ON THE FORMATION OF TRAPPED SURFACES

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1. Introduction

1.1. **Main Goals.** In a recent important breakthrough D. Christodoulou [Chr] has solved a long standing problem of General Relativity of evolutionary formation of trapped surfaces in the Einstein-vacuum space-times. He has identified an open set of regular initial conditions on a finite outgoing null hypersurface leading to a formation a trapped surface in the corresponding vacuum space-time to the future of the initial outgoing hypersurface and another incoming null hypersurface with the prescribed Minkowskian data. He also gave a version of the same result for data given on part of past null infinity. His proof, which we outline below, is based on an inspired choice of the initial condition, an ansatz which he calls *short pulse*, and a complex argument of propagation of estimates, consistent with the ansatz, based, largely, on the methods used in the global stability of the Minkowski space [Chr-Kl]. Once such estimates are established in a sufficiently large region of the space-time the actual proof of the formation of a trapped surface is quite straightforward.

The goal of the present paper is to give a simpler proof by enlarging the admissible set of initial conditions and, consistent with this, relaxing the corresponding propagation estimates just enough that a trapped surface still forms. We also reduce the number of derivatives needed in the argument from two derivatives of the curvature to just one. More importantly, the proof, which can be easily localized with respect to angular sectors, has the potential for further developments. We prove in fact another result, concerning the formation of *pre-scarred* surfaces, i.e surfaces whose outgoing expansion is negative in an open angular sector. We only concentrate here on the finite problem, the problem from past null infinity can be treated in the same fashion as in [Chr] once the finite problem is well understood. The problem from past null infinity has been subsequently considered in a recent preprint by Reiterer and Trubowitz, [R-T].

We start by providing the framework of double null foliations in which Christodoulou's result is formulated. We then present, in subsection 1.3, the heuristic argument for the formation of a trapped surface. In subsection 1.4 we then introduce Christodolou's *short pulse* ansatz and discuss the propagation estimates which it entails.

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1.2. **Double null foliations.** We consider a region $\mathcal{D} = \mathcal{D}(u_*, \underline{u}_*)$ of a vacuum spacetime (M, g) spanned by a double null foliation generated by the optical functions $(u, \underline{u})$ increasing towards the future, $0 \le u \le u_*$ and $0 \le \underline{u} \le \underline{u}_*$. We denote by H_u the outgoing null hypersurfaces generated by the level surfaces of u and by $\underline{H}_{\underline{u}}$ the incoming null hypersurfaces generated level hypersurfaces of $\underline{u}$. We write $S_{u,\underline{u}} = H_u \cap \underline{H}_{\underline{u}}$ and denote by $H_u^{(\underline{u}_1,\underline{u}_2)}$, and $\underline{H}_{\underline{u}}^{(u_1,u_2)}$ the regions of these null hypersurfaces defined by $\underline{u}_1 \le \underline{u} \le \underline{u}_2$ and respectively $u_1 \le u \le u_2$. Let $L, \underline{L}$ be the geodesic vectorfields associated to the two foliations and define,

$$\frac{1}{2}\Omega^2 = -g(L, \underline{L})^{-1} \tag{1}$$

Observe that the flat value[1] of Ω is 1. As well known, our space-time slab $\mathcal{D}(u_*, \underline{u}_*)$ is completely determined (for small values of $u_*, \underline{u}_*$) by data along the null, characteristic, hypersurfaces H_0, $\underline{H}_0$ corresponding to $\underline{u} = 0$, respectively $u = 0$. Following [Chr] we assume that our data is trivial along $\underline{H}_0$, i.e. assume that H_0 extends for $\underline{u} < 0$ and the spacetime (M, g) is Minkowskian for $\underline{u} < 0$ and all values of $u \ge 0$. Moreover we can construct our double null foliation such that $\Omega = 1$ along H_0, i.e.,

$$\Omega(0, \underline{u}) = 1, \qquad 0 \le \underline{u} \le \underline{u}_*. \tag{2}$$

Throughout this paper we work with the normalized null pair (e_3, e_4),

$$e_3 = \Omega \underline{L}, \quad e_4 = \Omega L, \qquad g(e_3, e_4) = -2.$$

Given a 2-surfaces $S(u, \underline{u})$ and $(e_a)_{a=1,2}$ an arbitrary frame tangent to it we define define the Ricci coefficients,

$$\Gamma_{(\lambda)(\mu)(\nu)} = g(e_{(\lambda)}, D_{e_{(\nu)}} e_{(\mu)}), \quad \lambda, \mu, \nu = 1, 2, 3, 4 \tag{3}$$

These coefficients are completely determined by the following components,

$$\begin{aligned}
\chi_{ab} &= g(D_a e_4, e_b), \qquad \underline{\chi}_{ab} = g(D_a e_3, e_b),\\
\eta_a &= -\frac{1}{2} g(D_3 e_a, e_4), \quad \underline{\eta}_a = -\frac{1}{2} g(D_4 e_a, e_3)\\
\omega &= -\frac{1}{4} g(D_4 e_3, e_4), \qquad \underline{\omega} = -\frac{1}{4} g(D_3 e_4, e_3),\\
\zeta_a &= \frac{1}{2} g(D_a e_4, e_3)
\end{aligned} \tag{4}$$

where $D_a = D_{e_{(a)}}$. We also introduce the null curvature components,

$$\begin{aligned}
\alpha_{ab} &= R(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} = R(e_a, e_3, e_b, e_3),\\
\beta_a &= \frac{1}{2} R(e_a, e_4, e_3, e_4), \quad \underline{\beta}_a = \frac{1}{2} R(e_a, e_3, e_3, e_4),\\
\rho &= \frac{1}{4} R(Le_4, e_3, e_4, e_3), \quad \sigma = \frac{1}{4} {}^*R(Le_4, e_3, e_4, e_3)
\end{aligned} \tag{5}$$

[1]Note that our normalization for Ω differ from that of [K-Ni]

Here ${}^*R$ denotes the Hodge dual of R. We denote by ∇ the induced covariant derivative operator on $S(u,\underline{u})$ and by ∇_3, ∇_4 the projections to $S(u,\underline{u})$ of the covariant derivatives D_3, D_4, see precise definitions in [K-Ni]. Observe that,

$$\begin{aligned} \omega &= -\frac{1}{2}\nabla_4(\log\Omega), \qquad \underline{\omega} = -\frac{1}{2}\nabla_3(\log\Omega), \\ \eta_a &= \zeta_a + \nabla_a(\log\Omega), \quad \underline{\eta}_a = -\zeta_a + \nabla_a(\log\Omega) \end{aligned} \tag{6}$$

The connection coefficients Γ verify equations which have, very roughly, the form,

$$\begin{aligned} \nabla_4\Gamma &= R + \nabla\Gamma + \Gamma\cdot\Gamma \\ \nabla_3\Gamma &= R + \nabla\Gamma + \Gamma\cdot\Gamma \end{aligned} \tag{7}$$

Similarly the Bianchi identities for the null curvature components verify, also very roughly,

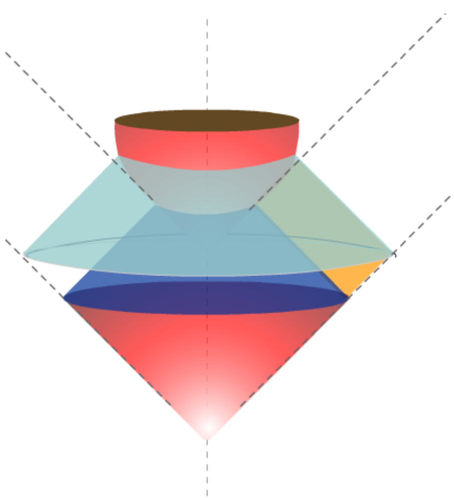
$$\begin{aligned} \nabla_4 R &= \nabla R + \Gamma\cdot R \\ \nabla_3 R &= R + \Gamma\cdot R \end{aligned} \tag{8}$$

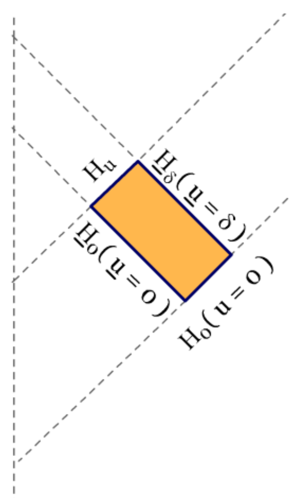
The precise form of these equations is given in the next section, see (47)–(50). Among these equations we note the following two, which play an essential role in Christodoulou's argument for the formation of trapped surfaces.

$$\nabla_4\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2 - 2\omega\mathrm{tr}\chi \tag{9}$$

$$\nabla_3\hat{\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} = \nabla\widehat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\underline{\hat{\chi}} + \eta\widehat{\otimes}\eta \tag{10}$$

1.3. **Heuristic argument.** We start by making some important simplifying assumptions. As mentioned above we assume that our data is trivial along $\underline{H}_0$, i.e. assume that H_0 extends for $\underline{u} < 0$ and the spacetime (M,g) is Minkowskian for $\underline{u} < 0$ and all values of $u \geq 0$. We introduce a small parameter $\delta > 0$ and restrict the values of $\underline{u}$ to $0 \leq \underline{u} \leq \delta$, i.e. $\underline{u}_* = \delta$.





The colored region on the right represents the domain $\mathcal{D}(u,\underline{u})$, $0 \leq \underline{u} \leq \delta$. The same picture is represented, more realistically on the left The lower red region on the left is the flat portion of H_0, $u = 0$, while the upper red region, corresponding to a large values of u, is trapped starting with $\underline{u} = \delta$.

We also make the following additional assumptions, assumed to hold in *the entire slab* $\mathcal{D}(u,\delta)$. We denote by $r = r(u,\underline{u})$ the radius of the 2-surfaces $S = S(u,\underline{u})$, i.e. $|S(u,\underline{u})| = 4\pi r^2$.

- For small δ, $u,\underline{u}$ are comparable with their standard values in flat space, i.e. $u \approx \frac{t-r+r_0}{2}$, $\underline{u} \approx \frac{t+r-r_0}{2}$. We also assume that $\Omega \approx 1$, $\frac{dr}{du} \approx -1$.
- Assume that $\mathrm{tr}\underline{\chi}$ is close to its value in flat space, i.e. $\mathrm{tr}\underline{\chi} \approx -\frac{2}{r}$.
- Assume that the term $E = \nabla\widehat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}} + \eta\widehat{\otimes}\eta$ on the right hand side of equation (10) is sufficiently small and can be neglected in a first approximation. Assume also that we can neglect the term $\mathrm{tr}\chi\omega$ on the right hand side of (9).

Given these assumptions we can rewrite (9),

$$\frac{d}{d\underline{u}}\mathrm{tr}\chi \lesssim -|\hat{\chi}|^2$$

or, integrating,

$$\begin{aligned}\mathrm{tr}\chi(u,\underline{u}) &\lesssim \mathrm{tr}\chi(u,0) - \int_0^{\underline{u}} |\hat{\chi}|(u,\underline{u}')^2 d\underline{u}' \\ &= \frac{2}{r(u,0)} - \int_0^{\underline{u}} |\hat{\chi}(u,\underline{u}')|^2 d\underline{u}'\end{aligned} \tag{11}$$

Multiplying (10) by $\hat{\chi}$ we deduce,

$$\frac{d}{du}|\hat{\chi}|^2 + \mathrm{tr}\underline{\chi}|\hat{\chi}|^2 = \hat{\chi}\cdot E$$

or, in view of our assumptions for $\mathrm{tr}\underline{\chi}$, and $\frac{dr}{du}$

$$\begin{aligned}\frac{d}{du}(r^2|\hat{\chi}|^2) &= r^2\frac{d}{du}|\hat{\chi}|^2 + 2r\frac{dr}{du}|\hat{\chi}|^2 = r^2|\hat{\chi}|^2\big(-\mathrm{tr}\underline{\chi} + \frac{2}{r}\frac{dr}{du}\big) + r^2\hat{\chi}\cdot E \\ &= r^2|\hat{\chi}|^2\big(-(\mathrm{tr}\underline{\chi} + \frac{2}{r}) + \frac{2}{r}(1 + \frac{dr}{du})\big) + r^2\hat{\chi}\cdot E := F\end{aligned}$$

i.e.

$$r^2|\hat{\chi}|^2(u,\underline{u}) = r^2(0,\underline{u})|\hat{\chi}|^2(0,\underline{u}) + \int_0^u F(u',\underline{u})du'$$

Therefore, as $\int_0^\delta |F|$ is negligible in $\mathcal{D}$, we deduce

$$r^2|\hat{\chi}|^2(u,\underline{u}) \approx r^2(0,\underline{u})|\hat{\chi}|^2(0,\underline{u})$$

We now freely prescribe $\hat{\chi}$ along the initial hypersurface $H_0^{(0,\delta)}$, i.e.

$$\hat{\chi}(0,\underline{u}) = \hat{\chi}_0(\underline{u}) \tag{12}$$

for some traceless 2 tensor $\hat{\chi}_0$. We deduce,

$$|\hat{\chi}|^2(u,\underline{u}) \approx \frac{r^2(0,\underline{u})}{r^2(u,\underline{u})}|\hat{\chi}_0|^2(\underline{u})$$

or, since $|\underline{u}| \le \delta$ and $r(u,\underline{u}) = r_0 + \underline{u} - u$,

$$|\hat{\chi}|^2(u,\underline{u}) \approx \frac{r_0^2}{(r_0-u)^2}|\hat{\chi}_0|^2(\underline{u})$$

Thus, returning to (11),

$$\mathrm{tr}\chi(u,\underline{u}) \quad \le \quad \frac{2}{r_0-u} - \frac{r_0^2}{(r_0-u)^2}\int_0^{\underline{u}} |\hat{\chi}_0|^2(\underline{u}')d\underline{u}' + \text{ error}$$

Hence, for small δ, the necessary condition to have $\mathrm{tr}\chi(u,\underline{u}) \le 0$ is,

$$\frac{2(r_0-u)}{r_0^2} < \int_0^\delta |\hat{\chi}_0|^2$$

Analyzing equation (9) along H_0 we easily deduce that the condition for the initial hypersurface H_0 not to contain trapped hypersurfaces is,

$$\int_0^\delta |\hat{\chi}_0|^2 < \frac{2}{r_0}$$

i.e. we are led to prescribe $\hat{\chi}_0$ such that,

$$\frac{2(r_0-u)}{r_0^2} < \int_0^\delta |\hat{\chi}_0|^2 < \frac{2}{r_0} \tag{13}$$

We thus expect, following Christodoulou, that trapped surfaces may form if (13) is verified.

1.4. **Short pulse data.** To prove such a result however we need to check that all the assumptions we made above can be verified. To start with, the assumption (13) requires, in particular, an L^∞ upper bound of the form,

$$|\hat{\chi}_0| \lesssim \delta^{-1/2}$$

If we can show that such a bound persist in $\mathcal{D}$ then, in order to control the error terms F we need, for some $c > 0$,

$$\begin{aligned} &\mathrm{tr}\underline{\chi} + \frac{2}{r} = O(\delta^c), \qquad \frac{dr}{du} + 1 = O(\delta^c), \qquad \eta = O(\delta^{-1/2+c}), \\ &\omega = O(\delta^{-1+c}), \qquad \nabla\eta = O(\delta^{-1/2+c}). \end{aligned} \tag{14}$$

Other bounds will be however needed as we have to take into account all null structure equations. We face, in particular, the difficulty that most null structure equations have curvature components as sources. Thus we are obliged to derive bounds not just for all Ricci coefficients $\chi, \omega, \eta, \underline{\eta}, \underline{\chi}, \underline{\omega}$ but also for all null curvature components $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$. In his work [Chr] Christodoulou has been able to derive such estimates starting with an ansatz (which he calls short pulse) for the initial data $\hat{\chi}_0$. More precisely he assumes, in addition to the triviality of the initial data along $\underline{H}_0$, that $\hat{\chi}_0$ verifies, relative to coordinates $\underline{u}$ and transported coordinates ω along H_0, (i.e. transported with respect to $\frac{d}{d\underline{u}}$),

$$\hat{\chi}_0(\underline{u}, \omega) = \delta^{-1/2} f_0(\delta^{-1}\underline{u},\, \omega) \tag{15}$$

where f_0 is a fixed traceless, symmetric S-tangent two tensor along H_0. This ansatz is consistent with the following more general condition, for sufficiently large number of derivatives N and sufficiently small $\delta > 0$.

$$\delta^{1/2+k} \|\nabla_4^k \nabla^m \hat{\chi}_0\|_{L^2(0,\underline{u})} < \infty, \qquad 0 \le k + m \le N, \;\; 0 \le \underline{u} \le \delta. \tag{16}$$

Notation. Here $\|\cdot\|_{L^2(u,\underline{u})}$ denotes the standard L^2 norm for tensorfields on $S(u, \underline{u})$. Whenever there is no possible confusion we will also denote these norms by $\|\cdot\|_{L^2(S)}$. We shall also denote by $\|\cdot\|_{L^2(H)}$ and $\|\cdot\|_{L^2(\underline{H})}$ the standard L^2 norms along the null hypersurfaces $H = H_u$ and $\underline{H} = \underline{H}_{\underline{u}}$.

Remark 1.5. In [Chr] Christodoulou also includes weights, depending on $|u|$, in his estimates. These allow him to derive not only a local result but also one with data at past null infinity. In our work here we only concentrate on the local result, for $|u| \lesssim 1$, and thus drop the weights.

Assumption (16), together with the null structure equations (7) and null Bianchi equations (8) leads to the following estimates for the null curvature components, along the initial null hypersurface H_0,

$$\delta\|\alpha\|_{L^2(H_0)} + \|\beta\|_{L^2(H_0)} + \delta^{-1/2}\|(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-3/2}\|\underline{\beta}\|_{L^2(H_0)} \;\; < \;\; \infty \tag{17}$$

Consistent with (16), the angular derivatives of $\alpha, \beta, \rho, \sigma, \beta$ obey the same scaling as in (17) while each ∇_4 derivative costs an additional power of δ.

$$\begin{aligned} \delta\|\nabla\alpha\|_{L^2(H_0)} + \|\nabla\beta\|_{L^2(H_0)} + \delta^{-\frac{1}{2}}\|\nabla(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-3/2}\|\nabla\underline{\beta}\|_{L^2(H_0)} &< \infty, \\ \delta^2\|\nabla_4\alpha\|_{L^2(H_0)} + \delta\|\nabla_4\beta\|_{L^2(H_0)} + \delta^{1/2}\|\nabla_4(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-1/2}\|\nabla_4\underline{\beta}\|_{L^2(H_0)} &< \infty \end{aligned} \tag{18}$$

Moreover one can derive estimates for the Ricci coefficients, in various norms, weighted by appropriated powers of δ. Note that if one were to neglect the quadratic terms in (8) than the expected scaling behavior in δ would have been,

$$\delta\|\alpha\|_{L^2(H_0)} + \|\beta\|_{L^2(H_0)} + \delta^{-1}\|(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-2}\|\underline{\beta}\|_{L^2(H_0)} \;\; < \;\; \infty$$

Most of the body of work in [Chr] is to prove that these estimates can be propagated in the entire space-time region $\mathcal{D}(u_*, \delta)$, with u_* of size one and δ sufficiently small, and thus fulfill the necessary conditions for the formation of a trapped surface along the lines of the heuristic argument presented above. The proof of such estimates, which follows the main outline of the proof of stability of Minkowski space, as in [Chr-Kl] and [K-Ni], requires a step by step analysis to make sure that all estimates are consistent with the assigned powers of δ. This task is made particularly taxing in view of the fact that there are many nonlinear interferences which have to be tracked precisely.

1.6. **Outline of Christodoulou's propagation estimates.** To see what this entails it pays to say a few words about the strategy of the proof. As in [Chr-Kl] and [K-Ni] the centerpiece of the entire proof consists in proving spacetime curvature estimates consistent with (17). In this case however the primary attention has to be given to the stratification of the estimates for different curvature components based on their δ-weights. This is done using the Bianchi identities,

$$D_{[\epsilon}R_{\alpha\beta]\gamma\delta} = 0,$$

the associated Bel-Robinson tensor Q and carefully chosen vectorfields X whose deformation tensors ${}^{(X)}\pi$ depend only on the Ricci coefficients $\chi, \omega, \eta, \underline{\eta}, \underline{\chi}, \underline{\omega}$. These vectorfields can be used either as commutation vectorfields or multipliers. In the latter case we would have,

$$D^\delta(Q_{\alpha\beta\gamma\delta}X^\alpha Y^\beta Z^\delta) = Q({}^{(X)}\pi, Y, Z) + \dots \tag{19}$$

As multipliers X, Y, Z we can chose the vectorfields e_3, e_4. The choice $X = Y = Z = e_4$ leads to, after integration on $\mathcal{D}(u, \underline{u})$,

$$\|\alpha\|^2_{L^2(H_u^{(0,\underline{u})})} + \|\beta\|^2_{L^2(\underline{H}_{\underline{u}}^{(0,u)})} = \|\alpha\|^2_{L^2(H_0^{(0,\underline{u})})} + \int\int_{\mathcal{D}(u,\underline{u})} 3Q({}^{(4)}\pi, e_4, e_4) \tag{20}$$

where π is the deformation tensor of e_4. Since the initial data at H_0 verifies (17) we write,

$$\delta^2\big(\|\alpha\|^2_{L^2(H_u^{(0,\underline{u})})} + \|\beta\|^2_{L^2(\underline{H}_{\underline{u}}^{(0,u)})}\big) = \delta^2\|\alpha\|^2_{L^2(H_0^{(0,\underline{u})})} + 3\delta^2\int\int_{\mathcal{D}(u,\underline{u})} Q({}^{(4)}\pi, e_4, e_4)$$

and expect to bound the double integral term on the right. One can derive similar identities for all other possible choices of X, Y, Z among the set $\{e_3, e_4\}$. This allows one to estimate both the $L^2(H)$ norms of $\alpha, \beta, \rho, \sigma, \underline{\beta}$ and the $L^2(\underline{H})$ of $\beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$, with appropriate δ weights, in terms of corresponding δ-weighted $L^2(H_0)$ norms of $\alpha, \beta, \rho, \sigma, \underline{\beta}$ and spacetime integrals of $Q({}^{(4)}\pi, e_\mu, e_\nu)$ and $Q({}^{(3)}\pi, e_\mu, e_\nu)$ with $\mu, \nu = 3, 4$. We can thus extend the initial estimates (17) to every null hypersurface H_u in our slab provided that we can bound all the double integrals on the right hand side of our integral identities. Now, both deformation tensors ${}^{(4)}\pi$ and ${}^{(3)}\pi$ can be expressed in terms of our connection coefficients $\chi, \omega, \eta, \underline{\eta}, \underline{\omega}, \underline{\chi}$. Since Q is quadratic in R, to be able to close estimates for our null curvature components we need to derive sup-norm estimates for all our Ricci coefficients. This leads us to the second pillar of the construction which is to derive estimates for Ricci coefficients in terms of the null curvature components, with the help of the null structure equations (7). Combining these equations with the constrained equations, on fixed 2 surfaces $S(u, \underline{u})$, and the null Bianchi identities we are lead to precise δ- weighted estimates of all Ricci coefficients in terms of δ- weighted $L^2(H)$ and $L^2(\underline{H})$ norms of all null curvature components and their derivatives. Thus, in a first approximation, the error terms in the above integral identities are quadratic in R and linear in their first derivatives. Therefore to be able to close one needs:

(1) Derive higher derivative estimates for the curvature components.
(2) Make sure that all error terms can be controlled in terms of the principal terms, in the corresponding energy inequality, or terms which have already been estimated at previous steps.

Note that 2) here seems counterintuitive in view of the large data character of the problem under consideration. Indeed, typically, in such situations one cannot expect to control the nonlinear error terms by the principal energy terms. The miracle here is that the error terms are either linear (in the main energy terms), or they contain factors which have been already estimated in previous steps, or are truly nonlinear, in which case they are small in powers of δ relative to the principal energy terms. This is due to the structure of the error terms, reminiscent of the null condition, in which the factors combine in such a way that the total weight in powers of δ is positive.

In his work Christodoulou derives estimates for the first two derivatives of the curvature tensor by commuting the Bianchi identities with the vectorfields L, $S = \frac{1}{2}(ue_3 + \underline{u}e_4)$ and rotation vectorfields O. This process leads to a proliferation of error terms. Moreover not all error terms which are generated this way verify the following essential requirement, alluded above; *that they lead to an overall factor of δ^c, with a positive exponent c, and thus can be absorbed on the left, for sufficiently small δ.* Due to nonlinear interactions, Christodoulou has to tackle anomalous error terms which are $O(1)$ in δ. Yet he is able to show, by a careful step by step analysis, that all such terms are, indeed, linear relative to terms which have already been estimated and thus only quadratic (i.e. linear in the principal energy norm) relative to the remaining components. They can therefore be absorbed by a standard Gronwall inequality. A similar phenomenon helps him to estimate, step by step, all Ricci coefficients.

1.7. **New initial conditions.** As explained above the main purpose of this paper is to embed the short-pulse ansatz of Christodoulou into a more general set of initial conditions, based on a different underlying scaling. The new scaling, which we incorporate into our basic norms, allows us to conceptualize the separation between the linear and nonlinear terms in the null Bianchi and null structure equations and explain the favorable appearance of additional positive powers of δ in the nonlinear error terms mentioned above. Though the initial conditions required to include Christodoulou's data do not quite satisfy this scaling, the generated anomalies are fewer and thus much easier to track.

We start with the observation that a natural alternative to (15) which comes to mind, related to the familiar parabolic scaling on null hyperplanes in Minkowski space, is

$$\hat{\chi}_0(\underline{u}, \omega) = \delta^{-1/2} f_0(\delta^{-1}\underline{u},\, \delta^{-1/2}\omega), \tag{21}$$

This does not quite make sense in our framework of compact 2-surfaces $S(u, \underline{u})$, unless of course one is willing to consider the initial data $\hat{\chi}_0(\underline{u}, \omega)$ supported in the angular sector ω of size $\delta^{\frac{1}{2}}$. Such a support assumption would be however in contradiction with the lower bound in (13) required to be satisfied for *each* $\omega \in \mathbb{S}^2$.

The following interpretation of (21) (compare with (16)) makes sense however.

$$\delta^{k+\frac{m}{2}} \|\nabla_4^k \nabla^m \hat{\chi}_0\|_{L^2(0,\delta)} < \infty, \qquad 0 \le k + m \le N \tag{22}$$

Just as in the derivation of (17) we can use null structure equations (7) and null Bianchi equations (8) to derive, from (22),

$$\begin{aligned}
&\delta^{1/2}\|\alpha\|_{L^2(H_0)} + \|\beta\|_{L^2(H_0)} + \delta^{-1/2}\|(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-1}\|\underline{\beta}\|_{L^2(H_0)} < \infty\\
&\delta\|\nabla\alpha\|_{L^2(H_0)} + \delta^{1/2}\|\nabla\beta\|_{L^2(H_0)} + \|\nabla(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-1/2}\|\nabla\underline{\beta}\|_{L^2(H_0)} < \infty,\\
&\delta^{3/2}\|\nabla_4\alpha\|_{L^2(H_0)} + \delta\|\nabla_4\beta\|_{L^2(H_0)} + \delta^{1/2}\|\nabla_4(\rho,\sigma)\|_{L^2(H_0)} + \|\nabla_4\underline{\beta}\|_{L^2(H_0)} < \infty
\end{aligned}\tag{23}$$

We refer to these conditions, consistent with the null parabolic scaling, as δ-*coherent* assumptions. Observe that, unlike in the Christodoulou's case, each ∇ derivative costs a $\delta^{-1/2}$. It turns out that proving the propagation of such estimates can be done easily and systematically without the need of the step by step procedure mentioned earlier. In fact one can show, in this case, that all error terms, generated in the process of the energy estimates are either quadratic in the curvature and can be easily taken care by Gronwall or, if cubic, they must come with a factor of $\delta^{1/2}$ and therefore can be all absorbed for small values of δ.

The main problem with the ansatz (21), as with initial conditions (22), however, is that it is inconsistent with the formation of trapped surfaces requirements discussed above. One can only hope to show that the expansion scalar $\mathrm{tr}\chi$ along H_u, at $S(u,\underline{u})$, for some $u \approx 1$, will become negative[2] only in a small angular sector of size $\delta^{1/2}$. This is because, consistent with (23), condition (13) may only be satisfied in such a sector.

At this point we abandon the ansatz formulation of the characteristic initial data problem for the Einstein-vacuum equations and replace with an hierarchy of bounds, which "interpolate" between the regular δ-coherent assumptions (23) and the estimates (17)-(18) following from Christodoulou's short pulse ansatz.

At the level of curvature the new assumptions correspond to:

$$\begin{aligned}
&\delta\|\alpha\|_{L^2(H_0)} + \|\beta\|_{L^2(H_0)} + \delta^{-1/2}\|(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-1}\|\underline{\beta}\|_{L^2(H_0)} < \infty\\
&\delta\|\nabla\alpha\|_{L^2(H_0)} + \delta^{1/2}\|\nabla\beta\|_{L^2(H_0)} + \|\nabla(\rho,\sigma)\|_{L^2(H_0)} + \delta^{-1/2}\|\nabla\underline{\beta}\|_{L^2(H_0)} < \infty,\\
&\delta^{2}\|\nabla_4\alpha\|_{L^2(H_0)} + \delta\|\nabla_4\beta\|_{L^2(H_0)} + \delta^{1/2}\|(\nabla_4\rho,\nabla_4\sigma)\|_{L^2(H_0)} + \|\nabla_4\underline{\beta}\|_{L^2(H_0)} < \infty
\end{aligned}\tag{24}$$

Observe that, by comparison with (23), the only anomalous terms are $\|\alpha\|_{L^2(H_0)}$ and $\|\nabla_4\alpha\|_{L^2(H_0)}$.

In the next section we make precise our initial data assumptions, state the main results and explain the strategy of the proof. We close the discussion here with a summary of our approach

(1) Replace the short pulse ansatz of Christodoulou with a larger class of data satisfying (24)
(2) Prove propagation of the curvature estimates consistent with (24) through the domain of existence and show that these (weaker) estimates are sufficient for the existence result

[2]We could call such a region locally trapped, or a pre-scar

(3) The propagation estimates involve only the L^2 based norms of curvature and its first derivatives but generate nonlinear terms involving both the Ricci coefficients and its first derivatives. To close such estimates requires addressing two major difficulties
   - Regularity problem: show that the L^2 propagation curvature estimates are sufficient to control the Ricci coefficients (in L^∞) and its first and even second derivatives in appropriate norms required by the nonlinear terms in the curvature estimates
   - δ-consistency problem: show that the nonlinear terms are either effectively linear in (curvature and its derivatives), and thus can be handled by the Gronwall inequality, or contain a smallness coefficient generated by an additional power of the parameter δ. Our approach, based on the weaker propagation estimates (24), is particularly suitable for dealing with this problem in that a) it generates fewer borderline terms of the first kind and b) it naturally lends itself to the introduction of a notion of *scale-invariant* norms relative to which the structure of the nonlinear terms and their δ-smallness become apparent and nearly universal.

(4) The propagation estimates consistent with (24), and the corresponding Ricci coefficient estimates which it generate, are not strong enough to prove the formation of a trapped surface. However, once such estimates have been proved in the entire domain $\mathcal{D}(u \approx 1, \underline{u} = \delta)$ it is straightforward to impose slightly stronger conditions on the initial data and show that they lead to spacetimes which satisfy all the necessary conditions to implement, rigorously, the informal argument presented above.

2. Main Results

2.1. **Initial data assumptions.** We define the initial data quantity,

$$\mathcal{I}^{(0)} = \sup_{0 \le \underline{u} \le \delta} \mathcal{I}^{(0)}(\underline{u}) \tag{25}$$

where, with the notation convention in (16),

$$\begin{aligned}\mathcal{I}^{(0)}(\underline{u}) &= \delta^{1/2}\|\hat{\chi}_0\|_{L^\infty} + \sum_{0\le k\le 2} \delta^{1/2}\|(\delta\nabla_4)^k\hat{\chi}_0\|_{L^2(0,\underline{u})} \\ &+ \sum_{0\le k\le 1}\sum_{1\le m\le 4} \delta^{1/2}\|(\delta^{1/2}\nabla)^{m-1}(\delta\nabla_4)^k\,\nabla\hat{\chi}_0\|_{L^2(0,\underline{u})}\end{aligned}$$

Our main assumption, replacing Christodoulou's ansatz, is

$$\mathcal{I}^{(0)} < \infty \tag{26}$$

We show that, under this assumption and for sufficiently small $\delta > 0$, the spacetime slab $\mathcal{D}(u, \delta)$ can be extended for values of $u \ge 1$, with precise estimates for all Ricci coefficients of the double null foliation and null components of the curvature tensor. We can then show, by a slight modification of this assumption together with Christodoulou's lower bound assumption on $\int_0^\delta |\hat{\chi}_0|^2$ (see equations

14, 15 in [Chr]), that a trapped surface must form in $\mathcal{D}(u \approx 1, \delta)$. As in the case of [Chr]) most of the work is required to prove the semi global result concerning the double null foliation. Once this is established the actual formation of trapped surfaces result is proved by making a slight modification of the main assumption (26) and following the heuristic argument outlined below. In addition we show that a small modification of the regular δ-coherence assumption leads to the formation of a pre-scar.

2.2. **Curvature norms.** To give a precise formulation of our result we need to introduce the following norms.

$$\begin{aligned}
\mathcal{R}_0(u,\underline{u}): &= \delta\|\alpha\|_{H_u^{(0,\underline{u})}} + \|\beta\|_{H_u^{(0,\underline{u})}} + \delta^{-1/2}\|(\rho,\sigma)\|_{H_u^{(0,\underline{u})}} + \delta^{-1}\|\underline{\beta}\|_{H_u^{(0,\underline{u})}} \\
\mathcal{R}_1(u,\underline{u}): &= \delta\|\nabla\alpha\|_{H_u^{(0,\underline{u})}} + \delta^{1/2}\|\nabla\beta\|_{H_u^{(0,\underline{u})}} + \|\nabla(\rho,\sigma)\|_{H_u^{(0,\underline{u})}} + \delta^{-1/2}\|\nabla\underline{\beta}\|_{H_u^{(0,\underline{u})}} \\
&+ \delta\|\nabla_4\alpha\|_{H_u^{(0,\underline{u})}} \qquad (27)\\
\underline{\mathcal{R}}_0(u,\underline{u}): &= \delta\|\beta\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \|(\rho,\sigma)\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \delta^{-1/2}\|\underline{\beta}\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \delta^{-1}\|\underline{\alpha}\|_{\underline{H}_{\underline{u}}^{(0,u)}} \\
\underline{\mathcal{R}}_1(u,\underline{u}): &= \delta\|\nabla\beta\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \delta^{1/2}\|\nabla(\rho,\sigma)\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \|\nabla\underline{\beta}\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \delta^{-1/2}\|\nabla\underline{\alpha}\|_{\underline{H}_{\underline{u}}^{(0,u)}} \\
&+ \delta^{-1}\|\nabla_3\underline{\alpha}\|_{\underline{H}_{\underline{u}}^{(0,u)}}
\end{aligned}$$

We also set $\mathcal{R}_0, \mathcal{R}_1$ to be the supremum over $u, \underline{u}$ in our spacetime slab of $\mathcal{R}_0(u,\underline{u})$ and respectively $\mathcal{R}_1(u,\underline{u})$ and similarly for the norms $\underline{\mathcal{R}}$. Also we write $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ and $\underline{\mathcal{R}} = \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_1$. Finally, $\mathcal{R}^{(0)}$ denotes the initial value for the norm $\mathcal{R}$ i.e.,

$$\mathcal{R}^{(0)} = \sup_{0\le\underline{u}\le\delta}\big(\mathcal{R}_0(0,\underline{u}) + \mathcal{R}_1(0,\underline{u})\big)$$

Remark that the only ∇_4 derivative appearing in the norms above is that of α. All other ∇_4 derivatives can be deduced from the null Bianchi equations and thus do not need to be incorporated in our norms. We denote the norms of a specific curvature component ψ by $\mathcal{R}_0[\psi]$ and $\mathcal{R}_1[\psi]$.

2.3. **Ricci coefficient norms.** : We introduce norms for the Ricci coefficients $\hat{\chi}, \mathrm{tr}\chi, \omega, \eta, \underline{\eta}, \underline{\omega}, \underline{\hat{\chi}}$ and $\widetilde{\mathrm{tr}\underline{\chi}} = \mathrm{tr}\underline{\chi} - \mathrm{tr}\underline{\chi}_0$, with $\mathrm{tr}\underline{\chi}_0 = -\frac{4}{\underline{u}-u+2r_0}$ the flat value of $\mathrm{tr}\underline{\chi}$ along the initial hypersurface $\underline{H}_0$.

For any $S = S(u, \underline{u})$ we introduce norms ${}^{(S)}\mathcal{O}(s,p)(u,\underline{u})$,

$$\begin{aligned}
{}^{(S)}\mathcal{O}_{0,\infty}(u,\underline{u}) &= \delta^{1/2}\big(\|\hat{\chi}\|_{L^\infty(S)} + \|\omega\|_{L^\infty(S)}\big) + \|\eta\|_{L^\infty(S)} + \|\underline{\eta}\|_{L^\infty(S)}\\
&+ \delta^{-1/2}\big(\|\underline{\hat{\chi}}\|_{L^\infty(S)} + \|\widetilde{\mathrm{tr}\underline{\chi}}\|_{L^\infty(S)} + \|\underline{\omega}\|_{L^\infty(S)}\big)\\
{}^{(S)}\mathcal{O}_{0,4}(u,\underline{u}) &= \delta^{1/2}\|\hat{\chi}\|_{L^4(S)} + \delta^{1/4}|\omega\|_{L^4(S)} + \delta^{-1/4}\big(\|\eta\|_{L^4(S)} + \|\underline{\eta}\|_{L^4(S)}\big)\\
&+ \delta^{-1/2}\|\underline{\hat{\chi}}\|_{L^4(S)} + \delta^{-3/4}\big(\|\widetilde{\mathrm{tr}\underline{\chi}}\|_{L^4(S)} + \|\underline{\omega}\|_{L^4(S)}\big)\\
{}^{(S)}\mathcal{O}_{1,4}(u,\underline{u}) &= \delta^{3/4}\big(\|\nabla\chi\|_{L^4(S)} + |\omega\|_{L^4(S)}\big) + \delta^{1/4}\big(\|\nabla\eta\|_{L^4(S)} + \|\nabla\underline{\eta}\|_{L^4(S)}\big)\\
&+ \delta^{-1/4}\big(\|\nabla\underline{\chi}\|_{L^4(S)} + \|\underline{\omega}\|_{L^4(S)}\big)\\
{}^{(S)}\mathcal{O}_{1,2}(u,\underline{u}) &= \delta^{1/2}\big(\|\nabla\chi\|_{L^2(S)} + |\omega\|_{L^4(S)}\big) + \|\nabla\eta\|_{L^2(S)} + \|\nabla\underline{\eta}\|_{L^2(S)}\\
&+ \delta^{-1/2}\big(\|\nabla\underline{\chi}\|_{L^2(S)} + \|\underline{\omega}\|_{L^2(S)}\big)
\end{aligned} \tag{28}$$

Also,

$$\begin{aligned}
{}^{(H)}\mathcal{O}(u,\underline{u}) &= \delta^{1/2}\big(\|\nabla^2\chi\|_{L^2(H_u^{(0,\underline{u})})} + \|\nabla^2\omega\|_{L^2(H_u^{(0,\underline{u})})}\big)\\
&+ \big(\|\nabla^2\eta\|_{L^2(H_u^{(0,\underline{u})})} + \|\nabla^2\underline{\eta}\|_{L^2(H_u^{(0,\underline{u})})}\big)\\
&+ \delta^{-1/2}\big(\|\nabla^2\underline{\hat{\chi}}\|_{L^2(H_u^{(0,\underline{u})})} + \|\nabla^2\underline{\omega}\|_{L^2(H_u^{(0,\underline{u})})}\big)
\end{aligned}$$

and,

$$\begin{aligned}
{}^{(\underline{H})}\mathcal{O}(u,\underline{u}) &= \delta^{1/2}\big(\|\nabla^2\chi\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})} + \|\nabla^2\omega\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})}\big)\\
&+ \big(\|\nabla^2\eta\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})} + \|\nabla^2\underline{\eta}\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})}\big)\\
&+ \delta^{-1/2}\big(\|\nabla^2\underline{\hat{\chi}}\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})} + \|\nabla^2\underline{\omega}\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})}\big)
\end{aligned}$$

We define the norms ${}^{(S)}\mathcal{O}_{0,4}$, ${}^{(S)}\mathcal{O}_{1,2}$, ${}^{(S)}\mathcal{O}_{1,4}$, ${}^{(H)}\mathcal{O}$, ${}^{(\underline{H})}\mathcal{O}$ to be the supremum over all values of $u, \underline{u}$ in our slab of the corresponding norms. Finally we set set total Ricci norm $\mathcal{O}$,

$$\mathcal{O} = {}^{(S)}\mathcal{O}_{0,\infty} + {}^{(S)}\mathcal{O}_{0,4} + {}^{(S)}\mathcal{O}_{1,2} + {}^{(S)}\mathcal{O}_{1,4} + {}^{(H)}\mathcal{O} + {}^{(\underline{H})}\mathcal{O}$$

and by $\mathcal{O}^{(0)}$ the corresponding norm of the initial hypersurface H_0. We further differentiate between the first order norms $\mathcal{O}_{[1]} = {}^{(S)}\mathcal{O}_{0,4} + {}^{(S)}\mathcal{O}_{1,2}$ and second order ones, $\mathcal{O}_{[2]} = {}^{(S)}\mathcal{O}_{1,4}$.

2.4. **Main Theorems.** We are now ready to state our main result. The first result follows from analyzing assumption (25) on the initial hypersurface H_0.

Proposition 2.5. *In view of our initial assumption* (25) *we have, for sufficiently small* $\delta > 0$*, along* H_0*,*

$$\mathcal{R}^{(0)} + \mathcal{O}^{(0)} \lesssim \mathcal{I}^{(0)} \tag{29}$$

The proof of the proposition follows by analyzing the null structure and null Bianchi equations restricted to the initial hypersurface H_0, as in chapter 2 of [Chr]. In view of this result we may replace assumption (25) with (29), as an initial data assumption. Alternatively we may assume only that $\mathcal{R}^{(0)} \lesssim \mathcal{I}^{(0)}$. It is not too hard to see, following roughly the same steps as in the proof of proposition 2.5, that, for small δ, we would also have $\mathcal{O}^{(0)} \lesssim \mathcal{I}^{(0)}$.

Theorem 2.6 (Main Theorem)**.** *Assume that $\mathcal{R}^{(0)} \lesssim \mathcal{I}^{(0)}$ for an arbitrary constant $\mathcal{I}^{(0)}$. Then, there exists a sufficiently small $\delta > 0$ such that,*

$$\mathcal{R} + \underline{\mathcal{R}} + \mathcal{O} \quad \lesssim \quad \mathcal{I}^{(0)}. \tag{30}$$

Theorem 2.7. *Assume that , in addition to* (25)*, we also have, for $2 \le k \le 4$*

$$\delta^{\frac{1}{2}} \|(\delta^{\frac{1}{2}}\nabla)^k \hat{\chi}_0\|_{L^2(0,\underline{u})} \le \epsilon \tag{31}$$

for a sufficiently small parameter ϵ such that $0 < \delta \ll \epsilon$. Assume also that $\hat{\chi}_0$ verifies (13)*. Then, for $\delta > 0$ sufficiently small, a trapped surface must form in the slab $\mathcal{D}(u \approx 1, \delta)$.*

Proof. We sketch below the proof of theorem 2.7.

Step 1. We reinterpret (31) in terms of the curvature norms according to the following:

Proposition 2.8. *Under the smallness condition* (31) *the initial curvature norms satisfy, in addition to the estimates of proposition 2.5,*

$$\delta^{1/2}\|\nabla\beta\|_{H_0^{(0,\delta)}} + \|\nabla(\rho,\sigma)\|_{H_0^{(0,\delta)}} + \delta^{-1/2}\|\nabla\underline{\beta}\|_{H_0^{(0,\delta)}} \le \epsilon. \tag{32}$$

The proof is standard and will be omitted.

Step 2. We show, see the end of section 15, that this condition can be propagated in the entire slab $\mathcal{D}(u \approx 1, \delta)$,

Proposition 2.9. *Under the assumptions* (31) *we have, uniformly in $u \lesssim 1, \underline{u} \le \delta$, for δ sufficiently small,*

$$\begin{aligned} &\delta^{1/2}\|\nabla\beta\|_{H_u^{(0,\underline{u})}} + \|\nabla(\rho,\sigma)\|_{H_u^{(0,\underline{u})}} + \delta^{-1/2}\|\nabla\underline{\beta}\|_{H_u^{(0,\underline{u})}} \le \epsilon. \\ &\delta^{1/2}\|\nabla(\rho,\sigma)\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \|\nabla\underline{\beta}\|_{\underline{H}_{\underline{u}}^{(0,u)}} + \delta^{-1/2}\|\nabla\underline{\alpha}\|_{\underline{H}_{\underline{u}}^{(0,u)}} \le \epsilon. \end{aligned} \tag{33}$$

Step 3. We return to the system (9)- (10),

$$\begin{aligned} \nabla_4 \mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 &= -|\hat{\chi}|^2 - 2\omega\mathrm{tr}\chi \\ \nabla_3\hat{\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} &= \nabla\widehat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\underline{\hat{\chi}} + \eta\widehat{\otimes}\eta \end{aligned}$$

responsible, as we have seen, for the formation of a trapped surface. Theorem 2.6 implies that the terms ignored in our heuristic derivation are negligible. Specifically, the bounds $|\omega\mathrm{tr}\chi| \lesssim \delta^{-\frac{1}{2}}$,

$|\underline{\omega}\hat{\chi}| + |\mathrm{tr}\chi\underline{\hat{\chi}}\,| + |\eta\widehat{\otimes}\eta| \lesssim 1$ should be compared to the principle terms of size δ^{-1} and $\delta^{-\frac{1}{2}}$ in the first and the second equation respectively. We can also easily verify the other bounds in (14) with the exception of that for $\nabla\widehat{\otimes}\eta$. The additional condition (31) is imposed in fact precisely in order to assure that the linear term $\nabla\widehat{\otimes}\eta$ in (10) is sufficiently small. To control this term we rely on the following proposition.

Proposition 2.10. *Under the assumptions of Theorem 2.7 the solution ${}^{(3)}\phi$ of the problem $\nabla_3^{(3)}\phi = \nabla\widehat{\otimes}\eta$, with trivial initial data on H_0, verifies,*

$$|{}^{(3)}\phi| \le C\delta^{-1/2}\epsilon^{\frac{1}{4}} \tag{34}$$

The proof of proposition 2.10, which appear is section 15.12, depends on the arguments of section 11, in particular proposition 11.12. The argument for the formation of a trapped surface then proceeds as above with a renormalized quantity $(\hat{\chi} -^{(3)} \phi)$ in place of $\hat{\chi}$. Note that in view of the estimate on ${}^{(3)}\phi$ the size of $(\hat{\chi} -^{(3)} \phi)$ is comparable to that $\hat{\chi}$. An important comment in this regard, is that our curvature propagation estimates does not allow us to control the L^∞ norm of $\nabla\widehat{\otimes}\eta$, let alone prove the bound stated in (14). This regularity problem, which is discussed in the two remarks below, is resolved with the help of the renormalized estimates for the Ricci coefficients in section 11, of which Proposition 2.10 is an important example. □

Remark 1. We remark that while a loss of derivatives occurs when passing from assumption (26) to assumption $\mathcal{R}^{(0)} \lesssim \mathcal{I}^{(0)}$ in the main theorem, no further derivative losses occurs in (30).
Remark 2. By contrast with [Chr], where two derivatives of the curvature and up to three derivatives of the Ricci coefficients are needed, here we need only one derivative of the curvature and two of the Ricci coefficients. This is due to our new refined estimates for the deformation tensor of the angular momentum vectorfields O. As mentioned above these vectorfields are needed to derive estimates for the angular derivatives of the null curvature components. These new estimates for the deformation tensor of the angular momentum vectorfields O are based on the *renormalized* estimates for the Ricci coefficients developed in Section 11. Together with the trace estimates for the curvature components, which serve as a replacement for the failed $H^1(S) \subset L^\infty(S)$ embedding on a 2-dimensional surface S, proved in Section 12, they allow us to limit the degree of differentiability required in the proof to the L^2 norms of curvature and its first derivatives. Similar ideas related to the gain of differentiability via renormalization and trace estimates were exploited in our earlier work [K-R:causal].

Our next and final result concerns the formation of a *pre-scar* in an angular sector of size $\delta^{\frac{1}{2}}$.

Theorem 2.11. *Let ϵ be a small parameter such that $0 < \delta \ll \epsilon$. Assume that the initial data $\hat{\chi}_0$ satisfies*

$$\delta^{1/2}\|\hat{\chi}_0\|_{L^\infty} + \sum_{0\le k\le 1}\sum_{0\le m\le 4} \epsilon\|(\epsilon^{-1}\delta^{\frac{1}{2}}\nabla)^m(\delta\nabla_4)^k\,\hat{\chi}_0\|_{L^2(0,\underline{u})} < \infty$$

and that the lower bound in (13) *is verified in angular sector* $\omega \in \Lambda$ *of size* $\delta^{\frac{1}{2}}$. *Then, for* $\delta > 0$ *sufficiently small, a pre-scar must form in the slab* $\mathcal{D}(u \approx 1, \delta)$*, i.e. the expansion scalar* $tr\chi(u, \underline{u}, \omega)$ *becomes stricly negative for some values of* $u \approx 1$, $\underline{u} = \delta$ *and all* $\omega \in \Lambda$.

Remark. Theorem 2.11 corresponds to the initial data consistent with the ansatz

$$\hat{\chi}_0(\underline{u}, \omega) = \delta^{-\frac{1}{2}} f_0(\delta^{-1}\underline{u},\ \delta^{-1/2}\epsilon\,\omega)$$

and localized in an angular sector of size $\delta^{\frac{1}{2}}\epsilon^{-1}$. This should be compared with the data discussed in (21). As in Theorem 2.7 additional smallness provided by the parameter ϵ is only needed to guarantee the formation of a pre-scar but not required for the proof of the existence result. A direct comparison shows that the data of Theorem 2.11 is significantly more regular than that of Theorems 2.6 and 2.7. In particular, it essentially corresponds to the δ-coherent assumptions, consistent with the natural null parabolic scaling discussed in (23). Thus the proof of Theorem 2.11 is significantly easier than that of our main result and will be omitted.

2.12. **Strategy of the proof.** We divide proof of the main theorem in three parts. In the first part we derive estimates for the Ricci coefficients norms $\mathcal{O}$ in terms of the initial data $\mathcal{I}^{(0)}$ and the curvature norms $\mathcal{R}$. More precisely we prove:

Theorem 2.13 (Theorem A)**.** *Assume that* $\mathcal{O}^{(0)} < \infty$ *and* $\mathcal{R} < \infty$. *There exists a constant* C *depending only on* $\mathcal{O}^{(0)}$ *and* $\mathcal{R}, \underline{\mathcal{R}}$ *such that,*

$$\mathcal{O} \lesssim C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}). \tag{35}$$

Moreover,

$${}^{(S)}\mathcal{O}_{0,4}[\underline{\hat{\chi}}\,] \lesssim \mathcal{O}^{(0)} + C(\mathcal{I}^{(0)}, \mathcal{R},\ \underline{\mathcal{R}})\,\delta^{1/4} \tag{36}$$

We prove the theorem by a bootstrap argument. We start by assuming that there exists a sufficiently large constant Δ_0 such that,

$${}^{(S)}\mathcal{O}_{0,\infty} \le \Delta_0. \tag{37}$$

Based on this assumption we show that, if δ is sufficiently small, estimate (35) also holds. This allows us to derive a better estimate than (37).

In the second part we need to define angular momentum operators O and show that their deformation tensors verify compatible estimates, stated in Theorem B, at the end of section 13 .

Finally in the last and main part we need to use the estimates of Theorems A and B to derive estimates for the curvature norms $\mathcal{R}$ and thus end the proof of the main theorem.

Theorem 2.14 (Theorem C)**.** *There exists* δ *sufficiently small such that,*

$$\mathcal{R} + \underline{\mathcal{R}} \lesssim \mathcal{I}_0 \tag{38}$$

Theorem C is proved in sections 14 and 15.

2.15. **Signature and Scaling.** Our norms are intimately tied with a natural scaling which we introduce below.

