ON EMERGING SCARRED SURFACES FOR THE EINSTEIN VACUUM EQUATIONS

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1. INTRODUCTION

This is a follow up on our work [K-R:trapped] in which we have presented a modified, simpler version of the remarkable recent result of Christodoulou, see [Chr:book], on the formation of evolutionary trapped surfaces in vacuum. The approach in [K-R:trapped], based on a different scaling[1] than that of [Chr:book], allowed us not only to reprove Christodoulou's trapped surface result, but also enabled us to localize with respect to small angular regions. This led us, in particular, to a simple result concerning the formation of pre-scarred surfaces[2]. Both results were based on the proof of a semi-global existence theorem which established the propagation of precise estimates, for both curvature and Ricci coefficients, starting with non-trivial initial conditions on an outgoing null hypersurface.

In this paper we provide a considerable extension of our result on pre-scared surfaces to allow for the formation of a surface with multiple pre-scared angular regions which, together, can cover an arbitrarily large portion of the surface. In a forthcoming paper we plan to show that once a significant part of the surface is pre-scared, it can be additionally deformed to produce a bona-fide trapped surface. This result implies, in particular, that Christodoulou's crucial uniform lower bound initial condition necessary for the formation of a trapped surface can be relaxed to an average condition, which requires only that the lower bound holds true only on a sufficiently large angular portion of the initial outgoing null hyper-surface.

In this paper we state and discuss three related results.

(1) We state an optimal propagation result, critical with respect to the natural null scaling of Einstein vacuum equations introduced in [K-R:trapped] (which dealt with the subcritical regime), see theorem 1.14. In this paper, prompted, in part, by our interest on pre-scarred surfaces and in part by reflecting on the scale transformation in the work of Reiterer and Trubowitz [R-T], we note that the argument of the main propagation theorem in [K-R:trapped] proves

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[1]The natural null, parabolic, scaling of the Einstein vacuum equations

[2]These are surfaces for which the outgoing expansion is negative in an open subset of the surface

in fact a stronger, indeed optimal result. We are happy to acknowledge that a related result is stated in theorem 8.1. of [R-T], in a different setting. We would like to thank Reiterer and Trubowitz for drawing our attention and making an effort to explain its formulation to us.

(2) We state, see theorem 2.4, an angular localized version of the global energy estimates for the null curvature components of theorem 1.14. The proof relies on a natural modification of the proof in theorem 1.14 and is discussed in section 5.

(3) We give a large class of critical, sufficient conditions on the initial data, which lead to the formation of pre-scarred surfaces. The main result is stated in theorem 2.8. The proof rests on theorem 1.14 as well as on a localized version of the Ricci coefficient estimates in [K-R:trapped]. As mentioned above, the importance of this result is due to the fact that once a significant part of a surface is pre-scarred, it can be deformed to a real trapped surface.

Concerning the new propagation result stated in theorem 1.14, we note that the main new idea is to use, in addition to the small parameter $\delta > 0$, originating in the short pulse method of [Chr:book], a new small parameter ϵ with $\delta^{1/2}\epsilon^{-1}$ sufficiently small. The parameter δ is used to define scale invariant norms, similar to those we have introduced in [K-R:trapped] but with one important modification. In the main result of [K-R:trapped], for example, the scaling was such that all null curvature components, except the component denoted by α, were bounded (in its scale invariant norms) . The behavior (in the scale invariant norm) of the *anomalous* component α, on the other hand, was $\delta^{-1/2}$. Here we choose the scaling with respect to δ such that the scale invariant norm of α is bounded, independent of the second parameter ϵ, and the scale invariant norms of all other curvature components are proportional to ϵ, i.e. small. All results in [K-R:trapped] correspond precisely to the case when ϵ is chosen to be proportional to $\delta^{1/2}$. It is quite remarkable that the proof of the stronger propagation result in theorem 1.14 is exactly the same as in [K-R:trapped]. This is surprising, especially considering that the initial data in theorem 1.14 is allowed to be $\delta^{-\frac{1}{2}}\epsilon$ times bigger[3] than that in [K-R:trapped] (as measured in absolute, unscaled norms). In [K-R:trapped] nonlinear non-anomalous interactions were controlled by the scale invariant Hölder estimates

$$\|\psi\cdot\phi\|_{\mathcal{L}^p_{(sc)}} \lesssim \delta^{\frac{1}{2}}\|\psi\|_{\mathcal{L}^r_{(sc)}}\|\phi\|_{\mathcal{L}^q_{(sc)}}, \qquad \frac{1}{r}+\frac{1}{q}=\frac{1}{p}.$$

In this work the new critical scaling does not generate a small factor of $\delta^{\frac{1}{2}}$ in such interactions. Instead we have

$$\|\psi\cdot\phi\|_{\mathcal{L}^p_{(sc)}} \lesssim \|\psi\|_{\mathcal{L}^r_{(sc)}}\|\phi\|_{\mathcal{L}^q_{(sc)}}, \qquad \frac{1}{r}+\frac{1}{q}=\frac{1}{p}.$$

For non-anomalous ψ and ϕ the scale invariant norms on the right hand side are both of size ϵ and so is the expected value of the left hand side norm. This analysis indicates that with the new scaling the factor $\delta^{\frac{1}{2}}$ of quadratic interactions is effectively replaced by the independent small parameter ϵ.

[3]More precisely all components of the curvature tensor , except α, are $\delta^{-\frac{1}{2}}\epsilon$ times bigger. The α component behaves exactly the same as in [K-R:trapped].

In the result on the formation of a pre-scarred surface we describe a set of initial data which lead to a space-time with a surface containing approximately $\delta^{-\frac{1}{2}}q$ angular regions of size $\delta^{\frac{1}{2}}q^{-1}$, each of which is pre-trapped for some sufficiently small parameter q.

We start by recalling the framework of double null foliations in which the results of both [Chr:book] and [K-R:trapped] are formulated.

1.1. **Double null foliations.** We consider a region $\mathcal{D} = \mathcal{D}(u_*, \underline{u}_*)$ of a vacuum spacetime (M, g) spanned by a double null foliation generated by the optical functions $(u, \underline{u})$ increasing towards the future, $0 \le u \le u_*$ and $0 \le \underline{u} \le \underline{u}_*$. We denote by H_u the outgoing null hypersurfaces generated by the level surfaces of u and by $\underline{H}_{\underline{u}}$ the incoming null hypersurfaces generated level hypersurfaces of $\underline{u}$. We write $S_{u,\underline{u}} = H_u \cap \underline{H}_{\underline{u}}$ and denote by $H_u^{(\underline{u}_1,\underline{u}_2)}$, and $\underline{H}_{\underline{u}}^{(u_1,u_2)}$ the regions of these null hypersurfaces defined by $\underline{u}_1 \le \underline{u} \le \underline{u}_2$ and respectively $u_1 \le u \le u_2$. Let $L, \underline{L}$ be the geodesic vectorfields associated to the two foliations and define the null lapse Ω and connection, or Ricci, coefficients, $\chi, \omega, \eta, \underline{\eta}, \underline{\chi}, \underline{\omega}$,

$$\frac{1}{2}\Omega^2 = -g(L, \underline{L})^{-1} \tag{1}$$

$$\begin{aligned} \chi_{ab} &= g(D_a e_4, e_b), \qquad & \underline{\chi}_{ab} &= g(D_a e_3, e_b), \\ \eta_a &= -\frac{1}{2}g(D_3 e_a, e_4), \quad & \underline{\eta}_a &= -\frac{1}{2}g(D_4 e_a, e_3) \\ \omega &= -\frac{1}{4}g(D_4 e_3, e_4), \qquad & \underline{\omega} &= -\frac{1}{4}g(D_3 e_4, e_3) \end{aligned} \tag{2}$$

where $e_3 = \Omega\underline{L}$, $e_4 = \Omega L$ and $D_a = D_{e_{(a)}}$. As usual we decompose the null second fundamental forms $\chi, \underline{\chi}$ into their traceless parts $\hat{\chi}, \hat{\underline{\chi}}$ and traceless parts, or *expansions*, $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}$. We also introduce the null curvature components,

$$\begin{aligned} \alpha_{ab} &= R(e_a, e_4, e_b, e_4), \qquad & \underline{\alpha}_{ab} &= R(e_a, e_3, e_b, e_3), \\ \beta_a &= \frac{1}{2}R(e_a, e_4, e_3, e_4), \quad & \underline{\beta}_a &= \frac{1}{2}R(e_a, e_3, e_3, e_4), \\ \rho &= \frac{1}{4}R(Le_4, e_3, e_4, e_3), \quad & \sigma &= \frac{1}{4}\,{}^*R(e_4, e_3, e_4, e_3) \end{aligned} \tag{3}$$

Here ${}^*R$ denotes the Hodge dual of R. We denote by ∇ the induced covariant derivative operator on $S(u, \underline{u})$ and by ∇_3, ∇_4 the projections to $S(u, \underline{u})$ of the covariant derivatives D_3, D_4. We note the formulas,

$$\omega = -\frac{1}{2}\nabla_4(\log\Omega), \qquad \underline{\omega} = -\frac{1}{2}\nabla_3(\log\Omega), \qquad \eta + \underline{\eta} = 2\nabla(\log\Omega) \tag{4}$$

We recall also the formula for the Gauss curvature K of $S(u, \underline{u})$,

$$K \;=\; -\rho + \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} - \frac{1}{4}\mathrm{tr}\chi\cdot\mathrm{tr}\underline{\chi} \tag{5}$$

As well known, our space-time slab $\mathcal{D}(u_*, \underline{u}_*)$ is completely determined (for small values of $u_*, \underline{u}_*$) by specifying, *freely*, the traceless parts of the null second fundamental forms $\hat{\chi}$, respectively $\underline{\hat{\chi}}$, along the null, characteristic, hypersurfaces H_0, respectively $\underline{H}_0$, corresponding to $\underline{u} = 0$, respectively $u = 0$, and prescribing $\text{tr}\chi$ together with $\text{tr}\underline{\chi}$ on $S(0,0)$. Following [Chr:book] we assume that our data is trivial along $\underline{H}_0$, i.e. assume that H_0 extends for $\underline{u} < 0$ and the spacetime (M, g) is Minkowskian for $\underline{u} < 0$ and all values of $u \geq 0$. Moreover we can construct our double null foliation such that $\Omega = 1$ along H_0, i.e.,

$$\Omega(0, \underline{u}) = 1, \qquad 0 \leq \underline{u} \leq \underline{u}_*. \tag{6}$$

We also introduce the notation,

$$\widetilde{\text{tr}\underline{\chi}} = \text{tr}\underline{\chi} - \text{tr}\underline{\chi}_0, \qquad \text{tr}\underline{\chi}_0 = -\frac{4}{\underline{u} - u + 2r_0} \tag{7}$$

where $\text{tr}\underline{\chi}_0$ is the flat value of $\text{tr}\underline{\chi}$ along the initial hypersurface $\underline{H}_0$. We denote by γ the induced metric on the surfaces $S(u, \underline{u})$ of intersection between H_u and $H_{\underline{u}}$. A space-time tensor tangent to $S(u, \underline{u})$ is called an $S-$ tensor, or horizontal tensor.

We define systems of, local, transported coordinates along the null hypersurfaces H and $\underline{H}$. Starting with a local coordinate system $\theta = (\theta^1, \theta^2)$ on $U \subset S(u, 0) \subset H_u$, we parametrize any point along the null geodesics starting in U by the the corresponding coordinate θ and affine parameter $\underline{u}$. Similarly, starting with a local coordinate system $\underline{\theta} = (\underline{\theta}^1, \underline{\theta}^2)$ on $V \subset S(0, \underline{u}) \subset \underline{H}_{\underline{u}}$ we parametrize any point along the null geodesics starting in V by the the corresponding coordinate $\underline{\theta}$ and affine parameter u.

