ON THE RADIUS OF INJECTIVITY OF NULL HYPERSURFACES.

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Abstract. We investigate the regularity of past boundaries of points in regular Einstein vacuum spacetimes We provide conditionsm compatible with bounded L^2 curvature, which are sufficient to ensure the local non-degeneracy of these boundaries. More precisely we provide a uniform lower bound on the radius of injectivity of the null boundaries $\mathcal{N}^-(p)$ of the causal past sets $\mathcal{J}^-(p)$ in terms of the Riemann curvature flux on $\mathcal{N}^-(p)$ and some other natural assumptions. Such lower bounds are essential in understanding the causal structure and the related propagation properties of solutions to the Einstein equations. They are particularly important in construction of an effective Kirchoff-Sobolev type parametrix for solutions of wave equations on $\mathbf{M}$, see [KR4]. Such parametrices are used in [KR5] to prove a large data break-down criterion for solutions of the Einstein-vacuum equations.

1. Introduction

This paper is concerned with the regularity properties of boundaries $\mathcal{N}^-(p) = \partial\mathcal{I}^-(p)$ of pasts (future) of points in a $3+1$ Lorentzian manifold $(\mathbf{M}, \mathbf{g})$. The past of a point p, denoted $\mathcal{I}^-(p)$, is the collection of points that can be reached by a past directed time-like curve from p. As it is well known the past boundaries $\mathcal{N}^-(p)$ play a crucial role in understanding the causal structure of Lorentzian manifolds and the propagation properties of linear and nonlinear waves, e.g. in flat space-time the null cone $\mathcal{N}^-(p)$ is exactly the propagation set of solutions to the standard wave equation with a Dirac measure source point at p. However these past boundaries fail, in general, to be smooth even in a smooth, curved, lorentzian space-time; one can only guarantee that $\mathcal{N}^-(p)$ is a Lipschitz, achronal, 3-dimensional manifold without boundary ruled by inextendible null geodesics from p, see [HE]. In fact $\mathcal{N}^-(p) \setminus \{p\}$ is smooth in a small neighborhood of p but fails to be so in the large because of conjugate points, resulting in formation of caustics, or because of intersections of distinct null geodesics from p. Providing a lower bound for the radius of injectivity of the sets $\mathcal{N}^-(p)$ is thus an essential step in understanding the more refined properties of solutions to linear and nonlinear wave equations a Lorentzian background.

The phenomenon described above is also present in Riemannian geometry in connection to geodesic coordinates relative to a point, yet in that case the presence of

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conjugate or cut-locus points has nothing to do with the regularity of the manifold itself. In that sense lower bounds for the radius of injectivity of a Riemannian manifold are important only in so far as geodesic normal coordinates, and their applications[1], are concerned. Thus, for example, lower bounds for the radius of injectivity can sometimes be replaced by lower bounds for the harmonic radius, which plays an important role in Cheeger-Gromov theory, see e.g. [A2].

In this paper we investigate regularity of past boundaries $\mathcal{N}^-(p)$ in Einstein vacuum space-times, i.e., Lorentzian manifolds $(\mathbf{M}, \mathbf{g})$ with the Ricci flat metric, $\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0$. We provide conditions on an Einstein vacuum space-time $(\mathbf{M}, \mathbf{g})$, compatible with bounded L^2 curvature, which are sufficient to ensure the local non-degeneracy of $\mathcal{N}^-(p)$. More precisely we provide a uniform lower bound on the radius of injectivity of the null boundaries $\mathcal{N}^-(p)$ of the causal past sets $\mathcal{J}^-(p)$ in terms of the Riemann curvature flux on $\mathcal{N}^-(p)$ and some other natural assumptions[2] on $(\mathbf{M}, \mathbf{g})$. Such lower bounds are essential in understanding the causal structure and the related propagation properties of solutions to the Einstein equations. They are particularly important in construction of an effective Kirchoff-Sobolev type parametrix for solutions of wave equations on $\mathbf{M}$, see [KR4]. Such parametrices are used in [KR5] to prove a large data break-down criterion for solutions of the Einstein-vacuum equations.

