

A NEW APPROACH TO STUDY THE VLASOV-MAXWELL SYSTEM

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ABSTRACT. We give a new proof based on Fourier Transform of the classical Glassey and Strauss [4] global existence result for the 3D relativistic Vlasov-Maxwell system, under the assumption of compactly supported particle densities. Though our proof is not substantially shorter than that of [4], we believe it adds a new perspective to the problem. In particular the proof is based on three main observations, see Facts 1-3 following the statement of Theorem 1.4, which are of independent interest

1. INTRODUCTION

We write the Vlasov-Maxwell system as

$$(1) \quad \begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + (E + \frac{\hat{v}}{c} \times B) \cdot \nabla_v f = 0 \\ c^{-1} E_t = \nabla \times B - 4\pi c^{-1} j, \quad \nabla \cdot E = 4\pi \rho \\ c^{-1} B_t = -\nabla \times E, \quad \nabla \cdot B = 0, \end{cases}$$

where $f(t, x, v)$ denotes the *particle density*¹, $j(t, x) = \int \hat{v} f dv$ the *current density*, $\rho = \int f dv$ the *charge density* and $\hat{v} = \frac{v}{(1+|v|)^{1/2}}$ the *relativistic velocity*. Consider then the initial value problem (IVP) given by (1) and

$$(2) \quad f(0, x, v) = f_0(x, v) > 0, E(0, x) = E_0(x), B(0, x) = B_0(x), \nabla \cdot E_0 = \rho_0, \nabla \cdot B_0 = 0.$$

The standard regularity for the data in (2) is $f_0 \in C_0^1$ and $E_0, B_0 \in C^2$. The question of global well-posedness has been considered by many authors. The reduced Vlasov-Poisson system has been tackled successfully, for large data, by Pfaffelmoser [6], Lions-Parthame [7], Schaeffer [8]. The outstanding result for the full 3D Maxwell-Vlasov system remains that of Glassey-Strauss [4], who were able to prove a global existence result under the hypothesis of compactly supported (for all time !) particle density. Later Glassey-Schaeffer [5] were able to remove the additional support hypothesis for the, so called², $2 + 1/2$ dimensional system.

We recall below the main result of Glassey and Strauss:

Theorem 1.1 (Glassey and Strauss). *Assume the above conditions on the initial data. Assume there exists a continuous function $\beta(t)$ such that for all $x \in \mathbb{R}^3$*

$$(3) \quad f(t, x, v) = 0 \quad \text{for } |v| > \beta(t).$$

Then there exists a unique C^1 solution of the system for all t .

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¹Here we assume that we are dealing with only one specie, for a more detailed introduction of the system we refer to [3].

²In which case $f = f(t, x, v)$ is a function of time t , $x \in \mathbb{R}^2$ and $v \in \mathbb{R}^3$.

The Glassey-Strauss proof relies on showing uniform bounds in time for the L^∞ norms of E , B , f as well as of all their first derivatives. They start by rewriting (1) and (2) as follows³:

$$(4) \quad \begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0, \\ E_{tt} - \Delta E = -(\partial_t j + \nabla_x \rho) = -\int_{\mathbb{R}^3} (\nabla_x f + \hat{v} \partial_t f) dv, \\ B_{tt} - \Delta B = \nabla_x \times j = \int_{\mathbb{R}^3} (\hat{v} \times \nabla_x f) dv, \\ f(0, x, v) = f_0(x, v) > 0, E(0, x) = E_0(x), B(0, x) = B_0(x). \end{cases}$$

Then they represent the fields E and B using the explicit form of the fundamental solution of $\square = \partial_t^2 - \Delta$ in physical space. For example for E they write

$$(5) \quad E(t, x) = E_0(t, x) + \frac{1}{2\pi} \int_0^t ds \int_{|y-x| \leq t-s} \frac{dy}{|y-x|} \left(\int (\nabla + \hat{v} \partial_t) f(s - |y-x|, y, v) dv \right),$$

where E_0 is a solution of the homogeneous equation $\square E_0 = 0$, $\hat{v} = \frac{v}{(1+|v|)^{1/2}}$ and $\nabla = (\partial_1, \partial_2, \partial_3)$. The presence of the derivative fields $\nabla + \hat{v} \partial_t$ in the integrand seems to create great difficulties⁴. The main new idea of Glassey and Strauss was to decompose $\nabla + \hat{v} \partial_t$ into fields $T_i = \partial_{y_i} - \omega_i \partial_t$, with $\omega_i = \frac{(y-x)_i}{|y-x|}$, tangent to the backward characteristic cone⁵ passing through (t, x) , and the vector field $S = \partial_t + \hat{v} \cdot \nabla$. Indeed we have,

$$\partial_{y_i} = \frac{\omega_i S}{1 + \hat{v} \cdot \omega} + \left(\delta_{i,j} - \frac{\omega_i \hat{v}_j}{1 + \hat{v} \cdot \omega} \right) T_j$$

and similarly for ∂_t . The crucial observation is that Sf can be reexpressed by using the transport equation, i.e.,

$$Sf = (\partial_t + \hat{v} \cdot \nabla) f = -(E + \hat{v} \times B) \cdot \nabla_v f.$$

One can get rid of the v derivative of f by integrating by parts with respect to the v -integration in (5). All this works, and thus allows the authors to estimate the L^∞ norm of E and B in terms of the L^∞ norm of f , as long as the denominator $1 + \hat{v} \cdot \omega$ is bounded away from zero. This important fact is guaranteed by the apriori assumption (3).

A similar, much more delicate argument, is needed to obtain the estimates for the derivatives of E and B . Glassey-Strauss accomplish this goal by proving in fact that for any t in a fixed interval of time $[0, T]$,

$$(6) \quad \|\nabla_x E(t)\|_\infty + \|\nabla_x B(t)\|_\infty \leq C_T \left(1 + \log^+ \left(\sup_{\tau \leq t} \|Df(\tau)\|_\infty \right) \right)$$

where⁶ ∇_x denotes the space-time derivatives in x and $D = (\nabla_x, \nabla_v)$. On the other hand, using the transport equation, they show by a straightforward argument that:

$$(7) \quad \|Df(t)\|_\infty \leq C_T \int_0^t (1 + \|\nabla_x E(\tau)\|_\infty + \|\nabla_x B(\tau)\|_\infty) \|Df(\tau)\|_\infty d\tau.$$

The combination of (6) and (7) followed by a straightforward application of Gronwall's inequality gives a bound for all the quantities involved. In proving (6) and (7) Glassey and Strauss had to deal with denominators of type $(1 + \hat{v} \cdot \omega)^n$, $n \leq 4$ whose possible singularities are avoided

³We get rid of the constants c and π by taking $c = 4\pi = 1$ in (1).

⁴Recall that we need to estimate the first derivatives of E and B . According to the above formula this seems to require two derivatives of f !.

⁵Which disappears by simple integration in the formula (5).

⁶Note that $\log^+ z := \log(2 + z)$ for nonnegative z .

by (3). The last step to complete the proof of Theorem 1.1 is based on a standard recursive method.

In this paper we are taking a different approach based on the Fourier representation of E and B . Observe that,

$$\begin{aligned} E_{tt} - \Delta E &= \int_{\mathbb{R}^3} \left(-\nabla_x f - \hat{v}(-\hat{v}\nabla_x f - (E + \hat{v} \times B)\nabla_v f) \right) dv, \\ B_{tt} - \Delta B &= \int_{\mathbb{R}^3} (\hat{v} \times \nabla_x) f dv \end{aligned}$$

For convenience we shall write the system (4) in the form:

$$(8) \quad \begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + \alpha(v)\Phi \cdot \nabla_v f = 0, \\ \Phi_{tt} - \Delta \Phi = J(t, x), \\ f(0, x, v) = f_0(x, v) > 0, \quad \Phi(0, x) = \Phi_0(x). \end{cases}$$

where

$$(9) \quad \Phi(t, x) = (E(t, x), B(t, x)),$$

$$(10) \quad \alpha(v)\Phi = (E, \hat{v} \times B),$$

$$(11) \quad J(t, x) = \int_{\mathbb{R}^3} M(v)\nabla_x f(t, x, v)dv + \int_{\mathbb{R}^3} N(v)\Phi(t, x) \cdot \nabla_v f(t, x, v) dv,$$

with $M(v)$ and $N(v)$ matrices depending only on v . In the first part of our proof we don't need the explicit representation of $\alpha(v)$, $M(v)$ and $N(v)$; shall only make use of the fact that all their components, and their derivatives with respect to v , are bounded for all $v \in \mathbb{R}^3$. When convenient we will make use of the explicit representation (4).

In this paper we prove the following version of Theorem 1.1:

Theorem 1.2. *Consider the IVP (8) with $f_0 \in C_0^1(\mathbb{R}^3)$ and $\Phi(0, x) \in C^1(\mathbb{R}^3)$. Assume that, on any fixed interval of time $[0, T]$,*

$$(12) \quad \|\Phi\|_{L_{x,t}^\infty([0, T] \times \mathbb{R}^3)} < C.$$

Then the system (8) admits unique $C^1([0, T] \times \mathbb{R}^3)$ solution.