Signature. To every null curvature component $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$, null Ricci coefficients components $\chi, \zeta, \eta, \underline{\eta}, \omega, \underline{\omega}$, and metric γ we assign a signature according to the following rule:

$$\mathrm{s}gn(\phi) = 1 \cdot N_4(\phi) + \frac{1}{2} \cdot N_a(\phi) + 0 \cdot N_3(\phi) - 1 \tag{39}$$

where $N_4(\phi), N_3(\phi), N_a(\phi)$ denote the number of times e_4, respectively e_3 and $(e_a)_{a=1,2}$, which appears in the definition of ϕ. Thus,

$$\mathrm{s}gn(\alpha) = 2, \quad \mathrm{s}gn(\beta) = 1 + 1/2, \quad \mathrm{s}gn(\rho, \sigma) = 1, \quad \mathrm{s}gn(\underline{\beta}) = 1/2, \quad \mathrm{s}gn(\underline{\alpha}) = 0.$$

Also,

$$\mathrm{s}gn(\chi) = \mathrm{s}gn(\omega) = 1, \quad \mathrm{s}gn(\zeta, \eta, \underline{\eta}) = 1/2, \quad \mathrm{s}gn(\underline{\chi}) = \mathrm{s}gn(\underline{\omega}) = \mathrm{s}gn(\gamma) = 0.$$

Consistent with this definition we have, for any given null component ϕ,

$$\mathrm{s}gn(\nabla_4 \phi) = 1 + \mathrm{s}gn(\phi), \quad \mathrm{s}gn(\nabla \phi) = \frac{1}{2} + \mathrm{s}gn(\phi), \quad \mathrm{s}gn(\nabla_3 \phi) = \mathrm{s}gn(\phi).$$

Also, based on our convention,

$$\mathrm{s}gn(\phi_1 \cdot \phi_2) = \mathrm{s}gn(\phi_1) + \mathrm{s}gn(\phi_2).$$

Remark. All terms in a given null structure or null Bianchi identity (see equations (47)–(53)) have the same overall signature.

We now introduce a notion of scale for any quantity ϕ which has a signature $\mathrm{s}gn(\phi)$, in particular for our basic null curvature quantities $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$ and null Ricci coefficients components $\chi, \zeta, \eta, \underline{\eta}, \omega, \underline{\omega}$. This scaling plays a fundamental role in our work.

Definition 2.16. For an arbitrary horizontal tensor-field ϕ, with a well defined signature $\mathrm{s}gn(\phi)$, we set:

$$\mathrm{sc}(\phi) = -\mathrm{s}gn(\phi) + \frac{1}{2} \tag{40}$$

Observe that $\mathrm{sc}(\nabla_L \phi) = \mathrm{sc}(\phi) - 1$, $\mathrm{sc}(\nabla \phi) = \mathrm{sc}(\phi) - \frac{1}{2}$, $\mathrm{sc}(\nabla_{\underline{L}} \phi) = \mathrm{sc}(\phi)$. For a given product of two horizontal tensor-fields we have,

$$\mathrm{sc}(\phi_1 \cdot \phi_2) = \mathrm{sc}(\phi_1) + \mathrm{sc}(\phi_2) - \frac{1}{2} \tag{41}$$

2.17. **Scale invariant norms.** For any horizontal tensor-field ψ with scale $sc(\psi)$ we define the following scale invariant norms along the null hypersurfaces $H = H_u^{(0,\delta)}$ and $\underline{H} = \underline{H}_{\underline{u}}^{(0,1)}$.

$$\|\psi\|_{\mathcal{L}^2_{(sc)}(H)} = \delta^{-\mathrm{sc}(\psi)-1}\|\psi\|_{L^2(H)}, \quad \|\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} = \delta^{-\mathrm{sc}(\psi)-\frac{1}{2}}\|\psi\|_{L^2(\underline{H})} \tag{42}$$

We also define the scale invariant norms on the 2 surfaces $S = S_{u,\underline{u}}$,

$$\|\psi\|_{\mathcal{L}^p_{(sc)}(S)} = \delta^{-\mathrm{sc}(\psi)-\frac{1}{p}}\|\psi\|_{L^p(S)} \tag{43}$$

In particular,

$$\|\psi\|_{\mathcal{L}^2_{(sc)}(S)} = \delta^{-\mathrm{sc}(\psi)-\frac{1}{2}}\|\psi\|_{L^2(S)}, \qquad \|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} = \delta^{-\mathrm{sc}(\psi)}\|\psi\|_{L^\infty(S)}$$

Observe that we have,

$$\|\psi\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} = \delta^{-1}\int_0^{\underline{u}}\|\psi\|^2_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}d\underline{u}', \qquad \|\psi\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} = \int_0^u \|\psi\|^2_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \tag{44}$$

We denote the scale invariant L^∞ norm in $\mathcal{D}$ by $\|\psi\|_{\mathcal{L}^\infty_{(sc)}}$.

Remark. Observe that the noms above are scale invariant if we take into account the scales of the L^2 noms along H and $\underline{H}$, given by,

$$\mathrm{sc}(\| \ \|_{L^2(H_u^{0,\delta})}) = 1, \quad \mathrm{sc}(\| \ \|_{L^2(\underline{H}_{\underline{u}}^{0,1})}) = \frac{1}{2}, \quad \mathrm{sc}(\| \ \|_{L^p(S)}) = \frac{1}{p}.$$

Moreover they are consistent to the following convention,

$$\nabla_4 \sim \delta^{-1}, \quad \nabla \sim \delta^{-\frac{1}{2}}, \quad \nabla_3 \sim 1$$

In view of (41) all standard product estimates in the usual L^p spaces translate into product estimates in $\mathcal{L}_{(sc)}$ spaces with a gain of $\delta^{1/2}$. Thus, for example,

$$\|\psi_1 \cdot \psi_2\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \delta^{1/2}\|\psi_1\|_{\mathcal{L}^\infty_{(sc)}(S)} \cdot \|\psi_2\|_{\mathcal{L}^2_{(sc)}(S)} \tag{45}$$

or,

$$\|\psi_1 \cdot \psi_2\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{1/2}\|\psi_1\|_{\mathcal{L}^\infty_{(sc)}(H)} \cdot \|\psi_2\|_{\mathcal{L}^2_{(sc)}(H)} \tag{46}$$

Remark 2.18. If f is a scalar function constant along the surfaces $S(u,\underline{u}) \subset \mathcal{D}$, we have

$$\|f \cdot \psi\|_{\mathcal{L}^p_{(sc)}(S)} \lesssim \|\psi\|_{\mathcal{L}^p_{(sc)}(S)}$$

or, if f is also bounded on H,

$$\|f \cdot \psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\psi\|_{\mathcal{L}^2_{(sc)}(H)}$$

This remark applies in particular to the constant $\mathrm{tr}\underline{\chi}_0 = \frac{4}{2r_0+\underline{u}-u}$.

We can reinterpret our main curvature and Ricci coefficient norms in light of the scale invariant norms. Thus (27) can be rewritten in the form[3],

$$
\begin{aligned}
\mathcal{R}_0(u,\underline{u}): &= \delta^{1/2}\|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(\beta,\rho,\sigma,\underline{\beta})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\
\mathcal{R}_1(u,\underline{u}): &= \delta^{1/2}\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\nabla(\alpha,\beta,\rho,\sigma,\underline{\beta})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\mathcal{R}}_0(u,\underline{u}): &= \delta^{1/2}\|\beta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})} + \|(\rho,\sigma,\underline{\beta},\underline{\alpha})\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}\\
\mathcal{R}_1(u,\underline{u}): &= \|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})} + \|\nabla(\beta,\rho,\sigma,\underline{\beta},\underline{\alpha})\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}
\end{aligned}
$$

Remark 2.19. All curvature norms are scale invariant except for the anomalous $\|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$, $\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$ and $\|\beta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(u,0)})}$. By abuse of language, in a given context, we refer to α, respectively β, as anomalous.

To rectify the anomaly of α we introduce an additional scale-invariant norm

$$\mathcal{R}_0^\delta[\alpha] := \sup_{{}^\delta H\subset H}\|\alpha\|_{\mathcal{L}^2_{(sc)}({}^\delta H)},$$

where ${}^\delta H$ is a piece of the hypersurface $H = H_u^{0,\delta}$ obtained by evolving a disc $S_\delta \subset S_{u,0}$ of radius $\delta^{\frac12}$ along the integral curves of the vectorfield e_4.

The Ricci coefficient norms (28) can be written,

$$
\begin{aligned}
{}^{(S)}\mathcal{O}_{0,\infty}(u,\underline{u}) &= \|(\hat\chi,\omega,\eta,\underline{\eta},\widetilde{\mathrm{tr}\underline{\chi}},\underline{\hat\chi},\underline{\omega})\|_{\mathcal{L}^\infty_{(sc)}(S)}\\
{}^{(S)}\mathcal{O}_{0,4}(u,\underline{u}) &= \delta^{1/4}\big(\|\hat\chi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\underline{\hat\chi}\|_{\mathcal{L}^4_{(sc)}(S)}\big)\\
&+ \|(\mathrm{tr}\chi,\omega,\eta,\underline{\eta},\widetilde{\mathrm{tr}\underline{\chi}},\underline{\omega})\|_{\mathcal{L}^2_{(sc)}(S)}\\
{}^{(S)}\mathcal{O}_{1,4}(u,\underline{u}) &= \|\nabla(\chi,\omega,\eta,\underline{\eta},\widetilde{\mathrm{tr}\underline{\chi}},\underline{\hat\chi},\underline{\omega})\|_{\mathcal{L}^4_{(sc)}(S)}\\
{}^{(S)}\mathcal{O}_{1,2}(u,\underline{u}) &= \|\nabla(\chi,\omega,\eta,\underline{\eta},\widetilde{\mathrm{tr}\underline{\chi}},\underline{\hat\chi},\underline{\omega})\|_{\mathcal{L}^2_{(sc)}(S)}\\
{}^{(H)}\mathcal{O}(u,\underline{u}) &= \|\nabla^2(\chi,\omega,\eta,\underline{\eta},\widetilde{\mathrm{tr}\underline{\chi}},\underline{\hat\chi},\underline{\omega})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}
\end{aligned}
$$

Remark 2.20. All quantities are scale invariant except for $\hat\chi,\underline{\hat\chi}$ in the $\mathcal{L}^4_{(sc)}(S)$ norm.

As before we complement the anomalous norms for $\hat\chi,\underline{\hat\chi}$ by the local, non-anomalous, scale-invariant norms

$$\mathcal{O}_0^\delta[\hat\chi](u,\underline{u}) = \sup_{{}^\delta S\subset S}\|\hat\chi\|_{\mathcal{L}^4_{(sc)}({}^\delta S)},\quad \mathcal{O}_0^\delta[\underline{\hat\chi}](u,\underline{u}) = \sup_{{}^\delta S\subset S}\|\underline{\hat\chi}\|_{\mathcal{L}^4_{(sc)}({}^\delta S)},$$

where ${}^\delta S$ is a disk of radius $\delta^{\frac12}$ obtained by transporting from the initial data embedded in $S_{u,0}$.

[3]We use the short hand notation $\|(\beta,\rho,\sigma,\underline{\beta})\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} = \|\beta\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\rho\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\sigma\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \dots$

3. Main equations. Preliminaries

3.1. **Null structure equations.** We recall the null structure equations (see section 3.1 in [K-Ni] or [Chr].)

$$
\begin{aligned}
\nabla_4\chi &= -\chi\cdot\chi - 2\omega\chi - \alpha\\
\nabla_3\underline{\chi} &= -\underline{\chi}\cdot\underline{\chi} - 2\underline{\omega}\underline{\chi} - \underline{\alpha}\\
\nabla_4\eta &= -\chi\cdot(\eta-\underline{\eta}) - \beta\\
\nabla_3\underline{\eta} &= -\underline{\chi}\cdot(\underline{\eta}-\eta) + \underline{\beta}\\
\nabla_4\underline{\omega} &= 2\omega\underline{\omega} + \frac{3}{4}|\eta-\underline{\eta}|^2 - \frac{1}{4}(\eta-\underline{\eta})\cdot(\eta+\underline{\eta}) - \frac{1}{8}|\eta+\underline{\eta}|^2 + \frac{1}{2}\rho\\
\nabla_3\omega &= 2\omega\underline{\omega} + \frac{3}{4}|\eta-\underline{\eta}|^2 + \frac{1}{4}(\eta-\underline{\eta})\cdot(\eta+\underline{\eta}) - \frac{1}{8}|\eta+\underline{\eta}|^2 + \frac{1}{2}\rho
\end{aligned}
\tag{47}
$$

and the constraint equations

$$
\begin{aligned}
\mathrm{div}\,\hat{\chi} &= \frac{1}{2}\nabla\mathrm{tr}\chi - \frac{1}{2}(\eta-\underline{\eta})\cdot(\hat{\chi}-\frac{1}{2}\mathrm{tr}\chi) - \beta,\\
\mathrm{div}\,\underline{\hat{\chi}} &= \frac{1}{2}\nabla\mathrm{tr}\underline{\chi} + \frac{1}{2}(\eta-\underline{\eta})\cdot(\underline{\hat{\chi}}-\frac{1}{2}\mathrm{tr}\underline{\chi}) + \underline{\beta}\\
\mathrm{curl}\,\eta &= -\mathrm{curl}\,\underline{\eta} = \sigma + \underline{\hat{\chi}}\wedge\hat{\chi}\\
K &= -\rho + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \frac{1}{4}\mathrm{tr}\chi\cdot\mathrm{tr}\underline{\chi}
\end{aligned}
\tag{48}
$$

with K the Gauss curvature of the surfaces S. The first two equation in (47) can also be written in the form,

$$
\begin{aligned}
\nabla_4\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 &= -|\hat{\chi}|^2 - 2\omega\mathrm{tr}\chi\\
\nabla_4\hat{\chi} + \mathrm{tr}\chi\hat{\chi} &= -2\omega\hat{\chi} - \alpha\\
\nabla_3\mathrm{tr}\underline{\chi} + \frac{1}{2}(\mathrm{tr}\underline{\chi})^2 &= -2\underline{\omega}\mathrm{tr}\underline{\chi} - |\underline{\hat{\chi}}|^2\\
\nabla_3\underline{\hat{\chi}} + \mathrm{tr}\underline{\chi}\,\underline{\hat{\chi}} &= -2\underline{\omega}\underline{\hat{\chi}} - \underline{\alpha}
\end{aligned}
\tag{49}
$$

Also, with $\check{\rho} = \rho - \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}$,

$$
\begin{aligned}
\nabla_4\mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} &= 2\omega\mathrm{tr}\underline{\chi} + 2\check{\rho} + 2\mathrm{div}\,\underline{\eta} + 2|\underline{\eta}|^2\\
\nabla_3\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi &= 2\underline{\omega}\mathrm{tr}\chi + 2\check{\rho} + 2\mathrm{div}\,\eta + 2|\eta|^2
\end{aligned}
\tag{50}
$$

and[4],

$$\begin{aligned}\nabla_3\hat\chi+\frac12\mathrm{tr}\underline{\chi}\hat\chi&=\nabla\widehat{\otimes}\eta+2\underline{\omega}\hat\chi-\frac12\mathrm{tr}\chi\underline{\hat\chi}+\eta\widehat{\otimes}\eta\\ \nabla_4\underline{\hat\chi}+\frac12\mathrm{tr}\chi\underline{\hat\chi}&=\nabla\widehat{\otimes}\underline{\eta}+2\omega\underline{\hat\chi}-\frac12\mathrm{tr}\underline{\chi}\hat\chi+\underline{\eta}\widehat{\otimes}\underline{\eta}\end{aligned}\tag{51}$$

Remark. The transport equations for ω and $\underline{\omega}$ in (47) are obtained from the null structure equation,

$$\nabla_4\underline{\omega}+\nabla_3\omega=\zeta\cdot(\eta-\underline{\eta})-\eta\cdot\underline{\eta}+4\omega\underline{\omega}+\rho$$

and the commutation relation, for a scalar f (see proposition 4.8.1 in [K-Ni])

$$[\nabla_3,\nabla_4]f=-2\omega\nabla_3f+2\underline{\omega}\nabla_4f+4\zeta\cdot\nabla f\tag{52}$$

applied to $f=\log\Omega$.

3.2. **Null Bianchi.** We record below the null Bianchi identities (Observe that we can eliminate $\zeta=\frac12(\eta-\underline{\eta})$ in the equations below),

$$\begin{aligned}
&\nabla_3\alpha+\frac12\mathrm{tr}\underline{\chi}\alpha=\nabla\widehat{\otimes}\beta+4\underline{\omega}\alpha-3(\hat\chi\rho+^*\hat\chi\sigma)+(\zeta+4\eta)\widehat{\otimes}\beta,\\
&\nabla_4\beta+2\mathrm{tr}\chi\beta=\mathrm{div}\ \alpha-2\omega\beta+\eta\alpha,\\
&\nabla_3\beta+\mathrm{tr}\underline{\chi}\beta=\nabla\rho+2\underline{\omega}\beta+^*\nabla\sigma+2\hat\chi\cdot\underline{\beta}+3(\eta\rho+^*\eta\sigma),\\
&\nabla_4\sigma+\frac32\mathrm{tr}\chi\sigma=-\mathrm{div}\ ^*\beta+\frac12\underline{\hat\chi}\cdot^*\alpha-\zeta\cdot^*\beta-2\underline{\eta}\cdot^*\beta,\\
&\nabla_3\sigma+\frac32\mathrm{tr}\underline{\chi}\sigma=-\mathrm{div}\ ^*\underline{\beta}+\frac12\hat\chi\cdot^*\underline{\alpha}-\zeta\cdot^*\underline{\beta}-2\eta\cdot^*\underline{\beta},\\
&\nabla_4\rho+\frac32\mathrm{tr}\chi\rho=\mathrm{div}\ \beta-\frac12\underline{\hat\chi}\cdot\alpha+\zeta\cdot\beta+2\underline{\eta}\cdot\beta,\\
&\nabla_3\rho+\frac32\mathrm{tr}\underline{\chi}\rho=-\mathrm{div}\ \underline{\beta}-\frac12\hat\chi\cdot\underline{\alpha}+\zeta\cdot\underline{\beta}-2\eta\cdot\underline{\beta},\\
&\nabla_4\underline{\beta}+\mathrm{tr}\chi\underline{\beta}=-\nabla\rho+^*\nabla\sigma+2\omega\underline{\beta}+2\underline{\hat\chi}\cdot\beta-3(\underline{\eta}\rho-^*\underline{\eta}\sigma),\\
&\nabla_3\underline{\beta}+2\mathrm{tr}\underline{\chi}\underline{\beta}=-\mathrm{div}\ \underline{\alpha}-2\underline{\omega}\underline{\beta}+\underline{\eta}\cdot\underline{\alpha},\\
&\nabla_4\underline{\alpha}+\frac12\mathrm{tr}\chi\underline{\alpha}=-\nabla\widehat{\otimes}\underline{\beta}+4\omega\underline{\alpha}-3(\underline{\hat\chi}\,\rho-^*\underline{\hat\chi}\,\sigma)+(\zeta-4\underline{\eta})\widehat{\otimes}\underline{\beta}
\end{aligned}\tag{53}$$

We record below commutation formulae between ∇ and ∇_4,∇_3:

Lemma 3.3. *For a scalar function f:*

$$[\nabla_4,\nabla]f=\frac12(\eta+\underline{\eta})D_4f-\chi\cdot\nabla f\tag{54}$$

$$[\nabla_3,\nabla]f=\frac12(\eta+\underline{\eta})D_3f-\underline{\chi}\cdot\nabla f,\tag{55}$$

[4]Recall the notation $(u\widehat{\otimes}v)_{ab}=u_av_b+u_bv_a-(u\cdot v)\delta_{ab}$.

For a 1-form tangent to S:

$$\begin{aligned}
[D_4,\nabla_a]U_b &= -\chi_{ac}\nabla_c U_b + \in_{ac}\ {}^*\beta_b U_c + \frac{1}{2}(\eta_a+\underline{\eta}_a)D_4 U_b\\
&- -\chi_{ac}\underline{\eta}_b U_c + \chi_{ab}\,\underline{\eta}\cdot U\\
[D_3,\nabla_a]U_b &= -\underline{\chi}_{ac}\nabla_c U_b + \in_{ac} *\underline{\beta}_b U_c + \frac{1}{2}(\eta_a+\underline{\eta}_a)D_3 U_b\\
&- \underline{\chi}_{ac}\eta_b U_c + \underline{\chi}_{ab}\,\eta\cdot U
\end{aligned}$$

In particular,

$$\begin{aligned}
[\nabla_4, div\,]U &= -\frac{1}{2}tr\chi\, div\, U - \hat{\chi}\cdot\nabla U - \beta\cdot U + \frac{1}{2}(\eta+\underline{\eta})\cdot\nabla_4 U\\
&- \underline{\eta}\cdot\hat{\chi}\cdot U - \frac{1}{2}tr\chi\underline{\eta}\cdot U + tr\chi\underline{\eta}\cdot U\\
[\nabla_3, div\,]U &= -\frac{1}{2}tr\underline{\chi}\, div\, U - \hat{\underline{\chi}}\cdot\nabla U + \underline{\beta}\cdot U + \frac{1}{2}(\eta+\underline{\eta})\cdot\nabla_4 U\\
&- \eta\cdot\hat{\chi}\cdot U - \frac{1}{2}tr\underline{\chi}\eta\cdot U + tr\underline{\chi}\eta\cdot U
\end{aligned}$$

3.4. **Integral formulas.** Given a scalar function f in $\mathcal{D}$ we have[5],

$$\begin{aligned}
\frac{d}{d\underline{u}}\int_{S(u,\underline{u})} f &= \int_{S(u,\underline{u})}\big(\frac{df}{d\underline{u}} + \Omega\mathrm{tr}\chi f\big) = \int_{S(u,\underline{u})}\Omega\big(e_4(f)+\mathrm{tr}\chi f\big)\\
\frac{d}{du}\int_{S(u,\underline{u})} f &= \int_{S(u,\underline{u})}\big(\frac{df}{du} + \Omega\mathrm{tr}\underline{\chi} f\big) = \int_{S(u,\underline{u})}\Omega\big(e_3(f)+\mathrm{tr}\underline{\chi} f\big)
\end{aligned}$$

As a consequence of these we deduce, for any horizontal tensorfield ψ,

$$\begin{aligned}
\|\,\psi\|^2_{L^2(S(u,\underline{u}))} &= \|\psi\|^2_{L^2(S(u,0))} + \int_{H_u^{(0,\underline{u})}} 2\Omega\big(\psi\cdot\nabla_4\psi + \frac{1}{2}\mathrm{tr}\chi|\psi|^2\big)\\
\|\,\psi\|^2_{L^2(S(u,\underline{u}))} &= \|\psi\|^2_{L^2(S(0,\underline{u}))} + \int_{H_{\underline{u}}^{(u,0)}} 2\Omega\big(\psi\cdot\nabla_3\psi + \frac{1}{2}\mathrm{tr}\underline{\chi}|\psi|^2\big)
\end{aligned} \tag{56}$$

Proof. The first formula in (56) is derived as follows,

$$\begin{aligned}
\|\,\psi\|^2_{L^2(S(u,\underline{u}))} &= \|\psi\|^2_{L^2(S(u,0))} + \int_0^{\underline{u}}\frac{d}{d\underline{u}}\big(\int_{S(u,\underline{u})}|\psi|^2\big)\\
&= \|\psi\|^2_{L^2(S(u,0))} + \int_{H_u^{(0,\underline{u})}} 2\Omega\big(\psi\cdot\nabla_4\psi + \frac{1}{2}\mathrm{tr}\chi|\psi|^2\big)
\end{aligned}$$

[5]see for example Lemma 3.1.3 in [K-Ni]

The second formula is proved in the same manner. □

3.5. **Hodge systems.** We work with the following Hodge operators acting on the leaves $S = S(u, \underline{u})$ of our double null foliation.

(1) The operator $\mathcal{D}_1$ takes any 1-form F into the pairs of functions $(\text{div } F\,,\, \text{curl } F)$
(2) The operator $\mathcal{D}_2$ takes any S tangent symmetric, traceless tensor F into the S tangent one form $\text{div } F$.
(3) The operator ${}^\star\mathcal{D}_1$ takes the pair of scalar functions (ρ, σ) into the S-tangent 1-form[6] $-\nabla\rho + {}^*\nabla\,\sigma$.
(4) The operator ${}^\star\mathcal{D}_2$ takes 1-forms F on S into the 2-covariant, symmetric, traceless tensors $-\frac{1}{2}\widehat{\mathcal{L}_F\gamma}$ with $\mathcal{L}_F\gamma$ the traceless part of the Lie derivative of the metric γ relative to F, i.e.

$$\widehat{(\mathcal{L}_F\gamma)}_{ab} = \nabla_b F_a + \nabla_a F_b - (\text{div } F)\gamma_{ab}.$$

The kernels of both $\mathcal{D}_1$ and $\mathcal{D}_2$ in $L^2(S)$ are trivial and that ${}^\star\mathcal{D}_1$, resp. ${}^\star\mathcal{D}_2$ are the L^2 adjoints of $\mathcal{D}_1$, respectively $\mathcal{D}_2$. The kernel of ${}^\star\mathcal{D}_1$ consists of pairs of constant functions (ρ, σ) while that of ${}^\star\mathcal{D}_2$ consists of the set of all conformal Killing vectorfields on S. In particular the L^2- range of $\mathcal{D}_1$ consists of all pairs of functions ρ, σ on S with vanishing mean. The L^2 range of $\mathcal{D}_2$ consists of all L^2 integrable 1-forms on S which are orthogonal to the Lie algebra of all conformal Killing vectorfields on S. Accordingly we shall consider the inverse operators $\mathcal{D}_1^{-1}$ and $\mathcal{D}_2^{-1}$ and implicitly assume that they are defined on the L^2 subspaces identified above.

Finally we record the following simple identities,

$$
\begin{aligned}
{}^\star\mathcal{D}_1 \cdot \mathcal{D}_1 &= -\Delta + K, \qquad \mathcal{D}_1 \cdot {}^\star\mathcal{D}_1 = -\Delta \qquad (57)\\
{}^\star\mathcal{D}_2 \cdot \mathcal{D}_2 &= -\frac{1}{2}\Delta + K, \qquad \mathcal{D}_2 \cdot {}^\star\mathcal{D}_2 = -\frac{1}{2}(\Delta + K) \qquad (58)
\end{aligned}
$$

Proposition 3.6. *Let (S, γ) be a compact manifold with Gauss curvature K.*

i.) *The following identity holds for vectorfields ψ on S:*

$$\int_S \big(|\nabla\psi|^2 + K|\psi|^2\big) = \int_S \big(|div\ \psi|^2 + |curl\ \psi|^2\big) = \int_S |\mathcal{D}_1\psi|^2 \qquad (59)$$

ii.) *The following identity holds for symmetric, traceless, 2-tensorfields ψ on S:*

$$\int_S \big(|\nabla\psi|^2 + 2K|\psi|^2\big) = 2\int_S |div\ \psi|^2 = 2\int_S |\mathcal{D}_2\psi|^2 \qquad (60)$$

[6]Here $({}^*\nabla\,\sigma)_a = \in_{ab} \nabla_b\sigma$.

iii.) *The following identity holds for pairs of functions* (ρ, σ) *on* S*:*

$$\int_S \big(|\nabla\rho|^2 + |\nabla\sigma|^2\big) = \int_S | - \nabla\rho + (\nabla\sigma)^\star|^2 = \int_S |\, {}^\star\mathcal{D}_1(\rho,\sigma)|^2 \tag{61}$$

iv.) *The following identity holds for vectors* ψ *on* S*,*

$$\int_S \big(|\nabla\psi|^2 - K|\psi|^2\big) = 2\int_S |\, {}^\star\mathcal{D}_2\psi|^2 \tag{62}$$

4. Preliminary estimates

As explained in the introduction the proof of Theorem A is based on the bootstrap assumption (37), i.e.

$${}^{(S)}\mathcal{O}_{0,\infty} \le \Delta_0.$$

In this section we use this bootstrap to prove various preliminary results. In the following three sections we then derive estimates for the Ricci coefficient norms ${}^{(S)}\mathcal{O}_{0,4}$, ${}^{(S)}\mathcal{O}_{1,2}$ and ${}^{(S)}\mathcal{O}_{1,4}$ respectively.

4.1. **Preliminary results.** We prove here results which follows easily from our bootstrap assumption. ${}^{(S)}\mathcal{O}_{0,\infty} \le \Delta_0$. We first derive an estimate for Ω. To do this we use the definition of $\underline{\omega} = -\frac{1}{2}\nabla_3 \log\Omega = \frac{1}{2}\Omega\nabla_3(\Omega)^{-1} = \frac{1}{2}\frac{d}{du}(\Omega)^{-1}$. Thus, since $\Omega^{-1} = 2$ on H_0,

$$\|\Omega^{-1} - 2\|_{L^\infty(u,\underline{u})} \lesssim \int_0^u \|\underline{\omega}\|_{L^\infty(u',\underline{u})} du' \lesssim \delta^{1/2}\ {}^{(S)}\mathcal{O}_{0,\infty}[\underline{\omega}] \lesssim \delta^{1/2}\Delta_0$$

Thus, if δ is sufficiently small we deduce that $|\Omega - \frac{1}{2}|$ is small and therefore,

$$\frac{1}{4} \le \Omega \le 4. \tag{63}$$

We now prove the following proposition.

Proposition 4.2. *Under assumption* (37) *we have the following estimates for an arbitrary horizontal tensor-field* ψ*,*

$$\begin{aligned}
\|\,\psi\|_{L^2(u,\underline{u})} &\lesssim \|\psi\|_{L^2(u,0)} + \int_0^{\underline{u}} \|\nabla_4\psi\|_{L^2(u,\underline{u}')}\, d\underline{u}' \\
\|\,\psi\|_{L^2(u,\underline{u})} &\lesssim \|\psi\|_{L^2(0,\underline{u})} + \int_0^{u} \|\nabla_3\psi\|_{L^2(u',\underline{u})}\, du'
\end{aligned} \tag{64}$$

More generally the same estimates hold in $L^p(S)$ *norms.*

Also,

$$\begin{aligned}\|\psi\|^2_{L^2(u,\underline{u})} &\lesssim \|\psi\|^2_{L^2(u,0)} + \|\psi\|_{L^2(H_u^{(0,\underline{u})})}\|\nabla_4\psi\|_{L^2(H_u^{(0,\underline{u})})}\\ \|\psi\|^2_{L^2(u,\underline{u})} &\lesssim \|\psi\|^2_{L^2(0,\underline{u})} + \|\psi\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})}\|\nabla_4\psi\|_{L^2(\underline{H}_{\underline{u}}^{(0,u)})}\end{aligned} \tag{65}$$

Corollary 4.3. *Under the same hypothesis,*

$$\begin{aligned}\| \psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\psi\|_{\mathcal{L}^2_{(sc)}(u,0)} + \int_0^{\underline{u}} \delta^{-1}\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}\, d\underline{u}'\\ \| \psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\psi\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \int_0^{u} \|\nabla_3\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}\, du'\end{aligned} \tag{66}$$

and,

$$\begin{aligned}\|\psi\|^2_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\psi\|^2_{\mathcal{L}^2_{(sc)}(u,0)} + \|\psi\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\ \|\psi\|^2_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\psi\|^2_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \|\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}\end{aligned} \tag{67}$$

More generally, let $S' \subset S_{u,\underline{u}}$ and $S'_{u',\underline{u}}$, $S'_{u,\underline{u}'}$ are obtained by evolving S' along the null generators of $\underline{H}_{\underline{u}}$, H_u respectively. Then

$$\begin{aligned}\| \psi\|_{\mathcal{L}^p_{(sc)}(S')} &\lesssim \|\psi\|_{\mathcal{L}^p_{(sc)}(S'_{u,0})} + \int_0^{\underline{u}} \delta^{-1}\|\nabla_4\psi + \frac{1}{p}tr\chi\psi\|_{\mathcal{L}^p_{(sc)}(S'_{u,\underline{u}'})}\, d\underline{u}'\\ \| \psi\|_{\mathcal{L}^p_{(sc)}(S')} &\lesssim \|\psi\|_{\mathcal{L}^p_{(sc)}(S'_{0,\underline{u}})} + \int_0^{u} \|\nabla_3\psi + \frac{1}{p}tr\underline{\chi}\psi\|_{\mathcal{L}^p_{(sc)}(S'_{u',\underline{u}})}\, du'\end{aligned} \tag{68}$$

Proof. The corollary follows immediately from the proposition and definition of the scale invariant norms. The last statement of the corollary follows by applying (66) to the function $\chi\psi$, where the cut-off function χ is first defined on $S_{u,\underline{u}}$ as the characteristic function of S' and then extended by solving the transport equations $\nabla_4\chi = 0$ and $\nabla_3\chi = 0$.

To prove the proposition we first make use of (63) and (37),

$$\|\mathrm{tr}\chi\|_{L^\infty} \quad\lesssim\quad \Delta_0\delta^{-\frac{1}{2}}$$

and deduce from the first equation in (56),

$$\begin{aligned}\| \psi\|^2_{L^2(S(u,\underline{u}))} \quad&\lesssim\quad \|\psi\|^2_{L^2(S(u,0))} + \int_0^{\underline{u}}\int_{S(u,\underline{u}')} |\psi\,|\,|\nabla_4\psi + \frac{1}{2}\mathrm{tr}\chi\psi|\\ &\lesssim\quad \|\psi\|^2_{L^2(S(u,0))} + \int_0^{\underline{u}} \|\psi\|_{L^2(S)}\big(\|\nabla_4\psi\|_{L^2(S)} + \Delta_0\delta^{-\frac{1}{2}}\|\psi\|_{L^2(S)}\big)\\ &\lesssim\quad \|\psi\|^2_{L^2(S(u,0))} + \int_0^{\underline{u}} \|\psi\|_{L^2(S)}\|\nabla_4\psi\|_{L^2(S)} + \Delta_0\delta^{-1/2}\int_0^{\underline{u}}\|\psi\|^2_{L^2(S)}\end{aligned}$$

Thus, by Gronwall, since $\underline{u} \le \delta$,

$$\|\psi\|^2_{L^2(S(u,\underline{u}))} \lesssim \|\psi\|^2_{L^2(S(u,0))} + \int_0^{\underline{u}} \|\nabla_4\psi\|_{L^2(u,\underline{u}')} \cdot \|\psi\|_{L^2(u,\underline{u}')}\, d\underline{u}' \tag{69}$$

from which we easily derive the ∇_4 equations in both (64) and (65).

To prove the ∇_3 estimates we need to take into account the anomalous character of $\mathrm{tr}\underline{\chi}$. From our bootstrap assumption we deduce (recall that $\mathrm{tr}\underline{\chi}_0 = -\frac{4}{\underline{u}-u+2r_0}$ is the flat value of $\mathrm{tr}\underline{\chi}$) ,

$$\|\mathrm{tr}\underline{\chi} - \mathrm{tr}\underline{\chi}_0\|_{L^\infty} \lesssim \Delta_0\delta^{1/2}$$

Thus,

$$\begin{aligned}
\|\,\psi\|^2_{L^2(S(u,\underline{u}))} &\lesssim \|\psi\|^2_{L^2(S(0,\underline{u}))} + \int_0^u \int_{S(u',\underline{u})} |\psi\,|\,|\nabla_3\psi + \frac{1}{2}\mathrm{tr}\underline{\chi}\psi| \\
&\lesssim \|\psi\|^2_{L^2(S(0,\underline{u}))} + \int_0^u \|\psi\|_{L^2(S)}\big(\|\nabla_3\psi\|_{L^2(S)} + \Delta_0\delta^{1/2}\|\psi\|_{L^2(S)}\big) \\
&+ \int_0^u \|\mathrm{tr}\underline{\chi}_0\|_{L^\infty}\|\psi\|^2_{L^2(S)} \\
&\lesssim |\psi\|^2_{L^2(S(0,\underline{u}))} + \int_0^u \|\psi\|_{L^2(S)}\big(\|\nabla_3\psi\|_{L^2(S)} + (1+\Delta_0\delta^{1/2})\|\psi\|_{L^2(S)}\big)
\end{aligned}$$

Thus, using Gronwall and smallness of $\delta^{1/2}\Delta_0$ we deduce,

$$\|\,\psi\|^2_{L^2(S(u,\underline{u}))} \lesssim \|\psi\|^2_{L^2(S(0,\underline{u}))} + \int_0^u \|\psi\|_{L^2(S)}\|\nabla_3\psi\|_{L^2(S)} \tag{70}$$

from which both (64) and (65) follow. □

We next prove an improved estimate for $\mathrm{tr}\chi$.

Proposition 4.4. *For $\delta^{1/2}\Delta_0$ sufficiently small we have for all $S = S(u,\underline{u})$,*

$$\|tr\chi\|_{L^\infty(S)} \lesssim \Delta_0^2 \tag{71}$$

Proof. We recall that $\mathrm{tr}\chi$ verifies the transport equation,

$$\nabla_4\mathrm{tr}\chi = -\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2 - 2\omega\mathrm{tr}\chi$$

or,

$$\frac{d}{d\underline{u}}\mathrm{tr}\chi = -\Omega\big(\frac{1}{2}(\mathrm{tr}\chi)^2 + |\hat{\chi}|^2 + 2\omega\mathrm{tr}\chi\big)$$

Thus, since $\|\chi,\omega\|_{L^\infty} \lesssim \delta^{-1/2}\Delta_0$,

$$\begin{aligned}\|\text{tr}\chi\|_{L^\infty(u,\underline{u})} &\lesssim \int_0^{\underline{u}} \|\chi\|_{L^\infty(u,\underline{u}')}\big(\|\chi\|_{L^\infty(u,\underline{u}')} + \|\omega\|_{L^\infty(u,\underline{u}')}\big)d\underline{u}' \\ &\lesssim \Delta_0^2 + \delta^{1/2}\Delta_0.\end{aligned}$$

□

4.5. **Transported coordinates.** We define systems of, local, transported coordinates along the null hypersurfaces H and $\underline{H}$. Staring with a local coordinate system $\theta = (\theta^1, \theta^2)$ on $U \subset S(u,0) \subset H_u$ we parametrize any point along the null geodesics starting in U by the the corresponding coordinate θ and affine parameter $\underline{u}$. Similarly, starting with a local coordinate system $\underline{\theta} = (\underline{\theta}^1, \underline{\theta}^2)$ on $V \subset S(0,\underline{u}) \subset \underline{H}_{\underline{u}}$ we parametrize any point along the null geodesics starting in V by the the corresponding coordinate $\underline{\theta}$ and affine parameter u. We denote the respective metric components by γ_{ab} and $\underline{\gamma}_{ab}$.

Proposition 4.6. *Let γ^0_{ab} denote the standard metric on $\mathbb{S}^2$. Then, for any $0 \le u \le 1$ and $0 \le \underline{u} \le \delta$ and sufficiently small $\delta^{\frac12}\Delta_0$*

$$|\gamma_{ab} - \gamma^0_{ab}| \le \delta^{\frac12}\Delta_0, \qquad |\underline{\gamma}_{ab} - \gamma^0_{ab}| \le \delta^{\frac12}\Delta_0.$$

In addition, the transported coordinates verify

$$\begin{aligned}|\nabla_3\theta^a| &\lesssim \delta\Delta_0, \qquad |\nabla\theta^a| \lesssim 1\\ \|\nabla_4\underline{\theta}^a| &\lesssim \delta\Delta_0, \qquad |\nabla\underline{\theta}^a| \lesssim 1\end{aligned}$$

for $a = 1,2$. The Christoffel symbols Γ_{abc} and $\underline{\Gamma}_{ab}$, obey the scale invariant estimates[7]

$$\|\Gamma_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \mathcal{O}_{[1]}, \qquad \|\partial_d\Gamma_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \mathcal{O}_{[2]}, \tag{72}$$

$$\|\underline{\Gamma}_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \mathcal{O}_{[1]}, \qquad \|\partial_d\underline{\Gamma}_{abc}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \mathcal{O}_{[2]}, \tag{73}$$

Proof. We will only show the argument in the case of γ_{ab}. In the transported coordinate system the metric γ_{ab} verifies

$$\frac{d}{d\underline{u}}\gamma_{ab} = 2\Omega\chi_{ab}.$$

Therefore,

$$|\gamma_{ab} - \gamma^0_{ab}| \le 2\int_0^{\underline{u}} |\chi_{ab}| \le \delta^{\frac12}\Delta_0,$$

where in the last inequality we used that $|\chi_{ab}| \le |\chi||\gamma^{-1}|$ and ran a simple bootstrap argument.

The transported system of coordinates θ^a satisfies the system of equations

$$\nabla_4\theta^a = 0.$$

[7]we can attach signature to Γ and $\underline{\Gamma}$ $sgn(\Gamma) = \frac12$, $\qquad sgn(\underline{\Gamma}) = \frac12$

Commuting these equations with ∇_3 and taking into account the commutation formula (52) we obtain

$$\nabla_4(\nabla_3\theta^a) = 2\omega\nabla_3\theta^a - 4\zeta\cdot\nabla\theta^a$$

Using the bootstrap assumptions (37), the inequality $|\nabla\theta^a| \lesssim 1$ and the triviality of the data for $\nabla_3\theta^a$ we obtain that

$$|\nabla_3\theta^a| \lesssim \delta\Delta_0.$$

To verify that $|\nabla\theta^a| \lesssim 1$ we commute the transport equation for θ^a with ∇ to obtain according to (54)

$$\nabla_4(\nabla\theta^a) = -\chi\cdot\nabla\theta^a,$$

which together with the bootstrap assumption (37) gives the desired result.

To prove (72) we differentiate the transport equation for γ_{ab} to obtain

$$\frac{d}{d\underline{u}}\left(\partial_c\gamma_{ab}\right) = 2\partial_c\Omega\chi_{ab} + 2\Omega\partial_c\chi_{ab}.$$

Taking into account that

$$|\partial_c\Omega| \lesssim |\nabla\Omega| \le |\eta| + |\underline{\eta}|, \qquad |\partial_c\chi_{ab}| \lesssim |\nabla\chi| + |\Gamma||\chi|$$

we derive

$$\begin{aligned}
\|\partial_c\gamma_{ab}\|_{L^2(u,\underline{u})} &\lesssim \int_0^{\underline{u}}\left(\|\eta\|_{L^4((u,\underline{u}')} + \|\underline{\eta}\|_{L^4(u,\underline{u}')}\right)\|\chi\|_{L^4(u,\underline{u}')}d\underline{u}' \\
&+ \int_0^{\underline{u}}\left(\|\nabla\chi\|_{L^2(u,\underline{u}')} + \|\Gamma\|_{L^2(u,\underline{u}')}\right)\|\chi\|_{L^\infty(u,\underline{u}')}d\underline{u}' \\
&\lesssim \delta^{\frac{3}{4}}\ {}^{(S)}\mathcal{O}_{0,4}[\chi]\ {}^{(S)}\mathcal{O}_{0,4}[\eta,\underline{\eta}] + {}^{(S)}\mathcal{O}_{1,2}[\chi] + \delta^{-\frac{1}{2}}\Delta_0\int_0^{\underline{u}}\|\Gamma\|_{L^2(u,\underline{u}')}d\underline{u}'.
\end{aligned}$$

Thus, by Gronwall,

$$\|\Gamma\|_{L^2(u,\underline{u})} \lesssim {}^{(S)}\mathcal{O}_{1,2} + \delta^{3/4}\ {}^{(S)}\mathcal{O}_{0,4}^2$$

The desired estimate for Γ follows by Gronwall. The second estimate of (72)can be derived by an additional differentiation of the transport equation. The estimates (73) are proved in the same manner. We omit the details. □

4.7. **Estimates for $\mathcal{R}_0^\delta[\alpha]$.** Using the transported coordinates of the previous subsection we now derive estimates for $\mathcal{R}_0^\delta[\alpha]$ norm of the anomalous curvature component α.

Proposition 4.8.

$$\mathcal{R}_0^\delta[\alpha](u) \lesssim \mathcal{R}_0^\delta[\alpha](0) + \mathcal{R}$$

Proof. Recall that, $\mathcal{R}_0^\delta[\alpha] := \sup_{{}^\delta H\subset H} \|\alpha\|_{\mathcal{L}^2_{(sc)}({}^\delta H)}$, where ${}^\delta H$ is the subset of H_u generated by transporting a disk ${}^\delta S$ of radius $\delta^{\frac12}$, embedded in the sphere $S_{u,0}$, along the integral curves of the vectorfield e_4. We denote by ${}^\delta S_{\underline{u}}$ the intersection between ${}^\delta H$ and the level hypersurfaces of $\underline{u}$ and by ${}^\delta S_{u',\underline{u}}$ the sets obtained by transporting ${}^\delta S_{\underline{u}}$ along the integral curves of e_3 According to (68)

$$\| \alpha\|_{\mathcal{L}^2_{(sc)}({}^\delta S_{\underline{u}})} \lesssim \|\alpha\|_{\mathcal{L}^2_{(sc)}({}^\delta S_{0,\underline{u}})} + \int_0^u \|\nabla_3\alpha + \frac12 \mathrm{tr}\underline{\chi}\alpha\|_{\mathcal{L}^2_{(sc)}({}^\delta S_{u',\underline{u}})}\, du'$$

We note that (72) implies that ${}^\delta S_{u',\underline{u}}$ are contained in the intersection of ${}^{2\delta}H_{u'}$ and the level hypersurface of $\underline{u}$. Therefore,

$$\| \alpha\|_{\mathcal{L}^2_{(sc)}({}^\delta H_u)} \lesssim \|\alpha\|_{\mathcal{L}^2_{(sc)}({}^{2\delta} H_0)} + \int_0^u \|\nabla_3\alpha + \frac12 \mathrm{tr}\underline{\chi}\alpha\|_{\mathcal{L}^2_{(sc)}({}^{2\delta} H_{u'})}\, du'$$

Using the equation for α

$$\nabla_3\alpha + \frac12\mathrm{tr}\underline{\chi}\alpha = \nabla\widehat{\otimes}\beta + 4\underline{\omega}\alpha - 3(\hat\chi\rho +^* \hat\chi\sigma) + (\zeta+4\eta)\widehat{\otimes}\beta$$

and the bootstrap assumptions (37) we obtain

$$\begin{aligned}\|\nabla_3\alpha + \frac12\mathrm{tr}\underline{\chi}\alpha\|_{\mathcal{L}^2_{(sc)}({}^{2\delta}H_{u'})} &\le \|\nabla_3\alpha + \frac12\mathrm{tr}\underline{\chi}\alpha\|_{\mathcal{L}^2_{(sc)}(H_{u'})}\\ &\le \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(H_{u'})} + \delta^{\frac12}\, {}^{(S)}\mathcal{O}_{0,\infty}\cdot\mathcal{R}_0 \lesssim \mathcal{R} + \delta^{\frac12}\Delta_0\mathcal{R}_0\end{aligned}$$

It remains to observe that

$$\|\alpha\|_{\mathcal{L}^2_{(sc)}({}^{2\delta}H_0)} \lesssim \mathcal{R}_0^\delta[\alpha](u=0),$$

which follows from a simple covering argument. □

4.9. **Calculus inequalities.**

Proposition 4.10. *Let (S,γ) be a compact 2-dimensional surface covered by local charts (disks) U_i in which the metric γ satisfies*

$$|\gamma_{ij} - \delta_{ij}| \le \frac12.$$

Let d denote the minimum between 1 and the smallest radius of the disks U_i. Then for any $p > 2$

$$\|\psi\|_{L^4(S)} \lesssim \|\psi\|_{L^2(S)}^{\frac12}\|\nabla\psi\|_{L^2(S)}^{\frac12} + d^{-\frac12}\|\psi\|_{L^2(S)}, \tag{74}$$

$$\|\psi\|_{L^\infty(S)} \lesssim \|\psi\|_{L^p(S)}^{\frac{p}{p+4}}\|\nabla\psi\|_{L^p(S)}^{\frac{4}{p+4}} + d^{-\frac{4}{p+4}}\|\psi\|_{L^p(S)}. \tag{75}$$

More generally,

$$\|\psi\|_{L^4(U_i)} \lesssim \|\psi\|_{L^2(U_i)}^{\frac12}\|\nabla\psi\|_{L^2(U_i')}^{\frac12} + d^{-\frac12}\|\psi\|_{L^2(U_i')}, \tag{76}$$

$$\|\psi\|_{L^\infty(S)} \lesssim \sup_{U_i}\left(\|\psi\|_{L^p(U_i)}^{\frac{p}{p+4}}\|\nabla\psi\|_{L^p(U_i')}^{\frac{4}{p+4}} + d^{-\frac{4}{p+4}}\|\psi\|_{L^p(U_i')}\right). \tag{77}$$

The disk U_i' is a doubled version of U_i.

We can combine the above proposition with Proposition 4.6 to obtain

Corollary 4.11. *Let $S = S_{u,\underline{u}}$ and $S_\delta \subset S$ denote a disk of radius $\delta^{\frac{1}{2}}$ relative to either θ or $\underline{\theta}$ coordinate system. Then for any horizontal tensor ψ*

$$\|\psi\|_{L^4(S)} \lesssim \|\psi\|_{L^2(S)}^{\frac{1}{2}} \|\nabla\psi\|_{L^2(S)}^{\frac{1}{2}} + \|\psi\|_{L^2(S)}, \tag{78}$$

$$\|\psi\|_{L^\infty(S)} \lesssim \|\psi\|_{L^p(S)}^{\frac{p}{p+4}} \|\nabla\psi\|_{L^p(S)}^{\frac{4}{p+4}} + \|\psi\|_{L^p(S)}. \tag{79}$$

and

$$\|\psi\|_{L^4(S_\delta)} \lesssim \delta^{\frac{1}{4}} \|\nabla\psi\|_{L^2(S_{2\delta})} + \delta^{-\frac{1}{4}} \|\psi\|_{L^2(S_{2\delta})}, \tag{80}$$

$$\|\psi\|_{L^\infty(S)} \lesssim \sup_{S_\delta \subset S} \left(\delta^{\frac{1}{4}} \|\nabla\psi\|_{L^4(S_{2\delta})} + \delta^{-\frac{1}{4}} \|\psi\|_{L^4(S_{2\delta})} \right). \tag{81}$$

Also, in the scale invariant norms

Corollary 4.12. *Let $S = S_{u,\underline{u}}$ and $S_\delta \subset S$ denote a disk of radius $\delta^{\frac{1}{2}}$ relative to either θ or $\underline{\theta}$ coordinate system. Then for any horizontal tensor ψ*

$$\|\psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\psi\|_{\mathcal{L}^2_{(sc)}(S)}^{\frac{1}{2}} \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)}^{\frac{1}{2}} + \delta^{\frac{1}{4}} \|\psi\|_{\mathcal{L}^2_{(sc)}(S)}, \tag{82}$$

$$\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \|\psi\|_{\mathcal{L}^p_{(sc)}(S)}^{\frac{p}{p+4}} \|\nabla\psi\|_{\mathcal{L}^p_{(sc)}(S)}^{\frac{4}{p+4}} + \delta^{\frac{1}{p}} \|\psi\|_{\mathcal{L}^p_{(sc)}(S)}. \tag{83}$$

and

$$\|\psi\|_{\mathcal{L}^4_{(sc)}(S_\delta)} \lesssim \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S_{2\delta})} + \|\psi\|_{\mathcal{L}^2_{(sc)}(S_{2\delta})}, \tag{84}$$

$$\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \sup_{S_\delta \subset S} \left(\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S_{2\delta})} + \|\psi\|_{\mathcal{L}^4_{(sc)}(S_{2\delta})} \right). \tag{85}$$

4.13. **Codimension 1 trace formulas.** We will use the $L^4(S)$ trace formulas[8] along the null hypersurfaces H and $\underline{H}$, see [Chr-Kl], [K-Ni], [K-R:LP].

Lemma 4.14. *The following formulas hold true for any two sphere $S = S(u,\underline{u}) = H(u) \cup \underline{H}(\underline{u})$ and any horizontal tensor ψ*

$$\begin{aligned} \|\psi\|_{L^4(S)} &\lesssim \big(\|\psi\|_{L^2(H)} + \|\nabla\psi\|_{L^2(H)}\big)^{1/2} \big(\|\psi\|_{L^2(H)} + \|\nabla_4\psi\|_{L^2(H)}\big)^{1/2} \\ \|\psi\|_{L^4(S)} &\lesssim \big(\|\psi\|_{L^2(\underline{H})} + \|\nabla\psi\|_{L^2(\underline{H})}\big)^{1/2} \big(\|\psi\|_{L^2(\underline{H})} + \|\nabla_3\psi\|_{L^2(\underline{H})}\big)^{1/2} \end{aligned}$$

Also, in scale invariant norms,

[8]Our bootstrap assumption are more than enough to verify the conditions of validity of these estimates.

