ON THE GLOBAL REGULARITY OF WAVE MAPS IN THE CRITICAL SOBOLEV NORM

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ABSTRACT. We extend the recent result of T.Tao [6] to wave maps defined from the Minkowski space $\mathbf{R}^{n+1}$, $n \geq 5$, to a target manifold $\mathcal{N}$ which possesses a "bounded parallelizable" structure. This is the case of Lie groups, homogeneous spaces as well as the hyperbolic spaces $\mathbf{H}^N$. General compact Riemannian manifolds can be imbedded as totally geodesic submanifolds in bounded parallelizable manifolds, see [1], and therefore are also covered, in principle, by our result. Compactness of the target manifold, which seemed to play an important role in [6], turns out however to play no role in our discussion. Our proof follows closely that of [6] and is based, in particular, on its remarkable microlocal gauge renormalization idea.

1. INTRODUCTION

Let $\phi : \mathbf{R}^{n+1} \longrightarrow (\mathcal{N}, h)$ with $(\mathcal{N}, h)$ a Riemannian manifold of dimension N. Here $\mathbf{R}^{n+1}$ denotes the standard Minkowski space endowed with the metric $m = \mathrm{diag}(-1, 1, \dots, 1)$. We denote by ∇ the Levi-Civitta connection on $T\mathcal{N}$, the tangent bundle of $\mathcal{N}$, and by $\overline{\nabla}$ the induced connection on $\phi^*(T\mathcal{N})$. Recall that the pull-back bundle $\phi^*(T\mathcal{N}) = \cup_{x \in \mathbf{R}^{n+1}} \; \{x\} \times T_{\phi(x)}\mathcal{N}$ is a vector bundle over the Minkowski space $\mathbf{R}^{n+1}$. The induced metric on $\phi^*(T\mathcal{N})$ is defined by

$$< V, W >= V^a(\phi(x)) W^b(\phi(x)) < e_a, e_b >_h$$

where $V = V^a(\phi(x))\, e_a$, $W = W^b(\phi(x))\, e_b$ are sections of $\phi^*(T\mathcal{N})$ and e_a is a frame of vectorfields on $\mathcal{N}$. The induced connection $\overline{\nabla}$ is defined according to the rule

$$\overline{\nabla}_X V = \nabla_{\phi_* X} V$$

where $X \in T(\mathbf{R}^{n+1})$ and $V \in \phi^*(T\mathcal{N})$. A map $\phi : \mathbf{R}^{n+1} \longrightarrow (\mathcal{N}, h)$ is said to be a wave map if

$$m^{\alpha\beta} \overline{\nabla}_{\partial_\beta} \phi_*(\partial_\alpha) = 0.$$

In local coordinates $y^I, I = 1, \dots, N$ on $\mathcal{N}$ the wave maps equation takes the familiar form

$$\partial^\alpha \partial_\alpha \phi^I + \Gamma^I_{JK} \partial_\beta \phi^J \partial_\gamma \phi^K m^{\beta\gamma} = 0. \tag{1}$$

where Γ^I_{JK} are the Christoffel coefficients of the Levi-Civitta connection ∇ on $\mathcal{N}$ and $\phi^I, I = 1, \dots, N$ the components of the map ϕ in local coordinates on $\mathcal{N}$.

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Let $e_a = e_a^I \frac{\partial}{\partial y^I}$ be an orthonormal frame of vectorfields and $\omega^a = \omega_I^a dy^I$ be the corresponding dual basis of 1-forms $\omega^a(e_b) = \delta_b^a$. Since $h(e_a, e_b) = \delta_{ab}$ we infer that $h_{IJ} = \sum_a \omega_I^a \omega_J^a$. Define,

$$\phi_\alpha^a = \omega_I^a \partial_\alpha \phi^I \tag{2}$$

where ϕ^I are the components of the map ϕ relative to the local coordinates y^I on $\mathcal{N}$. Clearly, $\partial_\alpha \phi^I = e_a^I \phi_\alpha^a$. Given a function F on $\mathcal{N}$ we write $\partial_\alpha F(\phi) = \frac{\partial}{\partial y^I} F(\phi) \partial_\alpha \phi^I = \frac{\partial}{\partial y^I} F(\phi) e_d^I \phi_\alpha^d = F_{,d}(\phi) \phi_\alpha^d$ where $F_{,d} = e_d(F)$.

We easily check that the functions $\phi_\alpha^a =< \partial_\alpha \phi, e_a >$ associated to a wave map ϕ verify[1] the following divergence-curl system,

$$\begin{aligned} \partial_\beta \phi_\alpha^a - \partial_\alpha \phi_\beta^a &= C_{bc}^a \; \phi_\alpha^b \, \phi_\beta^c & (3) \\ \partial^\alpha \phi_\alpha^a &= -\Gamma_{bc}^a \, \phi_\beta^b \, \phi_\gamma^c \, m^{\beta\gamma} & (4) \end{aligned}$$

where, C_{bc}^a and Γ_{bc}^a are respectively the structure and connection coefficients of the frame,

$$\begin{aligned} [e_b, e_c] &= C_{bc}^a e_a \\ \nabla_{e_b} e_c &= \Gamma_{bc}^a e_a \end{aligned}$$

In view of the formula $[e_b, e_c] = \nabla_{e_b} e_c - \nabla_{e_c} e_b$ we infer that,

$$C_{bc}^a = \Gamma_{bc}^a - \Gamma_{cb}^a.$$

Since the frame e_a is orthonormal we have $\Gamma_{bc}^a =< \nabla_{e_b} e_c \,;\, e_a >= - < e_c \,; \nabla_{e_b} e_a >$ and therefore

$$\Gamma_{bc}^a = -\Gamma_{ba}^c. \tag{5}$$

Also,

$$\Gamma_{bc}^a = \frac{1}{2}\Big(C_{bc}^a + C_{ac}^b + C_{ab}^c \Big).$$

Definition: *We say that a Riemannian manifold $\mathcal{N}$ has a " bounded parallelizable" structure if there exists an orthonormal frame $(e_a)_{a=1}^N$ on $\mathcal{N}$ relative to which the structure coefficients C_{bc}^a and their frame derivatives $C_{bc\,,\,d_1 d_2 \ldots d_k}^a$ are uniformly bounded on $\mathcal{N}$.*

Remark 1.1. There are plenty of examples of bounded parallelizable manifolds. To start with on any Lie group we can construct an orthonormal basis of left invariant vectorfields e_a relative to which the structure constants C_{bc}^a are constant[2]. The constant negative curvature manifolds $\mathbf{H}^N$, $N > 2$ are bounded parallelizable. Moreover $\mathbf{H}^2$, i.e. the hyperbolic plane, is a Lie group, see the relevant discussion in section 3.1 of [1]. In addition any compact Riemannian manifold can be embedded as a totally geodesic submanifold in a bounded parallelizable Riemannian manifold, see [1].

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[1] Our description of wave maps expressed relative to an ortonormal frame follows closely that of [1]. A similar formalism has been used earlier by Helein [2] in his well known work on 2-dimensional weak harmonic maps

[2] We refer to these as "constant parallelizable".

Proposition 1.2. *Let $\mathcal{N}$ be a Riemannian manifold and $\phi : \mathbf{R}^{n+1} \longrightarrow \mathcal{N}$ a wave map. The 1-forms $\phi^a_\alpha =< \partial_\alpha \phi, e_a >$ verify the equations,* (3),(4) *as well as the system of wave equations,*

$$\Box \Phi = -2R_\mu \cdot \partial^\mu \Phi + E \tag{6}$$

with $\Phi = (\phi^a_\alpha)$, $R_\mu = (R^a_{b\mu})^N_{a,b=1}$ and $R^a_{b\mu} = \Gamma^a_{cb}\phi^c_\mu$. The components of $E = (E^a_\alpha)$ are homogeneous polynomial of degree three relative to the components of $\Phi = (\phi^a_\alpha)$ with coefficients depending only on the structure functions C^a_{bc} and their derivatives $C^a_{bc,d}$ with respect to the frame.