1.2. **Signature.** To every null curvature component $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$, null Ricci coefficients components $\chi, \zeta, \eta, \underline{\eta}, \omega, \underline{\omega}$, and metric γ we assign a signature according to the following rule:

$$sgn(\phi) = 1 \cdot N_4(\phi) + \frac{1}{2} \cdot N_a(\phi) + 0 \cdot N_3(\phi) - 1 \tag{8}$$

where $N_4(\phi), N_3(\phi), N_a(\phi)$ denote the number of times e_4, respectively e_3 and $(e_a)_{a=1,2}$, which appears in the definition of ϕ. Thus,

$$sgn(\alpha) = 2, \quad sgn(\beta) = 1 + 1/2, \quad sgn(\rho, \sigma) = 1, \quad sgn(\underline{\beta}) = 1/2, \quad sgn(\underline{\alpha}) = 0.$$

Also,

$$sgn(\chi) = sgn(\omega) = 1, \quad sgn(\zeta, \eta, \underline{\eta}) = 1/2, \quad sgn(\underline{\chi}) = sgn(\underline{\omega}) = sgn(\gamma) = 0.$$

Consistent with this definition we have, for any given null component ϕ,

$$sgn(\nabla_4\phi) = 1 + sgn(\phi), \quad sgn(\nabla\phi) = \frac{1}{2} + sgn(\phi), \quad sgn(\nabla_3\phi) = sgn(\phi).$$

Also, based on our convention,

$$sgn(\phi_1 \cdot \phi_2) = sgn(\phi_1) + sgn(\phi_2). \tag{9}$$

1.3. **Main equations.** As in [K-R:trapped] we denote all Ricci coefficients $\{\chi, \omega, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}, \underline{\omega}\}$ by $\psi^{(s)}$, with s the signature of the specific component. We further differentiate between the components $\psi_4^{(s)} \in \{\chi, \eta, \underline{\omega}\}$, which verify transport equations in the e_4 direction, and $\psi_3^{(s)} \in \{\omega, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}\}$ which verify transport equations in the e_3 direction. We denote by $\Psi^{(s)}$ the null curvature components of signature s. With these notation the null structure equations, see precise equations in section 3 of [K-R:trapped], take the form,

$$\nabla_4 \psi_4^{(s)} = \sum_{s_1+s_2=s+1} \psi^{(s_1)} \cdot \psi^{(s_2)} + \Psi^{(s+1)} \tag{10}$$

$$\nabla_3 \psi_3^{(s)} = \mathrm{tr}\underline{\chi}_0 \cdot \psi_3^{(s)} + \sum_{s_1+s_2=s} \psi^{(s_1)} \cdot \psi^{(s_2)} + \Psi^{(s)} \tag{11}$$

Similarly we write the null Bianchi identities in the from,

$$\nabla_4 \Psi_4^{(s)} = \nabla \Psi^{(s+\frac{1}{2})} + \sum_{s_1+s_2=s+1} \psi^{(s_1)} \cdot \Psi^{(s_2)} \tag{12}$$

$$\nabla_3 \Psi_3^{(s)} = \nabla \Psi^{(s-\frac{1}{2})} + \sum_{s_1+s_2=s} \psi^{(s_1)} \cdot \Psi^{(s_2)} \tag{13}$$

where $\Psi_4 \in \{\alpha, \beta, \rho, \sigma\}$ and $\Psi_3 \in \{\beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}$.

1.4. **Scale invariant norms.** For any horizontal tensor-field ψ with signature $\mathrm{s}gn(\psi)$ we define the following scale invariant norms along the null hypersurfaces $H = H_u^{(0,\delta)}$ and $\underline{H} = \underline{H}_{\underline{u}}^{(0,1)}$.

$$\|\psi\|_{\mathcal{L}^2_{(sc)}(H)} = \delta^{\mathrm{s}gn(\psi)-1} \|\psi\|_{L^2(H)}, \quad \|\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} = \delta^{\mathrm{s}gn(\psi)-\frac{1}{2}} \|\psi\|_{L^2(\underline{H})} \tag{14}$$

We also define the scale invariant norms on the 2 surfaces $S = S_{u,\underline{u}}$,

$$\|\psi\|_{\mathcal{L}^p_{(sc)}(S)} = \delta^{\mathrm{s}gn(\psi)-\frac{1}{p}} \|\psi\|_{L^p(S)} \tag{15}$$

We have,

$$\|\psi\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} = \delta^{-1} \int_0^{\underline{u}} \|\psi\|^2_{\mathcal{L}^2_{(sc)}(u,\underline{u}')} d\underline{u}', \qquad \|\psi\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} = \int_0^u \|\psi\|^2_{\mathcal{L}^2_{(sc)}(u',\underline{u})} du' \tag{16}$$

We denote the scale invariant L^∞ norm in $\mathcal{D}$ by $\|\psi\|_{\mathcal{L}^\infty_{(sc)}}$.

Remark 1.5. These norms correspond to a different scaling than that introduced in [K-R:trapped]. Indeed in [K-R:trapped] the scale invariant norms were based on the definition of the scale of an horizontal component of scale $\mathrm{sc}(\psi) = -\mathrm{s}gn(\psi) + \frac{1}{2}$. The norms introduced here would correspond to a new definition of scale give by $\mathrm{sc}(\psi) = -\mathrm{s}gn(\psi)$. To distinguish between them we denote the old scaling by $\dot{sc}$. Thus, for example,

$$\|\psi\|_{\mathcal{L}^p_{(sc)}(S)} = \delta^{-1/2} \|\psi\|_{\mathcal{L}^p_{(\dot{sc})}(S)}$$

Remark 1.6. With the new scale invariant norms introduced here we have,

$$\|\psi_1 \cdot \psi_2\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\psi_1\|_{\mathcal{L}^\infty_{(sc)}(S)} \cdot \|\psi_2\|_{\mathcal{L}^2_{(sc)}(S)} \tag{17}$$

or,

$$\|\psi_1 \cdot \psi_2\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\psi_1\|_{\mathcal{L}^\infty_{(sc)}(H)} \cdot \|\psi_2\|_{\mathcal{L}^2_{(sc)}(H)} \tag{18}$$

These differ from the situation in [K-R:trapped] where the corresponding estimates (with (sc) replaced by (s$\dot{c}$)) had an additional power of $\delta^{1/2}$ on the right.

Curvature norms. We introduce our main curvature norms

$$\begin{aligned}
\mathcal{R}_0(u,\underline{u}) &:= \|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \mathcal{R}_0'(u,\underline{u}')\\
\mathcal{R}_0'(u,\underline{u}') &:= \epsilon^{-1}\|(\beta,\rho,\sigma,\underline{\beta},K)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\
\mathcal{R}_1(u,\underline{u}) &:= \|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \mathcal{R}_1'(u,\underline{u})\\
\mathcal{R}_1'(u,\underline{u}) &:= \epsilon^{-1}\|\nabla(\alpha,\beta,\rho,\sigma,\underline{\beta},K)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\
\\
\underline{\mathcal{R}}_0(u,\underline{u}) &:= \|\beta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})} + \underline{\mathcal{R}}_0'(u,\underline{u}')\\
\underline{\mathcal{R}}_0'(u,\underline{u}) &= \epsilon^{-1}\|(\rho,\sigma,\underline{\beta},\underline{\alpha},K)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}\\
\underline{\mathcal{R}}_1(u,\underline{u}) &:= \|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})} + \underline{\mathcal{R}}_1'(u,\underline{u})\\
\underline{\mathcal{R}}_1'(u,\underline{u}) &:= \epsilon^{-1}\|\nabla(\beta,\rho,\sigma,\underline{\beta},\underline{\alpha},K)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}
\end{aligned} \tag{19}$$

Also,

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1, \qquad \underline{\mathcal{R}} = \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_1 \tag{20}$$

Remark 1.7. We have included the Gauss curvature K with the null components. Since $K = -\rho + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}$ we easily deduce that,

$$\epsilon^{-1}\|K\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \epsilon^{-1}\|\rho\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \big(1+(\epsilon^{-2}\delta)^{\frac{1}{2}}\big)\ {}^{(S)}\mathcal{O}_{0,\infty}\ {}^{(S)}\mathcal{O}_{0,2}.$$

Remark 1.8. All curvature norms above have a factor of ϵ^{-1} in front of them except for $\|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$, $\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$ and $\|\beta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})}$. These correspond exactly to the *anomalous* curvature norms of [K-R:trapped].

To rectify the anomaly of α we introduce, as in [K-R:trapped], an additional scale-invariant norm,

$$\mathcal{R}_0^{(\epsilon)}[\alpha](u,\underline{u}) := \sup_{{}^{(\epsilon)}H\subset H} \epsilon^{-1}\|\alpha\|_{\mathcal{L}^2_{(sc)}({}^{(\epsilon)}H)},$$

where ${}^{(\epsilon)}H$ is a piece of the hypersurface $H = H_u^{(0,\delta)}$ obtained by evolving an angular disc $S_\epsilon \subset S_{u,0}$ of radius ϵ relative to our transported coordinates. We define the initial quantity $\mathcal{R}^{(0)}$ by,

$$\mathcal{R}^{(0)} = \sup_{0 \le \underline{u} \le \delta} \big(\mathcal{R}(0, \underline{u}) + \mathcal{R}_0^{(\epsilon)}[\alpha](0, \underline{u}) \big) \tag{21}$$

1.9. **Connection coefficients norms.** We introduce the Ricci coefficient norms, with the supremum taken over all surfaces $S = S(u', \underline{u}')$, $0 \le u' \le u$, $0 \le \underline{u}' \le \underline{u}$,

$$\begin{aligned}
{}^{(S)}\mathcal{O}_{0,\infty}(u,\underline{u}) &= \epsilon^{-1} \sup_S \|(\hat{\chi}, \omega, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}, \underline{\omega})\|_{\mathcal{L}^\infty_{(sc)}(S)} \\
{}^{(S)}\mathcal{O}_{0,2}(u,\underline{u}) &= \sup_S \big(\|\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S)} \big) + {}^{(S)}\mathcal{O}'_{0,2}(u,\underline{u}) \\
{}^{(S)}\mathcal{O}'_{0,2}(u,\underline{u}) &= \epsilon^{-1} \sup_S \|(\mathrm{tr}\chi, \omega, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \underline{\omega})\|_{\mathcal{L}^2_{(sc)}(S)} \\
{}^{(S)}\mathcal{O}_{0,4}(u,\underline{u}) &= \epsilon^{-1/2} \sup_S \big(\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} \big) + {}^{(S)}\mathcal{O}'_{0,4}(u,\underline{u}) \\
{}^{(S)}\mathcal{O}'_{0,4}(u,\underline{u}) &= \epsilon^{-1} \sup_S \|(\mathrm{tr}\chi, \omega, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \underline{\omega})\|_{\mathcal{L}^4_{(sc)}(S)} \\
{}^{(S)}\mathcal{O}_{1,4}(u,\underline{u}) &= \epsilon^{-1} \sup_S \|\nabla(\chi, \omega, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}, \underline{\omega})\|_{\mathcal{L}^4_{(sc)}(S)} \\
{}^{(S)}\mathcal{O}_{1,2}(u,\underline{u}) &= \epsilon^{-1} \sup_S \|\nabla(\chi, \omega, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}, \underline{\omega})\|_{\mathcal{L}^2_{(sc)}(S)} \\
{}^{(H)}\mathcal{O}(u,\underline{u}) &= \epsilon^{-1} \|\nabla^2(\chi, \omega, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}, \underline{\omega})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}
\end{aligned} \tag{22}$$

and,

$$\mathcal{O} \;=\; {}^{(S)}\mathcal{O}_{0,2} + {}^{(S)}\mathcal{O}_{0,4} + {}^{(S)}\mathcal{O}_{0,\infty} + {}^{(S)}\mathcal{O}_{1.4} + {}^{(H)}\mathcal{O} \tag{23}$$

Remark 1.10. Note that the only norms which do not contain powers of ϵ^{-1} are the $\mathcal{L}^2_{(sc)}(S)$ norms of $\hat{\chi}$ and $\hat{\underline{\chi}}$. This anomaly is also manifest in the $\mathcal{L}^4_{(sc)}(S)$ norms of the same quantities. These are precisely the same quantities which were anomalous in [K-R:trapped], with respect to the $\dot{sc}$ scaling.

To cure the above anomaly we define the auxiliary norms,

$${}^{(S)}\mathcal{O}^{(\epsilon)}_{0,4}(u,\underline{u}) = \epsilon^{-1} \sup_S \sup_{S_\epsilon \subset S} \|(\hat{\chi}, \hat{\underline{\chi}})\|_{\mathcal{L}^4_{(sc)}(S_\epsilon)}$$

with S_ϵ - an angular subset of S of size ϵ relative to our transported coordinates.

Finally we define the initial data quantity:

$$\mathcal{O}^{(0)} = \sup_{0 \le \underline{u} \le \delta} \big(\mathcal{O}(0, \underline{u}) + {}^{(S)}\mathcal{O}^{(\epsilon)}_{0,4}(0, \underline{u}) \big) \tag{24}$$

1.11. **Initial conditions.** Define the main initial data quantity,

$$\begin{aligned}\mathcal{I}^{(0)}(\underline{u}) = &\sum_{0\le k\le 2}\|\nabla_4^k\hat{\chi}_0\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})}\\ &+\epsilon^{-1}\Big(\|\hat{\chi}_0\|_{\mathcal{L}^\infty_{(sc)}(0,\underline{u})}+\sum_{0\le k\le 1}\sum_{1\le m\le 4}\|\nabla^{m-1}\nabla_4^k\nabla\hat{\chi}_0\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})}\Big)\end{aligned}\tag{25}$$

or, in the natural norms,

$$\begin{aligned}\mathcal{I}^{(0)}(\underline{u}) = &\sum_{0\le k\le 2}\delta^{k+1/2}\|\nabla_4^k\hat{\chi}_0\|_{L^2(0,\underline{u})}\\ &+\epsilon^{-1}\Big(\delta\|\hat{\chi}_0\|_{L^\infty(0,\underline{u})}+\sum_{0\le k\le 1}\sum_{1\le m\le 4}\delta^{\frac{m+1}{2}+k}\|\nabla^{m-1}\nabla_4^k\nabla\hat{\chi}_0\|_{L^2(0,\underline{u})}\Big)\end{aligned}$$

1.12. **Main propagation result.** The first result establishes the boundedness of the initial curvature and Ricci coefficent scale invariant norms $\mathcal{R}^{(0)}$, $\mathcal{O}^{(0)}$ in terms of $\mathcal{I}^{(0)}$.