Proposition 4.15. *The following formulas hold true for a fixed $S = S(u, \underline{u}) = H(u) \cap \underline{H}(\underline{u}) \subset \mathcal{D}$ and any horizontal tensor ψ*

$$\begin{aligned} \|\psi\|_{\mathcal{L}^4_{(sc)}(S)} &\lesssim \big(\delta^{1/2}\|\psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)}\big)^{1/2}\big(\delta^{1/2}\|\psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(H)}\big)^{1/2} \\ \|\psi\|_{\mathcal{L}^4_{(sc)}(S)} &\lesssim \big(\delta^{1/2}\|\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)^{1/2}\big(\delta^{1/2}\|\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla_3\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)^{1/2} \end{aligned}$$

4.16. **Estimates for Hodge systems.** Consider a Hodge system,

$$\mathcal{D}\psi = F$$

with $\mathcal{D}$ one of the operators in section 3.5. In view of proposition 3.6,

$$\int_S |\nabla\psi|^2 + \int_S K|\psi|^2 \lesssim \|F\|^2_{L^2(S)}$$

where,

$$K = -\rho + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}$$

is the Gauss curvature of S. Hence,

$$\|\nabla\psi\|^2_{L^2(S)} \lesssim \|K\|_{L^2(S)}\|\psi\|^2_{L^4(S)} + \|F\|^2_{L^2(S)}$$

Making use of the calculus inequality on S,

$$\|\psi\|^2_{L^4(S)} \lesssim \|\nabla\psi\|_{L^2(S)}\,\|\psi\|_{L^2(S)}$$

we deduce,

$$\|\nabla\psi\|^2_{L^2(S)} \lesssim \|K\|_{L^2(S)}\|\nabla\psi\|_{L^2(S)}\,\|\psi\|_{L^2(S)} + \|F\|^2_{L^2(S)}$$

and consequently,

$$\|\nabla\psi\|_{L^2(S)} \lesssim \|K\|_{L^2(S)}\|\psi\|_{L^2(S)} + \|F\|_{L^2(S)}$$

We state below the same result in scale invariant norms

Proposition 4.17. *Let ψ verify the Hodge system*

$$\mathcal{D}\psi = F \tag{86}$$

Then,

$$\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \delta^{1/2}\|K\|_{\mathcal{L}^2_{(sc)}(S)}\|\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|F\|_{\mathcal{L}^2_{(sc)}(S)} \tag{87}$$

To obtain the second derivative estimates for the Hodge system $\mathcal{D}\psi = F$ we apply the operator $\mathcal{D}^*$ and write the resulting equation schematically in the form

$$\Delta\psi = K\psi + D^*F.$$

Multiplying the equation by $\Delta\psi$, integrating over S and using that $\|D^*F\|_{L^2(S)} \lesssim \|\nabla\psi\|_{L^2(S)}$ we obtain

$$\|\Delta\psi\|_{L^2(S)} \lesssim \|K\|_{L^2(S)}\|\psi\|_{L^\infty(S)} + \|\nabla F\|_{L^2(S)}$$

Using Böchner's identity, see e.g. [K-R:LP],

$$\|\nabla^2\psi\|_{L^2(S)} \lesssim \|K\|_{L^2(S)}\|\psi\|_{L^\infty(S)} + \|K\|^{\frac{1}{2}}_{L^2(S)}\|\nabla\psi\|_{L^4(S)} + \|\Delta\psi\|_{L^2(S)}. \tag{88}$$

we then obtain

Proposition 4.18. *Let ψ verify the Hodge system*

$$\mathcal{D}\psi = F \tag{89}$$

Then,

$$\begin{aligned}\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{\frac{1}{2}}\|K\|_{\mathcal{L}^2_{(sc)}(S)}\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} + \delta^{\frac{1}{4}}\|K\|^{\frac{1}{2}}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)}\\ &+ \|\nabla F\|_{\mathcal{L}^2_{(sc)}(S)}\end{aligned} \tag{90}$$

5. ${}^{(S)}\mathcal{O}_{0,4}$ AND ${}^{(S)}\mathcal{O}_{0,2}$ ESTIMATES

5.1. **Estimates for $\chi, \eta, \underline{\omega}$.** The null Ricci coefficients χ, η and $\underline{\omega}$ verify transport equations of the form,

$$\nabla_4\psi^{(s)} = \sum_{s_1+s_2=s+1}\psi^{(s_1)}\cdot\psi^{(s_2)} + \Psi^{(s+1)} \tag{91}$$

Here $\psi^{(s)}$ denotes an arbitrary Ricci coefficient component of signature s while $\Psi^{(s)}$ denotes a null curvature component of signature s. In view of proposition 4.2 we have

$$\|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,0)} + \int_0^{\underline{u}}\delta^{-1}\|\nabla_4\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}$$

To estimate $\|\nabla_4\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}$ we make us of the scale invariant estimates

$$\|\phi\cdot\psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \delta^{1/2}\|\phi\|_{\mathcal{L}^\infty_{(sc)}(S)}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}$$

Hence,

$$\|\nabla_4\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(S)} + \delta^{\frac{1}{2}}\sum_{s_1+s_2=s+1}\|\psi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}(S)}\|\psi^{(s_2)}\|_{\mathcal{L}^4_{(sc)}(S)}$$

At this point we remark that if all Ricci coefficient and curvature norms ${}^{(S)}\mathcal{O}_{0,4}, \mathcal{R}_0$ were scale invariant we would proceed in a straightforward manner as follows,

$$\begin{aligned}\|\nabla_4\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(S)} &\lesssim \|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(S)} + \delta^{1/2}\,{}^{(S)}\mathcal{O}_{0,\infty}\cdot\,{}^{(S)}\mathcal{O}_{0,4}\\ &\lesssim \|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(S)} + \delta^{1/2}\Delta_0\cdot\,{}^{(S)}\mathcal{O}_{0,4}\end{aligned}$$

Hence

$$\begin{aligned}\|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,0)} + \int_0^{\underline{u}} \delta^{-1}\|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')} + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4}\\ &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,0)} + \mathcal{R}_0^{\frac12}\mathcal{R}_1^{\frac12} + \delta^{\frac14}\mathcal{R}_0 + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4},\end{aligned}$$

where in the last step we used the interpolation inequality (82) for the curvature Ψ^{s+1}. Thus, since the initial data is trivial along $\underline{u}=0$,

$$\|(\underline{\omega},\eta)\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \mathcal{R}_0^{\frac12}\mathcal{R}_1^{\frac12} + \delta^{\frac14}\mathcal{R}_0 + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4}$$

We only have to be more careful with the cases when $\|\Psi^{(s+1)}\|_{\mathcal{L}^4_{(sc)}(S)}$ is anomalous, i.e. $\Psi=\alpha$, and both $\psi^{(s_1)}$, $\psi^{(s_2)}$ are anomalous. The first situation (but not second) appear only in the case of the transport equation for $\hat{\chi}$ while the second appear only in the transport equation for $\mathrm{tr}\chi$.

$$\begin{aligned}\nabla_4\hat{\chi} + \mathrm{tr}\chi\hat{\chi} &= -2\omega\hat{\chi} - \alpha\\ \nabla_4\mathrm{tr}\chi + \frac12(\mathrm{tr}\chi)^2 &= -|\hat{\chi}|^2 - 2\omega\mathrm{tr}\chi\end{aligned}$$

Thus, for fixed u, we estimate with ${}^{\delta}S_{\underline{u}}$ denoting a disc of radius $\delta^{\frac12}$ transported from the data at $S_{u,0}$(recall also the triviality of the initial data on H_0),

$$\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}({}^{\delta}S_{\underline{u}})} \lesssim \int_0^{\underline{u}} \delta^{-1}\|\alpha\|_{\mathcal{L}^4_{(sc)}({}^{\delta}S_{u,\underline{u}'})}d\underline{u}' + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4}$$

Using (84) we obtain

$$\int_0^{\underline{u}} \delta^{-1}\|\alpha\|_{\mathcal{L}^4_{(sc)}({}^{\delta}S_{u,\underline{u}'})}d\underline{u}' \lesssim \|\alpha\|_{\mathcal{L}^2_{(sc)}({}^{2\delta}H_u^{(0,\underline{u})})} + \|\nabla\alpha\|_{\mathcal{L}^2_{(sc)}({}^{2\delta}H_u^{(0,\underline{u})})} \lesssim \mathcal{R}_0^{\delta}[\alpha] + \mathcal{R}_1[\beta]$$

Therefore,

$$\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}({}^{\delta}S_{\underline{u}})} \lesssim \|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}({}^{\delta}S_0)} + \mathcal{R}_0^{\delta}[\alpha] + \mathcal{R}_1[\alpha] + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4}$$

from which we derive both the scale invariant δ estimate for $\hat{\chi}$,

$${}^{(S)}\mathcal{O}^{\delta}_{0,4}[\hat{\chi}] \lesssim \mathcal{R}_0^{\delta}[\alpha] + \mathcal{R}_1[\alpha] + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4}. \tag{92}$$

We can also estimate directly the anomalous ${}^{(S)}\mathcal{O}_{0,4}[\hat{\chi}]$ from,

$$\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S_{\underline{u}})} \lesssim \int_0^{\underline{u}} \delta^{-1}\|\alpha\|_{\mathcal{L}^4_{(sc)}({}^{\delta}S_{u,\underline{u}'})}d\underline{u}' + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4}$$

Using the scale invariant interpolation inequality (74) we deduce,

$$\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S_{\underline{u}})} \lesssim \|\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}} \cdot \|\nabla\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}} + \delta^{1/4}\|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}} + \delta^{1/2}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4}$$

Taking into account the anomalous character of $\mathcal{R}_0[\alpha]$ and the definition of ${}^{(S)}\mathcal{O}_{0,4}[\hat{\chi}]$, we deduce,

$${}^{(S)}\mathcal{O}_{0,4}[\hat{\chi}] \lesssim \mathcal{R}_0[\alpha]^{1/2}\big(\mathcal{R}_1[\alpha] + \mathcal{R}_0[\alpha]\big)^{1/2} + \delta^{1/4}\Delta_0 \cdot \ {}^{(S)}\mathcal{O}_{0,4} \tag{93}$$

On the other hand,

$$\begin{aligned}\|\mathrm{tr}\chi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|\mathrm{tr}\chi\|_{\mathcal{L}^4_{(sc)}(u,0)} + \int_0^{\underline{u}} \delta^{-\frac12}\Delta_0\|\mathrm{tr}\chi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}d\underline{u}' \\ &+ \delta^{-\frac12}\Delta_0\int_0^{\underline{u}}\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u}')}d\underline{u}' + \delta^{1/2}\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4} \\ &\lesssim \|\mathrm{tr}\chi\|_{\mathcal{L}^4_{(sc)}(u,0)} + \delta^{\frac14}\Delta_0\ {}^{(S)}\mathcal{O}_{0,4}\end{aligned}$$

We summarize the results of the section in the following[9].

Proposition 5.2. *Under the bootstrap assumption* ${}^{(S)}\mathcal{O}_{0,\infty} \le \Delta_0$ *and assuming that* $\delta^{1/2}\Delta_0$ *is sufficiently small we derive,*

$$\begin{aligned}{}^{(S)}\mathcal{O}_{0,4}[\underline{\omega},\eta] &\lesssim \mathcal{R}_0 + \mathcal{R}_0^{\frac12}\mathcal{R}_1^{\frac12} + \delta^{\frac14}\mathcal{R}_0 + \delta^{1/2}\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4} \\ {}^{(S)}\mathcal{O}_{0,4}[tr\chi] &\lesssim 1 + \delta^{\frac14}\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4}, \\ {}^{(S)}\mathcal{O}_{0,4}[\hat{\chi}] &\lesssim \mathcal{R}_0[\alpha]^{1/2}\big(\mathcal{R}_1[\alpha]+\mathcal{R}_0[\alpha]\big)^{1/2} + \delta^{1/4}\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4}\end{aligned}$$

Also,

$$\mathcal{O}_0^\delta[\hat{\chi}] \lesssim \mathcal{R}_{[1]} + \delta^{1/2}\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4}$$

5.3. **Estimates for $\underline{\chi},\underline{\eta},\omega$.** The Ricci coefficients $\underline{\eta},\underline{\chi}$ and $\underline{\omega}$ verify equations of the form,

$$\nabla_3\psi^{(s)} = -\frac12 k\,\mathrm{tr}\underline{\chi}\psi^{(s)} + \sum_{s_1+s_2=s}\psi^{(s_1)}\cdot\psi^{(s_2)} + \Psi^{(s)}$$

with k a positive integer. Writing $\mathrm{tr}\underline{\chi} = \mathrm{tr}\underline{\chi}_0 + \widetilde{\mathrm{tr}\underline{\chi}}$, with $\mathrm{tr}\underline{\chi}_0 = -\frac{4}{\underline{u}-u+2r_0}$, we derive

$$\nabla_3\psi^{(s)} = -\frac12 k\,\mathrm{tr}\underline{\chi}_0\psi^{(s)} + \sum_{s_1+s_2=s}\psi^{(s_1)}\cdot\psi^{(s_2)} + \Psi^{(s)} \tag{94}$$

In this case we observe that the curvature term $\Psi^{(s)}$ is never anomalous and the only time when both $\psi^{(s_1)}$ and $\psi^{(s_2)}$ are anomalous is in the case of the transport equations for $\hat{\underline{\chi}}$ and $\mathrm{tr}\underline{\chi}$. In all other cases we can write, proceeding exactly as before,

$$\|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \int_0^u \|\nabla_3\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u',\underline{u})}$$

and,

$$\|\nabla_3\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} + \|\Psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} + \delta^{1/2}\ {}^{(S)}\mathcal{O}_{0,\infty}\cdot\ {}^{(S)}\mathcal{O}_{0,4}$$

[9]Recall the triviality of our initial conditions at $\underline{u}=0$.

Thus, in these cases,

$$\begin{aligned}\|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \int_0^u \|\Psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(u',\underline{u})} + \delta^{1/2}\ {}^{(S)}\mathcal{O}_{0,\infty}\cdot\ {}^{(S)}\mathcal{O}_{0,4}\\ &\lesssim \|\psi^{(s)}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \underline{\mathcal{R}}_0^{\frac12}\underline{\mathcal{R}}_1^{\frac12} + \delta^{\frac14}\underline{\mathcal{R}}_0 + \delta^{1/2}\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4} \end{aligned} \tag{95}$$

Similarly,

$$\|\psi^{(s)}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\psi^{(s)}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \underline{\mathcal{R}}_0 + \delta^{1/2}\Delta_0\cdot\ {}^{(S)}\mathcal{O}_{0,4} \tag{96}$$

It thus only remains to estimate $\mathrm{tr}\underline{\chi}, \underline{\hat{\chi}}$. We first estimate $\mathcal{O}_0^\delta[\underline{\chi}]$ from the equation,

$$\nabla_3\underline{\hat{\chi}} = -\underline{\alpha} + \mathrm{tr}\underline{\chi}_0\,\underline{\hat{\chi}} - \widetilde{\mathrm{tr}\underline{\chi}}\,\underline{\hat{\chi}} - 2\underline{\omega}\underline{\hat{\chi}}$$

Clearly, for fixed $\underline{u}$

$$\|\nabla_3\underline{\hat{\chi}} + \frac12\mathrm{tr}\underline{\chi}\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_u)} \lesssim \|\underline{\alpha}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_u)} + \|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_u)} + \delta^{1/2}\ {}^{(S)}\mathcal{O}_{0,\infty}\cdot\ {}^{(S)}\mathcal{O}_{0,4}$$

and thus, after a standard application of the Gronwall inequality,

$$\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_u)} \lesssim \|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_0)} + \int_0^u \|\underline{\alpha}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_{u'})}$$

Taking into account the scale invariant interpolation inequality (82) we deduce,

$$\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_u)} \lesssim \|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_0)} + \underline{\mathcal{R}}_0^{\frac12}[\underline{\alpha}]\cdot\underline{\mathcal{R}}_1^{\frac12}[\underline{\alpha}] + \delta^{\frac14}\underline{\mathcal{R}}_0[\underline{\alpha}] + \delta^{1/2}\Delta_0\ {}^{(S)}\mathcal{O}_{0,4}$$

or, since $\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_0)} \lesssim \mathcal{O}^{(0)}$,

$$\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}({}^\delta S_u)} \lesssim \mathcal{O}^{(0)} + \underline{\mathcal{R}}_0^{\frac12}[\underline{\alpha}]\big(\underline{\mathcal{R}}_1^{\frac12}[\underline{\alpha}] + \delta^{\frac14}\underline{\mathcal{R}}_0[\underline{\alpha}]\big) + \delta^{1/2}\Delta_0\ {}^{(S)}\mathcal{O}_{0,4} \tag{97}$$

Proceeding in the same fashion,

$$\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_u)} \lesssim \|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_0)} + \underline{\mathcal{R}}_0^{\frac12}[\underline{\alpha}]\cdot\underline{\mathcal{R}}_1^{\frac12}[\underline{\alpha}] + \delta^{\frac14}\underline{\mathcal{R}}_0[\underline{\alpha}] + \delta^{1/2}\Delta_0\ {}^{(S)}\mathcal{O}_{0,4}$$

Now, observe that the only anomaly on the right hand side is due to $\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_0)}$. In fact

$$\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S_0)} \lesssim \delta^{-1/4}\mathcal{O}^{(0)} \tag{98}$$

Thus,

$${}^{(S)}\mathcal{O}_{0,4}[\underline{\hat{\chi}}] \lesssim \mathcal{O}^{(0)} + \delta^{1/4}\underline{\mathcal{R}}_0^{\frac12}[\underline{\alpha}]\cdot\underline{\mathcal{R}}_1^{\frac12}[\underline{\alpha}] + \delta^{\frac12}\underline{\mathcal{R}}_0 + \delta^{3/4}\Delta_0\ {}^{(S)}\mathcal{O}_{0,4} \tag{99}$$

To estimate $\widetilde{\mathrm{tr}\underline{\chi}} = \mathrm{tr}\underline{\chi} - \mathrm{tr}\underline{\chi}_0$ we start with the equation

$$D_3\mathrm{tr}\underline{\chi} + \frac12(\mathrm{tr}\underline{\chi})^2 = -2\underline{\omega}\mathrm{tr}\underline{\chi} - |\underline{\hat{\chi}}|^2.$$

Since, $D_3 u = \Omega^{-1}$, $D_3 \underline{u} = 0$ we have, since $\mathrm{tr}\underline{\chi}_0 = -\frac{4}{\underline{u}-u+2r_0}$,

$$D_3 \mathrm{tr}\underline{\chi}_0 = -\Omega^{-1}\frac{1}{4}\mathrm{tr}\underline{\chi}_0^2$$

Hence, using $\widetilde{\mathrm{tr}\underline{\chi}} = \mathrm{tr}\underline{\chi} - \mathrm{tr}\underline{\chi}_0$,

$$\nabla_3 \widetilde{\mathrm{tr}\underline{\chi}} + \mathrm{tr}\underline{\chi}_0 \cdot \widetilde{\mathrm{tr}\underline{\chi}} = -\frac{1}{2\Omega}(\Omega - \frac{1}{2})\mathrm{tr}\underline{\chi}_0^2 + 2\underline{\omega}\mathrm{tr}\underline{\chi}_0 - 2\underline{\omega}\widetilde{\mathrm{tr}\underline{\chi}} - |\hat{\underline{\chi}}|^2 \tag{100}$$

Now, taking into account the anomalous scaling of ${}^{(S)}\mathcal{O}_{0,4}[\hat{\underline{\chi}}]$ and estimate, $\|\Omega - \frac{1}{2}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S)}$ (which can be easily derived using the transport equation $\nabla_3\Omega = \underline{\omega}$) we derive,

$$\|\nabla_3 \widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\underline{\omega}\|_{\mathcal{L}^4_{(sc)}(S)} + \delta^{\frac{1}{4}}\ {}^{(S)}\mathcal{O}_{0,\infty} \cdot\ {}^{(S)}\mathcal{O}_{0,4}.$$

from which,

$$\begin{aligned}
\|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} &\lesssim \|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \int_0^u \|\nabla_3 \widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(u',\underline{u})} \\
&\lesssim \|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \int_0^u \|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(u',\underline{u})} du' \\
&+ \int_0^u \|\underline{\omega}\|_{\mathcal{L}^4_{(sc)}(u',\underline{u})} du' + \delta^{\frac{1}{4}}\Delta_0 \cdot\ {}^{(S)}\mathcal{O}_{0,4}.
\end{aligned}$$

By Gronwall, and using the estimate for $\underline{\omega}$ derived in the previous section,

$$\|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(0,\underline{u})} + \mathcal{R}_0^{\frac{1}{2}}\mathcal{R}_1^{\frac{1}{2}} + \delta^{\frac{1}{4}}\mathcal{R}_0 + \delta^{\frac{1}{4}}\Delta_0 \cdot\ {}^{(S)}\mathcal{O}_{0,4}.$$

Thus,

$${}^{(S)}\mathcal{O}_{0,4}[\widetilde{\mathrm{tr}\underline{\chi}}] \lesssim \mathcal{O}^{(0)} + \mathcal{R}_0\mathcal{R}_1 + \delta^{\frac{1}{4}}\mathcal{R}_0 + \delta^{\frac{1}{4}}\Delta_0 \cdot\ {}^{(S)}\mathcal{O}_{0,4} \tag{101}$$

We summarize the result of this subsection in the following

Proposition 5.4. *We have, for sufficiently small δ,*

$$\begin{aligned}
{}^{(S)}\mathcal{O}_{0,4}[\underline{\eta}, \omega] &\lesssim \mathcal{O}^{(0)} + \underline{\mathcal{R}}_0 + \underline{\mathcal{R}}_0^{\frac{1}{2}}\underline{\mathcal{R}}_1^{\frac{1}{2}} + \delta^{\frac{1}{4}}\underline{\mathcal{R}}_0 + \delta^{\frac{1}{2}}\Delta_0 \cdot\ {}^{(S)}\mathcal{O}_{0,4} \\
{}^{(S)}\mathcal{O}_{0,4}[\hat{\underline{\chi}}] &\lesssim \mathcal{O}^{(0)} + \delta^{1/4}\underline{\mathcal{R}}_0^{\frac{1}{2}} \cdot \underline{\mathcal{R}}_1^{\frac{1}{2}} + \delta^{\frac{1}{2}}\underline{\mathcal{R}}_0 + \delta^{3/4}\Delta_0\ {}^{(S)}\mathcal{O}_{0,4} \\
{}^{(S)}\mathcal{O}_{0,4}[\widetilde{tr\underline{\chi}}] &\lesssim \mathcal{O}^{(0)} + \mathcal{R}_0\mathcal{R}_1 + \delta^{\frac{1}{4}}\mathcal{R}_0 + \delta^{\frac{1}{4}}\Delta_0 \cdot\ {}^{(S)}\mathcal{O}_{0,4}
\end{aligned}$$

Also,

$$\mathcal{O}^{\delta}[\hat{\underline{\chi}}] \lesssim \mathcal{O}^{(0)} + \underline{\mathcal{R}}_0^{\frac{1}{2}}\underline{\mathcal{R}}_1^{\frac{1}{2}} + \delta^{\frac{1}{4}}\underline{\mathcal{R}}_0 + \delta^{\frac{1}{2}}\Delta_0 \cdot\ {}^{(S)}\mathcal{O}_{0,4}$$

5.5. **Summary of ${}^{(S)}\mathcal{O}_{0,4}$ estimates.** Putting together the results of the last two propositions we deduce the following.

Proposition 5.6. *There exists a constant C depending only on $\mathcal{O}^{(0)}$ and $\mathcal{R}$ such that, if $\delta^{1/2}\Delta_0$ is sufficiently small, we have,*

$$ {}^{(S)}\mathcal{O}_{0,4} \lesssim C \tag{102}$$

Moreover,

$$ {}^{(S)}\mathcal{O}_{0,4}[\hat{\chi}] \lesssim \mathcal{R}_0[\alpha]^{1/2}\big(\mathcal{R}_1[\alpha]+\mathcal{R}_0[\alpha]\big)^{1/2}+\delta^{1/4}C \tag{103}$$

$$ {}^{(S)}\mathcal{O}_{0,4}[\underline{\hat{\chi}}] \lesssim \mathcal{O}^{(0)}+\delta^{1/4}C \tag{104}$$

5.7. **${}^{(S)}\mathcal{O}_{0,2}$ estimates.** The following estimates will also be needed.

Proposition 5.8. *There exists a constant C depending only on $\mathcal{O}^{(0)}$ and $\mathcal{R}$ such that, if $\delta^{1/2}\Delta_0$ is sufficiently small, we have,*

$$ {}^{(S)}\mathcal{O}_{0,2} \lesssim C \tag{105}$$

Proof. These are similar but somewhat simpler, once we already have the ${}^{(S)}\mathcal{O}_{0,4}$ estimates. Indeed, starting with (91), (dropping indices for simplicity) we write as before,

$$\|\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\psi\|_{\mathcal{L}^2_{(sc)}(u,0)}+\int_0^{\underline{u}}\delta^{-1}\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}$$

and, assuming the worst case scenario when both terms in $\psi\cdot\psi$ are anomalous, i.e. both satisfy $\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}\lesssim C\delta^{-1/4}$,

$$\begin{aligned}\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}+\delta^{\frac12}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}\\ &\lesssim \|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}+{}^{(S)}\mathcal{O}^2_{0,4}\\ &\lesssim \|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}+C^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim +\int_0^{\underline{u}}\delta^{-1}\|\Psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}+C^2\\ &\lesssim \|\Psi\|_{\mathcal{L}^2_{(sc)}(H_u)}+C^2\end{aligned}$$

Ψ can only be the anomalous α in the case of the transport equation for $\hat{\chi}$. Thus,

$$\begin{aligned}\|(\underline{\omega},\eta)\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \mathcal{R}_0+C^2\\ \|\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \delta^{-1/2}\mathcal{R}_0[\alpha]\end{aligned}$$

or, with a constant $C=C(\mathcal{O}^{(0)},\mathcal{R},\underline{\mathcal{R}})$,

$$ {}^{(S)}\mathcal{O}_{0,2}[\mathrm{tr}\chi,\hat{\chi},\underline{\omega},\eta] \lesssim C$$

The estimates for $\mathrm{tr}\underline{\chi}, \underline{\hat{\chi}}, \omega, \underline{\eta}$ are proved in the same manner.

$\square$

6. $\mathcal{O}_1$ ESTIMATES

6.1. **General Strategy.** To get the first and second derivative estimates for the Ricci coefficients we cannot proceed as we did in the previous section. Following a path first pursued in [Chr-Kl] and continued in [K-Ni], [K-R:causal] and [Chr] we introduce new quantities[10] $\Theta^{(s)}$, with signature s, depending on first derivative of the Ricci coefficients and which verify transport equations of the form[11]

$$\begin{aligned}\nabla_4\Theta^{(s)} &= \mathrm{tr}\chi\big(\Theta^{(s)} + \nabla\psi^{(s-\frac{1}{2})}\big) + \sum_{s_1+s_2+\frac{1}{2}=s+1}\psi^{(s_1)}\big(\nabla\psi^{(s_2)} + \Psi^{(s_2)}\big)\\ &+ \sum_{s_1+s_2=s+1}\mathrm{tr}\underline{\chi}_0\cdot\psi^{(s_1)}\cdot\psi^{(s_2)} + \sum_{s_1+s_2+s_3=s+1}\psi^{(s_1)}\cdot\psi^{(s_2)}\cdot\psi^{(s_3)}\end{aligned} \tag{106}$$

$$\begin{aligned}\nabla_3\Theta^{(s)} &= \mathrm{tr}\underline{\chi}\big(\Theta^{(s)} + \nabla\psi^{(s-\frac{1}{2})}\big) + \sum_{s_1+s_2+\frac{1}{2}=s}\psi^{(s_1)}\big(\nabla\psi^{(s_2)} + \Psi^{(s_2)}\big)\\ &+ \sum_{s_1+s_2=s}\mathrm{tr}\underline{\chi}_0\cdot\psi^{(s_1)}\cdot\psi^{(s_2)} + \sum_{s_1+s_2+s_3=s}\psi^{(s_1)}\cdot\psi^{(s_2)}\cdot\psi^{(s_3)}\end{aligned} \tag{107}$$

Here $\psi^{(s)}$ are components of all the Ricci coefficients $(\mathrm{tr}\chi, \hat{\chi}, \underline{\omega}, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \underline{\hat{\chi}})$ with signature s, while $\Psi^{(s)}$ are curvature components with signature s.

The main idea behind our strategy is to show that once we control the $\mathcal{L}^2_{(sc)}(S)$ norms of these quantities Θ we derive all $\mathcal{O}_1$ estimates by using the elliptic Hodge systems. The most general form of such systems is given by

$$\mathcal{D}\psi^{(s)} = \Theta^{(s+\frac{1}{2})} + \Psi^{(s+\frac{1}{2})} + \mathrm{tr}\underline{\chi}_0\cdot\psi^{(s+\frac{1}{2})} + \sum_{s_1+s_2=s+\frac{1}{2}}\psi^{(s_1)}\psi^{(s_2)}. \tag{108}$$

where $\mathcal{D}$ is one of the Hodge systems of section 3.5. Observe also that both Hodge systems have non- anomalous curvature source terms, β, respectively $\underline{\beta}$ and no quadratic anomalies in ψ (relative to the $\mathcal{O}_0$ norm).

[10]Different components Θ appear in (106) and (107). It may in fact be more appropriate to call Θ the components which appear on the left of the ∇_4 equation and by $\underline{\Theta}$ those appearing on the left of the ∇_3 equations.

[11]We neglect to write possible constants in front of each term on the right of our equations

6.2. **Explicit Θ variables and Hodge systems.** In this section we introduce explicit variables $\Theta^{(s)}$ and derive transport equations of the type (106), (107).

Transport-Hodge systems for $\chi, \underline{\chi}$. First observe that the Codazzi equations

$$\begin{aligned} \text{div}\,\hat{\chi} &= \frac{1}{2}\nabla \text{tr}\chi - \frac{1}{2}(\eta - \underline{\eta})\cdot(\hat{\chi} - \frac{1}{2}\text{tr}\chi) - \beta, && (109)\\ \text{div}\,\underline{\hat{\chi}} &= \frac{1}{2}\nabla \text{tr}\underline{\chi} + \frac{1}{2}(\eta - \underline{\eta})\cdot(\underline{\hat{\chi}} - \frac{1}{2}\text{tr}\underline{\chi}) + \underline{\beta} && (110)\end{aligned}$$

can be written as Hodge systems of type (108). with $\mathcal{D}$ the Hodge operator $\mathcal{D}_2$, discussed in section 3.5, and $\Theta = \nabla\text{tr}\chi$, resp. $\Theta = \nabla\text{tr}\underline{\chi}$.

We now derive a ∇_4 transport equation for $\nabla\text{tr}\chi$. Using commutation formula, $[\nabla_4, \nabla]f = \frac{1}{2}(\eta + \underline{\eta})D_4 f - \chi\cdot\nabla f$, we obtain,

$$\begin{aligned} \nabla_4\nabla\text{tr}\chi &= -\nabla\text{tr}\chi\text{tr}\chi - 2\text{tr}\chi\nabla\omega - 2\omega\nabla\text{tr}\chi - 2\nabla\hat{\chi}\cdot\hat{\chi} && (111)\\ &+ \frac{1}{2}(\eta+\underline{\eta})\big(-\frac{1}{2}(\text{tr}\chi)^2 - 2\omega\text{tr}\chi - |\hat{\chi}|^2\big) - \chi\cdot\nabla\text{tr}\chi \end{aligned}$$

which is clearly of the form (106) with no curvature terms present and no triple anomalies (relative to the $\mathcal{O}_0$ norm, i.e. among the cubic terms at least one of the factors are not anomalous).

To derive a transport equation for $\nabla\text{tr}\underline{\chi}$ we start with the transport equation,

$$\nabla_3\text{tr}\underline{\chi} = -\frac{1}{2}(\text{tr}\underline{\chi})^2 + F, \quad F = -2\underline{\omega}\text{tr}\underline{\chi} - |\underline{\hat{\chi}}|^2 = -2\underline{\omega}\text{tr}\underline{\chi}_0 - 2\underline{\omega}\widetilde{\text{tr}\underline{\chi}} - |\underline{\hat{\chi}}|^2$$

Using the commutator formula, $[\nabla_3, \nabla]f = -\underline{\chi}\cdot\nabla f + \frac{1}{2}(\eta+\underline{\eta})D_3 f$ we deduce,

$$\nabla_3(\nabla\text{tr}\underline{\chi}) = -\underline{\hat{\chi}}\cdot\nabla\text{tr}\underline{\chi} - \frac{3}{2}\text{tr}\underline{\chi}\nabla\text{tr}\underline{\chi} - \big(\nabla + \frac{1}{2}(\eta+\underline{\eta})\big)F$$

Or, writing $\text{tr}\underline{\chi} = \text{tr}\underline{\chi}_0 + \widetilde{\text{tr}\underline{\chi}}$, we deduce,

$$\nabla_3(\nabla\text{tr}\underline{\chi}) = -\underline{\hat{\chi}}\cdot\nabla\text{tr}\underline{\chi} - \frac{3}{2}\text{tr}\underline{\chi}_0\nabla\text{tr}\underline{\chi} - \frac{3}{2}\widetilde{\text{tr}\underline{\chi}}\nabla\text{tr}\underline{\chi} - \big(\nabla + \frac{1}{2}(\eta+\underline{\eta})\big)F \qquad (112)$$

This is clearly a system of the form (107) with no curvature terms present and no anomalous cubic terms.

Transport- Hodge systems for $\mu, \underline{\mu}, \nabla\eta, \nabla\underline{\eta}$. We start with equation

$$\text{curl}\,\eta = \text{curl}\,\underline{\eta} = \sigma + \underline{\hat{\chi}}\wedge\hat{\chi}$$

We derive equations for div η and div $\underline{\eta}$ by taking he divergence of the transport equations

$$\begin{aligned}\nabla_4\eta &= -\tfrac{1}{2}\mathrm{tr}\chi(\eta-\underline{\eta}) - \hat{\chi}\cdot(\eta-\underline{\eta}) - \beta\\ \nabla_3\underline{\eta} &= -\tfrac{1}{2}\mathrm{tr}\underline{\chi}(\underline{\eta}-\eta) - \underline{\hat{\chi}}\cdot(\underline{\eta}-\eta) + \underline{\beta}\end{aligned}$$

Using commutation lemma (3.3) we derive,

$$\begin{aligned}\nabla_4(\mathrm{div}\ \eta) &= \mathrm{div}\ (-\frac{1}{2}\mathrm{tr}\chi(\eta-\underline{\eta}) - \hat{\chi}\cdot(\eta-\underline{\eta}) - \beta)\\ &- \frac{1}{2}\mathrm{tr}\chi\,\mathrm{div}\ \eta - \hat{\chi}\cdot\nabla\eta - \eta\cdot\beta + \frac{1}{2}(\eta+\underline{\eta})\cdot\nabla_4\eta\\ &= -\mathrm{div}\ \beta - \frac{1}{2}\mathrm{tr}\chi(2\mathrm{div}\ \eta - \mathrm{div}\ \underline{\eta}) - (\eta-\underline{\eta})\cdot\big(\frac{1}{2}\nabla\mathrm{tr}\chi + \mathrm{div}\ \hat{\chi}\big)\\ &- \hat{\chi}\cdot\nabla(2\eta-\underline{\eta}) - \eta\cdot\beta + \frac{1}{2}(\eta+\underline{\eta})\cdot\nabla_4\eta\end{aligned}$$

Using the null Codazzi equation,

$$\frac{1}{2}\nabla\mathrm{tr}\chi + \mathrm{div}\ \hat{\chi} = \nabla\mathrm{tr}\chi + \frac{1}{2}\zeta\mathrm{tr}\chi - \beta$$

we derive,

$$\begin{aligned}\nabla_4(\mathrm{div}\ \eta) &= -\mathrm{div}\ \beta - \frac{1}{2}\mathrm{tr}\chi(2\mathrm{div}\ \eta - \mathrm{div}\ \underline{\eta}) - \hat{\chi}\cdot\nabla(2\eta-\underline{\eta}) - (\eta-\underline{\eta})\cdot\nabla\mathrm{tr}\chi\\ &- \underline{\eta}\cdot\beta - \frac{1}{4}\mathrm{tr}\chi(\eta-\underline{\eta})^2 + \frac{1}{2}(\eta+\underline{\eta})\big(-\frac{1}{2}\mathrm{tr}\chi(\eta-\underline{\eta}) - \hat{\chi}\cdot(\eta-\underline{\eta}) - \beta\big)\\ &= -\mathrm{div}\ \beta - \frac{1}{2}\mathrm{tr}\chi(2\mathrm{div}\ \eta - \mathrm{div}\ \underline{\eta}) - \hat{\chi}\cdot\nabla(2\eta-\underline{\eta}) - (\eta-\underline{\eta})\cdot\nabla\mathrm{tr}\chi\\ &- \frac{1}{2}(3\underline{\eta}+\eta)\cdot\beta - \frac{1}{2}\mathrm{tr}\chi(|\eta|^2 - \eta\cdot\underline{\eta}) - \frac{1}{2}(\eta+\underline{\eta})\cdot\hat{\chi}\cdot(\eta-\underline{\eta})\end{aligned}$$

or,

$$\begin{aligned}\nabla_4(\mathrm{div}\ \eta) + \mathrm{tr}\chi\mathrm{div}\ \eta &= -\mathrm{div}\ \beta + \frac{1}{2}\mathrm{tr}\chi\mathrm{div}\ \underline{\eta} - \hat{\chi}\cdot\nabla(2\eta-\underline{\eta}) - (\eta-\underline{\eta})\cdot\nabla\mathrm{tr}\chi\\ &- \frac{1}{2}(3\underline{\eta}+\eta)\cdot\beta - \frac{1}{2}\mathrm{tr}\chi(|\eta|^2 - \eta\cdot\underline{\eta}) - \frac{1}{2}(\eta+\underline{\eta})\cdot\hat{\chi}\cdot(\eta-\underline{\eta})\end{aligned}$$

On the other hand,

$$\nabla_4\rho + \frac{3}{2}\mathrm{tr}\chi\rho = \mathrm{div}\ \beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha + \zeta\cdot\beta + 2\underline{\eta}\cdot\beta$$

Adding the two equations and setting,

$$\mu = -\mathrm{div}\ \eta - \rho$$

we derive,

$$\begin{aligned}\nabla_4\mu + \mathrm{tr}\chi\mu &= -\frac{1}{2}\mathrm{tr}\chi\mathrm{div}\,\underline{\eta} + (\eta - \underline{\eta})\nabla\mathrm{tr}\chi + \hat{\chi}\cdot\nabla(2\eta - \underline{\eta}) + \frac{1}{2}\,\underline{\hat{\chi}}\cdot\alpha - (\eta - 3\underline{\eta})\cdot\beta + \frac{1}{2}\mathrm{tr}\chi\rho\\ &+ \frac{1}{2}\mathrm{tr}\chi(|\eta|^2 - \eta\cdot\underline{\eta}) + \frac{1}{2}(\eta + \underline{\eta})\cdot\hat{\chi}\cdot(\eta - \underline{\eta})\end{aligned}$$

Similarly, setting

$$\underline{\mu} = -\mathrm{div}\,\underline{\eta} - \rho$$

we derive,

$$\begin{aligned}\nabla_3\underline{\mu} + \mathrm{tr}\underline{\chi}\underline{\mu} &= -\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{div}\,\eta + (\underline{\eta} - \eta)\nabla\mathrm{tr}\underline{\chi} + \underline{\hat{\chi}}\cdot\nabla(2\underline{\eta} - \eta) + \frac{1}{2}\,\hat{\chi}\cdot\underline{\alpha} - (\underline{\eta} - 3\eta)\cdot\underline{\beta} + \frac{1}{2}\mathrm{tr}\underline{\chi}\rho\\ &+ \frac{1}{2}\mathrm{tr}\underline{\chi}(|\underline{\eta}|^2 - \underline{\eta}\cdot\eta) + \frac{1}{2}(\eta + \underline{\eta})\cdot\underline{\hat{\chi}}\cdot(\underline{\eta} - \eta)\end{aligned}$$

We summarize the results above in the following.

Lemma 6.3. *The reduced mass aspect functions,*

$$\begin{aligned}\mu &= -div\,\eta - \rho\\ \underline{\mu} &= -div\,\underline{\eta} - \rho\end{aligned}$$

verify the transport equations,

$$\begin{aligned}\nabla_4\mu + tr\chi\mu &= -\frac{1}{2}tr\chi div\,\underline{\eta} + (\eta - \underline{\eta})\nabla tr\chi + \hat{\chi}\cdot\nabla(2\eta - \underline{\eta}) + \frac{1}{2}\,\underline{\hat{\chi}}\cdot\alpha - (\eta - 3\underline{\eta})\cdot\beta + \frac{1}{2}tr\chi\rho\\ &+ \frac{1}{2}tr\chi(|\eta|^2 - \eta\cdot\underline{\eta}) + \frac{1}{2}(\eta + \underline{\eta})\cdot\hat{\chi}\cdot(\eta - \underline{\eta}) \qquad (113)\end{aligned}$$

$$\begin{aligned}\nabla_3\underline{\mu} + tr\underline{\chi}\underline{\mu} &= -\frac{1}{2}tr\underline{\chi}div\,\eta + (\underline{\eta} - \eta)\nabla tr\underline{\chi} + \underline{\hat{\chi}}\cdot\nabla(2\underline{\eta} - \eta) + \frac{1}{2}\,\hat{\chi}\cdot\underline{\alpha} - (\underline{\eta} - 3\eta)\cdot\underline{\beta} + \frac{1}{2}tr\underline{\chi}\rho\\ &+ \frac{1}{2}tr\underline{\chi}(|\underline{\eta}|^2 - \underline{\eta}\cdot\eta) + \frac{1}{2}(\eta + \underline{\eta})\cdot\underline{\hat{\chi}}\cdot(\underline{\eta} - \eta) \qquad (114)\end{aligned}$$

Remark 6.4. Observe that our mass aspect functions differ from those of [Chr-Kl] or [K-Ni]. Thus, in [K-Ni]), $\mu = -\mathrm{div}\,\eta - \rho + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}$ verifies (see equation 4.3.32 in [K-Ni]),

$$\begin{aligned}\nabla_4\mu + \mathrm{tr}\chi\mu &= \hat{\chi}\cdot(\nabla\widehat{\otimes}\eta) + (\eta - \underline{\eta})\cdot(\nabla\mathrm{tr}\chi + \mathrm{tr}\chi\zeta) + \frac{1}{2}\mathrm{tr}\chi\big(\mu + \mathrm{div}\,(\eta - \underline{\eta})\big)\\ &- \frac{1}{4}\mathrm{tr}\underline{\chi}|\hat{\chi}|^2 + \frac{1}{2}\mathrm{tr}\chi(\hat{\chi}\cdot\underline{\hat{\chi}} + 2\rho - |\underline{\eta}|^2) + 2(\eta\cdot\hat{\chi}\cdot\underline{\eta} - \eta\cdot\beta)\end{aligned}$$

The reason we prefer our definition here is to avoid the presence of triple anomalous terms on the right hand side of the transport equations for $\mu, \underline{\mu}$.

We write (113) symbolically in the form,

$$\nabla_4\mu = \psi\cdot(\nabla\psi + \Psi_g) + \underline{\hat{\chi}}\cdot\alpha + \psi\cdot\psi\cdot\psi_g \qquad (115)$$

which is of the form (106), with $\psi_g \in \{\mathrm{tr}\chi, \hat{\chi}, \eta, \underline{\eta}, \omega, \underline{\omega}, \mathrm{tr}\underline{\chi}\}$ and $\Psi_g \in \{\beta, \rho, \sigma\, \underline{\beta}\}$. We can also write, in shorter form,

$$\nabla_4 \mu = \psi \cdot (\nabla \psi + \Psi) + \psi \cdot \psi \cdot \psi_g$$

and recall that $\psi \cdot \Psi$ contains the more difficult term $\underline{\hat{\chi}} \cdot \alpha$ anomalous in both ψ and Ψ.

We also rewrite (114) symbolically. In this case we have to keep track of the terms proportional to $\mathrm{tr}\underline{\chi} = \mathrm{tr}\underline{\chi}_0 + \widetilde{\mathrm{tr}\underline{\chi}}$. We thus write symbolically,

$$\nabla_3 \underline{\mu} = \mathrm{tr}\underline{\chi}_0 (\nabla \psi + \underline{\mu}) + \psi \cdot (\nabla \psi + \Psi_g) + \psi_g \cdot \underline{\beta} + \mathrm{tr}\underline{\chi}_0 \psi \cdot \psi_g + \psi \cdot \psi \cdot \psi_g \tag{116}$$

Here $\Psi_g \in \{\rho, \sigma, \underline{\beta}, \underline{\alpha}\}$. Observe that at least one of the factors ψ in $\mathrm{tr}\underline{\chi}_0 \psi \cdot \psi_g$ and $\psi \cdot \psi \cdot \psi_g$ can be anomalous. Unlike in the case of $\nabla_4 \mu$ equation, there are no terms of the form $\psi \cdot \beta$ with ψ also anomalous (recall that β is anomalous for $\underline{\mathcal{R}}_0$).

We combine the transport equations (115) and (116) with the Hodge systems,

$$\begin{aligned} \mathrm{div}\, \eta &= -\mu - \rho \\ \mathrm{curl}\, \eta &= \sigma - \frac{1}{2} \hat{\chi} \wedge \underline{\hat{\chi}} \end{aligned} \tag{117}$$

and,

$$\mathrm{div}\, \underline{\eta} = -\underline{\mu} - \rho \tag{118}$$

$$\mathrm{curl}\, \underline{\eta} = \sigma - \frac{1}{2} \hat{\chi} \wedge \underline{\hat{\chi}} \tag{119}$$

They are both systems of type (108). Note that the quadratic term $\hat{\chi} \cdot \underline{\hat{\chi}}$ is anomalous with respect to both factors.

Transport-Hodge systems for $\underline{\kappa}, \kappa, \nabla \underline{\omega}, \nabla \omega$. We look for transport equations for quantities connected to $\nabla \omega$ and $\nabla \underline{\omega}$. Recall that

$$\begin{aligned} \nabla_4 \underline{\omega} &= \frac{1}{2} \rho + F \\ F &= 2\omega \underline{\omega} + \frac{3}{4} |\eta - \underline{\eta}|^2 - \frac{1}{4} (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{8} |\eta + \underline{\eta}|^2 \end{aligned} \tag{120}$$

and,

$$\begin{aligned} \nabla_3 \omega &= \frac{1}{2} \rho + \underline{F} \\ \underline{F} &= 2\omega \underline{\omega} + \frac{3}{4} |\eta - \underline{\eta}|^2 + \frac{1}{4} (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) - \frac{1}{8} |\eta + \underline{\eta}|^2 \end{aligned} \tag{121}$$

We introduce the auxiliary quantities $\underline{\omega}^\dagger$ and $\omega^\dagger$ as follows.

$$\nabla_4\underline{\omega}^\dagger = \frac{1}{2}\sigma \tag{122}$$

$$\nabla_3\omega^\dagger = \frac{1}{2}\sigma \tag{123}$$

with zero boundary conditions along $\underline{H}_0$, respectively H_0. We introduce the pair of scalars $<\underline{\omega}>= (\underline{\omega},\underline{\omega}^\dagger)$ and $<\omega>=(-\omega,\omega^\dagger)$ and apply the Hodge operator ${}^\star\mathcal{D}_1$ (see subsection 3.5),

$${}^\star\mathcal{D}_1 <\underline{\omega}> = -\nabla\underline{\omega} + {}^*\nabla\underline{\omega}^\dagger, \quad {}^\star\mathcal{D}_1 <\omega>=\nabla\omega + {}^*\nabla\omega^\dagger.$$

Next we derive a ∇_4 equation for $<\underline{\omega}>$ and a ∇_3 equation for $<\omega>$. To do this we write the commutation relation (55) in the form,

$$\begin{aligned}
[\nabla_4,\nabla]f &= -\frac{1}{2}\mathrm{tr}\chi\nabla f - \hat{\chi}\cdot\nabla f + \frac{1}{2}(\eta+\underline{\eta})D_4 f\\
[\nabla_4,{}^*\nabla]g &= -\frac{1}{2}\mathrm{tr}\chi\,{}^*\nabla g + \hat{\chi}\cdot{}^*\nabla g + \frac{1}{2}(\eta^*+\underline{\eta}^*)D_4 g
\end{aligned}$$

Thus, for a pair of scalars (f,g),

$$[\nabla_4,{}^\star\mathcal{D}_1](f,g) = -\frac{1}{2}\mathrm{tr}\chi\,{}^\star\mathcal{D}_1(f,g) + \hat{\chi}\cdot(\nabla f + {}^*\nabla g) - \frac{1}{2}(\eta+\underline{\eta})\nabla_4 f + \frac{1}{2}(\eta^*+\underline{\eta}^*)D_4 g$$

Therefore,

$$\begin{aligned}
\nabla_4\,{}^\star\mathcal{D}_1<\underline{\omega}> &= {}^\star\mathcal{D}_1(\rho,\sigma) - \nabla F + [\nabla_4,{}^\star\mathcal{D}_1]<\underline{\omega}>\\
&= {}^\star\mathcal{D}_1(\rho,\sigma) - \nabla F - \frac{1}{2}\mathrm{tr}\chi\,{}^\star\mathcal{D}_1<\underline{\omega}> + \hat{\chi}\cdot(\nabla\underline{\omega} + {}^*\nabla\underline{\omega}^\dagger)\\
&- \frac{1}{2}(\eta+\underline{\eta})(\rho+F) + \frac{1}{2}(\eta^*+\underline{\eta}^*)\sigma
\end{aligned}$$

On the other hand, we have the Bianchi equation,

$$D_4\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = \mathcal{D}_1^*(\rho,\sigma) + 2\omega\underline{\beta} + 2\underline{\hat{\chi}}\cdot\beta - 3(\underline{\eta}\rho -{}^*\underline{\eta}\sigma),$$

Thus,introducing the new horizontal vector,

$$\underline{\kappa} := {}^\star\mathcal{D}_1<\underline{\omega}> - \frac{1}{2}\underline{\beta} = {}^\star\mathcal{D}_1(\underline{\omega},\underline{\omega}^\dagger) - \frac{1}{2}\underline{\beta} = -\nabla\underline{\omega} + {}^*\nabla\underline{\omega}^\dagger - \frac{1}{2}\underline{\beta} \tag{124}$$

we deduce,

$$\begin{aligned}
\nabla_4\underline{\kappa} &= -\mathrm{tr}\chi\cdot\underline{\kappa} - \omega\underline{\beta} - \underline{\hat{\chi}}\cdot\beta + \frac{3}{2}(\underline{\eta}\rho - {}^*\underline{\eta}\sigma) - \frac{1}{2}(\eta+\underline{\eta})\rho + \frac{1}{2}(\eta^*+\underline{\eta}^*)\sigma\\
&+ \hat{\chi}\cdot(\nabla\underline{\omega} + {}^*\nabla\underline{\omega}^\dagger) - \nabla F - \frac{1}{2}(\eta+\underline{\eta})\,F
\end{aligned} \tag{125}$$

Similarly we set,

$$\kappa := {}^\star\mathcal{D}_1<\omega> - \frac{1}{2}\beta = {}^\star\mathcal{D}_1(-\omega,\omega^\dagger) - \frac{1}{2}\beta = \nabla\omega + {}^*\nabla\omega^\dagger - \frac{1}{2}\beta \tag{126}$$

and, using the Bianchi equations,

$$D_3\beta + \mathrm{tr}\underline{\chi}\beta = \mathcal{D}_1^*(-\rho, \sigma) + 2\underline{\omega}\beta + 2\hat{\chi}\cdot\underline{\beta} + 3(\eta\rho +^* \eta\sigma),$$

we derive,

$$\begin{aligned}\nabla_3\kappa &= -\mathrm{tr}\underline{\chi}\cdot\kappa - \underline{\omega}\beta - \hat{\chi}\cdot\underline{\beta} + \frac{3}{2}(\eta\rho + {}^*\eta\sigma) - \frac{1}{2}(\underline{\eta}+\eta)\rho + \frac{1}{2}(\underline{\eta}^* + \eta^*)\sigma \\ &+ \hat{\underline{\chi}}\cdot(-\nabla\omega + {}^*\nabla\omega^\dagger) + \nabla\underline{F} + \frac{1}{2}(\underline{\eta}+\eta)\underline{F} \qquad (127)\end{aligned}$$

To estimate $\nabla\underline{\omega}$ we combine the ∇_4 equation (125) with the Hodge system,

$$^\star\mathcal{D}_1(\underline{\omega}, \underline{\omega}^\dagger) = \underline{\kappa} + \frac{1}{2}\underline{\beta} \qquad (128)$$

To estimate $\nabla\omega$ we combine the ∇_3 equation (127) with the Hodge system,

$$^\star\mathcal{D}_1(-\omega, \omega^\dagger) = \kappa + \frac{1}{2}\beta \qquad (129)$$

Clearly transport equations for $\underline{\kappa}$ and κ are of the form (106) and (107) provided that we extend the set of Ricci coefficients ψ to also include the new scalars $\underline{\omega}^\dagger$ and $\omega^\dagger$. We observe that $\underline{\omega}^\dagger$ has the same signature as $\underline{\omega}$ and $\omega^\dagger$ has the same signature as ω. Moreover $\underline{\omega}^\dagger$, $\omega^\dagger$ they satisfy equations similar to those satisfied by $\underline{\omega}, \omega$. Thus, for example, we can easily derive both $\mathcal{L}^2_{(sc)}$ and $\mathcal{L}^4_{(sc)}$ estimates for them. Indeed, from (122) we easily derive,

$$\|\underline{\omega}^\dagger\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \int_0^{\underline{u}} \delta^{-1}\|\sigma\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')} d\underline{u}' \lesssim \mathcal{R}_0[\sigma].$$

Similarly, from (123),

$$\|\underline{\omega}^\dagger\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \int_0^{u} \|\sigma\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})} d\underline{u}' \lesssim \underline{\mathcal{R}}_0[\sigma]$$

It thus make perfect sense to extend the definition of the set of Ricci coefficients as well as the definition of the norms $^{(S)}\mathcal{O}_\infty$, $^{(S)}\mathcal{O}_{0,4}$, $^{(S)}\mathcal{O}_{1,2}$, $^{(S)}\mathcal{O}_{1,4}$ to include them. We thus also assume, from now on, that the main bootstrap assumption (37) includes $\underline{\omega}^\dagger, \omega^\dagger$.

