\nabla'_4, \nabla')^{\leq i-1} \left(f \cdot (\underline{\eta}, f \cdot \underline{\chi}) \cdot \psi \right). \tag{2.7}$$

2.3 Basic facts about the Kerr spacetime $\mathcal{K}(a, m)$

2.3.1 Boyer-Lindquist (BL) coordinates

Relative to the BL coordinates, the Kerr metric $\mathbf{g} = \mathbf{g}_{a,m}$ takes the form

$$\mathbf{g} \;=\; -\frac{(\Delta - a^2 \sin^2\theta)}{|q|^2} dt^2 - \frac{4amr}{|q|^2} \sin^2\theta \, dt d\phi + \frac{|q|^2}{\Delta} dr^2 + |q|^2 d\theta^2 + \frac{\Sigma^2}{|q|^2} \sin^2\theta \, d\phi^2,$$

where $q = r + ia\cos\theta$ and

$$\begin{cases} \Delta & = & r^2 - 2mr + a^2, \\ |q|^2 & = & r^2 + a^2 (\cos\theta)^2, \\ \Sigma^2 & = & (r^2 + a^2)|q|^2 + 2mra^2(\sin\theta)^2 = (r^2 + a^2)^2 - a^2 (\sin\theta)^2 \Delta. \end{cases}$$

The function $\Delta = r^2 - 2mr + a^2$ has two zeros $r_\pm = m \pm \sqrt{m^2 - a^2}$ with $\{r = r_+\}$ the event horizon of $\mathcal{K}(a, m)$. The ingoing principal null (PN) frame, regular towards the future for all $r > 0$, is given by

$$e_4 = \frac{r^2 + a^2}{|q|^2} \partial_t + \frac{\Delta}{|q|^2} \partial_r + \frac{a}{|q|^2} \partial_\phi, \qquad e_3 = \frac{r^2 + a^2}{\Delta} \partial_t - \partial_r + \frac{a}{\Delta} \partial_\phi. \tag{2.8}$$

The canonical horizontal basis is given by

$$e_1 = \frac{1}{|q|} \partial_\theta, \qquad e_2 = \frac{a \sin\theta}{|q|} \partial_t + \frac{1}{|q| \sin\theta} \partial_\phi. \tag{2.9}$$

2.3.2 Ingoing Eddington–Finkelstein coordinates

The ingoing Eddington–Finkelstein coordinates (v, r, θ, φ) are given by

$$v := t + f(r), \quad f'(r) = \frac{r^2 + a^2}{\Delta}, \qquad \varphi := \phi + h(r), \quad h'(r) = \frac{a}{\Delta},$$

such that,

$$e_3(r) = -1, \quad e_3(v) = e_3(\theta) = e_3(\varphi) = 0, \qquad e_1(r) = e_2(r) = 0, \quad e_1(v) = 0, \quad e_2(v) = \frac{a\sin\theta}{|q|}.$$

Thus,

$$e_4 = \frac{2(r^2 + a^2)}{|q|^2}\partial_v + \frac{\Delta}{|q|^2}\partial_r + \frac{2a}{|q|^2}\partial_\varphi, \qquad e_3 = -\partial_r, \qquad e_1 = \frac{1}{|q|}\partial_\theta, \qquad e_2 = \frac{a\sin\theta}{|q|}\partial_v + \frac{1}{|q|\sin\theta}\partial_\varphi.$$

The metric expression in the ingoing Eddington–Finkelstein coordinates is given by

$$\mathbf{g}_{a,m} = -\left(1 - \frac{2mr}{|q|^2}\right)(dv - a\sin^2\theta\, d\varphi)^2 + 2(dv - a\sin^2\theta\, d\varphi)dr + |q|^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{2.10}$$

In the language of [16] (chapters 2,3), the nonvanishing Ricci and curvature coefficients are given by

$$\mathrm{tr}X = \frac{2\overline{q}\Delta}{|q|^4}, \quad \mathrm{tr}\underline{X} = -\frac{2}{\overline{q}}, \quad Z = H = \frac{aq}{|q|^2}\mathfrak{J}, \quad \underline{H} = -\frac{a\overline{q}}{|q|^2}\mathfrak{J}, \quad \omega = -\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right), \quad P = -\frac{2m}{q^3},$$

where $X = \chi + i\,{}^*\chi$, $\underline{X} = \underline{\chi} + i\,{}^*\underline{\chi}$, $H = \eta + i\,{}^*\eta$, $\underline{H} = \underline{\eta} + i\,{}^*\underline{\eta}$, $Z = \zeta + i\,{}^*\zeta$, $P = \rho + i\,{}^*\rho$ are complexified Ricci coefficients and curvature components, and $\mathfrak{J}$ is a horizontal 1-form satisfying $\mathfrak{J}_1 = i\sin\theta\,|q|^{-1}$, $\mathfrak{J}_2 = \sin\theta\,|q|^{-1}$ in the canonical horizontal basis.

2.4 The $O_N(\epsilon)$-interior perturbation of Kerr

2.4.1 Quantitative bounds in ${}^{(int)}\mathcal{M}$

By our definition of an $O_N(\epsilon)$-interior perturbation of the Kerr spacetime $\mathcal{K}(a, m)$, we have $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$, where ${}^{(int)}\mathcal{M}$ is endowed with an ingoing principal geodesic[12] (PG) structure. In [16], ${}^{(int)}\mathcal{M}$ has a future spacelike boundary $\{r = r_+ - \delta_{\mathcal{H}}\}$, where $\delta_{\mathcal{H}} > 0$ is a small constant. We make a corresponding assumption that the region $\{r_+ - \delta_{\mathcal{H}} \le r \le r_0\}$ is contained in ${}^{(int)}\mathcal{M}$. Note that

$$r_+ = m + \sqrt{m^2 - a^2}, \quad \Delta = r^2 - 2mr + a^2, \quad q = r + ia\cos\theta,$$

where r, θ are the coordinates from the ingoing PG structure.

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[12] See Section 2.3.1 in [16] for the definition of PG structures. Note that Kerr has a canonical ingoing PG structure given by the Eddington–Finkelstein coordinates and associated frame.

The ingoing PG structure of ${}^{(int)}\mathcal{M}$ consists of coordinates (v, r, θ, φ), (v, r, x^1, x^2), and a null frame $\{e_3, e_4, e_a\}$, satisfying

$$D_3 e_3 = 0, \quad e_3(r) = -1, \quad e_a(r) = 0, \quad e_3(v) = e_3(\theta) = e_3(\varphi) = e_3(x^1) = e_3(x^2) = 0.$$

The intersecting spheres of level sets of v and r are denoted by $S(v, r)$. The two coordinate systems $S(v, r)$ are related by

$$x^1 = \sin\theta\cos\varphi, \quad x^2 = \sin\theta\sin\varphi.$$

The coordinates (x^1, x^2) are regular near the north and south pole where the coordinates (θ, φ) become irregular.

To state the estimates for the Ricci and curvature coefficients, we need to define a complex horizontal 1-form $\mathfrak{J}$ as in [16]. In ${}^{(int)}\mathcal{M}$, $\mathfrak{J}$ satisfies

$$\nabla_3 \mathfrak{J} = \frac{1}{\overline{q}}\mathfrak{J}, \qquad {}^*\mathfrak{J} = -i\mathfrak{J}, \qquad \mathfrak{J} \cdot \overline{\mathfrak{J}} = \frac{2(\sin\theta)^2}{|q|^2}.$$

We have $\mathfrak{J} = j + i\,{}^*j$ where j is a real horizontal 1-form, which satisfies

$$\nabla_3 j = |q|^{-2}(rj - a\cos\theta\,{}^*j), \quad \nabla_3\,{}^*j = |q|^{-2}(r\,{}^*j + a\cos\theta j), \quad |j|^2 = \frac{(\sin\theta)^2}{|q|^2}. \tag{2.11}$$

Definition 2.4. *We define $\check{\Gamma}$ to be the set of the following quantities*[13]

$$\xi,\ \check{\omega},\ \widecheck{\mathrm{tr}\,\chi},\ \widecheck{{}^{(a)}\mathrm{tr}\,\chi},\ \widehat{\chi},\ \widehat{\underline{\chi}},\ \check{\zeta},\ \check{\underline{\eta}},\ \widecheck{\mathrm{tr}\,\underline{\chi}},\ \widecheck{{}^{(a)}\mathrm{tr}\,\underline{\chi}},\ r\check{\rho},\ r\,{}^*\rho,\ r\beta,\ r\alpha,\ r\underline{\beta},\ \underline{\alpha},\ r\widecheck{e_4(r)},\ r\widecheck{e_4(v)},\ \widecheck{\nabla} v,$$
$$r\mathrm{div} j,\ r\widecheck{\mathrm{curl}} j,\ \widecheck{\nabla_4\, j},\ \widecheck{\nabla_4\,{}^*j},\ r\nabla\widehat{\otimes} j,\ \widecheck{\nabla(\cos\theta)},\ re_4(\cos\theta),$$

where the checked, linearized quantities are defined by

$$\begin{aligned}
&\check{\omega} := \omega + \frac{1}{2}\partial_r\Big(\frac{\Delta}{|q|^2}\Big), \quad \widecheck{\mathrm{tr}\,\chi} := \mathrm{tr}\,\chi - \frac{2r\Delta}{|q|^4}, \quad \widecheck{{}^{(a)}\mathrm{tr}\,\chi} := {}^{(a)}\mathrm{tr}\,\chi - \frac{2a\cos\theta\Delta}{|q|^4}, \quad \check{\zeta} = \zeta - \frac{rj - a\cos\theta\,{}^*j}{|q|^2}a,\\
&\check{\underline{\eta}} = \underline{\eta} + \frac{rj + a\cos\theta\,{}^*j}{|q|^2}a, \quad \widecheck{\mathrm{tr}\,\underline{\chi}} := \mathrm{tr}\,\underline{\chi} + \frac{2r}{|q|^2}, \quad \widecheck{{}^{(a)}\mathrm{tr}\,\underline{\chi}} := {}^{(a)}\mathrm{tr}\,\underline{\chi} - \frac{2a\cos\theta}{|q|^2}, \quad \check{\rho} = \rho + \Re\Big(\frac{2m}{q^3}\Big)\\
&\widecheck{e_4(r)} = e_4(r) - \frac{\Delta}{|q|^2}, \quad \widecheck{e_4(v)} = e_4(v) - \frac{2(r^2+a^2)}{|q|^2}, \quad \widecheck{\nabla} v = \nabla v - aj, \quad \widecheck{\nabla(\cos\theta)} := \nabla(\cos\theta) + {}^*j\\
&\widecheck{\mathrm{curl}}\, j = \mathrm{curl}\, j - \frac{2(r^2+a^2)\cos\theta}{|q|^4}, \quad \widecheck{\nabla_4 j} = \nabla_4 j - \frac{\Delta(rj + a\cos\theta{}^*j)}{|q|^4}, \quad \widecheck{\nabla_4\,{}^*j} = \nabla_4\,{}^*j + \frac{\Delta(r\,{}^*j - a\cos\theta\, j)}{|q|^4}.
\end{aligned} \tag{2.12}$$

Note that under an ingoing PG structure, we have $\underline{\xi} = 0$, $\underline{\omega} = 0$, $\eta = \zeta$.

The following smallness estimate in ${}^{(int)}\mathcal{M}$ is our main assumption:

$$|(\nabla_3, \nabla_4, \nabla)^{\leq N}\check{\Gamma}| \leq \epsilon. \tag{2.13}$$

---

[13] This real form of the linearized quantities is equivalent to the complex form in [16]. Note that the formulation in [16] works in the full subextremal case.

Remark 2.5. *In [16, Section 3.4.3], the following decay estimate holds:*

$$|(\nabla_3, \nabla_4, \nabla)^{\le N} \check{\Gamma}| \le \epsilon\, v^{-1-\delta_{dec}}, \tag{2.14}$$

where $\delta_{dec} > 0$. Our arguments do not require decay and are valid in the full sub-extremal case $|a| < m$.

Since the metric can be given by the null frame, we have:[14]

$$\begin{aligned} g &= \mathbf{g}_{a,m} + (dv, dr, r d\theta, r\sin\theta\, d\varphi)^2\, O(\epsilon), \qquad && \frac{\pi}{4} < \theta < \frac{3\pi}{4}, \\ g &= \mathbf{g}_{a,m} + (dv, dr, r dx^1, r dx^2)^2\, O(\epsilon), && 0 \le \theta < \frac{\pi}{3} \text{ or } \frac{2\pi}{3} < \theta \le \pi. \end{aligned} \tag{2.15}$$

Remark 2.6 (Main constants)**.** *Here ϵ is a small constant satisfying $\epsilon \ll \min\{\delta_{\mathcal{H}}, \delta_* := 1 - |a|/m\}$. We choose a small constant $\delta = \epsilon\delta_*^{-\frac{1}{2}}$ such that $\epsilon \le \delta \le \min\{\delta_{\mathcal{H}}/2, \delta_*\}$.*

We will need to deal with derivatives of the real 1-form j when changing frames.

Lemma 2.7. *We have*

$$|(\nabla, \nabla_3, \nabla_4)^{\le N}(j, {}^*j)| \lesssim 1. \tag{2.16}$$

Proof. In view of the linearization of $\nabla_4 j$, $\operatorname{curl} j$, see (2.12), and (2.11), along with (2.13), in order to control $(\nabla, \nabla_3, \nabla_4)^{\le k}(j, {}^*j)$, it suffices to control $(\nabla, \nabla_3, \nabla_4)^{\le k}\cos\theta$ and $(\nabla, \nabla_3, \nabla_4)^{\le k-1}(j, {}^*j)$. The boundedness of $(j, {}^*j)$ has been established in (2.11), so by induction, we assume $|(\nabla, \nabla_3, \nabla_4)^{\le k-1}(j, {}^*j, \cos\theta)| \lesssim 1$. Then we have

$$|(\nabla, \nabla_3, \nabla_4)^k \cos\theta| \lesssim |(\nabla, \nabla_3, \nabla_4)^{\le k-1}(\nabla_4\cos\theta, \nabla_3\cos\theta, \nabla\cos\theta)| \lesssim |(\nabla, \nabla_3, \nabla_4)^{\le k-1}(\check{\Gamma}, {}^*j)| \lesssim 1,$$

where we used $e_3(\cos\theta) = 0$, $re_4(\cos\theta) \in \check{\Gamma}$, and $\widecheck{\nabla(\cos\theta)} := \nabla(\cos\theta) + {}^*j$. ☐

Remark 2.8. *From the estimates, we see that while the linearized quantities are ϵ-small, some original quantities ζ, $\underline{\eta}$, and ∇v have sizes that can only be bounded by $|a|$. This is a key difference to address for the full subextremal case, along with the fact that the effect of ω decays to zero as $|a|/m \to 1$, as can be seen from Lemma 2.9 below.*

2.4.2 The quantity ω

Lemma 2.9. *Assuming that (2.13) holds for some $|a| < m$, we have, in the region $\{|r - r_+| \le \delta\}$ with δ a small constant satisfying $\delta \le \delta_* := 1 - |a|/m$, we have*

$$\omega \le -\frac{1}{32m}\delta_*^{\frac{1}{2}}, \qquad |\nabla^{\le N}\omega| \lesssim O(\delta_*^{\frac{1}{2}}). \tag{2.17}$$

---

[14] In this paper, by $A \lesssim B$ we mean $A \le CB$ for some $C > 0$ independent of A, B, and the location in the spacetime. If the implicit constant is dependent on a parameter s, we denote it by $A \lesssim_s B$. We also use $O(B)$ to denote a quantity A satisfying $|A| \lesssim |B|$.

Proof. In the exact Kerr $\mathcal{K}(a,m)$, under the principal ingoing PG frame, we have

$$\omega_{a,m} = -\frac{1}{2}\partial_r\left(\frac{\Delta}{|q|^2}\right) = -\frac{a^2\cos^2\theta(r-m)+mr^2-a^2r}{|q|^4}.$$

Recall that $r_+ = m+\sqrt{m^2-a^2} = m+m\sqrt{1-(a/m)^2} = m+m\sqrt{\delta_*(2-\delta_*)} = m+O(\delta_*^{1/2})$. For $r=r_+$,

$$\begin{aligned} a^2\cos^2\theta(r-m)+mr^2-a^2r \geq &= ma^2\cos^2\theta\sqrt{\delta_*(2-\delta_*)} + r_+\big(\big(m+\sqrt{\delta_*(2-\delta_*)}\big)m-a^2\big) \\ &= \sqrt{\delta_*(2-\delta_*)}\big(ma^2\cos^2\theta+r_+m\big)+r_+\big(m^2-a^2\big) \\ &= \sqrt{\delta_*(2-\delta_*)}\big(ma^2\cos^2\theta+r_+m\big)+r_+m^2\delta_*(2-\delta_*) \\ &= mr_+\sqrt{\delta_*(2-\delta_*)}\big(1+\frac{a^2\cos^2\theta}{r_+}+m\sqrt{\delta_*(2-\delta_*)}\big) \\ &\geq m^2r_+\sqrt{\delta_*(2-\delta_*)}. \end{aligned}$$

We deduce,

$$-\omega_{a,m}|_{r=r_+} \geq \frac{m^2r_+\sqrt{\delta_*(2-\delta_*)}}{|q|^4} \geq \frac{\delta_*^{1/2}}{16m}.$$

Therefore, with a perturbation of size $O(\epsilon)$, $\epsilon \ll \delta_*$, in the region $\{|r-r_+|\leq\delta\}$, we have

$$\omega \leq -\frac{\delta_*^{1/2}}{32m}.$$

Also, since $\nabla(r)=0$, for $|r-r_+|\leq\delta\leq\delta_*$,

$$|\nabla^{\leq N}\omega| = O(|r-m|,|mr^2-a^2r|) = O\left(\sqrt{1-|a|/m}\right) = O(\delta_*^{\frac{1}{2}}).$$

With the assumption that similar estimates in the last subsection hold, we see that the estimates regarding ω here also hold under an $O(\epsilon)$ perturbation. $\square$

2.5 Adapted integrable frame in ${}^{(int)}\mathcal{M}$

In addition to the non-integrable ingoing PG frame in ${}^{(int)}\mathcal{M}$, we introduce a related integrable frame adapted to the spheres $S(v,r)$.

2.5.1 Adapted integrable frame in Kerr

Lemma 2.10. *For the ingoing PG structure in Kerr, there exists a canonical integrable horizontal structure (* ${}^{(S)}e_3$, ${}^{(S)}e_4$, ${}^{(S)}H$*) compatible*[15] *with the sphere* $S(v,r)$ *with transformation parameters* $(F,\underline{F},\lambda=1)$ *such that*

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[15] i.e. ${}^{(S)}H$ tangent to $S(v,r)$.

- *We have*

$$F = -\frac{4e_4(r)\nabla v}{e_4(v) + \sqrt{|e_4(v)|^2 - 4e_4(r)|\nabla v|^2}}, \qquad \underline{F} = -\frac{2\nabla v}{\sqrt{|e_4(v)|^2 - 4e_4(r)|\nabla v|^2}}. \tag{2.18}$$

  where $e_4(r) = \frac{\Delta}{|q|^2}$ *and* $e_4(v) = \frac{2(r^2+a^2)}{|q|^2}$.

- *The spheres* $S(v, r_+)$ *are marginally outer trapped surfaces (MOTS), i.e.* ${}^{(S)}\mathrm{tr}\,\chi = 0$.

Proof. Denote the frame transformation from $\{e_3, e_4, e_a\}$ to $\{{}^{(S)}e_3, {}^{(S)}e_4, {}^{(S)}e_a\}$ by $(F, \underline{F})$. Recall that $e_3(r) = -1$, $e_3(v) = 0$, $e_a(r) = 0$. According to (2.1) we look for a horizontal structure spanned by

$${}^{(S)}e_a \quad = e_a + \tfrac{1}{2}\underline{F}_a F^b e_b + \tfrac{1}{2}\underline{F}_a e_4 + \left(\tfrac{1}{2}F_a + \tfrac{1}{8}|F|^2\underline{F}_a\right) e_3,$$

such that ${}^{(S)}e_a(v) = 0$, ${}^{(S)}e_a(r) = 0$. Therefore

$$\begin{aligned} 0 &= e_a(v) + \frac{1}{2}\underline{F}_a F^b e_b(v) + \frac{1}{2}\underline{F}_a e_4(v) \\ 0 &= \frac{1}{2}\underline{F}_a e_4(r) - \left(\frac{1}{2}F_a + \frac{1}{8}|F|^2\underline{F}_a\right). \end{aligned}$$

From the second equation we deduce

$$\Big(e_4(r) - \frac{1}{4}|F|^2\Big)\underline{F} = F, \qquad \text{with } e_4(r) = \frac{\Delta}{|q|^2}. \tag{2.19}$$

The first equation takes the form

$$\nabla v + \frac{1}{2}\underline{F}(F\cdot\nabla v) + \frac{1}{2}\underline{F}e_4(v) = 0.$$

We look for an F of the form $F = h\nabla v$. Thus,

$$\nabla v + \frac{1}{2}h|\nabla v|^2\underline{F} + \frac{1}{2}\underline{F}e_4(v) = 0.$$

Multiplying by $e_4(r) - \frac{1}{4}|F|^2$ and using (2.19) we deduce

$$\Big(e_4(r) - \frac{1}{4}h|\nabla v|^2\Big)h\nabla v + \frac{1}{2}h|\nabla v|^2 h\nabla v + \frac{1}{2}(h\nabla v)e_4(v) = 0.$$

We deduce

$$\frac{1}{4}|\nabla v|^2 h^2\nabla v + \Big(\frac{1}{2}e_4(v)\nabla v + he_4(r)\Big)\nabla v = 0.$$

and therefore, as long as $|\nabla v| \neq 0$,

$$|\nabla v|^2 h^2 + 2e_4(v)h + 4e_4(r) = 0, \tag{2.20}$$

or, taking the smaller root,[16]

$$h = \frac{-e_4(v) + \sqrt{|e_4(v)|^2 - 4e_4(r)|\nabla v|^2}}{|\nabla v|^2} = -\frac{4e_4(r)}{e_4(v) + \sqrt{|e_4(v)|^2 - 4e_4(r)|\nabla v|^2}}.$$

Recall that $e_4(r) = \frac{\Delta}{|q|^2}$, $e_4(v) = 2\frac{r^2+a^2}{|q|^2}$. Also, whenever $e_4(r) \neq 0$, we have

$$\underline{F} = \frac{1}{e_4(r) - \frac{1}{4}|F|^2} F = \frac{h\nabla v}{e_4(r) - \frac{1}{4}h^2|\nabla v|^2} = \frac{h\nabla v}{2e_4(r) + \frac{1}{2}e_4(v)h} = \frac{-2\nabla v}{\sqrt{|e_4(v)|^2 - 4e_4(r)|\nabla v|^2}},$$

where we used (2.20). When $e_4(r) = 0$, there is in fact no constraint on $\underline{F}$, so we can simply use the same expression for $\underline{F}$.