Remark 1.3. It is essential to remark that the matrices R_μ are antisymmetric i.e.

$$R^a_{b\mu} = -R^b_{a\mu} \tag{7}$$

This is an immediate consequence of (5). This shows that the well known "Helein trick" of antisymmetrizing the form of the wave maps equations in the particular case when $\mathcal{N}$ is a standard sphere, a trick which plays a fundamental role in [6], is due in fact to a general feature of the connection coefficients on *any Riemannian manifold*, expressed relative to *orthonormal frames.*

Proof :

Differentiating (3) and using (4) we derive:

$$\begin{aligned}
\partial^\beta \partial_\beta \phi^a_\alpha &= -\partial_\alpha(\Gamma^a_{bc}\,\phi^b_\mu\,\phi^c_\nu\, m^{\mu\nu}) + \partial^\beta(C^a_{bc}\,\phi^b_\alpha\,\phi^c_\beta) \\
&= m^{\mu\nu}\Big(-\Gamma^a_{bc}\,(\partial_\alpha\phi^b_\mu)\,\phi^c_\nu - \Gamma^a_{bc}\,\phi^b_\mu(\,\partial_\alpha\phi^c_\nu\,) + C^a_{bc}(\,\partial_\mu\phi^b_\alpha)\,\phi^c_\nu\Big) \\
&+ C^a_{bc}\,\phi^b_\alpha(\,\partial^\beta\phi^c_\beta\,) - \partial_\alpha(\Gamma^a_{bc})\,\phi^b_\mu\,\phi^c_\nu\, m^{\mu\nu} + \partial^\beta(C^a_{bc})\,\phi^b_\alpha\,\phi^c_\beta
\end{aligned}$$

Setting $A^a_\alpha = -m^{\mu\nu}\Big(\Gamma^a_{bc}\,(\partial_\alpha\phi^b_\mu)\,\phi^c_\nu + \Gamma^a_{bc}\,\phi^b_\mu(\,\partial_\alpha\phi^c_\nu\,) - C^a_{bc}(\,\partial_\mu\phi^b_\alpha)\,\phi^c_\nu\Big)$ and using (3) we write

$$\begin{aligned}
A^a_\alpha &= -m^{\mu\nu}\Big(\Gamma^a_{bc}\,(\partial_\mu\phi^b_\alpha + C^b_{mn}\phi^m_\alpha\phi^n_\mu)\,\phi^c_\nu + \Gamma^a_{bc}\,\phi^b_\mu(\partial_\nu\phi^c_\alpha + C^c_{mn}\phi^m_\alpha\phi^n_\nu) \\
&- C^a_{bc}(\,\partial_\mu\phi^b_\alpha)\,\phi^c_\nu\Big) \\
&= -m^{\mu\nu}\Big(\Gamma^a_{bc} + \Gamma^a_{cb} - C^a_{bc}\Big)\phi^c_\nu\partial_\mu\phi^b_\alpha + m^{\mu\nu}\Gamma^a_{bc}\Big(C^b_{mn}\phi^m_\alpha\phi^n_\mu\phi^c_\nu + C^c_{mn}\phi^m_\alpha\phi^n_\nu\phi^b_\mu\Big)
\end{aligned}$$

or since $\Gamma^a_{bc} + \Gamma^a_{cb} - C^a_{bc} = \Gamma^a_{bc} + \Gamma^a_{cb} - (\Gamma^a_{bc} - \Gamma^a_{cb}) = 2\Gamma^a_{cb}$,

$$A^a_\alpha = -2m^{\mu\nu}\Gamma^a_{cb}\phi^c_\nu\partial_\mu\phi^b_\alpha + m^{\mu\nu}\Gamma^a_{bc}\Big(C^b_{mn}\phi^m_\alpha\phi^n_\mu\phi^c_\nu + C^c_{mn}\phi^m_\alpha\phi^n_\nu\phi^b_\mu\Big)$$

Therefore,

$$\begin{aligned}
\partial^\beta\partial_\beta\phi^a_\alpha &= -2m^{\mu\nu}\Gamma^a_{cb}\phi^c_\nu\partial_\mu\phi^b_\alpha + m^{\mu\nu}\Gamma^a_{bc}\Big(C^b_{mn}\phi^m_\alpha\phi^n_\mu\phi^c_\nu + C^c_{mn}\phi^m_\alpha\phi^n_\nu\phi^b_\mu\Big) \\
&- C^a_{bc}\,\phi^b_\alpha(\,\Gamma^c_{mn}\phi^m_\mu\phi^n_\nu m^{\mu\nu}\,) - \Gamma^a_{bc,d}\,\phi^b_\mu\,\phi^c_\nu\phi^d_\alpha\, m^{\mu\nu} + C^a_{bc,d}\,\phi^b_\alpha\,\phi^c_\beta\phi^d_\beta
\end{aligned}$$

Finally we write

$$\Box\phi^a_\alpha = -2\Gamma^a_{cb}\,\phi^c_\nu\,\partial_\mu\phi^b_\alpha m^{\mu\nu} + E^a_\alpha$$

where $E = E^a_\alpha$ denote the terms above which are cubic in $\Phi = (\phi^a_\alpha)$ with coefficients depending only on the structure functions C^a_{bc} and their derivatives $C^a_{bc,d}$ with respect to the frame. This is precisely the equation (6). ∎

We study the evolution of wave maps subject to the initial value problem

$$\phi(0) = \varphi, \qquad \partial_t \phi(0) = \psi = \psi^a_0 e_a \tag{8}$$

φ is an arbitrary smooth map defined from $\mathbf{R}^n$ with values in $\mathcal{N}$ and $\psi = \psi^a_0 e_a$ and arbitrary smooth map from $\mathbf{R}^{n+1}$ to $T N$. Let $\varphi^a_i = < \partial_i \varphi, e_a >$.

Definition 1.4. We shall say that the initial data $\phi[0] = (\varphi, \psi)$ belongs to the Sobolev space $\dot{H}^s(\mathbf{R}^n)$, resp. $H^s(\mathbf{R}^n)$, if all components φ^a_i, ψ^a_i belong to the space $\dot{H}^{s-1}(\mathbf{R}^n)$, resp. $H^{s-1}(\mathbf{R}^n)$. We write

$$\|\phi[0]\|_{\dot{H}^s} = \sum_{a,i} \left(\|\varphi^a_i\|_{\dot{H}^{s-1}} + \|\psi^a_i\|_{\dot{H}^{s-1}} \right)$$

and similarly for $\|\phi[0]\|_{H^s}$.

We are now ready to state our main theorem.

Main Theorem *Let $\mathcal{N}$ be a Riemannian manifold endowed with a bounded parallelizable structure. Assume $n \geq 5$ and that the initial data $\phi[0] = (\varphi, \psi = \psi^a_i e_a)$ is in H^s for some $\frac{n}{2} < s$. We make also the critical smallness assumption:*

$$\|\phi[0]\|_{\dot{H}^{\frac{n}{2}}} \leq \varepsilon$$

Then the wave map ϕ with initial data $\phi[0]$ can be uniquely continued in H^s norm globally in time.

Our theorem provides an extension of the result in [6] from the case when the target manifold is a standard sphere to that of bounded, parallelizable manifolds. The restriction on the dimension, $n \geq 5$, is the same as in [6]; this allows us to rely only on Strichartz estimates. The dimensional restriction, for the case of the standard sphere, was removed in [7] with the additional help of bilinear estimates[3], $H^{s,\theta}$ spaces, and the refined methods of [8]. Even in light of [7] the extension of our result to two dimensions does not seem to be straightforward[4].

The proof of the Main Theorem relies on a local well-posedness result in H^s, $s > \frac{n}{2}$. We state the precise result below:

Theorem 1.5. *Assume that the initial data $\phi[0] \in H^s(\mathbf{R}^n)$ for some $s \geq s_0 > \frac{n}{2}$. There exists a $T > 0$, depending only on the size of $\|\phi[0]\|_{H^{s_0}}$, and a unique solution ϕ of the system (3) , (4) defined on the slab $[0, T] \times \mathbf{R}^n$ verifying,*

$$\|\phi[t]\|_{H^s} \leq C \|\phi[0]\|_{H^s}$$

for all $t \in [0, T]$ and C a constant depending only on T and s, $s_0 - \frac{n}{2}$ and n.

[3]These were used to take advantage of the presence of the special null quadratic form $Q_0(u, v) = m^{\alpha\beta} \partial_\alpha u \partial_b v$ in the special expression of wave maps to the standard sphere used in [6], [7].

[4]This is due to the fact that one needs to treat other other types of null quadratic forms than Q_0. Also, the fact that we use a wave equation, see (6), in Φ corresponding to the first derivatives of the map rather than the map itself, adds additional complications.