Finally we observe that equations (125), (127) can be written in the form,

$$\begin{aligned}\nabla_4\underline{\kappa} &= -\mathrm{tr}\chi\cdot\underline{\kappa} + \psi\cdot(\Psi_g + \nabla\psi) + \psi\cdot\psi\cdot\psi_g \\ \nabla_3\kappa &= -\mathrm{tr}\underline{\chi}\cdot\kappa + \psi\cdot(\Psi_g + \nabla\psi) + \psi\cdot\psi\cdot\psi_g\end{aligned}$$

with $\Psi_g \in \{\beta, \rho, \sigma, \underline{\beta}\}$ and $\psi_g \in \{\mathrm{tr}\chi, \underline{\omega}, \underline{\omega}^\dagger, \eta, \underline{\eta}, \omega, \omega^\dagger, \widetilde{\mathrm{tr}\underline{\chi}}\}$. Since $\underline{\kappa}$ can be expressed in terms of $\nabla\omega, \nabla\omega^\dagger$ and $\underline{\beta}$ we can also write the first equation in the form

$$\nabla_4\underline{\kappa} = \psi\cdot(\Psi_g + \nabla\psi) + \psi\cdot\psi\cdot\psi_g$$

The second equation can be written in the form,

$$\nabla_3\kappa = -\mathrm{tr}\underline{\chi}_0\cdot\kappa + \psi\cdot(\Psi_g + \nabla\psi) + \psi\cdot\psi\cdot\psi_g \qquad (130)$$

6.5. **Main $\mathcal{O}_1$ estimates.** We start by rewriting systems (106), (107) and (108) in short form, dropping the reference to signature.

$$\nabla_4\Theta = \psi\cdot\big(\nabla\psi+\Psi\big)+\mathrm{tr}\underline{\chi}_0\cdot\psi\cdot\psi_g+\psi\cdot\psi\cdot\psi_g \tag{131}$$

$$\nabla_3\Theta = \mathrm{tr}\underline{\chi}_0\cdot\nabla\psi+\psi\cdot\big(\nabla\psi+\Psi\big)+\mathrm{tr}\underline{\chi}_0\cdot\psi\cdot\psi_g+\psi\cdot\psi\cdot\psi_g \tag{132}$$

where ψ_g denotes an extended Ricci coefficient term (i.e. including $\underline{\omega}^\dagger,\omega^\dagger$ defined below.) which is not anomalous in the ${}^{(S)}\mathcal{O}_{0,4}$-norm.). Also,

$$\mathcal{D}\psi = \Theta+\Psi+\mathrm{tr}\underline{\chi}_0\cdot\psi_g+\psi\cdot\psi. \tag{133}$$

Remark 1. In reality equation (132) should also contain a term of the form $\mathrm{tr}\underline{\chi}_0\Theta$ as seen in (112), (116) and (130). We observe however that such terms can be easily eliminated by a standard Gronwall inequality.

Remark 2. The curvature terms Ψ appearing on the right hand side of (131) belong to the admissible[12] set $\{\alpha,\beta,\rho,\sigma,\underline{\beta}\}$. Special attention needs to be given to terms of the form[13] $\hat{\underline{\chi}}\cdot\alpha$.

Remark 3. The curvature terms Ψ appearing on the right hand side of (132) belong to the admissible[14] set $\{\beta,\rho,\sigma,\underline{\beta},\underline{\alpha}\}$. Special attention needs to be given to terms of the form $\psi\cdot\beta$, since $\underline{\mathcal{R}}_0[\beta]$ is anomalous. We observe however that among all possible terms of the form $\psi\cdot\beta$, ψ is never anomalous.

Remark 4. The curvature terms Ψ appearing on the right hand side of (133) belong to the set $\{\beta,\rho,\sigma,\underline{\beta}\}$.

Remark 5. ψ_g denotes an extended Ricci coefficient which is not anomalous in the $\mathcal{O}_0$ norm. Whenever we write simply ψ we allow for the possibility that it may be anomalous. For example the terms of the form $\psi\cdot\psi$ in (133) may be both anomalous (as happens to be the case for the div -curl systems for $\eta,\underline{\eta}$, due to $\hat{\chi}\cdot\hat{\underline{\chi}}$).

Remark 6. Due to the triviality of our initial data at $\underline{u}=0$ we have

$$\|\Theta\|_{\mathcal{L}^2_{(sc)}(u,0)}=0.$$

In view of the definition of the Θ we have,

$$\|\Theta\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} \lesssim \mathcal{O}^{(0)}+\mathcal{R}^{(0)}. \tag{134}$$

We start deriving estimates for (131). As in the proof of the $\mathcal{O}_0$ estimates,

$$\|\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\Theta\|_{\mathcal{L}^2_{(sc)}(u,0)}+\int_0^{\underline{u}}\delta^{-1}\|\nabla_4\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}$$

[12]This are the curvature components appearing in the main curvature norms $\mathcal{R}_0,\mathcal{R}_1$.

[13]such a term appear in the transport equation for μ.

[14]This are the curvature components appearing in the main curvature norms $\underline{\mathcal{R}}_0,\underline{\mathcal{R}}_1$.

Recall that none of the $\mathcal{L}^\infty_{(sc)}(S)$ norms of the Ricci coefficients ψ or the $\mathcal{L}^2_{(sc)}(S)$ norms of their derivatives $\nabla\psi$ are anomalous. Moreover,

$$\|\psi_g\|_{\mathcal{L}^4_{(sc)}(S)} + \delta^{1/4}\|\psi_g\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim {}^{(S)}\mathcal{O}_{0,4}(S) \lesssim C$$

where C is the constant in proposition 5.6. Also,

$$\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \delta^{1/2}\Delta_0, \qquad \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim {}^{(S)}\mathcal{O}_{1,2},$$

Now, according to (131), for $\delta^{1/2}\Delta_0 \lesssim 1$,

$$\begin{aligned}\|\nabla_4\Theta\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\psi\cdot\Psi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)}\cdot\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)}\\ &+ \delta^{1/2}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}\|\psi_g\|_{\mathcal{L}^4_{(sc)}(S)} + \delta\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}\|\psi_g\|_{\mathcal{L}^4_{(sc)}(S)}\\ &\lesssim \|\psi\cdot\Psi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\Delta_0\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/4}C^2\end{aligned}$$

Recalling the triviality of the initial conditions at $\underline{u}=0$, we deduce,

$$\begin{aligned}\|\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \int_0^{\underline{u}}\delta^{-1}\|\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}\,d\underline{u}'\\ &\lesssim \delta^{-1}\int_0^{\underline{u}}\|\psi\cdot\Psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}\,d\underline{u}' + \Delta_0\,\delta^{1/2}\ {}^{(S)}\mathcal{O}_{1,2} + \delta^{1/4}C^2\end{aligned}$$

Among the terms of the form $\psi\cdot\Psi$ the most dangerous[15] is $\hat{\underline{\chi}}\cdot\alpha$ which is anomalous in both ψ and Ψ. In this case, recalling estimate (102),

$$\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \delta^{-1/4}C$$

we deduce,

$$\begin{aligned}\|\hat{\underline{\chi}}\cdot\alpha\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2}\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)}\cdot\|\alpha\|_{\mathcal{L}^4_{(sc)}(S)}\\ &\lesssim \delta^{1/4}C\left(\|\nabla\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)}\cdot\|\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\alpha\|_{\mathcal{L}^2_{(sc)}(S)}\right)\end{aligned}$$

All other terms are better in powers of δ, i.e.,

$$\|\psi\cdot\Psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \delta^{1/4}C\left(\|\Psi\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)}\cdot\|\nabla\Psi\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}\right)$$

[15]This is the case for the ∇_4 equation for μ.

Therefore, recalling Remark 2 and the definition of the scale invariant norms $\mathcal{L}^2_{(sc)}(H_u)$,

$$\begin{aligned}
\delta^{-1}\int_0^{\underline{u}} \|\psi\cdot\Psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}\,d\underline{u}' &\lesssim C\delta^{-3/4}\int_0^{\underline{u}} \|\Psi\|^{1/2}_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}\|\nabla\Psi\|^{1/2}_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}\\
&\lesssim C\delta^{-1/4}\Big(\int_0^{\underline{u}} \|\Psi\|^2_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}d\underline{u}'\cdot\int_0^{\underline{u}} \|\nabla\Psi\|^2_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}d\underline{u}'\Big)^{1/2}\\
&\lesssim C\,\mathcal{R}_0^{1/2}\cdot\big(\mathcal{R}_0+\mathcal{R}_1\big)^{1/2}
\end{aligned}$$

We have thus established,

$$\|\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \delta^{1/2}\Delta_0\cdot\,{}^{(S)}\mathcal{O}_{1,2}+C\,\mathcal{R}_0^{1/2}\cdot\big(\mathcal{R}_0+\mathcal{R}_1\big)^{1/2}+\delta^{1/4}C^2 \tag{135}$$

We next estimate the Θ components which verify the ∇_3 equation (132). The only terms which do not appear in (131) are of the form, $\mathrm{tr}\underline{\chi}_0\nabla\psi$. Thus, exactly as before,

$$\|\nabla_3\Theta\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\psi\cdot\Psi\|_{\mathcal{L}^2_{(sc)}(S)}+(1+\delta^{1/2}\Delta_0)\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)}+\delta^{1/4}C^2$$

and,

In view of Remark 3 $\Psi\in\{\beta,\rho,\sigma,\underline{\beta},\underline{\alpha}\}$ and there are no double anomalous terms $\psi\cdot\Psi$. Thus, proceeding exactly as above,

$$\begin{aligned}
\|\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\Theta\|_{\mathcal{L}^2_{(sc)}(u,0)}+\int_0^u \|\nabla_3\Theta\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du'\\
&\lesssim \int_0^u \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}\,du'+\delta^{1/2}\Delta_0\cdot\,{}^{(S)}\mathcal{O}_{1,2}\\
&+ C\delta^{1/4}\underline{\mathcal{R}}_0^{1/2}(\underline{\mathcal{R}}_1+\underline{\mathcal{R}}_0)^{1/2}+C^2\delta^{1/4}
\end{aligned}$$

Combining with (135) we deduce, for a constant $C=C(\mathcal{O}^{(0)},\mathcal{R},\underline{\mathcal{R}})$ and sufficiently small δ,

$$\|\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim C+\int_0^u \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du'+\delta^{\frac12}\Delta_0\mathcal{O}_1 \tag{136}$$

It remains to discuss estimates for the Hodge systems (133). The following proposition will be needed.

Proposition 6.6. *There exists a constant $C=C(\mathcal{O}^{(0)},\mathcal{R},\underline{\mathcal{R}})$ such that if δ is sufficiently small, the following estimates hold true:*

$$\|\beta,\rho,\sigma,\underline{\beta}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C \tag{137}$$

$$\|K\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C \tag{138}$$

In view of proposition 4.17 we derive from (133),

$$\begin{aligned}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{\frac14}\|K\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}+\|\Theta\|_{\mathcal{L}^2_{(sc)}(S)}\\ &+ \|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}+\|\psi_g\|_{\mathcal{L}^2_{(sc)}(S)}+\|\psi\cdot\psi\|_{\mathcal{L}^2_{(sc)}(S)}.\end{aligned}$$

According to proposition 6.6, $\|K\|_{\mathcal{L}^2_{(sc)}(S)}\lesssim C$. Thus even if the term $\|\psi\|_{\mathcal{L}^2_{(sc)}(S)}$ multiplying $\|K\|_{\mathcal{L}^2_{(sc)}(S)}$ is anomalous[16], i.e. $\|\psi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})}\lesssim\delta^{-1/4}\ {}^{(S)}\mathcal{O}_{0,4}\lesssim C\delta^{-1/4}$ we deduce, for some $C=C(\mathcal{O}^{(0)},\mathcal{R},\underline{\mathcal{R}})$,

$$\delta^{\frac14}\|K\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C$$

Also, since $\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}\lesssim C$ for $\Psi\in\{\beta,\rho,\sigma,\underline{\beta}\}$ and $\|\psi_g\|_{\mathcal{L}^2_{(sc)}(S)}\lesssim\mathcal{O}_0[\psi_g]\lesssim C$ we deduce,

$$\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C+\|\Theta\|_{\mathcal{L}^2_{(sc)}(S)}+\|\psi\cdot\psi\|_{\mathcal{L}^2_{(sc)}(S)}.$$

Among the remaining quadratic terms $\|\psi\cdot\psi\|_{\mathcal{L}^2_{(sc)}(S)}$ we can have terms such as $\hat{\chi}\cdot\underline{\hat{\chi}}$, in which both factors are anomalous[17] . For such terms

$$\|\psi\cdot\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}\cdot\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}\lesssim C^2$$

Henceforth,

$$\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C^2+\|\Theta\|_{\mathcal{L}^2_{(sc)}(S)}$$

Combining this with (136) we deduce,

$$\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim C^2+\int_0^u\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du'+\delta^{\frac12}\Delta_0\mathcal{O}_1$$

from which, by Gronwall,

$$\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim C^2+\delta^{\frac12}\Delta_0\ {}^{(S)}\mathcal{O}_{1,2}.$$

and thus

$${}^{(S)}\mathcal{O}_{1,2}+\|\Theta\|_{\mathcal{L}^2_{(sc)}(S)}\lesssim C^2$$

as desired. We summarize the results in the following

Proposition 6.7. *Consider systems of the form* (106)*,* (107)*,* (108) *verifying the properties discussed in the Remarks 1-5 below. There exists a constant* $C=C(\mathcal{O}^{(0)},\mathcal{R},\underline{\mathcal{R}})$ *such that,*

$$\|\Theta\|_{\mathcal{L}^2_{(sc)}(S)}+\ {}^{(S)}\mathcal{O}_{1,2} \lesssim C. \tag{139}$$

[16]This situation occur only for the Hodge system div $\hat{\chi}$, see (109), since $\mathcal{O}_0[\hat{\chi}]$ is anomalous.
[17]In fact $\hat{\chi}\cdot\underline{\hat{\chi}}$ appears in the Hodge systems for η and $\underline{\eta}$, see formulas (117) and (118).

6.8. **Curvature Estimates.** In this subsection we prove proposition 6.6 concerning $\mathcal{L}^2_{(sc)}(S)$ estimates for the curvature components $\beta, \rho, \sigma, \underline{\beta}$. We also provide estimates for $\alpha, \underline{\alpha}$ which will be needed later. Recall the Bianchi identities,

$$\begin{aligned}
\nabla_4\beta + 2\text{tr}\chi\beta &= \text{div }\alpha - 2\omega\beta - (2\zeta + \underline{\eta})\alpha \\
\nabla_4\rho + \frac{3}{2}\text{tr}\chi\rho &= -\text{div }\beta + \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha - \zeta\cdot\beta - 2\underline{\eta}\cdot\beta, \\
\nabla_4\sigma + \frac{3}{2}\text{tr}\chi\sigma &= -\text{div }{}^*\beta + \frac{1}{2}\underline{\hat{\chi}}\cdot{}^*\alpha - \zeta\cdot{}^*\beta - 2\underline{\eta}\cdot{}^*\beta, \\
\nabla_4\underline{\beta} + \text{tr}\chi\underline{\beta} &= -\nabla\rho + {}^*\nabla\sigma + 2\omega\underline{\beta} + 2\underline{\hat{\chi}}\cdot\beta - 3(\underline{\eta}\rho - {}^*\underline{\eta}\sigma)
\end{aligned}$$

Thus $\beta, \rho, \sigma, \underline{\beta}$ verify equations of the form:

$$\nabla_4\Psi^{(s)} = \nabla\Psi^{(s+\frac{1}{2})} + \sum_{s_1+s_2=s+1}\psi^{(s_1)}\cdot\Psi^{(s_2)}$$

Among the curvature terms on the right we have to play special attention to multiples of the curvature term α with signature 2. We write schematically,

$$\nabla_4\Psi_g = \nabla\Psi + \psi\cdot\Psi \tag{140}$$

with $\Psi_g \in \{\beta, \rho, \sigma, \underline{\beta}\}$ while $\Psi \in \{\alpha, \beta, \rho, \sigma, \underline{\beta}\}$.

Thus,

$$\|\nabla_4\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\alpha\cdot\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)}$$

Now, as in the estimates for Θ in the previous section the worst case scenario estimate for $\|\alpha\cdot\psi\|_{\mathcal{L}^2_{(sc)}(S)}$, for anomalous ψ, has the form

$$\|\psi\cdot\alpha\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C\delta^{\frac{1}{4}}\left(\|\nabla\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)}\cdot\|\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac{1}{4}}\|\alpha\|_{\mathcal{L}^2_{(sc)}(S)}\right)$$

We deduce,

$$\begin{aligned}
\|\nabla_4\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac{1}{2}}\Delta_0\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)} \\
&+ C\delta^{\frac{1}{4}}\left(\|\nabla\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)}\cdot\|\alpha\|^{1/2}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac{1}{4}}\|\alpha\|_{\mathcal{L}^2_{(sc)}(S)}\right)
\end{aligned}$$

from which,

$$\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\Psi_g\|_{\mathcal{L}^2_{(sc)}(u,0)} + \mathcal{R}_1 + \delta^{\frac{1}{2}}\Delta_0\mathcal{R}_0 + C\mathcal{R}_0^{\frac{1}{2}}[\alpha]\cdot\mathcal{R}_1^{\frac{1}{2}}[\alpha] + C\mathcal{R}_0[\alpha]$$

Thus, since the initial data $\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(u,0)}$ is trivial

$$\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \mathcal{R}_1 + \delta^{\frac{1}{2}}\Delta_0\mathcal{R}_0 + C\mathcal{R}_0^{\frac{1}{2}}[\alpha]\mathcal{R}_1^{\frac{1}{2}}[\alpha] + C\mathcal{R}_0[\alpha]$$

or, with a new constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$,

$$\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim C \tag{141}$$

as desired.

It remains to estimate the $\mathcal{L}^2_{(sc)}(S)$ norm of the Gauss curvature

$$K = -\rho + \frac{1}{2}\hat{\chi} \cdot \underline{\hat{\chi}} - \frac{1}{4}\mathrm{tr}\chi \cdot \mathrm{tr}\underline{\chi} = \rho + \frac{1}{2}\hat{\chi} \cdot \underline{\hat{\chi}} - \frac{1}{4}\mathrm{tr}\chi \cdot \mathrm{tr}\underline{\chi}_0 - \frac{1}{4}\mathrm{tr}\chi \cdot \widetilde{\mathrm{tr}\underline{\chi}}$$

Thus,

$$\begin{aligned} \|K\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\rho\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S)} \cdot \|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} \\ &+ \|\mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}} + \delta^{1/2}\Delta_0\|\widetilde{\mathrm{tr}\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}} \\ &\lesssim C + \delta^{1/2}\Delta_0\mathcal{R}_0 \end{aligned}$$

from which the desired estimate follows.

$$\|K\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$$

as desired.

In the next proposition we derive estimates for the remaining curvature components.

Proposition 6.9. *There exists a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ such that for $\delta^{\frac{1}{2}}\Delta_0$ sufficiently small*

$$\|\alpha\|_{\mathcal{L}^2_{(sc)}(S)} \le C\delta^{-\frac{1}{2}}, \qquad \|\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(S)} \le C$$

Proof. To prove the estimate for α we use the Bianchi equation for $\nabla_3\alpha$, which can be written schematically in the form

$$\nabla_3\alpha = \mathrm{tr}\underline{\chi}_0 \cdot \alpha + \psi \cdot \alpha + \nabla\Psi + \psi \cdot \Psi$$

with Ψ from the set not containing α. We therefore obtain

$$\begin{aligned} \|\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} &\lesssim \|\alpha\|_{\mathcal{L}^2_{(sc)}(S_{0,\underline{u}})} + (1 + \delta^{\frac{1}{2}}\Delta_0)\int_0^u \|\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u',\underline{u}})}du' \\ &+ \underline{\mathcal{R}}_1 + \delta^{\frac{1}{2}}\Delta_0\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}. \end{aligned}$$

In the worst case when $\Psi = \beta$, which is anomalous, we have, $\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})} \lesssim \delta^{-\frac{1}{2}}\underline{\mathcal{R}}_0$. Thus, by Gronwall,

$$\|\alpha\|_{\mathcal{L}^2_{(sc)}(S_{u,\underline{u}})} \lesssim \|\alpha\|_{\mathcal{L}^2_{(sc)}(S_{0,\underline{u}})} + \underline{\mathcal{R}}$$

Similarly, the equation for $\nabla_4\underline{\alpha}$ has the form

$$\nabla_4\underline{\alpha} = \nabla\Psi + \psi \cdot \Psi,$$

where the curvature term in $\nabla\Psi$ is not $\underline{\alpha}$ and $\Psi \neq \alpha$ in the nonlinear term. Therefore, using the triviality of initial data

$$\|\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \mathcal{R}_1 + \delta^{\frac{1}{2}}\Delta_0 \int_0^u \big(\|\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})} + \|\Psi_g\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}\big)du'$$

with $\Psi_g \in \rho, \sigma, \underline{\beta}$. The result follows then easily by Gronwall and the $\mathcal{L}^2_{(sc)}(H)$ curvature bounds for Ψ_g. $\square$

7. Second angular derivative estimates for the Ricci coefficients

To derive second angular derivative estimates for the Ricci coefficients we differentiate (106), (107) and (108) with respect to ∇.

7.1. **Basic equations.** Based on the experience with the first derivative estimates we expect that the ∇_3 equation for $\nabla\Theta$ is slightly more challenging as it contains a lot more $\mathrm{tr}\underline{\chi}$ terms. Thus, differentiating (132) we derive,

$$\begin{aligned}
\nabla_3\nabla\Theta &= \mathrm{tr}\underline{\chi}_0\big(\nabla\Theta + \nabla\Psi + \nabla^2\psi\big) + \psi\cdot\big(\nabla\Theta + \nabla\Psi + \nabla^2\psi\big) + \nabla\psi\cdot\big(\Theta + \Psi + \nabla\psi\big)\\
&+ \mathrm{tr}\underline{\chi}_0\psi\cdot\nabla\psi + \psi\cdot\psi\cdot\nabla\psi + [\nabla_3,\nabla]\Theta
\end{aligned}$$

According to commutation formulae of lemma (3.3) we write symbolically,

$$\begin{aligned}
[\nabla_3,\nabla]\Theta &= \mathrm{tr}\underline{\chi}\cdot\nabla\Theta + \hat{\underline{\chi}}\cdot\nabla\Theta + \Psi\cdot\Theta + \mathrm{tr}\underline{\chi}\cdot\psi\cdot\Theta + \psi\cdot\psi\cdot\Theta + \psi\cdot\nabla_3\Theta\\
&= \mathrm{tr}\underline{\chi}_0\nabla\Theta + \psi\cdot\nabla\Theta + \Psi\cdot\Theta + \mathrm{tr}\underline{\chi}_0\cdot\psi\cdot\Theta + \psi\cdot\psi\cdot\Theta + \psi\cdot\nabla_3\Theta
\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla_3\nabla\Theta &= \mathrm{tr}\underline{\chi}_0\big(\nabla\Theta + \nabla\Psi + \nabla^2\psi\big) + \psi\cdot\big(\nabla\Theta + \nabla\Psi + \nabla^2\psi\big) + \nabla\psi\cdot\big(\Theta + \Psi + \nabla\psi\big)\\
&\quad + \Theta\cdot\Psi + \mathrm{tr}\underline{\chi}_0\big(\psi\cdot\nabla\psi + \psi\cdot\Theta\big) + \psi\cdot\big(\psi\cdot\nabla\psi + \psi\cdot\Theta + \nabla_3\Theta\big)
\end{aligned}$$

Ignoring the term of the form $\mathrm{tr}\underline{\chi}_0\nabla\Theta$ which can be easily eliminated by Gronwall, and observing that Θ and $\nabla\Theta$ on the left can be can be expressed in terms of $\nabla\psi$ and Ψ, respectively, $\nabla^2\psi$ and $\nabla\Psi$, we write,

$$\begin{aligned}
\nabla_3\nabla\Theta &= (\mathrm{tr}\underline{\chi}_0 + \psi)(\nabla\Psi + \nabla^2\psi) + \mathrm{tr}\underline{\chi}_0\big(\psi\cdot\nabla\psi + \psi\cdot\Theta\big) + \nabla\psi\cdot\big(\Psi + \nabla\psi\big)\\
&\quad + \Theta\cdot\Psi + \psi\cdot\big(\psi\cdot\nabla\psi + \psi\cdot\Theta + \nabla_3\Theta\big)\\
&= F_1 + F_2 + F_3 + F_4 + F_5
\end{aligned} \tag{142}$$

Similarly,

$$\nabla_4\nabla\Theta = \psi\cdot\big(\nabla\Psi + \nabla^2\psi\big) + \nabla\psi\cdot\big(\Psi + \nabla\psi\big) + \Theta\cdot\Psi + \psi\cdot\psi\cdot\nabla\psi + [\nabla_4,\nabla]\Theta$$

and

$$[\nabla_4,\nabla]\Theta = \psi\cdot\nabla\Theta + \Psi\cdot\Theta + \psi\cdot\psi\cdot\Theta + \psi\cdot\nabla_4\Theta$$

so that[18]

$$\nabla_4 \nabla \Theta = \psi \cdot \big(\nabla \Psi + \nabla^2 \psi\big) + \nabla \psi \cdot \big(\Psi + \nabla \psi\big) + \psi \cdot (\psi \cdot \nabla \psi + \psi \cdot \Theta + \nabla_4 \Theta) \\ = G_1 + G_3 + G_4 + G_5 \tag{143}$$

Equations (142),(143) will be combined with the differentiated Hodge system for ψ in (133):

$$\mathcal{D}^* \mathcal{D} \psi \;\; = \;\; \mathcal{D}^* \Big(\Theta + \Psi + \psi \cdot \psi + \mathrm{tr} \underline{\chi}_0 \cdot \psi \Big), \tag{144}$$

which can be schematically written in the form

$$\Delta \psi = K \psi + \nabla \Theta + \nabla \Psi + \nabla \psi \cdot \psi + \mathrm{tr} \underline{\chi}_0 \cdot \nabla \psi$$

7.2. **Estimates for $\nabla \Theta$, $\nabla^2 \psi$.** We now collect estimates for the terms on the right hand side of the transport equations (142),(143):

$$\begin{aligned}
\|F_1\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim (1 + \delta^{1/2} \Delta_0)\big(\|\nabla^2 \psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla \Psi\|_{\mathcal{L}^2_{(sc)}(S)}\big) \\
\|F_2\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \Delta_0 (\|\Theta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla \psi\|_{\mathcal{L}^2_{(sc)}(S)}) \\
\|F_3\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \|\nabla \psi\|_{\mathcal{L}^4_{(sc)}(S)} \cdot \big(\|\nabla \psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\Psi\|_{\mathcal{L}^4_{(sc)}(S)}\big) \\
\|F_4\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \|\Theta\|_{\mathcal{L}^4_{(sc)}(S)} \cdot \|\Psi\|_{\mathcal{L}^4_{(sc)}(S)} \\
\|F_5\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \Delta_0 \big(\delta^{\frac{1}{2}} \Delta_0 \|(\nabla \psi, \Theta)\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3 \Theta\|_{\mathcal{L}^2_{(sc)}(S)}\big)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|G_1\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \Delta_0 \left(\|\nabla \Theta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla^2 \psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla \Psi\|_{\mathcal{L}^2_{(sc)}(S)}\right) \\
\|G_3\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \|\nabla \psi\|_{\mathcal{L}^4_{(sc)}(S)} \cdot \big(\|\nabla \psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\Psi\|_{\mathcal{L}^4_{(sc)}(S)}\big) \\
\|G_4\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \|\Theta\|_{\mathcal{L}^4_{(sc)}(S)} \cdot \|\Psi\|_{\mathcal{L}^4_{(sc)}(S)} \\
\|G_6\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{1/2} \Delta_0 \cdot \big(\delta^{\frac{1}{2}} \Delta_0 \cdot \|(\nabla \psi, \Theta)\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4 \Theta\|_{\mathcal{L}^2_{(sc)}(S)}\big)
\end{aligned}$$

We note that the curvature terms Ψ present in the F terms belong to the admissible set $\{\beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}$ while the curvature terms Ψ appearing in the G terms belong to the set $\{\alpha, \beta, \rho, \sigma, \underline{\beta}\}$. We also recall that according to the ${}^{(S)}\mathcal{O}_{1,2}$ estimates and their consequences proved in the previous section

$$\|\nabla \psi\|_{L^2(S)} + \|\Theta\|_{L^2(S)} + \|\nabla_4 \Theta\|_{L^2(H)} + \|\nabla_3 \Theta\|_{L^2(\underline{H})} \le C$$

[18]Observe that the structure of

Using the L^4 interpolation estimate from (82) which imply that

$$\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\nabla\psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla^2\psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C\|\nabla^2\psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}C,$$
$$\|\Theta\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\Theta\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\Theta\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\Theta\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C\|\nabla\Theta\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}C,$$
$$\|\Psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\Psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\Psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}$$

we obtain for $\delta^{\frac12}\Delta_0$ sufficiently small

$$\begin{aligned}\|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \int_0^u \|\nabla_3\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \\ &\lesssim \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + C\int_0^u \big(\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}} + \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}}\big)du' \\ &+ \delta^{\frac12}C\int_0^u \left(\|\nabla^2\psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\Psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\Psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\nabla^2\psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\Psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\right)du' \\ &+ \delta^{\frac12}C\int_0^u \left(\|\nabla\Theta\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\Psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\Psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\nabla\Theta\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}\right)du' + \delta^{\frac12}C\end{aligned}$$

We kept track of the terms containing $\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)}$ as they may lead to the potentially anomalous norm $\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$ in the case of $\Psi = \beta$. However, even in that case

$$\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \delta^{-\frac12}\underline{\mathcal{R}}_0$$

By Gronwall, and recalling the definition[19] of $\underline{\mathcal{R}}_1$

$$\|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + C\int_0^u \|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + C\underline{\mathcal{R}}_1. \tag{145}$$

In view of the estimates for the G terms we similarly obtain

$$\|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,0)} + C\int_0^{\underline{u}} \|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u}')}d\underline{u}' + C\mathcal{R}_1. \tag{146}$$

We now couple this with the second derivative estimates for the Hodge system

$$D\psi = \Theta + \Psi + \mathrm{tr}\underline{\chi}_0\psi + \psi\cdot\psi.$$

Using Proposition 4.18 we deduce

$$\begin{aligned}\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(S)} \;&\lesssim\; \delta^{\frac12}\|K\|_{\mathcal{L}^2_{(sc)}(S)}\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} + \delta^{\frac14}\|K\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} \\ &+\; \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\mathrm{tr}\underline{\chi}_0\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\psi\cdot\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)}\end{aligned}$$

[19] note again that $\underline{\alpha}$ does not appear among the Ψ's

By Proposition 6.6, $\|K\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C$ with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$. Therefore,

$$\begin{aligned}\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \delta^{\frac{1}{2}} C\Delta_0 + \delta^{\frac{1}{4}} C\left(\|\nabla^2\psi\|^{\frac{1}{2}}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\psi\|^{\frac{1}{2}}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac{1}{4}}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)}\right)\\ &+ \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac{1}{2}} C\Delta_0\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)}\end{aligned}$$

Using Cauchy-Schwarz and the boundedness of the ${}^{(S)}\mathcal{O}_{1,2}$ norm we then obtain

$$\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C + \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(S)}. \tag{147}$$

We note that the curvature terms Ψ involved in the above inequality belong to the set $\{\beta, \rho, \sigma, \underline{\beta}\}$. In particular,

$$\|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \mathcal{R}_1, \qquad \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \underline{\mathcal{R}}_1.$$

Thus, substituting the estimate for $\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(S)}$ into (145) and (146) and using Gronwall we obtain

$$\begin{aligned}\|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + C\underline{\mathcal{R}}_1,\\ \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,0)} + C\mathcal{R}_1.\end{aligned}$$

This, together with (147), in turn, implies

Proposition 7.3. *There exists a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ such that all second derivatives $\nabla^2\psi$ of the Ricci coefficients $\psi \in \{tr\chi, \hat{\chi}, \eta, \underline{\eta}, \omega, \underline{\omega}, \underline{\hat{\chi}}, \widetilde{tr\underline{\chi}}\}$ and the first derivatives of the quantities $\Theta \in \{\nabla tr\chi,\ div\,\eta + \rho,\ div\,\underline{\eta} + \rho,\ \nabla\omega + {}^*\nabla\omega^\dagger - \frac{1}{2}\beta,\ -\nabla\underline{\omega} + {}^*\nabla\underline{\omega}^\dagger - \frac{1}{2}\underline{\beta},\ \nabla tr\underline{\chi}\}$ verify,*

$$\|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(H_u)} + \|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}})} \lesssim C.$$

7.4. ${}^{(S)}\mathcal{O}_{1,4}$ **estimates.** As a corollary of proposition 7.3, together with corollary 4.12 we also have,

Corollary 7.5. *There exists a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ such that, for $\delta^{1/2}\Delta_0$ sufficiently small,*

$${}^{(S)}\mathcal{O}_{1,4} \lesssim C. \tag{148}$$

We end this section by deriving a slightly more refined estimate on the second angular derivatives of η. These estimates are needed in the application to the problem of formation of a trapped surface. We review the system of equations for η, written schematically it has the form

$$\begin{aligned}&\text{curl}\ \eta = \sigma + \psi\cdot\psi, \qquad \text{div}\ \eta = -\mu - \rho,\\ &\nabla_4\mu = \psi\cdot(\nabla\psi + \Theta + \Psi + \psi\cdot\psi).\end{aligned}$$

We note the absence of $\text{tr}\underline{\chi}_0$ terms in this system. Applying $\mathcal{D}^*$ to the Hodge system for η and commuting the equation for μ with ∇ we obtain

$$\begin{aligned}&\Delta\eta = \nabla\sigma + \nabla\rho + \nabla\mu + \nabla\psi\cdot\psi + K\eta,\\ &\nabla_4\nabla\mu = \nabla\psi\cdot(\nabla\psi + \Theta + \Psi + \psi\cdot\psi) + \psi\cdot(\nabla^2\psi + \nabla\Theta + \nabla\Psi + \nabla\psi\cdot\psi)\end{aligned}$$

The absence of $\mathrm{tr}\underline{\chi}_0$ terms allows us to estimate $\nabla\mu$ in terms of its (trivial) data on H_0 and an error term of size $\delta^{\frac{1}{2}}$. To show that we bound

$$\begin{aligned}
&\|\nabla\psi\cdot(\nabla\psi+\Theta+\Psi+\psi\cdot\psi)\|_{\mathcal{L}^2_{(sc)}(H_u)}\\
&\lesssim \delta^{\frac{1}{2}}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)}\Big(\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)}+\|\Theta\|_{\mathcal{L}^4_{(sc)}(S)}+\|\Psi\|_{\mathcal{L}^4_{(sc)}(S)}+\delta^{\frac{1}{2}}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)}\Big)\\
&+\delta^{\frac{1}{2}}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\Big(\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(H_u)}+\delta^{\frac{1}{2}}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H_u)}\Big)\\
&\lesssim \delta^{\frac{1}{2}}C
\end{aligned}$$

In the final estimate the only dangerous term is $\|\Psi\|_{\mathcal{L}^4_{(sc)}(S)}$, which may be $\delta^{-\frac{1}{4}}$ anomalous in the case of $\Psi=\alpha$. It is not difficult to check however that $\Psi=\alpha$ does not appear in this system but even if it did the size of the error term would have been $\delta^{\frac{1}{4}}$ instead of $\delta^{\frac{1}{2}}$. As a result of this estimate and the trivial data for $\nabla\mu$ we obtain

$$\|\nabla\mu\|_{\mathcal{L}^2_{(sc)}(S)}\lesssim\delta^{\frac{1}{2}}C.$$

To estimate η we remember that $K=\rho+\mathrm{tr}\underline{\chi}_0\cdot\psi_g+\psi\cdot\psi$. Therefore,

$$\begin{aligned}
&\|\Delta\eta\|_{\mathcal{L}^2_{(sc)}(H_u)}\lesssim\|\nabla\rho\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\nabla\sigma\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\nabla\mu\|_{\mathcal{L}^2_{(sc)}(H_u)}\\
&+\delta^{\frac{1}{2}}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\Big(\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\rho\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\psi_g\|_{\mathcal{L}^2_{(sc)}(H_u)}+\delta^{\frac{1}{2}}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\cdot\|\psi\|_{\mathcal{L}^2_{(sc)}(H_u)}\Big)\\
&\lesssim\|\nabla\rho\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\nabla\sigma\|_{\mathcal{L}^2_{(sc)}(H_u)}+\delta^{\frac{1}{2}}C.
\end{aligned}$$

Using the Böchner identity we obtain

$$\begin{aligned}
\|\nabla^2\eta\|_{\mathcal{L}^2_{(sc)}(H_u)}&\lesssim\|\Delta\eta\|_{\mathcal{L}^2_{(sc)}(H_u)}+\delta^{\frac{1}{2}}\|K\|_{\mathcal{L}^2_{(sc)}(H_u)}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}+\delta^{\frac{1}{4}}\|K\|^{\frac{1}{2}}_{\mathcal{L}^2_{(sc)}(H_u)}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)}\\
&\lesssim\|\nabla\rho\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\nabla\sigma\|_{\mathcal{L}^2_{(sc)}(H_u)}+\delta^{\frac{1}{4}}C.
\end{aligned}$$

The same estimates also hold along the $\underline{H}_{\underline{u}}$ hypersurfaces.

We summarize this in a proposition.

Proposition 7.6. *The Ricci coefficient η verifies the estimate*

$$\begin{aligned}
\|\nabla^2\eta\|_{\mathcal{L}^2_{(sc)}(H_u)}&\lesssim\|\nabla\rho\|_{\mathcal{L}^2_{(sc)}(H_u)}+\|\nabla\sigma\|_{\mathcal{L}^2_{(sc)}(H_u)}+\delta^{\frac{1}{4}}C,\\
\|\nabla^2\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}&\lesssim\|\nabla\rho\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}+\|\nabla\sigma\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}+\delta^{\frac{1}{4}}C.
\end{aligned}$$

8. Remaining first and second derivative estimates

In the previous sections we have derived estimates on the first and second angular derivatives of the Ricci coefficients. In this section examine their ∇_3, ∇_4, $\nabla\nabla_4$ and $\nabla\nabla_3$ derivatives.

8.1. **Direct ∇_3, ∇_4 estimates.** These are derived directly from the null structure equations (see section 3.1).

Proposition 8.2. *There exists a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ such that for $\delta^{\frac{1}{2}}\Delta_0$ sufficiently small and any $S = S_{u,\underline{u}}$*

$$\|\nabla_4 tr\chi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\eta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4 tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} \le C,$$
$$\|\nabla_3 \widetilde{tr\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3\omega\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3 tr\chi\|_{\mathcal{L}^2_{(sc)}(S)} \le C,$$
$$\|\nabla_4\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(S)} \le C\,\delta^{-\frac{1}{2}}.$$

Remark. Note the anomalous estimates of the last line. The anomaly of $\nabla_4\hat{\chi}$ is due to the curvature term α in the second equation in (49). The anomaly of $\nabla_3\underline{\hat{\chi}}$ is due to the term $\mathrm{tr}\underline{\chi}\cdot\underline{\hat{\chi}}$ in the fourth equation in (49) . The anomalies for $\nabla_3\hat{\chi}$ and $\nabla_4\underline{\hat{\chi}}$ are explained by the presence of $\mathrm{tr}\underline{\chi}\hat{\chi}$ in both equations of (50).

Proof. The claimed estimates follow directly from all the estimates derived so far. We need the full set of $\|\Psi\|_{L^2(S)}$ estimates for all null curvature components Ψ which were derived in propositions 6.6 and 6.9. We also need to make use of the ${}^{(S)}\mathcal{O}_{0,2}$ estimates of proposition 5.8. As an example we prove the estimate for $\nabla_4\hat{\chi}$ in more detail. We start with $\nabla_4\hat{\chi} = -\mathrm{tr}\chi\hat{\chi} - 2\omega\hat{\chi} - \alpha$ which we write in the form,

$$\nabla_4\hat{\chi} = \psi_g\cdot\hat{\chi} + \alpha$$

As a result,

$$\begin{aligned}\|\nabla_4\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\psi_g\cdot\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\alpha\|_{\mathcal{L}^2_{(sc)}(S)}\\ &\lesssim \delta^{1/2}\|\psi_g\|_{\mathcal{L}^4_{(sc)}(S)}\cdot\|\hat{\chi}\|_{\mathcal{L}^4_{(sc)}} + \|\alpha\|_{\mathcal{L}^2_{(sc)}(S)}\\ &\lesssim \delta^{1/4}\ {}^{(S)}\mathcal{O}^2_{0,4} + C\delta^{-1/2} \lesssim C\delta^{-1/2}\end{aligned}$$

as desired. Similarly we write,

$$\nabla_3\hat{\chi} \quad = \quad \mathrm{tr}\underline{\chi}_0\cdot\psi_b + \psi_g\cdot\psi_b + \nabla\psi + \Psi_g,$$

with ψ_g, Ψ_g non- anomalous and ψ_b anomalous. Hence,

$$\begin{aligned}\|\nabla_3\hat{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\psi_b\|_{\mathcal{L}^2_{(sc)}(S)} + \|\psi_g\|_{\mathcal{L}^4_{(sc)}(S)}\cdot\|\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)}\\ &\lesssim \delta^{-1/2}C + \delta^{-1/4}C^2 + C\end{aligned}$$

More generally, all of our null structure equations have the form

$$\nabla_4\psi = \mathrm{tr}\underline{\chi}_0\cdot\psi + \psi\cdot\psi + \nabla\psi + \Psi,$$
$$\nabla_3\psi = \mathrm{tr}\underline{\chi}_0\cdot\psi + \psi\cdot\psi + \nabla\psi + \Psi,$$

and one can easily see that the only anomalies occur for ∇_3, ∇_4 of $\chi, \underline{\hat{\chi}}$. □

8.3. **Estimates for $\nabla_3\eta, \nabla_4\underline{\eta}, \nabla_3\underline{\omega}, \nabla_4\omega$.** The above proposition does not address the fate of $\nabla_3\eta, \nabla_4\underline{\eta}, \nabla_3\underline{\omega}$ and $\nabla_4\omega$ derivatives which do not appear in the null structure equations. These can be estimated by commuting the valid transport equations for these quantities with the desired derivative.

Proposition 8.4. *There exists a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ such that for $\delta^{\frac12}\Delta_0$ sufficiently small*

$$\|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\omega\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3\eta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(S)} \le C.$$

Proof. As all the arguments are similar we will only derive the estimate for $\nabla_4\underline{\eta}$. Commuting the transport equation

$$\nabla_3\underline{\eta} = -\frac{1}{2}\mathrm{tr}\underline{\chi}(\underline{\eta}-\eta) - \underline{\hat{\chi}}\cdot(\underline{\eta}-\eta) + \underline{\beta}$$

with ∇_4 (according to Lemma 3.3) we obtain

$$\begin{aligned}
\nabla_3(\nabla_4\underline{\eta}) &= -\frac{1}{2}\nabla_4\mathrm{tr}\underline{\chi}(\underline{\eta}-\eta) - \frac{1}{2}\mathrm{tr}\underline{\chi}\nabla_4(\underline{\eta}-\eta)\\
&- \nabla_4\underline{\chi}\cdot(\underline{\eta}-\eta) - \underline{\chi}\cdot\nabla_4(\underline{\eta}-\eta) + \nabla_4\underline{\beta}\\
&- 2(\underline{\eta}-\eta)\cdot\nabla\underline{\eta} + 2\underline{\omega}\nabla_4\underline{\eta} - 2\omega\nabla_3\underline{\eta} - 2(\eta_a\underline{\eta}_b - \underline{\eta}_b\underline{\eta}_a - \in_{ab}\sigma)\underline{\eta}_b
\end{aligned}$$

which we write symbolically,

$$\begin{aligned}
\nabla_3(\nabla_4\underline{\eta}) &= \mathrm{tr}\underline{\chi}_0\cdot(\nabla_4\psi_g + \nabla_4\underline{\eta} + \psi\cdot\psi_g) + \psi\cdot(\nabla_4\psi + \nabla_4\underline{\eta})\\
&+ \psi\cdot(\nabla\psi + \Psi_g + \psi\cdot\psi_g) + \nabla_4\underline{\beta}
\end{aligned}$$

Remark. In the above expression, $\nabla_4\psi$ denotes quantities already controlled according to the previous proposition and, among them, $\nabla_4\psi_g$ denote those which are not anomalous. Also Ψ_g is a curvature component different from α. Furthermore we can eliminate $\nabla_4\beta$ according to the null Bianchi equations

$$\nabla_4\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\nabla\rho +^* \nabla\sigma + 2\omega\underline{\beta} + 2\underline{\hat{\chi}}\cdot\beta - 3(\underline{\eta}\rho -^* \underline{\eta}\sigma)$$

Thus,

$$\begin{aligned}
\nabla_3(\nabla_4\underline{\eta}) &= \mathrm{tr}\underline{\chi}_0\cdot(\nabla_4\psi_g + \nabla_4\underline{\eta} + \psi\cdot\psi_g) + \psi\cdot(\nabla_4\psi + \nabla_4\underline{\eta})\\
&+ \psi\cdot(\nabla\psi + \Psi_g + \psi\cdot\psi_g) + \nabla\Psi_g.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\nabla_3(\nabla_4\underline{\eta})\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim (1+\delta^{1/2}\Delta_0)\|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\psi_g\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\Delta_0|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(S)}\\
&+ \|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)}\big(\|\psi_g\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)}\big) + \|\nabla\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)}\\
&\lesssim (1+\delta^{1/2}\Delta_0)\|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)} + C
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S(0,\underline{u})} + \int_0^u \|\nabla_3\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \\
&\lesssim \|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + (1+\delta^{\frac{1}{2}}\Delta_0)\int_0^u \|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \\
&+ \int_0^u \|\nabla\Psi_g\|_{\mathcal{L}_{(sc)}(u',\underline{u})}du' + C \\
&\lesssim \mathcal{O}^{(0)} + (1+\delta^{\frac{1}{2}}\Delta_0)\int_0^u \|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + \underline{\mathcal{R}}_1 + C
\end{aligned}$$

Thus by Gronwall,

$$\|\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \mathcal{O}^{(0)} + C.$$

□

8.5. **Direct angular derivative estimates.** Here we derive angular derivative estimates for all the quantities which appear in proposition 8.2. We shall first prove the following:

Lemma 8.6. *If $\delta^{1/2}\Delta_0$ is small we have with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$, for all Ricci coefficients ψ,*

$$\begin{aligned}
\|[\nabla_4,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim C \\
\|[\nabla_4,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim C
\end{aligned}$$

As a corollary we also have,

$$\begin{aligned}
\|[\nabla_4,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|[\nabla_4,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C, \\
\|[\nabla_3,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|[\nabla_3,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C
\end{aligned}$$

Proof. We write,

$$\begin{aligned}
&[\nabla_4,\nabla]\psi = \psi\cdot\nabla\psi + \beta\cdot\psi + \psi_g\nabla_4\psi, \\
&[\nabla_3,\nabla]\psi = \mathrm{tr}\underline{\chi}_0\cdot\nabla\psi + \psi\cdot\nabla\psi + \underline{\beta}\cdot\psi + \psi_g\nabla_3\psi,
\end{aligned}$$

Hence, in view of the previous estimates ${}^{(S)}\mathcal{O}_{1,2} \lesssim C$, $\|\beta\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C$ and the possibly anomalous estimate $\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C\delta^{-1/2}$, we derive,

$$\|[\nabla_4,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \delta^{\frac{1}{2}}\Delta_0\Big(\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\beta\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(S)}\Big) \lesssim C$$

Similarly,

$$\begin{aligned}\|[\nabla_3,\nabla]\psi\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim (1+\delta^{1/2}\Delta_0)\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)}\\ &+ \delta^{\frac12}\Delta_0\Big(\|\underline{\beta}\|_{\mathcal{L}^2_{(sc)}(S)}+\|\nabla_3\psi\|_{\mathcal{L}^2_{(sc)}(S)}\Big)\lesssim C.\end{aligned}$$

from which the estimates of the lemma quickly follow by integration. □

Proposition 8.7. *There exists a constant* $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ *such that for* $\delta^{\frac12}\Delta_0$ *sufficiently small*

$$\begin{aligned}&\|\nabla\nabla_4\chi\|_{\mathcal{L}^2_{(sc)}(H)}+\|\nabla\nabla_4\eta\|_{\mathcal{L}^2_{(sc)}(H)}+\|\nabla\nabla_4\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(H)}+\|\nabla\nabla_4\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)}\le C,\\ &\|\nabla\nabla_4 tr\chi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\nabla\nabla_4\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\nabla\nabla_4\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\nabla\nabla_4\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\le C,\\ &\|\nabla\nabla_3\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\nabla\nabla_3\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\nabla\nabla_3\omega\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\nabla\nabla_3\chi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\le C,\\ &\|\nabla\nabla_3\widetilde{tr\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(H)}+\|\nabla\nabla_3\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H)}+\|\nabla\nabla_3\omega\|_{\mathcal{L}^2_{(sc)}(H)}+\|\nabla\nabla_3\chi\|_{\mathcal{L}^2_{(sc)}(H)}\le C\end{aligned}$$

Remark 8.8. Note the absence of anomalies. This is analogous to the situation with ${}^{(S)}\mathcal{O}_{1,2}$ estimates: additional ∇ derivatives eliminate the anomalies due α and Ricci coefficients $\hat\chi, \underline{\hat\chi}$.

Remark 8.9. The quantities $\nabla\nabla_4\hat\chi$ and $\nabla\nabla_3\underline{\hat\chi}$ are controlled only along H and $\underline{H}$ respectively. This is due to the absence of the corresponding estimates for $\nabla\alpha$ and $\nabla\underline{\alpha}$ along $\underline{H}$ and H respectively.

Remark 8.10. As a consequence of the Lemma above the same estimates hold true if we reverse the order of differentiation.

Proof. Consider the ∇_4 transport equations verified by $\psi \in \{\mathrm{tr}\chi, \hat\chi, \underline{\omega}, \eta, \widetilde{\mathrm{tr}\underline{\chi}}, \underline{\hat\chi}\}$

$$\nabla_4\psi = \mathrm{tr}\underline{\chi}_0\cdot\psi + \psi\cdot\psi + \nabla\psi + \Psi_4,$$

with curvature components $\Psi_4 \in \{\alpha, \beta, \rho, \sigma\}$. Clearly,

$$\begin{aligned}\|\nabla\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(H)} &\lesssim \big(\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(H)}+\|\nabla\Psi_4\|_{\mathcal{L}^2_{(sc)}(H)}\big)+(1+\delta^{\frac12})\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)}\\ &\lesssim C.\end{aligned}$$

Also, along $\underline{H}$,

$$\begin{aligned}\|\nabla\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim \big(\|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\nabla\Psi_4\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)+(1+\delta^{\frac12})\|\nabla\psi\||_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &\lesssim C\end{aligned}$$

provided that $\Psi_4 \neq \alpha$, (i.e. the original ψ on the left is not $\hat\chi$).