Finally in view of the transformation formula (2.4), from the standard frame to the S-frame, using the fact that F and $\operatorname{tr}\chi$ vanish everywhere on $\{r = r_+\}$, we have, on $S(v, r_+)$,

$$\begin{aligned} {}^{(S)}\operatorname{tr}\chi &= \delta^{ab} g(D_{{}^{(S)}e_a} {}^{(S)}e_4, {}^{(S)}e_b) = \delta^{ab} g\Big(D_{e_a + \frac{1}{2}\underline{F}_a e_4}\big(e_4 + F^b e_b + \frac{1}{4}|F|^2 e_3\big), e_a + \frac{1}{2}\underline{F}_a e_4\Big)\Big|_{r=r_+} \\ &= \operatorname{tr}\chi + \frac{1}{2}\underline{F}\cdot\xi = \operatorname{tr}\chi = 0, \end{aligned}$$

as stated. □

2.5.2 Adapted integrable frame in ${}^{(int)}\mathcal{M}$

Similarly, we can obtain an adapted integrable frame from the PG frame in the perturbed spacetime.

Lemma 2.11. *Consider the perturbation of Kerr spacetime $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$ constructed in [16], with the ingoing PG structure in ${}^{(int)}\mathcal{M}$. Then there exists a null frame $({}^{(S)}e_3, {}^{(S)}e_4, {}^{(S)}e_a)$ on ${}^{(int)}\mathcal{M}$, with an integrable horizontal structure adapted to the spheres $S(v, r)$. The transformation parameters $(F, \underline{F}, \lambda = 1)$ from the ingoing PG frame to the new integrable frame can be written as*

$$F = -\frac{4e_4(r)\nabla v}{e_4(v) + \sqrt{|e_4(v)|^2 - 4e_4(r)|\nabla v|^2}}, \qquad \underline{F} = -\frac{2\nabla v}{\sqrt{|e_4(v)|^2 - 4e_4(r)|\nabla v|^2}}. \tag{2.21}$$

Proof. The procedure is identical to the one in Lemma 2.10. □

2.6 Estimates of the integrable frame near the event horizon

We restrict our discussion in a neighborhood of $\mathcal{H}^+$, i.e. the region

$$\mathcal{R}_\delta := \{|r - r_+| \le \delta\}, \quad \text{where} \quad \epsilon \ll \delta \le \min\{\delta_*, \delta_{\mathcal{H}}/2\}, \tag{2.22}$$

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[16] It is straightforward to verify that the discriminant is strictly positive in ${}^{(int)}\mathcal{M}$.

with $\delta_* = 1 - \frac{|a|}{m}$.

We have the following bounds for the transformation parameters $(F, \underline{F})$.

Proposition 2.12. *In the region $\mathcal{R}_\delta$, we have*

$$|(\nabla_4, \nabla)^{\leq N} F| \leq C_N \delta, \quad |(\nabla_3, \nabla_4, \nabla)^{\leq N} F| \leq C_N \max\{\delta, |a|\}, \qquad |(\nabla_3, \nabla_4, \nabla)^{\leq N} \underline{F}| \leq C_N \max\{\delta, |a|\}.$$

Proof. The bounds proportional to a are direct corollaries of the estimates (2.13), (2.16) and the formulas for $F, \underline{F}$ in (2.21), which are both proportional to $\nabla v = \widecheck{\nabla v} + aj$. To check the $O(\delta)$ bound for F it suffices to notice that its expression is proportional to $e_4(r) = \frac{\Delta}{|q|^2} + \widecheck{e_4(r)} = O(\delta)$. Moreover this bound is preserved by taking higher derivatives in e_4, e_a. Indeed,

$$\begin{aligned}
e_4\left(e_4(r)\right) &= e_4\left(\widecheck{e_4(r)}\right) + e_4\left(\frac{\Delta}{|q|^2}\right) = e_4\left(\widecheck{e_4(r)}\right) + e_4(r)\left(\partial_r \Delta\right)\left(\frac{1}{|q|^2}\right) + \Delta\, e_4\left(\frac{1}{|q|^2}\right), \\
e_a\left(e_4(r)\right) &= e_a\left(\widecheck{e_4(r)}\right) + \Delta\, e_a\left(\frac{1}{|q|^2}\right),
\end{aligned}$$

so there is always a Δ or $e_4(r)$ factor, ensuring the δ-smallness. Note that $e_4(\Delta)$ gives $e_4(r)$, so along with $e_a(r) = 0$, we see that a similar estimate holds with higher order differentiations in ∇_4 and ∇. ☐

Corollary 2.13. *In the region $\mathcal{R}_\delta$, we have $|(\nabla_{{}^{(S)}e_a}, \nabla_{{}^{(S)}e_4})^{\leq N} F| \leq \delta$.*

Note that we are still using the H-horizontal operator ∇. Recall that ${}^{(S)}e_a = e_a + \frac{1}{2}\underline{F}_a F^b e_b + \frac{1}{2}\underline{F}_a e_4 + (\frac{1}{2}F_a + \frac{1}{8}|F|^2 \underline{F}_a)e_3$, so $\nabla_{{}^{(S)}e_a}$ is defined by $\nabla_{{}^{(S)}e_a} = \nabla_a + \frac{1}{2}\underline{F}_a F^b \nabla_b + \frac{1}{2}\underline{F}_a \nabla_4 + (\frac{1}{2}F_a + \frac{1}{8}|F|^2 \underline{F}_a)\nabla_3$. Similarly $\nabla_{{}^{(S)}e_4} = \nabla_4 + F^a \nabla_a + \frac{1}{4}|F|^2 \nabla_3$.

Proof. The only derivative that does not give δ-smallness is the e_3 direction, but one can see from the expression of $\nabla_{{}^{(S)}e_a}$ and $\nabla_{{}^{(S)}e_4}$ that ∇_3 is always paired with a factor in F, which provides a δ-smallness factor. Therefore, the estimate is a direct corollary of Proposition 2.12. ☐

2.7 The time function τ

We endow the region $\{|r - r_+| < 2\delta\}$ with a spacelike foliation Σ_τ, given by the level hypersurfaces of the function $\tau := v - r$. One may extend Σ_τ to the whole spacetime, but here it suffices to focus on $\{|r - r_+| < 2\delta\}$. To see that τ is indeed a time function and therefore Σ_τ are spacelike, we compute (for $|r - r_+| \leq 2\delta$),

$$\begin{aligned}
g(\operatorname{grad}\tau, \operatorname{grad}\tau) &= \partial^\mu \tau\, \partial_\mu \tau = -e_3(\tau)e_4(\tau) + e_a(\tau)e_a(\tau) = e_3(r)e_4(v - r) + e_a(v)e_a(v) \\
&= -(e_4(v) + O(\delta)) + |\nabla v|^2 \\
&= -\frac{2(r^2 + a^2)}{|q|^2} + \frac{a^2 \sin^2\theta}{|q|^2} + O(\delta) + O(\epsilon) \\
&= -\frac{2r^2 + a^2 \cos^2\theta}{|q|^2} + O(\delta) \leq -1 < 0,
\end{aligned}$$

so τ is indeed a time function.

The coordinate system on Σ_τ. From the spacetime coordinates (v, r, θ, φ) (and (v, r, x^1, x^2)), we see that (r, θ, φ) (and (r, x^1, x^2)) become coordinate systems on $\Sigma_\tau = \{v - r = \tau\}$ for each τ. The coordinate basis can be expressed by the spacetime coordinate basis by $(\overline{\partial}_r, \overline{\partial}_\theta, \overline{\partial}_\varphi) = (\partial_r + \partial_v, \partial_\theta, \partial_\varphi)$ (similarly for the (r, x^1, x^2) coordinate).

In fact, by (2.15), there exists a diffeomorphism

$$\Phi_{\Sigma_\tau} \colon (r_+ - 2\delta, r_+ + 2\delta) \times \mathbb{S}^2 \to \Sigma_\tau \cap \{|r - r_+| < 2\delta\}, \tag{2.23}$$

where (θ, φ) (and $(x^1 = \sin\theta\cos\varphi, x^2 = \sin\theta\sin\varphi)$) are the spherical coordinates on $\mathbb{S}^2$, where the pullback metric, denoted by $\Phi^*_{\Sigma_\tau}\overline{g}$, or simply $\overline{g}$, satisfies

$$\overline{g} = \overline{\mathbf{g}}_{a,m} + O(\epsilon).$$

Here $\overline{\mathbf{g}}_{a,m}$ is the corresponding pullback metric induced on $\{\tau = v - r = \text{const}\}$ in exact Kerr. One can compute from the expression (2.10) that

$$\overline{\mathbf{g}}_{a,m} = \left(1 + \frac{2mr}{|q|^2}\right) dr^2 - \frac{4amr\sin^2\theta}{|q|^2} dr d\varphi + |q|^2 d\theta^2 + \frac{\Sigma^2}{|q|^2}\sin^2\theta d\varphi^2.$$

One can also derive the expression in (r, x^1, x^2) coordinates by straightforward calculation:

$$\begin{aligned}\overline{\mathbf{g}}_{a,m} = &\left(1 + \frac{2mr}{|q|^2}\right) dr^2 + \frac{4amr}{|q|^2}\left(x^2 dr dx^1 - x^1 dr dx^2\right)\\ &+ \left(a^2\left(1 + \frac{2mr}{|q|^2}\right)(x^2)^2 + \frac{1 - (x^2)^2}{1 - |x|^2}|q|^2\right)(dx^1)^2\\ &+ 2\left(-a^2\left(1 + \frac{2mr}{|q|^2}\right)x^1x^2 + \frac{x^1x^2}{1 - |x|^2}|q|^2\right)dx^1dx^2\\ &+ \left(a^2\left(1 + \frac{2mr}{|q|^2}\right)(x^1)^2 + \frac{1 - (x^1)^2}{1 - |x|^2}|q|^2\right)(dx^2)^2,\end{aligned}$$

which is regular away from $\theta = \pi/2$.

In particular, we see that

$$1 \leq g(\overline{\partial}_r, \overline{\partial}_r) = 1 + \frac{2mr}{|q|^2} + O(\delta) \leq 3. \tag{2.24}$$

It is also clear from the metric expression that there exists a constant only dependent on a, m, δ such that

$$C^{-1}\overline{g}_0 \leq \overline{g} \leq C\overline{g}_0, \quad |r - r_+| \leq 2\delta. \tag{2.25}$$

Here $\overline{g}_0$ denotes the induced Euclidean metric when we embed $(r_+ - 2\delta, r_+ + 2\delta) \times \mathbb{S}^2$ into $\mathbb{R}^3$ using the standard spherical coordinates.

We will frequently use this diffeomorphism Φ_{Σ_τ} from (2.23). Denote the natural projection $(r_+ - 2\delta, r_+ + 2\delta) \times \mathbb{S}^2 \to \mathbb{S}^2$ by $P_{\mathbb{S}^2}$. Then we also define

$$P_{\Sigma_\tau;\mathbb{S}^2} := P_{\mathbb{S}^2} \circ \Phi^{-1}_{\Sigma_\tau} \colon \Sigma_\tau \cap \{|r - r_+| < 2\delta\} \to \mathbb{S}^2. \tag{2.26}$$

This assigns to each point on $\Sigma_\tau \cap \{|r - r_+| < 2\delta\}$ a coordinate value on $\mathbb{S}^2$.

2.8 Causal relations

Definition 2.14. *Given a set S in the spacetime, we define its chronological future $I^+(S)$ and causal future $\mathcal{J}^+(S)$ by*

$$I^+(S) := \{q\colon \text{There exists a point } p \in S \text{ and a future-directed timelike curve from } p \text{ to } q\}.$$

$$\mathcal{J}^+(S) := \{q\colon \text{There exists a point } p \in S \text{ and a future-directed causal curve from } p \text{ to } q\}.$$

One can also define its chronological past $I^-(S)$ and causal past $\mathcal{J}^-(S)$ by replacing "future" with "past".

By a causal curve we mean a C^1 curve with velocity always timelike or null. We have the transitivity properties [21, Propositions 2.5, 2.18]:[17]

Proposition 2.15. *The following statements are true:*

- *(i) If $a \in \mathcal{J}^+(b)$, $b \in \mathcal{J}^+(c)$, then $a \in \mathcal{J}^+(c)$;*
- *(ii) If $a \in \mathcal{J}^+(b)$, $b \in I^+(c)$ or $b \in I^+(b)$, $b \in \mathcal{J}^+(c)$, then $a \in I^+(c)$.*

The spacetime we study here is globally hyperbolic. In this case, we have the following properties ([24, Theorem 8.3.11])

Proposition 2.16. *In any globally hyperbolic spacetime, $\mathcal{J}^+(K)$ is closed if K is compact. Moreover, we have $\overline{I^+(K)} = \mathcal{J}^+(K)$. Same things hold for $\mathcal{J}^-(K)$ and $I^-(K)$.*

Definition 2.17. *A set is called achronal, if any timelike curve cannot intersect it more than once.*[18]

2.8.1 Null cones over spheres

Consider a spacetime $\mathcal{M}$ foliated by a family of spacelike hypersurfaces $\{\Sigma_\tau\}_{\tau\in\mathbb{R}}$, and each Σ_τ is diffeomorphic to $\mathbb{R}^3\backslash B$, where B is the unit ball in $\mathbb{R}^3$. Any embedded sphere on Σ_τ corresponds to a sphere on $\mathbb{R}^3\backslash B$ through a diffeomorphism map and divides[19] $\Sigma_\tau\backslash B$ into two regions, the interior, and the exterior (the latter containing the spatial infinity). This then defines the outgoing and incoming null direction perpendicular to S.

Definition 2.18. *The future outgoing (incoming) null cone of a sphere S, denoted by $C^+(S)$ ($C^-(S)$), is the null hypersurface generated by the congruence of future outgoing (incoming) null geodesics perpendicular to S. We can define the past null cones in the same way.*

---

[17] Strictly speaking, the set $I^+(S)$ and $\mathcal{J}^+(S)$ is defined differently by piecewise C^1 geodesics in [21], but the equivalence was also discussed in the same chapter.

[18] A typical example is the light cone from a point in Minkowski space.

[19] The Jordan-Brouwer separation theorem applies, for example.

Definition 2.19. *A (portion of) null cone $C(S) = C^{\pm}(S)$, generated by an embedded sphere S, is said to be regular if it is smooth and embedded, i.e. it is the image of a smooth embedding from $[0,T] \times \mathbb{S}^2$ to the spacetime $\mathcal{M}$.*

The following definition is important in determining the regularity of $C(S)$, see Remark 3.4.

Definition 2.20. *Let L be a null geodesic vectorfield of $C(S)$, i.e. tangent to the null generators of $C(S)$ and $D_L L = 0$, and define the null expansion along each null generator to be $\operatorname{tr}\chi = \delta^{ab} g(D_{E_a} L, E_b)$ where E_a is an arbitrary choice of spacelike orthonormal frame perpendicular to L. Note that $\operatorname{tr}\chi$ is invariant, i.e., it does not depend on the choice of E_a.*[20]

We now give some causal properties when the future outgoing null cone $C^+(S)$ is regular. Clearly similar properties also hold true for a regular past incoming cone of S (i.e. $C^+(S)$ when extended towards the past).

Definition 2.21. *Assume that $C = C^+(S)$ is regular, so $S_\tau = C \cap \Sigma_\tau$ is an embedded sphere on Σ_τ. In this case, S_τ defines the interior and exterior region of Σ_τ, denoted by Σ^i_τ and Σ^e_τ. Unless specifically mentioned otherwise, we assume that they include the boundaries.*

Proposition 2.22. *Assume $C = C^+(S)$ is a regular null cone with Σ^i_τ, Σ^e_τ defined as above. Then there is no future-directed timelike curve*[21] *emanating from Σ^i_τ that intersects C.*

Proof. Suppose there is such a curve γ initiating on Σ_τ, and denote the first intersecting point $p \in S_{\tau'} \subset C$ for some $\tau' > \tau$. Take a null frame $\{L, \underline{L}, E_1, E_2\}$ so that E_1, E_2 are tangent to $S_{\tau'}$ (hence $\Sigma_{\tau'}$), L is tangent to the null generators of C, and both L, $\underline{L}$ are pointing to the future side of $\Sigma_{\tau'}$. Then the velocity vector V along γ can be written as $V = aL + b\,\underline{L} + V_s$, where $V_s \in \operatorname{span}\{E_1, E_2\}$ is spacelike. Since p is the first intersecting point, we must have $b \leq 0$ as otherwise, V points to the future side of C, and hence there exist points on γ in the other side (where initially γ does not reside) prior to p, implying that there must be an earlier intersection of γ and C, a contradiction. Note that

$$g(V,V) = -4ab + g(V_s, V_s) \geq -4ab.$$

Since V is timelike, we must have $ab > 0$. Therefore $a, b < 0$, which contradicts the assumption that V is pointing to the future of $\Sigma_{\tau'}$. □

Similarly, one can show that the past incoming null cone, if regular, does not intersect the chronological past of Σ^e_τ.

Definition 2.23. *For any τ_1, τ_2 with $\tau_1 < \tau_2$, denote the portion of $C = C^+(S)$ between Σ_{τ_1} and Σ_{τ_2} by $C(\tau_1, \tau_2)$. Assume that $C(\tau_1, \tau_2)$ is regular, then from above, the set $\Sigma^i_{\tau_1} \cup C(\tau_1, \tau_2) \cup \Sigma^e_{\tau_2}$ is an achronal set, i.e., no timelike curve can intersect with it more than once. We denote it by $\widehat{C}(\tau_1, \tau_2)$.*

---

[20] Note that when L is fixed, any two null frames with horizontal structure perpendicular to L can be related by a null frame transformation $(f = 0, \underline{f}, \lambda = 1)$ (with L viewed as e_4). In view of the transformation formulas, see Section 2.2, we easily see that $\operatorname{tr}\chi$ is invariant.

[21] By convention, a single point is not a timelike curve; it is however often counted as a causal curve.

Recall that in a globally hyperbolic spacetime, the closure of $I^+(S)$ is $\mathcal{J}^+(S)$. Therefore we have

Corollary 2.24. *Under the assumption of Definition 2.23, we have* $\mathcal{J}^+(\Sigma^i_{\tau_1}) \cap \Sigma^e_{\tau_2} = S_{\tau_2}$, $\mathcal{J}^-(\Sigma^e_{\tau_2}) \cap \Sigma^i_{\tau_1} = S_{\tau_1}$.

3 Strategy of the proof of Theorem 1.1

The goal of the section is to describe the main ideas of the proof of Theorem 1.1, restated below:

Theorem 3.1. *For the spacetime $\mathcal{M}$, described in Section 2, satisfying the smallness condition* (2.13), *the event horizon $\mathcal{H}^+$, defined as the boundary of the region $\mathcal{J}^-(\mathcal{I}^+)$, is a regular null hypersurface in the future of the spacelike hypersurface Σ_0.*[22]

Remark 3.2. *In the proof, we only need to assume the smallness estimate* (2.13) *in* $\mathcal{R}_\delta = \{|r - r_+| \le \delta\}$ *and the characterization* (3.2) *of the past of null infinity.*

3.1 Regular null cones over spheres

We start with the following definition.

Definition 3.3. *An immersion $i\colon X \to Y$ is a smooth map whose tangent map is injective at all points of X. An embedding is an injective immersion where the map is a homeomorphism onto its image, with the image $i(X)$ viewed in the subspace topology of Y.*

In the case when X is compact, an injective immersion map is an embedding.

We now assume that S is an embedded sphere on Σ_0 in a spacelike foliation $\{\Sigma_\tau\}$ of $\mathcal{M}$. Consider the outgoing (or incoming) cone of S, denoted by $C(S)$. Denote the null generator emanating from $p \in S$ by γ_p, and its unique intersection with Σ_τ by $\gamma_{p,\tau}$. Define the following map:

$$i_\tau\colon S \to \Sigma_\tau, \quad p \mapsto \gamma_{p,\tau}. \tag{3.1}$$

The regularity of the null cone $C(S)$, see Definition 2.18, is crucially dependent on whether i_τ is an embedding into Σ_τ. Since S is diffeomorphic to $\mathbb{S}^2$, we can, by abuse of language, write this map as $i_\tau\colon \mathbb{S}^2 \to \Sigma_\tau$.