Strictly speaking such a sharp local existence result for div-curl systems of type (3), (4) does not exist in the literature. Nevertheless we are confident that the methods discussed in [4] in connection to a special model problem related to (3),(4)(see also [3]) do apply[5]. A proof of this fact will appear elsewhere. Alternatively we can avoid Theorem 1.5 and rely instead on the known sharp local existence result for the Wave Maps system written in local coordinates (1), see [4] and the references therein. Indeed any H^s data[6], $s > \frac{n}{2}$, is also H^s with respect to local coordinates on $\mathcal{N}$. Using the finite propagation speed property of wave equations we can therefore construct a local in time H^s solution for the system (3)-(4) which is unique, as a solution of (1), in any local chart on $\mathcal{N}$. This is the solution for which the Main Theorem applies.

For simplicity we shall present the proof of the Main Theorem in the particular case of constant parallelizable target manifolds. The general case complicates matters only in so far as the number of terms we need to treat is larger, there are however no conceptual differences. We shall thus assume that the C^a_{bc} are constant and indicate whenever needed what additional steps are required to treat the general case.

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2. Notation, Strichartz estimates and main proposition

We use the Littlewood-Paley notation of [6]. Thus, for a function $\phi(t,x)$ we denote the projections $P_{\le k}\phi(t,x) = \int e^{ix\cdot\xi}\chi(2^{-k}\xi)\hat{\phi}(t,\xi)d\xi$ where $\hat{\phi}(t,\xi)$ is the space Fourier transform of ϕ and $\chi(\xi) = \eta(\xi) - \eta(2\xi)$ with η a non-negative smooth bump function supported on $|\xi| \le 2$ and equal to 1 on the ball $|\xi| \le 1$. Therefore $\chi(\xi)$ is supported in $\{\frac{1}{2} \le |\xi| \le 2\}$ and $\sum_{k\in\mathbf{Z}}\chi(2^{-k}\xi) = 1$ for all $\xi \ne 0$. We also define $P_k = P_{\le k} - P_{<k}$. Also for any interval $I \subset \mathbf{Z}$ we define P_I in an obvious fashion, see [6].

Notation: *We shall frequently use the notation $A \lesssim B$ to denote $A \le cB$ for some constant $c > 0$ which does not depend on any of the important parameters used in our estimates.*

Following [6] we introduce the notation

$$\|\Phi\|_{S_k} = \sup_{q,r\in\mathcal{A}} 2^{k(\frac{1}{q}+\frac{n}{r}-1)}\left(\|\Phi\|_{L^q_tL^r_x} + 2^{-k}\|\partial_t\Phi\|_{L^q_tL^r_x}\right) \tag{9}$$

where $\mathcal{A} = \{(q,r)/2 \le q,r \le \infty,\ \frac{1}{q} + \frac{n-1}{2r} \le \frac{n-1}{4}\}$ is the set of admissible Strichartz exponents. Recall that,

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[5]In [3] one proves a related result for a model problem in dimension $n = 3$. The proof was vastly simplified and extended to all dimensions $n \ge 3$ in [5] and [4]. The higher dimensional case is in fact a lot simpler.

[6]According to the global definition 1.4.

Theorem 2.1. *For any fixed integer k and $\phi(t,x)$ a function on $\mathbf{R}\times\mathbf{R}^n$ such that the support of $\hat{\phi}(t,\xi)$ is included in the dyadic region $2^{k-1}\le|\xi|\le 2^{k+1}$ we have the estimate,*

$$\|\phi\|_{S_k}\lesssim\|\phi(0)\|_{H^{\frac{n-2}{2}}}+\|\partial_t\phi(0)\|_{H^{\frac{n-4}{2}}}+2^{k\frac{n-4}{2}}\|\Box\phi\|_{L^1_tL^2_x}.$$

In what follows we recall the definition of frequency envelope given in [6].

Definition 2.2. A frequency envelope is an l^2 sequence $c=(c_k)_{k\in\mathbf{Z}}$ verifying

$$c_k\lesssim 2^{\sigma|k-k'|}c_{k'} \tag{10}$$

for all $k,k'\in\mathbf{Z}$. Here σ is a fixed positive constant; as in [6] we take $0<\sigma<\frac{1}{2}$. In addition we shall also need $0<\sigma<\frac{n-4}{4}$ and $0<\sigma<\frac{n-3}{4(n-1)}$.

We say that the $\dot{H}^s$ norm of a function f on $\mathbf{R}^n$ lies underneath an envelope c if, for all $k\in\mathbf{Z}$, $\|P_kf\|_{\dot{H}^s}\le c_k$. We shall write $f<<_s c$ or simply $f<<c$ when there is no danger of confusion. Recall, see [6] section 3, that if $\|f\|_{\dot{H}^s}\le\varepsilon$ then there exists an envelope $c\in l^2$ such that $\|c\|_{l^2}\lesssim\varepsilon$ and $f<<_s c$. Indeed we can simply take, $c_k=\sum_{k'\in\mathbf{Z}}2^{-\sigma|k-k'|}\|P_k\|_{\dot{H}^s}$.

Definition 2.3. Fix $0<\sigma<\min(\frac{1}{2},\frac{n-4}{4},\frac{n-3}{4(n-1)})$ and c a frequency envelope. We say that the initial data $\phi[0]=\Big(\phi(0)=\varphi,\partial_t\phi(0)=\psi=\psi^a_0e_a\Big)$ lies underneath c if, relative to our frame e_a we have for all $k\in\mathbf{Z}$,

$$\|P_k\phi[0]\|_{\dot{H}^{\frac{n}{2}}}\le c_k.$$

We shall use the short hand notation $\phi[0]<<c$.

Following the same arguments as in section 3 of [6] we can reduce the proof of our main theorem to the following[7]:

Proposition 2.4. *(Main Proposition) Let c be a frequency envelope[8] with $\|c\|_{l^2}\le\varepsilon$, $0<T<\infty$ and $\Phi=(\phi^a_\alpha)$ verify the equations (3), (4), and therefore also (6). Assume that, according to definition 2.3, the initial data verifies the smallness condition $\phi[0]<<c$. Assume also the bootstrap assumption,*

$$\|P_k\Phi\|_{S_k([0,T]\times\mathbf{R}^n)}\le 2Cc_k \tag{11}$$

for all $k\in\mathbf{Z}$. Then in fact, for sufficiently small ε, and all $k\in\mathbf{Z}$,

$$\|P_k\Phi\|_{S_k([0,T]\times\mathbf{R}^n)}\le Cc_k. \tag{12}$$

Remark 2.5. We may assume that ε is small enough, so that $C^2\varepsilon\le\frac{1}{2}$.

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[7] Our main proposition below corresponds to Proposition 3.3 in [6]. The reduction relies on Theorem 1.5.

[8] verifying (10) with $\sigma<\min(\frac{1}{2},\frac{n-4}{4},\frac{n-3}{4(n-1)})$.

Returning to the definition (9) we make explicit all the useful estimates contained in the bootstrap assumption (11),

$$\begin{aligned}
\|P_k\Phi\|_{L_t^2 L_x^{\frac{2(n-1)}{n-3}}} + 2^{-k}\|\partial_t P_k\Phi\|_{L_t^2 L_x^{\frac{2(n-1)}{n-3}}} &\leq 2^{\frac{k}{2}-\frac{nk}{2}+\frac{nk}{n-1}}\cdot(2Cc_k)\\
\|P_k\Phi\|_{L_t^2 L_x^4} + 2^{-k}\|\partial_t P_k\Phi\|_{L_t^2 L_x^4} &\leq 2^{\frac{k}{2}-\frac{nk}{4}}\cdot(2Cc_k)\\
\|P_k\Phi\|_{L_t^2 L_x^{n-1}} + 2^{-k}\|\partial_t P_k\Phi\|_{L_t^2 L_x^{n-1}} &\leq 2^{\frac{k}{2}-\frac{nk}{n-1}}\cdot(2Cc_k)\\
\|P_k\Phi\|_{L_t^2 L_x^\infty} + 2^{-k}\|\partial_t P_k\Phi\|_{L_t^2 L_x^\infty} &\leq 2^{\frac{k}{2}}\cdot(2Cc_k)\\
\|P_k\Phi\|_{L_t^\infty L_x^2} + 2^{-k}\|\partial_t P_k\Phi\|_{L_t^\infty L_x^2} &\leq 2^{\frac{k}{2}(2-n)}\cdot(2Cc_k)\\
\|P_k\Phi\|_{L_t^\infty L_x^\infty} + 2^{-k}\|\partial_t P_k\Phi\|_{L_t^\infty L_x^\infty} &\leq 2^{k}\cdot(2Cc_k)\\
\|P_k\Phi\|_{L_t^4 L_x^{2(n-1)}} + 2^{-k}\|\partial_t P_k\Phi\|_{L_t^4 L_x^{2(n-1)}} &\leq 2^{\frac{3k}{4}-\frac{nk}{2(n-1)}}\cdot(2Cc_k)
\end{aligned}$$

Lemma 2.6. *The assumptions* (11) *imply*

$$\|\Box P_k\Phi\|_{L_t^2 L_x^{n-1}} \leq 2^{k(2+\frac{1}{2}-\frac{n}{n-1})}Cc_k \tag{13}$$

We prove this estimate at the end of section 3.