Proposition 1.13. *Assume that the initial data along $\underline{H}_0$ is flat and that $\mathcal{I}^{(0)}<\infty$ along $H_0^{(0,\delta)}$. Then, for $\delta^{1/2}\epsilon^{-1}$ and $\epsilon>0$ sufficiently small we have, with C a fixed super-linear polynomial*

$$\mathcal{R}^{(0)}+\mathcal{O}^{(0)}\lesssim\mathcal{I}^{(0)}+C(\mathcal{I}^{(0)})\tag{26}$$

Also, starting with $\mathcal{R}^{(0)}<\infty$ and $\delta^{1/2}\epsilon^{-1}$, ϵ sufficiently small, we have, with C a fixed super-linear polynomial,

$$\mathcal{O}^{(0)}\ \lesssim\ \mathcal{R}^{(0)}+C(\mathcal{R}^{(0)})\tag{27}$$

We can now state our main propagation result.

Theorem 1.14 (Main Theorem I). *Under the assumption $\mathcal{R}^{(0)}<\infty$, if $\delta^{1/2}\epsilon^{-1}$ and ϵ are sufficiently small then, for $0\le u\le 1$, $0\le\underline{u}\le\delta$, with C a fixed super-linear polynomial,*

$$(\mathcal{R}+\underline{\mathcal{R}}+\mathcal{O})(u,\underline{u})\lesssim\mathcal{R}^{(0)}+C(\mathcal{R}^0)$$

Remark 1. The results presented extends all the results of [K-R:trapped]. Indeed, to derive the results of propositions 2.5, theorems 2.6, and 2.7 there, it suffices to choose $\epsilon=\mu\delta^{1/2}$ with μ sufficiently small.

Remark 2. The additional smallness assumption on $\delta^{1/2}\epsilon^{-1}$ is due to the lower order terms which appear in some of the calculus inequalities presented in the next section.

In the remaining part of this section we introduce norms for the deformation tensors of the geodesic null generators $L,\underline{L}$ and rotation vectorfields O and give a short sketch of the proof of theorem 1.14.

1.15. **Deformation tensors norms for $L, \underline{L}$.** If π is the deformation tensor of either L or $\underline{L}$ we denote by $\pi^{(s)}$ its null component of of signature s. We now introduce the norms for ${}^{(L)}\pi$ and ${}^{(\underline{L})}\pi$ as follows,

$$\Pi_0 = \Pi_{0,4} + \Pi_{0,\infty}, \qquad \underline{\Pi}_0 = \underline{\Pi}_{0,4} + \underline{\Pi}_{0,\infty} \tag{28}$$

with,

$$\begin{aligned}
\Pi_{0,4} &= \epsilon^{-1} \sum_{s\in\{0,\frac12\}} \|\,{}^{(L)}\pi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^{-\frac12}\|\,{}^{(L)}\pi^{(1)}\|_{\mathcal{L}^4_{(sc)}(S)},\\
\Pi_{0,\infty} &= \epsilon^{-1} \sum_{s\in\{0,\frac12,1\}} \|\,{}^{(L)}\pi^{(s)}\|_{\mathcal{L}^\infty_{(sc)}(S)},\\
\underline{\Pi}_{0,4} &= \epsilon^{-1} \sum_{s\in\{\frac12,1\}} \|\,{}^{(\underline{L})}\pi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^{-\frac12}\|\,{}^{(\underline{L})}\pi^{(0)}\|_{\mathcal{L}^4_{(sc)}(S)},\\
\underline{\Pi}_{0,\infty} &= \epsilon^{-1} \sum_{s\in\{0,\frac12,1\}} \|\,{}^{(\underline{L})}\pi^{(s)}\|_{\mathcal{L}^\infty_{(sc)}(S)}
\end{aligned} \tag{29}$$

We introduce also the first derivative norms,

$$\begin{aligned}
\Pi_1 &= \|\nabla_4\,{}^{(L)}\pi^{(0)}\|_{\mathcal{L}^4_{(sc)}(S)} + \sum_{s\in\{\frac12,1\}} \epsilon^{-1}\|\bar{\nabla}\,{}^{(L)}\pi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)},\\
\underline{\Pi}_1 &= \|\nabla_4\,{}^{(\underline{L})}\pi^{(0)}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_3\,{}^{(\underline{L})}\pi^{(0)}\|_{\mathcal{L}^4_{(sc)}(S)}\\
&\quad + \epsilon^{-1}\|\bar{\nabla}\,{}^{(\underline{L})}\pi^{(\frac12)}\|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^{-1}\|(\nabla,\nabla_3)\,{}^{(\underline{L})}\pi^{(1)}\|_{\mathcal{L}^4_{(sc)}(S)},
\end{aligned} \tag{30}$$

We also set,

$$\Pi = \Pi_0 + \Pi_1, \qquad \underline{\Pi} = \underline{\Pi}_0 + \underline{\Pi}_1$$

1.16. **Deformation tensor norms for O.** We recall the rotation vectorfields ${}^{(i)}O$ obeying the commutation relations

$$[{}^{(i)}O, {}^{(j)}O] = \in_{ijk}\ {}^{(k)}O,$$

were defined, see section 13 in [K-R:trapped], by parallel transport starting with the standard rotation vectorfields on $\mathbb{S}^2 = S_{u,0} \subset H_{u,0}$ along the integral curves of e_4. Suppressing the index ${}^{(i)}$ we have,

$$\nabla_4 O_b = \chi_{bc} O_c. \tag{31}$$

The only non-trivial components of the deformation tensor $\pi_{\alpha\beta} = \frac12(\nabla_\alpha O_\beta + \nabla_\beta O_\alpha)$ are given below:

$$\begin{aligned}
\pi_{34} &= -2(\eta + \underline{\eta})_a O_a,\\
\pi_{ab} &= \frac12(\nabla_a O_b + \nabla_b O_a) = \frac12(H_{ab} + H_{ba}),\\
\pi_{3a} &= \frac12(\nabla_3 O_a - \underline{\chi}_{ab} O_b) := \frac12 Z_a.
\end{aligned}$$

The quantities, H and Z can be assigned signature and scaling, (consistent with those for the Ricci coefficients and curvature components) according to.

$$sgn(H) = 0, \qquad sgn(Z) = -\frac{1}{2}. \tag{32}$$

Similarly, assigning signatures to all other components of ${}^{(O)}\pi$, we introduce the norms,

$$\begin{aligned} {}^{(O)}\Pi_0 &= \epsilon^{-1}\| {}^{(O)}\pi \|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^{-1}\| {}^{(O)}\pi \|_{\mathcal{L}^\infty_{(sc)}(S)}, \\ {}^{(O)}\Pi_1 &= \sum_{(\mu,s)\neq(3,0)} \epsilon^{-1}\|D_\mu {}^{(O)}\pi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)} \\ &\quad + \epsilon^{-1}\|D_3 {}^{(O)}\pi^{(0)} - \nabla_3 Z\|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^{-1}\|\sup_{\underline{u}} |\nabla_3 Z|\|_{\mathcal{L}^2_{(sc)}(S)}, \end{aligned} \tag{33}$$

1.17. **Proof of Main Theorem I.** To prove the theorem we start by making a bootstrap assumption on the Ricci coefficient norm $\mathcal{O}$. More precisely we assume that,

$$\mathcal{O} \lesssim \Delta_0 \tag{34}$$

Based on this assumption we state various preliminary estimates in section 3, which are simple adaptation of results proved in [K-R:trapped]. It is interesting to remark that this is the only place when we need to make a restriction for the size of $\delta^{1/2}\epsilon^{-1}$. Using these preliminary estimates we then indicate how, by a simple adjustment of the curvature estimates in [K-R:trapped] we can prove, see section 4, the following.

Theorem 1.18 (Theorem A)**.** *There exists a positive constant $a > \frac{1}{8}$ such that, for $\delta^{1/2}\epsilon^{-1}$ and ϵ sufficiently small,*

$$\mathcal{R}(u, \underline{u}) + \underline{\mathcal{R}}(u, \underline{u}) \lesssim \mathcal{R}^{(0)} + C\epsilon^a(\mathcal{R} + \underline{\mathcal{R}}) \tag{35}$$

with $C = C(\Pi, \underline{\Pi}, {}^{(O)}\Pi, \mathcal{R}, \underline{\mathcal{R}})$.

Next we rely on a theorem which bounds the norms $\Pi, \underline{\Pi}$ and ${}^{(O)}\Pi$, for the deformation tensors of $L, \underline{L}$ and O, to the Ricci coefficients norms $\mathcal{O}$.

Theorem 1.19 (Theorem B)**.** *Under the assumptions $\delta^{1/2}\epsilon^{-1}$ and ϵ sufficiently small we have,*

$$\Pi + \underline{\Pi} + {}^{(O)}\Pi \lesssim C(\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}}) \tag{36}$$

Finally we state the theorem which relates the norms $\mathcal{O}$ to the curvature norms $\mathcal{R}, \underline{\mathcal{R}}$.

Theorem 1.20 (Theorem C)**.** *Under the assumptions $\delta^{1/2}\epsilon^{-1}$ and ϵ sufficiently small we have, with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$,*

$$\mathcal{O} \;\lesssim\; C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}) \tag{37}$$

Combining theorems B and C with theorem A we deduce, under the bootstrap assumption 34,

$$\mathcal{R}(u,\underline{u}) + \underline{\mathcal{R}}(u,\underline{u}) \lesssim \mathcal{R}^{(0)} + \epsilon^a C(\mathcal{R},\underline{\mathcal{R}})(\mathcal{R}+\underline{\mathcal{R}}),$$

from which, for ϵ sufficiently small,

$$\mathcal{R}(u,\underline{u}) + \underline{\mathcal{R}}(u,\underline{u}) \lesssim \mathcal{R}^{(0)}. \tag{38}$$

Thus, back to (37) and using also proposition 1.13,

$$\mathcal{O} \lesssim C(\mathcal{R}^{(0)})$$

which allows us to remove the bootstrap assumption and confirm the result of the main theorem I.

2. Formation of pre-scars

Relying on the results of theorem 1.14 we prove a new result concerning the formation of pre-scars. Throughout this section we assume that the assumptions and conclusions of theorem 1.14 hold true.