On the other hand the ∇_3 transport equations verified by $\psi \in \{\mathrm{tr}\chi, \hat\chi, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \underline{\hat\chi}, \omega\}$ are of the form,

$$\nabla_3\psi = \mathrm{tr}\underline{\chi}_0\cdot\psi + \psi\cdot\psi + \nabla\psi + \Psi_3,$$

with the curvature components $\Psi_3 \in \{\rho, \sigma, \underline{\beta}, \underline{\alpha}\}$. The corresponding estimates follow precisely in the same manner. □

8.11. **Estimates for $\nabla\nabla_3\eta, \nabla\nabla_4\underline{\eta}, \nabla\nabla_3\underline{\omega}$, $\nabla\nabla_4\omega$.** In this subsection we prove the following:

Proposition 8.12. *There exists a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ such that*

$$\|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla\nabla_3\eta\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla\nabla_3\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \le C,$$

Remark 8.13. Together with the previous proposition, this proposition allows us to control all angular derivatives of all ∇_3, ∇_4 derivatives of all the Ricci coefficients $\mathrm{tr}\underline{\chi}, \hat{\chi}, \underline{\omega}, \eta, \underline{\eta}, \widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}, \omega$ (in some $\mathcal{L}^2_{(sc)}(H)$ or $\mathcal{L}^2_{(sc)}(\underline{H})$ or both) except for $\nabla\nabla_4\omega$ and $\nabla\nabla_3\underline{\omega}$.

Proof. To control $\nabla\nabla_3\eta, \nabla\nabla_4\underline{\eta}$ we make use of lemma 6.3. Recall that reduced mass aspect functions μ and $\underline{\mu}$ verify equations of the form,

$$\begin{aligned} \nabla_4\mu &= \psi\cdot\big(\nabla\mathrm{tr}\chi + \nabla\psi + \Psi_4\big) + \psi\cdot\psi\cdot\psi_g \\ \nabla_3\underline{\mu} &= \mathrm{tr}\underline{\chi}_0\cdot\big(\nabla\mathrm{tr}\underline{\chi} + \nabla\psi\big) + \psi\cdot\big(\nabla\mathrm{tr}\underline{\chi} + \nabla\psi + \Psi_3\big) \\ &+ \mathrm{tr}\underline{\chi}_0\cdot\psi\cdot\psi_g + \psi\cdot\psi\cdot\psi_g \end{aligned} \tag{149}$$

which are to be coupled with the Hodge systems of the form

$$\mathcal{D}(\eta, \underline{\eta}) = (\mu, \underline{\mu}) + \rho + \sigma + \psi\cdot\psi. \tag{150}$$

Here $\Psi_4 = \{\alpha\,\beta, \rho, \sigma\}$ and $\Psi_3 = \{\underline{\alpha}, \underline{\beta}, \rho, \sigma\}$.

Remark. We note absence of the Ricci coefficients $\omega, \underline{\omega}$ among the ψ variables in the above equations, in particular among the terms of the form $\nabla\psi$. This fact is very important in view of the lack of estimates for $\nabla\nabla_4\omega$ and $\nabla\nabla_3\underline{\omega}$. Equally important is the absence of the terms $\mathrm{tr}\underline{\chi}_0\cdot\psi$ with $\psi = \{\hat{\chi}, \hat{\underline{\chi}}\}$ in equation (149). Such terms would lead to an unmanageable double anomaly.

To estimate $\nabla\nabla_4\underline{\eta}$ we need to commute the above equations for $\underline{\eta}, \underline{\mu}$ with ∇_4. Making use of lemma 3.3 we derive,

$$\begin{aligned} \nabla_3(\nabla_4\underline{\mu}) &= \nabla_4\mathrm{tr}\underline{\chi}_0\cdot\big(\nabla\mathrm{tr}\underline{\chi} + \nabla\psi\big) + \mathrm{tr}\underline{\chi}_0\cdot\big(\nabla_4\nabla\mathrm{tr}\underline{\chi} + \nabla_4\nabla\psi\big) + \psi\cdot\nabla\underline{\mu} \\ &+ \nabla_4\psi\cdot\big(\nabla\mathrm{tr}\underline{\chi} + \nabla\psi + \Psi_3\big) + \psi\cdot\big(\nabla_4\nabla\mathrm{tr}\underline{\chi} + \nabla_4\nabla\psi + \nabla_4\Psi_3 + \nabla_4\underline{\eta}\big) \\ &+ \nabla_4\mathrm{tr}\underline{\chi}_0\cdot\psi\cdot\psi_g + \mathrm{tr}\underline{\chi}_0\cdot\nabla_4\psi\cdot\psi + \nabla_4\psi\cdot\psi\cdot\psi + \underline{\omega}\nabla_4\underline{\mu} + \omega\nabla_3\underline{\mu} \\ \mathcal{D}(\nabla_4\underline{\eta}) &= \nabla_4\underline{\mu} + \nabla_4(\rho, \sigma) + \psi\cdot(\nabla_4\psi + \nabla\underline{\eta} + \Psi_4) \end{aligned}$$

Proceeding as many times before, we write,

$$\|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \int_0^u \|\nabla_3\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}$$

and (with $\underline{H}(u,\underline{u}) = H_u^{0,\underline{u}}$)

$$\begin{aligned}\int_0^u \|\nabla_3\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' &\lesssim \int_0^u \|\nabla_4\nabla\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + \int_0^u \|\underline{\omega}\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u}))}du'\\ &+ \|\nabla_4\psi\cdot\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H}(u,\underline{u}))} + \|\nabla_4\psi\cdot\nabla\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H}(u,\underline{u}))}\\ &+ \|\psi\cdot\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H}(u,\underline{u}))} + \|\psi\cdot\nabla_4\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})(u,\underline{u})}\\ &+ \|\psi\cdot\nabla\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H}(u,\underline{u}))} + \|\omega\nabla_3\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H}(u,\underline{u}))}\cdots\end{aligned}$$

We have kept on the right only the most problematic terms. We now write,

$$\|\nabla_4\psi\cdot\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \delta^{1/2}\|\nabla_4\psi\|_{\mathcal{L}^4_{(sc)}(\underline{H})}\cdot\|\Psi_3\|_{\mathcal{L}^4_{(sc)}(\underline{H})}$$

Using the interpolation estimates of corollary 4.12,

$$\begin{aligned}\|\nabla_4\psi\|_{\mathcal{L}^4_{(sc)}(\underline{H})} &\lesssim \|\nabla\nabla_4\psi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(\underline{H})}\|\nabla_4\psi\|^{\frac12}_{\mathcal{L}^4_{(sc)}(\underline{H})} + \delta^{\frac14}\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ \|\Psi_3\|_{\mathcal{L}^4_{(sc)}(\underline{H})} &\lesssim \|\nabla\Psi_3\|^{\frac12}_{\mathcal{L}^2_{(sc)}(\underline{H})}\|\Psi_3\|^{\frac12}_{\mathcal{L}^2_{(sc)}(\underline{H})} + \delta^{\frac14}\|\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

Taking into account the possible anomaly of $\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(S)}$ (recalling also that ψ here differs from $\omega, \underline{\omega}$!) we deduce,

$$\|\nabla_4\psi\|_{\mathcal{L}^4_{(sc)}(\underline{H})} \lesssim C\delta^{-1/4} \qquad \|\Psi_3\|_{\mathcal{L}^4_{(sc)}(\underline{H})} \lesssim C$$

Therefore,

$$\|\nabla_4\psi\cdot\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C\delta^{1/4}.$$

Similarly, taking into account the estimates for ${}^{(S)}\mathcal{O}_{1,4}$ of corollary 7.5,

$$\|\nabla_4\psi\cdot\nabla\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \delta^{1/2}\|\nabla_4\psi\|_{\mathcal{L}^4_{(sc)}(\underline{H})}\cdot\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(\underline{H})} \lesssim C\delta^{1/4}$$

To estimate $\|\psi\nabla_4\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$ we write, using the Bianchi equations,

$$\nabla_4\Psi_3 = \nabla\Psi_g + \psi\cdot\Psi + \omega\cdot\Psi,$$

where $\nabla\Psi_g \in \{\nabla\beta, \nabla\rho, \nabla\sigma, \nabla\underline{\beta}\}$. Recalling the estimate $\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C\delta^{-1/2}$ encountered before and $\|\nabla\Psi_g\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \underline{\mathcal{R}}$,

$$\|\psi\cdot\nabla_4\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}(\underline{H})}\Big(\|\nabla\Psi_g\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\Big) \lesssim C$$

The term $\|\nabla_4\psi\cdot\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$ may contain a double anomaly. We estimate it as follows:

$$\|\nabla_4\psi\cdot\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \delta^{1/2}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C$$

All other terms in $\mathcal{L}^2_{(sc)}(\underline{H})$ can be estimated in the same manner to derive,

$$\begin{aligned}\|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \int_0^u \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \\ &+ \int_0^u \|\nabla_4\nabla\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + C\end{aligned}$$

or, by Gronwall,

$$\|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \int_0^u \|\nabla_4\nabla\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + C$$

Now,

$$\int_0^u \|\nabla_4\nabla\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \lesssim \int_0^u \|\nabla_4\nabla\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + \|\nabla_4\nabla\psi_g\|_{\mathcal{L}^2_{(sc)}(\underline{H}(u,\underline{u}))}$$

where $\psi_g \in \{\mathrm{tr}\chi, \hat{\chi}, \eta, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}\}$. Thus, in view of the estimates of proposition 8.7 and commutator lemma 8.5,

$$\int_0^u \|\nabla_4\nabla\psi\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \lesssim \int_0^u \|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + C$$

and therefore,

$$\|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \int_0^u \|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + C \tag{151}$$

Using the elliptic estimates of proposition 4.17 applied to the Hodge system for $\nabla_4\psi$ we derive,

$$\begin{aligned}\|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4(\rho,\sigma)\|_{\mathcal{L}^2_{(sc)}(S)} \\ &+ \delta^{\frac12}\Delta_0\big(\|\nabla_4\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\Psi_4\|_{\mathcal{L}^2_{(sc)}(S)}\big)\end{aligned}$$

Now,

$$\nabla_4(\rho,\sigma) = \nabla\beta + \psi\cdot\Psi_4 + \omega\cdot\Psi_4,$$

with $\Psi_4 \in \{\alpha,\beta,\rho,\sigma\}$, Now,

$$\|\nabla_4(\rho,\sigma)\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac12}\Delta_0\|\Psi_4\|_{\mathcal{L}^2_{(sc)}(S)},$$

In the particular case when $\Psi_4 = \alpha$, (recall that α component is not allowed in the definition of the curvature norms $\underline{\mathcal{R}}$) we recall (see proposition 6.9) the estimate $\|\alpha\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \delta^{-1/2}C$. Therefore, in all cases,

$$\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C\delta^{-1/2}$$

and consequently,

$$\|\nabla_4(\rho,\sigma)\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(S)} + C$$

with $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$. Therefore,

$$\|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(S)} + C \tag{152}$$

Integrating,

$$\begin{aligned}\int_0^u \|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' &\lesssim \int_0^u \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + \int_0^u \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + C\\ &\lesssim \int_0^u \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + \underline{\mathcal{R}} + C.\end{aligned}$$

i.e.,

$$\int_0^u \|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' \lesssim \int_0^u \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + C. \tag{153}$$

Therefore, combining with (151) and applying Gronwall again, we deduce,

$$\|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + C$$

It is easy to check on the initial hypersurface H_0,

$$\|\nabla_4\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} \lesssim \mathcal{O}^{(0)}.$$

On the other hand, returning to (152), we deduce

$$\|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C + \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(S)}.$$

Hence,

$$\|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C$$

as desired.

The remaining estimate

$$\|\nabla\nabla_3\eta\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla\nabla_3\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$$

is proved in exactly the same manner. □

9. $\mathcal{O}_\infty$ estimates and proof of Theorem A

In this section we combine the estimates obtained so far to derive L^∞ estimates for all our Ricci coefficients and thus verify the bootstrap assumption (37). This would also allow us to conclude the proof of theorem A 2.13. To achieve this we combine the ${}^{(S)}\mathcal{O}_{0,4}, \mathcal{O}_{1,2}$, ${}^{(H)}\mathcal{O}$, ${}^{(\underline{H})}\mathcal{O}$ and the remaining second derivative estimates with the interpolation results of Proposition 4.15. We will only require results before and culminating with Proposition 8.7. In particular it does need the estimates of Proposition 8.12.

For the Ricci coefficients $\psi \in \{\mathrm{tr}\chi, \hat{\chi}, \eta, \omega\}$ we make use of the interpolation estimate of Proposition 4.15 together with ${}^{(S)}\mathcal{O}_{1,2} + {}^{(H)}\mathcal{O} \lesssim C$ and $\|\nabla_4 \nabla \psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C$ of Proposition 8.7 in the previous section, to derive

$$\begin{aligned}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} &\lesssim \left(\delta^{1/2}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(H)}\right)^{1/2} \\ &\cdot \left(\delta^{1/2}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)}\right)^{1/2} \\ &\lesssim C\end{aligned}$$

Similarly, for $\psi \in \{\widetilde{\mathrm{tr}\underline{\chi}}, \hat{\underline{\chi}}, \underline{\eta}, \underline{\omega}\}$, using the estimates ${}^{(S)}\mathcal{O}_{1,2} + {}^{(\underline{H})}\mathcal{O} \lesssim C$ and estimate $\|\nabla_3\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C$ of Proposition 8.7 in the previous section

$$\begin{aligned}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} &\lesssim \left(\delta^{1/2}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\right)^{1/2} \\ &\cdot \left(\delta^{1/2}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla_3\nabla\psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\right)^{1/2} \\ &\lesssim C\end{aligned}$$

Next, for the non-anomalous coefficients $\psi \in \{\mathrm{tr}\chi, \eta, \underline{\eta}, \omega, \underline{\omega}, \widetilde{\mathrm{tr}\underline{\chi}}\}$ we use the interpolation inequality

$$\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \|\nabla\psi\|^{\frac{1}{2}}_{\mathcal{L}^4_{(sc)}(S)}\|\psi\|^{\frac{1}{2}}_{\mathcal{L}^4_{(sc)}(S)} + \delta^{\frac{1}{4}}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)},$$

which leads to the desired estimate,

$$\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim C.$$

In the anomalous case of $\psi = \{\hat{\chi}, \hat{\underline{\chi}}\}$ we use the interpolation inequality (85)

$$\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \sup_{{}^\delta S}\left(\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\psi\|_{\mathcal{L}^4_{(sc)}({}^\delta S)}\right),$$

which gives

$$\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim C.$$

as desired. We deduce,

Proposition 9.1. *There exists a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$ such that, for $\delta^{1/2}\Delta_0$ sufficiently small we have,*

$$ {}^{(S)}\mathcal{O}_{0,\infty} \lesssim C. \tag{154}$$

In particular, choosing $\Delta_0 \approx C$, and $\delta > 0$ sufficiently small, depending only on C we dispense of the bootstrap assumption and derive the conclusion of Theorem A.

10. $\mathcal{L}^4_{(sc)}(S)$ estimates for curvature and the first derivatives of the Ricci coefficients

In this section we establish $\mathcal{L}^4_{(sc)}(S)$ estimates for all first derivatives of the Ricci coefficients ψ. In the previous section we have already established such estimates for $\nabla\psi$. The Ricci coefficients satisfy the structure equations

$$\nabla_4\psi = \operatorname{tr}\underline{\chi}_0 \cdot \psi + \psi\cdot\psi + \nabla\psi + \Psi,$$
$$\nabla_3\psi = \operatorname{tr}\underline{\chi}_0 \cdot \psi + \psi\cdot\psi + \nabla\psi + \Psi.$$

We note that the double anomalous terms $\operatorname{tr}\underline{\chi}_0\cdot\hat{\chi}$ and $\operatorname{tr}\underline{\chi}_0\cdot\hat{\underline{\chi}}$ appear only in the $\nabla_4\hat{\underline{\chi}}$, $\nabla_3\hat{\chi}$ and $\nabla_3\hat{\underline{\chi}}$ equations. Similarly the anomalous α curvature component only appears in the $\nabla_4\hat{\chi}$ equation.

For the remaining equations we estimate

$$\begin{aligned}\|\nabla_4\psi\|_{\mathcal{L}^4_{(sc)}(S)} &\lesssim \|\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}(S)}\|\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\Psi\|_{\mathcal{L}^4_{(sc)}(S)}\\ &\lesssim \mathcal{O}_{0,4} + \delta^{\frac14}\mathcal{O}_{0,\infty}\mathcal{O}_{0,4} + \mathcal{O}_{1,4} + \|\Psi\|_{\mathcal{L}^4_{(sc)}(S)},\end{aligned}$$

where the $\delta^{\frac14}$ takes into account a potential anomaly of the $\|\psi\|_{\mathcal{L}^4_{(sc)}(S)}$ term. To estimate $\|\Psi\|_{\mathcal{L}^4_{(sc)}(S)}$ we use the interpolation estimates

$$\|\Psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \Big(\delta^{\frac12}\|\Psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(H)}\Big)^{\frac12}\Big(\delta^{\frac12}\|\Psi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4\Psi\|_{\mathcal{L}^2_{(sc)}(H)}\Big)^{\frac12},$$
$$\|\Psi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \Big(\delta^{\frac12}\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\Big)^{\frac12}\Big(\delta^{\frac12}\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla_3\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\Big)^{\frac12}$$

Each of the null curvature components Ψ satisfies either ∇_4 or ∇_3 equation. These equations can be written schematically in the form

$$\nabla_4\Psi^{(s)} = \nabla\Psi^{(s+\frac12)} + \sum_{s_1+s_2=s+1}\psi^{(s_1)}\cdot\Psi^{(s_2)},$$
$$\nabla_3\Psi^{(s)} = \nabla\Psi^{(s-\frac12)} + \operatorname{tr}\underline{\chi}_0\cdot\Psi^s + \sum_{s_1+s_2=s}\psi^{(s_1)}\cdot\Psi^{(s_2)}$$

Let us consider the ∇_3 equation since the presence of the $\operatorname{tr}\underline{\chi}_0$ makes it more difficult to handle. We estimate

$$\|\nabla_3\Psi^{(s)}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \|\nabla\Psi^{(s-\frac12)}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\Psi^s\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \delta^{\frac12}\sum_{s_1+s_2=s}\|\psi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}}\|\Psi^{(s_2)}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$$

Note that the terms $\|\Psi^s\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$ and $\|\Psi^{s_2}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$ are anomalous only for $s = s_2 = 2$, that is in the case of the estimate for α. We summarize these estimates in the following

Lemma 10.1. *For a constant $C = C(\mathcal{I},\mathcal{O},\mathcal{R},\underline{\mathcal{R}})$ and $\Psi\in\{\beta,\rho,\sigma,\underline{\beta},\underline{\alpha}\}$*

$$\delta^{\frac14}\|\alpha\|_{\mathcal{L}^4_{(sc)}(S)} + \|\Psi\|_{\mathcal{L}^4_{(sc)}(S)} \le C$$

Combining this result with $\nabla_4\psi$ and $\nabla_3\psi$ equations, as described above, gives us the

$$\|\nabla_4\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_3\psi\|_{\mathcal{L}^4_{(sc)}(S)} \le C$$

estimates for those derivatives, with the exception of $\psi = \hat{\chi}, \underline{\hat{\chi}}$. On the other hand, the anomalies present in their respective equations lead to the anomalous estimates

$$\|\nabla_4\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_3\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_4\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_3\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} \le C\delta^{-\frac{1}{4}}$$

It remains to estimate $\nabla_3\eta, \nabla_4\underline{\eta}, \nabla_3\underline{\omega}, \nabla_4\omega$ which do not satisfy direct equations. We argue as in sections 8.3 and 8.11. Using the interpolation estimates stated in the beginning of this section and the bounds

$$\|\nabla\nabla_3\eta\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4\nabla_3\eta\|_{\mathcal{L}^2_{(sc)}(H)} \le C,$$
$$\|\nabla\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_3\nabla_4\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(H)} \le C$$

of sections 8.3 and 8.11, we obtain the desired $\mathcal{L}^4_{(sc)}(S)$ estimates for $\nabla_3\eta$ and $\nabla_4\underline{\eta}$. However, we can not obtain the corresponding estimates for $\nabla_4\omega$ and $\nabla_3\underline{\omega}$. We summarize the second main result of this section.

Lemma 10.2.

$$\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_{3,4}\eta\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_{3,4}\underline{\eta}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_4\underline{\omega}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_3\omega\|_{\mathcal{L}^4_{(sc)}(S)} \le C,$$
$$\|\nabla_4\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_3\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_4\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla_3\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} \le C\delta^{-\frac{1}{4}}$$

11. Renormalized estimates

11.1. **Trace theorems.** The results of this section rely on sharp trace theorems which we discuss below. We introduce the following new norms for an S tangent tensor ϕ with scale $\mathrm{sc}(\phi)$ along $H = H_u^{(0,\underline{u})}$, relative to the transported coordinates $(\underline{u}, \theta)$ of proposition 4.6:

$$\|\phi\|_{Tr_{(sc)}(H)} = \delta^{-\mathrm{sc}(\phi)-\frac{1}{2}}\Big(\sup_{\theta\in S(u,0)}\int_0^{\underline{u}} |\phi(u,\underline{u}',\theta)|^2 d\underline{u}'\Big)^{1/2}$$

Also, along $\underline{H} = \underline{H}_{\underline{u}}^{(0,u)}$ relative to the transported coordinates $(u, \underline{\theta})$ of proposition 4.6

$$\|\phi\|_{Tr_{(sc)}(\underline{H})} = \delta^{-\mathrm{sc}(\phi)}\Big(\sup_{\underline{\theta}\in S(\underline{u},0)}\int_0^{u} |\phi(u',\underline{u},\underline{\theta})|^2 du'\Big)^{1/2}$$

Proposition 11.2. *For any horizontal tensor ϕ along $H = H_u^{(0,\underline{u})}$,*

$$\begin{aligned}\|\nabla_4\phi\|_{Tr_{(sc)}(H)} \lesssim & \Big(\|\nabla_4^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \delta^{\frac12}C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla_4\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\ &\times\Big(\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \delta^{\frac12}C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\ &+\|\nabla_4\nabla\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \delta^{\frac12}C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)}) + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(H)}\end{aligned} \tag{155}$$

where C is a constant which depends on $\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}$.

Also, for any horizontal tensor ϕ along $\underline{H} = H_{\underline{u}}^{(u,0)}$, and a similar constant C,

$$\begin{aligned}\|\nabla_3\phi\|_{Tr_{(sc)}(\underline{H})} \lesssim & \Big(\|\nabla_3^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \delta^{\frac12}C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla_3\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\ &\times\Big(\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \delta^{\frac12}C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\ &+\|\nabla_3\nabla\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \delta^{\frac12}C(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)}) + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned} \tag{156}$$

The proof relies on the classical (euclidean) trace inequality formulated in (u,θ) or $(\underline{u},\theta)$ coordinates

Lemma 11.3. *For any scalar function ϕ along $H = H_u^{(0,\underline{u})}$, supported in a coordinate chart, we have*

$$\begin{aligned}\Big(\int_0^{\underline{u}} |\partial_{\underline{u}}\phi(u,\underline{u}',\theta)|^2 d\underline{u}'\Big)^{1/2} \lesssim & \big(\|\partial_{\underline{u}}^2\phi\|_{L^2(H)} + \delta^2\|\phi\|_{L^2(H)}\big)^{1/2}\cdot\|\partial_\theta^2\phi\|_{L^2(H)}^{1/2}\\ &+ \|\partial_\theta\partial_{\underline{u}}\phi\|_{L^2(H)} + \delta\|\partial_\theta\phi\|_{L^2(H)}\end{aligned} \tag{157}$$

For any scalar function ϕ along $\underline{H} = \underline{H}_{\underline{u}}^{(0,u)}$, supported in a neighborhood patch,

$$\begin{aligned}\Big(\int_0^{u} |\partial_u\phi(u',\underline{u},\underline{\theta})|^2 du'\Big)^{1/2} \lesssim & \big(\|\partial_u^2\phi\|_{L^2(\underline{H})} + \|\partial_u^2\phi\|_{L^2(\underline{H})}\big)^{1/2}\cdot\|\partial_{\underline{\theta}}^2\phi\|_{L^2(\underline{H})}^{1/2}\\ &+ \|\partial_{\underline{\theta}}\partial_u\phi\|_{L^2(\underline{H})} + \|\partial_{\underline{\theta}}\phi\|_{L^2(\underline{H})}\end{aligned} \tag{158}$$

In scale invariant norms we have,

$$\begin{aligned}\|\partial_{\underline{u}}\phi\|_{Tr_{(sc)}(H)} \lesssim & \big(\|\partial_{\underline{u}}^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\phi\|_{\mathcal{L}^2_{(sc)}(H)}\big)^{1/2}\cdot\|\partial_\theta^2\phi\|_{\mathcal{L}^2_{(sc)}(H)}^{1/2}\\ &+ \|\partial_\theta\partial_{\underline{u}}\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\partial_\theta\phi\|_{\mathcal{L}^2_{(sc)}(H)}\end{aligned}$$

and,

$$\begin{aligned}\|\partial_u\phi\|_{Tr_{(sc)}(\underline{H})} \lesssim & \big(\|\partial_u^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)^{1/2}\cdot\|\partial_\theta^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}^{1/2}\\ &+ \|\partial_\theta\partial_{\underline{u}}\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\partial_\theta\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

Proof. We start by making the additional assumption that $\phi(\underline{u}, \theta)$ is compactly supported for $\underline{u}' \in (0, \underline{u})$.

Integrating by parts in $\theta = (\theta^1, \theta^2)$,

$$\begin{aligned} \Big| \int_0^{\underline{u}} |\partial_{\underline{u}} \phi(\underline{u}, \theta)|^2 \Big| &= \Big| \int_{\theta^1}^{\infty} \int_{\theta^2}^{\infty} d\theta^1 d\theta^2 \; \partial_{\theta^1} \partial_{\theta^2} \int \partial_{\underline{u}} \phi(\underline{u}', \theta) \cdot \partial_{\underline{u}} \phi(\underline{u}', \theta) d\underline{u}' \Big| \\ &\lesssim \int_D \Big| \partial_{\theta^1} \partial_{\theta^2} \int_0^{\underline{u}} \partial_{\underline{u}} \phi(\underline{u}', \theta) \cdot \partial_{\underline{u}} \phi(\underline{u}', \theta) d\underline{u}' \Big| \, d\theta^1 d\theta^2 \\ &\lesssim \int_D \Big| \int_0^{\underline{u}} \partial_{\theta^1} \partial_{\theta^2} \partial_{\underline{u}} \phi(\underline{u}, \theta) \cdot \partial_{\underline{u}} \phi(\underline{u}, \theta) d\underline{u} \Big| d\theta \\ &+ \int_D \int_0^{\underline{u}} \big| \partial_\theta \partial_{\underline{u}} \phi(\underline{u}, \theta) \big|^2 d\underline{u}' d\theta \end{aligned}$$

Now, integrating by parts in $\underline{u}$,

$$\int_0^{\underline{u}} \partial_{\theta^1} \partial_{\theta^2} \partial_{\underline{u}} \phi(\underline{u}', \theta) \cdot \partial_{\underline{u}} \phi(\underline{u}', \theta) d\underline{u}' = -\int_0^{\underline{u}} \partial_{\theta^1} \partial_{\theta^2} \phi(\underline{u}', \theta) \cdot \partial_{\underline{u}}^2 \phi(\underline{u}', \theta)$$

Hence,

$$\int_0^{\underline{u}} |\partial_{\underline{u}} \phi(\underline{u}, \theta)|^2 \lesssim \|\partial_\theta^2 \phi\|_{L^2(H)} \cdot \|\partial_{\underline{u}}^2 \phi\|_{L^2(H)} + \|\partial_\theta \partial_{\underline{u}} \phi\|_{L^2(H)}^2. \tag{159}$$

To remove our additional assumption concerning the compact support in $(0, \underline{u})$ we simply extend the original ϕ to $-\delta \le \underline{u} \le 2\delta$ such that all norms on the right hand side of (157), on the extended interval, are bounded by a constant multiple of the same norms restricted to the original interval $(0, \underline{u})$. We then apply a cut-off to make the extended ϕ compactly supported in the interval $(-\delta, 2\delta)$ and finally use (159) in the extended interval to get the desired result. The proof of (158) is exactly the same. The scale version of these estimates is immediate. □

We now pass to the proof of proposition 11.2. It suffices to prove (155), the proof of (156) is exactly the same.

One can easily pass from the coordinate dependent form of the trace inequalities to a covariant form with the help of the estimates of proposition 4.6.

According to that proposition we have, for $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$,

$$\|\Gamma\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla \Gamma\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C$$

Thus,

$$\nabla_4 \phi_a = \Omega^{-1} \partial_{\underline{u}} \phi_a - \chi_{ab} \phi_b$$

As a consequence, along $H = H_u$,

$$\begin{aligned}\|\nabla_4\phi\|_{Tr_{(sc)}(H)} &\lesssim \|\partial_{\underline{u}}\phi\|_{Tr_{(sc)}(H)} + \delta^{1/2}\|\chi\|_{\mathcal{L}^\infty_{(sc)}}\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\\ &\lesssim \|\partial_{\underline{u}}\phi\|_{Tr_{(sc)}(H)} + C\delta^{1/2}\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\end{aligned}$$

Also, schematically, ignoring factors of Ω (which are bounded in L^∞), we have with $\psi \in \{\chi, \omega\}$,

$$\nabla_4^2\phi = \partial_{\underline{u}}^2\phi + \psi\cdot\partial_{\underline{u}}\phi + \alpha\cdot\phi + \psi\cdot\psi\cdot\phi$$

Thus, in view of our estimates for the Ricci coefficients ψ, we have

$$\begin{aligned}\|\partial_{\underline{u}}^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} &\lesssim \|\nabla_4^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \delta^{1/2}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\cdot\|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(H)}\\ &+ \delta^{1/2}\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\big(\|\alpha\|_{\mathcal{L}^2_{(sc)}(H)} + \|\psi\|^2_{\mathcal{L}^\infty_{(sc)}}\big)\\ &\lesssim \|\nabla_4^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + C\delta^{1/2}\big(\|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\phi\|_{\mathcal{L}^\infty_{(sc)}}\big)\end{aligned}$$

We next note that for a horizontal tensor we can convert ∂_θ into a covariant ∇ derivative according to the formula $\partial_\theta = \nabla + \Gamma$. Therefore,

$$\begin{aligned}\|\partial_\theta\phi_a\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\|\Gamma\|_{\mathcal{L}^2_{(sc)}(S)}\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\\ &\lesssim \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}C\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\end{aligned}$$

and,

$$\begin{aligned}\|\partial_\theta^2\phi_a\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\|\partial\Gamma\|_{\mathcal{L}^2_{(sc)}(S)}\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \delta^{1/2}\|\Gamma\|_{\mathcal{L}^4_{(sc)}(S)}\|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)}\\ &\lesssim \|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}C\big(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)}\big)\end{aligned}$$

Also,

$$\begin{aligned}\|\partial_\theta\partial_{\underline{u}}\phi_a\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\nabla\nabla_4\phi\|_{L^2(S)} + \delta^{1/2}\|\partial\Gamma\|_{L^2(S)}\|\phi\|_{L^\infty} + \delta^{1/2}\|\Gamma\|_{L^4(S)}\|\nabla_4\phi\|_{L^4(S)}\\ &\lesssim \|\nabla\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}C\big(\|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla_4\phi\|_{\mathcal{L}^4_{(sc)}(S)}\big)\end{aligned}$$

According , to the the scale invariant estimate of lemma 11.3,

$$\begin{aligned}\|\partial_{\underline{u}}\phi\|_{Tr_{(sc)}(H)} &\lesssim \big(\|\partial_{\underline{u}}^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\phi\|_{\mathcal{L}^2_{(sc)}(H)}\big)^{1/2}\cdot\|\partial_\theta^2\phi\|^{1/2}_{\mathcal{L}^2_{(sc)}(H)}\\ &+ \|\partial_\theta\partial_{\underline{u}}\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\partial_\theta\phi\|_{\mathcal{L}^2_{(sc)}(H)}\end{aligned}$$

Combining this with the previous estimates we obtain the desired result, which can be clearly extended to any ϕ along H_u, not necessarily restricted to a coordinate patch, by a simple partition of unity argument. This proves the desired estimate (155). Estimate (156) is proved in exactly the same manner.

11.4. **Estimate for the trace norms of $\nabla\chi$, $\nabla\underline{\chi}$.** Our main goal in this subsection is to derive estimates for the trace norms $\|\nabla\chi\|_{Tr_{(sc)}(H)}$ and $\|\nabla\underline{\chi}\|_{Tr_{(sc)}(\underline{H})}$. In view of proposition 11.2 we could achieve this goal if we could write $\nabla\hat{\chi} = \nabla_4\phi$ and $\nabla\underline{\hat{\chi}} = \nabla_3\underline{\phi}$ where ϕ, respectively $\underline{\phi}$ are such that the norms on the right hand side of (155), respectively (156), are finite. We prove the following proposition.

Proposition 11.5. *Consider the following transport equations along $H = H_u$, respectively $\underline{H} = \underline{H}_{\underline{u}}$*

$$\nabla_4\phi \;=\; \nabla\hat{\chi}, \qquad \phi(0,\underline{u}) = 0 \tag{160}$$

and

$$\nabla_3\underline{\phi} \;=\; \nabla\underline{\hat{\chi}}, \qquad \underline{\phi}(0,\underline{u}) = 0 \tag{161}$$

(1) *Solution ϕ of* (160) *verifies the estimates,*

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\phi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(S)} \;\lesssim\; C \tag{162}$$

$$\|\nabla\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} \;\lesssim\; C \tag{163}$$

with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$. Moreover,

$$\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} \;\lesssim\; \|\nabla^3 tr\chi\|_{\mathcal{L}^2_{(sc)}(H)} + C \tag{164}$$

As a consequence (see calculus inequalities of subsection 4.9) we also have,

$$\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \;\lesssim\; \|\nabla^3 tr\chi\|_{\mathcal{L}^2_{(sc)}(H)} + C \tag{165}$$

and as a consequence of the trace estimate (155),

$$\|\nabla_4\phi\|_{Tr_{(sc)}(H)} \;\lesssim\; \|\nabla^3 tr\chi\|_{\mathcal{L}^2_{(sc)}(H)} + C \tag{166}$$

(2) *Solution $\underline{\phi}$ of* (161) *verifies the estimates,*

$$\|\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\underline{\phi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} \;\lesssim\; C \tag{167}$$

$$\|\nabla\nabla_3\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla_3^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \;\lesssim\; C \tag{168}$$

with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$. Moreover,

$$\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \;\lesssim\; \|\nabla^3 tr\chi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C \tag{169}$$

As a consequence (see calculus inequalities of subsection 4.9) we also have,

$$\|\underline{\phi}\|_{\mathcal{L}^\infty_{(sc)}} \;\lesssim\; \|\nabla^3 tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C \tag{170}$$

and as a consequence of the trace estimate (155),

$$\|\nabla_3\underline{\phi}\|_{Tr_{(sc)}(\underline{H})} \;\lesssim\; \|\nabla^3 tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} + C \tag{171}$$

Proof. Estimates (162)-(163) and respectively (167)-(168) follow easily from (160), respectively (161) in view of our estimates for $\hat{\chi}$, respectively $\underline{\hat{\chi}}$, and their first two derivatives derived in the previous sections. The second ∇ derivative estimates are subtle; they require a non-trivial renormalization procedure, nothing less than another series miracles. As always we expect the estimates for $\underline{\phi}$ to be somewhat more demanding in view of the presence of $\mathrm{tr}\underline{\chi} = \mathrm{tr}\chi_0 + \widetilde{\mathrm{tr}\underline{\chi}}$. We shall thus concentrate on them in what follows. No other anomalies occur at this high level of differentiability. The idea is to derive first a transport equation for $\Delta\underline{\phi}$ and hope somehow that the principal term on the right, i.e. $\nabla\Delta\underline{\hat{\chi}}$, can be re-expressed a ∇_4 derivative of another quantity depending only on two derivatives of a Ricci coefficient. We write,

$$\nabla_3\Delta\underline{\phi} \;\; = \;\; \Delta\nabla\underline{\hat{\chi}} + [\nabla_3, \Delta]\phi$$

Now, recalling commutation lemma 3.3, we write schematically (we eliminate $\underline{\beta}$ using the Codazzi equation)

$$\begin{aligned}
[\nabla_3, \nabla]\underline{\phi} &= \underline{\chi} \cdot \nabla\phi + \nabla\psi_3 \cdot \phi + \psi_3 \cdot \nabla_3\phi + \underline{\chi} \cdot \psi_3 \cdot \phi \\
[\nabla_3, \nabla^2]\underline{\phi} &= \underline{\chi} \cdot \nabla^2\phi + \nabla\psi_3 \cdot (\nabla\phi + \nabla_3\phi) + \nabla^2\psi_3 \cdot \phi + \psi_3 \cdot \nabla\nabla_3\phi + \nabla(\underline{\hat{\chi}} \cdot \psi_3 \cdot \phi) \\
&+ \psi_3 \cdot \nabla_3\nabla\phi + \underline{\hat{\chi}} \cdot \psi_3 \cdot \nabla\phi
\end{aligned}$$

where $\psi_3 \in \{\widetilde{\mathrm{tr}\underline{\chi}}, \underline{\hat{\chi}}, \eta, \underline{\eta}\}$.

Hence, using our estimates for ψ_3 as well as the estimates (167)-(168) for $\underline{\phi}$ we can write,

$$[\nabla_3, \Delta]\underline{\phi} \;\; = \;\; \mathrm{tr}\underline{\chi}_0\nabla^2\underline{\phi} + \underline{\hat{\chi}} \cdot \nabla^2\underline{\phi} + \mathrm{Err}_\phi \tag{172}$$

$$\|\mathrm{Err}_\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \;\; \lesssim \;\; C\delta^{1/2}\big(C + \|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big) \tag{173}$$

Indeed, we have, for example,

$$\begin{aligned}
\|\nabla^2\psi_3 \cdot \underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim \delta^{1/2}\|\underline{\phi}\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla^2\psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \delta^{1/2}C\|\underline{\phi}\|_{\mathcal{L}^\infty_{(sc)}} \\
&\lesssim C\delta^{1/2}\big(\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla\nabla_3\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big) \\
&\lesssim C\delta^{1/2}\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C^2\delta^{1/2}.
\end{aligned}$$

Consequently,

$$\nabla_3\Delta\underline{\phi} \;\; = \;\; \Delta\nabla\underline{\hat{\chi}} + \mathrm{tr}\underline{\chi}_0\nabla^2\underline{\phi} + \underline{\hat{\chi}} \cdot \nabla^2\underline{\phi} + \mathrm{Err}_\phi \tag{174}$$

Since,

$$[\Delta, \nabla]\phi \;\; = \;\; K\nabla\phi + \nabla K \cdot \phi$$

we have,

$$\begin{aligned}
\|[\Delta, \nabla]\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim \|K\|_{\mathcal{L}^4_{(sc)}(\underline{H})} \cdot \|\nabla\phi\|_{Lsc^4(\underline{H})} + \|\nabla K\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \\
&\lesssim C\delta^{1/2}\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C^2\delta^{1/2}
\end{aligned}$$

Hence, also,

$$\begin{aligned} \nabla_3 \Delta \underline{\phi} &= \Delta \nabla \underline{\hat{\chi}} + \mathrm{tr}\underline{\chi}_0 \nabla^2 \underline{\phi} + \underline{\hat{\chi}} \cdot \nabla^2 \underline{\phi} + \mathrm{Err}_\phi \\ \|E\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C\delta^{1/2}\big(C + \|\nabla^2 \underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big) \end{aligned} \tag{175}$$

Now, according to the Codazzi equations,

$$\mathcal{D}_2 \underline{\hat{\chi}} = -\underline{\beta} - \tfrac{1}{2}\nabla \mathrm{tr}\underline{\chi} + \mathrm{tr}\underline{\chi}\psi_3 + \psi_4 \cdot \psi_3$$

Thus,

$$^\star\mathcal{D}_2 \mathcal{D}_2 \underline{\hat{\chi}} = {}^\star\mathcal{D}_2 \underline{\beta} - \tfrac{1}{2}\, {}^\star\mathcal{D}_2 \nabla \mathrm{tr}\underline{\chi} + {}^\star\mathcal{D}_2(\mathrm{tr}\underline{\chi}\psi_3 + \psi_3 \cdot \psi_3)$$

or, making use of (58),

$$-\frac{1}{2}\Delta \underline{\hat{\chi}} + K \underline{\hat{\chi}} = {}^\star\mathcal{D}_2 \underline{\beta} - \frac{1}{2}\, {}^\star\mathcal{D}_2 \nabla \mathrm{tr}\chi + {}^\star\mathcal{D}_2(\mathrm{tr}\underline{\chi}\psi_3 + \psi_3 \cdot \psi_3).$$

Thus, differentiating once more,

$$\begin{aligned} \nabla \Delta \underline{\hat{\chi}} &= \nabla^2 \underline{\beta} + \nabla^3 \mathrm{tr}\underline{\chi} + K \nabla \underline{\hat{\chi}} + \mathrm{Err} \\ \mathrm{Err} &= \nabla K \cdot \underline{\hat{\chi}} + \mathrm{tr}\underline{\chi} \nabla^2 \psi_3 + \nabla^2(\psi_3 \cdot \psi_3) \end{aligned} \tag{176}$$

Here, and in what follows, Err denotes an error term of the form,

$$\|\mathrm{Err}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C$$

On the other hand we recall the structure equation,

$$\nabla_3 \underline{\eta} = \underline{\beta} + \underline{\chi} \cdot (\eta - \underline{\eta})$$

Thus, commuting, and writing as before,

$$\begin{aligned} [\nabla_3, \nabla]\underline{\eta} &= \underline{\chi} \cdot \nabla \underline{\eta} + \nabla \psi_3 \cdot \underline{\eta} + \psi_3 \cdot \nabla_3 \underline{\eta} + \underline{\chi} \cdot \psi_3 \cdot \underline{\eta} \\ [\nabla_3, \nabla^2]\underline{\eta} &= \underline{\chi} \cdot \nabla^2 \underline{\eta} + \nabla \psi_3 \cdot (\nabla \underline{\eta} + \nabla_3 \underline{\eta}) + \nabla^2 \psi_3 \cdot \underline{\eta} + \psi_3 \cdot \nabla \nabla_3 \underline{\eta} + \nabla(\underline{\hat{\chi}} \cdot \psi_3 \cdot \underline{\eta}) \\ &+ \psi_3 \cdot \nabla_3 \nabla \underline{\eta} + \underline{\hat{\chi}} \cdot \psi_3 \cdot \nabla \underline{\eta} \end{aligned}$$

Observe that,

$$\|[\nabla_3, \nabla]\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C$$

and consequently,

$$\begin{aligned} \nabla^2 \underline{\beta} &= \nabla_3(\nabla^2 \underline{\eta}) + \mathrm{Err} \\ \mathrm{Err} &= \nabla^2\big(\underline{\chi} \cdot (\eta - \underline{\eta})\big) + [\nabla_3, \nabla^2]\underline{\eta} \end{aligned} \tag{177}$$

Clearly,

$$\|\mathrm{Err}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C \tag{178}$$

Therefore, we deduce,

$$\nabla \Delta \underline{\hat{\chi}} = -\nabla_3(\nabla^2 \underline{\eta}) + \nabla^3 \mathrm{tr}\underline{\chi} + K \nabla \underline{\hat{\chi}} + \mathrm{Err}$$

Commuting ∇ with Δ again,

$$\Delta\nabla\underline{\hat{\chi}} \quad = \quad \nabla\Delta\underline{\hat{\chi}} + K\nabla\underline{\hat{\chi}} + \nabla K\underline{\hat{\chi}}$$

Hence, since $\nabla\underline{\hat{\chi}} = \nabla_3\underline{\phi}$,

$$\Delta\nabla\underline{\hat{\chi}} \quad = \quad \nabla_3(\nabla^2\underline{\eta}) + \nabla^3\mathrm{tr}\underline{\chi} + K\nabla_3\underline{\phi} + \mathrm{Err} \tag{179}$$

Back to (175) we rewrite,

$$\begin{aligned}\nabla_3\Delta\underline{\phi} \quad &= \quad -\nabla_3(\nabla^2\underline{\eta}) + \nabla^3\mathrm{tr}\underline{\chi} + \mathrm{tr}\underline{\chi}_0\cdot\nabla^2\underline{\phi} + K\cdot\nabla + 3\underline{\phi} + \mathrm{Err}_\phi\\ \|\mathrm{Err}_\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \quad &\lesssim \quad C\big(1 + \delta^{1/2}\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)\end{aligned}$$

which we could rewrite in the form,

$$\nabla_3\big(\Delta\underline{\phi} + \nabla^2\underline{\eta} - K\underline{\phi}\big) \quad = \quad \nabla^3\mathrm{tr}\underline{\chi} + \mathrm{tr}\underline{\chi}_0\cdot\nabla^2\phi - \nabla_3 K\cdot\underline{\phi} + \mathrm{Err}_\phi \tag{180}$$

Recall that $K = \rho - \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}$. Hence, we easily find,

$$\|\nabla_3 K\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \quad \lesssim \quad C$$

Thus,

$$\begin{aligned}\|\nabla_3\big(\Delta\underline{\phi} + \nabla^2\underline{\eta} - K\underline{\hat{\chi}}\big)\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \quad &\lesssim \quad \|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + |\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\\ &+ \quad \|\mathrm{Err}_\phi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\end{aligned}$$

i.e.,

$$\begin{aligned}\|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \quad &\lesssim \quad \|\nabla^2\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + C\delta^{1/2}\|K\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \quad (1 + \delta^{1/2}C)\int_0^u \|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' + \|E_1\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

Now, using the elliptic estimates discussed in subsection 4.16, we have and our estimates for K, we deduce

$$\begin{aligned}\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} \quad &\lesssim \quad \|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)}\\ &+ \quad \delta^{1/2}\big(\|\nabla K\|_{\mathcal{L}^2_{(sc)}(S)}\|\underline{\phi}\|_{\mathcal{L}^\infty_{(sc)}(S)} + \|K\|_{\mathcal{L}^4_{(sc)}(S)}\|\nabla\underline{\phi}\|_{\mathcal{L}^4_{(sc)}(S)}\big)\\ &\lesssim \quad \|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\big(\|\underline{\phi}\|_{\mathcal{L}^\infty_{(sc)}(S)} + \|\nabla\underline{\phi}\|_{\mathcal{L}^4_{(sc)}(S)}\big)\\ &\lesssim \quad \|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\big(C + \|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)\end{aligned} \tag{181}$$

Thus,

$$\begin{aligned}\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \quad &\lesssim \quad \|\nabla^2\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + C\delta^{1/2}\|K\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \quad (1 + \delta^{1/2}C)\int_0^u \|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du'\\ &+ \quad C(1 + \delta^{1/2})\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

Using Gronwall,

$$\begin{aligned} \|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla^2\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + C\delta^{1/2}\|K\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \\ &+ C(1+\delta^{1/2})\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \end{aligned} \tag{182}$$

Integrating we deduce, for $C\delta^{1/2}$ sufficiently small,

$$\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C + \|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$$

as desired. □

To close the estimates of proposition 11.5 it remains to estimate $\|\nabla^3\mathrm{tr}\chi\|_{\mathcal{L}^2_{(sc)}(H)}$ and $\|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$. To achieve this we start with the transport equation for $\mathrm{tr}\underline{\chi}$,

$$\nabla_3(\mathrm{tr}\underline{\chi}) = -\frac{1}{2}\mathrm{tr}\underline{\chi}^2 - |\hat{\underline{\chi}}|^2 - 2\underline{\omega}\hat{\underline{\chi}}$$

which we rewrite in the form,

$$\begin{aligned} \nabla_4(\mathrm{tr}\underline{\chi}') &= -\frac{1}{2}\Omega^{-1}\mathrm{tr}\underline{\chi}^2 - \Omega^{-1}|\hat{\underline{\chi}}|^2 \\ \mathrm{tr}\underline{\chi}' &= \Omega^{-1}\mathrm{tr}\underline{\chi} \end{aligned}$$

The plan is to derive a transport equation for the quantity $\Delta\nabla\mathrm{tr}\underline{\chi}'$. We make use of the following commutation formulae, written schematically, for an arbitrary scalar f verifying the equation $\nabla_3 f = F$,

$$\begin{aligned} \nabla_3(\nabla f) &= \nabla F + \underline{\chi}\cdot\nabla f + \psi_3\cdot F \\ \nabla_3(\nabla^2 f) &= \nabla\big(\nabla F + \underline{\chi}\cdot\nabla f + \psi_3\cdot F\big) + \underline{\chi}\cdot\nabla^2 f + \underline{\beta}\cdot\nabla f + \psi_3\cdot\nabla_3(\nabla f) \\ &= \nabla^2 F + \psi_3\cdot\nabla F + \nabla\psi_3\cdot F + \underline{\chi}\cdot\nabla^2 f + \nabla\underline{\chi}\cdot\nabla f \\ &+ \psi_3\cdot\nabla_3(\nabla f) + \hat{\underline{\chi}}\cdot\psi_3\cdot\nabla f \\ \nabla_3(\nabla^3 f) &= \nabla^3 F + \psi_3\cdot\nabla^2 F + \nabla\psi_3\cdot\nabla F + \nabla^2\psi_3\cdot F \\ &+ \underline{\chi}\cdot\nabla^3 f + \nabla\underline{\chi}\cdot\nabla^2 f + \nabla^2\underline{\chi}\cdot\nabla f \\ &+ \nabla\big(\psi_3\nabla_3(\nabla f) + \hat{\underline{\chi}}\cdot\psi_3\cdot\nabla f\big) + \underline{\beta}\cdot\nabla^2 f + \psi_3\cdot\nabla_3(\nabla^2 f) \end{aligned}$$

or,

$$\begin{aligned} \nabla_3(\nabla^3 f) &= \nabla^3 F + \psi_3\cdot\nabla^2 F + \nabla\psi_3\cdot\nabla F + \nabla^2\psi_3\cdot F \\ &+ \underline{\chi}\cdot\nabla^3 f + \nabla\underline{\chi}\cdot\nabla^2 f + \nabla^2\underline{\chi}\cdot\nabla f \\ &+ \psi_3\cdot\nabla_3(\nabla^2 f) + \nabla\psi_3\cdot\nabla_3(\nabla f) + \psi_3[\nabla,\nabla_3](\nabla f) + \nabla(\hat{\underline{\chi}}\cdot\psi_3\cdot\nabla f) \end{aligned}$$

Applying the calculations above to $f = \Omega^{-1}\mathrm{tr}\chi$, $F = -\frac{1}{2}\Omega^{-1}\mathrm{tr}\chi^2 - \Omega^{-1}|\chi|^2$ and using $\nabla(\Omega^{-1}) = -\Omega^{-2}\nabla\Omega = -\frac{1}{2}\Omega^{-2}(\eta - \underline{\eta})$ we derive, omitting factors of Ω which are bounded in L^∞,

$$\begin{aligned}\nabla_4(\Delta\nabla\mathrm{tr}\underline{\chi}') &= \hat{\underline{\chi}} \cdot \Delta\nabla\hat{\underline{\chi}} + \underline{\chi}\cdot\nabla^3\mathrm{tr}\underline{\chi} + \nabla\hat{\underline{\chi}}\cdot\nabla^2\hat{\underline{\chi}} + \nabla\underline{\chi}\cdot\nabla^2\mathrm{tr}\underline{\chi} + F\\ F &= \mathrm{tr}\underline{\chi}_0\big(\psi_3\cdot\nabla^2\psi_3 + \nabla\psi_3\cdot\nabla\psi_3 + \psi_3\cdot\psi_3\cdot\nabla\psi_3\big)\\ &+ \psi_3\cdot\psi_3\cdot\nabla^2\psi_3 + \psi_3\cdot\nabla\psi_3\cdot\nabla\psi_3 + \psi_3\cdot\psi_3\cdot\psi_3\cdot\nabla\psi_3\end{aligned}$$

Making use of our estimates for ψ_3 we easily derive, with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$,

$$\|F\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \delta^{1/2}C$$

Thus,

$$\begin{aligned}\nabla_3(\Delta\nabla\mathrm{tr}\underline{\chi}') &= \hat{\underline{\chi}} \cdot \Delta\nabla\hat{\underline{\chi}} + \underline{\chi}\cdot\nabla^3\mathrm{tr}\underline{\chi} + \nabla\hat{\underline{\chi}}\cdot\nabla^2\hat{\underline{\chi}} + \nabla\underline{\chi}\cdot\nabla^2\mathrm{tr}\underline{\chi} + F_1\\ \|F_1\|_{\mathcal{L}^2_{(sc)}(H)} &\lesssim \delta^{1/2}C\end{aligned} \tag{183}$$