Remark 1.3. Observe that we have substituted the support assumption (3) with the boundedness assumption (12). The two assumptions are, essentially, equivalent. In fact (3) implies (12) through a straightforward application of the Glassey-Strauss decomposition idea mentioned above. On the other hand, (12) together with the assumption of compact support in v for $f_0(t, x, v)$ implies the Glassey-Strauss assumption (3). This follows easily from the properties of characteristics, see Lemma 2.1. As a consequence of (12) we also have $|\hat{v}| \leq \delta < 1$ on the support of $f(t, x, v)$ for all $(t, x) \in [0, T] \times \mathbb{R}^3$, fact on which we shall rely heavily in our proof.

Our strategy is to prove directly that the characteristics associated to the transport equation in (8) are Lipschitz. To do this we look at the problem not just in the physical space, but also in the frequency space.

Denote, as in [4], the characteristics⁷,

$$(X(s, t, x, y), V(s, t, x, y)), \quad s, t \in \mathbb{R}, x, y \in \mathbb{R}^3$$

⁷We shall call them forward characteristics for $s > t$ and backward for $s < t$.

of the transport equation in (8). They are solutions of the system of ordinary differential equations in s , with initial data at $s = t$,

$$(13) \quad \begin{cases} \frac{dX}{ds} = \frac{V}{\sqrt{1+|V|^2}} =: \hat{V} \\ \frac{dV}{ds} = \alpha(V)\Phi(s, X) \\ X(t, t, x, v) = x \quad \text{and} \quad V(t, t, x, v) = v \end{cases}$$

Our main theorem can easily be reduced to the following:

Theorem 1.4. *Consider the IVP (8) on a fixed interval $[0, T]$ and assume that $f_0 \in C_0^1$, $\Phi(0, x) \in C^1$ and Φ satisfies (12). Then for every $t \in [0, T]$ and $i = 1, 2, 3$*

$$(14) \quad \sup_{s \in [0, T]} [\sup_{x, v} |\partial_{x_i} X(s, t, x, v)|] \leq C$$

$$(15) \quad \sup_{s \in [0, T]} [\sup_{x, v} |\partial_{v_i} X(s, t, x, v)|] \leq C$$

$$(16) \quad \sup_{s \in [0, T]} [\sup_{x, v} |\partial_{x_i} V(s, t, x, v)|] \leq C$$

$$(17) \quad \sup_{s \in [0, T]} [\sup_{x, v} |\partial_{v_i} V(s, t, x, v)|] \leq C.$$

The proof is based on three main observations.

- **Fact 1:** When we solve $\square\Phi = J$ by the Fourier method we are led to integrals of the form $\int_{\mathbb{R}^3} H(v) \left(\int_0^t e^{\pm i\sigma|\xi|} \hat{f}(\sigma, \xi, v) d\sigma \right) dv$. These are smoother than may be apparent at first glance as can be seen integrating first by parts in σ , making use of the transport equation (8) for f and then integrating by parts once more in v . This fact, which follows easily from Lemma 2.2 below⁸, seems to be the Fourier counterpart of the Glassey-Strauss idea of decomposition of general derivatives into S and T , integration by parts and use of the transport equation mentioned above.
- **Fact 2:** The time integral of a wave solution Φ along characteristics is smoother than Φ itself. More precisely if Φ is solution of the inhomogeneous wave equation

$$(18) \quad \square\Phi = F, \quad \Phi(0) = \partial_t\Phi = 0,$$

and $(s, X(s, x))$ is a time-like characteristic curve⁹ then $\int_0^t \Phi(s, X(s, x)) ds$, is one derivative smoother than Φ . This fact can be easily seen in Fourier space, i.e.

$$(19) \quad \begin{aligned} \Phi &= \frac{1}{2i}(\Phi_+ + \Phi_-) \\ \Phi_{\pm}(t, x) &= (2\pi)^{-n} \int_0^t ds \int \frac{1}{|\xi|} e^{\pm i(t-s)|\xi|} e^{ix \cdot \xi} \hat{F}(s, \xi) d\xi. \end{aligned}$$

⁸A similar “smoothing” lemma has been recently obtained by Bouchut, Golse and Pallard [2].

⁹i.e. $|\frac{dX}{ds}| < 1$, $X(0, x) = x$.

Thus we are led to integrals of the form $J = \int_0^t e^{i(\pm s|\xi| + X(s,x)\cdot\xi)} H(s) ds$. Integrating by parts and using the characteristic equations (13) we find,

$$\begin{aligned} J &= \int_0^t \frac{1}{(\pm|\xi| + \dot{X}(s,x)\cdot\xi)} \frac{d}{ids} e^{i(\pm s|\xi| + X(s,x)\cdot\xi)} H(s) ds \\ &= - \int_0^t \frac{1}{(\pm|\xi| + \dot{X}(s,x)\cdot\xi)} e^{i(\pm s|\xi| + X(s,x)\cdot\xi)} \frac{d}{ids} H(s) ds \\ &\quad + \int_0^t \frac{\frac{d}{ids} \hat{V}(s,x)}{(\pm|\xi| + \hat{V}(s,x)\cdot\xi)^2} e^{i(\pm s|\xi| + X(s,x)\cdot\xi)} H(s) ds \end{aligned}$$

Thus, introducing the 3×3 matrix $\beta(v) = \frac{\partial \hat{v}}{\partial v}$, we have

$$\frac{d}{ds} \hat{V}(s,x) = \beta(V) \frac{d}{ds} V(s,x) = \beta(V(s,x)) \alpha(V(s,x)) \Phi(s, X(s,x))$$

and therefore,

$$\begin{aligned} \int_0^t e^{i(\pm s|\xi| + X(s,x)\cdot\xi)} H(s) ds &= - \int_0^t \frac{1}{(\pm|\xi| + \dot{X}(s,x)\cdot\xi)} e^{i(\pm s|\xi| + X(s,x)\cdot\xi)} \frac{d}{ids} H(s) ds \\ &\quad + \int_0^t \frac{\beta(V(s,x)) \alpha(V(s,x)) \Phi(s, X(s,x))}{(\pm|\xi| + \hat{V}(s,x)\cdot\xi)^2} e^{i(\pm s|\xi| + X(s,x)\cdot\xi)} H(s) ds \end{aligned} \tag{20}$$

and the gain of differentiability is obvious in view of $\hat{V} \leq \delta < 1$. Observe also that the matrices $\beta(V), \alpha(V)$ are bounded.

- **Fact 3:** In the process of estimating the sup-norms of our main quantities, expressed as Fourier integrals, we need an extension of the well known Beale-Kato-Majda lemma in [1] to Fourier integral operators. This is the content of the Lemma 2.5 below. To understand its usefulness consider the standard initial value problem in \mathbb{R}^{3+1}

$$\square \phi = 0, \quad \phi(0) = 0, \quad \partial_t \phi(0) = g.$$

It is well known, from the explicit form of the fundamental solution in physical space, that $\|\phi(t)\|_{L^\infty} \lesssim t \|g\|_{L^\infty}$. As discussed above, in the context of the inhomogeneous wave equation, this fact plays a fundamental role in the proof of Glassey-Strauss. Unfortunately this estimate seems very unstable. Indeed consider the Fourier representation of the solution:

$$\phi(t,x) \approx \int e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} \hat{g}(\xi) d\xi$$

an decompose it $\phi = \phi_+ + \phi_-$ with

$$\phi_\pm(t,x) \approx \int e^{ix\cdot\xi} \frac{e^{\pm it|\xi|}}{|\xi|} \hat{g}(\xi) d\xi.$$

Can we still bound the L^∞ norm of $\phi_+(t, \cdot)$ in terms of $\|g\|_{L^\infty}$? The answer is no; the best we can hope for is an estimate of the form

$$\|\phi_+(t)\|_{L^\infty} \lesssim \|g\|_{L^\infty} \log(2 + \|\nabla g\|_{L^p}) + 1 + \|g\|_{L^2}$$

with $t \in [0, T]$ and $p > 1$. We prove a more general estimate of this type in Lemma 2.5, for Fourier integral operators of the form

$$Tg(t,x) = \int e^{ix\cdot\xi} \frac{e^{\pm it|\xi|}}{|\xi|} m_0(t,x,\xi) \hat{g}(\xi) d\xi$$

with m_0 homogeneous of degree zero in ξ .

Throughout the paper we will denote with \hat{f} the Fourier transform¹⁰ of the function f :

$$\hat{g}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} g(x) dx.$$

We will also write for $1 \leq p < \infty$

$$\begin{aligned} \|g\|_{L^p} &= \left(\int_{\mathbb{R}^3} |g|^p(z) dz \right)^{1/p} \\ \|g\|_{\infty} &= \sup_{z \in \mathbb{R}^3} |g(z)|. \end{aligned}$$

If the function g has two variables, then for $1 \leq p, q < \infty$ we write¹¹

$$\|g\|_{L_x^p L_v^q} = \left(\int \left(\int |g(x, v)|^q dv \right)^{p/q} dx \right)^{1/p}$$

and if $p = q$ we simplify the notation by writing

$$\|g\|_{L_x^p L_v^q} = \|g\|_p.$$

Very often we will use the notation $B \approx C$, if there exists $\beta \in \mathbb{C}, \beta \neq 0$, such that $B = \beta C$. We will also write $B \lesssim C$, if there exists $m > 0$ such that $B \leq mC$.

2. PROOF OF THEOREM 1.4

During the process of proving Theorem 1.4 we will introduce few lemmas, but in order not to distract the reader from the main flow of ideas, we postpone their proofs in the Appendix.