This work complements our series of papers [KR1]-[KR3]. The methods of [KR1]-[KR3] can be adapted[3] to prove lower bounds on the geodesic radius of conjugacy of the congruence of past null geodesics from p which depends only on the geodesic (reduced) flux of curvature, i.e. an L^2 integral norm along $\mathcal{N}^-(p)$ of tangential components of the Riemann curvature tensor $\mathbf{R} = \mathbf{R}(\mathbf{g})$, see section 5.5 for a precise definition. It is however possible that the radius of conjugacy of the null congruence is bounded from below and yet there are past null geodesics form a point p intersecting again at points arbitrarily close to p. Indeed, this can happen on a flat Lorentzian manifold such as $\mathbf{M} = \mathbb{T}^3 \times \mathbb{R}$ where $\mathbb{T}^3$ is the torus obtained by identifying the opposite sides of a lattice of period L and metric induced by the standard Minkowski metric. Clearly there can be no conjugate points for the congruence of past or future null geodesics from a point and yet there are plenty of distinct null geodesics from a point p in $\mathbf{M}$ which intersect on a time scale proportional to L. There can be thus no lower bounds on the null radius of injectivity expressed only in terms of bounds for the curvature tensor $\mathbf{R}$. This problem occurs, of course, also in Riemannian geometry where we can control the radius of conjugacy in terms of uniform bounds for the curvature tensor, yet, in order to control the radius of injectivity we need to make other geometric assumptions such as, in the case of compact Riemannian manifolds, lower bounds on volume and upper bounds for its diameter. It should thus come as no surprise that we also need, in addition to bounds for the curvature flux, other assumptions on the geometry of solutions to

[1] such as, for example, Sobolev inequalities or the finiteness theorem of Cheeger.

[2] arising specifically in applications to the the problem of a break-down criteria in General Relativity discussed in [KR5].

[3] In [KR1]-[KR3] we have considered the case of the congruence of outgoing future null geodesics initiating on a 2-surface S_0 embedded in a space-like hypersurface Σ_0. The extension of our results to null cones from a point forms the subject of Qian Wang's Princeton 2006 PhD thesis.

the Einstein equations in order to ensure control on the null cut-locus of points in $\mathbf{M}$ and obtain lower bounds for the null radius of injectivity.

In this paper we give sufficient conditions, expressed relative to a space-like foliation Σ_t given by the level surfaces of a regular time function t with unit future normal $\mathbf{T}$. We discuss two related results. Both are based on a space-time assumption on the uniform boundedness of the deformation tensor ${}^{(\mathbf{T})}\pi = \mathcal{L}_{\mathbf{T}}\mathbf{g}$ and boundedness of the L^2 norm of the curvature tensor on a fixed slice $\Sigma_0 = \Sigma_{t_0}$ of the foliation. Standard energy estimates, based on the Bel-Robinson tensor, allows to get a uniform control on the L^2 norm of the curvature tensor on all slices. In the first result we also assume that every point of the space-time admits a sufficiently large coordinate patch with a system of coordinates in which the Lorentz metric $\mathbf{g}$ is close to a flat Minkowski metric. In the second result we dispense of the latter condition by showing how such coordinates can be constructed, dynamically, from a given coordinate system on the initial slice Σ_0. Though the second result (Main Theorem II), is more appropriate for applications, the main new ideas of the paper appear in section 2 related to the proof of the first result (Main Theorem I).

The energy estimates mentioned above also provide uniform control on the geodesic (reduced) curvature flux along the null boundaries $\mathcal{N}^-(p)$ and thus, according to [KR1]-[KR3], give control on the radius of conjugacy of the corresponding null congruence. There is however an important subtlety involved here. Past points of intersection, distinct null geodesics from p are no longer on the boundary of $\mathcal{J}^-(p)$ and therefore the energy estimates mentioned above do not apply. Consequently we cannot simply apply the results [KR1]-[KR3] and estimate the null radius of conjugacy independent of the cut-locus, but have to treat them together by a delicate boot-strap argument. The main new ideas of this paper concern estimates for the cut locus, i.e. establishing lower bounds, with respect to the time parameter t, for the points of intersection of distinct past null geodesics from p. Though our results, as formulated, hold only for Einstein vacuum manifolds our method of proof in section 2 can be extended to general Lorentzian manifolds if we make, in addition to the assumptions mentioned above, uniform norm assumptions for the curvature tensor $\mathbf{R}$. Our results seem to be new even in this vastly simplified case, indeed we are not aware of any non-trivial results concerning the null radius of injectivity for Lorentzian manifolds.