Observe that neither the principal term $\hat{\underline{\chi}} \cdot \nabla\Delta\hat{\underline{\chi}}$ or the lower order term $\nabla\hat{\underline{\chi}}\cdot\nabla^2\hat{\underline{\chi}}$ appear to satisfy an $\mathcal{L}^2_{(sc)}(H)$ estimate. The principal terms seems particularly nasty since we can't possible expect to estimate three derivatives of $\hat{\underline{\chi}}$ using norms which involve only one derivative of curvature components. Clearly another renormalization is needed. In fact we make use of equation (174) which we write in the form,

$$\Delta\nabla\hat{\underline{\chi}} = \nabla_3\Delta\underline{\phi} - \mathrm{tr}\underline{\chi}_0\nabla^2\underline{\phi} - \hat{\underline{\chi}}\cdot\nabla^2\underline{\phi} - E$$

We can thus replace the dangerous term $\Delta\nabla\hat{\underline{\chi}}$ in (183) and obtain,

$$\begin{aligned}\nabla_3(\Delta\nabla\mathrm{tr}\underline{\chi}') &= \hat{\underline{\chi}}\cdot\nabla_3\Delta\underline{\phi} + \underline{\chi}\cdot\nabla^3\mathrm{tr}\underline{\chi} + \nabla\hat{\underline{\chi}}\cdot\nabla^2\hat{\underline{\chi}} + \nabla\underline{\chi}\cdot\nabla^2\mathrm{tr}\underline{\chi} + F_2\\ F_2 &= F_1 - (\mathrm{tr}\underline{\chi}_0\nabla^2\underline{\phi} - \hat{\underline{\chi}}\cdot\nabla^2\underline{\phi} - E)\cdot\hat{\underline{\chi}}\end{aligned}$$

In view of our estimates for $\underline{\phi}$ we have,

$$\|F_2\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C\delta^{1/2}(1+\delta^{1/2}C)\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$$

Now, recalling also the definition of $\underline{\phi}$,

$$\begin{aligned}\nabla_3(\Delta\nabla\mathrm{tr}\underline{\chi}' - \hat{\underline{\chi}}\cdot\Delta\underline{\phi}) &= -\nabla_3\hat{\underline{\chi}}\cdot\Delta\underline{\phi} + \mathrm{tr}\underline{\chi}_0\nabla^3\mathrm{tr}\underline{\chi} + \psi_3\cdot\nabla^3\mathrm{tr}\underline{\chi} + \nabla_3\underline{\phi}\cdot\nabla^2\underline{\chi}\\ &+ \nabla\mathrm{tr}\underline{\chi}\cdot\nabla^2\mathrm{tr}\underline{\chi} + F_2\end{aligned}$$

Hence,

$$\begin{aligned}\|\Delta\nabla\text{tr}\underline{\chi}'\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\Delta\nabla\text{tr}\underline{\chi}'\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + C\delta^{1/2}\|\underline{\hat{\chi}}\|_{\mathcal{L}^\infty_{(sc)}} \cdot \|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\\ &+ (1+C\delta^{1/2})\int_0^u \|\nabla^3\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du'\\ &+ C\delta^{1/2}\|\nabla_3\underline{\hat{\chi}}\|_{Tr_{(sc)}(\underline{H})}\cdot\|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \delta^{1/2}\|\nabla_3\underline{\phi}\|_{Tr_{(sc)}(\underline{H})}\cdot\|\nabla^2\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \delta^{1/2}\|\nabla\text{tr}\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\cdot\|\nabla^2\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \|F_2\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

Using the calculus inequalities of subsection 4.9 and our estimates for $\nabla^2\nabla_3\text{tr}\underline{\chi}$,

$$\|\nabla\text{tr}\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C + \|\nabla^3\text{tr}\underline{\chi}\|_{\mathcal{L}_{(sc)}(\underline{H})}$$

Also, in view of the trace estimate (171),

$$\|\nabla_3\underline{\phi}\|_{Tr_{(sc)}(\underline{H})} \lesssim C + \|\nabla^3\text{tr}\underline{\chi}\|_{\mathcal{L}_{(sc)}(\underline{H})}$$

Hence,

$$\begin{aligned}\|\Delta\nabla\text{tr}\underline{\chi}'\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\Delta\nabla\text{tr}\underline{\chi}'\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + C\delta^{1/2}\|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\\ &+ (1+C\delta^{1/2})\int_0^u \|\nabla^3\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du'\\ &+ C\delta^{1/2}\|\nabla_3\underline{\hat{\chi}}\|_{Tr_{(sc)}(\underline{H})}\cdot\|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ C\delta^{1/2}\|\nabla^3\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C^2\delta^{1/2}\end{aligned}$$

Now,

$$\|\Delta\nabla\text{tr}\underline{\chi}'\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\Delta\nabla\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \delta^{1/2}C\big(\|\nabla^2\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + C\delta^{1/2}\big)$$

Now, using the elliptic estimates discussed in subsection 4.16, we have and our estimates for K, we deduce

$$\begin{aligned}\|\nabla^3\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\Delta\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S)}\\ &+ \delta^{1/2}\big(\|\nabla K\|_{\mathcal{L}^2_{(sc)}(S)}\|\nabla\text{tr}\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}(S)} + \|K\|_{\mathcal{L}^4_{(sc)}(S)}\|\nabla^2\text{tr}\underline{\chi}\|_{\mathcal{L}^4_{(sc)}(S)}\big)\\ &\lesssim \|\Delta\nabla\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\big(\|\nabla\text{tr}\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}(S)} + \|\nabla^2\text{tr}\underline{\chi}\|_{\mathcal{L}^4_{(sc)}(S)}\big)\\ &\lesssim \|\Delta\nabla\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{1/2}\big(C + \|\nabla^3\text{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)\end{aligned}$$

Hence, after using Gronwall,

$$\begin{aligned}\|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + C\delta^{1/2}\big(\|\nabla^2\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \|\nabla^2\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\big)\\ &+ C\delta^{1/2}\|\nabla_3\hat{\underline{\chi}}\|_{Tr_{(sc)}(\underline{H})}\cdot\|\Delta\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ C\delta^{1/2}\|\nabla^3\mathrm{tr}\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C^2\delta^{1/2}\end{aligned}$$

Thus, after integration,

$$\|\nabla^3\mathrm{tr}\underline{\chi}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}} \lesssim C^2 + C^2\delta\int_0^u \|\nabla_3\hat{\underline{\chi}}\|^2_{Tr_{(sc)}(\underline{H}^{(0,u')})}\cdot\|\Delta\underline{\phi}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u')})}du' \tag{184}$$

It remains to estimate the trace norm $\|\nabla_3\hat{\underline{\chi}}\|_{Tr_{(sc)}(\underline{H}^{(0,u')})}$. We claim the following,

Lemma 11.6. *There exists a constant C depending only on $\mathcal{O}^{(0)},\mathcal{R},\underline{\mathcal{R}}$ as well as $\|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$ such that,*

$$\|\nabla_3\hat{\underline{\chi}}\|_{Tr_{(sc)}(\underline{H})} \lesssim C\delta^{-1/2}. \tag{185}$$

Proof. in view of the trace estimate (156), we have for $\underline{H}=\underline{H}^{(0,u')}$,

$$\begin{aligned}\|\nabla_3\underline{\chi}\|_{Tr_{(sc)}(\underline{H})} &\lesssim \|\nabla_3^2\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla\nabla_3\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \|\nabla^2\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C\delta^{1/2}\|\hat{\underline{\chi}}\|_{\mathcal{L}^\infty_{(sc)}}\end{aligned}$$

Observe that,

$$\|\nabla_3\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C\delta^{-1/2}$$

We claim also that,

$$\|\nabla_3^2\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C\delta^{-1/2} + \|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}.$$

Indeed, differentiating,

$$\nabla_3\hat{\underline{\chi}} = -\underline{\alpha} - \mathrm{tr}\underline{\chi}\hat{\underline{\chi}} - 2\underline{\omega}\,\hat{\underline{\chi}}$$

Thus,

$$\nabla_3^2\hat{\underline{\chi}} = -\nabla_3\underline{\alpha} - \nabla_3\mathrm{tr}\underline{\chi}\cdot\hat{\underline{\chi}} - \mathrm{tr}\underline{\chi}\cdot\nabla_3\hat{\underline{\chi}} - 2\nabla_3\underline{\omega}\cdot\hat{\underline{\chi}} - \underline{\omega}\cdot\nabla_3\hat{\underline{\chi}}$$

Hence,

$$\begin{aligned}\|\nabla_3^2\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim \|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C\delta^{1/2}\big(\|\nabla_3\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \nabla_3\hat{\underline{\chi}}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)\\ &\lesssim C\delta^{-1/2} + \|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

which completes the proof of our estimate. □

Returning to (184), we have with a constant C depending on $\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}$, as well as $\|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$),

$$\|\nabla^3 \text{tr}\underline{\chi}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}} \lesssim C^2 + C^2 \int_0^u \|\nabla^2 \underline{\phi}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u')})} du' \lesssim C^2 \big(1 + \int_0^u \|\nabla^3 \text{tr}\underline{\chi}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u')})} du'\big)$$

Thus, applying Gronwall once more we derive,

$$\|\nabla^3 \text{tr}\underline{\chi}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}} \lesssim C^2$$

This finishes the proof of the second part of the following.

Proposition 11.7. *The following estimates hold true with a constant C depending on $\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}$ as well as $\sup_u \|\nabla_4 \alpha\|_{\mathcal{L}^2_{(sc)}(H_u)}$ and $\sup_{\underline{u}} \|\nabla_3 \underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}$*

(1) *We have along $H = H_u$,*

$$\|\nabla^3 tr\chi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla tr\chi\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$$
$$\sup_S \|\nabla\hat{\chi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\hat{\chi}\|_{Tr_{(sc)}(H)} \lesssim C.$$

(2) *We have along $\underline{H} = \underline{H}_{\underline{u}}$,*

$$\|\nabla^3 tr\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla tr\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$$
$$\sup_S \|\nabla\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\underline{\hat{\chi}}\|_{Tr_{(sc)}(\underline{H})} \lesssim C.$$

11.8. **Estimates for the trace norms of $\nabla\eta, \nabla\underline{\eta}$.** As in the previous subsection we need a series of renormalization. The proof follows, however, the same outline as above. We first prove the following,

Proposition 11.9. *Consider the following transport equations along $H = H_u$, respectively $\underline{H} = \underline{H}_{\underline{u}}$*

$$\nabla_4 {}^{(4)}\phi = \nabla\eta, \qquad {}^{(4)}\phi(0, \underline{u}) = 0 \tag{186}$$
$$\nabla_4 {}^{(4)}\underline{\phi} = \nabla\underline{\eta} \qquad {}^{(4)}\underline{\phi}(0, \underline{u}) = 0 \tag{187}$$

and

$$\nabla_3 {}^{(3)}\phi = \nabla\eta, \qquad {}^{(3)}\phi(0, \underline{u}) = 0 \tag{188}$$
$$\nabla_3 {}^{(3)}\underline{\phi} = \nabla\underline{\eta}, \qquad {}^{(3)}\underline{\phi}(0, \underline{u}) = 0 \tag{189}$$

(1) *Solutions $\phi = ({}^{(4)}\phi, {}^{(4)}\underline{\phi})$ of (186) -(187) verify the estimates,*

$$\|\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\phi\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C \tag{190}$$
$$\|\nabla\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C \tag{191}$$

with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$. *Moreover,*

$$\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\nabla^2\mu\|_{\mathcal{L}^2_{(sc)}(H)} + C \tag{192}$$

As a consequence (see calculus inequalities of subsection 4.9) we also have,

$$\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \lesssim \|\nabla^2\mu\|_{\mathcal{L}^2_{(sc)}(H)} + C \tag{193}$$

and as a consequence of the trace estimate (155),

$$\|\nabla_4\phi\|_{Tr_{(sc)}(H)} \lesssim \|\nabla^2\mu\|_{\mathcal{L}^2_{(sc)}(H)} + C \tag{194}$$

(2) *Solutions* $\underline{\phi} = ({}^{(3)}\phi, {}^{(3)}\underline{\phi})$ *of* (188), (189) *verify the estimates,*

$$\|\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\underline{\phi}\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C \tag{195}$$

$$\|\nabla\nabla_3\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla_3^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C \tag{196}$$

with a constant $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$. *Moreover,*

$$\|\nabla^2({}^{(3)}\phi, {}^{(3)}\underline{\phi})\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C \tag{197}$$

As a consequence (see calculus inequalities of subsection 4.9) we also have,

$$\|({}^{(3)}\phi, {}^{(3)}\underline{\phi})\|_{\mathcal{L}^\infty_{(sc)}} \lesssim \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C \tag{198}$$

and as a consequence of the trace estimate (156),

$$\|\nabla_3({}^{(3)}\phi, {}^{(3)}\underline{\phi})\|_{Tr_{(sc)}(\underline{H})} \lesssim \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C \tag{199}$$

Proof. We start with

$$\nabla_3\, {}^{(3)}\phi = \eta, \qquad \nabla_3\, {}^{(3)}\underline{\phi} = \underline{\eta}$$

Commuting both equations with Δ and proceeding exactly as in the derivation of (175) we derive

$$\nabla_3\,\Delta\,{}^{(3)}\phi = \nabla\Delta\eta + \mathrm{tr}\underline{\chi}_0\nabla^2\,{}^{(3)}\phi + \hat{\underline{\chi}}\cdot\nabla^2\,{}^{(3)}\phi + E \tag{200}$$

$$\nabla_3\,\Delta\,{}^{(3)}\underline{\phi} = \nabla\Delta\underline{\eta} + \mathrm{tr}\underline{\chi}_0\nabla^2\,{}^{(3)}\underline{\phi} + \hat{\underline{\chi}}\cdot\nabla^2\,{}^{(3)}\underline{\phi} + \underline{E} \tag{201}$$

$$\|E\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C\delta^{1/2}\big(C + \|\nabla^2\,{}^{(3)}\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)$$

$$\|\underline{E}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C\delta^{1/2}\big(C + \|\nabla^2\,{}^{(3)}\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)$$

Recall that, see (117), (118),

$$\mathrm{div}\,\eta = -\mu - \rho, \qquad \mathrm{curl}\,\eta = \sigma - \frac{1}{2}\hat{\chi}\wedge\hat{\underline{\chi}}$$

$$\mathrm{div}\,\underline{\eta} = -\underline{\mu} - \rho, \qquad \mathrm{curl}\,\eta = \sigma - \frac{1}{2}\hat{\chi}\wedge\hat{\underline{\chi}}$$

i.e., schematically,

$$\begin{aligned} {}^\star\mathcal{D}_1\mathcal{D}_1\eta &= {}^\star\mathcal{D}_1(-\mu-\rho,\sigma-\hat{\chi}\wedge\underline{\hat{\chi}}) \\ {}^\star\mathcal{D}_1\mathcal{D}_1\underline{\eta} &= {}^\star\mathcal{D}_1(-\underline{\mu}-\rho,\sigma-\hat{\chi}\wedge\underline{\hat{\chi}}) \end{aligned}$$

Prceeding as in the derivation of (176) we find, schematically,

$$\begin{aligned} \nabla\Delta\eta &= \nabla^2\mu+\nabla^2(\rho,\sigma)+F_1 \\ \nabla\Delta\underline{\eta} &= \nabla^2\underline{\mu}+\nabla^2(\rho,\sigma)+F_1 \\ \|F_1\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C \end{aligned}$$

We now make use of the equations, see equations (121) and (123),

$$\begin{aligned} \nabla_3\omega &= \frac{1}{2}\rho+2\omega\underline{\omega}+\frac{3}{4}|\eta-\underline{\eta}|^2+\frac{1}{4}(\eta-\underline{\eta})\cdot(\eta+\underline{\eta})-\frac{1}{8}|\eta+\underline{\eta}|^2 \\ \nabla_3\omega^\dagger &= \frac{1}{2}\sigma \end{aligned}$$

Proceeding now exactly as in the derivation of (177) and (178), we deduce,

$$\begin{aligned} \nabla^2(\rho,\sigma) &= \nabla_3\nabla^2(\omega,\omega^\dagger)+F_2 \\ \|F_2\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C. \end{aligned}$$

Therefore, just as before for the derivation of $\nabla\Delta\hat{\chi}$, schematically,

$$\begin{aligned} \nabla\Delta\eta &= \nabla_3\nabla^2(\omega,\omega^\dagger)+\nabla^2\mu+F && (202) \\ \nabla\Delta\underline{\eta} &= \nabla_3\nabla^2(\omega,\omega^\dagger)+\nabla^2\underline{\mu}+\underline{F} && (203) \\ \|F,\underline{F}\|_{\mathcal{L}_{(sc)}(\underline{H})} &\lesssim C. \end{aligned}$$

Thus, back to (200) and (201) we deduce (just as in (180)

$$\begin{aligned} \nabla_3\big((\Delta\,{}^{(3)}\phi-\nabla^2(\omega,\omega^\dagger)\big) &= \nabla^2\mu+\mathrm{tr}\underline{\chi}_0\nabla^{2\,(3)}\phi+\underline{\hat{\chi}}\cdot\nabla^{2\,(3)}\phi+E && (204) \\ \|E\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C\big(1+\delta^{1/2}\|\nabla^{2\,(3)}\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big) \end{aligned}$$

and,

$$\begin{aligned} \nabla_3\big((\Delta\,{}^{(3)}\underline{\phi}-\nabla^2(\omega,\omega^\dagger)\big) &= \nabla^2\underline{\mu}+\mathrm{tr}\underline{\chi}_0\nabla^{2\,(3)}\underline{\phi}+\underline{\hat{\chi}}\cdot\nabla^{2\,(3)}\underline{\phi}+\underline{E} && (205) \\ \|\underline{E}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C\big(1+\delta^{1/2}\|\nabla^{2\,(3)}\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big) \end{aligned}$$

We then proceed with elliptic $\mathcal{L}^2_{(sc)}$ estimates, exactly as in (181) and, after using also Gronwall, we find (as in (182))

$$\begin{aligned} \|\nabla^{2\,(3)}\phi\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla^2(\omega,\omega^\dagger)\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}+\int_0^u\|\nabla^2\mu\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})}du' && (206) \\ &+ C(1+\delta^{1/2})\|\nabla^{2\,(3)}\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \end{aligned}$$

and

$$\begin{aligned}\|\nabla^{2\,(3)}\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\nabla^2(\omega,\omega^\dagger)\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \int_0^u \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u',\underline{u})} du' \\ &+ C(1+\delta^{1/2})\|\nabla^{2\,(3)}\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned} \tag{207}$$

Integrating we deduce, for $C\delta^{1/2}$ sufficiently small,

$$\begin{aligned}\|\nabla^{2\,(3)}\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C + \|\nabla^2\mu\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \\ \|\nabla^{2\,(3)}\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C + \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

as desired. □

It remains to estimate $\|\nabla^2\mu\|_{\mathcal{L}^2_{(sc)}(H)}$ and $\|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$. As before we treat only the estimate for the slightly more difficult case of $\underline{\mu}$. In view of the proof of the previous proposition we have (neglecting signs and constants, as before),

$$\nabla\Delta\eta = \nabla_3\,\Delta^{(3)}\phi + \mathrm{tr}\underline{\chi}_0\nabla^{2\,(3)}\phi + \hat{\underline{\chi}}\cdot\nabla^{2\,(3)}\phi + E \tag{208}$$

$$\nabla\Delta\underline{\eta} = \mathrm{tr}\underline{\chi}_0\nabla^{2\,(3)}\underline{\phi} + \hat{\underline{\chi}}\cdot\nabla^{2\,(3)}\underline{\phi} + \underline{E} \tag{209}$$

$$\begin{aligned}\|E\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C\delta^{1/2}\big(C + \|\nabla^{2\,(3)}\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big) \\ \|\underline{E}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C\delta^{1/2}\big(C + \|\nabla^{2\,(3)}\underline{\phi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)\end{aligned}$$

We start with the transport equation (114),

$$\begin{aligned}\nabla_3\underline{\mu} + \mathrm{tr}\underline{\chi}\underline{\mu} &= -\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{div}\,\eta + (\underline{\eta}-\eta)\nabla\mathrm{tr}\underline{\chi} \\ &+ \hat{\underline{\chi}}\cdot\nabla(2\underline{\eta}-\eta) + \frac{1}{2}\,\hat{\chi}\cdot\underline{\alpha} - (\underline{\eta}-3\eta)\cdot\underline{\beta} + \frac{1}{2}\mathrm{tr}\underline{\chi}\rho \\ &+ \frac{1}{2}\mathrm{tr}\underline{\chi}(|\underline{\eta}|^2 - \underline{\eta}\cdot\eta) + \frac{1}{2}(\eta+\underline{\eta})\cdot\hat{\underline{\chi}}\cdot(\underline{\eta}-\eta)\end{aligned}$$

Commuting with the laplacean, we derive

$$\begin{aligned}\nabla_3\Delta\underline{\mu} &= \hat{\underline{\chi}}\cdot\Delta\nabla(\underline{\eta}+\eta) + \mathrm{tr}\underline{\chi}\Delta\mathrm{div}\,\eta + (\nabla\eta+\nabla\underline{\eta})\cdot\nabla^2\hat{\underline{\chi}} + \mathrm{tr}\underline{\chi}\Delta\underline{\mu} \\ &+ (\nabla^2\eta + \nabla^2\underline{\eta})\cdot\nabla\hat{\underline{\chi}} + \frac{1}{2}\,\hat{\chi}\cdot\Delta\underline{\alpha} - (\underline{\eta}-3\eta)\cdot\Delta\underline{\beta} + \frac{1}{2}\mathrm{tr}\underline{\chi}\Delta\rho \\ &+ \mathrm{Err}\end{aligned}$$

Here, and in what follows, Err denotes any term which allows a bound of the form,

$$\|\mathrm{Err}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C \tag{210}$$

Using equation, $\nabla_3\underline{\hat{\chi}} = -\underline{\alpha} - \mathrm{tr}\underline{\chi}_0\,\underline{\hat{\chi}} + \psi_3\cdot\psi_3$ we write,

$$\Delta\underline{\alpha} = -\nabla_3\Delta\underline{\hat{\chi}} + \mathrm{Err}.$$

Using equation, $\nabla_3\underline{\eta} = \underline{\beta} + \underline{\chi}\cdot(\eta - \underline{\eta})$ we can write

$$\not{\Delta}\underline{\beta} \;=\; \nabla_3\Delta\underline{\eta} + \mathrm{Err}$$

Using equation $\nabla_3\omega = \frac{1}{2}\rho + \psi\cdot\psi$ we can write

$$\Delta\rho \;=\; 2\nabla_3\Delta\omega + \mathrm{Err}$$

Therefore we can write,

$$\begin{aligned}\nabla_3\Delta\underline{\mu} &= \underline{\hat{\chi}}\cdot\nabla_3\Delta({}^{(3)}\phi + {}^{(3)}\underline{\phi}) + \mathrm{tr}\underline{\chi}\nabla_3\Delta({}^{(3)}\phi + {}^{(3)}\underline{\phi})\\ &+ \nabla_3({}^{(3)}\phi + {}^{(3)}\underline{\phi})\cdot\nabla^2\underline{\hat{\chi}} + \nabla^2(\eta + \underline{\eta})\cdot\nabla\underline{\hat{\chi}}\\ &+ \mathrm{tr}\underline{\chi}\nabla_3\Delta\omega + (\eta + \underline{\eta})\nabla_3\Delta\underline{\eta} + \underline{\hat{\chi}}\cdot\nabla_3\Delta\underline{\hat{\chi}} + \mathrm{Err}_\phi\end{aligned}$$

with Err_ϕ verifying,

$$\begin{aligned}\|\mathrm{Err}_\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} &\lesssim C\big(1 + \|\nabla^2({}^{(3)}\phi + {}^{(3)}\underline{\phi})\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)\\ &\lesssim C\big(1 + \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\big)\end{aligned}$$

Therefore, introducing the renormalized quantity

$$\underline{\not{\mu}} = \underline{\mu} - \underline{\chi}\cdot\Delta({}^{(3)}\phi + {}^{(3)}\underline{\phi}) - \mathrm{tr}\underline{\chi}\cdot\Delta\omega - (\eta + \underline{\eta})\cdot\Delta\underline{\eta} - \underline{\hat{\chi}}\cdot\Delta\underline{\hat{\chi}} \tag{211}$$

we have,

$$\begin{aligned}\nabla_3\underline{\not{\mu}} &= -\nabla_3\underline{\chi}\cdot\Delta({}^{(3)}\phi + {}^{(3)}\underline{\phi} + \underline{\hat{\chi}}) - \nabla_3\mathrm{tr}\underline{\chi}\cdot\Delta({}^{(3)}\phi + {}^{(3)}\underline{\phi})\\ &+ \nabla_3({}^{(3)}\phi + {}^{(3)}\underline{\phi})\cdot\nabla^2\underline{\hat{\chi}} + \nabla^2(\eta + \underline{\eta})\cdot\nabla\underline{\hat{\chi}}\\ &+ \nabla_3\mathrm{tr}\underline{\chi}\cdot\Delta\omega + \nabla_3(\eta + \underline{\eta})\cdot\Delta\underline{\eta} + \mathrm{Err}_\phi\end{aligned}$$

Consequently,

$$\begin{aligned}\|\underline{\not{\mu}}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \delta^{1/2}\|\nabla_3\underline{\hat{\chi}}\|_{Tr_{(sc)}(\underline{H})}\cdot\|\nabla^2({}^{(3)}\phi + {}^{(3)}\underline{\phi} + \underline{\hat{\chi}})\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \delta^{1/2}\|\nabla_3({}^{(3)}\phi + {}^{(3)}\underline{\phi})\|_{Tr_{(sc)}(\underline{H})}\cdot\|\nabla^2({}^{(3)}\phi + {}^{(3)}\underline{\phi})\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \delta^{1/2}\|\nabla_3(\eta + \underline{\eta})\|_{Tr_{(sc)}(\underline{H})}\cdot\|\nabla^2\underline{\eta}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\\ &+ \delta^{1/2}\|\nabla\underline{\hat{\chi}}\|_{Tr_{(sc)}(\underline{H})}\cdot\|\nabla^2(\eta + \underline{\eta})\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \mathrm{Err}_\phi\end{aligned}$$

We recall from the previous subsection, see lemma 11.6, that

$$\|\nabla_3\underline{\hat{\chi}}\|_{Tr_{(sc)}(\underline{H})} \;\lesssim\; C\delta^{-1/2}$$

with a constant C depending only on $\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}$ as well as $\|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$. Also, from the previous section, we have (see proposition 11.7)

$$\|\nabla\underline{\hat{\chi}}\|_{Tr_{(sc)}(\underline{H})} \;\lesssim\; C$$

Also, in view of (199),

$$\|\nabla_3({}^{(3)}\phi, {}^{(3)}\underline{\phi})\|_{Tr_{(sc)}(\underline{H})} \lesssim \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C$$

Also, we can easily show, with the help of the trace estimates of proposition 11.2 and our Ricci coefficient estimates,

$$\|\nabla_3(\eta, \underline{\eta})\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim C$$

Consequently,

$$\|\underline{\not{\mu}}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} \lesssim \|\underline{\not{\mu}}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + (1 + C\delta^{1/2}\|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$$

On the other hand,

$$\begin{aligned}\|\underline{\not{\mu}}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\Delta\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + \|\nabla^2\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + C\delta^{1/2}\|\nabla^2\eta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\\ &+ C\delta^{1/2}\|\nabla^2\underline{\hat{\chi}}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\end{aligned}$$

Hence,

$$\begin{aligned}\|\Delta\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} &\lesssim \|\underline{\not{\mu}}\|_{\mathcal{L}^2_{(sc)}(0,\underline{u})} + \|\nabla^2\underline{\omega}\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})} + C\delta^{1/2}\|\nabla^2\eta\|_{\mathcal{L}^2_{(sc)}(u,\underline{u})}\\ &+ \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C\delta^{1/2}\|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

We can now proceed precisely as in the last part of the proof of proposition 11.7 to deduce, after applying elliptic estimates and integrating,

$$\|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} \lesssim \mathcal{O}^{(0)} + (1 + C\delta^{1/2})\int_0^u \|\nabla^2\underline{\mu}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u')})}du' + C$$

from which the desired estimate follows. We have thus proved the second part of the following:

Proposition 11.10. *The following estimates hold true with a constant C depending on $\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}}$ as well as $\sup_u \|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u)}$ and $\sup_{\underline{u}} \|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}$.*

(1) *We have along $H = H_u$,*

$$\|\nabla(\eta, \underline{\eta})\|_{Tr_{(sc)}(H)} \lesssim C$$

(2) *We have along $\underline{H} = \underline{H}_{\underline{u}}$,*

$$\|\nabla(\eta, \underline{\eta})\|_{Tr_{(sc)}(\underline{H})} \lesssim C$$

(3) *Also,*

$$\sup_S \|\nabla(\eta, \underline{\eta})\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C$$

11.11. **Refined estimate for ${}^{(3)}\phi$.** We end this section by establishing a more refined estimate on ${}^{(3)}\phi$. This estimate is needed in the argument for the formation of a trapped surface described in our introduction. We examine the equation

$$\nabla_3\,{}^{(3)}\phi = \nabla\eta.$$

Commuting with ∇ we obtain

$$\nabla_3\nabla\,{}^{(3)}\phi = (\mathrm{tr}\underline{\chi}_0 + \psi)\cdot\nabla\,{}^{(3)}\phi + (\Psi + \psi\cdot\psi)\,{}^{(3)}\phi + \nabla^2\eta$$

Taking into account triviality of the data for $\nabla\,{}^{(3)}\phi$, non-anomalous estimates for Ψ appearing in this equation, and Gronwall we obtain

$$\|\nabla\,{}^{(3)}\phi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\nabla^2\eta\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})} + \delta^{\frac12}C.$$

Using Proposition 7.6 we obtain

$$\|\nabla\,{}^{(3)}\phi\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\nabla\rho\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})} + \|\nabla\sigma\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})} + \delta^{\frac14}C.$$

Combining with the interpolation estimates

$$\|{}^{(3)}\phi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \|{}^{(3)}\phi\|^{\frac12}_{\mathcal{L}^4_{(sc)}(S)}\|\nabla{}^{(3)}\phi\|^{\frac12}_{\mathcal{L}^4_{(sc)}(S)} + \delta^{\frac14}\|{}^{(3)}\phi\|_{\mathcal{L}^4_{(sc)}(S)},$$

$$\|\nabla{}^{(3)}\phi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim \|\nabla{}^{(3)}\phi\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)}\|\nabla^2({}^{(3)}\phi)\|^{\frac12}_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac14}\|\nabla{}^{(3)}\phi\|_{\mathcal{L}^2_{(sc)}(S)}$$

we conclude

Proposition 11.12. *The solution ${}^{(3)}\phi$ of the problem $\nabla_3\,{}^{(3)}\phi = \nabla\eta$ with trivial initial data satisfies*

$$\|{}^{(3)}\phi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim C\left(\|\nabla\rho\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})} + \|\nabla\sigma\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}\right)^{\frac14} + C\delta^{\frac18}.$$

12. Trace estimates for curvature

Proposition 12.1. *Under the assumptions of the finiteness of the norms $\mathcal{R}$ and $\underline{\mathcal{R}}$, which include $\|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})}$ and the anomalous norm $\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u)}$ we have*

$$\|\alpha\|_{Tr_{sc}(H)} \le \delta^{-\frac14}C,$$
$$\|(\beta,\rho,\sigma)\|_{Tr_{sc}(H)} \le C,$$
$$\|(\rho,\sigma,\underline{\beta})\|_{Tr_{sc}(\underline{H})} \le C,$$
$$\|\underline{\alpha}\|_{Tr_{sc}(\underline{H})} \le \delta^{-\frac14}C$$

The proof is based on the application of the trace inequalities of Proposition 11.2 and the null structure equations (47), (49)-(51). According to these the curvature components $\Psi_4 = \{\alpha,\beta,\rho,\sigma\}$ can be expressed in the form

$$\Psi_4 = \nabla_4\phi_4 + \phi\cdot\phi,$$

while $\Psi_3=\{\rho,\sigma,\underline{\beta},\underline{\alpha}\}$ can be represented as

$$\Psi_3=\nabla_3\phi_3+\mathrm{tr}\underline{\chi}_0\cdot\psi+\phi\cdot\phi,$$

with[20] $\phi_4\in\{\hat\chi,\eta,<\underline{\omega}>\}$ and $\phi_3\in\{\underline{\hat\chi},\underline{\eta},<\omega>\}$.

Therefore,

$$\|\Psi_4\|_{\mathrm{Tr}\,sc(H)}\lesssim\|\nabla_4\phi_4\|_{\mathrm{Tr}\,sc(H)}+\delta^{\frac12}\|\phi\|^2_{\mathcal{L}^\infty_{(sc)}},$$

$$\|\Psi_3\|_{\mathrm{Tr}\,sc(\underline{H})}\lesssim\|\nabla_3\phi_3\|_{\mathrm{Tr}\,sc(H)}+(1+\delta^{\frac12}\|\phi\|_{\mathcal{L}^\infty_{(sc)}})\|\phi\|_{\mathcal{L}^\infty_{(sc)}}.$$

By Proposition 11.2

$$\begin{aligned}\|\nabla_4\phi_4\|_{\mathrm{Tr}\,_{(sc)}(H)}\lesssim&\Big(\|\nabla_4^2\phi_4\|_{\mathcal{L}^2_{(sc)}(H)}+\|\phi_4\|_{\mathcal{L}^2_{(sc)}(H)}+\delta^{\frac12}C(\|\phi_4\|_{\mathcal{L}^\infty_{(sc)}}+\|\nabla_4\phi_4\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\&\times\Big(\|\nabla^2\phi_4\|_{\mathcal{L}^2_{(sc)}(H)}+\delta^{\frac12}C(\|\phi_4\|_{\mathcal{L}^\infty_{(sc)}}+\|\nabla\phi_4\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\&+\|\nabla_4\nabla\phi_4\|_{\mathcal{L}^2_{(sc)}(H)}+\delta^{\frac12}C(\|\phi_4\|_{\mathcal{L}^\infty_{(sc)}}+\|\nabla_4\phi_4\|_{\mathcal{L}^4_{(sc)}(S)})+\|\nabla\phi_4\|_{\mathcal{L}^2_{(sc)}(H)}\end{aligned}$$

$$\begin{aligned}\|\nabla_3\phi_3\|_{\mathrm{Tr}\,_{(sc)}(\underline{H})}\lesssim&\Big(\|\nabla_3^2\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\|\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\delta^{\frac12}C(\|\phi_3\|_{\mathcal{L}^\infty_{(sc)}}+\|\nabla_3\phi_3\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\&\times\Big(\|\nabla^2\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\delta^{\frac12}C(\|\phi_3\|_{\mathcal{L}^\infty_{(sc)}}+\|\nabla\phi_3\|_{\mathcal{L}^4_{(sc)}(S)})\Big)^{\frac12}\\&+\|\nabla_3\nabla\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+\delta^{\frac12}C(\|\phi_3\|_{\mathcal{L}^\infty_{(sc)}}+\|\nabla_3\phi_3\|_{\mathcal{L}^4_{(sc)}(S)})+\|\nabla\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}\end{aligned}$$

We observe that all the involved norms with the exception of $\|\nabla_4^2\phi_4\|_{\mathcal{L}^2_{(sc)}(H)}$ and $\|\nabla_3^2\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$ have been already estimated.

Recall that the derivatives with no estimates are the $\mathcal{L}^4_{(sc)}(S)$ norms of $\nabla_4\omega,\nabla_3\underline{\omega}$ and either $\mathcal{L}^2_{(sc)}(H)$ and $\mathcal{L}^2_{(sc)}(\underline{H})$ norms of $\nabla\nabla_4\omega$ and $\nabla\nabla_3\underline{\omega}$, while $\nabla\nabla_4\hat\chi$ and $\nabla\nabla_3\underline{\hat\chi}$ are controlled only along H and $\underline{H}$ respectively. Finally, the $\mathcal{L}^2_{(sc)}(S)$ and $\mathcal{L}^4_{(sc)}(S)$ estimates for $\hat\chi,\underline{\hat\chi}$, $\nabla_{3,4}\hat\chi$, $\nabla_{3,4}\underline{\hat\chi}$ are $\delta^{-\frac12}$ and $\delta^{-\frac14}$ anomalous. Therefore, for $\phi_4=\hat\chi$, i.e. $\Psi_4=\alpha$

$$\|\nabla_4\hat\chi\|_{\mathrm{Tr}\,_{(sc)}(H)}\lesssim C(\|\nabla_4^2\hat\chi\|_{\mathcal{L}^2_{(sc)}(H)}+C\delta^{-\frac12})^{\frac12}+C,$$

for $\phi_3=\underline{\hat\chi}$, i.e. $\Psi_3=\underline{\alpha}$

$$\|\nabla_3\underline{\hat\chi}\|_{\mathrm{Tr}\,_{(sc)}(\underline{H})}\lesssim C(\|\nabla_3^2\underline{\hat\chi}\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+C\delta^{-\frac12})^{\frac12}+C.$$

The remaining ϕ_4,ϕ_3 satisfy

$$\|\nabla_4\phi_4\|_{\mathrm{Tr}\,_{(sc)}(H)}\lesssim C(\|\nabla_4^2\phi_4\|_{\mathcal{L}^2_{(sc)}(H)}+C)^{\frac12}+C,$$

$$\|\nabla_3\phi_3\|_{\mathrm{Tr}\,_{(sc)}(\underline{H})}\lesssim C(\|\nabla_4^2\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})}+C)^{\frac12}+C.$$

[20]Recall that $<\underline{\omega}>=(\underline{\omega},\underline{\omega}^\dagger)$ and $<\omega>=(-\omega,\omega^\dagger)$, see (122) and (123).

We now express

$$\nabla_4^2\phi_4 = \nabla_4\Psi_4 + \nabla_4\phi\cdot\phi,$$
$$\nabla_3^2\phi_3 = \nabla_3\Psi_3 + \nabla_3\phi\cdot\psi.$$

Therefore,

$$\|\nabla_4^2\phi_4\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\nabla_4\Psi_4\|_{\mathcal{L}^2_{(sc)}(H)} + \delta^{\frac12}\|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(H)}\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \lesssim \|\nabla_4\Psi_4\|_{\mathcal{L}^2_{(sc)}(H)} + C,$$
$$\|\nabla_3^2\phi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \|\nabla_3\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \delta^{\frac12}\|\nabla_3\phi\|_{\mathcal{L}^2_{(sc)}(H)}\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \lesssim \|\nabla_4\Psi_4\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + C,$$

where we took into account possible $\delta^{-\frac12}$ anomalies of $\|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(H)}$ and $\|\nabla_3\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})}$. These immediately yield the desired trace estimates for α and $\underline{\alpha}$. For the remaining components Ψ_4, Ψ_3 we may express from Bianchi

$$\nabla_4\Psi_4 = \nabla\Psi^4 + \phi\cdot\Psi,$$
$$\nabla_3\Psi_3 = \nabla\Psi^3 + \mathrm{tr}\underline{\chi}_0\cdot\Psi + \phi\cdot\Psi,$$

where $\Psi^4 \in \{\alpha,\beta\}$ and $\Psi^3 \in \{\underline{\alpha},\underline{\beta}\}$. Therefore,

$$\|\nabla_4\Psi_4\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\nabla\Psi^4\|_{\mathcal{L}^2_{(sc)}(H)} + \delta^{\frac12}\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\|\Psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \mathcal{R} + C,$$
$$\|\nabla_3\Psi_3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \|\nabla\Psi^3\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + (1+\delta^{\frac12}\|\phi\|_{\mathcal{L}^\infty_{(sc)}})\|\Psi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} \lesssim \mathcal{R} + C.$$

In the last step we have to be careful to avoid the double anomalous term $\mathrm{tr}\underline{\chi}_0\cdot\alpha$. Its appearance is prohibited by the signature considerations, according to which

$$1 \geq sgn(\nabla_3\Psi_3) = sgn(\mathrm{tr}\underline{\chi}_0\cdot\alpha) = 2.$$

13. Estimates for the Rotation Vectorfields

We define the algebra of rotation vectorfields ${}^{(i)}O$ obeying the commutation relations

$$[{}^{(i)}O, {}^{(j)}O] = \in_{ijk}\ {}^{(k)}O,$$

obtained by parallel transport of the standard rotation vectorfields on $\mathbb{S}^2 = S_{u,0} \subset H_{u,0}$ along the integral curves of e_4. Suppressing the index ${}^{(i)}$ we obtain that

$$\nabla_4 O_b = \chi_{bc}O_c.$$

Commuting with ∇ and ∇_3 we obtain

$$\nabla_4(\nabla O) = \chi\cdot\nabla O + \beta\cdot O + \nabla\chi\cdot O + \chi\cdot\underline{\eta}\cdot O,$$
$$\nabla_4(\nabla_3 O) = (\underline{\eta}-\eta)\cdot\nabla O + (\chi+\omega)\nabla_3 O + \sigma\cdot O + (\underline{\omega}\cdot\chi + \eta\cdot\underline{\eta})\cdot O + \nabla_3\chi\cdot O$$

The only non-trivial components of the deformation tensor $\pi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha O_\beta + \nabla_\beta O_\alpha)$ are given below:

$$\begin{aligned}
\pi_{34} &= -2(\eta + \underline{\eta})_a O_a,\\
\pi_{ab} &= \frac{1}{2}(\nabla_a O_b + \nabla_b O_a),\\
\pi_{3a} &= \frac{1}{2}(\nabla_3 O_a - \underline{\chi}_{ab} O_b) := \frac{1}{2} Z_a.
\end{aligned}$$

13.1. **Estimates for** H, Z**.** The quantity Z verifies the following transport equation[21], written schematically,

$$\nabla_4 Z = \nabla(\eta + \underline{\eta}) \cdot O + (\underline{\eta} - \eta) \cdot \nabla O + \omega Z + (\sigma + \rho) \cdot O + (\eta - \underline{\eta}) \cdot (\eta + \underline{\eta}) \cdot O$$

Let $H_{ab} = \nabla_a O_b$ denote the non-symmetrized derivative of O. Then,

$$\nabla_4 H = \chi \cdot H + \beta \cdot O + \nabla \chi \cdot O + \chi \cdot \underline{\eta} \cdot O$$

We now rewrite these equations schematically in the form

$$\begin{aligned}
\nabla_4 Z &= \nabla \psi_{34} \cdot O + \psi_{34} \cdot H + (\chi + \omega) Z + \Psi_g \cdot O + \psi_{34} \cdot \psi_{34} \cdot O,\\
\nabla_4 H &= \psi \cdot H + (\Theta_4 + \nabla \psi_4) \cdot O + \psi \cdot \psi_{34} \cdot O.
\end{aligned} \tag{212}$$

Here $\psi_{34} \in \{\eta, \underline{\eta}\}$, $\Psi_g \in \{\rho, \sigma\}$. In what follows $\psi_3 4$ will be treated either as a ψ_3 or a ψ_4 quantity, depending on the situation. The quantities, H and Z can be assigned signature and scaling, (consistent with those for the Ricci coefficients and curvature components) according to.

$$sgn(H) - \frac{1}{2} = sc(H) = 0, \qquad sgn(Z) - \frac{1}{2} = sc(Z) = -\frac{1}{2}. \tag{213}$$

In view of equations (212) we derive, by integration,

$$\|Z\|_{\mathcal{L}^\infty_{(sc)}} \lesssim \|\nabla \psi_4\|_{Tr_{(sc)}} + \|\Psi_g\|_{Tr_{(sc)}} + \delta^{\frac{1}{2}} \|\psi\|_{\mathcal{L}^\infty_{(sc)}} (\|\psi\|_{\mathcal{L}^\infty_{(sc)}} + \|H\|_{\mathcal{L}^\infty_{(sc)}} + \|Z\|_{\mathcal{L}^\infty_{(sc)}})$$

Thus, according to the trace estimates of proposition 11.10 for $\psi_4 \in \{\eta, \underline{\eta}\}$ and proposition 12.1 for Ψ_g we derive,

$$\|Z\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C + \delta^{\frac{1}{2}} C (\|H\|_{\mathcal{L}^\infty_{(sc)}} + \|Z\|_{\mathcal{L}^\infty_{(sc)}})$$

Similarly,

$$\begin{aligned}
\|H\|_{\mathcal{L}^\infty_{(sc)}} &\lesssim \|\nabla \psi_4\|_{Tr_{(sc)}} + \|\Theta_4\|_{\mathcal{L}^\infty_{(sc)}} + \delta^{\frac{1}{2}} \|\psi\|^2_{\mathcal{L}^\infty_{(sc)}} (\|\psi\|_{\mathcal{L}^\infty_{(sc)}} + \|H\|_{\mathcal{L}^\infty_{(sc)}})\\
&\lesssim C + \delta^{\frac{1}{2}} C (C + \|H\|_{\mathcal{L}^\infty_{(sc)}}),
\end{aligned}$$

Therefore we have proved[22] the following.

[21] Note the absence of $\underline{\chi}$ and $\underline{\omega}$.

[22] Note the triviality of the data for Z on $\underline{H}_0$. Otherwise the term $\underline{\chi} \cdot O$ in the definition of Z might have caused an $\mathcal{L}^\infty_{(sc)}$ anomaly. The data for H however is not trivial. Initially $\|H\|_{L^\infty} \sim 1$, which means that while it is anomalous in $\mathcal{L}^2_{(sc)}(S)$ it is not in $\mathcal{L}^\infty_{(sc)}$.

Proposition 13.2. *The quantities Z and H verify the estimates*

$$\|H\|_{\mathcal{L}^\infty_{(sc)}} + \|Z\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C,$$

with a constant $C = C(\mathcal{I}^{(0)}, \mathcal{R}_{[1]}, \underline{\mathcal{R}}_{[1]})$.

We add a small remark concerning the symmetrized ∇ derivatives of O.

Proposition 13.3. *Let* $H'_{ab} := \nabla_a O_b + \nabla_a O_b = H_{ab} + H_{ba}$. *Then in addition to all the estimates for* H, H' *also enjoys a non-anomalous* $\mathcal{L}^2_{(sc)}(S)$ *estimate*

$$\|H'\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C.$$

Similarly,

$$\|Z\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C.$$

The result follows easily from the transport equation for H', which is virtually the same as for H, and crucially, triviality of the initial data for H^s. The claim for Z follows from the same considerations.

13.4. $\mathcal{L}^2_{(sc)}(S)$ **estimates for** $\nabla H, \nabla Z$**.** We prove below the following,

Proposition 13.5. *The following estimates hold true with* $C = C(\mathcal{I}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$,

$$\begin{aligned}
&\|\nabla H\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla Z\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C,\\
&\|\nabla_4 \nabla H\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4 \nabla Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C
\end{aligned}$$

Proof. We first commute the transport equations for H and Z with ∇.

$$\begin{aligned}
\nabla_4(\nabla H) &= \psi \cdot \nabla H + \nabla \psi \cdot H + (\nabla \Theta_4 + \nabla^2 \psi_4) \cdot O + (\Theta_4 + \Psi_g) \cdot H\\
&\quad + \psi \cdot \nabla \psi \cdot O + \psi_{34} \cdot \nabla_4 H + \psi \cdot \psi_g \cdot H,\\
\nabla_4(\nabla Z) &= \nabla^2 \psi_{34} \cdot O + (\nabla \psi + \Psi_g) \cdot (H + Z) + \psi \cdot (\nabla H + \nabla Z) + \nabla \Psi_g \cdot O\\
&\quad + \psi \cdot \nabla \psi \cdot O + \psi \cdot \psi \cdot (H + Z) + \psi_{34} \nabla_4 Z
\end{aligned}$$

The term $\nabla\Psi_g$ is in fact $\nabla(\sigma+\rho)$. The estimate for ∇H follows immediately from the following:

$$\begin{aligned}
&\|\psi\cdot\nabla H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C\|\nabla H\|_{\mathcal{L}^2_{(sc)}(H)}\\
&\|\nabla\psi\cdot H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|H\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C\\
&\|\nabla\Theta\cdot O\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\nabla\Theta\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C\\
&\|\nabla^2\psi\cdot O\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\nabla^2\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C\\
&\|(\Theta+\Psi_g)\cdot H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|H\|_{\mathcal{L}^\infty_{(sc)}}(\|\Theta\|_{\mathcal{L}^2_{(sc)}(H)}+\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(H)}) \lesssim \delta^{\frac12}C\\
&\|\psi\cdot\nabla\psi\cdot O\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C\\
&\|\psi\cdot\psi_g\cdot H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|H\|_{\mathcal{L}^\infty_{(sc)}}\|\psi_g\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta C,\\
&\|\psi\cdot\nabla_4 H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_4 H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C.
\end{aligned}$$

The estimates for ∇Z are proved in exactly the same manner. □

13.6. $\mathcal{L}^4_{(sc)}(S)$ **estimates for** $\nabla H, \nabla Z$**.** The results of the previous proposition can be strengthened to give the following,

Proposition 13.7. *The following hold true,*

$$\|\nabla H\|_{\mathcal{L}^4_{(sc)}(S)} + \|\nabla Z\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C$$

Proof. The arguments can be followed almost verbatim, as in the last proposition, with the exception of the analysis of the two terms:

$$\nabla^2\psi_{34}\cdot O, \quad \nabla\Psi_g\cdot O = \nabla(\sigma+\rho)\cdot O$$

We recall that $\psi_{34}=\{\eta,\underline{\eta}\}$ and according to Proposition 11.9 we can write,

$$\nabla\psi_{43} = \nabla_4\phi$$

with ϕ satisfying the estimates

$$\begin{aligned}
&\|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla^2\phi\|_{\mathcal{L}^2_{(sc)}(\underline{H})} + \|\nabla\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\phi\|_{\mathcal{L}^2_{(sc)}(S)} \le C,\\
&\|\nabla_4\phi\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\nabla\phi_4\|_{\mathcal{L}^2_{(sc)}(S)} + \|\phi\|_{\mathcal{L}^\infty_{(sc)}} + \|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C
\end{aligned}$$

We now the write

$$\begin{aligned}
\nabla^2\psi_{43}\cdot O &= \nabla_4(\nabla\phi\cdot O) - \nabla\phi\cdot\chi\cdot O - [\nabla_4,\nabla]\phi\cdot O\\
&= \nabla_4(\nabla\phi\cdot O) + \chi\cdot\nabla\phi\cdot O + \Psi_g\cdot\phi\cdot O + \psi\cdot\nabla\psi\cdot O + \psi\cdot\psi\cdot\phi\cdot O
\end{aligned}$$

We estimate

$$\delta^{-1}\int_0^{\underline{u}}\|\nabla\phi\cdot\chi\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u}\lesssim\delta^{\frac12}\sup_{\underline{u}}\|\nabla\phi\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}\|\chi\|_{\mathcal{L}^\infty_{(sc)}}\lesssim\delta^{\frac12}C,$$

$$\delta^{-1}\int_0^{\underline{u}}\|\Psi_g\cdot\phi\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u}\lesssim\delta^{\frac12}\left(\|\nabla\Psi_g\|^{\frac12}_{\mathcal{L}^2_{(sc)}(H)}\|\Psi_g\|^{\frac12}_{\mathcal{L}^2_{(sc)}(H)}+\delta^{\frac14}\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(H)}\right)\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\lesssim\delta^{\frac12}C,$$

$$\delta^{-1}\int_0^{\underline{u}}\|\nabla\psi\cdot\psi\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u}\lesssim\delta^{\frac12}\sup_{\underline{u}}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\lesssim\delta^{\frac12}C,$$

$$\delta^{-1}\int_0^{\underline{u}}\|\psi\cdot\psi\cdot\phi\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u}\lesssim\delta\sup_{\underline{u}}\|\phi\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}\|\psi\|^2_{\mathcal{L}^\infty_{(sc)}}\lesssim\delta C.$$

On the other hand, the null structure equations give for $<\underline{\omega}>=(\underline{\omega},\underline{\omega}^\dagger)$

$$\nabla_4<\underline{\omega}>=(\rho,\sigma)+\psi_g\cdot\psi_g.$$

As a result,

$$\nabla(\rho,\sigma)\cdot O=\nabla_4(\nabla<\underline{\omega}>\cdot O)+(\psi\cdot\nabla\psi+\chi\cdot\nabla<\underline{\omega}>+\Psi_g\cdot<\underline{\omega}>+\psi_g\cdot\Psi_g+\psi\cdot\psi_g\cdot(<\underline{\omega}>+\psi_g))\cdot O$$

We can estimate

$$\begin{aligned}
\delta^{-1}\int_0^{\underline{u}}\|\nabla<\underline{\omega}>\cdot\chi\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u} &\lesssim \delta^{\frac12}\sup_{\underline{u}}\|\nabla<\underline{\omega}>\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}\|\chi\|_{\mathcal{L}^\infty_{(sc)}}\lesssim\delta^{\frac12}C,\\
\delta^{-1}\int_0^{\underline{u}}\|\Psi_g\cdot\psi\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u} &\lesssim \delta^{\frac12}\left(\|\nabla\Psi_g\|^{\frac12}_{\mathcal{L}^2_{(sc)}(H)}\|\Psi_g\|^{\frac12}_{\mathcal{L}^2_{(sc)}(H)}+\delta^{\frac14}\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(H)}\right)\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\\
\delta^{-1}\int_0^{\underline{u}}\|\nabla\psi\cdot\psi\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u} &\lesssim \delta^{\frac12}\sup_{\underline{u}}\|\nabla\psi\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\lesssim\delta^{\frac12}C,\\
\delta^{-1}\int_0^{\underline{u}}\|\psi\cdot\psi\cdot(<\underline{\omega}>+\psi_g)\cdot O\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}})}d\underline{u} &\lesssim \delta\sup_{\underline{u}'\le\underline{u}}\|<\underline{\omega}>+\psi_g\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}'})}\|\psi\|^2_{\mathcal{L}^\infty_{(sc)}}\lesssim\delta C.
\end{aligned}$$

These allow us to conclude that,

$$\delta^{-1}\int_0^{\underline{u}}\|\nabla_4\left[(\nabla H,\nabla Z)-\nabla\phi\cdot O-\nabla<\underline{\omega}>\cdot O\right]\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}'})}d\underline{u}''\quad\lesssim\quad\delta^{\frac12}\sup_{\underline{u}'\le\underline{u}}\|(\nabla H,\nabla Z)\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}'})}+\delta^{\frac12}C.$$

Making use of the $\mathcal{L}^4_{(sc)}(S)$ bounds on both $\nabla\phi$ and $\nabla<\underline{\omega}>$ we finally obtain the estimate $\delta^{-1}\int_0^u\|\nabla_4(\nabla H,\nabla Z)\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}'})}du'\lesssim\delta^{\frac12}\sup_{\underline{u}'\le\underline{u}}\|(\nabla H,\nabla Z)\|_{\mathcal{L}^4_{(sc)}(S_{u,\underline{u}'})}+C\delta^{1/2}$, from which the conclusion of the proposition easily follows. □

13.8. **Estimates for $\nabla_3 Z$.** We now examine the equation for $\nabla_3 Z$.

$$\begin{aligned}
\nabla_4(\nabla_3 Z) &= \nabla_3\nabla\psi_{34}+\nabla\psi_{34}\cdot Z+\nabla\psi_{34}\cdot\underline{\chi}+\nabla_3\psi_{34}\cdot H+\psi_{34}\cdot\nabla_3 H\\
&\quad+(\nabla_3\chi+\nabla_3\omega)\cdot Z+\omega\cdot\nabla_3 Z\\
&\quad+\nabla_3\Psi_g\cdot O+(\rho+\sigma)\cdot Z+\Psi_g\cdot\underline{\chi}+\nabla_3\psi_{34}\cdot\psi_{34}+\psi_{34}\cdot\psi_{34}\cdot Z+\psi_{34}\cdot\psi_{34}\cdot\underline{\chi},
\end{aligned}$$

To estimate the right hand side of this equation we will need to use the first and second derivative estimates for ψ of Propositions 8.2,8.4,8.7 and 8.12, keeping in mind possible anomalies of $\underline{\chi}$, $\nabla_4\hat{\chi}, \nabla_3\hat{\chi}, \nabla_3\underline{\hat{\chi}}$, the relationship

$$\nabla_3(\rho+\sigma) = \nabla\underline{\beta} + (\mathrm{tr}\underline{\chi}_0 + \psi)\cdot\Psi,$$

given by the null Bianchi identities and the $\mathcal{L}^2_{(sc)}(S)$ curvature estimate[23] $\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)} \le C$ of Propositions 6.6 and 6.9. Thus,

$$\begin{aligned}
&\|\nabla_3\nabla\psi_{34}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C,\\
&\|\nabla\psi\cdot Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|Z\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\
&\|\nabla\psi\cdot\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C,\\
&\|\nabla_3\psi_{34}\cdot H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|H\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_3\psi_{34}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\
&\|\psi\cdot\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(H)},\\
&\|\nabla_3\omega\cdot Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|Z\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_3\omega\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\
&\|\nabla_3\chi\cdot Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|Z\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_3\chi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C,\\
&\|\omega\cdot\nabla_3 Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\omega\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_3 Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C\|\nabla_3 Z\|_{\mathcal{L}^2_{(sc)}(H)},\\
&\|\nabla_3(\rho+\sigma)\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\nabla\underline{\beta}\|_{\mathcal{L}^2_{(sc)}(H)} + \|(\mathrm{tr}\underline{\chi}_0+\psi)\cdot\Psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \mathcal{R}_1 + C,\\
&\|\Psi_g\cdot Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|Z\|_{\mathcal{L}^\infty_{(sc)}}\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\
&\|\Psi_g\cdot\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\|\Psi_g\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C,\\
&\|\nabla_3\psi_{34}\cdot\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_3\psi_{34}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\
&\|\psi\cdot\psi\cdot Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta\|Z\|_{\mathcal{L}^\infty_{(sc)}}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\psi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\
&\|\psi\cdot\psi_{34}\cdot\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta\|\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\psi_g\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C
\end{aligned}$$

13.9. **Estimates for $\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(H)}$.** The only quantity still requiring an estimate is $\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(H)}$. We use the relation[24]

$$\nabla_3 H = \nabla_3\nabla O = \nabla\nabla_3 O + [\nabla,\nabla_3]O = \nabla Z + \nabla\underline{\chi}\cdot O + \underline{\beta}\cdot O + \psi_{34}\cdot Z + \psi_{34}\cdot\underline{\chi}\cdot O$$

[23]Note that Ψ in the nonlinear term may contain an $\underline{\alpha}$ component but not the anomalous α term.