From the definition of characteristics we have

$$(21) \quad X(s, t, x, v) = x + \int_t^s \hat{V}(\sigma, t, x, v) d\sigma$$

$$(22) \quad V(s, t, x, v) = v + \int_t^s \alpha(V) \Phi(\sigma, X(\sigma, t, x, v)) d\sigma.$$

We first observe the following support property for f .

Lemma 2.1. *Assume that f_0 is supported on the set $S_k = \{|x| \leq k, |v| \leq k\}$. Then if f solves the transport equation in (8) and Φ satisfies (12), then for any $t \in [0, T]$*

$$(23) \quad f(t, x, v) = 0 \quad \text{for all } (x, v) \in S_{(k,t)},$$

where

$$(24) \quad S_{(k,t)} = \{(x, v) / |x| \geq 2(t+k) \text{ and } |v| \geq 2(t\|\Phi\|_{L^\infty} \|\alpha(v)\|_\infty + k)\}.$$

The proof follows directly from (21) and (22).

Since Φ solves the wave equation (8), we write

$$(25) \quad \Phi(t, x) = \Phi_0(t, x) + (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^3} \frac{\sin[(t-\sigma)|\xi|]}{|\xi|} \hat{J}(\sigma, \xi) d\xi d\sigma$$

$$(26) \quad = \Phi_0 + \frac{1}{2i}(\Phi_+ - \Phi_-)$$

¹⁰All $\hat{\cdot}$ quantities refer to Fourier transform with the exception of $\hat{v} = \frac{v}{(1+|v|)^{1/2}}$ which denotes the relativistic velocity!

¹¹The definition involving the L^∞ norm is the obvious one.

where Φ_0 is the solution of the homogeneous problem $\square\Phi_0 = 0$ and

$$(27) \quad \Phi_{\pm}(t, x) = (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^3} \frac{e^{\pm i(t-\sigma)|\xi|}}{|\xi|} e^{ix \cdot \xi} \hat{J}(\sigma, \xi) d\sigma d\xi$$

In what follows we neglect Φ_0 , which contributes only a trivial term in the estimates below. As Φ_+ and Φ_- are treated in the same way we may assume that $\Phi = \Phi_+$. Therefore, ignoring constants,

$$(28) \quad \begin{aligned} \Phi(t, x) &\approx \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i(t-\sigma)|\xi|}}{|\xi|} e^{ix \cdot \xi} M(v) \xi \hat{f}(\sigma, \xi, v) d\sigma d\xi dv \\ &+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i(t-\sigma)|\xi|}}{|\xi|} e^{ix \cdot \xi} N(v) \nabla_v (\widehat{\Phi} * \hat{f})(\sigma, \xi, v) d\sigma d\xi dv. \end{aligned}$$

The first integral can be written in the form,

$$\begin{aligned} I(t, x) &= \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i(t-\sigma)|\xi|}}{|\xi|} e^{ix \cdot \xi} M(v) \xi \hat{f}(\sigma, \xi, v) d\sigma d\xi dv \\ &= \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} \int_{\mathbb{R}^3} M(v) \xi \left(\int_0^t e^{-\sigma i|\xi|} \hat{f}(\sigma, \xi, v) d\sigma \right) dv d\xi \end{aligned}$$

At this point we use Fact 1 listed at the end of Section 1. The precise statement is contained in the following lemma:

Lemma 2.2. *Assume that f solves the equation in (8). Then for any fixed v and ξ*

$$\begin{aligned} D(v, \xi) \int_0^t e^{-i\sigma|\xi|} \hat{f}(\sigma, \xi, v) d\sigma &= i \left[\frac{e^{-i\sigma|\xi|}}{|\xi|} \hat{f} \right]_0^t + i \int_0^t \frac{e^{-i\sigma|\xi|}}{|\xi|} \alpha(v) \cdot \nabla_v (\widehat{\Phi} * \hat{f})(\sigma, \xi, v) d\sigma. \\ D(v, \xi) &= \left(1 + \frac{\hat{v} \cdot \xi}{|\xi|} \right) \end{aligned}$$

It is now recognizable in the left hand side of this equality the smoothing effect represented by the factor $\frac{1}{|\xi|}$. We also observe that condition (23) guarantees that $\left| 1 \pm \frac{\hat{v} \cdot \xi}{|\xi|} \right| \approx 1$. We postpone the proof of this lemma to the appendix. Returning to I we write

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} \int_{\mathbb{R}^3} M(v) \xi D(v, \xi)^{-1} \left[\frac{e^{-i\sigma|\xi|}}{|\xi|} \hat{f} \right]_0^t dv d\xi \\ &+ \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} \int_{\mathbb{R}^3} M(v) \xi D(v, \xi)^{-1} \int_0^t \frac{e^{-i\sigma|\xi|}}{|\xi|} \alpha(v) \cdot \nabla_v (\widehat{\Phi} * \hat{f})(\sigma, \xi, v) d\sigma dv d\xi. \end{aligned}$$

Therefore, back to (28), ignoring constants and integrating by parts in v ,

$$\begin{aligned}
\Phi(t, x) &\approx \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} \int_{\mathbb{R}^3} M(v)\xi D(v, \xi)^{-1} \left[\frac{e^{\mp i\sigma|\xi|}}{|\xi|} \hat{f} \right]_0^t dv d\xi \\
&+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i(t-\sigma)|\xi|}}{|\xi|} e^{ix\cdot\xi} N(v) \nabla_v (\widehat{\Phi} * \text{hat}f)(\sigma, \xi, v) d\sigma d\xi dv. \\
&+ \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} \int_{\mathbb{R}^3} M(v)\xi D(v, \xi)^{-1} \int_0^t \frac{e^{-i\sigma|\xi|}}{|\xi|} \alpha(v) \cdot \nabla_v (\widehat{\Phi} * \hat{f})(\sigma, \xi, v) d\sigma \\
&\approx \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} \int_{\mathbb{R}^3} M(v)\xi D(v, \xi)^{-1} \left[\frac{e^{-i\sigma|\xi|}}{|\xi|} \hat{f} \right]_0^t dv d\xi \\
&+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i(t-\sigma)|\xi|}}{|\xi|} e^{ix\cdot\xi} \nabla_v [N(v) + M(v) \frac{\xi}{|\xi|} D(v, \xi)^{-1}] (\widehat{\Phi} * \hat{f})(\sigma, \xi, v) d\sigma d\xi dv.
\end{aligned}$$

Finally, differentiating in x ,

$$\begin{aligned}
(29) \quad \nabla_x \Phi(t, x) &\approx \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix\xi} M_0(v, \xi) \xi \hat{f}(t, \xi, v) dv d\xi \\
&+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} M_1(v, \xi) \xi \hat{f}(0, \xi, v) dv d\xi \\
&+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i((t-\sigma)|\xi|+x\xi)}}{|\xi|} M_2(v, \xi) \xi (\widehat{\Phi} * \hat{f})(\sigma, \xi, v) d\sigma dv d\xi
\end{aligned}$$

where

$$(30) \quad \frac{M(v)\xi D(v, \xi)^{-1}}{|\xi|^2} = M_0(v, \xi)$$

$$(31) \quad \frac{M(v)\xi D(v, \xi)^{-1}}{|\xi|} = M_1(v, \xi)$$

$$(32) \quad \nabla_v [N(v) + M(v) \frac{\xi}{|\xi|} D(v, \xi)^{-1}] = M_2(v, \xi)$$

It is not hard to show that $M_0 \in S^{-1}$ and $M_1, M_2 \in S^0$, uniformly with respect to v , where S^m is the class of usual symbols¹² of order m .

We now start the estimates for the derivatives of the characteristics. We first assume that all characteristics start at time $t = 0$. We begin with $\nabla_x V$. From (22)

$$\begin{aligned}
(33) \quad \nabla_x V(s, 0, x, v) &= \int_0^s \alpha'(V) \partial_x V(\sigma, 0, x, v) \Phi d\sigma \\
&+ \int_0^s \alpha(V) \cdot (\nabla \Phi)(\sigma, X(\sigma, 0, x, v)) \nabla_x X(\sigma, 0, x, v) d\sigma
\end{aligned}$$

where, as defined in (9) and (10), $\Phi = (E, B)$ and $\alpha(v) = (1, \hat{v} \times)$.