We now want to make a comparison with the corresponding picture in Riemannian geometry. In general, all known lower bounds on the radius of injectivity require some pointwise control of the curvature. The gold standard in this regard is a theorem of Cheeger providing a lower bound on the radius of injectivity in terms of pointwise bounds on the sectional curvature and diameter, and a lower bound on the volume of a compact manifold, see [Ch]. Similar to our case, the problem in Riemannian geometry splits into a lower bound on the radius of conjugacy and an estimate on the length of the shortest geodesic loop. The radius of conjugacy is intimately tied to point-wise bounds on curvature, via the Jacobi equation. The estimate for the length of the shortest geodesic loop relied, traditionally, on the Toponogov's Theorem, which again needs pointwise bounds on the curvature. These two problems can be naturally separated in Riemannian geometry[4] and while the radius of conjugacy requires pointwise assumptions on curvature, a lower bound on the length of the shortest geodesic can be given under weaker, integral, assumptions on curvature. The best result in the latter direction, to our knowledge, is due to Petersen-Steingold-Wei, which in addition to the usual diameter and volume conditions on an n-dimensional compact manifold, requires *smallness* of the L^p-norm of sectional curvature, for $p > n-1$, [PSW]. Once more, we want to re-emphasize the fact that in Riemannian geometry lower bounds on the radius of injectivity require pointwise bounds for the curvature, yet this restriction can be often overcome in applications by replacing it with bounds for more flexible geometric quantities. A case in point is the Anderson-Cheeger result [AC] which proves a finiteness theorem under pointwise assumptions on the Ricci curvature and $L^{\frac{n}{2}}$ bounds on the full Riemann curvature tensor. Unlike the classical result of Cheeger, see [Ch], the radius of injectivity need not, and cannot in general, be estimated.

We should note that the works in Riemannian geometry, cited above, have been largely stimulated by applications to the Cheeger-Gromov theory. Applications of this theory to General Relativity have been pioneered by M. Anderson, [A2] and [A3], see also [KR5] for further applications. M. Anderson has, in particular, been interested in the possibility of transferring some aspects of Cheeger-Gromov theory to the Lorentzian setting. With this in mind he has proved existence of a special space-time coordinate system for Einstein vacuum space-times, under pointwise assumptions on the space-time Riemann curvature tensor, see [A1]. Another example of a global Riemannian geometric which has been successfully transplanted to Lorentzian setting is Galloway's null splitting theorem, see [G].

We note that, in the Riemannian setting, the radius of injectivity and shortest geodesic loop estimates depend crucially on lower bounds for the volume of the manifold, as confirmed by the example of a thin flat torus. By contrast, the notion of volume of a null hypersurface in Lorentzian geometry is not well-defined, as the restriction of the space-time metric to a null hypersurface is degenerate. We are forced to replace the condition on the volume of $\mathcal{N}^-(p)$ with the condition on the volume of the 3-dimensional domain obtained by intersecting the causal past of p with the level hypersurfaces of a time function t. To be more precise our assumption on existence of a coordinate system in which the metric $\mathbf{g}$ is close to the Minkowski metric will allow us to prove that the volume of these domains, at time $t < t(p)$, are close to the volume of the Euclidean ball of radius $t(p) - t$, where $t(p)$ denotes the value of the parameter t at p. Another ingredient of our proof is an argument showing that, at the first time of intersection q, past null geodesics from a point p meet each other at angle π, viewed with respect to the tangent space of the space-like hypersurface $t = t(q)$. We also show that we can find a point p, such that the above property holds both at p and the first intersection point q. Finally, we give a geometric comparison argument showing that an existence of a pair of null geodesics from a point p meeting each other at angle π both at p and at the point of first intersection violates the structure of the past set $\mathcal{J}^-(p)$, if the time of intersection is too close to the value $t(p)$.

[4] something that we do not know how to do in the Lorentzian case

2. Main results

We consider a space-time $(\mathbf{M}, \mathbf{g})$ verifying the Einstein -vacuum equations,

$$\mathbf{R}_{\alpha\beta} = 0. \tag{1}$$

and assume that a part of space-time $\mathcal{M}_I \subset \mathbf{M}$ is foliated by the level hypersurfaces of a time function t, monotonically increasing towards future in the interval $I \subset \mathbb{R}$. Without loss of generality we shall assume that the length of I, verifies,

$$|I| \geq 1.$$

Let Σ_0 be a fixed leaf of the t foliation. Starting with a local coordinates chart $U \subset \Sigma_0$ and coordinates (x^1, x^2, x^3) we parametrize the domain $I \times U \subset \mathcal{M}_I$ with transported coordinates (t, x^1, x^2, x^3) obtained by following the integral curves of $\mathbf{T}$, the future unit normal to Σ_t. The space-time metric $\mathbf{g}$ on $I \times U$ then takes the form

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j, \tag{2}$$

where n is the lapse function of the t foliation and g is the restriction of the metric $\mathbf{g}$ to the surfaces Σ_t of constant t. We shall assume that the space-time region $\mathcal{M}_I$ is globally hyperbolic, i.e. every causal curve from a point $p \in \mathcal{M}_I$ intersects Σ_0 at precisely one point.