In view of the scale invariance of both our equations and the smallness condition $\phi[0] << c$ it suffices to prove (12) for $k=0$. Let $\Psi = P_0\Phi$. We need to prove that,

$$\|\Psi\|_{S_0([0,T]\times\mathbf{R}^n)} \leq Cc_0 \tag{14}$$

To prove (14) we would like to apply Theorem 2.1 to the equation obtained by applying the projection P_0. to (6) i.e.,

$$\Box\Psi = P_0(R_\mu\cdot\partial^\mu\Phi + E).$$

A straightforward application of the Strichartz inequalities will not work however. Indeed according to Theorem 2.1

$$\|\Psi\|_{S_0} \leq c_0 + \|P_0(R_\mu\cdot\partial^\mu\Phi + E)\|_{L_t^1 L_x^2}$$

The cubic term E presents no difficulty, the problem comes up when we try to estimate $P_0(R_\mu\cdot\partial^\mu\Phi)$ more precisely the part of it which corresponds to the interaction between low frequencies of R and frequencies of Φ comparable to those of Ψ. More precisely the most dangerous terms are of the form $\tilde{R}\cdot\partial\Psi$ with $\tilde{R} = P_{\leq -10}R$. To estimate $\|\tilde{R}\cdot\partial\Psi\|_{L_t^1 L_x^2}$ relative to the available Strichartz norms we are forced to take Ψ in the energy norm $L_t^\infty L_x^2$. This leaves us with $\tilde{R}$ in the norm $L_t^1 L_x^\infty$ for which we don't have any Strichartz estimates. It is precisely this difficulty which led Tao to introduce his remarkable renormalization idea which we reproduce below in section 5.

The organization of the paper follows closely that of [6]. In the next section we reduce the proof of the main proposition to estimates for the linearized equation:

$$\Box\Psi = -2\tilde{R}_\mu\cdot\partial^\mu\Psi.$$

This corresponds to isolating the worst part of $P_0(R_\alpha \cdot \partial^\alpha \Phi)$ to which we have alluded above. In section 4, which represents the main contribution of this paper, we show how to replace the term $\tilde{R}_\mu$ by the perfect derivative $\partial_\mu \tilde{\Delta}$ of an antisymmetric potential $\tilde{\Delta}$. This fact plays a crucial role in carrying out Tao's renormalization procedure in section 5.

We shall use, throughout the paper, Tao's convention to call an acceptable error any function, or matrix valued function, F on $[0,T] \times \mathbf{R}^n$ such that

$$\|F\|_{L^1_t L^2_x([0,T]\times \mathbf{R}^n)} \le C^3 \varepsilon c_0 \tag{15}$$

3. Reduction to a linear equation

Proposition 3.1. *The matrix valued function $P_0\Phi = \Psi$ verifies the equation*

$$\Box \Psi = -2\tilde{R}_\mu \cdot \partial^\mu \Psi + \textit{error} \tag{16}$$

where $\tilde{R}_\mu = P_{\le -10} R_\mu = \Gamma \cdot \tilde{\Phi}_\mu$ and $\tilde{\Phi}_\alpha = P_{\le -10}\Phi_\alpha$. Here "error" refers to an acceptable error term in the sense of (15)*.*

Remark 3.2. Written in components $\Psi = (\psi^a_\alpha)$ with $\psi^a_\alpha = P_0 \phi^a_\alpha$ the equation (16) has the form

$$\Box \psi^a_\alpha = -2\tilde{R}^a_{b\nu} \cdot \partial_\mu \psi^b_\alpha m^{\mu\nu} + \text{error},$$

where $\tilde{R}^a_{b\nu} = \Gamma^a_{bc}\tilde{\phi}^c_\nu$ and $\tilde{\phi}^c_\nu = P_{\le -10}\phi^c_\nu$. Observe that the $N \times N$ matrices $\tilde{R}_\mu$ are antisymmetric i.e. $\tilde{R}^t_\mu = -\tilde{R}_\mu$.

Proof : We start with the equation (6) to which we apply the projection P_0. Therefore,

$$\Box \Psi = P_0(R_\mu \cdot \partial^\mu \Phi + E)$$

The proof of Proposition 3.1 is an immediate consequence of the following Lemmas

Lemma 3.3. *We have,*

$$P_0(R_\mu \cdot \partial^\mu \Phi) = \tilde{R}_\mu \cdot \partial^\mu \Psi + \textit{error}$$

where $\tilde{R}_\mu = P_{\le -10} R_\mu$.

Lemma 3.4. *The term $P_0 E$ is an acceptable error term.*

■

We sketch below the proofs of Lemmas 3.3 and 3.4.

Proof of Lemma 3.3 We start by decomposing $R_\mu = \sum_k P_k R_\mu = \sum_k R_{\mu,k}$ and $\Phi = \sum_k P_k \Phi = \sum \Phi_k$. Thus,

$$P_0(R_\mu \cdot \partial^\mu \Phi) = P_0(E_1 + E_2 + E_3 + E_4 + E_5 + E_6)$$

where,

$$\begin{aligned}
E_1 &= \sum_{\max(k_1,k_2)>10\,,\,|k_1-k_2|\le 5} R_{\mu,k_1}\cdot\partial^\mu\Phi_{k_2}\\
E_2 &= \sum_{\max(k_1,k_2)>10\,,\,|k_1-k_2|>5} R_{\mu,k_1}\cdot\partial^\mu\Phi_{k_2}\\
E_3 &= (P_{\le -10}R_\mu)\cdot\partial^\mu P_{\le -10}\Phi\\
E_4 &= (P_{(-10,10)}R_\mu)\cdot\partial^\mu P_{(-10,10)}\Phi\\
E_5 &= \tilde{R}_\mu\cdot\partial^\mu P_{(-10,10)}\Phi\\
E_6 &= (P_{(-10,10)}R_\mu)\partial^\mu P_{<-10}\Phi
\end{aligned}$$

Recall that the matrices R_μ are products between the constant matrices[9] Γ and Φ. Thus each P_kR_μ can be estimated in the same way as $P_k\Phi=\Phi_k$ according to (11). Using (11) and the envelope property (10),

$$\begin{aligned}
\|E_1\|_{L^1_tL^2_x} &\le \sum_{\max(k_1,k_2)>10\,,\,|k_1-k_2|\le 5} \|R_{\mu,k_1}\|_{L^2_tL^4_x}\cdot\|\partial^\mu\Phi_{k_2}\|_{L^2_tL^4_x}\\
&\lesssim C^2\sum_{k\ge 5}2^k2^{k-\frac{nk}{2}}c_k^2 \lesssim C^2c_0^2\sum_{k\ge 5}2^k\,2^{k-\frac{nk}{2}}\,2^{2\sigma k}\\
&\lesssim C^2\,c_0^2\sum_{k\ge 5}2^{k(2+2\sigma-\frac{n}{2})}\lesssim C^2\,c_0^2
\end{aligned}$$

provided that $\sigma<\frac{n-4}{4}$.