Thus

$$(34) \quad \nabla_x V(s, 0, x, v) \approx I_0 + I_1 + I_2,$$

¹²See for example [9], page 232, for a precise definition.

where

$$\begin{aligned} I_0 &= \int_0^s \alpha'(V) \partial_x V(\sigma, 0, x, v) \Phi d\sigma \\ I_1 &= \int_0^s \nabla_x X(\sigma, 0, x, v) (\nabla E)(\sigma, X(\sigma, 0, x, v)) d\sigma \\ I_2 &= \int_0^s \nabla_x X(\sigma, 0, x, v) \hat{V} \times (\nabla B)(\sigma, X(\sigma, 0, x, v)) d\sigma. \end{aligned}$$

Because $\alpha'(V)$ is uniformly bounded the estimate for I_0 is trivial. We first estimate I_1 . Using (29) we write

$$(35) \quad I_1 = I_{11} + I_{12} + I_{13}$$

where

$$\begin{aligned} I_{11} &\approx \int_0^s \nabla_x X(\sigma, 0, x, v) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{iX(\sigma, 0, x, v) \cdot \xi} M_0(w, \xi) \xi \hat{f}(\sigma, \xi, w) dw d\xi d\sigma \\ I_{12} &\approx \int_0^s \int_{\mathbb{R}^3} \frac{e^{i(\sigma|\xi| + X(\sigma, 0, x, v) \cdot \xi)}}{|\xi|} \nabla_x X(\sigma, 0, x, v) \int_{\mathbb{R}^3} M_1(w, \xi) \xi \hat{f}(0, \xi, w) dw d\xi d\sigma \\ I_{13} &\approx \int_0^s \int_{\mathbb{R}^3} \frac{e^{i(\sigma|\xi| + X(\sigma, 0, x, v) \cdot \xi)}}{|\xi|} \nabla_x X(\sigma, 0, x, v) \int_0^\sigma \int_{\mathbb{R}^3} e^{-i\tau|\xi|} M_2(w, \xi) \xi (\hat{E} * \hat{f})(\tau, \xi, w) d\tau dw d\xi d\sigma \end{aligned}$$

The integral I_{11} can be directly estimated since its symbol $K_{11}(w, \xi) = M_0(w, \xi) \xi \in S^0$. Before estimating I_{12} and I_{13} we need to do some more work. To get additional smoothing we need in fact to appeal to Fact 2 mentioned at the end of Section 1. In view of (20) we can write,

$$\begin{aligned} (36) \quad I_{12} &= I_{121} + I_{122} + I_{123} + I_{124} \\ I_{121} &\approx -\nabla_x X(s, 0, x, v) \int_{\mathbb{R}^3} \frac{e^{i(s|\xi| + X(s, 0, x, v) \cdot \xi)}}{|\xi|} \int_{\mathbb{R}^3} K_{121}(w, \xi) \hat{f}(0, \xi, w) dw d\xi \\ I_{122} &\approx \nabla_x X(0, 0, x, v) \int_{\mathbb{R}^3} e^{ix\xi} \int_{\mathbb{R}^3} K_{122}(w, \xi) \hat{f}(0, \xi, w) dw d\xi \\ I_{123} &\approx \int_0^s \nabla_x \hat{V}(\sigma, 0, x, v) \int_{\mathbb{R}^3} \frac{e^{i(\sigma|\xi| + X(\sigma, 0, x, v) \cdot \xi)}}{|\xi|} \int_{\mathbb{R}^3} K_{123}(w, \xi) \hat{f}(0, \xi, w) dw d\xi d\sigma \\ I_{124} &\approx \int_0^s \nabla_x X(\sigma, 0, x, v) \int_{\mathbb{R}^3} \frac{e^{i(\sigma|\xi| + X(\sigma, 0, x, v) \cdot \xi)}}{|\xi|} \\ &\quad \times \int_{\mathbb{R}^3} \beta(V(\sigma)) \alpha(V(\sigma)) \Phi(\sigma, X(\sigma)) \cdot K_{124}(w, \xi) \hat{f}(0, \xi, w) dw d\xi d\sigma \end{aligned}$$

and

$$(37) \quad K_{121}(w, \xi) = \frac{M_1(w, \xi) \xi}{(|\xi| + \hat{V}(s) \cdot \xi)}$$

$$(38) \quad K_{122}(w, \xi) = \frac{M_1(w, \xi) \xi}{(|\xi| + \hat{v} \cdot \xi) |\xi|}$$

$$(39) \quad K_{123}(w, \xi) = \frac{M_1(w, \xi) \xi}{(|\xi| + \hat{V}(\sigma) \cdot \xi)}$$

$$(40) \quad K_{124}(w, \xi) = \frac{|\xi|^2 M_1(w, \xi)}{(|\xi| + \hat{V}(\sigma) \cdot \xi)^2}.$$

In a similar fashion we also have,

$$\begin{aligned}
I_{13} &= I_{131} + I_{132} + I_{133} + I_{134} \\
I_{131} &\approx -\nabla_x X(s, 0, x, v) \int_{\mathbb{R}^3} \int_0^s \int_{\mathbb{R}^3} \frac{e^{i[(s-\tau)|\xi| + X(s, 0, x, v) \cdot \xi]}}{|\xi|} K_{131}(w, \xi) (\widehat{E} * \hat{f})(\tau, \xi, w) d\tau dw d\xi \\
I_{132} &\approx \int_0^s \nabla_x \hat{V}(\sigma, 0, x, v) \int_{\mathbb{R}^3} \int_0^\sigma \frac{e^{i[(\sigma-\tau)|\xi| + X(\sigma, 0, x, v) \cdot \xi]}}{|\xi|} \int_{\mathbb{R}^3} K_{132}(w, \xi) (\widehat{E} * \hat{f})(\tau, \xi, w) dw d\tau d\xi d\sigma \\
I_{133} &\approx \int_0^s \nabla_x X(\sigma, 0, x, v) \int_{\mathbb{R}^3} e^{iX(\sigma, 0, x, v) \cdot \xi} \int_{\mathbb{R}^3} K_{133}(w, \xi) (\widehat{E} * \hat{f})(\sigma, \xi, w) dw d\xi d\sigma \\
I_{134} &\approx \int_0^s \nabla_x X(\sigma, 0, x, v) \int_0^\sigma \int_{\mathbb{R}^3} \frac{e^{i[(\sigma-\tau)|\xi| + X(\sigma, 0, x, v) \cdot \xi]}}{|\xi|} \\
&\quad \cdot \int_{\mathbb{R}^3} \beta(V(\sigma)) \alpha(V(\sigma)) \Phi(\sigma, X(\sigma)) \times K_{134}(w, \xi) (\widehat{E} * \hat{f})(\tau, \xi, w) d\tau dw d\xi d\sigma.
\end{aligned}$$

where

$$(41) \quad K_{131}(w, \xi) \approx \frac{M_2(w, \xi) \xi}{(|\xi| + \hat{V}(s) \cdot \xi)}$$

$$(42) \quad K_{132}(w, \xi) \approx \frac{M_2(w, \xi) \xi}{(|\xi| + \hat{V}(\sigma) \cdot \xi)}$$

$$(43) \quad K_{133}(w, \xi) \approx \frac{M_2(w, \xi) \xi}{(|\xi| + \hat{V}(\sigma) \cdot \xi) |\xi|}$$

$$(44) \quad K_{134}(w, \xi) \approx \frac{|\xi|^2 M_2(w, \xi)}{(|\xi| + \hat{V}(\sigma) \cdot \xi)^2}$$

In a similar manner one can express also I_2 . In fact the only difference will be the presence of \hat{V} and $\partial_t \hat{V}$. But both functions are uniformly bounded thanks to (12) and (23).

Remark 2.3. We have intentionally suppressed the dependence of the symbols K on σ, x, v . This dependence, through \hat{V} , is trivial in view of the fact that $\hat{V} < 1$. We also observe that K_{122} and K_{133} belong uniformly to S^{-1} , while all the remaining K_{ijk} belong uniformly to the symbol class S^0 . They are in fact all homogeneous in $\xi \in \mathbb{R}^3$.

To estimate the terms involving these multipliers we use Fact 3 introduced at the end of Section 1. We formulate it explicitly in the following lemmas:

Lemma 2.4. *Assume that $m_k = m_k(x, \lambda; \xi) \in S^k$, uniformly with respect to $x \in \mathbb{R}^3$ and some family of parameters λ . Let P_k be the associated pseudodifferential operator,*

$$P_k f(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} m_k(x, \lambda; \xi) \hat{f}(\xi) d\xi$$

i.) For any $p \in [1, 3)$

$$(45) \quad \|P_{-1}g\|_\infty \lesssim (\|g\|_\infty + \|g\|_{L^p}).$$

ii.) For any $p > 3$ and $1 < q < \infty$,

$$(46) \quad \|P_0g\|_q \lesssim \|g\|_{L^q}$$

$$(47) \quad \|P_0g\|_\infty \lesssim \|g\|_\infty \log^+(\|\nabla g\|_{L^p}) + \max(1, \|g\|_{L^q}).$$

where we denote $\log^+ x = \log(2 + x)$.

We should need a similar lemma for Fourier integral operators.

Lemma 2.5. *Assume that $m_0 = m_0(t, x, \lambda; \xi) \in S^0$ uniformly with respect to $t \in \mathbb{R}$, $x \in \mathbb{R}^3$ and a family of parameters λ . Assume also that m_0 is homogeneous of degree zero in ξ . Let $T = T_{m_0}$ be the associated Fourier integral operator, defined by*

$$Tg(t, x, \lambda) = \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x \cdot \xi)}}{|\xi|} m_0(t, x, \lambda; \xi) \hat{g}(\xi) d\xi$$

for $g \in S(\mathbb{R}^3)$. Then for any $p \in (1, \infty]$

$$(48) \quad \|Tg\|_\infty \lesssim \max(1, \|g\|_{L^2}) + \|g\|_\infty \log^+(\|\nabla g\|_{L^p}).$$