The second fundamental form of Σ_t is defined by,

$$k(X, Y) = \mathbf{g}(\mathbf{D}_X \mathbf{T}, Y), \quad \forall X, Y \in T(\Sigma_t),$$

with $\mathbf{D}$ the Levi-Civita covariant derivative. Observe that,

$$\partial_t g_{ij} = -\frac{1}{2} n\, k_{ij}. \tag{3}$$

We denote by $\mathrm{tr}k$ the trace of k relative to g, i.e. $\mathrm{tr}k = g^{ij} k_{ij}$. We also assume the surfaces Σ_t are either compact or asymptotically flat.

Given a unit time-like normal $\mathbf{T}$ we can define a pointwise norm $|\Pi(p)|$ of any space-time tensor Π with the help of the decomposition

$$X = X^0 \mathbf{T} + \underline{X}, \quad X \in T\mathcal{M}, \quad \underline{X} \in T\Sigma_t$$

The norm $|\Pi(p)|$ is then defined relative to the Riemannian metric,

$$\bar{\mathbf{g}}(X, Y) = X^0 \cdot Y^0 + g(\underline{X}, \underline{Y}). \tag{4}$$

We denote by $\|\Pi(t)\|_{L^p}$ the L^p norm of Π on Σ_t. More precisely,

$$\|\Pi(t)\|_{L^p} = \int_{\Sigma_t} |\Pi|^p dv_g$$

with dv_g the volume element of the metric g of Σ_t.

Let ${}^{(\mathbf{T})}\pi = \mathcal{L}_{\mathbf{T}} \mathbf{g}$ be the deformation tensor of $\mathbf{T}$. The components of ${}^{(\mathbf{T})}\pi$ are given by

$${}^{(\mathbf{T})}\pi_{00} = 0, \quad {}^{(\mathbf{T})}\pi_{0i} = \nabla_i \log n, \quad {}^{(\mathbf{T})}\pi_{ij} = -2k_{ij}$$

2.1. **Main assumptions.** We make the space-time assumption,

$$N_0^{-1} \quad \leq \quad n \ \leq \ N_0 \tag{5}$$

$$|I| \cdot \sup_{t \in I} \|\pi(t)\|_{L^\infty} \quad \leq \quad \mathcal{K}_0 < \infty. \tag{6}$$

where $N_0, \mathcal{K}_0 > 0$ are given numbers and $|I|$ denotes the length of the time interval $I \subset \mathbb{R}$. We also make the following assumptions on the initial hypersurface Σ_0,

I1. There exists a covering of Σ_0 by charts U such that for any fixed chart, the induced metric g verifies

$$I_0^{-1}|\xi|^2 \leq g_{ij}(x)\xi_i\xi_j \leq I_0|\xi|^2, \qquad \forall x \in U \tag{7}$$

with I_0 a fixed positive number. Moreover there exists a number $\rho_0 > 0$ such that every point $y \in \Sigma_0$ admits a neighborhood B, included in a neighborhood chart U, such that B is precisely the Euclidean ball $B = B^{(e)}_{\rho_0}(y)$ relative to the local coordinates in U.

I2. The Ricci curvature of the initial slice Σ_0 verifies,

$$\|\mathbf{R}\|_{L^2(\Sigma_0)} \leq \mathcal{R}_0 < \infty. \tag{8}$$

Remark. If Σ_0 is compact the existence of $\rho_0 > 0$ is guaranteed by the existence of the coordinates charts U verifying (7). More precisely we have:

Lemma 2.2. *If Σ_0 is compact and has a system of coordinate charts U verifying* (7)*, there must exist a number $\rho_0 > 0$ such that every point $y \in \Sigma_0$ admits a neighborhood B, included in a neighborhood chart U, such that B is precisely the Euclidean ball $B = B^{(e)}_{\rho_0}(y)$ relative to the local coordinates in U.*