Remark 3.4. *A null cone $C(S)$ generated by a smooth embedded sphere S, in a smooth spacetime $\mathcal{M}$, is regular, if and only if:*

- *Nearby null geodesic generators do not intersect. This is equivalent to saying that i_τ is an immersion for each τ.*

---

[22] In the case of the stability result of [16] this would be the initial Cauchy hypersurface, which lies in the initial data layer.

- *No null geodesic generators of $C(S)$ intersect. This is equivalent to saying that i_τ is in fact an embedding for each τ.*

3.1.1 Immersion Criterion

Given a smooth choice of the vector field L on S, one can extend it as a geodesic vector along the null generators of $C(S)$. Then the null expansion $\operatorname{tr}\chi$ is well-defined, see Definition 2.20.

Proposition 3.5 (Immersion Criterion)**.** *The map i_τ is an immersion for all $\tau \in [0,T]$ if and only if the null expansion $\operatorname{tr}\chi$ is bounded away from $-\infty$ for $\tau \in [0,T]$. This also means that the map $[0,T]\times S \to \mathcal{M}$, $(\tau, p) \mapsto \gamma_{p,\tau}$ is an immersion.*

Proof. See Appendix B. □

3.1.2 Embedding Criterion

Lemma 3.6. *Suppose $i_{\tau_0} : \mathbb{S}^2 \to \Sigma_{\tau_0}$ is an embedding for some τ_0. Then there exists a sufficiently small ϵ such that the maps $i_\tau : \mathbb{S}^2 \to \Sigma_\tau$ remain embeddings for all τ, $|\tau - \tau_0| \leq \epsilon$.*

Proof. Denote by S the image of i_{τ_0}, i.e. $i_{\tau_0}(\mathbb{S}^2) = S$. If the lemma does not hold true, then there exists a sequence $\tau_n \searrow \tau_0$, and there exist p_1^n, $p_2^n \in \mathbb{S}^2$, distinct for each n, satisfying $i_{\tau_n}(p_1^n) = i_{\tau_n}(p_2^n)$. By compactness, we can assume[23] that $p_1^n \to p_1$, $p_2^n \to p_2$. Since S is compact, the pointwise convergence $i_{\tau_n}(p) \to i_{\tau_0}(p)$[24] as $n \to \infty$ is uniform over $p \in \mathbb{S}^2$, and therefore, $i_{\tau_n}(p_1^n) \to i_{\tau_0}(p_1)$, $i_{\tau_n}(p_2^n) \to i_{\tau_0}(p_2)$, so $i_{\tau_0}(p_1) = i_{\tau_0}(p_2)$, i.e. $p_1 = p_2$. Since S is embedded, the null expansion is clearly bounded near S, so by Proposition 3.5, we know that for each p, there exist a $\epsilon_p > 0$ and a neighborhood $\mathcal{O}_p$ on $\mathbb{S}^2$ such that $(-\epsilon_p, \epsilon_p) \times \mathcal{O}_p \to \mathcal{M}$, $(\tau, p') \mapsto i_\tau(p')$ is injective. Taking n large such that $i_{\tau_n}(p_1^n)$, $i_{\tau_n}(p_2^n)$ lie in this injective neighborhood for $p_1 = p_2$ gives a contradiction. □

Clearly for τ far away from τ_0, i_τ may fail to be an embedding, even if it remains an immersion. See Figure 2 for an illustration of the situation.

The following gives a useful global criterion. In the following, the spacelike foliation $\{\Sigma_\tau\}$ is the one defined in Section 2.7, and recall r is a coordinate function on Σ_τ.

Proposition 3.7 (Embedding Criterion)**.** *Fix any $\tau \geq 0$. Let $i : \mathbb{S}^2 \to \Sigma_\tau$ be a map that verifies the following properties:*

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[23] Or just extract a subsequence.

[24] Evaluated e.g. with respect to the 4-dimensional Riemannian metric associated to g under the spacelike foliation Σ_τ through the $1+3$-decomposition.

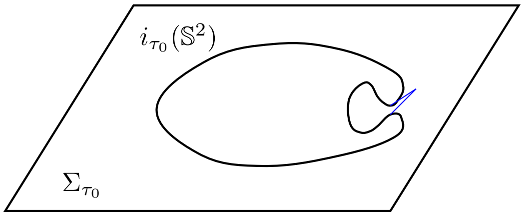

Figure 2: A picture in $2+1$ dimension illustrating the case where there will be $\tau > \tau_0$ such that i_τ is an immersion but not an embedding. The short blue lines represent intersecting null generators.

1. *For each point $p \in \mathbb{S}^2$, there is a neighborhood $\mathcal{O}_p$ such that*[25] $i|_{\mathcal{O}_p}$ *is an embedding to* Σ_τ.
2. *There exists a small number $\mathring{\delta} > 0$ such that every $\mathcal{O}_p$ admits an orthonormal frame $\{{}^{(0)}e_a\}$, tangent to the embedded submanifold $i(\mathcal{O}_p)$, for which $|{}^{(0)}e_a(r)| \leq \mathring{\delta}$.*

Then the i is an embedding. Moreover, there exists a smooth map

$$R\colon \mathbb{S}^2 \to [r_+ - \delta, r_+ + \delta], \quad p \mapsto R(p),$$

such that $\Phi_\Sigma(\{(R(p), p)\colon p \in \mathbb{S}^2\}) = i(\mathbb{S}^2)$ (with $\Phi_\Sigma = \Phi_{\Sigma_\tau}$ defined in (2.23)).

Remark 3.8. *The map R can be thought of as a graph function over $\mathbb{S}^2$. We will then establish uniform bounds of the graph function for all intersecting spheres on Σ_τ and apply the Arzela–Ascoli lemma to obtain the limit.*

Proof. See Appendix A. ☐

Corollary 3.9. *For any $T > 0$, if i_τ are embeddings for all $\tau \in [0, T]$, then the null cone $C(S)$ is regular between Σ_0 and Σ_T.*

Proof. Note that by causal structure, each null generator intersects a spacelike hypersurface only once. Therefore the injectivity w.r.t. the time variable is verified and the regularity of the cone follows. ☐

3.2 Main steps in the proof of the regularity theorem

Recall that we assume ${}^{(ext)}\mathcal{M}$ lies in the past of null infinity, denoted by $\mathcal{J}^-(\mathcal{I}^+)$, and we say $p \in \mathcal{J}^-(\mathcal{I}^+)$ if $p \in \mathcal{J}^-({}^{(ext)}\mathcal{M})$.[26]

Definition 3.10. *The event horizon of $\mathcal{M}$, denoted by $\mathcal{H}^+$, is defined as the boundary of $\mathcal{J}^-(\mathcal{I}^+)$.*

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[25] That is i is a local embedding.

[26] We have not defined what $\mathcal{I}^+$ means, so we also need to implicitly assume transitivity of $\mathcal{J}^-$ holds, but this is clearly natural.

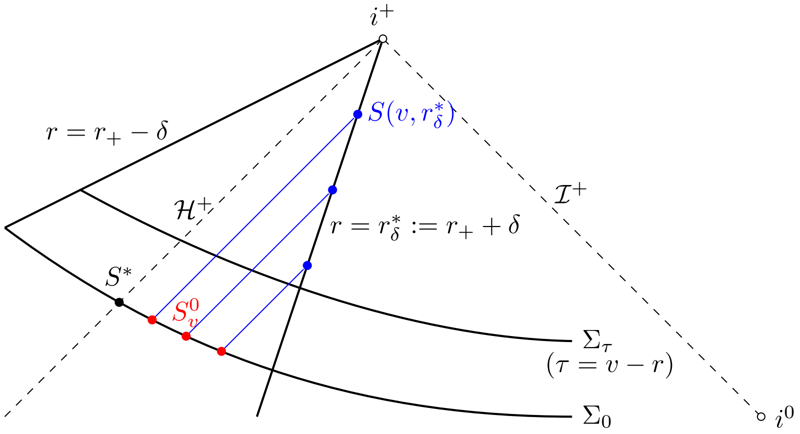

Figure 3: Idea of the proof of Theorem 1.1

We now give a characterization of $\mathcal{J}^{-1}(\mathcal{I}^+)$.

Proposition 3.11 (Characterization of $\mathcal{J}^-(\mathcal{I}^+)$)**.** *We have*

$$\mathcal{J}^-(\mathcal{I}^+) = \{p : \mathcal{J}^+(p) \cap \{r \geq r_+ + \delta\} \neq \varnothing\}. \tag{3.2}$$

In particular, any point in the set $\{r \geq r_+ + \delta\}$ *belongs to* $\mathcal{J}^-(\mathcal{I}^+)$.

Proof. Clearly, the left is included in the right. For the converse, if p is in the set on the right hand side, then there is a point $p' \in \mathcal{J}^+(p)$ with r-value of p' no less than $r_+ + \delta$. Consider the integral curve of e_4 starting from p'. This is a causal curve, and we know that, for all $r \geq r_+ + \delta$,

$$e_4(r) = \frac{\Delta}{|q|^2} + \widetilde{e_4(r)} > c\delta,$$

for some constant $c > 0$. This implies that along this curve one can reach arbitrarily large values of r. Therefore, $\sup_{\mathcal{J}^+(p')} r \geq r_0$, which meets our definition for $p' \in \mathcal{J}^-(\mathcal{I}^+)$. □

To prove Theorem 1.1, we consider a spacelike hypersurface Σ_0 given as the level set $\{\tau = 0\}$, where τ is the time function defined in Section 2.7, the outgoing (past incoming) null cones C_v generated backward from a sphere $S(v, r^*_\delta) \subset \{r = r^*_\delta := r_+ + \delta\}$, and the intersections $S^0_v = \Sigma_0 \cap C_v$. To prove the theorem we need to implement the following steps (see Figure 3 for a graphic representation of the proof):

Step 1. *Monotonicity.* Assuming that the hypersurfaces C_v are regular, we show that for each $v_1 < v_2$, the sphere $S^0_{v_1} = C_{v_1} \cap \Sigma_0$ is included in the exterior of the sphere $S^0_{v_2} = C_{v_2} \cap \Sigma_0$ on Σ_0.

Step 2. *Regularity of C_v.* This, the heart of the proof, is given in Section 4.1. The argument depends in an essential way on the bounds (2.13), Lemma 2.9 and the transformation formulas of Section 2.2. The proof shows moreover that for each τ, S^τ_v (the Σ_τ-section of C_v) is an embedded sphere that can be written as $r = R_v(\theta, \varphi)$ (switched to $r = R_v(x^1, x^2)$ when needed) on Σ_τ. This fact relies heavily on the Embedding Criterion (Proposition 3.7). Note that the main assumption 2. of the Embedding Criterion is verified in view of Lemma 4.5.

Step 3. *Regularity of the spheres* $S_v^0 = C_v \cap \Sigma_0$. Using the estimates in Step 2 we derive, in Section 4.3.3, geometric estimates for the spheres $S_v^0 = C_v \cap \Sigma_0$, in particular uniform bounds on the derivatives of the extrinsic curvature of S_v^0 on Σ_0.

Step 4. *Limiting sphere* S^*. Using Step 1 and the trivial lower bound for the functions R_v, $|R_v| \geq r_+ - \delta$, it is easy to show that the induced surfaces S_v^0 on Σ_0 have a limit S^* as $v \to \infty$. In Section 4.5 we rely on the bounds obtained in Step 3 to show that S^* is a smooth sphere.

Step 5. *Future Horizon.* In Section 4.6 we show that the future outgoing null cone generated from S^* coincides with the future horizon $\mathcal{H}^+ \cap \mathcal{J}^+(\Sigma_0)$.

3.3 Monotonicity property

We first prove a monotonicity property under the assumption that each C_v is regular. For $v_2 > v_1$, each point on $S(v_1, r_\delta^*)$ can be connected by a point on $S(v_2, r_\delta^*)$ through the integral curve of ∂_v, which is always timelike on $\{r = r_\delta^*\}$ (recall that $r_\delta^* = r_+ + \delta$), i.e., $S(v_1, r_\delta^*) \subset I^-(S(v_2, r_\delta^*))$. Therefore, the past of $S(v_1, r_\delta^*)$ is contained in the chronological past of the achronal set $\widehat{C}_{v_2}(0, v_2 - r_\delta^*)$ (see Definition 2.17 and 2.23). Therefore, by Proposition 2.15(ii), we have $\mathcal{J}^-(S(v_1, r_\delta^*)) \cap \widehat{C}_{v_2}(0, v_2 - r_\delta^*) = \varnothing$. In particular, $S_{v_1}^0 \subset \mathcal{J}^-(S(v_1, r_\delta^*))$ does not contain any point in the interior of $S_{v_2}^0$ on Σ_0, hence must be in the exterior of $S_{v_2}^0$ on Σ_0.

4 Proof of Theorem 1.1

In this section, we follow Steps 2-5 and prove Theorem 1.1.

4.1 Geometry of the null cones C_v

We now study the geometry of C_v. We only care about the part of C_v that is in the past of $S(v, r_\delta^*)$, so in what follows C_v only refers to this part.

4.1.1 Null generators

Recall that C_v is spanned by the null geodesics emanating from points on $S(v, r_\delta^*)$ orthogonal to the sphere at these points, called the null generators of C_v. To determine the tangent vector field e_4' corresponding to these null generators[27] we consider a frame transformation of the form

$$e_4' = \lambda \left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3 \right), \quad e_a' = e_a + \frac{1}{2} f_a e_3, \quad e_3' = \lambda^{-1} e_3, \tag{4.1}$$

---

[27] For simplicity, we do not label the vector with v, but clearly this is dependent on v.

and impose the condition (recall that ${}^{(S)}e_4$ is orthogonal to the spheres $S(v,r)$)

$$\omega' = 0, \quad \xi' = 0, \quad e'_4|_{S(v,r^*_\delta)} = {}^{(S)}e_4|_{S(v,r^*_\delta)}, \tag{4.2}$$

which ensures that e'_4 is geodesic. The third condition in (4.2) can be rewritten as

$$f|_{S(v,r^*_\delta)} = F|_{S(v,r^*_\delta)}, \quad \lambda|_{S(v,r^*_\delta)} = 1,$$

where F is defined in (2.21). Recall that $|F| \lesssim \min\{\delta, a\}$.

Remark 4.1. *Note that the parameters (f,λ) also depend on v, so they should in fact be denoted by (f_v,λ_v). We omit however the dependence on v as long as there is no danger of confusion.*

In view of the transformation formulas (2.4), we have

$$0 = \lambda^{-2}\xi' = \xi + \frac{1}{2}\lambda^{-1}\nabla'_4 f + \frac{1}{4}\mathrm{tr}\,\chi f + \omega f + \frac{1}{2} f\cdot\widehat{\chi} - \frac{1}{4}\,{}^{(a)}\mathrm{tr}\,\chi\,{}^*f + O(|f|^2),$$

$$0 = \lambda^{-1}\omega' = \omega - \frac{1}{2}\lambda^{-1}e'_4(\log\lambda) + \frac{1}{2}f\cdot(\zeta - \underline{\eta}) + O(|f|^2). \tag{4.3}$$

The first equation can be written as

$$\nabla'_{e_4+f^b e_b+\frac{1}{4}|f|^2 e_3} f + \frac{1}{2}\mathrm{tr}\,\chi f + 2\omega f = -2\xi - \widehat{\chi}\cdot f + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\chi\,{}^*f + O(|f|^2), \quad f|_{S(v,r^*_\delta)} = F, \tag{4.4}$$

and it is manifestly independent of λ as the horizontal structure $\{e'_a\}$ is also independent of λ.

Let s denote a time parameter along the null geodesics spanning C_v such that

$$\lambda^{-1}e'_4(s) = \Big(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\Big)(s) = 1, \qquad s|_{S(v,r^*_\delta)} = v. \tag{4.5}$$

Proposition 4.2. *For each null generator γ of C_v, assume that γ belongs to $\mathcal{R}_\delta = \{r \colon |r - r_+| \le \delta\}$, (see (2.22)) for $s_* \le s \le v$. Then, on this portion of γ,*

$$|f| \lesssim \delta, \qquad e^{C_1\delta_*^{\frac{1}{2}}(v-s)} \le \lambda \le e^{C_2(v-s)}. \tag{4.6}$$

Proof. On $\mathcal{R}_\delta$ we have $|\mathrm{tr}\,\chi| \le C\delta$ and, in view of Lemma 2.9, $\omega \le -C\delta_*^{\frac{1}{2}}$. Therefore, for $\delta \le \delta_*$ sufficiently small,

$$\phi := 4\omega + \mathrm{tr}\,\chi < -C\delta_*^{\frac{1}{2}}.$$

In view of (4.4), along each null geodesic, we have

$$\frac{d}{ds}\left(|f|^2\right) + \mathrm{tr}\,\chi|f|^2 + 4\omega|f|^2 = -4\xi\cdot f + O(|\widehat{\chi}, {}^{(a)}\mathrm{tr}\,\chi||f|^2) + O(|f|^3).$$

Therefore,

$$\frac{d}{ds}\left(e^{\int_{s_*}^{s}\phi ds'}|f|^2\right) = -4e^{\int_{s_*}^{s}\phi ds'}\xi\cdot f + e^{\int_{s_*}^{s}\phi ds'}\left(O(|\widehat{\chi},\ {}^{(a)}\mathrm{tr}\,\chi||f|^2) + O(|f|^3)\right).$$

We now make the bootstrap assumption

$$|f| \le C_b\delta, \quad \text{for } s \in [s_1, v], \tag{4.7}$$

which holds when $s = v$ with C_b replaced by some C_0, using the bound of F. Now, for any $s_0 \in [s_1, v]$, we integrate the equation over $s \in [s_0, v]$ to get

$$e^{\int_{s_*}^{v}\phi\, ds'}|f|^2\Big|_{s=v} - e^{\int_{s_*}^{s_0}\phi\, ds'}|f|^2\Big|_{s=s_0} = -\int_{s_0}^{v} 4e^{\int_{s_*}^{s'}\phi ds''}\xi\cdot f ds' + \int_{s_0}^{v} e^{\int_{s_*}^{s'}\phi ds''}\left(O(|\widehat{\chi},\ {}^{(a)}\mathrm{tr}\,\chi||f|^2) + O(|f|^3)\right)ds'.$$

We now make use of the bounds $\phi \le -C\delta_*^{1/2}$, $|\xi|, |\widehat{\chi}|, |{}^{(a)}\mathrm{tr}\,\chi| \le C\epsilon$ in $\mathcal{R}_\delta$, and the bound for $f = F$ on $S(v, r_\delta^*)$ provided by Proposition 2.12 to deduce, in $\mathcal{R}_\delta$,

$$\begin{aligned}
|f|^2\Big|_{s=s_0} &= e^{\int_{s_0}^{v}\phi ds}|f|^2\Big|_{s=v} + \int_{s_0}^{v} 4e^{\int_{s_0}^{s'}\phi ds''}\xi\cdot f ds + \int_{s_0}^{v} e^{\int_{s_0}^{s'}\phi ds''}\left(O(|\widehat{\chi},\ {}^{(a)}\mathrm{tr}\,\chi||f|^2) + O(|f|^3)\right)ds' \\
&\le C_0\delta^2 + \int_{s_0}^{v} e^{-C\delta_*^{\frac{1}{2}}(s'-s_0)}\left(C\epsilon|f| + C\delta|f|^2 + |f|^3\right)ds'.
\end{aligned}$$

Using the bootstrap bound (4.7), we deduce

$$\begin{aligned}
|f|^2\Big|_{s=s_0} &\le C_0\delta^2 + C\int_{s_0}^{v} e^{-C\delta_*^{\frac{1}{2}}(s'-s_0)}\epsilon|f|\, ds' + C_b\int_{s_0}^{v} e^{-C\delta_*^{\frac{1}{2}}(s'-s_0)}\delta|f|^2\, ds' \\
&\le C_0\delta^2 + C\int_{s_0}^{v} e^{-C\delta_*^{\frac{1}{2}}(s'-s_0)}\left(\frac{M\epsilon^2}{\delta_*^{\frac{1}{2}}} + \frac{\delta_*^{\frac{1}{2}}}{M}|f|^2\right)ds' + C_b\int_{s_0}^{v} e^{-C\delta_*^{\frac{1}{2}}(s'-s_0)}\delta|f|^2\, ds' \\
&\le C_0\delta^2 + CM\epsilon^2\delta_*^{-1} + C(\frac{1}{M} + C_b\delta\delta_*^{-\frac{1}{2}})\sup_{s\in[s_1,v]}|f|^2,
\end{aligned}$$

where M is a constant to be determined. Therefore, for appropriately large $M > 0$, using the relations $\epsilon \le \delta\delta_*^{\frac{1}{2}}$, $\delta \le \min\{\delta_*, \delta_{\mathcal{H}}/2\} \ll 1$, and taking the supremum over $s_0 \in [s_1, v]$, we obtain the bound

$$\sup_{s\in[s_1,v]}|f|^2 \lesssim (1 + M + C_b\delta^{-\frac{1}{2}})\delta^2,$$

which improves the bootstrap assumption (4.7) once we pick up appropriate $C_b > 0$. Therefore $|f| \lesssim \delta$ for all $s \in [s_*, v]$.