2.1. **Local scale invariant norms.** Consider a partition of $S_0 = S(0,0)$ into angular sectors Λ of a given size $|\Lambda|$. Let ${}^{(\Lambda)}f_{(0)}$ be a partition of unity associated to this partition, They can be extend trivially, first along $\underline{H}_0$ and then along each H_u, to be constant along the corresponding null generators. In particular we have,

$$\nabla_L {}^{(\Lambda)}f = 0, \qquad {}^{(\Lambda)}f|_{\underline{H}_0} = {}^{(\Lambda)}f_{(0)} \tag{39}$$

Then, under the assumptions and conclusions of theorem 1.14 we can easily deduce,

Lemma 2.2. *We have,*

$$\sum_\Lambda {}^{(\Lambda)}f = 1 \tag{40}$$

Also,

$$|\nabla {}^{(\Lambda)}f|_{L^\infty} \lesssim |\Lambda|^{-1}, \qquad |\nabla_{\underline{L}} {}^{(\Lambda)}f|_{L^\infty} \lesssim \epsilon\delta^{1/2}|\Lambda|^{-1} \tag{41}$$

or, in scale invariant norms (assigning to f signature 0),

$$|\nabla {}^{(\Lambda)}f|_{\mathcal{L}^\infty_{(sc)}} \lesssim \delta^{1/2}|\Lambda|^{-1}, \qquad |\nabla_{\underline{L}} {}^{(\Lambda)}f|_{\mathcal{L}^\infty_{(sc)}} \lesssim \epsilon\delta^{1/2}|\Lambda|^{-1}$$

We now introduce the localized curvature norms,

$$
\begin{aligned}
{}^{(\Lambda)}\mathcal{R}_0(u,\underline{u}) &:= \|{}^{(\Lambda)}f\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + {}^{(\Lambda)}\mathcal{R}_0'(u,\underline{u}')\\
{}^{(\Lambda)}\mathcal{R}_0'(u,\underline{u}) &:= \epsilon^{-1}\|{}^{(\Lambda)}f(\beta,\rho,\sigma,\underline{\beta},K)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\
{}^{(\Lambda)}\mathcal{R}_1(u,\underline{u}) &:= \|{}^{(\Lambda)}f\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + {}^{(\Lambda)}\mathcal{R}_1'(u,\underline{u})\\
{}^{(\Lambda)}\mathcal{R}_1'(u,\underline{u}) &:= \epsilon^{-1}\|{}^{(\Lambda)}f\nabla(\alpha,\beta,\rho,\sigma,\underline{\beta},K)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\
{}^{(\Lambda)}\underline{\mathcal{R}}_0(u,\underline{u}) &:= \|{}^{(\Lambda)}f\beta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})} + \underline{\mathcal{R}}_0'(u,\underline{u}')\\
{}^{(\Lambda)}\underline{\mathcal{R}}_0'(u,\underline{u}) &= \epsilon^{-1}\|{}^{(\Lambda)}f(\rho,\sigma,\underline{\beta},\underline{\alpha},K)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}\\
{}^{(\Lambda)}\underline{\mathcal{R}}_1(u,\underline{u}) &:= \|{}^{(\Lambda)}f\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})} + \underline{\mathcal{R}}_1'(u,\underline{u})\\
\underline{\mathcal{R}}_1'(u,\underline{u}) &:= \epsilon^{-1}\|{}^{(\Lambda)}f\nabla(\beta,\rho,\sigma,\underline{\beta},\underline{\alpha},K)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}
\end{aligned}
\tag{42}
$$

and,

$$
\begin{aligned}
{}^{[\Lambda]}\mathcal{R}_0(u,\underline{u}) &:= \sup_\Lambda {}^{(\Lambda)}\mathcal{R}_0, \qquad {}^{[\Lambda]}\mathcal{R}_1(u,\underline{u}) := \sup_\Lambda {}^{(\Lambda)}\mathcal{R}_1\\
{}^{[\Lambda]}\underline{\mathcal{R}}_0(u,\underline{u}) &:= \sup_\Lambda {}^{(\Lambda)}\underline{\mathcal{R}}_0, \qquad {}^{[\Lambda]}\underline{\mathcal{R}}_1(u,\underline{u}) := \sup_\Lambda {}^{(\Lambda)}\underline{\mathcal{R}}_1
\end{aligned}
\tag{43}
$$

with the supremum taken with respect to all elements of the partition. and,

$$
{}^{[\Lambda]}\mathcal{R} = {}^{[\Lambda]}\mathcal{R}_0 + {}^{[\Lambda]}\mathcal{R}_1, \qquad {}^{[\Lambda]}\underline{\mathcal{R}} = {}^{[\Lambda]}\underline{\mathcal{R}}_0 + {}^{[\Lambda]}\underline{\mathcal{R}}_1 \tag{44}
$$

2.3. **Angular localized curvature estimates.** Using a variation of our main energy estimates, with an additional angular localization, we can prove the following.

Theorem 2.4. *Under the assumptions and conclusions of theorem 1.14, if in addition $\delta^{\frac12}|\Lambda|^{-1}$ is sufficiently small, then, for $0 \le u \le 1$, $0 \le \underline{u} \le \delta$,*

$$
({}^{[\Lambda]}\mathcal{R} + {}^{[\Lambda]}\underline{\mathcal{R}})(u,\underline{u}) \lesssim {}^{[\Lambda]}\mathcal{R}^{(0)}
$$

Moreover,

$$
({}^{(\Lambda)}\mathcal{R} + {}^{(\Lambda)}\underline{\mathcal{R}})(u,\underline{u}) \lesssim {}^{(\Lambda)}\mathcal{R}^{(0)} + \delta^{\frac12}|\Lambda|^{-1}\,{}^{[\Lambda]}\mathcal{R}^{(0)} \tag{45}
$$

Remark 2.5. By the standard domain of dependence argument the energy estimate can not fully localized to individual sectors ${}^{(\Lambda)}H_u$ and ${}^{(\Lambda)}\underline{H}_{\underline{u}}$ contained in the support of the function ${}^{(\Lambda)}f$. This explains the need for the supremum in Λ in the definition of the ${}^{[\Lambda]}\mathcal{R}$, ${}^{[\Lambda]}\underline{\mathcal{R}}$ norms for the first part of the theorem. The second part of the theorem gives a bound for each sector individual Λ with the second term on the right hand side of (45) accounting for the defect of localization.

A proof of the theorem is sketched in section 5.

2.6. **Emerging scars.**

Definition 2.7. We say that the data $\mathcal{R}^{(0)}$ is uniformly distributed on the scale $\delta^{\frac{1}{2}}\varpi^{-1}$ if there exists a partition $\{\Lambda\}$ such that $|\Lambda| \approx \delta^{\frac{1}{2}}\varpi^{-1}$ and

$$^{[\Lambda]}\mathcal{R}^{(0)} \lesssim \delta^{\frac{1}{2}}\varpi^{-1}\mathcal{R}^{(0)} \tag{46}$$

Our second main result of this paper is the following.

Theorem 2.8 (Main theorem II)**.** *Assume that, in additions to the conditions of validity of theorem 1.14, the data $\mathcal{R}^{(0)}$ is uniformly distributed on the scale $\delta^{\frac{1}{2}}\varpi^{-1}$ for some constant $\varpi << 1$ and $\epsilon\varpi^{-1}$ sufficiently small. Let Λ be a fixed angular sector of size $|\Lambda| = q^{-1}\delta^{\frac{1}{2}}$ with $q = \epsilon\varpi^{-1}$ sufficiently small. Then, if*

$$\inf_{\theta\in\Lambda} \int_0^\delta |\hat{\chi}_0|^2(\underline{u},\theta)d\underline{u} > \frac{2(r_0-u)}{r_0^2} \tag{47}$$

the Λ-angular section $^{(\Lambda)}S_{u,\delta}$ of the surface $S_{u,\delta}$ must be trapped, i.e. $tr\chi < 0$ there.

Alternatively, if for some constant $c > 0$ independent of $\delta, \epsilon, q, \varpi$,

$$\sup_{\theta\in\Lambda} \int_0^\delta |\hat{\chi}_0|^2(\underline{u},\theta)d\underline{u} < \frac{2(r_0-u)}{r_0^2} - c \tag{48}$$

then $tr\chi > 0$ throughout the angular sector $^{(\Lambda)}S_{u,\delta}$.

We postpone a discussion of the proof of this theorem to the last section of the paper.

Remark 2.9. Observe that the parameters δ, ϵ, ϖ in theorem 2.8 verify the conditions:

$$0 < \delta^{1/2} < \epsilon < \varpi < 1, \qquad \delta^{1/2}\epsilon^{-1} << 1, \quad q = \epsilon\varpi^{-1} << 1.$$

3. Preliminary estimates

3.1. **Transported coordinates.** As mentioned in the previous section we define systems of, local, transported coordinates along the null hypersurfaces H and $\underline{H}$. Staring with a local coordinate system $\theta = (\theta^1, \theta^2)$ on $U \subset S(u,0) \subset H_u$ we parametrize any point along the null geodesics starting in U by the the corresponding coordinate θ and affine parameter $\underline{u}$. Similarly, starting with a local coordinate system $\underline{\theta} = (\underline{\theta}^1, \underline{\theta}^2)$ on $V \subset S(0,\underline{u}) \subset \underline{H}_{\underline{u}}$ we parametrize any point along the null geodesics starting in V by the the corresponding coordinate $\underline{\theta}$ and affine parameter u. We denote the respective metric components by γ_{ab} and $\underline{\gamma}_{ab}$.

Proposition 3.2. *Let γ^0_{ab} denote the standard metric on $\mathbb{S}^2$. Then, for any $0 \le u \le 1$ and $0 \le \underline{u} \le \delta$ and sufficiently small $\delta^{\frac{1}{2}}\Delta_0$*

$$|\gamma_{ab} - \gamma^0_{ab}| \le \delta^{\frac12}\Delta_0, \qquad |\underline{\gamma}_{ab} - \gamma^0_{ab}| \le \delta^{\frac12}\Delta_0.$$

The Christoffel symbols Γ_{abc} and $\underline{\Gamma}_{ab}$, obey the scale invariant estimates[4]

$$\|\Gamma_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \epsilon\ {}^{(S)}\mathcal{O}_{[1]}, \qquad \|\partial_d \Gamma_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \epsilon\ {}^{(S)}\mathcal{O}_{[2]}, \tag{49}$$

$$\|\underline{\Gamma}_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \epsilon\ {}^{(S)}\mathcal{O}_{[1]}, \qquad \|\partial_d \underline{\Gamma}_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \epsilon\ {}^{(S)}\mathcal{O}_{[2]}, \tag{50}$$

The proof is a trivial adaptation of proposition 4.6 in [K-R:trapped].

3.3. **Calculus inequalities.** We simply adapt here the results of section 4.9 in [K-R:trapped].

Proposition 3.4. *Let $S = S_{u,\underline{u}}$ and let $S_\epsilon \subset S$ denote a disk of radius ϵ relative to either θ or $\underline{\theta}$ coordinate system. Then for any horizontal tensor ϕ and any $p > 2$*

$$\|\phi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} \|\nabla\phi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\phi\|_{\mathcal{L}^2_{(sc)}(S)}, \tag{51}$$

$$\|\phi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \|\psi\|^{\frac{p}{p+4}}_{\mathcal{L}^p_{(sc)}(S)} \|\nabla\phi\|^{\frac{4}{p+4}}_{\mathcal{L}^p_{(sc)}(S)} + \delta^{\frac1p}\|\phi\|_{\mathcal{L}^p_{(sc)}(S)}. \tag{52}$$

and

$$\|\phi\|_{\mathcal{L}^4_{(sc)}(S_\epsilon)} \lesssim \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(S_{2\epsilon})} + (\epsilon^{-2}\delta)^{\frac14}\|\phi\|_{\mathcal{L}^2_{(sc)}(S_{2\epsilon})}, \tag{53}$$

$$\|\phi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \sup_{S_\epsilon\subset S}\Big(\|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S_{2\epsilon})} + (\epsilon^{-2}\delta)^{\frac14}\|\phi\|_{\mathcal{L}^4_{(sc)}(S_{2\epsilon})}\Big). \tag{54}$$

As a consequence of the proposition we derive.

Corollary 3.5.

$${}^{(S)}\mathcal{O}_{0,\infty} \lesssim\ {}^{(S)}\mathcal{O}^{\frac12}_{1,2}\cdot\ {}^{(S)}\mathcal{O}^{\frac12}_{2,2} + (\epsilon^{-2}\delta)^{\frac14}\ {}^{(S)}\mathcal{O}^{\epsilon}_{0,4}$$

3.6. **Codimension 1 trace formulas.** The following is a straightforward adaptation of proposition 4.15 in [K-R:trapped]

Proposition 3.7. *The following formulas hold true for a fixed $S = S(u,\underline{u}) = H(u)\cap \underline{H}(\underline{u}) \subset \mathcal{D}$ and any horizontal tensor ϕ*

$$\|\phi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \big(\delta^{1/2}\|\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(H)}\big)^{1/2}\big(\delta^{1/2}\|\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(H)}\big)^{1/2}$$

$$\|\phi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \big(\delta^{1/2}\|\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)^{1/2}\big(\delta^{1/2}\|\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla_3\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)^{1/2}$$

[4] We attach signature 1/2 to both Γ and $\underline{\Gamma}$.

3.8. **Estimates for Hodge systems.** Here we make straightforward adaptations of the results (more precisely propositions 4.17 and 4.17) in section 4.16 of [K-R:trapped] for Hodge systems.

Proposition 3.9. *Let ψ verify the Hodge system*

$$\mathcal{D}\psi = F, \tag{55}$$

with $\mathcal{D}$ one of the Hodge operators defined in section 3.5 of [K-R:trapped]. *Then,*

$$\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|K\|_{\mathcal{L}^2_{(sc)}(S)}\|\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|F\|_{\mathcal{L}^2_{(sc)}(S)} \tag{56}$$

Also,

Proposition 3.10. *Let ψ verify the Hodge system*

$$\mathcal{D}\psi = F \tag{57}$$

Then,

$$\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|K\|_{\mathcal{L}^2_{(sc)}(S)}\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} + \|K\|^{\frac{1}{2}}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla F\|_{\mathcal{L}^2_{(sc)}(S)} \tag{58}$$

3.11. **Trace theorems.** We state the straightforward adaptations of the results of section 11 in [K-R:trapped] concerning sharp trace theorems.