Also

$$(49) \quad \|Tf\|_{L^6(\mathbb{R}_x^3)} \lesssim \|f\|_{L^2}.$$

The proof of these lemmas can be found in the appendix. We now use (45)-(48) to estimate each of the terms I_{ijk} . For example,

$$\|I_{11}(s)\|_\infty \lesssim \int_0^s \|\nabla_x X(\sigma, 0, x, v)\|_\infty \int_{\mathbb{R}^3} \left(\|f(\sigma, w)\|_\infty \log^+(\|\nabla_x f(\sigma, w)\|_{L^p}) + \|f(\sigma, w)\|_{L^\infty} + 1 \right) dw d\sigma$$

follows from the fact that the symbol $K_{11} = \xi M_0 \in S^0$ and hence (47) can be applied for any $p > 3$. Using (45) we estimate

$$\|I_{122}(s)\|_\infty \lesssim \|\nabla_x X(0, 0, x, v)\|_\infty \int_{\mathbb{R}^3} \left(\|f_0(w)\|_\infty + \|f_0(w)\|_{L^2} \right) dw$$

and

$$\|I_{133}(s)\|_\infty \lesssim \int_0^s \|\nabla_x X(\sigma, 0, x, v)\|_\infty \int_{\mathbb{R}^3} \left(\|Ef(\sigma, w)\|_\infty + \|Ef(\sigma, w)\|_{L^2} \right) dw d\sigma.$$

To estimate the remaining I_{ijk} terms, we introduce:

$$T[H](\sigma, y) = \int_0^\sigma \int_{\mathbb{R}^3} \frac{e^{i[(\sigma-\tau)|\xi|+y \cdot \xi]}}{|\xi|} K_{ijk}(w, \xi) \hat{H}(\tau, \xi) d\tau d\xi,$$

where $\hat{H}(\tau, \xi) = \int_{\mathbb{R}^3} e^{iy \cdot \xi} H(\tau, y) dy$ and $K_{ijk} \in S^0$ uniformly. Clearly the Fourier integral $T[H]$ can be estimated by using (48). Using this notation we now write

$$I_{121}(s) = \nabla_x X(s, 0, x, v) \int_{\mathbb{R}^3} T[f_{(0,w)}](0, X(\sigma, 0, x, v)) dw$$

where $f_{(0,w)}(y) = f_0(y, w)$. Consequently,

$$\|I_{121}(s)\|_\infty \lesssim \|\nabla_x X(s, 0, x, v)\|_\infty \int_{\mathbb{R}^3} \left(\max(\|f_0(w)\|_{L^2}, 1) + \log^+(\|\nabla f_0(w)\|_{L^p}) \right) dw$$

for $p \in (1, \infty]$. Similarly,

$$I_{123}(s) = \int_0^s \int_{\mathbb{R}^3} \nabla_x V(\sigma, 0, x, v) T[f_{(0,w)}](\sigma, X(\sigma, 0, x, v)) \beta(V(\sigma)) dw d\sigma,$$

and using the boundedness of β on the time interval $[0, T]$,

$$\|I_{123}(s)\|_\infty \lesssim \int_0^s \|\nabla_x V(\sigma, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} \left(\log^+(\|\nabla f_0(w)\|_{L^p}) + \|f_0(w)\|_{L^2} + 1 \right) dw d\sigma,$$

and similarly, thanks to the boundedness of α , β and Φ

$$\|I_{124}(s)\|_\infty \lesssim \int_0^s \|\nabla_x X(\sigma, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} \left(\log^+(\|\nabla f_0(w)\|_{L^p}) + \|f_0(w)\|_{L^2} + 1 \right) dw d\sigma,$$

again for $p \in (1, \infty]$. We use a similar argument to bound I_{13j} , with $j = 1, 2, 4$. Here we only show the calculations for the case $j = 4$. We write

$$I_{134} = \int_0^s \int_{\mathbb{R}^3} \nabla_x X(\sigma, 0, x, v) T[Ef_w](\sigma, X(\sigma, 0, x, v)) \beta(V(\sigma)) \Phi(\sigma, X(\sigma)) dw d\sigma$$

where $(Ef_w)(\tau, y) = E(\tau, y)f(\tau, y, w)$. Consequently, since α , β and Φ are uniformly bounded, in our time interval $[0, T]$,

$$\begin{aligned} \|I_{134}(s)\|_{L^\infty} &\lesssim \int_0^s \|\nabla_x X(\sigma, 0, x, v)\|_{L^\infty} \\ &\times \int_{\mathbb{R}^3} \left(\log^+(\|Ef(\sigma, w)\|_\infty \|\nabla(Ef)(\sigma, w)\|_{L^6}) + \|Ef(\sigma, w)\|_{L^2} + 1 \right) dw d\sigma \\ &\lesssim \int_0^s \|\nabla_x X(\sigma, 0, x, v)\|_{L^\infty} \\ &\times \int_{\mathbb{R}^3} \left(\|Ef(\sigma, w)\|_\infty \log^+(\|\nabla E(\sigma)\|_{L^6} + \|\nabla f(\sigma, w)\|_{L^6}) + \|Ef(\sigma, w)\|_{L^2} + 1 \right) dw d\sigma. \end{aligned}$$

We now summarize the above estimates for $\partial_x V$,

$$\begin{aligned} (50) \quad \|\nabla_x V(s, 0, x, v)\|_\infty &\lesssim \int_0^s \|\nabla_x X(\sigma, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_1 + W_2 + W_3 + W_4)(w) dw d\sigma \\ &+ \int_0^s \|\nabla_x V(\sigma, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw d\sigma \\ &+ \|\nabla_x X(s, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw + 1, \end{aligned}$$

where,

$$\begin{aligned} W_1(w) &= \|f(\sigma, w)\|_\infty \log^+(\|\nabla_x f(\sigma, w)\|_{L^p}) + \|f(\sigma, w)\|_{L^\infty} + 1 \\ W_2(w) &= \|\Phi f(\sigma, w)\|_\infty + \|\Phi f(\sigma, w)\|_{L^2} \\ W_3(w) &= \log^+(\|\nabla f_0(w)\|_{L^p}) + \|f_0(w)\|_{L^2} + 1 \\ W_4(w) &= \log^+(\|\Phi f(\sigma, w)\|_\infty \|\nabla \Phi(\sigma)\|_{L^6} + \|\nabla f(\sigma, w)\|_{L^6}) + \|\Phi f(\sigma, w)\|_{L^2} + 1, \end{aligned}$$

for $p > 3$. In a similar manner one also obtains

$$\begin{aligned} (51) \quad \|\nabla_v V(s, 0, x, v)\|_\infty &\lesssim \int_0^s \|\nabla_v X(\sigma, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_1 + W_2 + W_3 + W_4)(w) dw d\sigma \\ &+ \int_0^s \|\nabla_v V(\sigma, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw d\sigma \\ &+ \|\nabla_v X(s, 0, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw + 1. \end{aligned}$$

On the other hand, from (21), we simply have

$$(52) \quad \|\nabla_x X(s, 0, x, v)\|_\infty \lesssim 1 + \int_0^s \|\nabla_x V(\sigma, 0, x, v)\|_\infty d\sigma,$$

and

$$(53) \quad \|\nabla_v X(s, 0, x, v)\|_\infty \lesssim \int_0^s \|\nabla_v V(\sigma, 0, x, v)\|_\infty d\sigma.$$

We need two more ingredients to be able to complete the proof. The first is the following lemma:

Lemma 2.6. *Assume that Φ and f are solutions for (8) and that Φ satisfies the uniform boundedness condition (12). Then for any $t \in [0, T]$*

$$(54) \quad \|f(t)\|_{L_x^p L_v^q} \leq C_T \|f_0\|_\infty, \text{ for } 1 \leq p, q \leq \infty,$$

$$(55) \quad \|\nabla_x f(t)\|_{L_x^p L_v^q} \leq C_T \|\nabla_x f(t)\|_\infty, \text{ for } 1 \leq q, p \leq \infty,$$

$$(56) \quad \|\nabla_x \Phi(t)\|_{L^6} \leq C_T \|\nabla_x f(t)\|_\infty.$$

Also the proof of this lemma can be found in the appendix ¹³. Next we define

$$\begin{aligned} x^*(s) &= \sup_{\tau \in [0, s]} \|\nabla_{x,v} X(\tau, 0)\|_\infty \\ v^*(s) &= \sup_{\tau \in [0, s]} \|\nabla_{x,v} V(\tau, 0)\|_\infty. \end{aligned}$$

We then go back to (50), (51) (52) and (53). Making use of Lemmas 2.6 and 2.1 as well as (12) and (23), we obtain:

$$(57) \quad \begin{aligned} x^*(s) &\lesssim 1 + \int_0^s v^*(\sigma) d\sigma \\ v^*(s) &\lesssim 1 + \int_0^s (x^*(\sigma) + v^*(\sigma)) \log^+(\|\nabla_x f(\sigma)\|_{L^\infty}) d\sigma \end{aligned}$$

$$(58) \quad + x^*(s) \int_0^s \log^+(\|\nabla_x f(\sigma)\|_{L^\infty}) d\sigma.$$

The last important ingredient of the proof is the fact that the derivatives of f can be estimated in terms of the derivatives of the forward characteristics $X(s, 0, x, v)$, $V(s, 0, x, v)$, $s \in [0, T]$. In fact, using the backward characteristics $X(0, s, x, v)$, $V(0, s, x, v)$, we write

$$f(s, x, v) = f_0(X(0, s, x, v), V(0, s, x, v)),$$

Therefore,

$$\|\nabla_x f(s)\|_{L^\infty} \lesssim \|\nabla_x f_0\|_{L^\infty} \left(\|\nabla_x X(0, s)\|_{L^\infty} + \|\nabla_x V(0, s)\|_{L^\infty} \right).$$

According to Lemma 3.1 we can invert the flow and derive,

$$(59) \quad \begin{aligned} \|\nabla_x f(s)\|_{L^\infty} &\lesssim \left(\|\nabla_x X(s, 0)\|_\infty + \|\nabla_x X(s, 0)\|_\infty \right. \\ &\quad \left. + \|\nabla_v V(s, 0)\|_\infty + \|\nabla_v V(s, 0)\|_\infty \right)^5. \end{aligned}$$

¹³Here we do not try to obtain the best L^p estimate for $\nabla_x \Phi$. We needed to have an estimate for some $p > 3$ and the one for $p = 6$ was already in the literature.