Proof : According to our assumption every point $x \in \Sigma_0$ belongs to a coordinate patch U. Let $r(x) > 0$ be such that the euclidean ball, with respect to the coordinates of U, centered at x of radius $r(x)$ is included in U. Due to compactness of Σ_0 we can find a finite number of points $x_1, \ldots x_N$ such that the balls $B^{(e)}_{r_j/2}(x_j)$, with $r_j = r(x_j)$ for $j = 1, \ldots, N$, cover Σ_0. Thus any $y \in \Sigma_0$ must belong to a ball $B^{(e)}_{r_j/2}(x_j) \subset B^{(e)}_{r_j}(x_j) \subset U$, for some U. Therefore the ball $B^{(e)}_{r_j/2}(y) \subset U$. We then choose $\rho_0 = \min_{j=1}^N r_j/2$. ∎

2.3. **Null boundaries of $\mathcal{J}^-(p)$.** Starting with any point $p \in \mathcal{M}_I \subset \mathbf{M}$, we denote by $\mathcal{J}^-(p)$ the causal past of p, by $\mathcal{I}^-(p)$ its interior and by $\mathcal{N}^-(p)$ its null boundary all restricted to the region $\mathcal{M}_I$ under consideration.

In general $\mathcal{N}^-(p)$ is an achronal, Lipschitz hypersurface, ruled by the set of past null geodesics from p. We parametrize these geodesics with respect to the future, unit, time-like vector $\mathbf{T}_p$. Then, for every direction $\omega \in \mathbb{S}^2$, with $\mathbb{S}^2$ denoting the standard sphere in $\mathbb{R}^3$, consider the null vector ℓ_ω in $T_p\mathbf{M}$,

$$\mathbf{g}(\ell_\omega, \mathbf{T}_p) = 1, \tag{9}$$

and associate to it the past null geodesic $\gamma_\omega(s)$ with initial data $\gamma_\omega(0) = p$ and $\dot\gamma_\omega(0) = \ell_\omega$. We further define a null vectorfield $\mathbf{L}$ on $\mathcal{N}^-(p)$ according to

$$\mathbf{L}(\gamma_\omega(s)) = \dot\gamma_\omega(s).$$

$\mathbf{L}$ may only be smooth almost everywhere on $\mathcal{N}^-(p)$ and can be multi-valued on a set of exceptional points. We can choose the parameter s in such a way so that $\mathbf{L} = \dot\gamma_\omega(s)$ is geodesic and $\mathbf{L}(s) = 1$.

For a sufficiently small $\delta > 0$ the exponential map $\mathcal{G}$ defined by,

$$\mathcal{G}: \ (s,\omega) \to \gamma_\omega(s) \tag{10}$$

is a diffeomorphism from $[0,\delta) \times \mathbb{S}^2$ to its image in $\mathcal{N}^-(p)$. Moreover for each $\omega \in \mathbb{S}^2$ either $\gamma_\omega(s)$ can be continued for all positive values of s or there exists a value $s_*(\omega)$ beyond which the points $\gamma_\omega(s)$ are no longer on the boundary $\mathcal{N}^-(p)$ of $\mathcal{J}^-(p)$ but rather in its interior, see [HE]. We call such points terminal points of $\mathcal{N}^-(p)$. We say that a terminal point $q = \gamma_\omega(s_*)$ is a conjugate terminal point if the map $\mathcal{G}$ is singular at (s_*,ω). A terminal point $q = \gamma_\omega(s_*)$ is said to be a cut locus terminal point if the map $\mathcal{G}$ is nonsingular at (s_*,ω) and there exists another null geodesic from p, passing through q.

Thus $\mathcal{N}^-(p)$ is a smooth manifold at all points except the vertex p and the terminal points of its past null geodesic generators. We denote by $\mathcal{T}^-(p)$ the set of all terminal points and by $\dot{\mathcal{N}}^-(p) = \mathcal{N}^-(p) \setminus \mathcal{T}^-(p)$ the smooth portion of $\mathcal{N}^-(p)$. The set $\mathcal{G}^{-1}(\mathcal{T}^-(p))$ has measure zero relative to the standard measure $dsda_{\mathbb{S}^2}$ of the cone $[0,\infty) \times \mathbb{S}^2$. We will denote by $dA_{\mathcal{N}^-(p)}$ the corresponding measure on $\mathcal{N}^-(p)$. Observe that the definition is not intrinsic, it depends in fact on the normalization condition (9).