Next we estimate λ. By (4.3) we have

$$\frac{d}{ds}(\log\lambda) = 2\omega + f\cdot(\zeta - \underline{\eta}) + O(|f|^2).$$

Using the above estimate for f, the estimates for $\underline{\eta}, \zeta$, and the upper bound for ω, we have for the right-hand side

$$-C_2 \le 2\omega + f\cdot(\zeta - \underline{\eta}) + O(|f|^2) \le -C_1\delta_*^{\frac{1}{2}}, \quad \text{when } |r - r_+| \le \delta.$$

Therefore, along each null generator, when $s_* \leq s \leq v$, we have

$$e^{C_1 \delta_*^{\frac{1}{2}}(v-s)} \leq \lambda \leq e^{C_2(v-s)}, \tag{4.8}$$

as stated. $\square$

4.1.2 Regularity of the null cones C_v

We now start to prove that C_v are regular null cones in $\mathcal{M}$. According to the transformation formula (2.4) for $\operatorname{tr}\chi$, we have

$$\lambda^{-1}\operatorname{tr}\chi' = \operatorname{tr}\chi + \operatorname{div}'f + f \cdot (\eta + \zeta) + O(|f|^2). \tag{4.9}$$

We already have estimates for λ and f, provided by Proposition 4.2, but we still need to estimate $\operatorname{div}'f$. To do this we need to commute the equation (4.4) for f with ∇', using the following standard commutation lemma, see Section 2.2.7 in [12].

Lemma 4.3 (Commutation formula)**.** *Given a null frame $\{e_3, e_4, e_a\}$ and a horizontal covariant tensor $\psi_A = \psi_{a_1\cdots a_k}$, we have*

$$\begin{aligned} [\nabla_4, \nabla_b]\psi_A = -\chi_{bc}\nabla_c\psi_A + (\underline{\eta}_b + \zeta_b)\nabla_4\psi_A + \sum_{i=1}^{k}\left(\chi_{ba_i}\underline{\eta}_c - \chi_{bc}\underline{\eta}_{a_i}\right)\psi_{a_1\cdots c\cdots a_k} \\ + \sum_{i=1}^{k}\left(\underline{\chi}_{ba_i}\xi_c - \underline{\chi}_{bc}\xi_{a_i} + \in_{a_i c}{}^*\beta_b\right)\psi_{a_1\cdots c\cdots a_k} + \xi_b\nabla_3\psi_A. \end{aligned} \tag{4.10}$$

We will apply this to the frame $\{e_3', e_4', e_a'\}$.

We now state the main result of this part.

Proposition 4.4. *For every v, the null geodesic congruence C_v is a regular null cone in $\mathcal{M} \cap \{0 \leq \tau \leq v - r_\delta^*\}$. Moreover, we have the estimate*

$$|\nabla'^{\leq i} f| \leq C_N\delta, \quad 0 \leq i \leq N. \tag{4.11}$$

Proof. The strategy of the proof is as follows:

Step 1. Fix a value of v. We assume that there exists a minimal value $\tau_* \in [0, v - r_\delta^*]$ such that

$$|\lambda^{-1}\operatorname{tr}\chi'| \leq \delta^{\frac{1}{2}}, \qquad \tau_* \leq \tau \leq v - r_\delta^*. \tag{4.12}$$

By the Immersion Criterion (Proposition 3.5), this shows that the intersection $S_v^\tau := C_v \cap \Sigma_\tau$, for $\tau_* \leq \tau \leq v - r_\delta^*$, is the image of an immersion $i_v^\tau \colon \mathbb{S}^2 \to \Sigma_\tau$. Thus every point $p \in \mathbb{S}^2$ has a neighborhood $\mathcal{O}_p \subset \mathbb{S}^2$ such that $i_v^\tau|_{\mathcal{O}_p}$ is an embedding into Σ_τ. It then makes sense to talk about a local orthonormal frame w.r.t. the induced metric on $i_v^\tau(\mathcal{O}_p) \subset \Sigma_\tau$. We have the following lemma.

Lemma 4.5. *Under the assumption* (4.12)*, consider a neighborhood* $\mathcal{O}_p$ *for which* $i^\tau_v|_{\mathcal{O}_p}$ *is an embedding. If*[28] $|f| \le C\delta$ *on* $i^\tau_v(\mathcal{O}_p)$*, then there exists an orthonormal basis* $\{{}^{(0)}e_a\}$ *on* $i^\tau_v(\mathcal{O}_p)$ *w.r.t the induced metric on* Σ_τ *satisfying*

$$|{}^{(0)}e_a(r)| \le C'\delta, \tag{4.13}$$

with C' *dependent on* C *but independent of* v*,* τ*,* τ^**,* p*.*

Proof. See Section 4.2. □

The bound (4.13) provided by the lemma verifies the condition in the Embedding Criterion (Proposition 3.7). Therefore we have the following:

Proposition 4.6. *Under the assumption* (4.12)*,* C_v *is a regular null cone in* $\mathcal{M} \cap \{\tau_* \le \tau \le v - r^*_\delta\}$*.*

Proof. Recall that $\tau = v - r$ and that the cones C_v initiate on $r = r^*_\delta = r_+ + \delta$. According to Proposition 3.5 and the assumption (4.12) on $\operatorname{tr}\chi'$, it suffices to show that the immersion $i^\tau_v : \mathbb{S}^2 \to \Sigma_\tau$ is an embedding for any v and $\tau \in [\tau_*, v - r^*_\delta]$. By compactness of $\mathbb{S}^2$, for it not to be an embedding, i^τ_v must fail to be injective. Denote $\tau'_* \ge \tau_*$ as the first (backward) value of τ such that i^τ_v fails to be injective. This means that the part of C_v strictly above (in the future of) $\Sigma_{\tau'_*}$ is regular, so all null generators are in $\mathcal{R}_\delta$.[29] By continuity, the intersection of these null generators with $\Sigma_{\tau'_*}$ remain in $\mathcal{R}_\delta$, so the estimate of $|f|$ in Proposition 4.2 applies for each of them, and hence we have $|f| \lesssim \delta$. Therefore, for δ small enough, applying Lemma 4.5 and the Embedding criterion (Proposition 3.7), we deduce that the intersection $C_v \cap \Sigma_{\tau'_*}$ must also be an embedded sphere on $\Sigma_{\tau'_*}$. This leads to a contradiction as we can now extend the cone from this embedded sphere on $\Sigma_{\tau'_*}$ regularly, for a small neighborhood of τ'_* in τ, by Lemma 3.6. □

Therefore, we conclude that for $\tau_* \le \tau \le v - r^*_\delta$, we have C_v is regular and $|f| \lesssim \delta$.

Step 2. We prove bounds for derivatives of f. We derive a higher order version which we will use later. Recall that from Step 1, we know that C_v is regular for $\tau \in [\tau_*, v - r^*_\delta]$.

Lemma 4.7. *Suppose the bootstrap assumption* (4.12) *holds. Then, on* C_v*, whenever* $\tau \in [\tau_*, v - r^*_\delta]$*, we have* $|\nabla'^{\le i} f| \le C_N \delta$ *for all* $i \le N$*.*

Proof. See Section 4.3. □

Step 3. We improve the bootstrap assumption (4.12) by showing that in fact $|\lambda^{-1}\operatorname{tr}\chi'| \le C\delta$ when $\tau \in [\tau_*, v - r^*_\delta]$, with the constant C independent of τ_* and v. This implies that C_v remains regular in $\mathcal{M}$ for all $\tau \in [0, v - r^*_\delta]$.

---

[28] Recall that we have defined f everywhere along each null generators, see (4.1) and (4.2).

[29] Indeed, $\{r = r_+ + \delta\}$ is a timelike hypersurface, spanned by timelike curves (the integral curves of ∂_v) from $S(v, r^*_\delta)$. The regular part of C_v lies on the boundary of $\mathcal{J}^-(S(v, r^*_\delta))$ and cannot intersect either $\{r = r_+ + \delta\}$ or $\{r = r_+ - \delta\}$.

Proof of Step 3. In view of the transformation formula (4.9), the estimate of $|f|$ in (4.6), and the estimates of η, ζ in (2.13), it suffices to show the bound for $|\mathrm{div}' f|$. This directly follows from the bound of $|\nabla' f|$ in Lemma 4.7. $\square$

This, modulo the proofs of the Lemmas 4.5 and 4.7, ends the proof of Proposition 4.4. $\square$

It remains to prove Lemma 4.5 and Lemma 4.7.

Remark 4.8. *From now on, since it is not relevant anymore, we consider transformations of type* (4.1) *with* $\lambda = 1$*, i.e.,*

$$e_4' = e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3, \quad e_a' = e_a + \frac{1}{2} f_a e_3, \quad e_3' = e_3.$$

In particular, this does not change the result of Lemma 4.7 since the horizontal structure H' *(and hence* ∇'*) remains unchanged.*

4.2 Proof of Lemma 4.5

In this subsection, we prove Lemma 4.5. We prove in fact the following more detailed version, in Lemma 4.9. Note that we only make use of things that have been proved before the statement of Proposition 4.4 and the bootstrap assumption (4.12).

Recall that for each fixed v and $\tau \in [\tau^*, v - r_\delta^*]$, S_v^τ is the image of an immersion $i_v^\tau \colon \mathbb{S}^2 \to \Sigma_\tau$, and for each $p \in \mathbb{S}^2$, there is an open neighborhood $\mathcal{O}_p$ such that $i_v^\tau|_{\mathcal{O}_p}$ is an embedding. In the following, we drop the dependence on τ and v and denote $S = S_v^\tau$, $i = i_v^\tau$. Moreover, since λ is not relevant, we adopt the consideration in Remark 4.8.

Lemma 4.9. *With the notation above, for each* $p \in \mathbb{S}^2$*, there exists a frame transformation* $({}^{(0)}f, {}^{(0)}\underline{f}, {}^{(0)}\lambda)$ *from* $\{e_3, e_4, e_a\}$ *on the embedded submanifold* $i(\mathcal{O}_p)$ *such that*

1. *The new frame* $\{{}^{(0)}e_3, {}^{(0)}e_4, {}^{(0)}e_a\}$ *is adapted to* $i(\mathcal{O}_p)$*, in the sense that* $\{{}^{(0)}e_a\}$ *is tangent to* $i(\mathcal{O}_p)$.
2. ${}^{(0)}N = \frac{1}{2}\left({}^{(0)}e_4 - {}^{(0)}e_3\right)$ *is the unit normal vector of* $i(\mathcal{O}_p)$ *on* Σ_τ.
3. *The transformation coefficients can be written as*

$$ {}^{(0)}f_a = f_a, \qquad {}^{(0)}\underline{f}_a = -\frac{f_a + 2e_a(v)}{e_4(\tau) + f^b e_b(v) + \frac{1}{4}|f|^2}, \tag{4.14}$$

$$ {}^{(0)}\lambda^2 = \frac{1 + \frac{1}{2} f \cdot {}^{(0)}\underline{f} + \frac{1}{16}|f|^2 |{}^{(0)}\underline{f}|^2 + ({}^{(0)}\underline{f}^b + \frac{1}{4}|{}^{(0)}\underline{f}|^2 f^b) e_b(v) + \frac{1}{4}|{}^{(0)}\underline{f}|^2 e_4(\tau)}{e_4(\tau) + f^b e_b(v) + \frac{1}{4}|f|^2},$$

where f *is the one defined in Section 3. Note that one also has*

$$ {}^{(0)}e_4 = {}^{(0)}\lambda e_4', \quad {}^{(0)}e_a = e_a' + \frac{1}{2}\,{}^{(0)}\underline{f} e_4', \quad e_3' = {}^{(0)}\lambda^{-1}\left(e_3 + {}^{(0)}\underline{f}^a e_a' + \frac{1}{4}|{}^{(0)}\underline{f}|^2 e_4'\right),$$

that is, the new frame can be obtained from the e'-frame through the transformation $(0, {}^{(0)}\underline{f}, {}^{(0)}\lambda)$.

4. *We have the estimates*
$$|{}^{(0)}\underline{f}| \leq C \max\{\delta, a\}, \quad C^{-1} \leq {}^{(0)}\lambda \leq C,$$
where $C > 0$ is independent of v, τ, τ^*, p.

Remark 4.10. *As a result, we have*

$$\begin{aligned} {}^{(0)}e_a(r) &= e_a(r) + \frac{1}{2}\,{}^{(0)}\underline{f}_a\,{}^{(0)}f^b e_b(r) + \frac{1}{2}\,{}^{(0)}\underline{f}_a e_4(r) + \left(\frac{1}{2}\,{}^{(0)}f_a + \frac{1}{8}|{}^{(0)}f|^2\,{}^{(0)}\underline{f}_a\right) e_3(r) \\ &= \frac{1}{2}\,{}^{(0)}\underline{f}_a e_4(r) + O(|{}^{(0)}f|) = O(\delta). \end{aligned}$$

This proves Lemma 4.5.

Proof. We first take ${}^{(0)}f_a = f_a$, which ensures that ${}^{(0)}e_4$ is tangent to the null generators. To have $\{{}^{(0)}e_a\}$ tangent to $i(\mathcal{O}_p) \subset \Sigma_\tau$, we impose (using $e_3(\tau) = 1$, $e_a(r) = 0$)

$$\begin{aligned} 0 = {}^{(0)}e_a(\tau) &= e_a(\tau) + \frac{1}{2}\,{}^{(0)}\underline{f}_a f^b e_b(\tau) + \frac{1}{2}\,{}^{(0)}\underline{f}_a e_4(\tau) + \Big(\frac{1}{2}f_a + \frac{1}{8}|f|^2\,{}^{(0)}\underline{f}_a\Big)e_3(\tau) \\ &= e_a(v) + \frac{1}{2}\,{}^{(0)}\underline{f}_a f^b e_b(v) + \frac{1}{2}\,{}^{(0)}\underline{f}_a e_4(\tau) + \Big(\frac{1}{2}f_a + \frac{1}{8}|f|^2\,{}^{(0)}\underline{f}_a\Big). \end{aligned}$$

Hence,

$${}^{(0)}\underline{f}_a = -\frac{f_a + 2e_a(v)}{e_4(\tau) + f^b e_b(v) + \frac{1}{4}|f|^2}. \tag{4.15}$$

This determines ${}^{(0)}\underline{f}$. Using the bound for f provided by Proposition 4.2 and the bounds for $\nabla v = \widecheck{\nabla v} + aj$ in Section 2.4.1 we deduce that

$$|{}^{(0)}\underline{f}| \lesssim \max\{\delta, a\}.$$

To determine ${}^{(0)}\lambda$, we impose the condition ${}^{(0)}e_4(\tau) = {}^{(0)}e_3(\tau)$. This gives

$${}^{(0)}\lambda^2 = \frac{1 + \frac{1}{2}f \cdot {}^{(0)}\underline{f} + \frac{1}{16}|f|^2|{}^{(0)}\underline{f}|^2 + ({}^{(0)}\underline{f}^b + \frac{1}{4}|{}^{(0)}\underline{f}|^2 f^b)e_b(v) + \frac{1}{4}|{}^{(0)}\underline{f}|^2 e_4(\tau)}{e_4(\tau) + f^b e_b(v) + \frac{1}{4}|f|^2}.$$

The denominator is away from zero since $e_4(\tau)$ is away from zero and $f = O(\delta)$. To see the numerator is also away from zero, we notice that, similar to Section 2.7,

$$e_4(\tau) = \frac{2(r^2 + a^2)}{r^2 + a^2\cos^2\theta} + O(\delta) \geq 2 + O(\delta), \quad {}^{(0)}\underline{f}_a = -\frac{2e_a(v)}{e_4(\tau)} + O(\delta), \quad |\nabla v|^2 = \frac{a^2\sin^2\theta}{|q|^2} + O(\delta) \leq 1 + O(\delta),$$

so the numerator is of the size

$$1 + {}^{(0)}\underline{f}^b e_b(v) + \frac{1}{4}|{}^{(0)}\underline{f}|^2 e_4(\tau) + O(\delta) = 1 - \frac{|\nabla v|^2}{e_4(\tau)} + O(\delta) \geq \frac{1}{2} + O(\delta).$$

Therefore ${}^{(0)}\lambda$ is well-defined, regular, and bounded from above and below (by a constant dependent on δ and a).

Note that ${}^{(0)}e_a = e'_a + \frac{1}{2}\,{}^{(0)}\underline{f}_a e'_4$, so ${}^{(0)}e_a$ is indeed tangent to $i(\mathcal{O}_p)$. It is also straightforward to verify that ${}^{(0)}N = \frac{1}{2}\left({}^{(0)}e_4 - {}^{(0)}e_3\right)$ is the unit normal of $i(\mathcal{O}_p)$ on Σ_τ. $\square$

4.3 Higher order estimates and proof of Lemma 4.7

4.3.1 Higher order estimates on C_v

We want to estimate higher-order derivatives of f and recall from Remark 4.8 that we now take $\lambda = 1$. Note that f is only defined on each null generator, hence on the null cone C_v, so we should commute the equation of f by e'_a derivatives, which are tangent to the null cone.

We first derive the bounds of H'-horizontal derivatives on spacetime geometric quantities in the original frame.

Proposition 4.11. *Let Γ denote all Ricci coefficients defined in the original e-frame. We have, for each $\psi \in \Gamma$,*

$$|(\nabla', \nabla'_4)^i \psi| \lesssim \left(1 + |(\nabla', \nabla'_4)^{\leq i-1} f|\right) |(\nabla, \nabla_4, f\nabla_3)^{\leq i}\psi|. \tag{4.16}$$

Proof. Recall the formula (2.5)

$$\nabla'_{a_1} \cdots \nabla'_{a_i} \psi = \nabla_{e'_{a_1}} \cdots \nabla_{e'_{a_i}} \psi + f \cdot \underline{\chi} \cdot (\nabla_{e'_a})^{\leq i-1} \psi + \nabla'^{\leq i-1}(f \cdot \underline{\chi} \cdot \psi). \tag{4.17}$$

Note that schematically

$$\nabla_{e'_{a_1}} \cdots \nabla_{e'_{a_i}} \psi = \left(1, (\nabla_{e'_a})^{\leq i-1} f\right) \cdot (f\nabla_3, \nabla)^{\leq i} \psi.$$

We first apply (4.17) to $\underline{\chi} = (\widehat{\underline{\chi}}, \operatorname{tr}\underline{\chi}, {}^{(a)}\operatorname{tr}\underline{\chi})$ to get

$$\nabla'_{a_1} \cdots \nabla'_{a_i} \underline{\chi} = \nabla_{e'_{a_1}} \cdots \nabla_{e'_{a_i}} \underline{\chi} + f \cdot \underline{\chi} \cdot (\nabla_{e'_a})^{\leq i-1} \underline{\chi} + \nabla'^{\leq i-1}(f \cdot \underline{\chi} \cdot \underline{\chi}),$$

then by induction, we see that

$$|\nabla'_{a_1} \cdots \nabla'_{a_i} \underline{\chi}| \lesssim \left(1 + O(|\nabla'^{\leq i-1} f|)\right) |(\nabla, f\nabla_3, \nabla_4)^{\leq i} \underline{\chi}| \lesssim 1 + |\nabla'^{\leq i-1} f|.$$

Then applying (2.5) to all other quantitiesin Γ we get similar estimates. The case with ∇'_4 can be obtained similarly using (2.7). $\square$

We now commute the equation (4.4) with ∇' for i times to get

$$\nabla'_4 \nabla'^i f + \nabla'^i \Big(\big(2\omega + \frac{1}{2}\operatorname{tr}\chi\big) f \Big) = [\nabla'_4, \nabla'^i] f - 2\nabla'^i \xi + \nabla'^i \big(-\widehat{\chi} \cdot f + \frac{1}{2}\,{}^{(a)}\operatorname{tr}\chi\, {}^*f + O(|f|^2) \big),$$

where $O(|f|^2)$ is an expression quadratic in f.

Lemma 4.12. *We have*

$$[\nabla'_4, \nabla'^i]U = -i\chi \cdot \nabla'^i U + \nabla'^{\le i} f \cdot \nabla'^{\le i} U + \nabla'^{\le i-1}(\Gamma, f) \cdot (\nabla'^{\le i-1}\nabla'_4 U + \nabla'^{\le i-1} U).$$

Proof. Recall the commutation formula (4.10) which can be written schematically as

$$[\nabla'_4, \nabla']U = -\chi' \cdot \nabla' U + (\underline{\eta}' + \zeta') \cdot \nabla'_4 U + (\chi', \underline{\eta}', \beta') \cdot U.$$

We also have (recall we are now transforming with $\lambda = 1$), schematically,

$$\chi' = \chi + \nabla' f + f \cdot (\eta, \zeta, f \cdot \widehat{\chi}), \quad \underline{\eta}' = \underline{\eta} + \frac{1}{2}\underline{\chi} \cdot f, \quad \zeta' = \zeta + \frac{1}{2}\underline{\chi} \cdot f, \quad \beta' = f \cdot (\rho, {}^*\rho, \beta),$$

so we obtain

$$[\nabla'_4, \nabla']U = -\nabla' f \cdot \nabla' U - \chi \cdot \nabla' U + (\Gamma, f) \cdot \nabla'_4 U + (\Gamma, \nabla' f, f) \cdot U.$$

Then, applying this formula recursively, we derive the desired formula in the lemma. □

Applying Lemma 4.12 to $U = f$,[30] and replacing $\nabla'_4 f$ by the right hand side of (4.4), which is schematically $\xi + f \cdot \Gamma$, we obtain

$$\nabla'_4 \nabla'^i f + \left(2\omega + \frac{i+1}{2}\mathrm{tr}\,\chi\right)\nabla'^i f = \nabla'^{\le i}(\widehat{\chi}, {}^{(a)}\mathrm{tr}\,\chi)\nabla'^{\le i} f + \nabla'^{\le i}\xi + \nabla'^{\le i-1}\big((\nabla'\omega, \nabla'\mathrm{tr}\,\chi, \Gamma) \cdot f\big) + O(|\nabla'^{\le i} f|^2). \tag{4.18}$$

4.3.2 Proof of Lemma 4.7

The following proposition concludes the proof of Lemma 4.7 and provides estimates of various quantities in the new frame.