Clearly $P_0E_2=0, P_0E_3=0$. The term E_4 is easy to estimate; it contains only a finite number of terms,

$$\|E_4\|_{L^1_tL^2_x}\le\|P_{(-10,10)}R\|_{L^2_tL^4_x}\|\partial P_{(-10,10)}\Phi\|_{L^2_tL^4_x}\lesssim C^2c_0^2.$$

For E_6 we write,

$$\begin{aligned}
\|E_6\|_{L^1_tL^2_x} &\lesssim \|\partial P_{\le -10}\Phi\|_{L^2_tL_x^{\frac{2(n-1)}{n-3}}}\|\partial P_{(-10,10)}\Phi\|_{L^2_tL_x^{n-1}}\\
&\lesssim C^2c_0\sum_{k\le -10}2^{k(\frac{3}{2}+\frac{1}{n-1})}c_k\\
&\lesssim C^2c_0^2\sum_{k\le -10}2^{k(\frac{3}{2}+\frac{1}{n-1}-\sigma)}\lesssim C^2c_0^2
\end{aligned}$$

It remains to consider the term $P_0E_5=P_0\left(\tilde{R}_\mu\cdot\partial^\mu P_{(-10,10)}\Phi\right)$. We use the standard commutator inequality for functions f,g in $\mathbf{R}^n$, see Lemma 4.3 in [6],

$$\|P_0(fg)-fP_0g\|_{L^r}\lesssim\|\nabla f\|_{L^p}\|g\|_{L^q},$$

[9] In the general case of a bounded parallelizable manifold one has to take into account the additional commutator terms. The commutators generate additional powers of Φ and therefore can be treated as easy error terms.

which holds for all $1 \le r, p, q \le \infty$, such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Taking $r = 2$, $p = n-1$ and $q = \frac{2(n-1)}{n-3}$ and proceeding precisely as for E_6 we derive,

$$\begin{aligned}\|P_0\big(\tilde{R}_\mu \cdot \partial^\mu P_{(-10,10)}\Phi\big) - \tilde{R}_\mu \cdot \partial^\mu \Psi\|_{L^1_t L^2_x} &\lesssim \|\partial \tilde{R}_\mu\|_{L^2_t L^{n-1}_x} \|\partial P_{(-10,10)}\Phi\|_{L^2_t L_x^{\frac{2(n-1)}{n-3}}} \\ &\lesssim C^2 c_0^2 \sum_{k \le -10} 2^{k(\frac{3}{2} + \frac{1}{n-1} - \sigma)} \lesssim C^2 c_0^2\end{aligned}$$

as desired. ■

Proof of Lemma 3.4 For simplicity we shall assume that the structure coefficients C^a_{bc} are constant. The general case, which is only slightly more involved, requires only their boundedness as well as that of their frame derivatives. We may thus assume that E has the form Φ^3. We split

$$\begin{aligned}P_0(\Phi^3) &\approx 3E_1 + E_2 + 3E_3 \\ E_1 &= P_0\Big(\sum_{k_1, k_2 > 5;\ |k_1 - k_2| \le 2;\ k_3 < 5} \Phi_{k_1}\Phi_{k_2}\Phi_{k_3}\Big) \\ E_2 &= P_0\Big(\sum_{k_1, k_2, k_3 > 5;\ |k_1 - k_2| \le 2;\ |k_2 - k_3| \le 2} \Phi_{k_1}\Phi_{k_2}\Phi_{k_3}\Big) \\ E_3 &= P_0\Big(\sum_{k_1, k_2 \le 5;\ |k_3| < 5} \Phi_{k_1}\Phi_{k_2}\Phi_{k_3}\Big)\end{aligned}$$

We have,

$$\begin{aligned}\|E_1\|_{L^1_t L^2_x} &\lesssim \big(\sum_{k > 5} \|\Phi_k\|^2_{L^2_t L^4_x}\big) \cdot \big(\sum_{k_3 \le 5} \|\Phi_{k_3}\|_{L^\infty_t L^\infty_x}\big) \\ &\lesssim C^3 c_0 \varepsilon^2 \sum_{k > 5} 2^{k(1 - \frac{n}{2} + \sigma)} \cdot \sum_{k_3 \le 5} 2^{k_3} \lesssim C^3 c_0 \varepsilon^2\end{aligned}$$

provided that $\sigma < \frac{n-2}{2}$. Now,

$$\begin{aligned}\|E_2\|_{L^1_t L^2_x} &\lesssim \sum_{k > 5} \|\Phi_k\|^2_{L^2_t L^4_x} \cdot \|\Phi_k\|_{L^\infty_t L^\infty_x} \\ &\lesssim C^3 c_0 \varepsilon^2 \sum_{k > 5} 2^{k(1 - \frac{n}{2} + \sigma)} \cdot 2^k \lesssim C^3 c_0 \varepsilon^2\end{aligned}$$

provided that $\sigma < \frac{n-4}{2}$. Finally,

$$\begin{aligned}\|E_3\|_{L^1_t L^2_x} &\lesssim \big(\sum_{k_1, k_2 \le 5} \|\Phi_{k_1}\|_{L^2_t L^\infty_x} \cdot \|\Phi_{k_2}\|_{L^2_t L^\infty_x}\big)\big(\sum_{|k_3| \le 5} \|\Phi_{k_3}\|_{L^\infty_t L^2_x}\big) \\ &\lesssim C^3 c_0 \varepsilon^2 \big(\sum_{k < 5} 2^{\frac{k}{2}}\big)^2 \lesssim C^3 c_0 \varepsilon^2\end{aligned}$$

■

In the remaining part of this section we shall sketch the proof of Lemma 2.6. By scale invariance it suffices to prove (13) for $k = 0$. In other words we have to prove,

$$\|\Box \Psi\|_{L^2_t L^{n-1}_x} \le C c_0 \tag{17}$$

According to equation (6) it suffices to prove that $\|P_0(R_\mu\cdot\partial^\mu\Phi+E)\|_{L^2_tL^{n-1}_x}\le Cc_0$. Once again, the cubic term P_0E is easier to treat and we concentrate on $P_0(R_\mu\cdot\partial^\mu\Phi)$. Observe that $R_\mu\cdot\partial^\mu\Phi=\partial^\mu(R_\mu\cdot\Phi)-\partial^\mu R_\mu\cdot\Phi$. Recall that $\Phi=(\phi^a_\alpha)$ and $R_\mu=(R^a_{b\mu})$, where $R^a_{b\mu}=\Gamma^a_{cb}\phi^c_\mu$. According to the equation (4) we have[10]

$$\partial^\mu R^a_{b\mu}=\partial^\mu(\Gamma^a_{cb}\phi^c_\mu)=\Gamma^a_{cb}\partial^\mu\phi^c_\mu=-\Gamma^a_{cb}\Gamma^c_{de}\phi^d_\beta\phi^e_\gamma m^{\beta\gamma}.$$

Thus $\partial^\mu R_\mu\cdot\Phi$ is another term cubic in Φ.

We split the remaining term

$$P_0\partial^\mu(R_\mu\cdot\Phi)\approx E_1+2E_2,$$

where

$$E_1=P_0\,\partial^\mu\sum_{k_1,k_2>5,\,|k_1-k_2|\le 2}R_{\mu,k_1}\cdot\Phi_{k_2},\qquad E_2=P_0\,\partial^\mu\sum_{k_1\le 5,|k_2|\le 5}R_{\mu,k_1}\cdot\Phi_{k_2}.$$

Since the Fourier supports of both E_1 and E_2 belong to the set $\{\frac12\le|\xi|\le 2\}$, it suffices to estimate $\|E_1,E_2\|_{L^2_tL^2_x}$. Using (11) we derive

$$\begin{aligned}\|E_1\|_{L^2_tL^2_x}&\le\sum_{k_1,k_2>5,\,|k_1-k_2|\le2}\|P_0\partial^\mu(R_{\mu,k_1}\cdot\Phi_{k_2})\|_{L^2_tL^2_x}\\&\le\sum_{k_1,k_2>5,\,|k_1-k_2|\le2}\|R_{k_1}\|_{L^2_tL^\infty_x}\|\Phi_{k_2}\|_{L^\infty_tL^2_x}\\&\lesssim C^2\sum_{k>5}2^{\frac k2}\cdot2^{\frac k2(2-n)}c_k^2\\&\lesssim C^2c_0\varepsilon\sum_{k>5}2^{\frac k2(3-n+2\sigma)}\lesssim C^2c_0\varepsilon\end{aligned}$$

provided that $\sigma<\frac{n-3}{2}$. We also have

$$\begin{aligned}\|E_2\|_{L^2_tL^2_x}\le\sum_{k_1\le5,\,|k_2|\le5}\|R_{\mu,k_1}\cdot\Phi_{k_2}\|_{L^2_tL^2_x}&\le\sum_{k_1\le5,\,|k_2|\le5}\|R_{k_1}\|_{L^2_tL^\infty_x}\|\Phi_{k_2}\|_{L^\infty_tL^2_x}\\&\lesssim C^2c_0\sum_{k\le5}2^{\frac k2}c_k\lesssim C^2c_0\varepsilon.\end{aligned}$$

The Lemma now follows by taking ε sufficiently small.