[24]Note a crucial cancellation of an anomalous term $\underline{\chi}\cdot H$.

Therefore,

$$\begin{aligned}\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(S)} &\lesssim \|\nabla Z\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla \underline{\chi}\|_{\mathcal{L}^2_{(sc)}(S)} + \|\Psi_g\|_{\mathcal{L}^2_{(sc)}(S)} + \delta^{\frac12}\|\psi_{34}\|_{\mathcal{L}^2_{(sc)}(S)}\|Z\|_{\mathcal{L}^\infty_{(sc)}}\\ &+ \delta^{\frac12}\|\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\|\psi_{34}\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim \|\nabla Z\|_{\mathcal{L}^2_{(sc)}(S)} + C\end{aligned}$$

This immediately implies the bounds

$$\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3 Z\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\nabla_3 Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C.$$

A similar argument allows us to immediately strengthen the $\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(S)}$ estimate (unlike the one for $\nabla_3 Z$) to the $\mathcal{L}^4_{(sc)}(S)$ norm

$$\|\nabla_3 H\|_{\mathcal{L}^4_{(sc)}(S)} \le C$$

Furthermore,

$$\begin{aligned}\nabla_4\nabla_3 H &= \nabla_4\nabla Z + \nabla_4\nabla\underline{\chi}\cdot O + \nabla\underline{\chi}\cdot\chi\cdot O + \nabla_4\underline{\beta}\cdot O + \Psi_g\cdot\chi\cdot O + \nabla_4\psi_{34}\cdot Z\\ &+ \psi_{34}\cdot\nabla_4 Z + \nabla_4\psi_{34}\cdot\underline{\chi}\cdot O + \psi_{34}\cdot\nabla_4\underline{\chi}\cdot O + \psi_{34}\cdot\underline{\chi}\cdot\chi\cdot O\end{aligned}$$

We once again remind the reader of the possible anomalies for $\hat{\chi}, \underline{\hat{\chi}}$ in $\mathcal{L}^2_{(sc)}(S)$, double anomaly for $\mathrm{tr}\underline{\chi}$ in $\mathcal{L}^2_{(sc)}(S)$ and a simple anomaly in $\mathcal{L}^\infty_{(sc)}$, anomalies for $\nabla_4\hat{\chi}$ and $\nabla_3\underline{\hat{\chi}}$. We estimate

$$\begin{aligned}&\|\nabla_4\nabla Z\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C,\\ &\|\nabla_4\nabla\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C,\\ &\|\nabla\underline{\chi}\cdot\chi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\chi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\ &\|\nabla_4\underline{\beta}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \|\nabla\Psi_g\|_{\mathcal{L}^2_{(sc)}(H)} + \|\psi\cdot\Psi_g\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \mathcal{R}_1 + \delta^{\frac12}C,\\ &\|\Psi_g\cdot\chi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\chi\|_{\mathcal{L}^\infty_{(sc)}}\|\Psi_g\|_{sc^2(H)} \lesssim \delta^{\frac12}C,\\ &\|\nabla_4\psi_{34}\cdot\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_4\psi_{34}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C,\\ &\|\psi_{34}\cdot\nabla_4\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}\|\psi\|_{\mathcal{L}^\infty_{(sc)}}\|\nabla_4\underline{\chi}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C,\\ &\|\psi_{34}\cdot\underline{\chi}\cdot\chi\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta\|\chi\|_{\mathcal{L}^\infty_{(sc)}}\|\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\|\psi_{34}\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim \delta^{\frac12}C.\end{aligned}$$

As a result we now established the following

Proposition 13.10. *There exists a constant $C = C(\mathcal{O}_{[2]}, \mathcal{O}_\infty, \mathcal{R}_{[1]}, \underline{\mathcal{R}}_{[1]})$ such that*

$$\|\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_3 Z\|_{\mathcal{L}^2_{(sc)}(S)} + \|\nabla_4\nabla_3 Z\|_{\mathcal{L}^2_{(sc)}(H)} + \|\nabla_4\nabla_3 H\|_{\mathcal{L}^2_{(sc)}(H)} \lesssim C.$$

13.11. **Derivatives of the deformation tensor.** We now compute the derivatives of the deformation tensor $D\pi$.

$$D_4\pi_{44}=0, \quad D_4\pi_{34}=-2\nabla_4(\eta+\underline{\eta})\cdot O-2(\eta+\underline{\eta})\cdot\chi\cdot O,$$
$$D_4\pi_{33}=\frac{1}{4}\underline{\eta}\cdot Z, \quad D_4\pi_{3a}=\frac{1}{2}\nabla_4 Z+\underline{\eta}\cdot(\eta+\underline{\eta})-\frac{1}{2}\underline{\eta}\cdot H^s, \quad D_4\pi_{4a}=0, \quad D_4\pi_{ab}=\nabla_4 H^s,$$
$$D_3\pi_{44}=0, \quad D_3\pi_{34}=-2\nabla_3(\eta+\underline{\eta})\cdot O-2(\eta+\underline{\eta})\cdot(Z-\underline{\chi}\cdot O)-\frac{1}{4}\eta\cdot Z,$$
$$D_3\pi_{33}=0, \quad D_3\pi_{3a}=\frac{1}{2}\nabla_3 Z, \quad D_3\pi_{4a}=-\eta\cdot(\eta+\underline{\eta})-\frac{1}{2}\eta\cdot H^s, \quad D_3\pi_{ab}=\nabla_3 H^s+\frac{1}{4}\eta\cdot Z,$$
$$D_c\pi_{44}=0, \quad D_c\pi_{34}=-2\nabla(\eta+\underline{\eta})\cdot O-2(\eta+\underline{\eta})\cdot H^s-\frac{1}{2}\chi\cdot Z,$$
$$D_c\pi_{33}=-\frac{1}{2}\underline{\chi}\cdot Z, \quad D_c\pi_{3a}=\frac{1}{2}\nabla Z-\underline{\chi}\cdot H^s-2\underline{\chi}(\eta+\underline{\eta})\cdot O,$$
$$D_c\pi_{4a}=-\chi\cdot H^s-2\chi(\eta+\underline{\eta})\cdot O, \quad D_c\pi_{ab}=\nabla H^s-\chi\cdot Z,$$

Based on the results of the previous section we then easily deduce the following result

Proposition 13.12. *There exists a constant $C=C(\mathcal{O}_{[2]},\mathcal{O}_\infty,\mathcal{R}_{[1]},\underline{\mathcal{R}}_{[1]})$ such that*

$$\|D\pi\|_{\mathcal{L}^2_{(sc)}(S)}\lesssim C$$

The only potentially problematic term is $\underline{\chi}\cdot H^s$, which can be estimated as follows:

$$\|\underline{\chi}\cdot H^s\|_{\mathcal{L}^2_{(sc)}(S)}\lesssim\delta^{\frac{1}{2}}\|\underline{\chi}\|_{\mathcal{L}^\infty_{(sc)}}\|H^s\|_{\mathcal{L}^2_{(sc)}}\lesssim C.$$

It is precisely this term that requires a non-anomalous $\mathcal{L}^2_{(sc)}(S)$ estimate for H^s, which incidentally does not hold for the non-symmetrized derivative H.

13.13. **Theorem B.** We are now ready to state the main result of this section, mentioned in the introduction.

Theorem 13.14 (Theorem B)**.** *The deformation tensors ${}^{(O)}\pi$ of the angular momentum operators O verify the following estimates, with a constant $C=C(\mathcal{I}^{(0)},\mathcal{R},\underline{\mathcal{R}})$,*

$$\|{}^{(O)}\pi\|_{\mathcal{L}^4_{(sc)}(S)}+\|{}^{(O)}\pi\|_{\mathcal{L}^\infty_{(sc)}(S)}\lesssim C \tag{214}$$

Also all null components of the derivatives $D\,{}^{(O)}\pi$, with the exception of $(D_3\,{}^{(O)}\pi)_{3a}$, verify the estimates,

$$\|D\,{}^{(O)}\pi\|_{\mathcal{L}^4_{(sc)}(S)}\lesssim C \tag{215}$$

Moreover,

$$\|(D_3\,{}^{(O)}\pi)_{3a}-\nabla_3 Z\|_{L^4(S)}+\|\sup_{\underline{u}}\nabla_3 Z\|_{L^2(S)}\lesssim C \tag{216}$$

14. Curvature estimates I.

In this section, in all the remaining sections of the paper C denotes a constant which depends on the initial data $\mathcal{I}_0$ all the curvature norms $\mathcal{R}, \underline{\mathcal{R}}$, including $\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$ and $\|\nabla_3\alpha\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})}$. Using the results of the previous sections we assume that the norms $\mathcal{O}$ of the Ricci coefficients are bounded by C.

14.1. **Preliminaries.** Let W be a Weyl tensorfield, with ${}^*W$ its Hodge dual verifying the Bianchi equations with sources

$$\mathrm{Div}\, W = J, \qquad \mathrm{Div}\ {}^*W = J^* \tag{217}$$

where J, ${}^*J$ are Weyl currents, i.e.

$$J_{[\alpha\beta\gamma]} \;=\; 0, \qquad J_{\alpha\beta\gamma} = -J_{\alpha\gamma\delta}, \qquad g^{\beta\gamma} J_{\beta\gamma\delta} = 0.$$

and $J^*_{\alpha\beta\gamma} = \frac{1}{2} J_{\alpha\mu\nu} \in^{\mu\nu}{}_{\beta\gamma}$ the right Hodge dual of J. Following the definitions of [Chr-Kl] we let $Q[W]$ be the Bel-Robinson tensor of W. As proved there we have,

Proposition 14.2. *Assume W verifies* (217)*. Given vectorfields X, Y, Z and $P[W] = P[W](X, Y, Z)$ defined by $P[W]^\alpha := Q[W]_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta$ we have,*

$$Div\,(P[W]) \;=\; Div\, Q[W](X, Y, Z) + \frac{1}{2}(Q[W] \cdot \pi)(X, Y, Z) \tag{218}$$

where,

$$\begin{aligned}(Q[W] \cdot \pi)(X, Y, Z) : \;&=\; Q[W]({}^{(X)}\pi, Y, Z) + Q[W]({}^{(Y)}\pi\,, X, Z)\\ &+\; Q[W]({}^{(Z)}\pi\,, X, Y)\end{aligned}$$

Thus, integrating on our fundamental domain $\mathcal{D} = \mathcal{D}(u, \underline{u})$,

$$\begin{aligned}&\int_{H_u} Q[W](L, X, Y, Z) + \int_{\underline{H}_{\underline{u}}} Q[W]X, Y, Z, (\underline{L})\\ &= \int_{H_0} Q[W](L, X, Y, Z) + \int_{\underline{H}_0}]Q[W](X, Y, Z, \underline{L})\\ &+ \int\int_{\mathcal{D}(u,\underline{u})} Div\, Q[W](X, Y, Z) + \frac{1}{2}\int\int_{\mathcal{D}(u,\underline{u})} Q[W] \cdot \pi(X, Y, Z)\end{aligned}$$

In the particular case when W is the curvature tensor R (and thus $J = J^* = 0$), recalling that the initial data on $\underline{\mathcal{H}}_0$ vanishes, we have

Corollary 14.3. *The following identity holds on our fundamental domain $\mathcal{D}(u,\underline{u})$,*

$$\int_{H_u} Q[R](L,X,Y,Z) + \int_{\underline{H}_{\underline{u}}} Q[R](X,Y,Z,\underline{L}) = \int_{H_0} Q[R](L,X,Y,Z) + \frac{1}{2}\int\int_{\mathcal{D}(u,\underline{u})} Q[R]\cdot\pi(X,Y,Z)$$

On the other hand, given a vectorfield O, we have

$$\mathrm{Div}\,(\widehat{\mathcal{L}}_O R) = J(O,R), \qquad \mathrm{Div}\,(\,^*\widehat{\mathcal{L}}_O R) = J^*(O,R). \tag{219}$$

where $J(O,R)$ is a Weyl current (calculated below in lemma 14.5) and $\widehat{\mathcal{L}}_O R$ denotes the modified Lie derivative of the curvature tensor R, i.e. (following [Chr-Kl]), $\widehat{\mathcal{L}}_O R = \mathcal{L}_O R - \frac{1}{8}\mathrm{tr}\,^{(O)}\pi\, R - \frac{1}{2}\,^{(O)}\hat{\pi}\,\cdot R$ and,

$$(\,^{(O)}\hat{\pi}\,\cdot R)_{\alpha\beta\gamma\delta} = \,^{(O)}\hat{\pi}\,_\alpha^{\ \mu} W_{\mu\beta\gamma\delta} + \,^{(O)}\hat{\pi}\,_\beta^{\ \mu} W_{\alpha\mu\gamma\delta} + \,^{(O)}\hat{\pi}\,_\gamma^{\ \mu} W_{\alpha\beta\mu\delta} + \,^{(O)}\hat{\pi}\,_\delta^{\ \mu} W_{\alpha\beta\gamma\mu}$$

with $^{(O)}\hat{\pi}$ is the traceless part of $^{(O)}\pi$, i.e. $^{(O)}\pi = \,^{(O)}\hat{\pi} + \frac{1}{4}\mathrm{tr}\,^{(O)}\pi\, g$. Observe that $\widehat{\mathcal{L}}_O R$ is also a Weyl field and that the modified Lie derivative commutes with the Hodge dual, i.e.,$\widehat{\mathcal{L}}_O(\,^*R\,) = \,^*\widehat{\mathcal{L}}_O R$. The following corollary of proposition 14.2 and proposition 7.1.1 in [Chr-Kl].

Corollary 14.4. *Let O be a vectorfield defined in our fundamental domain $\mathcal{D}(u,\underline{u})$, tangent to $\underline{\mathcal{H}}_{\,0}$. Then, with $H_u = H_u([0,\underline{u}])$,*

$$\int_{H_u} Q[\widehat{\mathcal{L}}_O R](L,X,Y,Z) + \int_{\underline{H}_{\underline{u}}} Q[\widehat{\mathcal{L}}_O R](X,Y,Z,\underline{L}) = \int_{H_0} Q[\widehat{\mathcal{L}}_O R](L,X,Y,Z)$$
$$+\frac{1}{2}\int\int_{\mathcal{D}(u,\underline{u})} Q[\widehat{\mathcal{L}}_O R]\cdot\hat{\pi}(X,Y,Z) + \int\int_{\mathcal{D}(u,\underline{u})} D(R,O)(X,Y,Z)$$

where, $D(O,R) := Div Q[\widehat{\mathcal{L}}_O R]$ is given by the formula,

$$D(O,R)_{\beta\gamma\delta} = (\widehat{\mathcal{L}}_O R)_\beta{}^\mu{}_\delta{}^\nu J(O,R)_{\mu\gamma\nu} + (\widehat{\mathcal{L}}_O R)_\beta{}^\mu{}_\gamma{}^\nu J(O,R)_{\mu\delta\nu}$$
$$+ \ ^*(\widehat{\mathcal{L}}_O R)_\beta{}^\mu{}_\gamma{}^\nu J^*(O,R)_{\mu\delta\nu} + \,^*(\widehat{\mathcal{L}}_O R)_\beta{}^\mu{}_\gamma{}^\nu J^*(O,R)_{\mu\delta\nu}$$

The Weyl current $J(O,R)$ is given by the following commutation formula, see proposition 7.1.2 and in [Chr-Kl],

Lemma 14.5. *We have,*

$$Div\,(\widehat{\mathcal{L}}_O R) = J(O;R) := J^1(O;R) + J^2(O;R) + J^3(O;R) \tag{220}$$

$$
\begin{aligned}
J^1(O,R)_{\beta\gamma\delta} &= \frac{1}{2}\,{}^{(O)}\hat{\pi}^{\,\mu\nu} D_\nu R_{\mu\beta\gamma\delta} \\
J^2(O,R)_{\beta\gamma\delta} &= \frac{1}{2}\,{}^{(O)}p_\lambda R^\lambda{}_{\beta\gamma\delta} \\
J^3(O,R)_{\beta\gamma\delta} &= \frac{1}{2}\big({}^{(O)}q_{\alpha\beta\lambda} R^{\alpha\lambda}{}_{\gamma\delta} + {}^{(O)}q_{\alpha\gamma\lambda} R^\alpha{}_\beta{}^\lambda{}_\delta + {}^{(O)}q_{\alpha\delta\lambda} R^\alpha{}_{\beta\gamma}{}^\lambda\big)
\end{aligned}
$$

where, ${}^{(O)}p_\gamma = D^\alpha({}^{(O)}\hat{\pi}_{\alpha\gamma})$. ${}^{(O)}q = D_\beta\,{}^{(O)}\hat{\pi}_{\gamma\alpha} - D_\gamma\,{}^{(O)}\hat{\pi}_{\beta\alpha} - \frac{1}{3}({}^{(O)}p_\gamma\, g_{\alpha\beta} - {}^{(O)}p_\beta\, g_{\alpha\gamma})$

In the remaining part of this section we should establish estimates for the norms $\mathcal{R}_0$ and $\underline{\mathcal{R}}_0$. We start with α.

14.6. **Estimate for α.** We apply corollary 14.3 to $X = Y = Z = e_4$ to derive,

$$
\int_{H_u^{(0,\underline{u})}} |\alpha|^2 + \int_{H_{\underline{u}}^{(0,u)}} |\beta|^2 \lesssim \int_{H_0^{(0,\underline{u})}} |\alpha|^2 + \int_{\mathcal{D}(u,\underline{u})} (Q[R]\cdot{}^{(4)}\pi)(e_4,e_4,e_4) \tag{221}
$$

Based on conservation of signature we write schematically,

$$
(Q[R]\cdot{}^{(4)}\pi)(e_4,e_4,e_4) = \sum_{s_1+s_2+s_3=4} \phi^{(s_1)}\cdot\Psi^{(s_2)}\cdot\Psi^{(s_3)} \tag{222}
$$

with Ricci coefficients $\phi\in\{\chi,\omega,\eta,\underline{\eta},\underline{\omega}\}$, null curvature components Ψ and labels s_1,s_2,s_3 denoting the signature of the corresponding component. In scale invariant norms we have,

$$
\|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\beta\|^2_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})} \lesssim \|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + I
$$

with,

$$
I = \delta^{1/2}\sum_{s_1+s_2+s_3=4} \|\phi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}} \int_0^u \|\Psi^{(s_2)}\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}\cdot\|\Psi^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} du'
$$

By far the worst term occur when $s_2 = s_3 = 2$ and $s_1 = 0$. Observe also that, since the signature of a Ricci coefficient $\phi^{(s_1)}$ may not exceed $s_1 = 1$, neither s_2 or s_3 can be zero, i.e. $\underline{\alpha}$ cannot occur among the curvature terms on the right. Using our estimates, $\|\phi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$, with $C = C(\mathcal{I}^0,\mathcal{R},\underline{\mathcal{R}})$ we deduce,

$$
\begin{aligned}
\|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\beta\|^2_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})} &\lesssim \|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + C\delta^{1/2}\|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \\
&+ C\mathcal{R}_0\delta^{1/2}\|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\delta^{1/2}\mathcal{R}_0^2
\end{aligned}
$$

Therefore, recalling the anomalous character of $\mathcal{R}_0[\alpha]$, $\underline{\mathcal{R}}_0[\beta]$ we deduce,

$$
\mathcal{R}_0[\alpha] + \underline{\mathcal{R}}_0[\beta] \lesssim \mathcal{I}^0 + C\delta^{3/4}\mathcal{R}_0 \tag{223}
$$

14.7. **Remaining estimates.** We follow the procedure outlined in the introduction. Define the energy quantities,

$$\begin{aligned}\mathcal{Q}_0(u,\underline{u}) &= \delta^2\int_{\mathcal{H}_u^{(0,\underline{u})}} Q[R](e_4,e_4,e_4,e_4)+\int_{\mathcal{H}_u^{(0,\underline{u})}} Q[R](e_3,e_4,e_4,e_4)\\ &+ \delta^{-1}\int_{\mathcal{H}_u^{(0,\underline{u})}} Q[R](e_3,e_3,e_4,e_4)+\delta^{-2}\int_{\mathcal{H}_u^{(0,\underline{u})}} Q[R](e_3,e_3,e_3,e_4)\end{aligned} \tag{224}$$

$$\begin{aligned}\underline{\mathcal{Q}}_0(u,\underline{u}) &= \delta^2\int_{\underline{\mathcal{H}}_{\underline{u}}^{(0,u)}} Q[R](e_4,e_4,e_4,e_3)+\int_{\underline{\mathcal{H}}_{\underline{u}}^{(0,u)}} Q[R](e_4,e_4,e_3,e_3)\\ &+ \delta^{-1}\int_{\underline{\mathcal{H}}_{\underline{u}}^{(0,u)}} Q[R](e_4,e_3,e_3,e_3)+\delta^{-2}\int_{\underline{\mathcal{H}}_{\underline{u}}^{(0,u)}} Q[R](e_3,e_3,e_3,e_3)\end{aligned} \tag{225}$$

According to corollary (14.3), for all possible choices of the vectorfields X,Y,Z in the set $\{e_4,e_3\}$ we are led to the identity,

$$\mathcal{Q}_0(u,\underline{u})+\underline{\mathcal{Q}}_0(u,\underline{u}) \approx \mathcal{Q}_0(0,\underline{u})+\mathcal{E}_0(u,\underline{u}) \tag{226}$$

where,

$$\begin{aligned}\mathcal{E}_0(u,\underline{u}) &= \delta^2\int\int_{\mathcal{D}(u,\underline{u})} Q[R](\,^{(4)}\pi,e_4,e_4)\\ &+ \int\int_{\mathcal{D}(u,\underline{u})} Q[R](\,^{(4)}\pi,e_3,e_4)+\int\int_{\mathcal{D}(u,\underline{u})} Q[R](\,^{(3)}\pi,e_4,e_4)\\ &+ \delta^{-1}\int\int_{\mathcal{D}(u,\underline{u})} Q[R](\,^{(4)}\pi,e_3,e_3)+\delta^{-1}\int\int_{\mathcal{D}(u,\underline{u})} Q[R](\,^{(3)}\pi,e_4,e_3)\\ &+ \delta^{-2}\int\int_{\mathcal{D}(u,\underline{u})} Q[R](\,^{(3)}\pi,e_3,e_3)\end{aligned}$$

with $^{(4)}\pi$, $^{(3)}\pi$ the deformation tensors of e_4,e_3. Every term appearing in the above integrands linear in $^{(4)}\pi$ or $^{(3)}\pi$ and quadratic with respect to R. Also all components of $^{(4)}\pi$ can be expressed in terms of our Ricci coefficients $\chi,\omega,\eta,\underline{\eta},\underline{\omega}$. In fact one can easily check the following, $^{(4)}\pi_{44}=\,^{(4)}\pi_{4a}=0$, $^{(4)}\pi_{34}=g(D_3e_4,e_4)+g(D_4e_4,e_3)=4\omega$, $^{(4)}\pi_{33}=2g(D_3e_4,e_3)=-8\underline{\omega}$, $^{(4)}\pi_{ab}=2\chi_{ab}$, $^{(4)}\pi_{a3}=g(D_ae_4,e_3)+g(D_3e_4,e_a)=2\zeta_a+2\eta_a$. A similar formula holds for $^{(3)}\pi$, with χ replaced by $\underline{\chi}$. Observe, in particular, that the term $\mathrm{tr}\underline{\chi}$ can only occur in connection to $^{(3)}\pi$. Thus, all terms appearing in the $\mathcal{E}$ integrand are of the form,

$$\phi\cdot\Psi_1\cdot\Psi_2$$

with ϕ one of the Ricci coefficients and Ψ_1,Ψ_2 null curvature components. Consider first the contribution to $\mathcal{Q}_0$ of the anomalous terms $\delta^2\int_{\mathcal{H}_u^{(0,\underline{u})}} Q[R](e_4,e_4,e_4,e_4)+\delta^2\int_{\underline{\mathcal{H}}_{\underline{u}}^{(0,u)}} Q[R](e_4,e_4,e_4,e_3)$ obtained in (19) in the case $X=Y=Z=e_4$. Since $Q[R](e_4,e_4,e_4,e_4)=|\alpha|^2$ and $Q[R](e_4,e_4,e_4,e_3)=|\beta|^2$

we derive,

$$\begin{aligned} \|\alpha\|^2_{L^2(H_u^{(0,\underline{u})})} + \|\beta\|^2_{L^2(\underline{H}_{\underline{u}}^{(0,u)})} &\approx \|\alpha\|^2_{L^2(H_0^{(0,\underline{u})})} + \mathcal{E}_{01}(u,\underline{u}) \\ \mathcal{E}_{01}(u,\underline{u}) &\approx \int\int_{\mathcal{D}(u,\underline{u})} Q({}^{(4)}\pi, e_4, e_4) \end{aligned}$$

Since all terms of the form $\phi \cdot \Psi_1 \cdot \Psi_2$ have the same overall signature 4. Thus, it is easy to derive the scale invariant norms estimate,

$$\|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\beta\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} \lesssim \|\alpha\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + \mathcal{E}_{01}$$

and,

$$\mathcal{E}_{01} \lesssim \delta^{1/2}\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \int_0^{\underline{u}} \|\Psi_1\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u}')})} \|\Psi_2\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u}')})} \tag{227}$$

The gain of $\delta^{1/2}$ is a reflection of the product estimates of type (46). Now, the only null curvature component which is anomalous with respect to the scale invariant norms $\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})$ is α. On the other hand the only Ricci coefficient which is anomalous in $\mathcal{L}^\infty_{(sc)}$ is $\mathrm{tr}\underline{\chi}$. Indeed we have to decompose $\mathrm{tr}\underline{\chi} = \widetilde{\mathrm{tr}\underline{\chi}} + \mathrm{tr}\underline{\chi}_0$, where $\mathrm{tr}\underline{\chi}_0$ is the flat value of $\mathrm{tr}\underline{\chi}_0$ and therefore independent of δ. This leads to a loss of $\delta^{1/2}$ in the corresponding estimates. Now, since $\mathrm{tr}\underline{\chi}$ cannot appear among the components of ${}^{(4)}\pi$, we can lose at most a power of δ on the right hand side of (227), which occurs only when $\Psi_1 = \Psi_2 = \alpha$. Fortunately the terms on the left of our integral inequality are also anomalous with respect to the same power of δ. Therefore, since $\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$, with $C = C(\mathcal{I}^0, \mathcal{R}, \underline{\mathcal{R}})$ we derive

$$\mathcal{R}_0^2[\alpha] + \underline{\mathcal{R}}_0^2[\beta] \lesssim (\mathcal{I}^{(0)})^2 + \delta^{1/2} \cdot C\mathcal{R}_0^2.$$

Therefore, for small $\delta > 0$, we derive the bound,

$$\mathcal{R}_0[\alpha] + \underline{\mathcal{R}}_0[\beta] \lesssim \mathcal{I}^{(0)} + \delta^{1/4}C(\mathcal{R}, \underline{\mathcal{R}}). \tag{228}$$

with C a universal constant depending only on the curvature norms $\mathcal{R}, \underline{\mathcal{R}}$. We would like to show that all other error terms can be estimated in the same fashion, i.e. we would like to prove an estimate of the form,

$$\mathcal{R}_0 + \underline{\mathcal{R}}_0 \lesssim \mathcal{I}^{(0)} + \delta^{1/4}C(\mathcal{R}, \underline{\mathcal{R}}). \tag{229}$$

Assuming that a similar estimate holds for $\mathcal{R}_1 + \underline{\mathcal{R}}_1$ we would thus conclude, for sufficiently small $\delta > 0$,

$$\mathcal{R} + \underline{\mathcal{R}} \lesssim \mathcal{I}_0. \tag{230}$$

To prove (229) we observe that all remaining terms in (226) are scale invariant (i.e. they have the correct powers of δ). In estimating the corresponding error terms, appearing on the right hand side, we only have to be mindful of those which contain $\mathrm{tr}\underline{\chi}$ and α. All other terms can be estimated by $\delta^{1/2}p(\mathcal{R},\underline{\mathcal{R}})$ exactly as above. It is easy to check that all terms involving $\mathrm{tr}\underline{\chi}$ can only appear through ${}^{(3)}\hat{\pi}_{34}$. Thus, it is easy to see that all such terms are of the form,

$$\begin{aligned} Q_{3444}\,{}^{(3)}\hat{\pi}^{34} &\approx -|\beta|^2\mathrm{tr}\underline{\chi}\\ Q_{3434}\,{}^{(3)}\hat{\pi}^{34} &\approx -(\rho^2+\sigma^2)\mathrm{tr}\underline{\chi}\\ Q_{3433}\,{}^{(3)}\hat{\pi}^{34} &= -|\underline{\beta}|^2\mathrm{tr}\underline{\chi}\end{aligned}$$

Thus, since $\mathrm{tr}\underline{\chi}=\widetilde{\mathrm{tr}\underline{\chi}}+\mathrm{tr}\underline{\chi}_0$, we easily deduce that all error terms containing $\mathrm{tr}\underline{\chi}$ can be estimated by,

$$\delta^{-1}\int_0^{\underline{u}}\mathcal{Q}_0(u,\underline{u}')d\underline{u}'+\delta^{1/2}C(\mathcal{R},\underline{\mathcal{R}}).$$

It is easy to check that the integral term can be absorbed on the left by a Gronwall type inequality. It thus remains to consider only the terms linear[25] in $\|\alpha\|_{\mathcal{L}_{(sc)}(H_u^{(0,\underline{u})})}$ which we have already estimated above. These lead to error terms with no excess powers of δ, which could be potentially dangerous. In fact we have to be a little more careful, because we would get an estimate of the form,

$$\mathcal{R}_0+\underline{\mathcal{R}}_0 \lesssim \mathcal{I}^{(0)}+C(\mathcal{R},\underline{\mathcal{R}})$$

which is useless for large curvature norms $\mathcal{R},\underline{\mathcal{R}}$. To avoid this problem we need to refine our use of the ${}^{(S)}\mathcal{O}_{0,\infty}$ norms. We observe that among all terms $\phi\cdot\Psi_1\cdot\Psi_2$ linear in α we can get better estimates for all, except those which contain a Ricci component ϕ which is anomalous in $\mathcal{L}^4_{(sc)}(S)$. All other terms gain a power of $\delta^{1/4}$. Indeed the corresponding error terms in $\mathcal{E}_1$ can be estimated by[26],

$$\begin{aligned} &\delta^{1/2}\|\phi\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})}\cdot\|\Psi\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}}\cdot\|\nabla\alpha\|^{1/2}_{\mathcal{L}_{(sc)}(H_u^{(0,\underline{u})})}\cdot\|\alpha\|^{1/2}_{\mathcal{L}_{(sc)}(H_u^{(0,\underline{u})})}\\ \lesssim\quad &\delta^{1/4}\;{}^{(S)}\mathcal{O}_{0,4}\cdot\mathcal{R}_0\cdot\mathcal{R}_0[\alpha]^{1/2}\cdot\mathcal{R}_1[\alpha]^{1/2}.\end{aligned}$$

Denoting by $\mathcal{E}_g$ all such error terms we thus have,

$$|\mathcal{E}_g| \lesssim \delta^{1/4}C(\mathcal{R},\underline{\mathcal{R}})$$

It remains to check the terms linear in α for which the Ricci coefficient is anomalous in the $\mathcal{L}^4_{(sc)}$ norm, i.e. terms for which ϕ is either $\hat{\chi}$ or $\underline{\hat{\chi}}$. It is easy to check that there are no terms linear in α which contain $\hat{\chi}$ and thus we only have to consider terms of the form $\underline{\hat{\chi}}\cdot\alpha\cdot\Psi$, which we denote by $\mathcal{E}_b$. Since $\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})}$ loses a power of $\delta^{1/4}$ we now have,

$$\begin{aligned} &\delta^{1/2}\|\underline{\hat{\chi}}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})}\cdot\|\Psi\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})}}\cdot\|\nabla\alpha\|^{1/2}_{\mathcal{L}_{(sc)}(H_u^{(0,\underline{u})})}\cdot\|\alpha\|^{1/2}_{\mathcal{L}_{(sc)}(H_u^{(0,\underline{u})})}\\ \lesssim\quad &{}^{(S)}\mathcal{O}_{0,4}[\underline{\hat{\chi}}]\cdot\mathcal{R}_0\cdot\mathcal{R}_0[\alpha]^{1/2}\cdot\mathcal{R}_1[\alpha]^{1/2}\end{aligned}$$

Since we are left with no positive power of δ we must now be mindful of the fact that the estimates for ${}^{(S)}\mathcal{O}_{0,4}$ depend at least linearly on the curvature norms $\mathcal{R},\underline{\mathcal{R}}$, in which case $\mathcal{E}_b$ is super-quadratic

[25]By signature considerations there can be no terms quadratic in α

[26]It follows from the Gagliardo-Nirenberg inequality $\|\alpha\|^2_{L^4(u,\underline{u})}\lesssim\|\nabla\alpha\|_{L^2(u,\underline{u})}\|\alpha\|_{L^2(u,\underline{u})}$

in $\mathcal{R}, \underline{\mathcal{R}}$. We can however trace back the $\delta^{1/4}$ loss of $\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})}$ to initial data, i.e. upon a careful inspection we find, see estimate (36) of theorem A,

$$\|\hat{\underline{\chi}}\|_{\mathcal{L}^4_{(sc)}(u,\underline{u})} \lesssim \delta^{-1/4}\mathcal{I}^{(0)} + C(\mathcal{R}, \underline{\mathcal{R}}) \tag{231}$$

Thus,

$$\mathcal{E}_b \lesssim \mathcal{I}^{(0)} \cdot \mathcal{R}_0 \cdot \mathcal{R}_0[\alpha]^{1/2} \cdot \mathcal{R}_1[\alpha]^{1/2} + \delta^{1/4} C(\mathcal{R}, \underline{\mathcal{R}})$$

The above considerations lead us to conclude, back to (226),

$$\mathcal{R}_0 + \underline{\mathcal{R}}_0 \lesssim \mathcal{I}^{(0)} + c\,\mathcal{R}_0[\alpha]^{1/2} \cdot \mathcal{R}_1[\alpha]^{1/2} + \delta^{1/8} C(\mathcal{R}, \underline{\mathcal{R}}). \tag{232}$$

with $c = c(\mathcal{I}^{(0)})$ a constant depending only on the initial data.

Remark In the analysis above we have not considered the possibility that, among the terms in the integrands of $\mathcal{E}_0$ we can have terms of the form $\phi \cdot \Psi_1 \cdot \Psi_2$ with at least one of the curvature term being the null component $\underline{\alpha}$, which cannot be estimated along H_u. Among these terms only those containing $\mathrm{tr}\underline{\chi}$ lead to terms which are $O(1)$ in δ. These can be treated by using $\underline{H}$ which leads to estimates of the form,

$$\mathcal{Q}_0(u,\underline{u}) + \underline{\mathcal{Q}}_0(u,\underline{u}) \lesssim \mathcal{I}_0^2 + \Big(\int_0^u \mathcal{Q}_0(u',\underline{u}) du' + \delta^{-1} \int_0^{\underline{u}} \underline{\mathcal{Q}}_0(u,\underline{u}') d\underline{u}' \Big) + C\delta^{1/2}$$

with $C = C(\mathcal{I}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$. The final estimate would follow from the following: lemma below(which can be easily proved by the method of continuity).

Lemma 14.8. *Let $f(x,y), g(x,y)$ be positive functions defined in the rectangle, $0 \le x \le x_0$, $0 \le y \le y_0$ which verify the inequality,*

$$f(x,y) + g(x,y) \lesssim J + a \int_0^x f(x',y) dx' + b \int_0^y g(x,y') dy'$$

for some nonnegative constants a, b and J. Then, for all $0 \le x \le x_0$, $0 \le y \le y_0$,

$$f(x,y), g(x,y) \lesssim J e^{ax+by}$$

We summarize the results of this section in the following.

Proposition 14.9. *The following estimate hold true with constants $C = C(\mathcal{I}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$, $c = c(\mathcal{I}^{(0))})$ and δ sufficiently small,*

$$\begin{aligned} \mathcal{R}_0[\alpha] + \underline{\mathcal{R}}_0[\beta] &\lesssim \mathcal{I}^{(0)} + C\delta^{3/4} \\ \mathcal{R}_0 + \underline{\mathcal{R}}_0 &\lesssim \mathcal{I}^{(0)} + c(\mathcal{I}^{(0)})\mathcal{R}^{1/2} + \delta^{1/8} C. \end{aligned}$$

15. Curvature estimates II.

We shall now estimate the first derivative of the null curvature components appearing in $\mathcal{R}_1, \underline{\mathcal{R}}_1$. We apply (14.4) for the angular momentum vectorfields O as well as for the vectorfields $L, \underline{L}$. We prefer to work here with the vectorfields $L, \underline{L}$ instead of e_4, e_3, as in the previous section, because their deformation tensors do not include ω, respectively $\underline{\omega}$. This will make a difference in this section because we don't have good estimates for $\nabla_4\omega$ and $\nabla_3\underline{\omega}$ which would appear among the derivatives of ${}^{(4)}\pi$ and ${}^{(3)}\pi$. On the other hand, since e_3, e_4 differ from $L, \underline{L}$ only by the bounded factor Ω no other estimates will be affected.

15.1. **Deformation tensors of the vectorfields L and $\underline{L}$.** Below we list the components of ${}^{L}\pi_{\alpha\beta}$ and ${}^{\underline{L}}\pi_{\alpha\beta}$.

$$
\begin{aligned}
&{}^{L}\pi_{44} = 0, \quad {}^{L}\pi_{43} = 0, \quad {}^{L}\pi_{33} = -2\Omega^{-1}\underline{\omega},\\
&{}^{L}\pi_{4a} = 0, \quad {}^{L}\pi_{3a} = \Omega^{-1}(\eta_a + \zeta_a) + \Omega^{-1}\nabla_a \log\Omega, \quad {}^{L}\pi_{ab} = \Omega^{-1}\chi_{ab}
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\underline{L}}\pi_{33} = 0, \quad {}^{\underline{L}}\pi_{43} = 0, \quad {}^{\underline{L}}\pi_{33} = -2\Omega^{-1}\omega,\\
&{}^{\underline{L}}\pi_{3a} = 0, \quad {}^{\underline{L}}\pi_{4a} = \Omega^{-1}(\underline{\eta}_a + \zeta_a) + \Omega^{-1}\nabla_a \log\Omega, \quad {}^{\underline{L}}\pi_{ab} = \Omega^{-1}\underline{\chi}_{ab}
\end{aligned}
$$

We start first with a sequence of lemmas:

15.2. **Preliminaries.** Given a vectorfield X we decompose both $\widehat{\mathcal{L}}_X R$ and $D_X R$ into their null components $\alpha(\widehat{\mathcal{L}}_X R), \beta(\widehat{\mathcal{L}}_X R), \ldots \underline{\alpha}(\widehat{\mathcal{L}}_X R)$ and $\alpha(D_X R), \beta(D_X R), \ldots \underline{\alpha}(D_X R)$. We consider these decompositions fo the vectorfields (note our discussion above concerning $X = L, \underline{L}$ and $e_a,\ a = 1, 2$. In the spirit of our discussion above we write e_4 and e_3 instead of $L, \underline{L}$. In the following lemma we estimate the null components of $D_X R$, for $X = e_3, e_4, e_a$, in terms of $\mathcal{R}$, $\underline{\mathcal{R}}$.

Lemma 15.3. *Denoting $\mathcal{R}_u$ and $\underline{\mathcal{R}}_{\underline{u}}$ the restriction of the norms $\mathcal{R}$ and $\underline{\mathcal{R}}$ to the interval $[0, u]$ and $[0, \underline{u}]$ respectively, we have with $C = C(\mathcal{O}^{(0)}, \mathcal{R}, \underline{\mathcal{R}})$, the following anomalous estimates,*

$$
\delta^{\frac12}\|\alpha(D_3 R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \delta^{\frac12}\|\beta(D_a R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{I}^{(0)} + \delta^{\frac14} C,
$$

We also have the regular estimates,

$$
\begin{aligned}
&\|\alpha(D_a R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\beta(D_3 R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\beta(D_4 R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\
+\quad &\|(\rho,\sigma)(D_4 R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(\rho,\sigma)(D_3 R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(\rho,\sigma)(D_a R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\
+\quad &\|\underline{\beta}(D_4 R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\underline{\beta}(D_a R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\underline{\alpha}(D_4 R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}_u + \delta^{\frac14} C
\end{aligned}
$$

and

$$\begin{aligned} & \|\beta(D_3R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} + \|(\rho,\sigma)(D_4R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} + \|(\rho,\sigma)(D_3R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} \\ + \quad & \|(\rho,\sigma)(D_4R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} + \|\underline{\beta}(D_4R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} + \|\underline{\beta}(D_3R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} \\ + \quad & \|\underline{\beta}(D_aR)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} + \|\underline{\alpha}(D_4R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} + \|\underline{\alpha}(D_aR)\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} \lesssim \underline{\mathcal{R}}_{\underline{u}} + \delta^{\frac{1}{4}}C \end{aligned}$$

Remark 15.4. We note the special nature of the anomalies in $\alpha(D_3R)$ and $\beta(D_aR)$. Specifically, we can show that both terms can be written in the form $G+F$ with $G = \mathrm{tr}\underline{\chi}_0 \cdot \alpha$ and F obeying the estimate

$$\|F\|_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} + \|F\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,\underline{u})}_{u})} \le C.$$

Proof. Let $\Psi^{(s)}(D_XR)$ denote the null components of D_XR and $\phi^{(s)}$ Ricci curvature components of signature s. Then, for $X = L, \underline{L}, e_1, e_2$, recalling that $\mathrm{s}gn(X) = 1, 1/2, 0$ for $X = L, e_a, \underline{L}$, we write,

$$\Psi^{(s)}(D_XR) \quad = \quad \nabla_X \Psi^{(s)} + \sum_{s_1+s_2=s+\mathrm{s}gn(X)} \phi^{(s_1)} \cdot \Psi^{(s_2)} \tag{233}$$

Ignoring possible anomalies we write,

$$\begin{aligned} \|\Psi^{(s)}(D_XR)\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u})} &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u})} + \delta^{1/2}\, {}^{(S)}\mathcal{O}_{0,\infty} \cdot \mathcal{R}_0 \\ &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u})} + C\delta^{1/2} \\ \|\Psi^{(s)}(D_XR)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}})} &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}})} + \delta^{1/2}\, {}^{(S)}\mathcal{O}_{0,\infty} \cdot \underline{\mathcal{R}}_0 \\ &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}})} + C\delta^{1/2} \end{aligned} \tag{234}$$

We only have to pay special attention to the case when $\phi^{(s_1)} = \mathrm{tr}\underline{\chi}$ and $\Psi^{(s_2)} = \alpha$. If $s_2 = 2$, i.e. $\Psi^{(s_2)} = \alpha$ then s_1 can be $1, 1/2$ and 0. The case $s_1 = 1$ occur only if $X = e_4$, which is not covered by the lemma. The case $s_2 = 2, s_1 = 1/2$ is regular. Indeed, in that case $s + \mathrm{s}gn(X) = 5/2$. Thus either $s = 2, X = e_a$ or $s = 3/2$, $X = L$. In both cases we simply estimate the worst quadratic term, on the right hand side of (233), with $s_2 = 2$, by

$$\begin{aligned} \|\phi\cdot\alpha\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u})} \quad &\lesssim \quad \delta^{\frac{1}{2}}\|\phi\|_{\mathcal{L}^4_{(sc)}{}_{u,\underline{u}}}\|\alpha\|_{\mathcal{L}^4_{(sc)}{}_{u,\underline{u}}} \lesssim \delta^{\frac{1}{2}}\, {}^{(S)}\mathcal{O}_{0,4}[\phi]\|\alpha\|^{\frac{1}{2}}_{\mathcal{L}^2_{(sc)}{}_{u,\underline{u}}}\|\nabla\alpha\|^{\frac{1}{2}}_{\mathcal{L}^2_{(sc)}{}_{u,\underline{u}}} \\ &\lesssim \quad \delta^{\frac{1}{4}}\, {}^{(S)}\mathcal{O}_{0,4}[\phi] \cdot \mathcal{R}_0[\alpha]^{\frac{1}{2}} \cdot \mathcal{R}_1[\alpha]^{\frac{1}{2}} \lesssim C\delta^{1/4}. \end{aligned}$$

The principal term is either $\nabla\alpha$ in the first case or $\nabla_L\beta$ in the second. In the second situation, using the null Bianchi identities, (proceeding as above with the term of the form $\phi\cdot\alpha$),

$$\|\nabla_L\beta\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u})} \quad \lesssim \quad \|\nabla\alpha\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}_{u})} + C\delta^{1/4}$$

In the case ($s_2 = 2$, $s_1 = 0$) $\mathrm{tr}\underline{\chi}$ can appear among the quadratic terms on the right. In that case $s + \mathrm{s}gn(X) = 2$. The $s = 2$ and $X = \underline{L}$ corresponds to the anomalous estimate for $\alpha(D_{\underline{L}}R)$. In that case the estimate is,

$$\|\alpha(D_{\underline{L}}R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \|\nabla_{\underline{L}}\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + (1+\delta^{1/2}C)\|\alpha\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}} + \delta^{1/2}C$$

Also, in view of the Bianchi identities, (53),

$$\|\nabla_{\underline{L}}\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\delta^{1/2}$$

Hence, in view of our estimate for α in the previous section

$$\begin{aligned}\delta^{1/2}\|\alpha(D_LR)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} &\lesssim \delta^{1/2}\|\nabla_L\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + (1+\delta^{1/2}C)\delta^{1/2}\|\alpha\|_{\mathcal{L}^2_{(sc)}(H^{(0,\underline{u})}}\\ &\lesssim \mathcal{I}^{(0)} + \delta^{1/4}C\end{aligned}$$

as desired. We need also to consider the case $s_2 = 2, s_1 = 0$, $s = 3/2$ and $X = e_a$. Then, due to the term $\mathrm{tr}\underline{\chi}_0 \cdot \alpha$ on the right hand side of (233) we have,

$$\|\beta(D_aR)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\delta^{1/4}$$

Thus,

$$\delta^{1/2}\|\beta(D_aR)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{I}^{(0)} + C\delta^{1/4}$$

as which is the second anomalous estimate.