Proposition 4.13. *Suppose f satisfies the equation (4.4) and is equal to F on $S(v, r^*_\delta)$. Then:*

1. *On the initial sphere $S(v, r^*_\delta)$ we have, for all $0 \le i \le N$,*

$$|\nabla'^i f| \lesssim_N \delta.$$

2. *The following bound holds true along C_v, for all $0 \le i \le N$,*

$$\sup_{C_v} |\nabla'^i f| \lesssim_N \delta. \tag{4.19}$$

---

[30] Here f is understood as an H'-horizontal tensor as explained in Definition 2.1.

3. We have, along C_v,

$$\sup_{C_v} |(\nabla', \nabla_4')^i \Gamma_1'| \lesssim_N 1, \quad i \le N-1, \tag{4.20}$$

where $\Gamma_1' = \{\chi', \underline{\chi}', \underline{\eta}', \zeta', \omega'\}$ are defined w.r.t. the frame $\{e_3', e_4', e_a'\}$ (transformed with $\lambda = 1$).

Proof of 1. For the initial condition, note that $f|_{S(v,r_\delta^*)} = F|_{S(v,r_\delta^*)}$ means that the derivatives of f in ${}^{(S)}e_a$ directions are the same as those of F. We define the vector field

$$\begin{aligned}{}^{(S,f)}e_a &:= e_a + \frac{1}{2}\underline{F}_a f^b e_b + \frac{1}{2}\underline{F}_a e_4 + \Big(\frac{1}{2}f_a + \frac{1}{8}|f|^2\underline{F}_a\Big)e_3 = \Big(e_a + \frac{1}{2}f_a e_3\Big) + \frac{1}{2}\underline{F}_a\Big(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3\Big)\\ &= e_a' + \frac{1}{2}\underline{F}_a e_4'.\end{aligned}$$

This vector field is the same as ${}^{(S)}e_a$ everywhere on $S(v, r_\delta^*)$ but has the advantage of being always tangent to the cone C_v.

We have, by (2.6), $\nabla_a' f = \nabla_{e_a'} f + f \cdot \underline{\chi} = \nabla_{{}^{(S,f)}e_a} f - \frac{1}{2}\underline{F}_a \nabla_{e_4'} f + f \cdot \underline{\chi}$ (the last term is written schematically), and

$$\begin{aligned}\nabla_a'\nabla_b' f &= \nabla_{e_a'}\nabla_{e_b'} f + \nabla'^{\le 1}(f\cdot\underline{\chi}) = \nabla_{{}^{(S,f)}e_a}\nabla_{e_b'} f - \frac{1}{2}\underline{F}_a\nabla_{e_4'}\nabla_{e_b'} f + \nabla'^{\le 1}(f\cdot\underline{\chi})\\ &= \nabla_{{}^{(S,f)}e_a}\nabla_{{}^{(S,f)}e_b} f - \nabla_{{}^{(S,f)}e_a}\Big(\frac{1}{2}\underline{F}_b\nabla_{e_4'} f\Big) - \frac{1}{2}\underline{F}_a\nabla_{e_4'}\nabla_{e_b'} f + \nabla'^{\le 1}(f\cdot\underline{\chi})\\ &= \nabla_{{}^{(S,f)}e_a}\nabla_{{}^{(S,f)}e_b} f + \nabla_{{}^{(S,f)}e_a}\big(\underline{F}\cdot(\nabla_4' f, \Gamma\cdot f)\big) + \underline{F}\cdot\big(\nabla_4'\nabla' f + (\nabla_4', \nabla')^{\le 1}(f\cdot\Gamma)\big) + \nabla'^{\le 1}(f\cdot\underline{\chi})\\ &= \nabla_{{}^{(S,f)}e_a}\nabla_{{}^{(S,f)}e_b} f + \sum_{i+j\le 1}(\nabla, \nabla_4)^{\le i}\underline{F}\cdot\nabla'^{\le j}(f, \xi),\end{aligned}$$

where we use many times (2.7), the boundedness of Ricci coefficients (4.16), and the fact that in view of (4.4) and (4.18), the expression ${\nabla_4'}^i \nabla'^j f$ can be written schematically as $\nabla'^{\le j}(f, \xi)$.

Therefore, repeating this, we obtain the schematic relation

$$\nabla_{a_1}' \cdots \nabla_{a_i}' f = \nabla_{{}^{(S,f)}e_{a_1}} \cdots \nabla_{{}^{(S,f)}e_{a_i}} f + \sum_{j_1+j_2\le i-1} (\nabla, \nabla_4)^{\le j_1}\underline{F}\cdot\nabla'^{\le j_2}(f, \xi).$$

When we evaluate this on $S(v, r_\delta^*)$, we have ${}^{(S,f)}e_a = {}^{(S)}e_a$ tangent to the sphere, so the first term on the right hand side can be replaced by $\nabla_{{}^{(S)}e_{a_1}} \cdots \nabla_{{}^{(S)}e_{a_i}} F$. Therefore, using Corollary 2.13, we obtain, on $S(v, r_\delta^*)$,

$$|\nabla'^i f| \lesssim \delta + |\nabla'^{\le i-1} f|,$$

so by induction we see that $|\nabla'^i f| \lesssim_N \delta$ for all $i \le N$. □

Proof of 2. We derive the estimate of $\nabla'^i f$ along the null cone. Using (2.17), we have

$$\nabla_a'\omega = \nabla_a\omega + \frac{1}{2}f_a\nabla_3\omega = O(\delta_*^{\frac{1}{2}}),$$

and by induction as before,

$$|\nabla'^i\omega| \lesssim \big(1+|\nabla'^{\leq i-1}f|\big)\,\delta_*^{\frac{1}{2}}.$$

Also, since $(\nabla,\nabla_4)^{\leq i}\mathrm{tr}\,\chi = (\nabla,\nabla_4)^{\leq i}(-\frac{2r\Delta}{|q|^2}) + O(\epsilon) \lesssim \min\{|\Delta|,|e_4(r)|,\epsilon\} \lesssim \delta$ in $\mathcal{R}_\delta$, using (4.16) we have

$$\begin{aligned}|\nabla'^i\mathrm{tr}\,\chi| &\lesssim (1+|\nabla'^{\leq i-1}f|)\Big(|(\nabla,\nabla_4)^{\leq i}\mathrm{tr}\,\chi| + O(|\nabla'^{\leq i-1}f|)|(\nabla,\nabla_4,\nabla_3)^{\leq i}\mathrm{tr}\,\chi|\Big)\\ &\lesssim (1+|\nabla'^{\leq i-1}f|)\left(\delta + O(|\nabla'^{\leq i-1}f|)\right).\end{aligned}$$

Therefore, (4.18) becomes

$$\nabla_4'\nabla'^i f + \left(2\omega + \frac{i+1}{2}\mathrm{tr}\,\chi\right)\nabla'^i f = O(\delta)\nabla'^{\leq i}f + O(\epsilon) + O(\delta_*^{\frac{1}{2}})\nabla'^{\leq i-1}f + O\left(|\nabla'^{\leq [i/2]}f||\nabla'^{\leq i}f|\right).$$

Then similar to the zero-order case, we have ($|\nabla'^i f|^2 := (\nabla'^i f)_{a_1\cdots a_i b}(\nabla'^i f)^{a_1\cdots a_i b}$)

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left(|\nabla'^i f|^2\right) + \left(2\omega + \frac{i+1}{2}\mathrm{tr}\,\chi\right)|\nabla'^i f|^2 \quad &= \quad O(\delta)|\nabla'^{\leq i}f|^2 + O(\epsilon)|\nabla'^{\leq i}f| + O(\delta_*^{\frac{1}{2}})|\nabla'^{\leq i-1}f||\nabla'^{\leq i}f|\\ &\quad + O\left(|\nabla'^{\leq [i/2]}f||\nabla'^{\leq i}f|^2\right).\end{aligned}$$

We now assume by induction that we have proved the estimate for $i \leq j-1$. Then for $i = j$, we integrate the equation as the zero-order case to get[31]

$$\begin{aligned}|\nabla'^j f|^2 &\leq C_0\delta^2 + C\int_s^v e^{-C\delta_*^{\frac{1}{2}}(s'-s)} + C\Big(\frac{M_1\epsilon^2}{\delta_*^{\frac{1}{2}}} + \frac{\delta_*^{\frac{1}{2}}}{M_1}|\nabla'^{\leq i}f|^2\Big)\\ &\quad + C\int_s^v e^{-C\delta_*^{\frac{1}{2}}(s'-s)}\left(O(\delta,|\nabla'^{\leq [j/2]}f|)|\nabla'^{\leq j}f|^2 + O(\delta_*^{\frac{1}{2}})|\nabla'^{\leq i-1}f||\nabla'^{\leq i}f|\right)ds'\\ &\leq C_0\delta^2 + C\Big(\int_0^\infty e^{-C\delta_*^{\frac{1}{2}}s'}\,ds'\Big)\cdot\sup_{s\in[s_*,v]}\left(C\Big(\frac{M_1\epsilon^2}{\delta_*^{\frac{1}{2}}} + \frac{\delta_*^{\frac{1}{2}}}{M_1}|\nabla'^{\leq i}f|^2\Big) + C\delta_*^{\frac{1}{2}}M_2|\nabla'^{\leq j-1}f|^2 + \frac{C\delta_*^{\frac{1}{2}}}{M_2}|\nabla'^{\leq j}f|^2\right)\\ &\quad + C\Big(\int_0^\infty e^{-C\delta_*^{\frac{1}{2}}s'}\,ds'\Big)\cdot\sup_{s\in[s_*,v]}C(\delta + |\nabla'^{\leq [j/2]}f|)|\nabla'^{\leq j}f|^2,\end{aligned} \tag{4.21}$$

where M_1, M_2 are positive constants to be determined. This is now similar to the zeroth-order estimate. For $j = 1$, one can make a bootstrap assumption that

$$|\nabla'^{\leq 1}f| \leq C_b\delta, \quad \text{for } s \in [s_1, v],$$

and derive, using the bound of $|f|$ we have established,

$$\begin{aligned}\sup_{s\in[s_1,v]}|\nabla'^{\leq 1}f|^2 &\leq C_0\delta^2 + C\delta_*^{-\frac{1}{2}}\cdot\left(\epsilon + \epsilon|\nabla'^{\leq j}f|^2 + M_2\delta_*^{\frac{1}{2}}|f|^2 + \delta_*^{\frac{1}{2}}\Big(\frac{1}{M_2} + C_b\delta\Big)\sup_{s\in[s_1,v]}|\nabla'^{\leq 1}f|^2\right)\\ &\leq C_0\delta^2 + CM_1\epsilon^2\delta_*^{-1} + CM_2\delta^2 + \left(\frac{C}{M_1} + \frac{C}{M_2} + C_b\delta\right)\sup_{s\in[s_1,v]}|\nabla'^{\leq 1}f|^2.\end{aligned}$$

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[31] For simplicity, we still use the parameter s; one can also use τ as the parameter. Note that along each null generator we have $ds/d\tau = 1/(e_4(v) + O(\delta))$ which is comparable to 1.

Recall the relations $\delta \leq \min\{\delta_*, \delta_{\mathcal{H}}/2\}$ and $\epsilon \leq \delta\delta_*^{-\frac{1}{2}}$. Therefore, picking suitably large constants M, we obtain the estimate

$$|\nabla'^{\leq 1} f|^2 \leq C\delta^2,$$

where C does not depend on C_b. Therefore, picking a larger C_b, we improve the bootstrap assumption. This finishes the estimate for $j = 1$.

For $j \geq 2$ we use the induction, and since in this case the bound of $\nabla'^{\leq [j/2]} f$ is given by the induction assumption, the estimate is linear. Therefore, the estimate $|\nabla'^{\leq N} f| \lesssim \delta$ is valid along each null generator until s reaches the value corresponding to the point on Σ_{τ_*} (recall the definition of τ_* from the bootstrap argument of Proposition 4.4). This concludes the proof of the Lemma 4.7. □

Proof of 3. As a consequence of the proof of Part 2, we also obtain the boundedness of the quantities $\Gamma_1' = \{\chi', \underline{\chi}', \underline{\eta}', \zeta', \omega'\}$ in the new frame $\{e_3', e_4', e_a'\}$:

$$|(\nabla', \nabla_4')^{\leq i} \Gamma_1'| \lesssim 1, \quad i \leq N - 1.$$

These estimates follow by applying ∇' and ∇_4' to the corresponding transformation formulas and using the estimates in Part 2. This ends the proof of Proposition 4.13. □

Remark 4.14. *Note that we have used $N\delta \ll 1$ in the estimate above. We have pointed out in Remark 1.8 that this is valid under the assumptions of [16]. Now assume that the estimate* (2.14) *holds for all large integers N and a given small number ϵ.*[32] *In this case, we would have to make use of the decay*[33] *in v. Indeed, for any positive $n \in \mathbb{N}$, we can find v_n such that the part $v \gtrsim v_n$ of the spacetime is an $O_n(\epsilon_n)$-perturbation satisfying that the constant $\delta_n := \epsilon_n \delta_*^{-\frac{1}{2}}$ verifies $n\delta_n \ll 1$. Then we can apply the arguments above and below, with ϵ and δ replaced by ϵ_n and δ_n, to show that the portion of $\mathcal{H}^+$ in the future of $\{v = v_n\}$ is C^n. In particular, the sphere $\mathcal{H}^+ \cap \{v = v_n\}$ is C^n. It remains to show that the portion of $\mathcal{H}^+$ between $v = v_0$ and $v = v_n$ is C^n. Since $[v_0, v_n]$ is a finite interval, it is standard to show, using the geodesic equation, that the portion of $\mathcal{H}^+$ between $v = v_0$ and $v = v_n$ is also C^n. This shows the C^n regularity of $\mathcal{H}^+$. Since this holds for each n, we have that $\mathcal{H}^+$ is C^∞ in this case.*

4.3.3 Higher order estimates on S_v^τ

Recall that we have shown that for fixed v, τ, $S_v^\tau = C_v \cap \Sigma_\tau$ is an embedded submanifold, so Lemma 4.5 becomes a global statement on S_v^τ, i.e., there exists a frame transformation $({}^{(0)}f, {}^{(0)}\underline{f}, {}^{(0)}\lambda)$ from $\{e_3, e_4, e_a\}$ on S_v^τ such that $\{{}^{(0)}e_a\}$ is tangent to S_v^τ, and ${}^{(0)}N = \frac{1}{2}({}^{(0)}e_4 - {}^{(0)}e_3)$ is the outward unit normal vector of S_v^τ on Σ_τ. In this subsection, since there is no danger of confusion, we denote for simplicity $\underline{f} = {}^{(0)}\underline{f}$, $\lambda = {}^{(0)}\lambda$, $\Sigma = \Sigma_v^\tau$, $S = S_v^\tau$.

We need to prove higher-order bounds of $(\underline{f}, \lambda)$ under the projected covariant derivatives adapted to the sphere S. Recall that the adapted frame $\{{}^{(0)}e_3, {}^{(0)}e_4, {}^{(0)}e_a\}$ is transformed from $\{e_3, e_4, e_a\}$ through

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[32] We note that the result of [16] does not directly imply this.

[33] which could be any slow rate, but cannot be merely boundedness.

$(f, \underline{f} = {}^{(0)}\underline{f}, \lambda = {}^{(0)}\lambda)$. Note that

$$ {}^{(0)}e_a = e'_a + \frac{1}{2}\underline{f}_a e'_4, $$

and we have estimates of various quantities in the e'-frame from Proposition 4.13. Using this expression of ${}^{(0)}e_a$, and that $\underline{f}$ is defined by f through (4.15), we immediately have, by the control of $(\nabla', \nabla'_4)^{\le i} f$,

$$ |(\nabla'_{{}^{(0)}e_a})^{\le i}\underline{f}| \lesssim 1 + |(\nabla'_{{}^{(0)}e_a})^{\le i-1}\underline{f}|, \text{ so by induction, } |(\nabla'_{{}^{(0)}e_a})^{\le i}\underline{f}| \lesssim_i 1. \tag{4.22} $$

We now prove

Proposition 4.15. *We have, on S,*

$$ |{}^{(0)}\nabla^{\le i-1}\chi'| \lesssim_N 1, \quad |{}^{(0)}\nabla^{\le i}(f, \underline{f})| \lesssim_N 1, \quad i \le N. \tag{4.23} $$

Here we use the convention $\nabla'^k\psi = 0$ if k is negative.

Proof. The estimates have been proved for $i = 0$. We now assume that the estimates hold with i replaced by $i-1$ and prove it for i.

As before, since the horizontal structure ${}^{(0)}H$ (given by the new ${}^{(0)}e$ frame, see (4.14)) does not depend on ${}^{(0)}\lambda$, we prove this under the frame given by $(f, \underline{f})$ with the simplifying assumption ${}^{(0)}\lambda = 1$, so

$$ {}^{(0)}e_4 = e'_4, \quad {}^{(0)}e_a = e'_a + \frac{1}{2}\underline{f}_a e'_4, \quad {}^{(0)}e_3 = e'_3 + \underline{f}^a e'_a + \frac{1}{4}|\underline{f}|^2 e'_3. $$

In this setting, by Lemma 2.2, for any H'-horizontal covariant tensor ψ', we have

$$ {}^{(0)}\nabla^i\psi' = (\nabla'_{{}^{(0)}e_a})^{\le i}\psi + \underline{f}\cdot\chi'\cdot(\nabla'_{{}^{(0)}e_a})^{\le i-1}\psi' + {}^{(0)}\nabla^{\le i-1}(\underline{f}\cdot\chi'\cdot\psi'). $$

Let $\psi' = \chi'$ first. This gives,

$$ \begin{aligned} |{}^{(0)}\nabla^{\le i-1}\chi'| &\lesssim |(\nabla'_{{}^{(0)}e_a})^{\le i-1}\chi'| + |{}^{(0)}\nabla^{\le i-2}\chi'|^2|{}^{(0)}\nabla^{\le i-2}\underline{f}| \\ &\lesssim |(\nabla', \nabla'_4)^{\le i-1}\chi'|\left(1 + |(\nabla_{{}^{(0)}e_a})^{\le i-2}\underline{f}|\right) + |{}^{(0)}\nabla^{\le i-2}\chi'|^2|{}^{(0)}\nabla^{\le i-2}\underline{f}|. \end{aligned} $$

Then by (4.22), the bounds of Ricci coefficients in e'-frame (4.20), and the induction assumptions, we obtain the boundedness of $|{}^{(0)}\nabla^{\le i-1}\chi'|$ for $i \le N$. Then, applying this estimate to $\psi' = (f, \underline{f})$ and using (4.22), we establish the required estimate for i and conclude the proof. $\square$

It is then also straightforward to derive, using (4.14) and (4.20),

$$ |{}^{(0)}\nabla^i(\lambda, \lambda^{-1}, \chi', \underline{\chi}', \omega', \zeta', \underline{\eta}')| \lesssim_N 1, \quad i \le N-1. \tag{4.24} $$

Note that those Ricci coefficients are still the ones in the e' frame.

We are now ready to derive the main estimate we need. Denote the Levi-Civita connection on Σ by $\overline{D}$. The extrinsic curvature of S on Σ is given by

$$ k_{ab} = g\left(\overline{D}_{{}^{(0)}e_a}{}^{(0)}N, {}^{(0)}e_b\right) = \frac{1}{2}g\left(\overline{D}_{{}^{(0)}e_a}({}^{(0)}e_4 - {}^{(0)}e_3), {}^{(0)}e_b\right) = \frac{1}{2}\left({}^{(0)}\chi_{ab} - {}^{(0)}\underline{\chi}_{ab}\right). $$

In view of the transformation formulas (2.4)

$$\lambda^{-1\,(0)}\chi = \chi', \quad \lambda\,^{(0)}\underline{\chi} = \underline{\chi}' + {}^{(0)}\nabla \underline{f} + \underline{f} \cdot (\zeta', \underline{\eta}') + \underline{f} \cdot \underline{f} \cdot (\omega', \chi'),$$

and the estimates (4.23), (4.24), we obtain the following estimate

Proposition 4.16. *For every fixed v, τ satisfying $0 \leq \tau \leq v - r_\delta^*$, the extrinsic curvature k of S_v^τ on Σ_τ satisfies $|\,^{(0)}\nabla^{\leq i} k| \lesssim_N 1$, $i \leq N-1$, where the implicit constant is independent of v, τ. Note that we lose one derivative due to the transformation formulas of $^{(0)}\underline{\chi}$ as well as the previous similar loss for the estimate of χ'.*

Remark 4.17. *It is interesting to point out the connection between our proof below and the well known Cheeger–Gromov (CG) convergence theorem (see e.g. [9], [13]). First, CG does not guarantee that the limit sphere is embedded in Σ_τ, while we want to study the regularity of the pointwise limit of a family of spheres $\{S_v^\tau\}$ on Σ_τ. Second, using our estimates, we can show that*

$$K(S_v^\tau) = -\,^{(0)}\rho + \frac{1}{2}\,^{(0)}\widehat{\chi} \cdot \,^{(0)}\underline{\widehat{\chi}} + \frac{1}{4}\,^{(0)}\mathrm{tr}\,\chi\,^{(0)}\mathrm{tr}\,\underline{\chi} = \Re\left(\frac{2m}{q^3}\right) + O(\delta) = \frac{2mr(r^2 - 3a^2\cos^2\theta)}{|q|^6} + O(\delta),$$

so Gauss curvature may not be positive for large a. This prevents direct use of the Myers theorem [20] to obtain the diameter estimate required in the convergence theorem.