Definition 2.4. We define $i_*(p)$ to be the supremum over all the values $s > 0$ for which the exponential map $\mathcal{G} : (s,\omega) \to \gamma_\omega(s)$ is a global diffeomorphism. We shall refer to $i_*(p)$ as the null radius of injectivity at p relative to the geodesic foliation defined by (9).

Remark. Unlike in Riemannian geometry where the radius of injectivity is defined with respect to the distance function, the definition above depends on the normalization (9).

Definition 2.5. We denote by $\ell_*(p)$ the smallest value of s for which there exist two distinct null geodesics $\gamma_{\omega_1}(s), \gamma_{\omega_2}(s)$ from p which intersect at a point for which the corresponding value (smallest for γ_{ω_1} and γ_{ω_2}) of the affine parameter is $s = \ell_*(p)$.

Definition 2.6. Let $s_*(p)$ denote the supremum over all values of $s > 0$ such that the exponential map is a local diffeomorphism on $[0,s) \times \mathbb{S}^2$. We shall refer to $s_*(p)$ as the null radius of conjugacy of the point p.

Clearly,

$$i_*(p) = \min(l_*(p), s_*(p)) \tag{11}$$

The first goal of this paper is to prove the following theorem concerning a lower bound for the radius of injectivity $i_*$of a space-time region $\mathcal{M}_I$, under the following assumption:

Assumption C. *Every point $p \in \mathcal{M}_I$ admits a coordinate neighborhood $I_p \times U_p$ such that U_p contains a geodesic ball $B_{r_0}(p)$ and*

$$\sup_{t,t' \in I_p} |t - t'| \geq r_0. \tag{12}$$

We assume that on $I_p \times U_p$ there exists a system of transported coordinates (2) *close to the flat Minkowski metric $-n(p)^2 dt^2 + \delta_{ij} dx^i dx^j$. More precisely,*

$$|n(t,x) - n(p)| < \epsilon \tag{13}$$

$$|g_{ij}(t,x) - \delta_{ij}| < \epsilon \tag{14}$$

where $n(p)$ denotes the value of the lapse n at p.

Theorem 2.7 (Main Theorem I)**.** *Assume that $\mathcal{M}_I$ is globally hyperbolic and verifies the main assumptions* (5), (6), (8) *as well as assumption* **C** *above. We also assume that $\mathcal{M}_I$ contains a future, compact set $\mathcal{D} \subset \mathcal{M}_I$ such that there exists a positive constant δ_0 for any point $q \in \mathcal{D}^c$ we have $\ell_*(q) > \delta_0$.*

Then, for sufficiently small $\epsilon > 0$, there exists a positive number $i_* > 0$, depending only on δ_0, r_0, N_0, $\mathcal{K}_0$ and $\mathcal{R}_0$, such that, for all $p \in \mathcal{M}_I$,

$$i_*(p) > i_* \tag{15}$$

Assumption **C** of theorem 2.7 can in fact be eliminated according to the following.

Theorem 2.8 (Main Theorem II)**.** *Assume that $\mathcal{M}_I$ is globally hyperbolic and verifies the assumptions* (5), (6) *as well as the initial assumptions* **I1** *and* **I2***. Assume also that $\mathcal{M}_I$ contains a future, compact set $\mathcal{D} \subset \mathcal{M}_I$ such that $\ell_*(q) > \delta_0$ for any point $q \in \mathcal{D}^c$, for some $\delta_0 > 0$.*

There exists a positive number $i_* > 0$, depending only on I_0, μ_0, δ_0, N_0, $\mathcal{K}_0$ and $\mathcal{R}_0$, such that, for all $p \in \mathcal{M}_I$,

$$i_*(p) > i_* \tag{16}$$

Remark. Observe that the last assumption of both theorems, concerning a lower bound for l_* outside a sufficiently large future compact set, is superfluous on a manifold with compact initial slice[5] Σ_0. Thus, for manifolds $\mathcal{M}_I = I \times \Sigma_0$, with Σ_0 compact i_* depends only on the constants I_0,N_0, $\mathcal{K}_0$ and $\mathcal{R}_0$.

The first key step in the proof of the Main Theorem is a lower bound on the radius of conjugacy $s_*(p)$.