We introduce the following trace norms for an S tangent tensor ϕ, with signature $sgn(\phi)$, along $H = H_u^{(0,\underline{u})}$, relative to the transported coordinates $(\underline{u}, \theta)$ of proposition 3.2:

$$\|\phi\|_{Tr_{(sc)}(H)} = \delta^{sgn(\phi)-\frac{1}{2}}\Big(\sup_{\theta\in S(u,0)}\int_0^{\underline{u}}|\phi(u,\underline{u}',\theta)|^2 d\underline{u}'\Big)^{1/2}$$

Also, along $\underline{H} = \underline{H}_{\underline{u}}^{(0,u)}$ relative to the transported coordinates $(u, \underline{\theta})$ of proposition 3.2

$$\|\phi\|_{Tr_{(sc)}(\underline{H})} = \delta^{sgn(\phi)}\Big(\sup_{\underline{\theta}\in S(\underline{u},0)}\int_0^{u}|\phi(u',\underline{u},\underline{\theta})|^2 du'\Big)^{1/2}$$

Proposition 3.12. *For any horizontal tensor ϕ along $H = H_u^{(0,\underline{u})}$,*

$$\begin{aligned}\|\nabla_4\phi\|_{Tr_{(sc)}(H)} \lesssim\ & \Big(\|\nabla_4^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \epsilon C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla_4\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac{1}{2}}\\ & \times\Big(\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \epsilon C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac{1}{2}}\\ & + \|\nabla_4\nabla\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(H)}\end{aligned} \tag{59}$$

where C is a constant which depends on $\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}$.

Also, for any horizontal tensor ϕ along $\underline{H} = H_{\underline{u}}^{(u,0)}$, and a similar constant C,

$$
\begin{aligned}
\|\nabla_3\phi\|_{Tr_{(sc)}(\underline{H})} \lesssim & \Big(\|\nabla_3^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \epsilon C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla_3\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac{1}{2}} \\
& \times \Big(\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \epsilon C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac{1}{2}} \\
& + \|\nabla_3\nabla\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}
\end{aligned}
\tag{60}
$$

4. Global Curvature Estimates

In this section we discuss the proof of theorem A, 1.18, which is a straightforward modification of the curvature estimates of sections 14 and 15 in [K-R:trapped].

4.1. **Zero order estimates.** As in [K-R:trapped] all curvature estimates are based on the energy identities for the Bel-Robinson tensor $\mathcal{Q}[W]$ of a Weyl field W which we take here to be either the Riemann curvature tensor R or its modified Lie derivatives $\widehat{\mathcal{L}}_U R = \mathcal{L}_U R - \frac{1}{8}\mathrm{tr}^U\pi R - \frac{1}{2}{}^U\hat{\pi}\cdot R$, relative to well chosen vectorfields U. Recall

Proposition 4.2. *The following identity holds on our fundamental domain $\mathcal{D}(u,\underline{u})$,*

$$
\begin{aligned}
\int_{H_u} Q[R](L,X,Y,Z) + \int_{\underline{H}_{\underline{u}}} Q[R](X,Y,Z,\underline{L}) &= \int_{H_0} Q[R](L,X,Y,Z) \\
&+ \frac{1}{2}\int\int_{\mathcal{D}(u,\underline{u})} Q[R]\cdot\pi(X,Y,Z),
\end{aligned}
$$

where $\pi(X,Y,Z)$ is a linear combination of the deformation tensors of the vectorfields X,Y,Z.

The global estimates corresponding to the norms $\mathcal{R}_0$ and $\underline{\mathcal{R}}_0$ are obtained, as in section 14 of [K-R:trapped] by making the choices $(X,Y,Z) = \{(L,L,L);(L,L,\underline{L});(L,\underline{L},\underline{L});(\underline{L},\underline{L},\underline{L})\}$. and following precisely the same steps as before. We summarize the result in the following,

Proposition 4.3. *There exists a positive constant $a > \frac{1}{8}$ such that, for $\delta^{1/2}\epsilon^{-1}$ and ϵ sufficiently small,*

$$
\mathcal{R}_0(u,\underline{u}) + \underline{\mathcal{R}}_0(u,\underline{u}) \le \mathcal{R}_0(0,\underline{u}) + \epsilon^a C(\Pi_0,\underline{\Pi}_0,\mathcal{R},\underline{\mathcal{R}})(\mathcal{R}+\underline{\mathcal{R}}) \tag{61}
$$

We sketch the proof in the particular case when $X = Y = Z = L$ in proposition 5.2. We obtain, schematically, by signature considerations,

$$\begin{aligned}\int_{H_u} |\alpha|^2 + \int_{\underline{H}_{\underline{u}}} |\beta|^2 &= \int_{H_0} |\alpha|^2 + \frac{3}{2}\int\int_{\mathcal{D}(u,\underline{u})} Q[R]({}^{(L)}\pi, L, L)\\ &\lesssim \int_{H_0} |\alpha|^2 + \sum_{s_1+s_2+s_3=4} {}^{(L)}\pi^{(s_1)} \cdot \Psi^{(s_2)} \cdot \Psi^{(s_3)}\end{aligned}$$

Passing to the scale invariant norms we have,

$$\|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}} + \|\beta\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)}} \le \|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})}} + \sum_{s_1+s_2+s_3=2s=4} \delta^2 \int\int_{\mathcal{D}(u,\underline{u})} {}^{(L)}\pi^{(s_1)} \cdot \Psi^{(s_2)} \cdot \Psi^{(s_3)}$$

The worst term occur when $s_2 = s_3 = 2$ and $s_1 = 0$. Observe also that, since the signature of a Ricci coefficient ${}^{(L)}\pi^{(s_1)}$ may not exceed $s_1 = 1$, neither s_2 or s_3 can be zero, i.e. $\underline{\alpha}$ cannot occur among the curvature terms on the right. We use the estimate $\|{}^{(L)}\pi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}} \le \epsilon\Pi_0$ to deduce,

$$\|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\beta\|^2_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})} \lesssim \|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + \epsilon\Pi_0 \cdot \mathcal{R}_0(u,\underline{u})^2$$

There other estimates are derived in the same manner, see [K-R:trapped]

4.4. **First derivative estimates.** As in [K-R:trapped] the first derivative curvature estimates are based on the following.

Proposition 4.5. *Let U be a vectorfield defined in our fundamental domain $\mathcal{D}(u,\underline{u})$, tangent to $\underline{H}_0$. Then, with $H_u = H_u([0,\underline{u}])$,*

$$\begin{aligned}&\int_{H_u} Q[\widehat{\mathcal{L}}_U R](L,X,Y,Z) + \int_{\underline{H}_{\underline{u}}} Q[\widehat{\mathcal{L}}_U R](X,Y,Z,\underline{L}) = \int_{H_0} Q[\widehat{\mathcal{L}}_U R](L,X,Y,Z)\\ &+\frac{1}{2}\int\int_{\mathcal{D}(u,\underline{u})} Q[\widehat{\mathcal{L}}_U R]\cdot\pi(X,Y,Z) + \int\int_{\mathcal{D}(u,\underline{u})} D(R,U)(X,Y,Z)\end{aligned}$$

with $D(U,R)$ the 3-tensor of the form, schematically.

$$D(U,R) = (\widehat{\mathcal{L}}_U R)\cdot\big(\pi\cdot DR + D\pi\cdot R\big)$$

We apply these estimate for the following the choice of vectorfields,

$$(U;X,Y,Z) = \{(L;L,L,L);(\underline{L};\underline{L},\underline{L},\underline{L});(O;L,L,L);(O;L,L,\underline{L});(O;L,\underline{L},\underline{L});(O;\underline{L},\underline{L},\underline{L})\},$$

As in [K-R:trapped], see section 15, we make the choice $(U;X,Y,Z) = (L;L,L,L)$ to the estimate $\nabla_4\alpha$ and the choice $(U;X,Y,Z) = (\underline{L};\underline{L},\underline{L},\underline{L})$ to estimate $\nabla_3\underline{\alpha}$. The four choices $U = O$ and $X,Y,Z \in \{L,\underline{L}\}$ lead to bounds for $\nabla\alpha, \nabla\beta, \nabla(\rho,\sigma), \nabla\underline{\beta}$, which coupled with the Bianchi identities are sufficient to estimate all first derivatives of the null curvature components. We outline below a typical estimate involving O. Let $\Psi^{(s)}(\widehat{\mathcal{L}}_O R)$ and $\Psi^{(s)}(DR)$ denote the null components of the Weyl field $\widehat{\mathcal{L}}_O R$ and DR of signature s. Then

$$\begin{aligned}\int_{H_u}|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)|^2+\int_{\underline{H}_{\underline{u}}}|\Psi^{(s-\frac12)}(\widehat{\mathcal{L}}_O R)|^2=&\int_{H_0}|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)|^2\\&+\sum_{s_1+s_2+s_3=2s}\int\int_{\mathcal{D}(u,\underline{u})}\left({}^{(L)}\pi^{(s_1)},\ {}^{(\underline{L})}\pi^{(s_1)}\right)\cdot\Psi^{(s_2)}(\widehat{\mathcal{L}}_O R)\cdot\Psi^{(s_3)}(\widehat{\mathcal{L}}_O R)\\&+\sum_{s_1+s_2+s_3=2s}\int\int_{\mathcal{D}(u,\underline{u})}{}^{(O)}\pi^{(s_1)}\cdot\Psi^{(s_2)}(DR)\cdot\Psi^{(s_3)}(\widehat{\mathcal{L}}_O R)\\&+\sum_{s_1+s_2+s_3=2s}\int\int_{\mathcal{D}(u,\underline{u})}(D\,{}^{(O)}\pi)^{(s_1)}\cdot\Psi^{(s_2)}\cdot\Psi^{(s_3)}(\widehat{\mathcal{L}}_O R)\end{aligned}$$

Using our scale invariant norms, and proceeding exactly as in section 15 of [K-R:trapped] we can easily derive the estimate, for some $a>\frac18$,

$$\begin{aligned}\sum_{s\in\{\frac12,1,\frac32,2\}}\epsilon^{-1}\left(\|\nabla\Psi^s\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}}+\|\nabla\Psi^{s-\frac12}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}\right)\lesssim&\sum_{s\in\{\frac12,1,\frac32,2\}}\epsilon^{-1}\|\nabla\Psi^s\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})}}\\&+\epsilon^a C(\Pi_0,\underline{\Pi}_0,\ {}^{(O)}\Pi_0,\mathcal{R},\underline{\mathcal{R}})(\mathcal{R}+\underline{\mathcal{R}}),\end{aligned}$$

Similarly,

$$\begin{aligned}\|\nabla_4\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}}&\lesssim\|\nabla_4\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})}}+\epsilon^a C(\Pi_0,\underline{\Pi}_0,\Pi_1,\mathcal{R},\underline{\mathcal{R}})(\mathcal{R}+\underline{\mathcal{R}}),\\ \|\nabla_3\underline{\alpha}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)}}&\lesssim\|\nabla_3\underline{\alpha}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_0^{(0,u)}}+\epsilon^a C(\Pi_0,\underline{\Pi}_0,\underline{\Pi}_1,\mathcal{R},\underline{\mathcal{R}})(\mathcal{R}+\underline{\mathcal{R}}),\end{aligned}$$

Combining, we derive the desired first derivative estimates

Proposition 4.6. *There exists a positive constant $a>\frac18$ such that, for $\delta^{1/2}\epsilon^{-1}$ and ϵ sufficiently small,*

$$\mathcal{R}_1(u,\underline{u})+\underline{\mathcal{R}}_1(u,\underline{u})\lesssim\mathcal{R}_1(0,\underline{u})+C\epsilon^a(\mathcal{R}+\underline{\mathcal{R}})$$

with $C=C(\Pi,\underline{\Pi},\ {}^{(O)}\Pi,\mathcal{R},\underline{\mathcal{R}})$.

Combining this with proposition 4.3 we derive,

$$\mathcal{R}(u,\underline{u})+\underline{\mathcal{R}}(u,\underline{u})\lesssim\mathcal{R}^{(0)}+C\epsilon^a(\mathcal{R}+\underline{\mathcal{R}})\tag{62}$$

which ends the proof of theorem 1.18.