4.4 Estimate of the graph function

From above we know that $i(\mathbb{S}^2)$, also denoted by $S = S_v^\tau$, is an embedded sphere on $\Sigma = \Sigma_\tau$. Furthermore it can be written as a graph function, i.e., there is a function $R \colon \mathbb{S}^2 \to [r_+ - \delta, r_+ + \delta]$ so that $\Phi_\Sigma\{(R(p), p) \colon p \in \mathbb{S}^2\} = S$.

Recall in (2.25) that pullback metric on $(r_+ - 2\delta, r_+ + 2\delta) \times \mathbb{S}^2$ through Φ_Σ, denoted by $\overline{g}$, is comparable with the natural metric $\overline{g}_0$ on $(r_+ - 2\delta, r_+ + 2\delta) \times \mathbb{S}^2$. We take a spherical coordinate (θ, φ), switched to (x^1, x^2) near the poles. In the following, the partial derivative ∂ should be understood in these two charts of $\mathbb{S}^2$.

Lemma 4.18. *We have the estimates*

$$|\partial R| \lesssim \delta, \quad |\partial^i R| \lesssim 1, \quad i \leq N.$$

In particular the constants do not depend on v, τ.

Proof. To prove the estimates, we focus on the chart (r, θ, φ); the argument for the other chart is the same. To start with, we derive the estimate of the first-order derivative.

First order derivatives. Let $\widetilde{\partial}_\theta := \partial_\theta + (\partial_\theta R)\overline{\partial}_r$, $\widetilde{\partial}_\varphi := \partial_\varphi + (\partial_\varphi R)\overline{\partial}_r$. Then $\widetilde{\partial}_\theta r = \partial_\theta R$, $\widetilde{\partial}_\varphi r = \partial_\varphi R$, so $\widetilde{\partial}_\theta, \widetilde{\partial}_\varphi$ are tangent to S as we get zero when applying them to $r - R(\theta, \varphi)$.

By Lemma 4.5, for any unit vector V tangent to S, we have $|V(r)| \lesssim \delta$, so in particular,

$$|\widetilde{\partial}_\theta|_{\overline{g}}^{-1}|\widetilde{\partial}_\theta r| \lesssim \delta, \quad |\widetilde{\partial}_\varphi|_{\overline{g}}^{-1}|\widetilde{\partial}_\varphi r| \lesssim \delta.$$

Since the metric coefficients on Σ under (r,θ,φ) are bounded, we have

$$|\widetilde{\partial}_\theta|_{\overline{g}} = |\partial_\theta + (\partial_\theta R)\overline{\partial}_r|_{\overline{g}} \lesssim 1 + |\partial_\theta R|, \text{ and similarly, } |\widetilde{\partial}_\varphi|_{\overline{g}} \lesssim 1 + |\partial_\varphi R|.$$

Therefore,

$$|\partial_\theta R| = |\widetilde{\partial}_\theta r| \leq C\delta(1 + |\partial_\theta R|),$$

which yields $|\partial_\theta R| \lesssim \delta$. Similar for $|\partial_\varphi R|$.

Second-order derivatives. We compare $\overline{g}$ with the Euclidean metric in the (r,θ,φ) coordinates, following [23]. Recall that in Euclidean space, the second fundamental form of a graph function is expressed in terms of the Hessian of the function. In the following, we use Greek letters $\alpha, \beta, \cdots$ to denote the indices on Σ, and Latin letters $i, j, \cdots$ to denote the indices on S. Recall the vector field $\widetilde{\partial}_\theta$, $\widetilde{\partial}_\varphi$ defined above. They are the natural coordinate vectors in the (θ,φ) coordinate on S. Nevertheless, we denote them by $\widetilde{\partial}_i$, $\widetilde{\partial}_j$, and reserve the notation ∂_i, ∂_j for ∂_θ, ∂_φ. We use $\mathcal{N}(a, b, \cdots)$ to denote schematically nonlinear functions of $a, b, \cdots$.

Denote the induced metric on Σ by $\overline{g}$. The vector field $N_{\overline{g}} = \mathrm{grad}_\Sigma(r - R(\theta,\varphi)) = \overline{g}^{\alpha\beta}\partial_\beta(r - R(\theta,\varphi))\partial_\alpha$ is normal to S (not necessarily of unit length). We have negative sign missed

$$|N_{\overline{g}}|k(\widetilde{\partial}_i, \widetilde{\partial}_j) = -\overline{g}(\overline{D}_{\widetilde{\partial}_i}\widetilde{\partial}_j, N_{\overline{g}}) = -(\overline{D}_{\widetilde{\partial}_i}\widetilde{\partial}_j)^\alpha\partial_\alpha(r - R(\theta,\varphi)).$$

Notice that $\overline{D}_{\widetilde{\partial}_i}\widetilde{\partial}_j$ differs from $\overline{D}^e_{\widetilde{\partial}_i}\widetilde{\partial}_j$, where $\overline{D}^e$ is the connection associated with the Euclidean metric with respect to (r,θ,φ) (i.e., $dr^2 + d\theta^2 + d\varphi^2$), by Christoffel symbols of $\overline{g}$ in the (r,θ,φ) coordinates. For the Euclidean connection, we have

$$\overline{D}^e_{\widetilde{\partial}_i}\widetilde{\partial}_j = \overline{D}^e_{\widetilde{\partial}_i}(\partial_j + (\partial_j R)\overline{\partial}_r) = (\partial_i\partial_j R)\overline{\partial}_r.$$

Therefore

$$(\overline{D}^e_{\widetilde{\partial}_i}\widetilde{\partial}_j)^\alpha\partial_\alpha(r - R(\theta,\varphi)) = (\partial_i\partial_j R(\theta,\varphi))\overline{\partial}_r(r - R(\theta,\varphi)) = \partial_i\partial_j R.$$

So we have shown that

$$\partial_i\partial_j R = -|N_{\overline{g}}|k(\widetilde{\partial}_i, \widetilde{\partial}_j) + \mathcal{N}(\Gamma_{\overline{g}}, \partial R),$$

where $\Gamma_{\overline{g}}$ represents the Christoffel symbol components of $\overline{g}$ in (r,θ,φ), which are independent of the sphere S. Since $|N_{\overline{g}}|$ is bounded as long as ∂R and $\overline{g}^{-1}$ are bounded, also using $\overline{g}(\widetilde{\partial}_i, \widetilde{\partial}_i) \lesssim 1 + |\partial_i R|^2$ we deduce $|k(\widetilde{\partial}_i, \widetilde{\partial}_j)| \lesssim |k|(1 + |\partial R|^2) \lesssim 1$. This establishes the boundedness of $|\partial^2 R|$.

Higher-order derivatives. For higher orders, we have (recall that the covariant derivative on the sphere S is denoted by ${}^{(0)}\nabla$)

$$\begin{aligned}
{}^{(0)}\nabla^m k(\widetilde{\partial}_i, \widetilde{\partial}_j; \widetilde{\partial}_{l_1}, \cdots, \widetilde{\partial}_{l_m}) &= {}^{(0)}\nabla_{\widetilde{\partial}_{l_m}}\, {}^{(0)}\nabla^{m-1}k(\widetilde{\partial}_i, \widetilde{\partial}_j; \widetilde{\partial}_{l_1}, \cdots, \widetilde{\partial}_{l_{m-1}}) \\
&= \widetilde{\partial}_{l_m}\big({}^{(0)}\nabla^{m-1}k(\widetilde{\partial}_i, \widetilde{\partial}_j; \widetilde{\partial}_{l_1}, \cdots, \widetilde{\partial}_{l_{m-1}})\big) - {}^{(0)}\nabla^{m-1}k({}^{(0)}\nabla_{\widetilde{\partial}_{l_m}}\widetilde{\partial}_i, \widetilde{\partial}_j; \widetilde{\partial}_{l_1}, \cdots, \widetilde{\partial}_{l_{m-1}}) \\
&\qquad - \cdots - k(\widetilde{\partial}_i, \widetilde{\partial}_j; \partial_{l_1}, \cdots, {}^{(0)}\nabla_{\widetilde{\partial}_{l_m}}\widetilde{\partial}_{l_{m-1}}).
\end{aligned} \tag{4.25}$$

To estimate ${}^{(0)}\nabla_{\widetilde{\partial}_i}\widetilde{\partial}_j$, note that the induced metric on the sphere S, denoted by $\not{g}$, satisfies

$$\not{g}(\widetilde{\partial}_i, \widetilde{\partial}_j) = \bar{g}(\partial_i + (\partial_i R)\overline{\partial}_r, \partial_j + (\partial_j R)\overline{\partial}_r) = \mathcal{N}(\partial R) \cdot \bar{g},$$

where $\bar{g}$ in the last term means coefficients of $\bar{g}$ in (r, θ, φ) coordinates. Therefore, since ${}^{(0)}\nabla_{\widetilde{\partial}_i}\widetilde{\partial}_j$ is given by Christoffel symbols, we have $|{}^{(0)}\nabla_{\widetilde{\partial}_i}\widetilde{\partial}_j| \lesssim \mathcal{N}(\partial^{\leq 2}R, \partial^{\leq 1}\bar{g})$.

We claim that for any $m \leq N-2$ we have

$$|N_{\bar{g}}|\,{}^{(0)}\nabla^m k(\widetilde{\partial}_i, \widetilde{\partial}_j; \widetilde{\partial}_{l_1}, \cdots, \widetilde{\partial}_{l_m}) = \partial_{l_m} \cdots \partial_{l_1}\partial_i\partial_j R + \mathcal{N}(\partial^{\leq m+1}R, \partial^{\leq m+1}\bar{g}).$$

In the proof above, we have shown that this is true when $m = 0$. Now suppose that this holds with m replaced by $m-1$. Then using (4.25), $|N_{\bar{g}}|^{-1} = \mathcal{N}(\bar{g}, \bar{g}^{-1}, \partial R)$ and the induction assumption, we have

$$\begin{aligned}{}^{(0)}\nabla^m k(\widetilde{\partial}_i, \widetilde{\partial}_j;\widetilde{\partial}_{l_1}, \cdots, \widetilde{\partial}_{l_m}) &= \widetilde{\partial}_{l_m}\left(|N_{\bar{g}}|^{-1}|N_{\bar{g}}|\,{}^{(0)}\nabla^{m-1}k(\widetilde{\partial}_i, \widetilde{\partial}_j; \widetilde{\partial}_{l_1}, \cdots, \widetilde{\partial}_{l_{N-1}})\right) \\ &\quad + \mathcal{N}({}^{(0)}\nabla^{m-1}k, \partial^{\leq 2}R, \partial^{\leq 1}\bar{g}) \\ &= |N_{\bar{g}}|^{-1}\widetilde{\partial}_{l_m}\left(\partial_{l_{m-1}} \cdots \partial_{l_1}\partial_i\partial_j R + \mathcal{N}(\partial^{\leq m}R, \partial^{\leq m}\bar{g})\right) + \mathcal{N}({}^{(0)}\nabla^{m-1}k, \partial^{\leq 2}R, \partial^{\leq 1}\bar{g}) \\ &= |N_{\bar{g}}|^{-1}\partial_{l_m} \cdots \partial_{l_1}\partial_i\partial_j R + \mathcal{N}(\partial^{\leq m+1}R, \partial^{\leq m+1}\bar{g}),\end{aligned}$$

which means that it also holds for $m \leq N-2$, concluding the induction. Again noting that $\bar{g}(\widetilde{\partial}_i, \widetilde{\partial}_i) \lesssim 1$, we obtain

$$|\partial^{\leq N}R| \leq C_N,$$

which is independent of v. This ends the proof of Lemma 4.18. □

4.5 Convergence of the spheres S_v^τ on Σ_τ

With the uniform bounds obtained in Lemma 4.18, we can now apply the Arzela–Ascoli lemma to get

Proposition 4.19. *On each Σ_τ, the family of spheres S_v^τ converges pointwise to a C^{N-1} sphere S_τ^* as $v \to \infty$.*

Proof. We have shown that under the coordinate charts $(\theta, \varphi; \frac{\pi}{4} < \theta < \frac{3\pi}{4})$ and $(x^1, x^2; 0 \leq \theta < \frac{\pi}{3}$ or $\frac{2\pi}{3} < \theta \leq \pi)$, each sphere S_v^τ can be expressed as $r = R_v(\theta, \varphi)$, $r = R'_v(x^1, x^2)$ and their C^N norms are uniformly bounded, independent of v. Therefore, we can find a subsequence $\{v_n\} \to \infty$ such that R_{v_n} and R'_{v_n} converges in C^{N-1} norm (over the corresponding region) to C^{N-1} functions $R^*(\theta, \varphi)$, $R'^*(x^1, x^2)$, which, of course, coincide with the pointwise limit. This in fact gives a diffeomorphism from $\mathbb{S}^2$ to S_τ^*. Therefore, S_τ^* is an embedded C^{N-1} submanifold. □

4.6 The event horizon

We have constructed a limit sphere S_τ^* on Σ_τ. In this section, we show that the union $\cup_{\tau \geq 0} S_\tau^*$ coincides with the future outgoing null cone of $S^* = S_0^*$ and gives the event horizon.

For any given τ_1, we consider the future outgoing (past incoming) null cone of $S^*_{\tau_1}$, denoted by $C^*_{\tau_1}$. Since $S^*_{\tau_1}$ is smoothly embedded and the background metric is smooth, by Lemma 3.6, $C^*_{\tau_1}$ is regular between $\Sigma_{\tau_1-\delta_{\tau_1}}$ and $\Sigma_{\tau_1+\delta_{\tau_1}}$ for some $\delta_{\tau_1} > 0$. For any $\tau \in (\tau_1 - \delta_{\tau_1}, \tau_1 + \delta_{\tau_1})$, we denote the intersection of $C^*_{\tau_1}$ and Σ_τ by $S^*_{\tau_1;\tau}$. We have

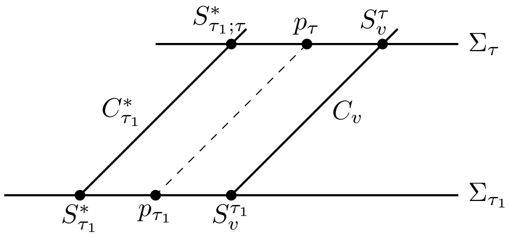
Proposition 4.20. *For any $\tau \in (\tau_1 - \delta_{\tau_1}, \tau_1 + \delta_{\tau_1})$, we have $S^*_{\tau_1;\tau} = S^*_\tau$.*

Proof. Throughout this proof, all discussions are restricted to $\tau \in (\tau_1 - \delta_{\tau_1}, \tau_1 + \delta_{\tau_1})$, where $C^*_{\tau_1}$ is regular.

Case $\tau \geq \tau_1$.

Step 1: Show that $S^*_{\tau_1;\tau}$ is in the interior[34] of S^*_τ.

Since $S^*_\tau = \lim_{v\to\infty} S^\tau_v$, it suffices to show that $S^*_{\tau_1;\tau}$ is in the interior of S^τ_v for each v. By construction, on Σ_{τ_1} we know that $S^{\tau_1}_v$ is in the exterior of $S^*_{\tau_1}$ for any v, by Corollary 2.24 applied to $\widehat{C}_v(\tau_1, \tau)$, we know that all points in $\mathcal{J}^+(S_{\tau_1^*})$ cannot be strictly in the exterior of S^τ_v on Σ_τ, i.e. $S^*_{\tau_1;\tau}$ must be in the interior of S^τ_v, for any v, as required.



Step 2: Show that they must coincide.

If they do not, then there is a point $p_\tau \in \Sigma_\tau$ in between, i.e., in the strict exterior of $S^*_{\tau_1;\tau}$, and the interior of every S^τ_v. For each v', there exists a point $p_{\tau_1,v'} \in \mathcal{J}^-(p_\tau)$ on Σ_{τ_1} that lies in the interior of $S^{\tau_1}_{v'}$. Then, we can pick a sequence of p_{τ_1,v_n}, lying in the interior of $S^{\tau_1}_{v_n}$ for each n. By compactness, there is a converging subsequence of p_{τ_1,v_n} on Σ_{τ_1}. On the other hand, by definition

$$S^*_{\tau_1} = \lim_{v\to\infty} S^{\tau_1}_v,$$

this subsequential limit must be on $S^*_{\tau_1}$. This means that a subsequence of $p_{\tau,v} \in \mathcal{J}^-(p_\tau)$ is converging to $S^*_{\tau_1}$. By Proposition 2.16, $\mathcal{J}^-(p_\tau)$ is closed, so this implies $S^*_{\tau_1} \cap \mathcal{J}^{-1}(p_\tau) \neq \varnothing$, i.e., $p_\tau \in \mathcal{J}^+(S^*_{\tau_1})$, which contradicts the assumption that p_τ is in the strict exterior of $S^*_{\tau_1;\tau}$, hence not on $S^*_{\tau_1;\tau}$.

Case $\tau \leq \tau_1$. The proof is similar. We first have that $S^\tau_v \subset \mathcal{J}^-(S^{\tau_1}_v)$ must be in the exterior $S^*_{\tau_1;\tau}$ since $\widehat{C}^*_\tau(\tau, \tau_1)$ is achronal. To prove they coincide, assume again a point p_τ in between (not on $S^*_{\tau_1;\tau}$ or S^τ_v and

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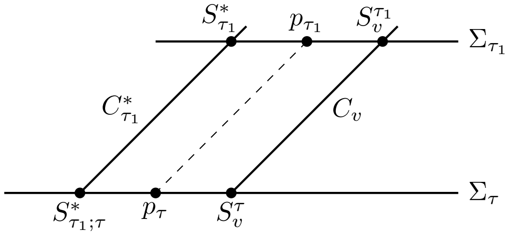
[34] Throughout the proof, the words "future", "past", "interior" and "exterior" of an object include the object itself. We will specify (e.g. by using the word "strict") otherwise.

for all v). Then by a similar argument, there must be a point $p_{\tau_1} \in \mathcal{J}^+(p_\tau) \cap \Sigma_\tau$ that lies in the exterior of $S^*_{\tau_1}$ on Σ_{τ_1}. Then since $S^*_{\tau_1}$ is the limit of $S^{\tau_1}_v$, there exists some v' such that p_{τ_1} is in the exterior of $S^{\tau_1}_{v'}$, so applying Corollary 2.24 to $C_{v'}(\tau, \tau_1)$ we see that p_τ cannot be in the strict interior $C_{v'}(\tau, \tau_1)$, a contradiction. □



We now show that the cone C^*_0 is globally regular towards the future and is the union of all S^*_τ with $\tau \geq 0$. In a neighborhood of S^*_0, C^*_0 is regular, and by Proposition 4.20, coincides with the union of S^*_τ. Now let τ^* be the supremum of τ' so that C^*_0 is regular for $\tau \in [0, \tau')$. Applying Proposition 4.20 with $\tau_1 = \tau^*$, we obtain a piece of null cone for $\tau \in (\tau^* - \delta_{\tau^*}, \tau^* + \delta_{\tau^*})$, which coincides with the union of S^*_τ. On the other hand, by the definition of τ^*, C^*_0 is regular for $\tau \in [0, \tau^* - \frac{1}{2}\delta_{\tau^*}]$, and, by Proposition 4.20, C^*_0 equals $\cup_\tau S^*_\tau$ on this interval. This extends C^*_0 regularly to $\tau \in [0, \tau_* + \delta')$ with $C^*_0 = \cup_\tau S^*_\tau$, so we must have $\tau^* = \infty$.

We have constructed a globally regular future outgoing null cone C^*_0 which equals the union of S^*_τ. We now show that it is the event horizon.

Proposition 4.21. *The future null cone C^*_0 coincides with the event horizon (in the future of Σ_0).*

Proof. Since $S^*_\tau \subset \mathcal{R}_\delta$, we see that the future of any points on C^*_0 is not reaching r-value larger than $r_+ + \delta$. Therefore, by Definition 1.6, points on C^*_0 are not in $\mathcal{J}^{-1}(\mathcal{I}^+)$.

By definition, to show that C^*_0 is indeed the event horizon, it remains to show that any neighborhood of a point on C^*_0 contains a point belonging to $\mathcal{J}^-(\mathcal{I}^+)$. To see this, notice that such a neighborhood contains a point p that lies in the strict exterior of S^*_τ on Σ_τ for some τ, so it must lie in the exterior of some S^τ_v for some large v, hence the exterior of C_v. This means that p is in the causal past of $S(v, r^*_\delta)$. Then by the characterization (3.2), we know that $p \in \mathcal{J}^{-1}(\mathcal{I}^+)$. □

5 Proof of the uniqueness theorem

We have shown that the event horizon $\mathcal{H}^+$ of the perturbed spacetime $\mathcal{M}$ is regular. According to the regularity proof, any point in the strict exterior side of $\mathcal{H}^+$ lies in the causal past of $S(v, r^*_\delta)$ for v

large enough, and hence belongs to $\mathcal{J}^-(\mathcal{I}^+)$. Hence, there cannot be another future inextendible null hypersurface (future horizon) containing a point in the exterior side of $\mathcal{H}^+$[35] that remains in $\mathcal{R}_\delta$.