Theorem 2.9. *There exists a sufficiently small constant $\delta_* > 0$, depending only on $\mathcal{K}_0$ and $\mathcal{R}_0$ such that, for any $p \in \mathcal{M}_I$ we must have,*

$$s_*(p) > \min(\ell_*(p), \delta_*).$$

[5]A similar statement can be made if Σ_0 is asymptotically flat.

The proof of Theorem 2.9 crucially relies on the results obtained in [KR1]-[KR3][6]. The discussion of these results and their reduction to Theorem 2.9 will be discussed in Section 5.

2.10. **Connection to t-foliation.** We reinterpret the result of theorem 2.9 relative to the Σ_t foliation. For this we first make the following definition.

Definition 2.11. Given $p \in \mathcal{M}_I$ we define $i_*(p,t)$ to be the supremum over all the values $t(p) - t$ for which the exponential map

$$\mathcal{G} : (t, \omega) \to \gamma_\omega(t) = \gamma_\omega(s(t))$$

is a global diffeomorphism. We shall refer to $i_*(p,t)$ as the null radius of injectivity at p relative to the t-foliation. We denote by $\ell_*(p,t)$ the supremum over all the values $t(p) - t$, $t < t(p)$, for which there exist two distinct null geodesics $\gamma_{\omega_1}, \gamma_{\omega_2}$, from p which intersect at a point on Σ_t. Similarly, we let $s_*(p,t)$ be the supremum of $t(p) - t$ for which the exponential map $\mathcal{G}(t,\omega)$ is a local diffeomorphism.

The results leading up to the proof of Theorem 2.9 also imply the following

Theorem 2.12. *There exists a sufficiently small constant $\delta_* > 0$, depending only on N_0, $\mathcal{K}_0$ and $\mathcal{R}_0$ such that, for any $p \in \mathcal{M}_I$ we must have,*

$$s_*(p,t) > \min(\ell_*(p,t), \delta_*).$$

Furthermore, for $0 \le t(p) - t \le \min(\ell_*(p,t), \delta_*)$ the foliation $S_t = \Sigma_t \cap \mathcal{N}^-(p)$ is smooth. For these values of t the metrics σ_t on $\mathbb{S}^2$, obtained by restricting the metric g_t on Σ_t to S_t and then pulling it back to $\mathbb{S}^2$ by the exponential map $\mathcal{G}(t,\cdot)$, and the the null lapse $\varphi^{-1} = \mathbf{g}(\mathbf{L}, \mathbf{T})$ satisfy

$$|\varphi - 1| \le \epsilon, \qquad |\sigma_t(X,X) - \sigma_0(X,X)| \le \epsilon \sigma_t(X,X), \quad \forall X \in T\mathbb{S}^2$$

where σ_0 is the standard metric on $\mathbb{S}^2$ and $\epsilon > 0$ is a sufficiently small constant dependent on δ_*.

Finally, there exists a universal constant $c > 0$ such that

$$i_*(p) \ge c \min(\ell_*(p,t), \delta_*).$$

We assume for the moment the conclusions of Theorem 2.12 and proceed with the proof of Main Theorem I.

3. Proof of theorem I

According to Theorem 2.12 the desired conclusion of Main Theorem I will follow after finding a small constant δ_* dependent only on $\delta_0, r_0, N_0, \mathcal{R}_0$ and $\mathcal{K}_0$ with the property that $\ell_*(p,t) \ge \delta_*$. We fix δ_*, to be chosen later, and assume that $\ell_*(p,t) < \delta_*$. Recall that g_t and σ_t denote the restrictions of the space-time metric $\mathbf{g}$ to respectively Σ_t and S_t, while σ_0 is the push-forward of the standard metric on

[6] and an extension in Q. Wang's thesis, Princeton University, 2006.

$\mathbb{S}^2$ by the exponential map $\mathcal{G}(t, \cdot)$. The latter is clearly well-defined for the values $t(p) - \ell_*(p,t) < t \le t(p)$.

We now record three statements consistent with the assumptions of the Main Theorem and conclusions of Theorem 2.12.

A1. There exists a constant $c_{\mathcal{N}} = c_{\mathcal{N}}(p) > \ell_*(p,t)$ such that $\mathcal{N}^-(p)$ has no conjugate terminal points in the time slab $[t(p) - c_{\mathcal{N}}, t(p)]$.