It remains to consider the cases $s_2 < 2$, $s_1 = 0$. In the worst case, when a quadratic term on the right hand side of (233) is of the form $\mathrm{tr}\underline{\chi}_0 \cdot \Psi^{(s_2)}$ we make the following correction to estimate (234),

$$\begin{aligned}\|\Psi^{(s)}(D_XR)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\Psi^{(s_2)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\delta^{1/4}\\ &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \mathcal{R}_u + C\delta^{1/4}\\ \|\Psi^{(s)}(D_XR)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} + \|\Psi^{(s_2)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} + C\delta^{1/4}\\ &\lesssim \|\nabla_X\Psi^{(s)}(R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} + \underline{\mathcal{R}}_{\underline{u}} + C\delta^{1/4}\end{aligned}$$

These imply the regular estimates of the Lemma for the case $X = e_a$. For the cases $X = L, \underline{L}$ we can express $\nabla_X\Psi^{(s)}(R)$ using the Bianchi identities,

$$\begin{aligned}\nabla_3\Psi^{(s)} &= \nabla\Psi^{(s-\frac12)} + \sum_{s_1+s_2=s}\phi^{(s_1)}\cdot\Psi^{(s_2)}, \qquad 0 < s < 2\\ \nabla_4\Psi^{(s)} &= \nabla\Psi^{(s+\frac12)} + \sum_{s_1+s_2=s+1}\phi^{(s_1)}\cdot\Psi^{(s_2)}, \qquad 0 \le s < 2.\end{aligned}$$

The worst quadratic terms which can appear on the right are of the form $\mathrm{tr}\underline{\chi}\cdot\Psi^{(s)}$ with $s < 2$ which can be easily estimated. We thus derive all the regular estimates of the Lemma. □

Lemma 15.5. *The following estimates for the Lie derivatives $\widehat{\mathcal{L}}_X R$, with respect to hold true $X = \{\underline{L}, L, O\}$.*

$$\|\alpha(\widehat{\mathcal{L}}_L R) - \nabla_L \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C \tag{235}$$

$$\delta^{1/2}\|\alpha(\widehat{\mathcal{L}}_{\underline{L}} R) - \nabla_{\underline{L}} \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}_0 + C\delta^{3/4} \tag{236}$$

Also,

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_L R) - (\nabla_L \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}, \qquad 1 \le s \le 5/2, \tag{237}$$

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_{\underline{L}} R) - (\nabla_{\underline{L}} \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}_0 + C\delta^{1/4}, \qquad 1 \le s \le 3/2 \tag{238}$$

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_{\underline{L}} R) - (\nabla_{\underline{L}} \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} \lesssim \mathcal{R}_0 + C\delta^{1/4}, \qquad s \le 1/2. \tag{239}$$

For $X = O$ we have the estimates.

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R) - (\nabla_O \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}, \qquad 1 \le s \le 5/2 \tag{240}$$

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R) - (\nabla_O \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} \lesssim C\delta^{1/4}, \qquad 1/2 \le s \le 2. \tag{241}$$

Proof. We will make use of the regular $\mathcal{L}^\infty_{(sc)}$ estimates for Ricci coefficients $\phi \in \{\chi, \omega, \eta, \underline{\eta}, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}, \underline{\omega}\}$. We also make use of the following estimates for ∇O and ${}^{(O)}\pi$.

We write, recalling the definition of the Lie derivative and with E denoting the set e_1, e_2, e_3, e_4,

$$\begin{aligned} \Psi^{(s)}(\mathcal{L}_X R) &= X(\Psi^{(s)}) - \sum_{s_1+s_2=s} \sum_{Y \in E} ([X,Y])^{(s_1)} \Psi^{(s_2)} \\ &= \not{\mathcal{L}}_X(\Psi^{(s)}) - \sum_{s_1+s_2=s} \sum_{Y \in E} (([X,Y])^{(s_1)})^{\perp} \cdot \Psi^{(s_2)} \end{aligned} \tag{242}$$

Here $\not{\mathcal{L}}_X(\Psi^{(s)})$ denotes the projection of the Lie derivative on the $S(u, \underline{u})$ surfaces and $[X,Y]^{\perp}$ the orthogonal component of $[X,Y]$ i.e.,

$$[X,Y]^{\perp} = -\frac{1}{2} g([X,Y], e_3) e_4 - \frac{1}{2} g([X,Y], e_4) e_3$$

Consider first the case when $X = L, \underline{L}$. In that case $[X,Y]^{\perp}$ depends only on the regular Ricci coefficients $\omega, \eta, \underline{\eta}, \underline{\omega}$ Therefore, taking into account the worst possible case when α appear among the quadratic terms (in which case we appeal to $\mathcal{L}^4_{(sc)}$ estimates), we derive,

$$\begin{aligned}
&\|\Psi^{(s)}(\mathcal{L}_L R) - (\not{\mathcal{L}}_L \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}, \qquad 1 \le s \le 3\\
&\|\Psi^{(s)}(\mathcal{L}_{\underline{L}} R) - (\not{\mathcal{L}}_{\underline{L}} \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}, \qquad 1 \le s \le 2.\\
&\|\Psi^{(s)}(\mathcal{L}_{\underline{L}} R) - (\not{\mathcal{L}}_{\underline{L}} \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} \lesssim C\delta^{1/4}, \qquad 0 \le s \le 1/2.
\end{aligned} \tag{243}$$

On the other hand, schematically,

$$\not{\mathcal{L}}_L \Psi^{(s)} = \nabla_L \Psi^{(s)} + \sum_{s_1+s_2=1+s} \phi^{(s_1)} \cdot \Psi^{(s_2)}$$

with $\phi^{(s_1)} \in \{\chi, \eta, \underline{\eta}\}$. In the particular case $s = 3$ we can have a double anomaly of the form, $\chi \cdot \alpha$. In that case,

$$\|\not{\mathcal{L}}_L \alpha - \nabla_L \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{\frac{1}{2}} \|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\delta^{1/2}$$

Therefore, $\|\not{\mathcal{L}}_L \alpha - \nabla_L \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C$, from which, combining with (243),

$$\|\alpha(\mathcal{L}_L R) - \nabla_L \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C$$

Recalling the definition of $\widehat{\mathcal{L}}_L R$ we deduce,

$$\delta^{1/2}\|\alpha(\widehat{\mathcal{L}}_L R) - \nabla_L \alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C$$

as desired.

We now consider all other cases, $1 \le s \le 5/2$. Since there are no double anomalies, we deduce, (using $\mathcal{L}^4_{(sc)}(S)$ estimates for the term containing α)

$$\|\not{\mathcal{L}}_L \Psi^{(s)} - (\nabla_L \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}$$

Hence, combining with (243),

$$\|\Psi^{(s)}(\mathcal{L}_L R) - (\nabla_L \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}$$

Recalling the definition of $\Psi^{(s)}(\mathcal{L}_L R)$ we deduce,

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_L R) - (\nabla_L \Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}, \qquad 1 \le s \le 5/2.$$

as desired.

We now consider the estimates for $\underline{L}$. We have,

$$\not{\mathcal{L}}_{\underline{L}} \Psi^{(s)} = \nabla_{\underline{L}} \Psi^{(s)} + \mathrm{tr}\underline{\chi}_0 \Psi^{(s)} + \sum_{s_1+s_2=s} \phi^{(s_1)} \cdot \Psi^{(s_2)}$$

with $\phi^{(s_1)} \in \{\eta, \underline{\eta}, \hat{\underline{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}\}$. Observe that the worst terms $\mathrm{tr}\underline{\chi}_0 \cdot \alpha$ can only appear for $s = 2$. In that case,

$$\|\not{\mathcal{L}}_{\underline{L}}\alpha - \nabla_{\underline{L}}\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \|\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\delta^{1/4} \lesssim \delta^{-1/2}\mathcal{R}_0 + C\delta^{1/4}$$

Thus, combining with (243),

$$\delta^{1/2}\|\alpha(\mathcal{L}_{\underline{L}}R) - \nabla_{\underline{L}}\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}_0 + C\delta^{3/4}$$

Finally, recalling the definition of $\alpha(\widehat{\mathcal{L}}_{\underline{L}}R)$ we deduce, $\delta^{1/2}\|\alpha(\widehat{\mathcal{L}}_{\underline{L}}R) - \nabla_{\underline{L}}\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}_0 + C\delta^{3/4}$ as desired.

In all other cases, $1 \le s \le \frac{3}{2}$ we have,

$$\begin{aligned}\|\not{\mathcal{L}}_{\underline{L}}\Psi^{(s)} - (\nabla_{\underline{L}}\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} &\lesssim \|\Psi^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\delta^{1/4} \\ &\lesssim \mathcal{R}_0 + C\delta^{1/4}\end{aligned}$$

Hence, combining with (243) and recalling the definition of $\widehat{\mathcal{L}}$ we deduce,

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_{\underline{L}}R) - (\nabla_{\underline{L}}\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \mathcal{R}_0 + C\delta^{1/4}$$

as desired.

We now consider the case when $X = O$. In view of (242),

$$\|\Psi^{(s)}(\mathcal{L}_O R) - (\not{\mathcal{L}}_O\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}$$

Indeed the projections of $[O, e_4]$, $[O, e_3]$ on e_3, e_4 depend only on O and the Ricci coefficients $\omega, \eta, \underline{\eta}, \underline{\omega}$ while $[O, e_a]$, $a = 1, 2$ are tangent to $S(u, \underline{u})$. On the other hand, $\not{\mathcal{L}}_O\Psi^{(s)}$ differs from $(\nabla_O\Psi)^{(s)}$ by terms quadratic in ∇O and Ψ. We recall that we have $\|\nabla O\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$, i.e. they are regular in the supremum norm. Thus, as before,

$$\|\not{\mathcal{L}}_O\Psi^{(s)} - (\nabla_O\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}.$$

Combining this with the estimate above and recalling the definition of $\widehat{\mathcal{L}}_O R$ as well as the estimates $\|\,^{(O)}\pi\,\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$ we derive, for all $s \ge 1/2$.

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R) - (\nabla_O\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/4}$$

Similarly we prove, for $s \le 3/2$

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R) - (\nabla_O\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} \lesssim C\delta^{1/4}$$

□

15.6. **Estimate for** $\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H)}$. It is important to observe throughout this section that the deformation tensors ${}^{(L)}\pi$ of L does not contain ω and ${}^{(\underline{L})}\pi$ of $\underline{L}$ does not contain either $\underline{\omega}$.

We apply corollary 14.4 to $O = L$ and $X = Y = Z = e_4$. and derive

$$\begin{aligned}\int_{H_u^{(0,\underline{u})}} |\alpha(\widehat{\mathcal{L}}_L R)|^2 &\lesssim \int_{H_0^{(0,\underline{u})}} |\alpha(\widehat{\mathcal{L}}_L R)|^2 + \int_{\mathcal{D}(u,\underline{u})} (Q[\widehat{\mathcal{L}}_L R]\cdot {}^{(4)}\pi)(e_4,e_4,e_4) \\ &+ \int_{\mathcal{D}(u,\underline{u})} D(L,R)(e_4,e_4,e_4) \end{aligned} \tag{244}$$

In view of the conservation of signature we can write schematically,

$$(Q[\widehat{\mathcal{L}}_L R]\cdot {}^{(4)}\pi)(e_4,e_4,e_4) = \sum_{s_1+s_2+s_3=6} \phi^{(s_1)}\cdot\Psi^{(s_2)}[\widehat{\mathcal{L}}_4 R]\cdot\Psi^{(s_3)}[\widehat{\mathcal{L}}_4 R] \tag{245}$$

$$D(L,R)(e_4,e_4,e_4) = \sum_{s_1+s_2+s_3=6} \Psi^{(s_2)}[\widehat{\mathcal{L}}_4 R]\cdot\big(\psi^{(s_1)}\cdot(D\Psi)^{(s_3)} + (D\psi)^{(s_1)}\cdot\Psi^{(s_3)}\big) \tag{246}$$

with Ricci coefficients $\phi \in \{\chi,\omega,\eta,\underline{\eta},\underline{\omega}\}$, $\psi\in\{\chi,\eta,\underline{\eta},\underline{\omega}\}$ null curvature components Ψ and labels s_1, s_2, s_3 denoting the signature of the corresponding component. Thus,

$$\|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + I_1 + I_2 + I_3$$

with

$$\begin{aligned} I_1 &= \delta^{1/2}\sum\|\phi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}}\int_0^u \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}\cdot\|\Psi^{(s_3)}(\widehat{\mathcal{L}}_4 R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} du' \\ I_2 &= \delta^{1/2}\sum\|\psi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}}\int_0^u \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}\cdot\|(D\Psi)^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} du' \\ I_3 &= \sum\int_0^u \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}\|(D\psi)^{(s_1)}\cdot\Psi^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} du' \end{aligned}$$

Among the terms I_1 the worst are those in which $s_2 = s_3 = 3$, in which case $s_1 = 0$. Since $\mathrm{tr}\underline{\chi}$ cannot appear among our Ricci coefficients here, and $\|\phi\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$, with $C = C(\mathcal{I}^0,\mathcal{R},\underline{\mathcal{R}})$

$$I_{11} \lesssim C\delta^{1/2}\int_0^u \|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} du'$$

All curvature terms $\|\Psi^{(s)}(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}$ with $s < 3$ can be estimated according to lemmas 15.3 and 15.5 to derive,

$$\|\Psi^{(s)}[\widehat{\mathcal{L}}_L R]\|_{\mathcal{L}^2_{(sc)}(H_u)} \lesssim \mathcal{R}_0 + \delta^{1/4}C \lesssim C, \qquad s<3.$$

Therefore, estimating all remaining terms in I_1 we deduce,

$$I_1(u,\underline{u}) \lesssim C\delta^{1/2}\int_0^u \big(\|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} + \|\alpha(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}\mathcal{R}\big)du' + \delta^{\frac{1}{2}}\mathcal{R}^2$$

The term I_2 can be estimated in exactly the same manner. Since $0 \le s_1 \le 1$ and $1 \le s_2 \le 3$ we have $2 \le s_3 \le 3$. This implies that the term $(D\Psi)^{s_3}$ may be estimated along H_u. With the exception of the term $\alpha(D_L R)$ these estimates are given in Lemma 15.3. Among those there are two anomalous terms $\alpha(D_3 R)$ and $\beta(D_a R)$. We then obtain

$$\begin{aligned} I_2(u,\underline{u}) &\lesssim C\delta^{1/2}\int_0^u \big(\|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} + (C\delta^{-\frac14} + \mathcal{I}^{(0)}\delta^{-\frac12})\|\alpha(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}\big)du' \\ &+ \mathcal{I}^{(0)}\delta^{-\frac12} + C\delta^{-\frac14} \\ &\lesssim C\delta^{1/2}\int_0^u \|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}du' + \mathcal{I}^{(0)}\delta^{-\frac12} + C\delta^{-\frac14} \end{aligned} \tag{247}$$

It remains to estimate I_3. We note that, in the worst case, the term $D\psi$ can be written in the form

$$(D\psi)^{(s_1)} = (\nabla\psi)^{s_1} + \mathrm{tr}\underline{\chi}_0 \cdot \psi^{(s_1)} + \sum_{s_{11}+s_{12}=s_1} \psi^{(s_{11})}\cdot\psi^{(s_{12})}.$$

Observe that $(\nabla\psi)^{s_1} \ne (\nabla_4\omega, \nabla_3\underline{\omega})$. Indeed $\nabla_4\omega$ cannot occur, since $\psi^{(s_1)} \in \{\chi, \eta, \underline{\eta}, \underline{\omega}$ On the other hand $\nabla_3\underline{\omega}$ cannot occur by signature considerations. Indeed in that case $s_1 = sgn(\nabla_3\underline{\omega}) = 0$, which is ruled out since $s_1 + s_2 + s_3 = 6$ while $s_2 \le 3$ and $s_3 \le 2$.

Thus, since $(\nabla\psi)^{s_1} \ne (\nabla_4\omega, \nabla_3\underline{\omega})$ (for which we do not have $\mathcal{L}^4_{(sc)}$ estimates !), we derive,

$$\begin{aligned} \|(D\psi)^{(s_1)}\cdot\Psi^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} &\lesssim \delta^{\frac12}\|(\nabla\psi)^{(s_1)}\|_{\mathcal{L}^4_{(sc)}(H_{u'}^{(0,\underline{u})})}\|\Psi^{(s_3)}\|_{\mathcal{L}^4_{(sc)}(H_{u'}^{(0,\underline{u})})} \\ &+ \left(\delta\sum_{s_{11}+s_{12}=s_1}\|\psi^{(s_{11})}\|_{\mathcal{L}^\infty_{(sc)}}\|\phi^{(s_{12})}\|_{\mathcal{L}^\infty_{(sc)}} + \delta^{\frac12}\|\psi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}}\right)\|\Psi^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} \\ &\lesssim C. \end{aligned}$$

Observe that in the last step we have used the $\mathcal{L}^4_{(sc)}$ estimates for the first derivatives of the Ricci coefficients $\psi \in \{\chi, \eta, \underline{\eta}\}$ and the null curvature components, and allowed for the worst possible scenario in which $(\Psi^{(s_3)} = \alpha)$,

$$\begin{aligned} &\|(\nabla\psi)^{(s_1)}\|_{\mathcal{L}^4_{(sc)}(H_{u'}^{(0,\underline{u})})} + \|\Psi^{(s_3)}\|_{\mathcal{L}^4_{(sc)}(H_{u'}^{(0,\underline{u})})} \le C\delta^{-\frac14}, \\ &\|\Psi^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} \lesssim C\delta^{-\frac12} \end{aligned}$$

As a consequence we derive,

$$I_3(u,\underline{u}) \lesssim C\int_0^u \|\alpha(\widehat{\mathcal{L}}_4 R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})}du' + C$$

Combining the estimates for I_1, I_2, I_3 we derive,

$$\|\alpha(\widehat{\mathcal{L}}_4 R)\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \|\alpha(\widehat{\mathcal{L}}_4 R)\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + C\big(1+\delta^{1/2}\big)\int_0^u \|\alpha(\widehat{\mathcal{L}}_4 R)\|_{\mathcal{L}^2_{(sc)}(H_{u'}^{0,\underline{u}})}du' + C\delta^{1/2}$$

Therefore, in view of the anomalous character of $\|\alpha(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_u)}$,

$$\delta\|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \delta\|\alpha(\widehat{\mathcal{L}}_L R)\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + C\delta^{\frac{3}{2}}$$

from which we infer that, for some $C = C(\mathcal{I}^0, \mathcal{R}, \underline{\mathcal{R}})$,

$$\begin{aligned}\delta^{1/2}\|\alpha(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} &\lesssim \delta^{1/2}\|\alpha(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + C\delta^{1/2}\\ &\lesssim \mathcal{I}^0 + C\delta^{1/2}\end{aligned}$$

On the other hand, in view of the definition of $\widehat{\mathcal{L}}_L R$ we have,

$$\alpha(\widehat{\mathcal{L}}_L R) = \nabla_L\alpha + \sum_{s_1+s_2=3}\phi^{(s_1)}\cdot\Psi^{(s_2)}$$

Hence,

$$\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \|\alpha(\widehat{\mathcal{L}}_L R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + C\mathcal{R}_0$$

Therefore we deduce,

Proposition 15.7. *The following estimate holds true for sufficiently small $\delta > 0$, with a constant $C = C(\mathcal{I}^0, \mathcal{R}, \underline{\mathcal{R}})$,*

$$\|\nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim \delta^{-1/2}\mathcal{I}^0 + C. \tag{248}$$

15.8. **Estimate for** $\|\nabla_3\underline{\alpha}\|_{\mathcal{L}^2_{(sc)}(H)}$. Applying corollary 14.4 to $O = e_3$ and $X = Y = Z = e_3$ we derive,

$$\begin{aligned}\int_{\underline{H}_{\underline{u}}^{(0,u)}}|\underline{\alpha}(\widehat{\mathcal{L}}_{\underline{L}}R)|^2 &\lesssim \int_{\underline{H}_0^{(0,u)}}|\underline{\alpha}(\widehat{\mathcal{L}}_{\underline{L}}R)|^2 + \int_{\mathcal{D}(u,\underline{u})}(Q[\widehat{\mathcal{L}}_{\underline{L}}R]\cdot{}^{(3)}\pi)(e_3,e_3,e_3)\\ &+ \int_{\mathcal{D}(u,\underline{u})}D(\underline{L},R)(e_3,e_3,e_3)\end{aligned} \tag{249}$$

In view of the conservation of signature we can write schematically (we need to take into account the signature associated to the integrals),

$$(Q[\widehat{\mathcal{L}}_{\underline{L}}R]\cdot{}^{(4)}\pi)(e_3,e_3,e_3) = \sum_{s_1+s_2+s_3=1}\psi^{(s_1)}\cdot\Psi^{(s_2)}[\widehat{\mathcal{L}}_3R]\cdot\Psi^{(s_3)}[\widehat{\mathcal{L}}_3R] \tag{250}$$

$$D(\underline{L},R)(e_3,e_3,e_3) = \sum_{s_1+s_2+s_3=1}\Psi^{(s_2)}[\widehat{\mathcal{L}}_{\underline{L}}R]\cdot\big(\psi^{(s_1)}\cdot(D\Psi)^{(s_3)} + (D\psi)^{(s_1)}\cdot\Psi^{(s_3)}\big) \tag{251}$$

with Ricci coefficients $\psi \in \{\omega, \eta, \underline{\eta}, \underline{\chi}\}$, null curvature components Ψ and labels s_1, s_2, s_3 denoting the signature of the corresponding component. We now need to be careful with terms which involve $\mathrm{tr}\underline{\chi}$ and $\nabla_3\mathrm{tr}\underline{\chi}$. In (250) the only terms which contain $\mathrm{tr}\underline{\chi}$ have the form $\mathrm{tr}\underline{\chi}\cdot|\underline{\beta}(\widehat{\mathcal{L}}_{\underline{L}}R)|^2$ which we write in the form

$$\mathrm{tr}\underline{\chi}_0\cdot|\underline{\beta}(\widehat{\mathcal{L}}_{\underline{L}}R)|^2 + \widetilde{\mathrm{tr}\underline{\chi}}\cdot|\underline{\beta}(\widehat{\mathcal{L}}_{\underline{L}}R)|^2$$

In (251) the only terms which contains $\nabla_3 \mathrm{tr}\underline{\chi}$, must be of the form

$$\nabla_3 \mathrm{tr}\underline{\chi} \cdot \Psi^{(s_2)}(\widehat{\mathcal{L}}_{\underline{L}} R) \cdot \Psi^{(s_3)}, \qquad s_2 + s_3 = 1.$$

Recall that,

$$\nabla_3 \mathrm{tr}\underline{\chi} \;=\; -\frac{1}{2}\mathrm{tr}\underline{\chi}^2 - 2\underline{\omega}\mathrm{tr}\underline{\chi} - |\hat{\underline{\chi}}|^2$$

Thus, writing, $\mathrm{tr}\underline{\chi} = \mathrm{tr}\underline{\chi}_0 + \widetilde{\mathrm{tr}\underline{\chi}}$, we have schematically,

$$\nabla_3 \mathrm{tr}\underline{\chi} \;=\; -\frac{1}{2}\mathrm{tr}\underline{\chi}_0^2 + \mathrm{tr}\underline{\chi}_0 \psi_g + \psi \cdot \psi$$

We have,

$$\|\underline{\alpha}(\widehat{\mathcal{L}}_{\underline{L}} R)\|^2_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_{\underline{u}})} \;\lesssim\; \|\underline{\alpha}(\widehat{\mathcal{L}}_{\underline{L}} R)\|^2_{\mathcal{L}^2_{(sc)}(H^{(0,u)}_0)} + P_1 + P_2 + P_3 + J_1 + J_2 + J_3$$

with, P_1, P_2, P_3 the terms corresponding to the terms in $\mathrm{tr}\underline{\chi}_0$,

$$\begin{aligned}
P_1 &= \sum_{s_2+s_3=1} \delta^{-1} \int_0^{\underline{u}} \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_{\underline{L}} R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \cdot \|\Psi^{(s_3)}(\widehat{\mathcal{L}}_{\underline{L}} R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}' \\
P_2 &= \sum_{s_2+s_3=1} \delta^{-1} \int_0^{\underline{u}} \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_{\underline{L}} R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \cdot \|(D\Psi)^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}' \\
P_3 &= \sum_{s_2+s_3=1} \delta^{-1} \int_0^{\underline{u}} \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_3 R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \cdot \|\Psi^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}'
\end{aligned}$$

and J_1, J_2, J_3 the remaining terms with Ricci terms $\psi \in \{\eta, \underline{\eta}, \hat{\underline{\chi}}\}$,

$$\begin{aligned}
J_1 &= \delta^{-1/2} \sum_{s_1+s_2+s_3=1} \|\psi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}} \int_0^{\underline{u}} \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_3 R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \cdot \|\Psi^{(s_3)}(\widehat{\mathcal{L}}_3 R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}' \\
J_2 &= \delta^{-1/2} \sum_{s_1+s_2+s_3=1} \|\psi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}} \int_0^{\underline{u}} \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_3 R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \cdot \|(D\Psi)^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}' \\
J_3 &= \sum_{s_1+s_2+s_3=1} \delta^{-1} \int_0^u \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_3 R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \|(D\psi)^{(s_1)} \cdot \Psi^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}'
\end{aligned}$$

It clearly suffices to estimate the principal terms P. Indeed the J terms can be treated exactly as in the previous subsection[27] . We have,

$$P_1 \;\lesssim\; \delta^{-1} \int_0^{\underline{u}} \|\underline{\beta}(\widehat{\mathcal{L}}_{\underline{L}} R)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}'$$

According to Lemma (15.5) we have,

$$\|\underline{\beta}(\widehat{\mathcal{L}}_{\underline{L}} R)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \;\lesssim\; \|\nabla_3 \underline{\beta}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} + \underline{\mathcal{R}}_0 + \delta^{1/4} C$$

[27] Remark that in J_2 $(D\Psi)^{(3)}$ differ from $\nabla_3 \underline{\omega}$, because $\Psi^{(s_3)} \in \{\omega, \eta, \underline{\eta}, \underline{\chi}\}$, and $\nabla_4 \omega$ by signature considerations.

In view of the Bianchi identities, for $\frac{1}{2} \le s \le 1$,

$$\nabla_3 \underline{\beta} = \quad \text{div } \underline{\alpha} - 2\text{tr}\underline{\chi} \cdot \underline{\beta} - 2\underline{\omega} \cdot \underline{\beta} + \underline{\eta} \cdot \underline{\alpha}$$

Therefore,

$$\|\underline{\beta}(\widehat{\mathcal{L}}_{\underline{L}} R)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \quad \lesssim \quad \|\nabla \underline{\alpha}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} + \mathcal{R}_0 + \delta^{1/4} C$$

Consequently,

$$\begin{aligned} P_1(u, \underline{u}) &\lesssim \delta^{-1} \int_0^{\underline{u}} \big(\|\nabla \underline{\alpha}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} + \mathcal{R}_0(u, \underline{u}') \big) d\underline{u}' + C\delta^{1/4} \\ &\lesssim \delta^{-1} \int_0^{\underline{u}} \underline{\mathcal{R}}^2(u, \underline{u}') d\underline{u}' + \delta^{1/4} C \end{aligned}$$

P_2, P_3 can be estimated exactly in the same manner. First, observe that in P_2 the terms of the form $(D\Psi)^{(s_3)}$ obey the bounds,

$$\|(D\Psi)^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \lesssim \underline{\mathcal{R}}(u, \underline{u}') + \delta^{\frac{1}{4}} C.$$

This follows from the restriction $s_3 \le 1$. Similarly, for $s_2 \le 1$

$$\|\Psi^{(s_2)}(\widehat{\mathcal{L}}_{\underline{L}} R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \lesssim \|\underline{\alpha}(\widehat{\mathcal{L}}_{\underline{L}} R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} + \underline{\mathcal{R}}(u, \underline{u}') + \delta^{\frac{1}{2}} C.$$

Therefore,

$$\begin{aligned} P_2(u, \underline{u}) &\lesssim \delta^{-1} \int_0^{\underline{u}} \|\underline{\alpha}(\widehat{\mathcal{L}}_{\underline{L}} R)\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} d\underline{u}' \\ &+ \delta^{-1} \int_0^{\underline{u}} \underline{\mathcal{R}}^2(u, \underline{u}') d\underline{u}' + \delta^{1/2} C \end{aligned}$$

Similarly,

$$\begin{aligned} P_3(u, \underline{u}) &\lesssim \delta^{-1} \int_0^{\underline{u}} \|\underline{\alpha}(\widehat{\mathcal{L}}_3 R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}'})} \underline{\mathcal{R}}(u, \underline{u}') d\underline{u}' \\ &+ \delta^{-1} \int_0^{\underline{u}} \underline{\mathcal{R}}^2(u, \underline{u}') d\underline{u}' + \delta^{1/2} C. \end{aligned}$$

Therefore using Lemma 15.5 we derive,

Proposition 15.9. *The following estimate holds true for sufficiently small $\delta > 0$, with a constant $C = C(\mathcal{I}^0, \mathcal{R}, \underline{\mathcal{R}})$,*

$$\|\nabla_3 \underline{\alpha}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_{\underline{u}})} \quad \lesssim \quad \|\nabla_3 \underline{\alpha}\|^2_{\mathcal{L}^2_{(sc)}(\underline{H}^{(0,u)}_0)} + \delta^{-\frac{1}{2}} \mathcal{R}_0 + \int_0^{\underline{u}} \mathcal{R}(u, \underline{u}')^2 d\underline{u}' + \delta^{\frac{1}{4}} C \tag{252}$$

15.10. **Estimates for the angular derivatives of R.** Applying corollary 14.4 to the angular momentum vectorfields O and $X, Y, Z \in \{e_3, e_4\}$ we derive,

$$\int_{H_u^{(0,\underline{u})}} |\Psi^{(s)}(\widehat{\mathcal{L}}_O R)|^2 + \int_{H_{\underline{u}}^{(0,u)}} |\Psi^{(s-\frac{1}{2})}(\widehat{\mathcal{L}}_O R)|^2 \lesssim \int_{H_0^{(0,\underline{u})}} |\Psi^{(s)}(\widehat{\mathcal{L}}_O R)|^2 + \int_{\mathcal{D}(u,\underline{u})} (Q[\widehat{\mathcal{L}}_O R]\cdot\pi)(X,Y,Z) + \int_{\mathcal{D}(u,\underline{u})} D(O,R)(X,Y,Z) \tag{253}$$

In view of the conservation of signature we can write schematically,

$$(Q[\widehat{\mathcal{L}}_O R]\cdot\pi)(X,Y,Z) = \mathrm{tr}\underline{\chi}_0 \cdot \sum_{s_2+s_3=2s} \Psi^{(s_2)}[\widehat{\mathcal{L}}_O R]\cdot\Psi^{(s_3)}[\widehat{\mathcal{L}}_O R] + \sum_{s_1+s_2+s_3=2s} \phi^{(s_1)}\cdot\Psi^{(s_2)}[\widehat{\mathcal{L}}_O R]\cdot\Psi^{(s_3)}[\widehat{\mathcal{L}}_O R] \tag{254}$$

with ϕ Ricci coefficients in $\{\chi, \omega, \eta, \underline{\eta}, \underline{\hat{\chi}}, \widetilde{\mathrm{tr}\underline{\chi}}, \omega\}$. Also, recalling that $\pi = \hat{\pi} + \frac{1}{4}\mathrm{tr}(\pi) g$,

$$D(O,R)(X,Y,Z) = \sum_{s_1+s_2+s_3=2s} \Psi^{(s_2)}[\widehat{\mathcal{L}}_O R]\cdot\big({}^{(O)}\pi^{(s_1)}\cdot(D\Psi)^{(s_3)} + (D\,{}^{(O)}\pi)^{(s_1)}\cdot\Psi^{(s_3)}\big) \tag{255}$$

with ${}^{(O)}\pi^{(s)}$ null components of the deformation tensor of O. Thus, for all $s > \frac{1}{2}$,

$$\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|^2_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\Psi^{(s-\frac{1}{2})}(\widehat{\mathcal{L}}_O R)\|^2_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})} \lesssim \|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|^2_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})} + I_1 + I_2 + I_3$$

- I_1 is the integral in $\mathcal{D}(u,\underline{u})$ whose integrand is given by (254),
- I_2 is the integral in $\mathcal{D}(u,\underline{u})$ whose integrand is given by

$$\sum_{s_1+s_2+s_3=2s} \Psi^{(s_2)}[\widehat{\mathcal{L}}_O R]\cdot{}^{(O)}\pi^{(s_1)}\cdot(D\Psi)^{(s_3)}.$$

- I_3 is the integral in $\mathcal{D}(u,\underline{u})$ whose integrand is given by

$$\sum_{s_1+s_2+s_3=2s} \Psi^{(s_2)}[\widehat{\mathcal{L}}_O R]\cdot(D\,{}^{(O)}\pi)^{(s_1)}\cdot\Psi^{(s_3)}.$$

In what follows we make use of the estimates for the deformation tensors of the angular momentum vectorfields established in theorem 13.14 O,

$$\|{}^{(O)}\pi\|_{\mathcal{L}^4_{(sc)}(S)} + \|{}^{(O)}\pi\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim C$$

Also all null components of the derivatives $D\,{}^{(O)}\pi$, with the exception of $(D_3\,{}^{(O)}\pi)_{3a}$, verify the estimates,

$$\|D\,{}^{(O)}\pi\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C \tag{256}$$

Moreover,

$$\|(D_3\,{}^{(O)}\pi\,)_{3a} - \nabla_3 Z\|_{L^4(S)} + \|\sup_{\underline{u}} |\nabla_3 Z|\|_{L^2(S)} \;\lesssim\; C \tag{257}$$

$$\tag{258}$$

The term I_1 can be easily estimated, since none of the curvature terms are anomalous. Indeed, in view of lemma 15.5 we have, for all $s > 1/2$

$$\begin{aligned}\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} &\lesssim \|\Psi^{(s)}(\widehat{\mathcal{L}}_O R) - \nabla_O\Psi^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|\nabla_O\Psi^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}\\ &\lesssim \mathcal{R}(u,\underline{u})\end{aligned}$$

while, for $s = \frac{1}{2}$,

$$\|\Psi^{(1/2)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} \;\lesssim\; \underline{\mathcal{R}}(u,\underline{u})$$

Consequently, for $s > 1/2$,

$$I_1 \;\lesssim\; \sum_{s\ge 1}\int_0^u \|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|^2_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} du' + \delta^{1/2} C$$

while for $s = 1/2$,

$$I_1 \;\lesssim\; \sum_{s\le 2}\delta^{-1}\int_0^{\underline{u}} \|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|^2_{\mathcal{L}^2_{(sc)}(H_{\underline{u}'}^{(0,u)})} d\underline{u} + \delta^{1/2} C$$

Therefore,

$$I_1 \;\lesssim\; \sum_{s\ge 1}\int_0^u \|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|^2_{\mathcal{L}^2_{(sc)}(H_{u'}^{(0,\underline{u})})} du' + \sum_{s\le 2}\delta^{-1}\int_0^{\underline{u}} \|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|^2_{\mathcal{L}^2_{(sc)}(H_{\underline{u}'}^{(0,u)})} d\underline{u}' + \delta^{1/2} C \tag{259}$$

Among the terms I_2 the only possible anomalies may be due to the case when $s_3 = 3$, i.e. $(D\Psi)^{(s_3)} = \alpha(D_4 R)$ or in the easier cases $(D\Psi)^{(s_3)} = \alpha(D_3 R)$ and $(D\Psi)^{(s_3)} = \beta(D_a R)$ (i.e. $s_3 = 2$). We denote by I_{21} all terms in I_2 except those which corresponds to these anomalous cases. For all other terms we have either $\|(D\Psi)^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}'}^{(0,u)})} \lesssim C$ or $\|(D\Psi)^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{u'}^{(0,\underline{u})})} \lesssim C$. Using also $\|\,{}^{(O)}\pi\,\|_{\mathcal{L}^\infty_{(sc)}} \lesssim C$ and,

$$\begin{aligned}\|\Psi^{(s_2)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}_{(sc)}(H_u^{(0,\underline{u})})} &\lesssim C, \qquad s_2 \ge 1\\ \|\Psi^{(s_2)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})} &\lesssim C, \qquad s_2 \le 2\end{aligned}$$

we derive,

$$I_{21} \;\lesssim\; \delta^{\frac{1}{4}} C$$

We now consider the terms I_{22} which contain $(D\Psi)^{(s_3)}\alpha(D_3R)$ and $(D\Psi)^{(s_3)} = \beta(D_aR)$ but not $\alpha(D_4R)$. In this case write, according to the Remark 15.4,

$$\begin{aligned}(D\Psi)^{(s_3)} &= G + F^{(s_3)},\\ \|F^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} &\lesssim C, \qquad s_3 > 1,\\ \|F^{(s_3)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}}^{(0,u)})} &\lesssim C, \qquad s_3 < 2.\end{aligned}$$

where $G = \operatorname{tr}\underline{\chi}_0 \cdot \alpha$. Clearly, the terms corresponding to $F^{(s_3)}$ can be estimated exactly as above. To estimate the terms corresponding to G we make use of the $\mathcal{L}^4_{(sc)}(S)$ estimate, $\|G\|_{\mathcal{L}^4_{(sc)}(S)} \le C\delta^{-\frac{1}{4}}$. Using also, $\|\,{}^{(O)}\pi\,\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C$ we obtain,

$$I_{22} \lesssim \delta^{\frac{1}{4}}C$$

It remains to estimate the terms in I_{23} which contain $\alpha(D_4R)$. The integrand, which contain $\alpha(D_4R)$ has the form,

$$D_{23} = \sum_{s_1+s_2=2s-3} {}^{(O)}\pi^{\,(s_1)} \cdot \Psi^{(s_2)}(\widehat{\mathcal{L}}_O R) \cdot \alpha(D_4R)$$

This term is potentially dangerous ! In view of lemma (15.5) $\Psi^{(s_2)}(\widehat{\mathcal{L}}_O R)$ differs from $(\nabla_O\Psi)^{(s_2)}$ by a lower order terms. It thus suffices to estimate,

$$D_{23} \equiv \sum_{s_1+s_2=2s-3} {}^{(O)}\pi^{\,(s_1)} \cdot (\nabla_O\Psi)^{(s_2)} \cdot \alpha(D_4R)$$

We also decompose

$$\alpha(D_4R) = \nabla_4\alpha + \sum_{s_3+s_4=3} \phi^{(s_3)} \cdot \Psi^{(s_4)}$$

where $\phi^{(s_3)} \in \{\omega, \eta, \underline{\eta}\}$. This forces $s_4 < 2$ and thus, since there are no anomalies we derive,

$$\|\alpha(D_4R) - \nabla_4\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} \lesssim C\delta^{1/2}$$

Therefore we can safely replace $\alpha(D_4R)$ by $\nabla_4\alpha$ and thus it remains to estimate,

$$D_{23} \equiv \sum_{s_1+s_2=2s-3} {}^{(O)}\pi^{\,(s_1)} \cdot (\nabla_O\Psi)^{(s_2)} \cdot \nabla_4\alpha$$

Because of the anomaly of $\nabla_4\alpha$ the best we can by a straightforward estimate is to derive an estimate of the form $I_{23} \lesssim \mathcal{I}^{(0)} + C$ which is not acceptable. Because of this we are forced to integrate by parts, Ignoring the boundary term $\int_{\underline{H}_{\underline{u}}} {}^{(O)}\pi^{\,(s_1)} \cdot (\nabla_O\Psi)^{(s_2)} \cdot \alpha$, for a moment

$$\int_{\mathcal{D}} {}^{(O)}\pi^{\,(s_1)} \cdot (\nabla_O\Psi)^{(s_2)} \cdot \nabla_4\alpha = -\int_{\mathcal{D}} \nabla_4\, {}^{(O)}\pi^{\,(s_1)} \cdot (\nabla_O\Psi)^{(s_2)} \cdot \alpha - \int_{\mathcal{D}} {}^{(O)}\pi^{\,(s_1)} \cdot \nabla_4(\nabla_O\Psi)^{(s_2)} \cdot \alpha \tag{260}$$

We write schematically, with $\phi^{(1/2)} \in \{\eta, \underline{\eta}\}$

$$\begin{aligned}
\nabla_4(\nabla_O\Psi)^{(s_2)} &= \nabla_4\nabla_O(\Psi)^{(s_2-\frac12)}\\
&= \nabla_O\nabla_4(\Psi)^{(s_2-\frac12)} + \sum_{s_3+s_4=s_2+1}\Psi^{(s_3)}\cdot\Psi^{(s_4)} + \sum_{s_4=s_2+1/2}\phi^{(1/2)}\cdot\Psi^{(s_4)}.
\end{aligned}$$

We can therefore replace the integrand D_{23} by,

$$\begin{aligned}
D_{23} &\equiv -D_{231} - D_{232} - D_{233} - D_{234}\\
D_{231} &= \sum_{s_1+s_2=2s-3} \nabla_4\, {}^{(O)}\pi^{(s_1)}\cdot(\nabla_O\Psi)^{(s_2)}\cdot\alpha\\
D_{232} &= \sum_{s_1+s_2=2s-3} {}^{(O)}\pi^{(s_1)}\cdot\nabla_O(\nabla_4\Psi^{(s_2+1/2)})\cdot\alpha\\
D_{233} &= \sum_{s_1+s_2=2s-3} {}^{(O)}\pi^{(s_1)}\cdot\Big(\sum_{s_3+s_4=s_2+1}\Psi^{(s_3)}\cdot\Psi^{(s_4)}\Big)\cdot\alpha\\
D_{234} &= \sum_{s_1+s_2=2s-3} {}^{(O)}\pi^{(s_1)}\cdot\Big(\sum_{s_4=s_2+1/2}\phi^{(1/2)}\cdot\Psi^{(s_4)}\Big)\cdot\alpha
\end{aligned}$$

Accordingly we decompose $I_{23} \equiv I_{231} + I_{232} + I_{233} + I_{234}$. Now,

$$\begin{aligned}
I_{231} &\lesssim \delta^{\frac12}\|\nabla_4\, {}^{(O)}\pi^{(s_1)}\|_{\mathcal{L}^4_{(sc)}(S)}\|\alpha\|_{\mathcal{L}^4_{(sc)}(S)}\cdot\delta^{-1}\int_0^{\underline{u}}\|(\nabla_O\Psi)^{(s_2)}\|_{\mathcal{L}_{(sc)}(H^{(0,u)}_{\underline{u}'})}d\underline{u}'\\
&\lesssim \delta^{\frac14}C.
\end{aligned}$$

The terms I_{233} and I_{234} are clearly lower order in δ, we derive

$$I_{233} + I_{234} \lesssim \delta^{\frac12}C$$

It remains to estimate I_{232} for which we need to perform another integration by parts. We write

$$\begin{aligned}
\int_{\mathcal{D}} {}^{(O)}\pi^{(s_1)}\cdot\nabla_O(\nabla_4\Psi^{(s_2+1/2)})\cdot\alpha &= -\int_{\mathcal{D}}\nabla_O\, {}^{(O)}\pi^{(s_1)}\cdot(\nabla_4\Psi^{(s_2+1/2)})\cdot\alpha\\
&- \int_{\mathcal{D}} {}^{(O)}\pi^{(s_1)}\cdot(\nabla_4\Psi^{(s_2+1/2)})\cdot\nabla_O\alpha\\
&- \int_{\mathcal{D}} {}^{(O)}\pi^{(s_1)}\cdot(\nabla_4\Psi^{(s_2+1/2)})\cdot(\nabla^a O_a)\alpha
\end{aligned}$$

By Bianchi, since $s_2 + 1/2 < 3$,

$$\|(\nabla_4\Psi)^{(s_2+1/2)}\|_{\mathcal{L}_{(sc)}(H^{(0,\underline{u})}_u)} \lesssim \|(\nabla\Psi)^{(s_2+1/2)}\|_{\mathcal{L}_{(sc)}(H^{(0,\underline{u})}_u)} + \delta^{\frac12}\|\phi\|_{\mathcal{L}^\infty_{(sc)}}\|\Psi\|_{\mathcal{L}^2_{(sc)}(S)} \le C.$$

Therefore,

$$|\int_{\mathcal{D}} \nabla_O\, {}^{(O)}\pi^{(s_1)} \cdot (\nabla_4 \Psi)^{(s_2+\frac12)} \cdot \alpha| \lesssim \delta^{\frac12} \|\nabla_O\, {}^{(O)}\pi^{(s_1)}\|_{\mathcal{L}^4_{(sc)}(S)} \|\alpha\|_{\mathcal{L}^4_{(sc)}(S)} \times \int_0^u \|(\nabla_4\Psi)^{(s_2+\frac12)}\|_{\mathcal{L}_{(sc)}(H_{u'}^{(0,\underline{u})})} du' \lesssim \delta^{\frac14} C.$$

Also,

$$|\int_{\mathcal{D}} {}^{(O)}\pi^{(s_1)} \cdot (\nabla_4\Psi)^{(s_2+\frac12)} \cdot \nabla_O \alpha| \lesssim \delta^{\frac12} \int_0^u \|\nabla_O \alpha\|_{\mathcal{L}_{(sc)}(H_{u'}^{(0,\underline{u})})} \|(\nabla_4\Psi)^{(s_2+\frac12)}\|_{\mathcal{L}_{(sc)}(H_{u'}^{(0,\underline{u})})} du' \times \|\, {}^{(O)}\pi^{(s_1)}\|_{\mathcal{L}^\infty_{(sc)}(S)} \lesssim \delta^{\frac12} C.$$

The remaining integral in I_{232} is clearly lower order in δ. For the boundary term in (260) we have,

$$|\int_{\underline{H}_{\underline{u}}} {}^{(O)}\pi^{(s_1)} \cdot (\nabla_O\Psi)^{(s_2)} \cdot \alpha| \quad\lesssim\quad \delta^{\frac12} \|(\nabla_O\Psi)^{(s_2)}\|_{\mathcal{L}^2_{(sc)}(\underline{H}_{\underline{u}})} \cdot \|\, {}^{(O)}\pi\, \|_{\mathcal{L}^4_{(sc)}(S)} \|\alpha\|_{\mathcal{L}^4_{(sc)}(S)} \le \delta^{\frac14} C.$$

We therefore deduce,

$$I_2 \quad\lesssim\quad C\delta^{1/4} \tag{261}$$

Consider now I_3. Ignoring powers of δ, we have to estimate the integral $\int_{\mathcal{D}} (D\, {}^{(O)}\pi\,)^{(s_1)} \cdot \Psi^{(s_2)} (\widehat{\mathcal{L}}_O R) \cdot \Psi^{(s_3)}$. Recall the estimates

$$\|(D\, {}^{(O)}\pi\,)^{(s_1)}\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C$$

for all components of $(D\, {}^{(O)}\pi\,)^{(s_1)}$ with the exception of the term $D_3\, {}^{(O)}\pi_{\,3a}$ which corresponds to the signature $s_1 = 0$. In this latter case we have,

$$\|D_3\, {}^{(O)}\pi_{\,3a} - \nabla_3 Z\|_{\mathcal{L}^4_{(sc)}(S)} \lesssim C, \qquad \|\sup_{\underline{u}} |(D_3\, {}^{(O)}\pi\,)_{3a}|\|_{\mathcal{L}^2_{(sc)}(S)} \lesssim C$$

In the case $(D\, {}^{(O)}\pi\,)^{(s_1)} \neq D_3\, {}^{(O)}\pi_{\,3a}$, we have

$$|\int_{\mathcal{D}} (D\, {}^{(O)}\pi\,)^{(s_1)} \cdot \Psi^{(s_2)} (\widehat{\mathcal{L}}_O R) \cdot \Psi^{(s_3)}| \lesssim \delta^{\frac12}\, \delta^{-1} \int_0^{\underline{u}} \|(\nabla_O\Psi)^{(s_2)}\|_{\mathcal{L}_{(sc)}(H_{\underline{u}'}^{(0,u)})} d\underline{u}' \cdot \times \|(D\, {}^{(O)}\pi\,)^{(s_1)}\|_{\mathcal{L}^4_{(sc)}(S)} \|\Psi^{(s_3)}\|_{\mathcal{L}^4_{(sc)}(S)} \le \delta^{\frac14} C,$$

where we considered the worst case in which $\Psi^{(s_3)} = \alpha$ and thus anomalous and $(\nabla_O\Psi)^{(s_2)}$ has to be estimated along $H_{\underline{u}'}^{(0,u)}$.

For the case we can replace, without loss of generality, $(D\nabla\, {}^{(O)}\pi\,)^{(s_1)}$ by $\nabla_3 Z$. Indeed the remaining error term can be estimated exactly as above. In this case, since $s_1 = 0$, signature considerations dictate that $s_3\ge 1$. It follows from the conditions $s_1+s_2+s_3=2s$, $s_2\in\{s,s-\frac12\}$ and $s\ge 1$. This implies that we may use the trace theorem along H_u

$$\|\Psi^{(s_3)}\|_{\mathrm{Tr}\,_{(sc)}(H)}\lesssim \delta^{\frac14}C,$$

where in fact $\delta^{\frac14}$ only occurs in the case $\Psi^{(s_3)}=\alpha$, for all other terms the behavior in δ is better. We thus give the argument only for $\Psi^{(s_3)}=\alpha$, other cases are even easier. Recalling also lemma 15.5,

$$\begin{aligned}|\int_{\mathcal{D}}\nabla_3 Z\cdot\Psi^{(s_2)}(\widehat{\mathcal{L}}_O R)\cdot\Psi^{(s_3)}| &\lesssim \delta^{\frac12}\,\delta^{-1}\int_0^{\underline{u}}\|(\nabla_O\Psi)^{(s_2)}\|_{\mathcal{L}_{(sc)}(H_{\underline{u}'}^{(0,u)})}d\underline{u}'\cdot\\ &\times\|\sup_{\underline{u}}|\nabla_3 Z|\|_{\mathcal{L}^2_{(sc)}(S)}\sup_u\|\Psi^{(s_3)}\|_{\mathrm{Tr}\,_{(sc)}(H_u)}\le\delta^{\frac14}C,\end{aligned}$$

Finally we observe that the only borderline terms, not resulting in positive powers of the parameter δ and arising from coupling to $\mathrm{tr}\underline{\chi}$, involve only β,ρ,σ and $\underline{\beta}$ components of curvature.

Combining all our estimates for I, I_2, I_3 and using lemma 14.8 we derive,

$$\begin{aligned}\sum_{1\le s\le 5/2}\big(\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\|\Psi^{(s-\frac12)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})}\big) &\lesssim \sum_{1\le s\le 2}\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_0^{(0,\underline{u})})}\\ &+\ \delta^{1/4}C\end{aligned}$$

More precisely, we easily check the following,

$$\|\alpha(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\|\beta(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})}\ \lesssim\ \|\alpha(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\delta^{1/4}C$$

For $s\le 2$ we have,

$$\sum_{s\le 2}\big(\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\|\Psi^{(s-\frac12)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})}\big)\ \lesssim\ \sum_{s\le 2}\|\Psi^{(s)}(\widehat{\mathcal{L}}_O R)\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\delta^{1/4}C$$

Using the estimates of lemma 15.5 we derive,

$$\|\nabla\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\|\nabla\beta\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})}\ \lesssim\ \|\nabla\alpha\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\delta^{1/4}C \tag{262}$$

For $s\le 2$ we have,

$$\sum_{s\le 2}\big(\|(\nabla\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\|(\nabla\Psi)^{(s-\frac12)}\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})}\big)\ \lesssim\ \sum_{s\le 2}\|(\nabla\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\delta^{1/4}C \tag{263}$$

We summarize the result above in the following.

Proposition 15.11. *The following estimates hold for δ sufficiently small and $C=C(\mathcal{I}^{(0)},\mathcal{R},\underline{\mathcal{R}})$.*

$$\sum_{1\le s\le 5/2}\big(\nabla\|\Psi^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})}+\|\nabla\Psi^{(s-\frac12)}\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})}\big)\ \lesssim\ \mathcal{I}_0+C\delta^{1/4} \tag{264}$$

Combining this proposition with propositions 15.9 and 15.7 we derive.

$$\mathcal{R}_1 + \underline{\mathcal{R}}_1 \quad \lesssim \quad \mathcal{I}_0 + C\delta^{1/4} \tag{265}$$

Finally combining this with proposition 14.9 we derive,

$$\mathcal{R} + \underline{\mathcal{R}} \quad \leq \quad \mathcal{I}_0 + C\delta^{1/4} \tag{266}$$

This ends the proof of our main theorem.

15.12. **Proof of propositions 2.9 and 2.10.** The proof of proposition 2.9 is an immediate consequence of estimate (263) together with the initial assumptions derived in proposition 2.8. Indeed, under initial assumptions (32) we derive,

$$\sum_{s\leq 2}\left(\|(\nabla\Psi)^{(s)}\|_{\mathcal{L}^2_{(sc)}(H_u^{(0,\underline{u})})} + \|(\nabla\Psi)^{(s-\frac{1}{2})}\|_{\mathcal{L}^2_{(sc)}(H_{\underline{u}}^{(0,u)})}\right) \quad \lesssim \quad \epsilon + \delta^{1/4}C$$

which gives, for sufficiently small δ, estimate (33).

We combine this result with proposition 11.12 to prove the following scale invariant version of proposition 2.10 of the introduction.

Proposition 15.13. *The solution ${}^{(3)}\phi$ of the problem $\nabla_3^{(3)}\phi = \nabla\eta$ with trivial initial data satisfies*

$$\|{}^{(3)}\phi\|_{\mathcal{L}^\infty_{(sc)}(S)} \leq C\epsilon^{\frac{1}{4}} + C\delta^{\frac{1}{8}}.$$

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