4. Can Replace $\tilde R_\mu$ by $\partial_\mu\tilde\Delta$

This reduction step is the main contribution of our paper. In order to apply Tao's renormalization procedure we express $\tilde R_\mu$ in terms of the space-time gradient of a potential $\tilde\Delta$ plus terms which lead to error terms. More precisely,

Proposition 4.1. *The matrix valued function Ψ verifies an equation of the form,*

$$\Box\Psi=-2\partial_\mu\tilde\Delta\cdot\partial^\mu\Psi+\textit{error}\tag{18}$$

where the potential $\tilde\Delta$ verifies the following properties:

[10] We assume here that the Γ's are constant; the general case introduces another cubic term.

i.) *The $N \times N$ matrix $\tilde{\Delta}$ is antisymmetric i.e. $\tilde{\Delta}^t = -\tilde{\Delta}$. The space Fourier transform of each component of $\tilde{\Delta}$ is supported in $|\xi| \le 2^{-10}$.*

ii.) *The following estimates hold for any $\tilde{\Delta}_k = P_k\tilde{\Delta}$:*

$$\|\tilde{\Delta}_k\|_{S_k} \lesssim 2^{-k}Cc_k \tag{19}$$
$$\|\partial\tilde{\Delta}_k\|_{S_k} \lesssim Cc_k \tag{20}$$

Also,

$$\|\Box\tilde{\Delta}_k\|_{S_k} \lesssim 2^k Cc_k \tag{21}$$

iii.) *Set $\bar{R}_\mu = \tilde{R}_\mu - \partial_\mu\tilde{\Delta}$. The following estimates hold for all $P_k\bar{R}$,*

$$\|P_k\bar{R}\|_{L^1_tL^\infty_x} \lesssim C^2c_k^2 \tag{22}$$
$$\|P_k\bar{R}\|_{L^\infty_tL^\infty_x} \lesssim 2^kC^2c_k^2 \tag{23}$$

Proof : We start with the equation

$$\Box\Psi = -2\tilde{R}_\mu\partial^\mu\Psi + \text{error}$$

We need to find the potential $\tilde{\Delta}$ such that $\bar{R}_\mu = \tilde{R}_\mu - \partial_\mu\tilde{\Delta}$ is small. Clearly[11]

$$\partial_\nu\bar{R}_\mu - \partial_\mu\bar{R}_\nu = \partial_\nu\tilde{R}_\mu - \partial_\mu\tilde{R}_\nu = P_{\le -10}\big(\Gamma\cdot(\partial_\nu\phi_\mu - \partial_\mu\phi_\nu)\big)$$

Thus according to the equation (3) and the constancy of the structure and connection coefficients,

$$\partial_\nu\bar{R}_\mu - \partial_\mu\bar{R}_\nu = P_{\le -10}\big(M\cdot\Phi\cdot\Phi\big).$$

with $M \approx C^2$ a matrix whose entries are quadratic in C^a_{bc}. Henceforth[12],

$$\begin{aligned}
\partial_\nu(P_k\bar{R}_\mu) - \partial_\mu(P_k\bar{R}_\nu) &= M\cdot P_k(\Phi\cdot\Phi) = E_1 + E_2\\
E_1 &\approx M\cdot P_k\sum_{k'<k-1}\Phi_{k'}\cdot\Phi_k\\
E_2 &\approx M\cdot P_k\sum_{k_1,k_2\ge k,\ |k_1-k_2|\le 2}\Phi_{k_1}\cdot\Phi_{k_2}
\end{aligned}$$

We now estimate, with the help of (11) and (10) with $\sigma < \frac{1}{2}$.

$$\begin{aligned}
\|E_1\|_{L^1_tL^\infty_x} &\lesssim \|\Phi_k\|_{L^2_tL^\infty_x}\sum_{k'<k}\|\Phi_{k'}\|_{L^2_tL^\infty_x}\\
&\lesssim 2^{k/2}Cc_k\sum_{k'<k}2^{k'/2}Cc_{k'} \lesssim 2^{k/2}C^2c_k^2\sum_{k'<k}2^{k'/2}2^{\sigma(k-k')}\\
&\lesssim 2^kC^2c_k^2
\end{aligned}$$

[11]In the general case we have additional terms of the form $\partial\Gamma(\phi)\cdot\Phi$. These have the form $C'\cdot\Phi\cdot\Phi$ with C' the first frame derivatives of the structure coefficients. They are therefore similar to the terms $M\cdot\Phi\cdot\Phi$ we treat in the text.

[12]In the general case of a bounded parallelizable manifold one would have an additional commutator term which contributes, roughly speaking, a cubic term in Φ.

Since the Fourier support of E_2 belongs to the set $\{2^{k-1} \le |\xi| \le 2^k\}$, we also have

$$\begin{aligned}
\|E_2\|_{L^1_t L^\infty_x} &\lesssim 2^{\frac{nk}{2}} \|E_2\|_{L^1_t L^2_x} \lesssim 2^{\frac{nk}{2}} \sum_{k_1 \ge k} \sum_{|k_2 - k_1| \le 2} \|\Phi_{k_1}\|_{L^2_t L^4_x} \|\Phi_{k_2}\|_{L^2_t L^4_x} \\
&\lesssim 2^{\frac{nk}{2}} \sum_{k_1 \ge k} \sum_{|k_2 - k_1| \le 2} 2^{k_1/2 - nk_1/4} C c_{k_1} 2^{k_2/2 - nk_2/4} C c_{k_2} \\
&\lesssim 2^k C^2 c_k^2 \sum_{k' > k} 2^{(1 - \frac{n}{2} + 2\sigma)(k' - k)} \lesssim 2^k C^2 c_k^2
\end{aligned}$$

provided that $\sigma < \frac{n-2}{4}$. Therefore all the components of the exterior derivative $F_{(k)} = d(P_k \bar{R})$ verify the estimates

$$\|F_{(k)}\|_{L^1_t L^\infty_x} \lesssim 2^k C^2 c_k^2 \tag{24}$$

Proceeding in precisely the same manner we find that

$$\|F_{(k)}\|_{L^\infty_t L^\infty_x} \lesssim 2^{2k} C^2 c_k^2 \tag{25}$$

We define $\tilde{\Delta}_k$ by requiring that the spatial components of $P_k \bar{R}$ verify the equation,

$$\partial^i (P_k \bar{R}_i) = 0 \tag{26}$$

Consider now the divergence -curl system,

$$\begin{aligned}
\partial_i (P_k \bar{R}_j) - \partial_i (P_k \bar{R}_j) &= F_{(k)ij} \\
\partial^i (P_k \bar{R}_i) = 0 &
\end{aligned}$$

By standard elliptic estimates, taking into account the fact that the Fourier support of $P_k \bar{R}$ is included in the dyadic region $|\xi| \approx 2^k$ and using (24) we infer that,

$$\|P_k \bar{R}_i\|_{L^1_t L^\infty_x} \lesssim 2^{-k} \|F_{(k)}\|_{L^1_t L^\infty_x} \lesssim C^2 c_k^2$$

On the other hand we also have good estimates for $F_{(k)0i} = \partial_t P_k \bar{R}_i - \partial_i P_k \bar{R}_0$. In view of the divergence condition $\partial^i (P_k \bar{R}_i) = 0$ we derive $\nabla^2 P_k \bar{R}_0 = -\partial^i F_{(k)0i}$, with ∇^2 the Laplacean in $\mathbf{R}^n$, $\nabla^2 = \sum_{i=1}^n \partial_i^2$. Therefore using standard elliptic estimates and (24) we infer that,

$$\|P_k \bar{R}_0\|_{L^1_t L^\infty_x} \lesssim 2^{-k} \|F_{(k)}\|_{L^1_t L^\infty_x} \lesssim C^2 c_k^2$$

We have thus derived the estimate (22). The estimate (23) follows in the same manner from (25). We now estimate $\tilde{\Delta}$. We first observe that the divergence equation $\partial^i (P_k \bar{R}_i) = 0$ takes the form $\nabla^2 \tilde{\Delta}_k = \partial^i (P_k \tilde{R}_i)$. This uniquely defines $\tilde{\Delta}_k$ and we have,

$$\|\tilde{\Delta}_k\|_{S_k} \lesssim 2^{-k} \|P_k \tilde{R}\|_{S_k} \lesssim 2^{-k} \|P_k \Phi\|_{S_k} \lesssim 2^{-k} C c_k.$$

which gives (19) and (20). To prove (21) we write $\nabla^2 \Box \tilde{\Delta}_k = \partial_i (P_k \Box \tilde{R}_i)$. Therefore, in view of (13), $\|\Box \tilde{\Delta}_k\|_{S_k} \lesssim 2^{-k} \|\Box P_k \tilde{R}\|_{S_k} \lesssim 2^{-k} \|\Box P_k \tilde{\Phi}\|_{S_k} \lesssim 2^k C c_k$ establishing the estimate (21).