We can now substitute (59) in (57) and (58) and finally derive

$$\begin{aligned} x^*(s) &\lesssim 1 + \int_0^s v^*(\sigma) d\sigma \\ v^*(s) &\lesssim 1 + \int_0^s (v^*(\sigma) + x^*(\sigma)) \log^+(x^*(\sigma) + v^*(\sigma)) d\sigma \\ &\quad + x^*(s) \int_0^s \log^+(x^*(\sigma) + v^*(\sigma)) d\sigma. \end{aligned}$$

It only remains to invoke Lemma 3.2, in the appendix, to conclude that for all $s \in [0, T]$,

$$(60) \quad \|\nabla_{x,v} X(s, 0)\|_\infty, \|\nabla_{x,v} V(s, 0)\|_\infty \leq C_T,$$

and Theorem 1.4 is proved for $t = 0$. To prove the theorem in full generality we first observe that if we insert (60) into (59) we obtain the uniform bound

$$(61) \quad \|\nabla_{x,v} f\|_\infty \leq C_T.$$

Then we repeat the argument presented above replacing $X(s, 0, x, v)$ with $X(s, t, x, v)$ and $V(s, 0, x, v)$ with $V(s, t, x, v)$. It is easy to see that (50) becomes

$$\begin{aligned} (62) \quad \|\nabla_x V(s, t, x, v)\|_\infty &\lesssim \int_0^s \|\nabla_x X(\sigma, t, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_1 + W_2 + W_3 + W_4)(w) dw d\sigma \\ &\quad + \int_0^s \|\nabla_x V(\sigma, t, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw d\sigma \\ &\quad + \|\nabla_x X(s, t, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw + 1, \end{aligned}$$

that (51) becomes

$$\begin{aligned} (63) \quad \|\nabla_v V(s, t, x, v)\|_\infty &\lesssim \int_0^s \|\nabla_v X(\sigma, t, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_1 + W_2 + W_3 + W_4)(w) dw d\sigma \\ &\quad + \int_0^s \|\nabla_v V(\sigma, t, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw d\sigma \\ &\quad + \|\nabla_v X(s, t, x, v)\|_{L^\infty} \int_{\mathbb{R}^3} (W_3 + W_4)(w) dw + 1, \end{aligned}$$

that (52) gives

$$(64) \quad \|\nabla_x X(s, t, x, v)\|_\infty \lesssim 1 + \int_0^s \|\nabla_x V(\sigma, t, x, v)\|_\infty d\sigma,$$

and similarly for (53). Thanks to (61), Lemma 2.6 and 2.1, (12) and (23) we have

$$(65) \quad \int_{\mathbb{R}^3} |W_i(w)| dw \leq C_T.$$

Then if we set

$$\begin{aligned} x^*(s, t) &= \sup_{\tau \in [0, s]} \|\nabla_{x,v} X(\tau, t)\|_\infty \\ v^*(s, t) &= \sup_{\tau \in [0, s]} \|\nabla_{x,v} V(\tau, t)\|_\infty, \end{aligned}$$

and we insert (65) into (62) and (63), we obtain

$$\begin{aligned} x^*(s, t) &\lesssim 1 + \int_0^s v^*(\sigma, t) d\sigma \\ v^*(s, t) &\lesssim 1 + \int_0^s (v^*(\sigma, t) + x^*(\sigma, t)) d\sigma + x^*(s, t), \end{aligned}$$

and again by Lemma 3.2

$$\|\nabla_{x,v} X(s, t)\|_\infty, \|\nabla_{x,v} V(s, t)\|_\infty \leq C_T,$$

for all $s, t \in [0, T]$. This concludes the proof of the theorem.

3. APPENDIX

Proof of Lemma 2.2

The proof is a simple consequence of integration by parts and the transport equation in (8). In fact

$$(66) \quad \int_0^t e^{-i\sigma|\xi|} \hat{f}(\sigma, \xi, v) d\sigma = \left[\frac{e^{-i\sigma|\xi|}}{-i|\xi|} \hat{f} \right]_0^t + \int \frac{e^{-i\sigma|\xi|}}{-i|\xi|} \partial_\sigma \hat{f}(\sigma, \xi, v) d\sigma.$$

The transport equation allows us to substitute $\partial_\sigma \hat{f}$ and continue the chain of equalities with

$$(67) \quad = \left[\frac{e^{-i\sigma|\xi|}}{-i|\xi|} \hat{f} \right]_0^t + \int_0^t e^{-i\sigma|\xi|} (\hat{v} \cdot \xi \hat{f})(\sigma, \xi, v) d\sigma + \int_0^t e^{-i\sigma|\xi|} (\alpha(v) \cdot \nabla_v (\widehat{\Phi} * \hat{f}))(\sigma, \xi, v) d\sigma.$$

Then the lemma is obtained by moving the second term of (67) to the left hand side of the equality.

Proof of Lemma 2.4

The Lemma is well known; we present its proof here for the sake of completeness and as an introduction to the more difficult proof of Lemma 2.5. For simplicity we shall pretend that our symbols m_k depend only on ξ . In view of the uniform boundedness of $m_k(t, x, \lambda; \xi)$ relative to the parameters t, x, λ the general case does not present any additional difficulties. We start by proving (45). First recall [9] that if

$$\bar{K}_{(\alpha)}(y) = \int_{\mathbb{R}^3} e^{iy \cdot \xi} m_\alpha(\xi) d\xi,$$

where $m_\alpha \in S^\alpha, \alpha \leq 0$, then

$$(68) \quad |\bar{K}_\alpha(y)| \lesssim \frac{1}{|y|^{3-\alpha}}.$$

Now write

$$P_{-1}g(x) = \int_{B_1(x)} \bar{K}_{-1}(x-y)g(y) dy + \int_{B_1(x)^c} \bar{K}_{-1}(x-y)g(y) dy,$$

where $B_1(x)$ is the ball in \mathbb{R}^3 centered at x and of radius 1, and $B_1(x)^c$ is its complement. We then use (68) to obtain, for any $p \in [1, 3)$,

$$|P_{-1}g(x)| \lesssim \|g\|_\infty + \|g\|_{L^p}.$$

The inequality (46) is a classic result of Harmonic Analysis; we refer to [9]. We now sketch the proof of (47). using a well known argument; see for example [1]. Fix $x \in \mathbb{R}^3$, $\epsilon > 0$ and write $g = g_1 + g_2$, where $g_1(y) = g(y)\chi_{B_\epsilon(x)}(y)$. Then if

$$m_{-1}(\xi) = \frac{m_0(\xi)}{|\xi|},$$

we can write

$$\begin{aligned} |P_0 g(x)| &\lesssim \left| \int_{|y-x| \leq \epsilon} \bar{K}_{-1}(y-x) \nabla g(y) dy \right| + \left| \int_{|y-x| > \epsilon} \bar{K}_0(y-x) g(y) dy \right| \\ &\lesssim \int_{|y-x| \leq \epsilon} \frac{1}{|y-x|^2} |\nabla g(y)| dy + \int_{1 > |y-x| > \epsilon} \frac{1}{|y-x|^3} |\nabla g(y)| dy \\ &\quad + \int_{|y-x| > 1} \frac{1}{|y-x|^3} |\nabla g(y)| dy \\ &\lesssim \epsilon^{3/p'-2} \|\nabla g\|_{L^p} + \log(\epsilon^{-1}) \|g\|_\infty + \|g\|_{L^q}, \end{aligned}$$

for any $p > 3$ and $1 < q < \infty$. Then pick $\epsilon^{-1/p'+2} = \|\nabla g\|_{L^p}$ and (47) is proved.

Proof of Lemma 2.5

As in the proof of the previous lemma we drop the dependence of m_0 on t, x, λ . Thus $m_0 = m_0(\xi)$ is an homogeneous symbol of degree zero in ξ . We first observe that it suffices to prove (48) for $t = 1$. In fact, by a simple change of variable and the homogeneity of m_0 , we have

$$(69) \quad (Tg)(t, y) = tT(g_t)(1, t^{-1}y)$$

where $g_t(y) = g(ty)$. Thus, assuming the estimate to be true for $t = 1$ we obtain

$$\begin{aligned} \|Tg(t)\|_\infty &= t\|(Tg_t)(1)\|_\infty \\ &\lesssim \|g\|_\infty \log^+(\|\nabla g\|_{L^p}) + \|g\|_{L^p} + 1. \end{aligned}$$

From now on we write $Tg(x) = Tg(1, x)$. For simplicity we may also assume $x = 0$. Then

$$(70) \quad Tg(0) = \int_{\mathbb{R}^3} K(y)g(y)dy$$

$$(71) \quad K(y) = \int_{\mathbb{R}^3} e^{i(|\xi|-y \cdot \xi)} \frac{m_0(\xi)}{|\xi|} d\xi,$$

By a simple limiting argument¹⁴ we can replace $K(y)$ in (71) by the kernel

$$(72) \quad K(y) = \int_{\mathbb{R}^3} e^{i(|\xi|-y \cdot \xi)} \frac{m_0(\xi)}{|\xi|} \chi(|\xi|) d\xi,$$

with, χ an arbitrary test function on $[0, \infty)$ verifying

$$(73) \quad 0 \leq \chi \leq 1, \quad \text{and} \quad \int |\chi'(u)| du \leq 10$$

Without loss of generality we may also assume that $\chi(|\xi|)$ is supported in the complement of $|\xi| \leq 1$. Indeed we can split $K = K_l + K_h$ with

$$K_l(y) = \int_{|\xi| \leq 1} e^{i(|\xi|+y \cdot \xi)} \frac{m_0(\xi)}{|\xi|} \chi(|\xi|) d\xi,$$

¹⁴Indeed $Tg(0) = \lim_{L \rightarrow \infty} T_L g(0)$ with $K_L(y) = \int_{\mathbb{R}^3} e^{i(|\xi|-y \cdot \xi)} \frac{m_0(\xi)}{|\xi|} \chi(|\xi|/L) d\xi$ where χ is a test function on $[0, \infty)$ with $\chi(u) = 1$ for $0 \leq u \leq 1$. It thus suffices to prove an estimate for all L . To do this it suffices to prove an estimate, for all test function $\chi(u)$ verifying (73).