However, it may still be the case that there exists such a future horizon in the interior side of $\mathcal{H}^+$ that does not exit $\mathcal{R}_\delta$. In this section, we eliminate this possibility by proving a stronger statement that any null geodesic in the interior side hits the spacelike hypersurface $\{r = r_+ - \delta\}$, hence entering the trapped region.

Recall that v, r are coordinate functions under the ingoing PG structure of $\mathcal{M}$, see Section 2.4.1. Given $V > 0$, we construct a foliation in the region $\mathcal{R}_\delta \cap \{v \leq V\}$ as follows: We define

$$u := -(r - r_+) \quad \text{ on } \{v = V\} \cap \mathcal{R}_\delta.$$

So $u = 0$ on $S(V, r_+)$, and $\partial_r u = -1$ on $\{v = V\}$. We then construct the null cone of $S(V, r)$, denoted by $C_{V,r}$, in the past incoming direction, for $-\delta \leq r - r_+ \leq \delta$. The spheres here are also $S(v, r)$-type spheres within $\mathcal{R}_\delta$, so by a similar argument as in Section 4, we can show that $C_{V,r}$ are regular null cones towards the past of $S(V, r)$, and there exists a null frame transformation

$$e_4' = e_4 + f^a e_a + \frac{1}{4}|f|^2 e_3, \quad e_a' = e_a + \frac{1}{2} f_a e_4, \quad e_3' = e_3,$$

such that e_4' is tangent to the null generators of the null cone, and the frame satisfies the estimates $|f|, |\nabla' f| \lesssim \delta$ (hence in particular, $|\mathrm{tr}\,\chi'| \lesssim \delta$) uniformly in V and r.

We extend u, from $\{v = V\}$, so that it is constant on each null cone $C_{V,r}$. We denote the level hypersurfaces of u by H_u.

Since e_a' is in the orthogonal complement of e_4', hence tangent to the null cone H_u, we have, using $\nabla' u = 0$, $\nabla_4' u = 0$,

$$\begin{aligned} \nabla_4'(e_3'(u)) &= e_4'(e_3'(u)) - e_3'(e_4'(u)) = (D_{e_4'} e_3')u - (D_{e_3'} e_4')u \\ &= 2(\underline{\eta}' - \eta') \cdot \nabla' u + 2\omega' \nabla_3' u - 2\underline{\omega}' \nabla_4' u = 2\omega' e_3'(u). \end{aligned} \tag{5.1}$$

Remark 5.1. *To be precise, we also need to show that η' is finite at each point. (The finiteness of other quantities $\underline{\omega}' = 0$, η', ω' are clear as they only depends on f itself, but not on any derivatives of f.)*

Proof of remark. One has the following transformation formula using that e_3 is geodesic:

$$\eta' = \eta + \frac{1}{2} \nabla_3' f.$$

Therefore, we need to estimate of $\nabla_3' f$. Commuting ∇_3' with the equation (4.4) we obtain

$$\nabla_4' \nabla_3' f + (\frac{1}{2}\mathrm{tr}\,\chi + 2\omega) \nabla_3' f + \nabla_3 (\frac{1}{2}\mathrm{tr}\,\chi + 2\omega) f = [\nabla_4', \nabla_3'] f + \nabla_3' \big(-2\xi - \widehat{\chi} \cdot f + \frac{1}{2} {}^{(a)}\mathrm{tr}\,\chi\, {}^* f + O(|f|^2) \big), \tag{5.2}$$

---

[35] The notions “exterior side” and “interior side” are well-defined in view of $\mathcal{H}^+ = \cup_\tau S_\tau^*$, see also Section 2.8.1.

with the initial condition $\nabla'_3 f = \nabla_3 F$ on $\{v = V\}$. In view of the formula

$$[\nabla'_4, \nabla'_3] f = 2(\eta' - \underline{\eta}') \cdot \nabla' f + 2\omega' \nabla'_3 f - 2\underline{\omega}' \nabla'_4 f + 2(\underline{\eta}' \cdot f)\eta' - 2(\eta' \cdot f)\underline{\eta}' - {}^*f \cdot {}^*\rho',$$

and the transformation formulas (2.4), we see that with f and $\nabla' f$ already controlled, there are no quadratic terms of $\nabla'_3 f$ in (5.2) (which is markedly different from the transport equation of $\nabla' f$ studied in Section 4), so $|\nabla'_3 f|$ grows at most exponentially. Therefore we see that $\nabla'_3 f$ remains finite at any given point and hence so is η'. □

We now prove the uniqueness theorem using this construction. Suppose there is a future null geodesic $\gamma \not\subseteq \mathcal{H}^+$ in the interior side of $\mathcal{H}^+$ that stays entirely in the region $\mathcal{R}_\delta$, see Figure 4 below. Take an arbitrary point $p_1 \in \gamma$ and denote its v-value by v_0. Fix a value v_0. Let p_1 denote a point of intersection of γ with $\{v = v_0\}$.[36] Consider the integral curve of $\partial_r = -e_3$ passing through p_1 along $v = v_0$. This is a transversal null geodesic, hence intersecting $\mathcal{H}^+$ at a unique point p_0. Since we assume p_1 is in the interior (future) side of $\mathcal{H}^+$ and $\partial_r = -e_3$ is past-directed null, we have $r(p_1) \leq r(p_0)$.

We want to show that on each such integral curve of $\partial_r = -e_3$, where r is of course an affine parameter, we have $r(p_1) = r(p_0)$, and hence $p_1 = p_0$. Then by the arbitrariness of v_0, we deduce that $\gamma \subset \mathcal{H}^+$, contradicting the above assumption.

To show this, we assume that for $p_1 \in \{v = v_0\}$, we have $p_1 \neq p_0$, i.e. $d(v_0) := |r(p_1) - r(p_0)|\big|_{v=v_0} \neq 0$. The proposition below shows that in this case, the difference $u(p_1) - u(p_0)$ would be exponentially amplified compared with $r(p_1) - r(p_0)$.

Proposition 5.2. *For $d(v_0) = |r(p_1) - r(p_0)|\big|_{v=v_0}$ and the function u constructed from $\{v = V\}$ above (for any given $V > v_0$), we have $u(p_1) - u(p_0) \geq d(v_0)\, e^{\frac{1}{2}C\delta_*^{\frac{1}{2}}(V - v_0)}$.*

Proof. For ω', from the transformation formula in (2.4), the bounds (2.17), (2.13), and $|f| \lesssim \delta$, we have

$$\omega' = \omega + \frac{1}{2} f \cdot (\zeta - \underline{\eta}) + O(|f|^2) \leq -C\delta_*^{\frac{1}{2}} + O(\delta),$$

so we also have $\omega' \leq -C\delta_*^{\frac{1}{2}}$ for some constant $C > 0$. Then, since $e'_3(u) = 1$ on $\{v = V\}$, and $e'_4(v) = e_4(v) + O(\delta) \sim 1$, using (5.1) we get $e'_3(u) \geq e^{\frac{1}{2}C\delta_*^{\frac{1}{2}}(V - v_0)}$ on $\{v = v_0\}$. Therefore,

$$u(p_1) - u(p_0) = \int_{r(p_1)}^{r(p_0)} (-\partial_r u)\, dr \geq |r(p_0) - r(p_1))|\big|_{v=v_0} \cdot \inf_{v=v_0} |e'_3(u)| \geq d(v_0)\, e^{C\delta_*^{\frac{1}{2}}(V - v_0)},$$

Need to show $r(p_1) \leq r(p_0)$. as stated. In particular, we can pick suitably large V (the above process works for any V) to make $u(p_1) - u(p_0)$ large (say, larger than δ^{-1}). □

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[36] Since $\{v = v_0\}$ is not achronal, it is possible that γ intersects with it more than once. Nevertheless, the argument holds for any points of intersection.

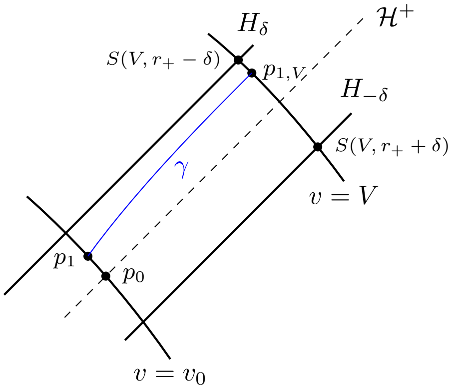

Figure 4: Idea of the proof of uniqueness. Note that the event horizon $\mathcal{H}^+$ does not correspond to any H_u, but it lies between $H_{-\delta}$ and H_δ.

On the other hand, let $p_{1,V}$ be the intersection of the null geodesic γ and $\{v = V\}$. By assumption, γ lies in $\mathcal{R}_\delta$, so we have $p_{1,V} \in \mathcal{R}_\delta$, and hence $-\delta \leq u(p_{1,V}) \leq \delta$. Since u is an optical function, it is non-increasing backward along any null geodesic, which means that $u(p_1) \leq \delta$. Moreover, since $p_0 \in \mathcal{H}^+$, by the characterization (3.2), we know that any point in the causal future of p_0 lies in $\{r < r_+ + \delta\}$, so in particular we must have $u(p_0) > -\delta$. Therefore,

$$0 \leq u(p_1) - u(p_0) \leq 2\delta,$$

In fact here the 0 on the left can be replaced by the lower bound $d(v_0)e^{C\delta_*^{\frac{1}{2}}(V-v_0)}$ we just proved. where the lower bound is due to the fact that p_1 and p_0 are connected by the integral curve of $-\partial_r = e_3$, which is a causal curve. Therefore, we get a contradiction on the size of $u(p_1) - u(p_0)$, so we must have $r(p_1) = r(p_0)$, hence $p_1 = p_0$.

A Proof of the Embedding Criterion

In this appendix, we prove the Embedding Criterion (Proposition 3.7). For simplicity, we denote $S = S_v^\tau$, $i = i_v^\tau$, $\Sigma = \Sigma_\tau$. We restate the proposition:

Proposition A.1. *Suppose that S is the image of an immersion $i : \mathbb{S}^2 \to \Sigma$ with $i(\mathbb{S}^2) \subset \Sigma \cap \mathcal{R}_\delta$, so for each point $p \in \mathbb{S}^2$, there exists a neighborhood $\mathcal{O}_p$ such that $i|_{\mathcal{O}_p}$ is an embedding into Σ. Then there exists a small number $\mathring{\delta} > 0$ such that the following statement holds:*

If for each $p \in \mathbb{S}^2$, there exists a tangent orthonormal frame $\{{}^{(0)}e_a\}$ (w.r.t. the induced metric from Σ) on $i(\mathcal{O}_p)$ verifying $|{}^{(0)}e_a(r)| \leq \mathring{\delta}$, then the immersion is in fact an embedding. Moreover, there exists a smooth map

$$R \colon \mathbb{S}^2 \to [r_+ - \delta, r_+ + \delta], \quad p \mapsto R(p),$$

such that $\Phi_\Sigma(\{(R(p), p) \colon p \in \mathbb{S}^2\}) = S$ (with $\Phi_\Sigma = \Phi_{\Sigma_\tau}$ defined by (2.23)).

Proof. We briefly explain the ideas. The smallness of ${}^{(0)}e_a(r)$ in the lemma will imply that locally the image of i on Σ can be written as a graph $r = R(\theta, \varphi)$ (or $R(x^1, x^2)$; we may omit this below).[37] If the image i fails to be injective, the pre-image of some point on $i(\mathbb{S}^2)$ will give isolated points, hence disconnected. Using the local graph property, we make a continuity argument with respect to a polar angle on Σ (constructed such that the point on Σ where the injectivity fails corresponds to the direction of the north pole), and reach a conclusion essentially saying that $\mathbb{S}^2$ becomes disconnected with finite points removed, which is a topological contradiction.

We proceed in steps as follows.

Step 1. We show that for each $p \in \mathbb{S}^2$, there is a smaller neighborhood $\tilde{\mathcal{O}}_p \subset \mathcal{O}_p$, such that $i(\tilde{\mathcal{O}}_p)$ can be written as $r = R(\theta, \varphi)$ on Σ. In other words, the map $P_{\mathbb{S}^2} \circ \Phi_\Sigma^{-1} \circ i|_{\tilde{\mathcal{O}}_p}$ is a diffeomorphism onto its image on $\mathbb{S}^2$ (See Figure 5). We will refer to this as the *local graph property at the point p.*

By the assumption that $|{}^{(0)}e_a(r)| \leq \mathring{\delta}$ for some orthonormal frame $\{{}^{(0)}e_a\}$, any unit tangent vector V of $i(\mathcal{O}_p)$ satisfies $|V(r)| \leq 2\mathring{\delta}$. Therefore, the vector field $\overline{\partial}_r$ is transversal to $i(\mathcal{O}_p)$ as $\overline{\partial}_r(r) = 1$ and its length is comparable to 1, see (2.24). In other words, assuming that $G(r, \theta, \varphi) = 0$ (or $G(r, x^1, x^2) = 0$) is a local description of the embedded submanifold $i(\mathcal{O}_p)$, then

$$\frac{\partial G}{\partial r}(r, \theta, \varphi) \neq 0.$$

Therefore, by the implicit function theorem, there exists a neighborhood $\tilde{\mathcal{O}}_p$, such that $i(\tilde{\mathcal{O}}_p)$ has the form $r = R(\theta, \varphi)$. In particular, the map $P_{\Sigma;\mathbb{S}^2} \circ i|_{\tilde{\mathcal{O}}_p}$[38] is an embedding to $\mathbb{S}^2$.

Step 2. The goal of this step is to contradict the following assumption.

A. Assume that there are two distinct points $p_1, p_2 \in \mathbb{S}^2$ such that $P_{\Sigma;\mathbb{S}^2}(i(p_1)) = P_{\Sigma;\mathbb{S}^2}(i(p_2)) := P_0$ are the same on $\mathbb{S}^2$.

Assuming **A**, we introduce some convenient notations. On $\mathbb{S}^2$ consider the spherical coordinates $(\theta_{P_0}, \varphi_{P_0})$ with P_0 being the north pole and θ_{P_0} being the polar angle. Denote the corresponding south pole by P_0', so $\theta_{P_0}(P_0) = 0$, $\theta_{P_0}(P_0') = \pi$. For each point p on Σ, we assign a ϑ-value to p, denoted by $\vartheta(p)$, defined to be the the pull back through $P_{\Sigma;\mathbb{S}^2}$ of the θ_{P_0}-coordinate value, i.e. the θ_{P_0} value of $P_{\Sigma;\mathbb{S}^2}(p)$. Clearly, ϑ is a continuous function on Σ.

Define $G_\kappa := i^{-1}(\{p \in \Sigma \colon \vartheta(p) \leq \kappa\})$, $0 \leq \kappa \leq \pi$. Also denote $H_\kappa := i^{-1}(\{p \in \Sigma \colon \vartheta(p) = \kappa\})$. They are closed subsets of $\mathbb{S}^2$, hence compact. Moreover, for each point p in H_0, using the local graph property, we see that $\tilde{\mathcal{O}}_p \cap H_0 = \{p\}$, where $\tilde{\mathcal{O}}_p$ is defined in Step 1. By compactness we deduce that H_0 only contains finite points. The same thing holds for H_π. We also remark that, since $\vartheta \circ i$ is a smooth function on $\mathbb{S}^2 \backslash (H_0 \cup H_\pi)$, the subspace $G_\kappa = \{\vartheta \circ i \leq \kappa\} \subseteq \mathbb{S}^2$ with $0 < \kappa < \pi$ are locally path-connected. Therefore, for $0 < \vartheta < \pi$, G_κ is connected implies that G_κ is path-connected.

We first prove a lemma.

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[37] Recall that (θ, φ) and (x^1, x^2) are coordinates on Σ given by the diffeomorphism Φ_Σ.

[38] Recall the definition of $P_{\Sigma;\mathbb{S}^2}$ in (2.26).

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "i"] & \Sigma \cap \{|r - r_+| < 2\delta\} \arrow[r, "\Phi_\Sigma^{-1}"] & (r_+ - 2\delta, r_+ + 2\delta) \times \mathbb{S}^2 \arrow[r, "P_{\mathbb{S}^2}"] & \mathbb{S}^2 \\
(G_\vartheta,\, H_\vartheta) & & & (P_0,\, P_0')
\end{tikzcd}$$

Figure 5: A diagram illustrating the maps involved in the local graph property. Note also the notation $P_{\Sigma;\mathbb{S}^2} = P_{\mathbb{S}^2} \circ \Phi_\Sigma^{-1}$. The local property says that the map from the left $\mathbb{S}^2$ to the right $\mathbb{S}^2$ is a local diffeomorphism.

Lemma A.2. *Given $\vartheta_0 \in [0, \pi)$, and an open neighborhood Ω of G_{ϑ_0} on $\mathbb{S}^2$, there exists a small $\delta' > 0$, such that $G_{\vartheta_0 + \delta'} \subset \Omega$. Similarly, for any open neighborhood of H_π, there exists a small $\delta'' > 0$ such that $H_{\pi - \delta'} \in \Omega$ for all $\delta' \leq \delta''$.*

Proof of lemma. If not, then there is a sequence $\vartheta_n \searrow \vartheta_0$ such that there are $\tilde{p}_n \in G_{\vartheta_n} \backslash \Omega$. Since $\mathbb{S}^2$ is compact, $\{\tilde{p}_n\}$ has a convergent subsequence, still denoted by $\tilde{p}_n$, to a point $\tilde{p} \in \mathbb{S}^2$. Since the immersion i is continuous, $i(\tilde{p}_n)$ is also convergent to $i(\tilde{p})$. Also notice that $i(\tilde{p}_n) \in i(G_{\vartheta_n})$, i.e., $\vartheta(i(\tilde{p}_n)) \leq \vartheta_n$, so we have $\vartheta(i(\tilde{p})) \leq \vartheta_0$, and by definition, $\tilde{p} \in G_{\vartheta_0}$. On the other hand, since $\tilde{p}_n \notin \Omega$, we have $\tilde{p} \notin \Omega$ since the complement of Ω is closed, a contradiction. The proof of the second statement is almost identical. ☐

Remark A.3. *By assumption* **A**, *$H_0 = G_0$ contains at least two points. Then since $H_0 = G_0$ is a set of finite distinct points on $\mathbb{S}^2$, it is disconnected. Using $\tilde{\mathcal{O}}_p \cap H_0 = \{p\}$ for each $p \in H_0$ and the lemma with $\vartheta_0 = 0$, it is easy to see that there exists a $\delta_0 > 0$ such that G_ϑ remains disconnected for all $\vartheta \leq \delta_0$.*

Now define $\vartheta^* := \sup\{\vartheta \in [0, \pi] \colon G_\kappa \text{ is disconnected for all } \kappa \in [0, \vartheta]\}$. In view of Remark A.3 we have $\vartheta^* \geq \delta_0$.

With this setup, we show the following statements, which lead to a contradiction with the assumption **A**:

A1 It is impossible to have $\vartheta^* = \pi$;

A2 If $\vartheta^* < \pi$, then G_{ϑ^*} is connected;

A3 If $\vartheta^* < \pi$, then G_{ϑ^*} is disconnected.

Once we show that **A1-A3** must all hold true, we obtain a contradiction with assumption **A**. Therefore, for different points p_1, p_2 on $\mathbb{S}^2$, $P_{\Sigma;\mathbb{S}^2}(i(p_1))$ and $P_{\Sigma;\mathbb{S}^2}(i(p_2))$ cannot be the same, i.e, $P_{\Sigma;\mathbb{S}^2} \circ i \colon \mathbb{S}^2 \to \mathbb{S}^2$ is injective. Since $P_{\Sigma;\mathbb{S}^2} \circ i \colon \mathbb{S}^2 \to \mathbb{S}^2$ is also a local diffeomorphism between $\mathbb{S}^2$, it is a global diffeomorphism from $\mathbb{S}^2$ onto its image. Now noticing that no proper subset of $\mathbb{S}^2$ has the same topology as $\mathbb{S}^2$,[39] we see that $P_{\Sigma;\mathbb{S}^2} \circ i$ must also be surjective as otherwise, it gives a homeomorphism between $\mathbb{S}^2$ and its proper subset. Hence, $S = i(\mathbb{S}^2)$ can be globally written as a graph of a function R on $\mathbb{S}^2$ stated in Proposition A.1. ☐

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[39] Recall that $\mathbb{S}^2$ with a point removed is homeomorphic to $\mathbb{R}^2$, so any proper subset of $\mathbb{S}^2$ is topologically a subset of $\mathbb{R}^2$.

We now show that **A** implies **A1-A3**.