5. Localized Energy estimates

5.1. **Localized zero order estimates.** We start by modifying slightly proposition 5.2,

Proposition 5.2. *The following identity holds on our fundamental domain* $\mathcal{D}(u, \underline{u})$*,*

$$\int_{H_u} {}^{(\Lambda)}f^2 Q[R](L, X, Y, Z) + \int_{\underline{H}_{\underline{u}}} {}^{(\Lambda)}f^2 Q[R](X, Y, Z, \underline{L}) = \int_{H_0} {}^{(\Lambda)}f^2 Q[R](L, X, Y, Z)$$
$$+\frac{1}{2}\int\int_{\mathcal{D}(u,\underline{u})} {}^{(\Lambda)}f^2 Q[R] \cdot \pi(X, Y, Z) + 2\int\int_{\mathcal{D}(u,\underline{u})} {}^{(\Lambda)}f Q[R](D\,{}^{(\Lambda)}f, X, Y, Z)$$

where $\pi(X, Y, Z)$ *is a linear combination of the deformation tensors of the vectorfields* X, Y, Z*.*

As in the derivation of the global estimates we make all the choices,

$$(X, Y, Z) = \{(L, L, L); (L, L, \underline{L}); (L, \underline{L}, \underline{L}); (\underline{L}, \underline{L}, \underline{L})\}.$$

In each case the only new term that needs to be estimated is due to $\int\int_{\mathcal{D}(u,\underline{u})} {}^{(\Lambda)}f Q[R](D\,{}^{(\Lambda)}f, X, Y, Z)$. Consider again the particular case $X = Y = Z = L$. Then,

$$\int_{H_u} |{}^{(\Lambda)}f\,\alpha|^2 + \int_{\underline{H}_{\underline{u}}} |{}^{(\Lambda)}f\,\beta|^2 = \int_{H_0} |{}^{(\Lambda)}f\,\alpha|^2 + \frac{3}{2}\int\int_{\mathcal{D}(u,\underline{u})} {}^{(\Lambda)}f^2\, Q[R]({}^{(L)}\pi, L, L)$$
$$+ \quad 2\int\int_{\mathcal{D}(u,\underline{u})} Q[R](D\,{}^{(\Lambda)}f, L, L, L)$$

Clearly, recalling (41),

$$|Q[R](D\,{}^{(\Lambda)}f, L, L, L)| \lesssim |\nabla_3\,{}^{(\Lambda)}f||\alpha|^2 + |\nabla\,{}^{(\Lambda)}f||\beta| \cdot |\alpha|$$
$$\lesssim \epsilon\delta^{1/2}|\Lambda|^{-1}|\alpha|^2 + |\Lambda|^{-1}|\beta| \cdot |\alpha|$$

Recalling also, $\sum_{\tilde{\Lambda}} {}^{(\tilde{\Lambda})}f = 1$, and ${}^{(\Lambda)}f \cdot {}^{(\tilde{\Lambda})}f = 0$ except for a a few neighboring $\tilde{\Lambda}$,

$$|{}^{(\Lambda)}f Q[R](D\,{}^{(\Lambda)}f, L, L, L)| \lesssim \epsilon\delta^{1/2}|\Lambda|^{-1}\sum_{\tilde{\Lambda}} |{}^{(\Lambda)}f\,\alpha| \cdot |{}^{(\tilde{\Lambda})}f\,\alpha| + |\Lambda|^{-1}|{}^{(\Lambda)}f\,\alpha| \cdot |{}^{(\tilde{\Lambda})}f\,\beta|$$
$$\lesssim \epsilon\delta^{1/2}|\Lambda|^{-1}|{}^{(\Lambda)}f\,\alpha| \cdot \sup_{\tilde{\Lambda}} |{}^{(\tilde{\Lambda})}f\,\alpha| + |\Lambda|^{-1}|{}^{(\Lambda)}f\,\alpha| \cdot \sup_{\tilde{\Lambda}} |{}^{(\tilde{\Lambda})}f\,\beta|$$

Therefore, passing to scale invariant norms, and treating the term in ${}^{(L)}\pi$ exactly as before,

$$\begin{aligned}\|{}^{(\Lambda)}f\,\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\|{}^{(\Lambda)}f\,\beta\|^2_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})} &\lesssim \|{}^{(\Lambda)}f\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})}+\epsilon\Pi_0\cdot{}^{(\Lambda)}\mathcal{R}_0^2(u,\underline{u})\\ &+\epsilon\delta^{1/2}|\Lambda|^{-1}\|{}^{(\Lambda)}f\,\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\cdot\sup_{\tilde\Lambda}\|{}^{(\tilde\Lambda)}f\,\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\ &+|\Lambda|^{-1}\|{}^{(\Lambda)}f\,\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\cdot\delta^{1/2}\sup_{\tilde\Lambda}\|{}^{(\tilde\Lambda)}f\,\beta\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\ &\lesssim \|{}^{(\Lambda)}f\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})}+\epsilon\Pi_0\ {}^{(\Lambda)}\mathcal{R}_0^2(u,\underline{u})\\ &+|\Lambda|^{-1}\delta^{1/2}\big(\epsilon\,{}^{(\Lambda)}\mathcal{R}_0[\alpha]\cdot{}^{[\Lambda]}\mathcal{R}_0[\alpha]+{}^{(\Lambda)}\mathcal{R}_0[\alpha]\cdot{}^{[\Lambda]}\mathcal{R}_0[\beta]\big)\end{aligned}$$

Therefore, taking the supremum over Λ on both sides, we derive

$$\begin{aligned}{}^{[\Lambda]}\mathcal{R}_0^2[\alpha](u,\underline{u})+{}^{[\Lambda]}\underline{\mathcal{R}}_0^2[\beta](u,\underline{u}) &\lesssim {}^{[\Lambda]}\mathcal{R}_0^2[\alpha](0,\underline{u})+\epsilon\,\Pi_0\cdot{}^{[\Lambda]}\mathcal{R}_0^2(u,\underline{u})\\ &+|\Lambda|^{-1}\delta^{1/2}\big(\epsilon\,{}^{[\Lambda]}\mathcal{R}_0^2[\alpha]+{}^{[\Lambda]}\mathcal{R}_0[\alpha]\cdot{}^{[\Lambda]}\mathcal{R}_0[\beta]\big)\end{aligned}$$

Proceeding in the same manner with all other curvature components we derive,

Proposition 5.3. *Consider a partition of unity ${}^{(\Lambda)}f$ of of length $|\Lambda|$ such that $\delta^{1/2}|\Lambda|^{-1}$ is sufficiently small. There exists a positive constant $a>\frac{1}{8}$ such that, for $\delta^{1/2}\epsilon^{-1}$ and ϵ sufficiently small,*

$$ {}^{[\Lambda]}\mathcal{R}_0(u,\underline{u})+{}^{[\Lambda]}\underline{\mathcal{R}}_0(u,\underline{u})\le{}^{[\Lambda]}\mathcal{R}_0(0,\underline{u})+\epsilon^aC(\Pi_0,\underline{\Pi}_0,{}^{[\Lambda]}\mathcal{R},{}^{[\Lambda]}\underline{\mathcal{R}})({}^{[\Lambda]}\mathcal{R}+{}^{[\Lambda]}\underline{\mathcal{R}}) \tag{63}$$

5.4. **Localized derivative estimates.** We start with a localized version of proposition 4.5.

Proposition 5.5. *Let U be a vectorfield defined in our fundamental domain $\mathcal{D}(u,\underline{u})$, tangent to $\underline{H}_0$. Then, with $H_u=H_u([0,\underline{u}])$,*

$$\begin{aligned}&\int_{H_u}{}^{(\Lambda)}f^2Q[\widehat{\mathcal{L}}_UR](L,X,Y,Z)+\int_{\underline{H}_{\underline{u}}}{}^{(\Lambda)}f^2Q[\widehat{\mathcal{L}}_UR](X,Y,Z,\underline{L})=\int_{H_0}{}^{(\Lambda)}f^2Q[\widehat{\mathcal{L}}_UR](L,X,Y,Z)\\ &+\frac12\int\int_{\mathcal{D}(u,\underline{u})}{}^{(\Lambda)}f^2Q[\widehat{\mathcal{L}}_UR]\cdot\pi(X,Y,Z)+\int\int_{\mathcal{D}(u,\underline{u})}{}^{(\Lambda)}f^2D(R,U)(X,Y,Z)\\ &+2\int\int_{\mathcal{D}(u,\underline{u})}{}^{(\Lambda)}fQ[\widehat{\mathcal{L}}_UR](D\,{}^{(\Lambda)}f,X,Y,Z)\end{aligned}$$

with $D(U,R)=(\widehat{\mathcal{L}}_UR)\cdot\big(\pi\cdot DR+D\pi\cdot R\big)$.

We apply these estimate, as before, for same choice of vectorfields,

$$(U;X,Y,Z)=\{(L;L,L,L);(\underline{L};\underline{L},\underline{L},\underline{L});(O;L,L,L);(O;L,L,\underline{L});(O;L,\underline{L},\underline{L});(O;\underline{L},\underline{L},\underline{L})\},$$

The only new terms which need to be treated are due to $\int\int_{\mathcal{D}(u,\underline{u})}{}^{(\Lambda)}fQ[\widehat{\mathcal{L}}_UR](D\,{}^{(\Lambda)}f,X,Y,Z)$. For all choices of vectorfields $U,,X,Y,Z$ we can proceed precisely as in the proof of proposition 4.5 and thus derive.

Proposition 5.6. *Given a partition of unity* ${}^{(\Lambda)}f$ *of length* $|\Lambda|$, *such that* $\delta^{1/2}|\Lambda|^{-1}$ *is sufficiently small, we can find* $a > \frac{1}{8}$ *such that, for* $\delta^{1/2}\epsilon^{-1}$ *and* ϵ *sufficiently small,*

$$ {}^{[\Lambda]}\mathcal{R}_1(u,\underline{u}) + {}^{[\Lambda]}\underline{\mathcal{R}}_1(u,\underline{u}) \lesssim \mathcal{R}_1(0,\underline{u}) + \epsilon^a C(\Pi, \underline{\Pi}, {}^{(O)}\Pi, {}^{[\Lambda]}\mathcal{R}, {}^{[\Lambda]}\underline{\mathcal{R}})({}^{[\Lambda]}\mathcal{R} + {}^{[\Lambda]}\underline{\mathcal{R}}) $$

Combining propositions 4.6 and 5.6 we derive,

Theorem 5.7. *Given a partition of unity* ${}^{(\Lambda)}f$ *of length* $|\Lambda|$, *such that* $\delta^{1/2}|\Lambda|^{-1}$ *is sufficiently small, we can find* $a > \frac{1}{8}$ *such that, for* $\delta^{1/2}\epsilon^{-1}$ *and* ϵ *sufficiently small, we have,*

$$ {}^{[\Lambda]}\mathcal{R}(u,\underline{u}) + {}^{[\Lambda]}\underline{\mathcal{R}}(u,\underline{u}) \lesssim {}^{[\Lambda]}\mathcal{R}^{(0)} + \epsilon^a C(\Pi, \underline{\Pi}, {}^{(O)}\Pi, {}^{[\Lambda]}\mathcal{R}, {}^{[\Lambda]}\underline{\mathcal{R}})({}^{[\Lambda]}\mathcal{R} + {}^{[\Lambda]}\underline{\mathcal{R}}) \tag{64} $$

6. Deformation tensor estimates

In this section we sketch the proof of the estimates which relate the norms Π of the deformation tensors for $L, \underline{L}$ and O to the Ricci coefficient norms $\mathcal{O}$, stated in theorem 1.19. Throughout the section we assume that $\delta^{1/2}\epsilon^{-1}$ and ϵ are sufficiently small.

6.1. **Estimates for Π and $\underline{\Pi}$.** The null components of ${}^{(L)}\pi$ and ${}^{(\underline{L})}\pi$ are simply expressed in terms of null Ricci coefficients according to the lemma.

Lemma 6.2. *Below we list the components of* ${}^{(L)}\pi_{\alpha\beta}$ *and* ${}^{(\underline{L})}\pi_{\alpha\beta}$.

$$ \begin{aligned} &{}^{(L)}\pi_{44} = 0, \quad {}^{(L)}\pi_{43} = 0, \quad {}^{L}\pi_{33} = -2\Omega^{-1}\underline{\omega}, \\ &{}^{(L)}\pi_{4a} = 0, \quad {}^{(L)}\pi_{3a} = \Omega^{-1}(\eta_a + \zeta_a) + \Omega^{-1}\nabla_a \log\Omega, \\ &{}^{(L)}\pi_{ab} = \Omega^{-1}\chi_{ab} \end{aligned} \tag{65} $$

and,

$$ \begin{aligned} &{}^{(\underline{L})}\pi_{33} = 0, \quad {}^{(\underline{L})}\pi_{43} = 0, \quad {}^{(\underline{L})}\pi_{33} = -2\Omega^{-1}\omega, \\ &{}^{(\underline{L})}\pi_{3a} = 0, \quad {}^{(\underline{L})}\pi_{4a} = \Omega^{-1}(\underline{\eta}_a + \zeta_a) + \Omega^{-1}\nabla_a \log\Omega, \\ &{}^{(\underline{L})}\pi_{ab} = \Omega^{-1}\underline{\chi}_{ab} \end{aligned} \tag{66} $$

As a result we can easily derive the estimates,

$$ \begin{aligned} \Pi_{0,4} &\lesssim {}^{(S)}\mathcal{O}_{0,4}, \qquad \Pi_{0,\infty} \lesssim {}^{(S)}\mathcal{O}_{0,\infty}, \\ \underline{\Pi}_{0,4} &\lesssim {}^{(S)}\mathcal{O}_{0,4}, \qquad \underline{\Pi}_{0,\infty} \lesssim {}^{(S)}\mathcal{O}_{0,\infty} \end{aligned} $$

Similarly we can estimate the first derivative norms,

$$ \Pi_1 \lesssim C({}^{(S)}\mathcal{O}_{1,4}, {}^{(S)}\mathcal{O}_0), \qquad \underline{\Pi}_1 \lesssim C({}^{(S)}\mathcal{O}_{1,4}, {}^{(S)}\mathcal{O}_0). $$

These can be summarized in the following:

Proposition 6.3. *The following estimates hold true for the deformation tensors* ${}^{(L)}\pi$ *and* ${}^{(\underline{L})}\pi$.

$$\Pi + \underline{\Pi} \lesssim C(\mathcal{O}) \tag{67}$$

which establishes half of theorem B (1.19).