Clearly, $\int_{\mathbb{R}^3} K_l(y)g(y)dy = \int_{|\xi|<1} \frac{m_0(\xi)}{|\xi|} \chi(|\xi|)\hat{g}(\xi)$ and therefore,

$$(74) \quad \left| \int_{\mathbb{R}^3} K_l(y)g(y)dy \right| = \left| \int_{|\xi|\leq 1} \frac{\hat{g}(\xi)}{|\xi|} d\xi \right| \lesssim \|g\|_{L^2}.$$

In what follows we shall therefore assume that $K(y) = K_h(y)$.

If m_0 were radially symmetric, then a sharp estimate for $K_h(y)$ was given by Stein¹⁵ in [9], 6.13, page 426. By expressing (72) in spherical coordinates we obtain

$$(75) \quad K(y) = \int_0^{2\pi} \int_1^\infty \int_0^\pi e^{i(u+|y|u \cos \theta)} m_0(\theta, w) u \chi(u) \sin \theta d\theta du dw.$$

Notice that because $m_0 \in S^0$ is homogeneous it will not depend on u . We now estimate the inner integral in θ . By integration by parts

$$\int_0^\pi e^{i|y|u \cos \theta} m_0(\theta, w) \sin \theta d\theta = \left[e^{i|y|u \cos \theta} \frac{m_0(\theta, w)}{i|y|u} \right]_0^\pi - \int_0^\pi e^{i|y|u \cos \theta} \frac{\partial_\theta m_0(\theta, w)}{i|y|u} d\theta$$

One can then use the method of stationary phase to estimate the asymptotic behavior of the integral in the right hand side (see for example [9] page 334 and (12) page 336) and prove that as $u \rightarrow \infty$

$$(76) \quad \int_0^\pi e^{i|y|u \cos \theta} m_0(\theta, w) \sin \theta d\theta \approx \frac{e^{-i|y|u}}{|y|u} \left(m_0(\pi, w) + \frac{1}{|y|^{1/2} u^{1/2}} \partial_\theta m_0(\pi, w) \right) - \frac{e^{i|y|u}}{|y|u} \left(m_0(0, w) + \frac{1}{|y|^{1/2} u^{1/2}} \partial_\theta m_0(0, w) \right).$$

If now we insert (76) into (75) we derive,

$$(77) \quad K(y) \approx \frac{H_z(y)}{|y|} \int_1^\infty e^{iu(1-|y|)} \chi(u) du - \frac{H_0(y)}{|y|} \int_1^\infty e^{iu(1+|y|)} \chi(u) du,$$

where

$$H_z(y) = \int_0^{2\pi} H_z(w, y) dw,$$

and

$$(78) \quad H_z(w, y) = \left(m_0(z, w) + \frac{1}{|y|^{1/2} u^{1/2}} \partial_\theta m_0(z, w) \right).$$

Observe that, for fixed z , both $H_z(y)$ and $\partial_y H_z(y)$ are uniformly bounded. Clearly the term in (77) with phase function $u(1+|y|)$ is easier to treat and we ignore it in what follows.

We next decompose the integral $I = \int_{\mathbb{R}^3} K(y)g(y) dy$ as follows:

$$(79) \quad \begin{aligned} I &= I_1 + I_2 + I_3 \\ I_1 &= \int_{|1-|y|| \leq 2\epsilon} \psi_\epsilon(y) K(y) g(y) dy \\ I_2 &= \int_{|1-|y|| > \epsilon; \frac{1}{2} < |y| < 10} (1 - \psi_\epsilon(y)) K(y) g(y) dy \\ I_3 &= \int_{\{|y| \leq 1/2\} \cup \{|y| \geq 10\}} K(y) g(y) dy \end{aligned}$$

¹⁵We would like to take the opportunity to thank E. Stein for pointing this out to us.

with $\psi_\epsilon(y)$ a smooth test function supported in the region $1 - |y| \leq 2\epsilon$ equal to 1 on $1 - |y| \leq \epsilon$ and such that $|\nabla_y \psi(y)| \lesssim \frac{1}{\epsilon}$.

We start by estimating I_3 , which is the easiest. In fact from (78) we obtain by a straightforward estimate that

$$|K(y)| \lesssim \frac{1}{|y|^{3/2}} \quad \text{for } |y| \leq 1/2.$$

On the other hand, for $|y| > 10$, we can integrate by parts in (78), and derive in view of (73),

$$\left| \int_1^\infty e^{iu(1-|y|)} \chi(u) du \right| \leq \frac{1}{|y| - 1} \int_1^\infty |\chi'(u)| du \lesssim \frac{1}{|y|}.$$

Therefore,

$$|K(y)| \lesssim \frac{1}{|y|^2} \quad \text{for } |y| > 10.$$

whence

$$(80) \quad |I_3| \lesssim \|g\|_\infty + \|g\|_{L^2}.$$

To estimate I_2 we proceed in the same way, by integration by parts and using (73), to derive

$$|K(y)| \approx \frac{1}{|1 - |y||} \quad \text{for any } ||y| - 1| \geq \epsilon, \quad \frac{1}{2} < |y| < 10.$$

Then clearly,

$$(81) \quad |I_2| \lesssim \|g\|_\infty \log(\epsilon^{-1}).$$

The estimate of I_1 is a little more delicate. Clearly, for $||y| - 1| \leq \epsilon$ we have

$$K(y) \approx H(y) \int_1^\infty e^{iu(1-|y|)} \chi(u) du dw,$$

Consider the new kernel

$$\tilde{K}_j(y) = \frac{y_j}{|y|} H(y) \int_1^\infty \frac{e^{iu(1-|y|)}}{u} \chi(u) du$$

Observe that,

$$\nabla \cdot \tilde{K}(y) \approx K(y) + \partial_j \left(\frac{y_j}{|y|} H(y) \right) \int_1^\infty \frac{e^{iu(1-|y|)}}{u} \chi(u) du dw,$$

Therefore,

$$(82) \quad K(y) \approx \nabla \cdot \tilde{K}(y) + \tilde{K}(y)$$

$$(83) \quad \tilde{K}(y) \approx \partial_j \left(\frac{y_j}{|y|} H(y) \right) \int_1^\infty \frac{e^{iu(1-|y|)}}{u} \chi(u) du$$

$$(84) \quad I_1 = \int \psi_\epsilon(y) K(y) g(y) \approx \int \psi_\epsilon(y) \tilde{K}(y) g(y) dy - \int \tilde{K}(y) \nabla_y (\psi_\epsilon(y) g(y)) dy$$

$$(85) \quad \approx I_{11} + I_{12} + I_{13}$$

where,

$$(86) \quad I_{11} = \int \bar{K}(y) \psi_\epsilon(y) \nabla_y g(y)$$

$$(87) \quad I_{12} = \int \bar{K}(y) \nabla \psi_\epsilon(y) g(y)$$

$$(88) \quad I_{13} = \int \tilde{K}(y) \psi_\epsilon(y) g(y) dy$$

We start by estimating the kernel $\bar{K}(y)$

$$\bar{K}(y) = \bar{K}_l(y) + \bar{K}_h(y)$$

where

$$(89) \quad \bar{K}_l(y) = \frac{y}{|y|} H(y) \int_1^M \frac{e^{i(|y|-1)u}}{u} \chi(u) du$$

$$(90) \quad \bar{K}_h(y) = \frac{y}{|y|} H(y) \int_M^\infty \frac{e^{i(|y|-1)u}}{u} \chi(u) du,$$

and $M = ||y| - 1|^{-1}$. Clearly,

$$\begin{aligned} |\bar{K}_l(y)| &\lesssim \int_1^M \frac{1}{u} du \leq \log M = -\log |1 - |y|| \\ |\bar{K}_h(y)| &\lesssim \int_M^\infty \left| \frac{d}{du} \left(\frac{\chi(u)}{u} \right) \right| du \leq M^{-1} = |1 - |y|| \end{aligned}$$

Therefore,

$$\begin{aligned} |I_{11}| &\leq \left| \int \bar{K}_l(y) \psi_\epsilon(y) \nabla_y g(y) dy \right| + \left| \int \bar{K}_h(y) \psi_\epsilon(y) \nabla_y g(y) dy \right| \\ &\lesssim \int \psi_\epsilon(y) \log \left(|1 - |y||^{-1} \right) \\ &\lesssim \|\nabla g\|_{L^p} \left(\int_{|1-|y|| < \epsilon} \log^{p'} \left(|1 - |y||^{-1} \right) dy \right)^{1/p'} \lesssim \epsilon^\gamma \|\nabla g\|_{L^p} \end{aligned}$$

for any $p > 1$ and some $0 < \gamma < 1$.