To end the proof of Proposition 4.1 it remains to observe that since each $\tilde{\Delta}_k$ is antisymmetric so is the $\tilde{\Delta} = \sum_{k \le -10} \tilde{\Delta}_k$. We also need to check that the terms

$\bar{R}_\mu \partial^\mu \Psi$ generated when we pass from the equation (16) to (18) are indeed error terms. We have, using (22)

$$\begin{aligned} \|\bar{R}_\mu \partial^\mu \Psi\|_{L^1_t L^2_x} &\le \|\bar{R}\|_{L^1_t L^\infty_x} \|\Psi\|_{L^\infty_t L^2_x} \lesssim C c_0 \|\bar{R}\|_{L^1_t L^\infty_x} \\ &\lesssim c_0 C \sum_{k\le -10} \|\bar{R}_k\|_{L^1_t L^\infty_x} \le c_0 C C^2 \sum_{k \le -10} c_k^2 \\ &\lesssim c_0 C^3 \varepsilon \end{aligned}$$

as desired.

■

5. Tao's renormalization procedure

This last step in our proof is a straightforward implementation of Tao's renormalization procedure. We repeat below the main arguments in his construction.

Let M be a large integer, depending on T, which will be chosen below. Define the real $N \times N$ matrix valued function U to be

$$U = I + \sum_{-M<k\le -10} U_k$$

with the U_k defined inductively as follows,

$$\begin{aligned} U_k &= 0 \qquad \text{for all} \quad k < -M \\ U_{-M} &= I \\ U_k &= \tilde{\Delta}_k \cdot U_{<k} \qquad \text{for all} \quad -M < k \le -10 \end{aligned} \tag{27}$$

with $U_{<k} = \sum_{k'<k} U_k$. Due to the fact that the matrices $\tilde{\Delta}_k = P_k \tilde{\Delta}$ are antisymmetric we find the identity

$$U_k^t \cdot U_{<k} + U_{<k}^t \cdot U_k = 0$$

whence,

$$U_{<k}^t \cdot U_{<k} - I = \sum_{k'<k} U_{k'}^t \cdot U_{k'} \tag{28}$$

Using this identity we can prove inductively that

$$\begin{aligned} \|U_{<k}\|_{L^\infty_t L^\infty_x} &\le 2 \\ \|U_k\|_{L^\infty_t L^\infty_x} &\lesssim C c_k \qquad \text{for} \quad k > -M \end{aligned} \tag{29}$$

as well as

$$\|U_k\|_{L^2_t L^\infty_x} \lesssim C 2^{-k/2} c_k \qquad \text{for} \quad k > -M. \tag{30}$$

Also,

$$\begin{aligned} \|\partial U_{<k}\|_{L^\infty_t L^\infty_x} &\lesssim 2^k C^2 c_k \\ \|\partial U_k\|_{L^\infty_t L^\infty_x} &\lesssim 2^k C^2 c_k \end{aligned} \tag{31}$$

and

$$\begin{aligned} \|\partial U_{<k}\|_{L_t^2 L_x^\infty} &\lesssim 2^{k/2} C c_k \\ \|\partial U_k\|_{L_t^2 L_x^\infty} &\lesssim 2^{k/2} C c_k \end{aligned} \tag{32}$$

as well as,

$$\begin{aligned} \|\Box U_{<k}\|_{L_t^2 L_x^{n-1}} &\lesssim 2^{k(\frac{3}{2}-\frac{n}{n-1})} C c_k \\ \|\Box U_k\|_{L_t^2 L_x^{n-1}} &\lesssim 2^{k(\frac{3}{2}-\frac{n}{n-1})} C c_k \end{aligned} \tag{33}$$

Indeed the first inequality of (29) holds for $k \le -M$. Assume that it holds up to some $-M < k < -10$. In view of part ii) of Proposition 4.1 we have $\|\tilde{\Delta}_k\|_{L_t^\infty L_x^\infty} \lesssim Cc_k$. Therefore,

$$\|U_k\|_{L_t^\infty L_x^\infty} = \|\tilde{\Delta}_k U_{<k}\|_{L_t^\infty L_x^\infty} \lesssim 2Cc_k$$

which proves the second part of (29). To complete the induction for the first inequality we use the identity (28) according to which

$$\|U_{\le k}\|^2_{L_t^\infty L_x^\infty} \le 1 + \sum_{-M<k'<k} \|U_{k'}\|^2_{L_t^\infty L_x^\infty} \le 2$$

provided that ε is sufficiently small.

To prove (31) and (32) we proceed once more by induction. Observe that the first estimate follows from the second. Assume that the second estimate holds for all $k' < k$ for some sufficiently large implicit constant A. Then, using (10),

$$\begin{aligned} \|\partial U_{<k}\|_{L_t^\infty L_x^\infty} &\le \sum_{k'<k} \|\partial U_{k'}\|_{L_t^\infty L_x^\infty} \le A\, C^2 \sum_{k'<k} 2^{k'} c_{k'} \\ &\le A\, C^2 c_k \sum_{k'<k} 2^{k'} 2^{\sigma(k-k')} = A\, C^2 c_k 2^k \sum_{k''<0} 2^{k''(1-\sigma)} \le 8A\, C^2 2^k c_k \end{aligned}$$

as desired. Also,

$$\begin{aligned} \|\partial U_{<k}\|_{L_t^2 L_x^\infty} &\le \sum_{k'<k} \|\partial U_{k'}\|_{L_t^2 L_x^\infty} \le A\, C^2 \sum_{k'<k} 2^{k'/2} c_{k'} \\ &\le A\, C^2 c_k \sum_{k'<k} 2^{k'/2} 2^{\sigma(k-k')} = A\, C^2 c_k 2^{k/2} \sum_{k''<0} 2^{k''(1/2-\sigma)} \le 8A\, C^2 2^k c_k \end{aligned}$$

since $0 < \sigma < \frac{1}{4}$.

The second estimate in (31) can be proved now by induction with the help of the definition $U_k = \tilde{\Delta}_k \cdot U_{<k}$. The result is clearly true for $k \le -M$. We may thus assume that the first estimate in (31) is verified for some $k < -10$ and a sufficiently large universal constant A. We can also assume that the implicit constant in the estimates (19), (20), and (21) of part ii) of Proposition 4.1 is dominated by $\frac{A}{8}$. Using the above estimates for $U_{<k}, \partial U_{<k}$, and the estimate $|c_k| \le \varepsilon$, we derive

$$\begin{aligned}\|\partial U_k\|_{L_t^\infty L_x^\infty} &\le \|\partial\tilde\Delta_k\|_{L_t^\infty L_x^\infty}\|U_{<k}\|_{L_t^\infty L_x^\infty} + \|\tilde\Delta_k\|_{L_t^\infty L_x^\infty}\|\partial U_{<k}\|_{L_t^\infty L_x^\infty}\\ &\le \frac{A}{8}C2^k c_k\cdot 2 + \frac{A}{8}Cc_k\cdot 8A2^kC^2c_k \le \frac{A}{4}C\,2^kc_k + \varepsilon A^2C^2 2^k c_k \le A\,C2^kc_k\end{aligned}$$

as desired. For the second estimate in (32) we have,

$$\begin{aligned}\|\partial U_k\|_{L_t^2 L_x^\infty} &\le \|\partial\tilde\Delta_k\|_{L_t^2 L_x^\infty}\|U_{<k}\|_{L_t^\infty L_x^\infty} + \|\tilde\Delta_k\|_{L_t^\infty L_x^\infty}\|\partial U_{<k}\|_{L_t^2 L_x^\infty}\\ &\le \frac{A}{8}\,C2^{k/2}c_k\cdot 2 + \frac{A}{8}\,Cc_k\cdot 8A\,2^{k/2}C^2c_k \le A\,C2^kc_k\end{aligned}$$

provided that ε is sufficiently small.