6.4. **Estimates for** ${}^{(O)}\Pi$**.** Recall that the only non-vanishing components of ${}^{(O)}\pi$ are given by

$$\begin{aligned}
\pi_{34} &= -2(\eta + \underline{\eta})_a O_a, \\
\pi_{ab} &= \frac{1}{2}(\nabla_a O_b + \nabla_b O_a) := H^{(s)}_{ab} = \frac{1}{2}(H_{ab} + H_{ba}), \\
\pi_{3a} &= \frac{1}{2}(\nabla_3 O_a - \underline{\chi}_{ab} O_b) := \frac{1}{2} Z_a.
\end{aligned}$$

The quantities Z and H verify the following transport equations, written schematically,

$$\begin{aligned}
\nabla_4 Z &= \nabla(\eta + \underline{\eta}) \cdot O + (\underline{\eta} - \eta) \cdot H + \omega Z + (\sigma + \rho) \cdot O + (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) \cdot O, \\
\nabla_4 H &= \chi \cdot H + \beta \cdot O + \nabla\chi \cdot O + \chi \cdot \underline{\eta} \cdot O
\end{aligned} \tag{68}$$

In view of equations (68) we derive, by integration,

$$\|Z\|_{\mathcal{L}^\infty_{(sc)}} \lesssim \|\nabla(\eta, \underline{\eta})\|_{Tr_{(sc)}} + \|(\rho, \sigma)\|_{Tr_{(sc)}} + \|\psi\|_{\mathcal{L}^\infty_{(sc)}} (\|\psi\|_{\mathcal{L}^\infty_{(sc)}} + \|H\|_{\mathcal{L}^\infty_{(sc)}} + \|Z\|_{\mathcal{L}^\infty_{(sc)}})$$

Using the trace estimates for $(\eta, \underline{\eta})$ and (ρ, σ) we derive,

$$\epsilon^{-1} \|Z\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim C + C(\|H\|_{\mathcal{L}^\infty_{(sc)}} + \|Z\|_{\mathcal{L}^\infty_{(sc)}})$$

with a constant $C = C(\mathcal{I}_0, {}^{(S)}\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}})$. Similarly,

$$\begin{aligned}
\epsilon^{-1} \|H\|_{\mathcal{L}^\infty_{(sc)}} &\lesssim \epsilon^{-1} \left(\|\nabla\hat{\chi}\|_{Tr_{(sc)}} + \|\nabla \mathrm{tr}\chi\|_{\mathcal{L}^\infty_{(sc)}} + \|\psi\|^2_{\mathcal{L}^\infty_{(sc)}} (\|\psi\|_{\mathcal{L}^\infty_{(sc)}} + \|H\|_{\mathcal{L}^\infty_{(sc)}}) \right) \\
&\lesssim C + C(C + \|H\|_{\mathcal{L}^\infty_{(sc)}}),
\end{aligned}$$

Following precisely the same steps as section 13 of in [K-R:trapped] we derive,

$$\epsilon^{-1} \| {}^{(O)}\pi \|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^{-1} \| {}^{(O)}\pi \|_{\mathcal{L}^\infty_{(sc)}(S)} + \epsilon^{-1} \|D\pi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C = C({}^{(S)}\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}}).$$

Also all null components of the derivatives $D\, {}^{(O)}\pi$, with the exception of $(D_3\, {}^{(O)}\pi)_{3a}$, verify the estimates,

$$\epsilon^{-1} \|D\, {}^{(O)}\pi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C$$

Moreover,

$$\epsilon^{-1} \|(D_3\, {}^{(O)}\pi)_{3a} - \nabla_3 Z\|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^{-1} \| \sup_{\underline{u}} |\nabla_3 Z| \|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C$$

Recalling the definition of the norms ${}^{(O)}\Pi$ we deduce,

Proposition 6.5. *The following estimates hold true with a constant* $C = C(\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}})$,

$$^{(O)}\Pi \ \lesssim \ C(\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}}). \tag{69}$$

This establishes the remaining part of theorem B(1.19).

7. Estimates for the Ricci coefficients

In this section we discuss the proof of theorem C(1.20). We make the point that the proof can be derived by a straightforward modification of the arguments in sections 5-10 of [K-R:trapped].

Relying on the bootstrap assumption the boot-strap assumption (34) we first derive, see section 4.1. in [K-R:trapped],

$$\|\Omega^{-1} - 2\|_{L^\infty(u,\underline{u})} \ \lesssim \ \int_0^u \|\underline{\omega}\|_{L^\infty(u',\underline{u})} du' \lesssim \epsilon \ ^{(S)}\mathcal{O}_{0,\infty}[\underline{\omega}] \lesssim \epsilon\, \Delta_0.$$

Thus, if ϵ is sufficiently small we deduce that $|\Omega - \frac{1}{2}|$ is small and therefore,

$$\frac{1}{4} \le \Omega \le 4.$$

Using this fact we can deduce, as in section 4.1. of [K-R:trapped],

Proposition 7.1.

$$\begin{aligned} \| \ \psi\|_{\mathcal{L}^p_{(sc)}(u,\underline{u})} &\lesssim \|\psi\|_{\mathcal{L}^p_{(sc)}(u,0)} + \int_0^{\underline{u}} \delta^{-1} \|\nabla_4 \psi\|_{\mathcal{L}^p_{(sc)}(u,\underline{u}')}\, d\underline{u}' \\ \| \ \psi\|_{\mathcal{L}^p_{(sc)}(u,\underline{u})} &\lesssim \|\psi\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \int_0^{u} \|\nabla_3 \psi\|_{\mathcal{L}^p_{(sc)}(u',\underline{u})}\, du'. \end{aligned} \tag{70}$$

7.2. **Estimates for** $\chi, \eta, \underline{\omega}$**.** The null Ricci coefficients χ, η and $\underline{\omega}$ verify transport equations of the form,

$$\nabla_4 \psi^{(s)} = \sum_{s_1+s_2=s+1} \psi^{(s_1)} \cdot \psi^{(s_2)} + \Psi^{(s+1)} \tag{71}$$

we have

$$\|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \ \lesssim \ \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,0)} + \int_0^{\underline{u}} \delta^{-1} \|\nabla_4 \psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}$$

To estimate $\|\nabla_4 \psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}$ we make us of the scale invariant estimates

$$\|\phi \cdot \psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\phi\|_{\mathcal{L}^\infty_{(sc)}(S)} \|\psi\|_{\mathcal{L}^4_{(sc)}(S)}$$

Hence,

$$\|\nabla_4\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(S)} + \sum_{s_1+s_2=s+1}\|\psi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}(S)}\|\psi^{(s_2)}\|_{\mathcal{L}^4_{(sc)}(S)}$$

If all scale invariant norms were small, i.e. $O(\epsilon)$, we would proceed in a straightforward manner as follows,

$$\begin{aligned}\|\nabla_4\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)} &\lesssim \|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^2\ {}^{(S)}\mathcal{O}_{0,\infty}\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\\ &\lesssim \|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(S)} + \epsilon^2\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\end{aligned}$$

This in fact works for $s<1$, i.e. for $\underline{\omega}$ and η. In that case we have, by integration,

$$\begin{aligned}\|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,0)} + \int_0^{\underline{u}}\delta^{-1}\|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')} + \epsilon^2\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\\ &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,0)} + \|\Psi^{(s+1)}\|^{1/2}_{\mathcal{L}_{(sc)}(H_u)}\|\nabla\Psi^{(s+1)}\|^{1/2}_{\mathcal{L}_{(sc)}(H_u)} + \epsilon^2\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\\ &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,0)} + \epsilon(\mathcal{R}'_0)^{1/2}(\mathcal{R}'_1)^{1/2} + \epsilon^2\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\end{aligned}$$

i.e.,

$$\epsilon^{-1}\|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \mathcal{I}'_0 + \mathcal{R} + \epsilon\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}$$

On the other hand, for $s=1$,

$$\begin{aligned}\|\chi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|\chi\|_{\mathcal{L}^4_{(sc)}(u,0)} + \|\alpha\|^{1/2}_{\mathcal{L}_{(sc)}(H_u)}\|\nabla\alpha\|^{1/2}_{\mathcal{L}_{(sc)}(H_u)}\\ &+ \epsilon\Delta_0\cdot\|\chi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} + \epsilon^2\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\\ &\lesssim \|\chi\|_{\mathcal{L}^4_{(sc)}(u,0)} + \epsilon^{1/2}\mathcal{R} + \epsilon\Delta_0\cdot\|\chi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} + \epsilon^2\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\end{aligned}$$

i.e., for small enough ϵ,

$$\epsilon^{-1/2}\|\chi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \epsilon^{-1/2}\|\chi\|_{\mathcal{L}^4_{(sc)}(u,0)} + \mathcal{R} + \epsilon^{3/2}\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}$$

Proposition 7.3. *Under the bootstrap assumption* ${}^{(S)}\mathcal{O}_{0,\infty}\le\Delta_0$ *and assuming that* $\epsilon\Delta_0$ *is sufficiently small we derive,*

$$\begin{aligned}{}^{(S)}\mathcal{O}_{0,4}[\underline{\omega},\eta] &\lesssim \mathcal{I}'_0 + \mathcal{R} + \epsilon\Delta_0\cdot\ {}^{(S)}\mathcal{O}'_{0,4}\\ {}^{(S)}\mathcal{O}_{0,4}[tr\chi] &\lesssim 1 + \mathcal{R} + \epsilon\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4},\\ {}^{(S)}\mathcal{O}_{0,4}[\chi] &\lesssim \mathcal{R} + \epsilon\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4}\end{aligned}$$

Remark. As in [K-R:trapped] we can get improved estimates for $\mathrm{tr}\chi$, i.e. $\|\mathrm{tr}\chi\|_{\mathcal{L}^\infty_{(sc)}}\lesssim\epsilon^2$ and $\|\mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}}\lesssim\epsilon$

7.4. **Estimates for $\underline{\chi}, \underline{\eta}, \omega$.** The Ricci coefficients $\underline{\eta}, \underline{\chi}$ and $\underline{\omega}$ verify equations of the form

$$\nabla_3 \psi^{(s)} = -\frac{1}{2} k \operatorname{tr}\underline{\chi}_0 \psi^{(s)} + \sum_{s_1+s_2=s} \psi^{(s_1)} \cdot \psi^{(s_2)} + \Psi^{(s)} \tag{72}$$

with k a positive integer. If $s \geq 1/2$ we have, after a simple Gronwall inequality, (since $\Psi^{(s)} \neq \alpha$),

$$\begin{aligned} \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \int_0^u \|\Psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u',\underline{u})} + \epsilon^2 \Delta \cdot {}^{(S)}\mathcal{O}'_{0,4} \\ &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \epsilon (\underline{\mathcal{R}}'_0)^{\frac{1}{2}} (\underline{\mathcal{R}}'_1)^{\frac{1}{2}} + \epsilon^2 \Delta_0 \cdot {}^{(S)}\mathcal{O}_{0,4} \end{aligned}$$

Hence,

$$\epsilon^{-1} \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \epsilon^{-1} \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \mathcal{R} + \epsilon \Delta_0 \cdot {}^{(S)}\mathcal{O}_{0,4} \tag{73}$$

To estimate $\hat{\underline{\chi}}$ we use the estimate,

$$\nabla_3 \hat{\underline{\chi}} = -\underline{\alpha} + \operatorname{tr}\underline{\chi}_0 \, \hat{\underline{\chi}} - \widetilde{\operatorname{tr}\underline{\chi}} \, \hat{\underline{\chi}} - 2\underline{\omega} \hat{\underline{\chi}}$$