Proof of A $\implies$ A1. Indeed, if $\vartheta^* = \pi$, then G_ϑ is disconnected for all $\vartheta < \pi$. We have argued above that the set $H_\pi = i^{-1}(\Phi_\Sigma(\{(r,p)\colon p = P_0'\}))$ consists of finite number of points, denoted by $\{p_1', p_2', \cdots, p_m'\}$, with neighborhoods $\tilde{\mathcal{O}}_{p_j'}$ satisfying the local graph property. Now take a sequence $\vartheta_n \nearrow \pi$. By Lemma A.2 above, for sufficiently large n, we have $i^{-1}\big(\{\vartheta > \vartheta_n\}\big) \subset \cup_{j=1}^k \tilde{\mathcal{O}}_{p_j'}$ and becomes a union of small disks on $\mathbb{S}^2$ with radius arbitraily small. Precisely, for any $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$, $\mathcal{O}_{p_j'} \cap i^{-1}(\{\vartheta > \vartheta_n\})$ lies inside an open disk on $\mathbb{S}^2$ with radius[40] less than ϵ for all $j = 1, \cdots, k$, and

$$G_{\vartheta_n} = \mathbb{S}^2 \backslash \big(\cup_{j=1}^k (\tilde{\mathcal{O}}_{p_j'} \cap i^{-1}(\{\vartheta > \vartheta_n\})\big),$$

is a set obtained by removing k disjoint open disks with radius less than ϵ from $\mathbb{S}^2$.

By the assumption $\theta^* = \pi$, G_{ϑ_n} is disconnected for all n. On the other hand, for any given positive integer k, there exists $\epsilon' > 0$ such that it is impossible to make $\mathbb{S}^2$ disconnected by taking away k geodesic disks with radius less than ϵ'. This gives a contradiction and proves **A1**.

Proof of A $\implies$ A2. If G_{ϑ^*} were disconnected, since it is a compact subset of $\mathbb{S}^2$, we can write $G_{\vartheta^*} = G^1_{\vartheta^*} \cup G^2_{\vartheta^*}$ with $G^1_{\vartheta^*}$, $G^2_{\vartheta^*}$ being two non-empty, disjoint compact subsets of $\mathbb{S}^2$. We can then find two disjoint open sets Ω_1, Ω_2 such that $G^1_{\vartheta^*} \subset \Omega_1$, $G^2_{\vartheta^*} \subset \Omega_2$. Let $\Omega = \Omega_1 \cup \Omega_2$. By Lemma A.2, for some $\delta' > 0$, $G_{\vartheta^* + \delta'} \subset \Omega$, and so is G_ϑ for all $\vartheta \le \vartheta^* + \delta'$. Also, for any $\vartheta^* \le \vartheta \le \vartheta^* + \delta'$, G_ϑ has a point in each connected component of $\Omega = \Omega_1 \cup \Omega_2$, so it is itself disconnected. By the definition of ϑ^*, we also know that G_ϑ is disconnected for $0 \le \vartheta < \vartheta^*$, hence for $0 \le \vartheta \le \vartheta^* + \delta'$. This contradicts the maximality of ϑ^*.

Proof of A $\implies$ A3. This is shown by the following proposition.

Proposition A.4. *If* $0 < \vartheta^* < \pi$, *and* G_{ϑ^*} *is connected, then there exists* $\vartheta'' < \vartheta^*$ *such that* $G_{\vartheta''}$ *is also connected.*

Proof. We construct the following cover of G_{ϑ^*}. Since ϑ^* is a given number in $(0, \pi)$, for each $p \in G_{\vartheta^*}$, we define

- If $\frac{|\vartheta(p) - \vartheta^*|}{\vartheta^*} \le \frac{1}{2}$, (i.e, $\vartheta(p)$ is not so close to 0), we pick a smaller neighborhood $\mathcal{O}_p' \subset \tilde{\mathcal{O}}_p$ satisfying the property that $P_{\Sigma;\mathbb{S}^2} \circ i(\mathcal{O}_p')$ is a disk $D_p(\epsilon_p)$, centered at $P_{\Sigma;\mathbb{S}^2}(i(p))$, of radius ϵ_p sufficiently small, in the coordinate chart of $(\theta_{P_0}, \varphi_{P_0})$.[41] This means that for each $\kappa \in (0, \pi)$, the set $\mathcal{O}_p' \cap i^{-1}(\{\vartheta \le \kappa\})$ is diffeomorphic to $D_p(\epsilon_p) \cap \{\theta_{P_0} \le \kappa\}$ through the map $P_{\Sigma;\mathbb{S}^2} \circ i$, so it is topologically the same as an upper half disk.
- If $\frac{|\vartheta(p) - \vartheta^*|}{\vartheta^*} > \frac{1}{2}$, we let $\mathcal{O}_p' \subset \tilde{\mathcal{O}}_p$ be a connected open set that lies in $G_{\frac{3}{4}\vartheta^*}$ and contains p.

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[40] Evaluated under the standard metric on $\mathbb{S}^2$.

[41] Note that $P_{\Sigma;\mathbb{S}^2} \circ i$ is already a diffeomorphism onto the image when restricted to $\mathcal{O}_p$, and $(\theta_{P_0}, \varphi_{P_0})$ is a regular coordinate chart away from P_0, P_0'.

Since $\{\mathcal{O}'_p\}_{p\in\mathbb{S}^2}$ is a cover of G_{ϑ^*}, we can extract a finite subcover $\{\mathcal{O}'_{p_1},\cdots,\mathcal{O}'_{p_l}\}$, or simply $\{\mathcal{O}'_1,\cdots,\mathcal{O}'_l\}$. Among them, we can further extract those that intersect with H_{ϑ^*}, and denote them by $\{\mathcal{U}_1,\cdots,\mathcal{U}_k\}$. This is a finite cover of H_{ϑ^*}.

We need a lemma that allows us to modify curves.

Lemma A.5. *Suppose that there is a curve $\gamma\colon[0,1]\to G_{\vartheta^*}$, with $\vartheta\circ i(\gamma(0)),\vartheta\circ i(\gamma(1))<\vartheta^*$. Then there exists another curve $\tilde{\gamma}\colon[0,1]\to G_{\vartheta^*}$ with $\tilde{\gamma}(0)=\gamma(0)$, $\tilde{\gamma}(1)=\gamma(1)$, and it satisfies $\sup_\gamma\vartheta<\vartheta^*$.*

Assuming the lemma to be true, we continue as follows:

For each $j\in\{1,\cdots,l\}$, we pick a representative $\mathring{p}_j\in\mathcal{O}'_j$ such that $\vartheta(\mathring{p}_j)<\vartheta^*$. We deduce that there is a $\vartheta'<\vartheta^*$ such that $\vartheta(\mathring{p}_j)<\vartheta'$ for all $j=1,\cdots,l$. Recall that we remarked above that when $0<\vartheta^*<\pi$, $G_{\vartheta*}$ is connected implies that it is path-connected. Therefore, by definition of ϑ^*, for $\mathring{p}_j$, $\mathring{p}_k$, there is a curve in G_{ϑ^*} connecting them. Modifying the curve as above, we see that we can connect $\mathring{p}_j$, $\mathring{p}_k$ by a curve in $G_{\vartheta_{j,k}}$ for some $\vartheta_{j,k}<\vartheta^*$. Do this for all pairs of $j,k=1,\cdots,l$, and define $\vartheta''=\max\{\frac{3}{4}\vartheta^*,\vartheta',\vartheta_{j,k}\}<\vartheta^*$. This ensures that all deformed curves connecting $\mathring{p}_j$ are in $G_{\vartheta''}$.

We claim that $G_{\vartheta''}$ has to be path connected, hence connected. To see this, take two points in $G_{\vartheta''}$. By our way of choosing $\mathcal{O}'_j$, we see that $\mathcal{O}'_j\cap i^{-1}(\{\vartheta\le\vartheta''\})$, if non-empty, is connected, so these two points can be connected to their representatives in $\mathcal{O}'_j\cap i^{-1}(\{\vartheta\le\vartheta''\})$ (using $\vartheta'\le\vartheta''$). Then since we have shown that for any $j,k=1,\cdots,l$, the points $\mathring{p}_j$, $\mathring{p}_k$ can be connected in $G_{\vartheta_{j,k}}\subset G_{\vartheta''}$, we see that $G_{\vartheta''}$ is indeed path-connected with $\vartheta''<\vartheta$. This ends the proof of Proposition A.4. □

Remark A.6. *Provided that we have shown that the map $P_{\Sigma;\mathbb{S}^2}\circ i\colon\mathbb{S}^2\to\mathbb{S}^2$ is a local diffeomorphism and a surjection, Step 2 can alternatively be done by noticing that a local diffeomorphism from $\mathbb{S}^2$ to $\mathbb{S}^2$ that is also surjective must be a covering map. Then, using that $\mathbb{S}^2$ is simply connected, one can deduce that the covering map can only be one-fold, hence an injection. Our proof provides an elementary argument that only requires point-set topology.*

A.1 Proof of Lemma A.5

We first illustrate the idea by assuming $k=2$, so $H_{\vartheta^*}\subset\mathcal{U}_1\cup\mathcal{U}_2$. Without loss of generality, we assume γ enters $\mathcal{U}_1$ first, and define $p_1\in\partial\mathcal{U}_1$ as the point of this first entry.[42] This implies that $p_1\notin H_{\vartheta^*}$, because otherwise it would be in $\mathcal{U}_2$ first. In the case when $\gamma(0)\in\mathcal{U}_1$, we define instead $p_1:=\gamma(0)$. We then define $q_1\in\partial\mathcal{U}_1$ to be the point where γ leaves $\mathcal{U}_1$ for the final time[43] (similarly, if $\gamma(1)\in\mathcal{U}_1$, we let $q_1:=\gamma(1)$ instead; in this case the process ends and there are no p_2 and q_2 defined below). While $p_1\notin H_{\vartheta^*}$, it is possible that $q_1\in H_{\vartheta^*}$. We now define p_2 as follows:

- If $q_1\in\mathcal{U}_2$ (which must be true if $q_1\in H_{\vartheta^*}$), we define $p_2=q_1$;

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[42] Strictly speaking, this should be described on the time interval $[0,1]$. We use the image point on G_{ϑ^*} for simplicity.

[43] This means that it is not entering $\mathcal{U}_1$ after q_1; we allow the possibility for it to intersect $\partial\mathcal{U}_1$.

- If $q_1 \notin \mathcal{U}_2$ (in this case $q_1 \notin H_{\vartheta^*}$), we define $p_2 \in \partial\mathcal{U}_2$ to be point where γ enters $\mathcal{U}_2$ for the first time after q_1 (if there is no such point, then end the process and there are no p_2 and q_2); by assumption, at p_2, γ has already left $\mathcal{U}_1$ for the final time, so $p_2 \notin \mathcal{U}_1$, and hence, $p_2 \notin H_{\vartheta^*}$.

We then define q_2 as the point where γ leaves $\mathcal{U}_2$ for the final time (similarly, if $\gamma(1) \in \mathcal{U}_2$, we define $q_2 = \gamma(1)$). By the assumption that γ has already left $\mathcal{U}_1$ for the final time, we see that $q_2 \notin H_{\vartheta^*}$. Also, γ will never be in $\mathcal{U}_1 \cup \mathcal{U}_2$, hence never intersect H_{ϑ^*}, after q_2.

Now by the local structure of $\mathcal{U}_1$ and $\mathcal{U}_2$, there exists a curve, lying in G_{ϑ^*}, and connecting p_1 and q_1 such that its only possible intersection with H_{ϑ^*} is q_1, and a curve connecting p_2 and q_2 that does not intersect H_{ϑ^*}. Therefore, if we replace the portion of γ between p_j and q_j ($j = 1, 2$) by such segments, we obtain a new curve which only possibly intersects H_{ϑ^*} at q_1. Since in this case $q_1 \in \mathcal{U}_2$, it is clear that one can further modify near q_1 so that the curve does not intersect H_{ϑ^*}. Therefore, there exists a curve lying entirely in G_{θ^*} and having the same endpoints as γ, but never hits H_{ϑ^*}. This finishes the proof with $k = 2$ in view of the compactness of the new curve.

For general k, we prove by induction. Assume that we have found $p_1, q_1, \cdots, p_j, q_j$ in time order on γ, and a subcollection $\{\mathcal{U}_1, \cdots, \mathcal{U}_j\}$ from $\{\mathcal{U}_1, \cdots, \mathcal{U}_k\}$ (with possibly a rearrangement of the indices), such that

- We have $p_1 \notin H_{\vartheta^*}$; it is possible that $q_{j'} \in H_{\vartheta^*}$ for some j' among $j' = 1, \cdots, j$, but for $j' \geq 2$ (when $j \geq 2$), if $p_{j'} \neq q_{j'-1}$, then $q_{j'-1} \notin H_{\vartheta^*}$ and $p_{j'} \notin H_{\vartheta^*}$;
- For all $j' \leq j$, the curve γ never intersects $\mathcal{U}_{j'}$ after $q_{j'}$;
- For all $j' \leq j$, there exists a curve $\gamma_{j'}$, having the same two endpoints as γ, only different from $\gamma_{j'-1}$ ($\gamma_0 := \gamma$) on the segments between $p_{j'}$ and $q_{j'}$. This new segment between $p_{j'}$ and $q_{j'}$ on $\gamma_{j'}$ is denoted by $\gamma_{j'}[p_{j'}; q_{j'}]$;
- The intersection $\gamma_j[p_{j'}; q_{j'}] \cap H_{\vartheta^*} \subset \{q_{j'}, q_{j'-1}\}$ (when $j' = 1$, only $\{q_1\}$) for all $j' = 1, \cdots, j$.

Following the same way of defining p_1 and q_1 above, we see that this holds for $j = 1$. We now define p_{j+1} in a similar way as we defined p_2 above:

- If q_j lies in some element in $\{\mathcal{U}_{j+1}, \cdots, \mathcal{U}_k\}$ (which must be true if $q_j \in H_{\vartheta^*}$; choose any one if there are multiple choices), without loss of generality denoted by $\mathcal{U}_{j+1}$, we define $p_{j+1} = q_j$;
- If q_j is not in any of $\{\mathcal{U}_{j+1}, \cdots, \mathcal{U}_k\}$, without loss of generality, we denote the next one γ enters by $\mathcal{U}_{j+1}$, and define $p_{j+1} \in \partial\mathcal{U}_{j+1}$ to be point of entry; this only happens when $q_j \notin H_{\vartheta^*}$. Also, this means that p_{j+1} is not in any among $\{\mathcal{U}_1, \cdots, \mathcal{U}_k\}$, so $p_{j+1} \notin H_{\vartheta^*}$.

Then, we define q_{j+1} as the point where γ (or equivalently, γ_j) leaves $\mathcal{U}_{j+1}$ for the final time (again, if $\gamma(1) \in \mathcal{U}_{j+1}$, define instead $p_{j+1} := \gamma(1)$ and end the process).

Note that it is possible that $q_{j+1} \in H_{\vartheta^*}$. As above, using the local structure of $\mathcal{U}_{j+1}$ we can find a curve segment connecting p_{j+1} and q_{j+1}, denoted by $\gamma_{j+1}[p_{j+1}; q_{j+1}]$ such that its only possible intersections with H_{ϑ^*} are q_{j+1} and, in the case $p_{j+1} = q_j$, the point q_j. We then replace the segment of γ_j between p_{j+1} and q_{j+1} by this curve segment, and denote the new curve by γ_{j+1}. One can verify that the induction assumptions now hold true with j replaced by $j+1$. This concludes the induction. The procedure will stop at $j = k'$ for some $k' \le k$, with γ does not intersect with any of $\{\mathcal{U}_1, \cdots, \mathcal{U}_k\}$ after $q_{k'}$, or $q_{k'} = \gamma(1)$. The curve $\gamma_{k'}$ satisfies

- It has the same two endpoints as γ, and is only different from γ on the segments between $p_{j'}$ and $q_{j'}$ for $j' \le k'$;
- The intersection $\gamma_{k'} \cap H_{\vartheta^*} \subset \{q_1, \cdots, q_{k'-1}\}$.

This shows that $\gamma(0)$ and $\gamma(1)$ can be connected by a curve $\gamma_{k'}$ that only possibly intersects with H_{ϑ^*} at $\{q_1, \cdots, q_{k'-1}\}$. Recall that if $q_{j'} \in H_{\vartheta^*}$, by our construction we have $q_{j'} \in \mathcal{U}_{j'+1}$, so clearly one can further modify the curve near $q_{j'}$ so that it does not intersect H_{ϑ^*} nearby. Since we have at most $k'-1$ such points, we can do this for finite times and obtain a curve $\tilde{\gamma}_{k'}$, lying in G_{ϑ^*} and connecting $\gamma(0)$ and $\gamma(1)$, that never intersects H_{ϑ^*}. By compactness we see that $\sup_{\tilde{\gamma}_{k'}} \vartheta < \vartheta^*$, which finishes the proof.

B Proof of the Immersion Criterion

For the benefit of the reader we provide below a short outline on the theory of focal points of null hypersurfaces; for a more complete treatment see [4].

In this part, it would be convenient to work with the foliation induced by the affine parameter of the null generators. Fix a smooth choice of L on S and consider the corresponding null cone $C(S)$. Define s by $L(s) = 1$, $s = 0$ on S. For a local coordinate (θ_p, φ_p) near p on S, one can then extend them near $\gamma(p)$ by $L(\theta_p) = L(\varphi_p) = 0$, which also defines ∂_{θ_p}, ∂_{φ_p} tangent to $\{s = \text{const}\}$ near each point, until the linear span of them becomes degenerate. The vector fields ∂_{θ_p}, ∂_{φ_p} are normal Jacobi fields in the sense that $[L, J] = 0$ for $J = \partial_{\theta_p}, \partial_{\varphi_p}$ ($L = \partial_s$), and they satisfy the Jacobi equation ($D_s := D_L$)

$$D_s^2 J + R(J, L)L = 0, \quad J|_S = \partial_{\theta_p}, \partial_{\varphi_p}, \quad D_s J|_S = D_J L.$$

The equation is linear when L is given, and the solution will be smooth if initially smooth.

Proposition B.1. *Fix a point $p \in S$. Let s^* be the first (minimal)*[44] *value of s' such that the span of ∂_{θ_p}, ∂_{φ_p} degenerates, then $\operatorname{tr}\chi \to -\infty$ towards this point along γ_p.*

Proof. We choose an orthonormal frame of S near p, denoted by $\{E_1, E_2\}$, and extend it as the Fermi frame by the equation (denote $E_4 = L$)

$$D_{E_4} E_a = -\zeta_a E_4, \tag{B.1}$$

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[44] This is towards the future; the treatment of the case towards the past is the same.

where $\zeta_a = g(D_{E_a}E_4, E_3)$ (with E_3 being the null companion of E_4 orthogonal to E_a). We have

$$E_4(E_i(s)) = [E_4, E_i]s = (D_{E_4}E_i - D_{E_i}E_4)(s) = (-\zeta_i E_4 - D_{E_i}E_4)s = -\chi(E_i, E_j)E_j(s),$$

so we see that if the time function s satisfies $E_i(s) = 0$ initially, then we always have $E_i(s) = 0$. Also,

$$E_4\, g(E_a, E_b) = g(-\zeta_a E_4, E_b) + g(E_a, -\zeta_b E_4) = 0, \quad E_4\, g(E_a, E_4) = g(-\zeta_a E_4, E_4) = 0.$$

Therefore, $\{E_a\}$ will always be an orthonormal frame orthogonal to E_4.

Since $X = \partial_{\theta_p}, \partial_{\varphi_p}$ are tangent to $\{s = \text{const}\}$, we write $X = X^a E_a$. The condition $[L, X] = 0$ implies

$$\frac{d}{ds}(X^a(s))E_a + X^a D_L E_a = X^a D_{E_a} L, \text{ hence } \frac{d}{ds}X^a(s) = \chi_{ab}X^b.$$

In particular, this determines the initial condition on $\frac{d}{ds}X^a(s)$.

The Jacobi equation now gives

$$\frac{d^2}{ds^2}X^a(s) = \alpha(E_a, E_b)X^b.$$

Note that E_a, α (defined under the frame $\{E_3, E_4, E_a\}$) are known quantities, so this is a linear equation of X^a. Therefore, X^a will be smooth when the initial data and spacetime are smooth.

Consider the deformation matrix M, defined by $X^a(s) = M^a_b(s)X^b(0)$. It satisfies

$$\frac{d}{ds}M^a_b(s) = \chi_{ac}M^c_b,$$

which implies

$$\frac{d}{ds}\big(\log\det M\big) = \operatorname{tr}\chi.$$

By assumption, we see that M becomes degenrate, i.e., $\det M \to 0^+$ along γ_p as $s \to s^*$. This implies that $\operatorname{tr}\chi \to -\infty$ towards the point. □

Therefore, if $\operatorname{tr}\chi$ is not diverging to $-\infty$ at any point, then M will always be a linear isomorphism. In this case, consider the map

$$i_\tau : S \to \Sigma_\tau,$$

where Σ_τ is a level hypersurface of a time function, and we define $i_\tau(p)$ to be the unique point of the intersection $\gamma_p \cap \Sigma_\tau$. The tangent map of i_τ is given by

$$\partial_{\theta_p}|_S \mapsto \partial_{\theta_p} - \frac{\partial_{\theta_p}\tau}{L(\tau)}L, \quad \partial_{\varphi_p}|_S \mapsto \partial_{\varphi_p} - \frac{\partial_{\varphi_p}\tau}{L(\tau)}L,$$

We know that $L(\tau) > 0$ when τ is a time function; it is then clear that we obtain two linearly independent vectors, so the tangent map is injective. Therefore, i_τ is an immersion. Since the map is also smooth in τ and L is transversal to Σ_τ, we also see that it gives an immersion from $[0, T] \times S \to \mathcal{M}$, $(\tau, p) \mapsto i_\tau(p)$.

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