To prove (33) assume the first estimate with the implicit constant $8A$ to be true. Then,

$$\Box U_k = \Box(\tilde\Delta_k U_{<k}) = (\Box\tilde\Delta_k)\cdot U_{<k} + 2\partial^\mu\tilde\Delta\cdot\partial_\mu U_{<k} + \tilde\Delta_k\cdot\Box U_{<k}.$$

Hence, using the induction hypothesis and the estimates we have for $U_{<k}$, $\partial U_{<k}$, $\tilde\Delta_k$ and $\partial\tilde\Delta_k$ we derive:

$$\begin{aligned}\|\Box U_k\|_{L_t^2 L_x^{n-1}} &\le \|\Box\tilde\Delta_k\|_{L_t^2L_x^{n-1}}\|U_{<k}\|_{L_t^\infty L_x^\infty}\\ &+ 2\|\partial\tilde\Delta_k\|_{L_t^2L_x^{n-1}}\|\partial U_{<k}\|_{L_t^\infty L_x^\infty}\\ &+ \|\tilde\Delta_k\|_{L_t^\infty L_x^\infty}\|\Box U_{<k}\|_{L_t^2L_x^{n-1}}\\ &\le \frac{A}{8}C2^{k(\frac32-\frac{n}{n-1})}c_k\cdot 2 + \frac{A}{4}Cc_k2^{k(\frac12-\frac{n}{n-1})}c_k\cdot 4A2^kC^2c_k\\ &+ \frac{A}{8}\,Cc_k\cdot 8A\,C2^{k(\frac32-\frac{n}{n-1})}c_k\\ &\le A\,C2^{k(\frac32-\frac{n}{n-1})}c_k\end{aligned}$$

Now, using (10),

$$\begin{aligned}\|\Box U_{\le k}\|_{L_t^2L_x^{n-1}} &\le A\,C\sum_{k'\le k}2^{k'(\frac32-\frac{n}{n-1})}c_{k'} \le A\,Cc_k\sum_{k'\le k}2^{\sigma k}2^{k'(\frac32-\frac{n}{n-1}-\sigma)}\\ &\le 8A\,C2^{k(\frac32-\frac{n}{n-1})}c_k\end{aligned}$$

provided that $\sigma < \frac{n-3}{4(n-1)}$.

We summarize the most important properties of $U = I + \sum_{-M<k\le -10}U_k$ in the following

Proposition 5.1. *Assume that ε is sufficiently small depending on C and M sufficiently large depending on T, C, ε. Then the matrices U verify the following properties:*

i.) Approximate orthogonality:

$$\|U^tU - I\|_{L_t^\infty L_x^\infty},\ \ \|\partial(U^tU-I)\|_{L_t^\infty L_x^\infty} \lesssim C^2\varepsilon \tag{34}$$

In particular, for small ε, U is invertible and we have,

$$\|U\|_{L_t^\infty L_x^\infty},\ \ \|U^{-1}\|_{L_t^\infty L_x^\infty} \lesssim 1 \tag{35}$$

ii.) *Approximate gauge condition:*

$$\|\partial_\mu U - \partial_\mu \tilde{\Delta} \cdot U\|_{L^1_t L^\infty_x} \lesssim C^2 \varepsilon \tag{36}$$

iii.) *We also have,*

$$\|\partial U\|_{L^\infty_t L^\infty_x}, \ \|\partial U\|_{L^1_t L^\infty_x} \lesssim C^2 \varepsilon \tag{37}$$

$$\|\Box U\|_{L^2_t L^{n-1}_x} \lesssim C^2 \varepsilon \tag{38}$$

Proof : The first part of the proposition is an easy consequence of the identity (28) as well as the estimates (31) and (32). To prove the crucial second part we write

$$\partial_\mu U - \partial_\mu \tilde{\Delta} \cdot U = \sum_{-M<k\le 10} \Big(\partial_\mu U_k - (\partial_\mu \tilde{\Delta}_{\le k} \cdot U_{\le k} - \partial_\mu \tilde{\Delta}_{<k} \cdot U_{<k}) \Big) - \partial_\mu \tilde{\Delta}_{\le -M}$$

We estimate $\partial_\mu \tilde{\Delta}_{\le -M}$ using Cauchy-Schwartz and (20) as follows

$$\|\partial_\mu \tilde{\Delta}_{\le -M}\|_{L^1_t L^\infty_x} \le T^{\frac{1}{2}} \|\partial_\mu \tilde{\Delta}_{\le -M}\|_{L^2_t L^\infty_x} \lesssim T^{\frac{1}{2}} \sum_{k \le -M} 2^{k/2} c_k \lesssim \varepsilon T^{\frac{1}{2}} 2^{-M/2}.$$

Thus, picking M sufficiently large,

$$\|\partial_\mu \tilde{\Delta}_{\le -M}\|_{L^1_t L^\infty_x} \le \varepsilon.$$

To end the proof of (36) it suffices to prove that for all $-M < k \le -10$ we have $\|E_k\|_{L^1_t L^\infty_x} \lesssim C^2 c_k^2$ where

$$\begin{aligned} E_k &= \partial_\mu U_k - (\partial_\mu \tilde{\Delta}_{\le k} \cdot U_{\le k} - \partial_\mu \tilde{\Delta}_{<k} \cdot U_{<k}) \\ &= \partial_\mu U_k - (\partial_\mu \tilde{\Delta}_{\le k} - \partial_\mu \tilde{\Delta}_{<k}) \cdot U_{<k} - \partial_\mu \tilde{\Delta}_{\le k} \cdot U_k \\ &= \partial_\mu U_k - \partial_\mu \tilde{\Delta}_k \cdot U_{<k} - \partial_\mu \tilde{\Delta}_{\le k} \cdot U_k \end{aligned}$$

Now using the definition $U_k = \tilde{\Delta}_k \cdot U_{<k}$,

$$\begin{aligned} E_k &= \partial_\mu (\tilde{\Delta}_k U_{<k}) - (\partial_\mu \tilde{\Delta}_k \cdot U_{<k} - \partial_\mu \tilde{\Delta}_{\le k} \cdot U_k) \\ &= \tilde{\Delta}_k \cdot \partial_\mu U_{<k} + \partial_\mu \tilde{\Delta}_{\le k} \cdot U_k \end{aligned}$$

Therefore, using (19), (20) as well as (30), (32)

$$\begin{aligned} \|E_k\|_{L^1_t L^\infty_x} &\le \|\tilde{\Delta}_k\|_{L^2_t L^\infty_x} \|\partial U_{<k}\|_{L^2_t L^\infty_x} + \|\partial \tilde{\Delta}_{\le k}\|_{L^2_t L^\infty_x} \|U_k\|_{L^2_t L^\infty_x} \\ &\lesssim C^2 c_k^2 \end{aligned}$$

as desired.

The estimates (37) of part iii.) of the proposition are immediate consequences of the estimates (31), (32). The inequality (38) can be derived immediately from (33). ■

Following [6] we are now ready to perform the gauge transformation

$$\Psi = U \cdot W \tag{39}$$

W verifies the equation

$$\begin{aligned}\Box W &= -2U^{-1}(\partial_\mu U - \partial_\mu \tilde{\Delta}\cdot U)\partial^\mu W \\ &- 2U^{-1}\partial_\mu\tilde{\Delta}\cdot(\partial^\mu U)U^{-1}\Psi - U^{-1}(\Box U)U^{-1}\Psi + \text{error}\end{aligned} \quad (40)$$

In view of Proposition 5.1 we derive,

Proposition 5.2. *The matrix valued function W verifies an equation of the form*

$$\Box W = error.$$

Therefore, if ε is sufficiently small,

$$\|\Psi\|_{S_0} \lesssim \|W\|_{S_0} \le \|\Psi[0]\|_{H^{\frac{n-2}{2}}} + CC^3\varepsilon c_0 \le Cc_0.$$

This is precisely (14) which ends the proof of the Main Proposition[13].

Remark 5.3. It is interesting to compare our results for the Hodge system (3) -(4) with the system obtained by considering Lorentz gauge, zero curvature connections in a general Lie algebra, see [3]:

$$\begin{aligned}\partial_\alpha A_\beta - \partial_\beta A_\alpha &= [A_\alpha, A_\beta] \\ \partial^\alpha A_\alpha = 0.\end{aligned} \quad (41)$$

Such systems can be written in the form (3) -(4) in the particular case when the structure constants C^a_{bc} verify, in addition to $C^a_{bc} = -C^a_{cb}$, the relations $C^a_{bc} = -C^b_{ac}$. In this case $\Gamma^a_{bc} = C^a_{bc}$. This corresponds to the case of a Lie group with a bi-invariant Riemannian metric such as S^3. The system (41) is interesting however in its own right, for general Lie algebras. The results of this paper can be extended to the case of classical Lie algebras such as $o(n), su(n)$ and probably more generally to Lie algebras of compact Lie groups. The compactness seems in this case to be essential[14], by contrast to the case of wave maps where the compactness of the target manifold is not important.

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[13]See also the more complete argument in section 7 of [6].
[14]In this case the transformation (39) should be replaced by the partial gauge transformation $A \longrightarrow UAU^{-1}$ with U an element of the group. Compactness of the group is needed to control the sup-norm of U.

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