Thus, after a standard application of the Gronwall inequality,

$$\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_u)} \lesssim \|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_0)} + \int_0^u \|\underline{\alpha}\|_{\mathcal{L}^4_{(sc)}(S_{u'})} + \epsilon^2 \cdot \Delta \cdot {}^{(S)}\mathcal{O}'_{0,4}$$

i.e.,

$$\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_u)} \lesssim \|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_0)} + \epsilon \underline{\mathcal{R}} + \epsilon^2 \cdot \Delta \cdot {}^{(S)}\mathcal{O}'_{0,4}$$

Now observe that,

$$\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_0)} \lesssim \epsilon^{1/2} \mathcal{I}^{(0)}$$

Indeed, along H_0 (where $\omega = 0$),

$$\nabla_4 \hat{\underline{\chi}} + \frac{1}{2} \operatorname{tr}\chi \hat{\underline{\chi}} = \nabla \widehat{\otimes} \underline{\eta} - \tfrac{1}{2} \operatorname{tr}\underline{\chi} \hat{\chi} + \underline{\eta} \widehat{\otimes} \underline{\eta}$$

or,

$$\nabla_4 \hat{\underline{\chi}} = -\frac{1}{2} \operatorname{tr}\underline{\chi}_0 \hat{\chi} + \nabla \widehat{\otimes} \underline{\eta} + \psi_g \cdot \psi$$

Hence,

$$\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} \lesssim \|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \epsilon^{3/2} C$$

i.e.,

$$\begin{aligned} \epsilon^{-1/2} \|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} &\lesssim \epsilon^{-1/2} \|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \epsilon C \\ &\lesssim \mathcal{I}^{(0)} + \epsilon C \end{aligned}$$

8. Proof of theorem 2.8

We denote by $\dot{\mathcal{R}}$ amd $\dot{\mathcal{O}}$ the curvature and Ricci coefficient norms which can be obtained by formally choosing $\epsilon = \delta^{1/2}$ in the definitions (20) and (23). These correspond precisely to the $\mathcal{R}, \mathcal{O}$ norms used in our paper [K-R:trapped]. Since we have assumed that the initial data quantity $\mathcal{R}^{(0)}$, defined in (21), is uniformly distributed on the scale $\delta^{1/2}\varpi^{-1}$,

$$ {}^{[\Lambda]}\mathcal{R}^{(0)} \lesssim \delta^{\frac{1}{2}}\varpi^{-1}\mathcal{R}^{(0)} \tag{74}$$

from which we also deduce, according to theorem 2.4,

$$ ({}^{[\Lambda]}\mathcal{R} + {}^{[\Lambda]}\underline{\mathcal{R}})(u, \underline{u}) \lesssim \delta^{\frac{1}{2}}\varpi^{-1}\mathcal{R}^{(0)} \tag{75}$$

Observe that, with respect to the old scaling $\dot{sc}$ in [K-R:trapped], we deduce, for all $0 \le u \le 1$,

$$\begin{aligned}
\epsilon\| {}^{(\Lambda)}f\alpha\|_{\mathcal{L}_{(\dot{sc})}(H_u)} + \| {}^{(\Lambda)}f(\beta,\rho,\sigma,\underline{\beta})\|_{\mathcal{L}_{(\dot{sc})}(H_u)} &\lesssim \epsilon\varpi^{-1}\mathcal{R}^{(0)}\\
\| {}^{(\Lambda)}f\nabla\alpha\|_{\mathcal{L}_{(\dot{sc})}(H_u)} + \| {}^{(\Lambda)}f\nabla(\beta,\rho,\sigma,\underline{\beta})\|_{\mathcal{L}_{(\dot{sc})}(H_u)} &\lesssim \epsilon\varpi^{-1}\mathcal{R}^{(0)}
\end{aligned}$$

or, for $\epsilon\varpi^{-1} \lesssim 1$

$$\begin{aligned}
\varpi\| {}^{(\Lambda)}f\alpha\|_{\mathcal{L}_{(\dot{sc})}(H_u)} + \| {}^{(\Lambda)}f(\beta,\rho,\sigma,\underline{\beta})\|_{\mathcal{L}_{(\dot{sc})}(H_u)} &\lesssim \mathcal{R}^{(0)}\\
\| {}^{(\Lambda)}f\nabla\alpha\|_{\mathcal{L}_{(\dot{sc})}(H_u)} + \| {}^{(\Lambda)}f\nabla(\beta,\rho,\sigma,\underline{\beta}\|_{\mathcal{L}_{(\dot{sc})}(H_u)} &\lesssim \epsilon\varpi^{-1}\mathcal{R}^{(0)}
\end{aligned}\tag{76}$$

In particular, if $\delta^{1/2} \lesssim \varpi$ we deduce, relative to the old scaling $\dot{sc}$,

$$ {}^{[\Lambda]}\dot{\mathcal{R}}(u, \underline{u}) \lesssim \mathcal{R}^{(0)}. \tag{77}$$

with ${}^{[\Lambda]}\dot{\mathcal{R}}$ the localized version of the norms $\dot{\mathcal{R}}$, i.e. ${}^{[\Lambda]}\dot{\mathcal{R}} = \sup_\Lambda {}^{(\Lambda)}\dot{\mathcal{R}}$.

Moreover, with $\epsilon\varpi^{-1} := q$ a small parameter, we have,

$$ \sup_\Lambda \| {}^{(\Lambda)}f\nabla(\beta,\rho,\sigma,\underline{\beta})\|_{\mathcal{L}_{(\dot{sc})}(H_u)} \lesssim q \tag{78}$$

which, restricted to $u = 0$, is precisely the localized version of estimate (32) of proposition 2.8 in [K-R:trapped]. In view of theorem 2.6 in [K-R:trapped], the global version of estimate (77), i.e. $\dot{\mathcal{R}} < \infty$, allows one to deduce the boundedness of the global $\dot{\mathcal{O}}$ norms, i.e. for some universal constant C, $\dot{\mathcal{O}} \lesssim C$. The global version of condition (78), with q sufficiently small (which corresponds[5] to condition (32) of proposition 2.8), was necessary in the proof of theorem 2.7 in [K-R:trapped] to insure the formation of a trapped surface. The proof of formation of a trapped surface was based, in addition, on the crucial lower bound condition,

$$\int_0^\delta |\hat{\chi}_0|^2(\underline{u},\theta)d\underline{u} > \frac{2(r_0 - u)}{r_0^2}.$$

[5]With the small quantity ϵ replaced by q here.

We note that the proof of formation of a trapped surface argument[6], is purely local in θ. More precisely, to show that $\mathrm{tr}\chi(u,\delta,\theta) < 0$ requires only the control of the Ricci coefficients in the domain[7] $\{(u',\underline{u},\theta) : 0 \le u' \le u,\ 0 \le \underline{u} \le \delta\}$. We further note that the global versions (unlocalized) of (77), (78) make no reference to the support of the quantities involved and in particular are entirely compatible with the possibility that most or even all of the norm is concentrated in the support of ${}^{(\Lambda)}f$ for some specific Λ.

In view of the above discussion we conclude that we could adapt the proof used in [K-R:trapped] for the formation of a trapped surface to our situation provided that we could derive bounds for the localized $\dot{\mathcal{O}}$ norms from the boundedness of the localized $\dot{\mathcal{R}}$ norms. More precisely we need to prove the following:

Proposition 8.1. *Let $\{\Lambda\}$ be a partition of S_0 of size $|\Lambda| \approx \delta^{\frac12} q^{-1}$ with q sufficiently small. Then assuming that ${}^{[\Lambda]}\dot{\mathcal{O}}^{(0)} < \infty$,*

$$ {}^{[\Lambda]}\dot{\mathcal{O}} \le C({}^{[\Lambda]}\dot{\mathcal{O}}^{(0)},\ {}^{[\Lambda]}\dot{\mathcal{R}},\ {}^{[\Lambda]}\dot{\underline{\mathcal{R}}}) $$

The proof of this proposition is based on the observation that all arguments used in sections 5-12 of [K-R:trapped] can be appropriately localized. This is particularly obvious for those estimates which are derived from transport equations. Consider, for example, the transport equations of the form (71) or, simply, $\nabla_4\psi = \psi\cdot\psi + \Psi$. Since $\nabla_4^{(\Lambda)} f = 0$ and $\sum_\Lambda {}^{(\Lambda)}f = 1$ we deduce,

$$ \begin{aligned} \nabla_4\, {}^{(\Lambda)}f\psi &= {}^{(\Lambda)}f\psi\cdot\psi + {}^{(\Lambda)}f\Psi \\ &= \sum_{\tilde\Lambda} {}^{(\Lambda)}f\psi\cdot {}^{(\tilde\Lambda)}f\psi + {}^{(\Lambda)}f\Psi \end{aligned} $$

Hence, with respect to the old scaling,

$$ \begin{aligned} \|{}^{(\Lambda)}f\psi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|{}^{(\Lambda)}f\psi\|_{\mathcal{L}^4_{(sc)}(u,0)} + \delta^{-1}\int_0^{\underline{u}} \delta^{1/2}\sup_\Lambda \|{}^{(\Lambda)}f\psi\|_{\mathcal{L}^\infty_{(sc)}(u,\underline{u}')}\sup_\Lambda\|{}^{(\Lambda)}f\psi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}d\underline{u}' \\ &+ \delta^{-1}\int_0^{\underline{u}}\delta^{1/2}\sup_\Lambda\|{}^{(\Lambda)}f\Psi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}d\underline{u}' \end{aligned} $$

Proceeding as in [K-R:trapped] we derive,

$$ \sup_\Lambda \|{}^{(\Lambda)}f\psi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim {}^{[\Lambda]}\dot{\mathcal{R}} + \delta^{1/2}\, {}^{[\Lambda]}\dot{\mathcal{O}}_{0,\infty}\cdot {}^{[\Lambda]}\dot{\mathcal{O}}_{0,4} $$

In the case of the transport equations of the form (72), i.e., $\nabla_3\psi = \psi\cdot\psi + \Psi$ we obtain,

$$ \nabla_3({}^{(\Lambda)}f\,\psi) = {}^{(\Lambda)}f\psi\cdot\psi + {}^{(\Lambda)}f\Psi + \epsilon\delta^{\frac12}|\Lambda|^{-1}\,{}^{(\Lambda)}\tilde f\,\psi, $$

[6] See the original argument in [Chr:book] and its outline in the introduction to [K-R:trapped].

[7] In actuality, because of the difference between the θ and $\underline{\theta}$ coordinates defined respectively by parallel transport along H_u and $\underline{H}_{\underline{u}}$, the domain has to be enlarged to include all angles θ' such that $|\theta'-\theta| \le \delta^{\frac12}$.

where $\tilde{f}$ is a function similar to f but with slightly large support. Using that $\delta^{\frac{1}{2}}|\Lambda|^{-1} \lesssim q$ we easily obtain the estimate, for the corresponding Ricci components,

$$\sup_{\Lambda} \| {}^{(\Lambda)} f\psi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim {}^{[\Lambda]}\dot{\underline{\mathcal{R}}} + \delta^{1/2}\, {}^{[\Lambda]}\dot{\mathcal{O}}_{0,\infty} \cdot {}^{[\Lambda]}\dot{\mathcal{O}}_{0,4}.$$

The angular localization also affects the elliptic estimates for the Ricci coefficients. For the Codazzi equation

$$\mathcal{D}\psi = \psi\cdot\psi + \Psi$$

we obtain

$$\| {}^{(\Lambda)} f\, \mathcal{D}\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \| {}^{(\Lambda)} f\psi\cdot\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \| {}^{(\Lambda)} f\Psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \delta^{\frac{1}{2}}\, {}^{[\Lambda]}\dot{\mathcal{O}}_{0,\infty} \cdot {}^{[\Lambda]}\dot{\mathcal{O}}_{0,2} + {}^{[\Lambda]}\dot{\mathcal{R}}_0 + {}^{[\Lambda]}\dot{\underline{\mathcal{R}}}_0$$

On the other hand, integrating by parts and using the identity $\Delta = \mathcal{D}^*\mathcal{D} \pm K$ we obtain

$$\begin{aligned}\| {}^{(\Lambda)} f\, \nabla\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \| {}^{(\Lambda)} f\, \mathcal{D}\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \|\nabla\, {}^{(\Lambda)} f\, \psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \| {}^{(\Lambda)} f\, K\cdot\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\\ &\lesssim \| {}^{(\Lambda)} f\, \mathcal{D}\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \delta^{\frac{1}{2}}|\Lambda|^{-1}\, {}^{[\Lambda]}\dot{\mathcal{O}}_{0,2} + \delta^{\frac{1}{2}}\, {}^{[\Lambda]}\dot{\mathcal{R}}\, {}^{[\Lambda]}\dot{\mathcal{O}}_{0,\infty}\end{aligned}$$

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