Now, since $|\nabla \psi_\epsilon(y)| \lesssim \epsilon^{-1}$,

$$(91) \quad |I_{12}| \lesssim \epsilon^{-1} \|g\|_{L^\infty} \int_{|1-|y|| \leq \epsilon} \log |1 - |y||^{-1} dy \lesssim \log \epsilon^{-1} \|g\|_{L^\infty}$$

Finally, since \tilde{K} admits the same estimate as \bar{K} , we find,

$$|I_{13}| \lesssim \|g\|_{L^\infty} \int_{|1-|y||} |\tilde{K}(y)| dy \lesssim \|g\|_{L^\infty}.$$

Therefore,

$$(92) \quad |I_1| \lesssim \epsilon^\gamma \|\nabla g\|_{L^p} + \log \epsilon^{-1} \|g\|_{L^\infty}$$

Combining this with the estimates for I_2, I_3 , see (79), (80), (81) as well as (74), we derive

$$(93) \quad |Tg(0)| = |I| \lesssim \epsilon^\gamma \|\nabla g\|_{L^p} + \log \epsilon^{-1} \|g\|_{L^\infty} + \|g\|_{L^2}$$

We now chose $\epsilon^{-1} = (2 + \|\nabla g\|_{L^p})^{\frac{1}{\gamma}}$ and infer that

$$|Tg(0)| \leq 1 + \|g\|_{L^2} + \frac{1}{\gamma} \|g\|_{L^\infty} \log^+(\|\nabla g\|_{L^p})$$

as desired.

The last inequality (49) of Lemma 2.5 for $t = 1$, can be found in [9], page 399.

Proof of Lemma 2.6

Proof. It is easy to see that (54) and (55) are a simple consequence of the fact that $\|f\|_\infty \leq \|f_0\|_\infty$ and (23). The proof of (56) requires more care. Recalling (29), in the proof of Theorem 1.4, we can write

$$\nabla_x \Phi = J_1 + J_2 + J_3,$$

with

$$\begin{aligned} J_1(t, x) &\approx \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix\xi} M_0(v, \xi) \xi \hat{f}(t, \xi, v) dv d\xi \\ J_2(t, x) &\approx \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{i(t|\xi|+x\xi)}}{|\xi|} M_1(v, \xi) \xi \hat{f}(0, \xi, v) dv d\xi \\ J_3(t, x) &\approx \int_0^t \int_{\mathbb{R}^3} \frac{e^{i((t-\sigma)|\xi|+x\xi)}}{|\xi|} \int_{\mathbb{R}^3} M_2(v, \xi) \xi \widehat{\Phi f}(\sigma, \xi, v) d\sigma dv d\xi, \end{aligned}$$

where $M_0 \in S^{-1}$, $M_1, M_2 \in S^0$, uniformly in v . Since $M_0(v, \xi)\xi \in S^0$, we can make use of the estimate (46), see Lemma 2.4, and the compactness of the support of f in v , see (23), to infer

$$(94) \quad \|J_1(t)\|_{L^p} \lesssim \|f(t)\|_{L^p}$$

for any $1 < p < \infty$. Making use of the estimate (49) of Lemma 2.5

$$(95) \quad \|J_2(t)\|_{L^6} \lesssim \int_{\mathbb{R}^3} \|\nabla_x f_0(t, v)\|_{L^2} dv$$

$$(96) \quad \|J_3(t)\|_{L^6} \lesssim \int_0^t \int_{\mathbb{R}^3} \|\nabla_x(\Phi f)(\sigma, v)\|_{L^2} d\sigma dv.$$

Then finally, using (54), (55) (12), Lemma 2.1 and Gronwall inequality, we derive (56). \square

Next lemma shows that the derivatives of the characteristics of the backwards flow can be estimated with respect to the derivatives of the characteristics of the forward flow.

Lemma 3.1. *Let $s \in [0, T]$ and denote by $(X(s, 0, x, v), V(s, 0, x, v))$ the forward characteristics, associated to (8) see also (21) and (22), initiating at $t = 0$ and by $X(0, s, x, v), V(0, s, x, v)$ the backward characteristics initiating at time $t = s$. Then,*

$$(97) \quad \|\nabla_{x,v}(X(0, s, x, v), V(0, s, x, v))\|_{L^\infty} \lesssim \left(\|\nabla_{x,v}(X(s, 0, x, v), V(s, 0, x, v))\|_{L^\infty} \right)^5$$

Proof. By definition

$$\begin{aligned} X\left(t, s, X(s, t, x, v), V(s, t, x, v)\right) &= x \\ V\left(t, s, X(s, t, x, v), V(s, t, x, v)\right) &= v \end{aligned}$$

Differentiating and applying Cramer's rule it is easy to see that $\nabla_{x,v}(X(0, s, x, v), V(0, s, x, v))$ can be expressed in terms of an homogeneous polynomial of degree 5 in the components of $\nabla_{x,v}(X(s, 0, x, v), V(s, 0, x, v))$ divided by the determinant $\Delta(s, 0, x, v)$ of the Jacobian of the transformation,

$$\Delta(s, 0, x, v) = |\nabla_{x,v}(X(s, 0, x, v), V(s, 0, x, v))|.$$

It thus remains to establish a bound from below for Δ in the interval $[0, T]$. To do this we shall make use of our uniform bounds for Φ . Recall that,

$$\begin{cases} \frac{dX}{ds} = \frac{V}{\sqrt{1+|V|^2}} =: \hat{V} \\ \frac{dV}{ds} = \alpha(V)\Phi(s, X) \\ X(t, 0, x, v) = x \quad \text{and} \quad V(t, 0, x, v) = v, \quad \text{for } t=0. \end{cases}$$

Differentiating,

$$(98) \quad \frac{d}{ds} \begin{pmatrix} \partial_x X, \partial_v X \\ \partial_x V, \partial_v V \end{pmatrix} = \begin{pmatrix} 0, & \beta(v) \\ \alpha(v)\partial_x \Phi, & \alpha'(V)\Phi \end{pmatrix} \begin{pmatrix} \partial_x X, \partial_v X \\ \partial_x V, \partial_v V \end{pmatrix}$$

Recall that $\Delta(s) = \Delta(0) \exp \int_0^s \text{tr} A(s) ds$ with $A(s)$ the first matrix on the right hand side of (98). Since the trace of A depends only on $\alpha'(V)\Phi$, which is bounded on our interval $[0, T]$, and $\Delta(0) = 1$ we infer that $\Delta(s) \approx 1$ as desired. \square

Lemma 3.2. *Let $x(s)$ and $v(x)$ be two nonnegative increasing functions defined for $s \in [0, T]$ and such that*

$$(99) \quad x(s) \leq c_1 + b_1 \int_0^s v(\sigma) d\sigma$$

$$(100) \quad v(s) \leq c_2 + b_2 x(s) \int_0^s \log^+(v(\sigma) + x(\sigma)) d\sigma$$

$$(101) \quad + b_3 \int_0^s (v(\sigma) + x(\sigma)) \log^+(v(\sigma) + x(\sigma)) d\sigma.$$

Then

$$v(s), x(s) \leq C_T, \quad \text{for all } s \in [0, T].$$

Proof. Without loss of generality we can assume that $x(\sigma) \geq M_0$ for all $\sigma \geq 0$ and M_0 is such that

$$(102) \quad \frac{M_0}{\log^+(M_0)} \geq M_0^{1/2}.$$

Combining (99) with (100) and (101) one easily obtains that

$$(103) \quad \begin{aligned} x(s) + v(s) &\leq c_3 + b_4 \int_0^s (x(\sigma) + v(\sigma)) d\sigma \int_0^s \log^+(v(\sigma) + x(\sigma)) d\sigma \\ &+ b_5 \int_0^s (x(\sigma) + v(\sigma)) \log^+(v(\sigma) + x(\sigma)) d\sigma. \end{aligned}$$

Again without loss of generality we can assume that $c_3 = 0, b_4 = b_5 = 1$. We set $x(s) + v(s) = w(s)$ and from (103) we obtain

$$(104) \quad \begin{aligned} w(s) &\leq + \int_0^s w(\sigma) d\sigma \int_0^s \log^+(w(\sigma)) d\sigma + \int_0^s w(\sigma) \log^+(w(\sigma)) d\sigma \\ &\leq \int_0^s w(\sigma) d\sigma \log^+(w(s)). \end{aligned}$$

We now set $y(\sigma) = \frac{w(\sigma)}{\log^+(w(\sigma))}$. Using (102) it is easy to see that for all $\sigma \in [0, T]$

$$w(\sigma) \leq 2 \frac{w(\sigma)}{\log^+(w(s))} \log^+ \left(\frac{w(\sigma)}{\log^+(w(s))} \right).$$

Then $y(s)$ satisfies the Gronwall's inequality

$$y(s) \leq \int_0^s y(\sigma) \log^+(y(\sigma)) d\sigma$$

from which we immediately deduce that $w(s) \leq C$ for all $s \in [0, T]$ and hence the lemma. \square

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