A2. The metric σ_t remains close to the metric σ_0, i.e. given any vector X in TS_t we have,

$$|\sigma_t(X,X) - \sigma_0(X,X)| < \epsilon\, \sigma_t(X,X) \tag{17}$$

uniformly for all $t(p) - \ell_*(p,t) < t \le t(p)$. The null lapse $\varphi = \mathbf{g}(\mathbf{L}, \mathbf{T})$ also verifies,

$$|\varphi - 1| \le \epsilon \tag{18}$$

A3. There exists a neighborhood $\mathcal{O} = I_p \times U_p$ of p and a system of coordinates x^α with $x^0 = t$, the time function introduced above, relative to which the metric $\mathbf{g}$ is close to the Minkowski metric $\mathbf{m}(p) = -n(p)^2 dt^2 + \delta_{ij} dx^i dx^j$,

$$|\mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta}(p)| < \epsilon \tag{19}$$

uniformly at all points in $\mathcal{O}$. The set U_p contains the geodesic ball $B_{r_0}(p)$ and $\sup_{t,t' \in I_p} |t - t'| \ge r_0$. We may assume that $r_0 >> \delta_*$. In particular, $B_{t,2(t(p)-t)} \subset \mathcal{O}$ for any $t \in [t(p) - r_0/3, t(p)]$, where $B_{t,a}$ denotes the Euclidean ball of radius a centered around the point on Σ_t with the same coordinates $x = (x^1, x^2, x^3)$ as the point p.

A4. The space-time region $\mathcal{M}_I = \cup_{t \in \mathbf{I}} \Sigma_t$ contains a future, compact set $\mathcal{D} \subset \mathcal{M}_I$ such that there exists a positive constant δ_0 with the property that, for any point $q \in \mathcal{D}^c$, we have $\ell_*(q,t) > \delta_0$.

Remark 1. As a first consequence of **A1** we infer that there must exist a largest value of $t > t(p) - c_{\mathcal{N}}$ with the property that two distinct null geodesics originating at p intersect at time t. Indeed let $t_0 \ge t(p) - c_{\mathcal{N}}$ be the supremum of such values[7] of t and let $(q_k, \lambda_k, \gamma_k)$ be a sequence of points $q_k \in \mathcal{N}^-(p)$ and distinct null geodesics $\lambda_k, \gamma_k$from p intersecting at q_k, with increasing $t(q_k) \ge t(p) - c_{\mathcal{N}}$. By compactness we may assume that $q_k \to q \in \Sigma_t$, $t(q_k) \to t = t(q) = t_0$ and $\lambda_k \to \lambda$, $\gamma_k \to \gamma$, with both λ, γ null geodesics passing through p and q. We claim that $\gamma \ne \lambda$ and that q is a cut locus terminal point of $\mathcal{N}^-(p)$. Indeed if $\gamma \equiv \lambda$ then for a sequence of positive constants $0 < \epsilon_0 \to 0$ we could find an increasing sequence of indices k such that for null geodesics γ_k, λ_k we have that

$$\mathbf{g}(\dot\gamma_k(0), \dot\lambda_k(0)) = \epsilon_0, \qquad \gamma_k(t(q_k)) = \lambda_k(t(q_k)), \quad t(q_k) > t(p) - c_{\mathcal{N}}.$$

This leads to a contradiction, as by assumption the exponential map $\mathcal{G}(t, \cdot)$ is a local diffeomorphism for all $t > t(p) - c_{\mathcal{N}}$.

[7]Note that $t_0 = t(p) - \ell_*(p,t) > t(p) - \delta_*$.

Remark 2. As consequence of **A2** we infer that, for all $t > t - c_{\mathcal{N}}$, the distances on S_t corresponding to the metrics σ_t and σ_0 are comparable,

$$(1-\epsilon)d_0(q_1,q_2) \le d_\sigma(q_1,q_2) \le (1+\epsilon)d_0(q_1,q_2) \qquad \forall\, q_1,q_2 \in S_t \tag{20}$$

or, equivalently,

$$(1+\epsilon)^{-1}d_\sigma(q_1,q_2) \le d_0(q_1,q_2) \le (1-\epsilon)^{-1}d_\sigma(q_1,q_2) \qquad \forall\, q_1,q_2 \in S_t \tag{21}$$

Remark 3. Similarly, as a consequence of **A3** the distances on $\Sigma_t \cap \mathcal{O}$ corresponding to the induced metric g and the euclidean metric e are also comparable,

$$(1-\epsilon)d_e(q_1,q_2) \le d_g(q_1,q_2) \le (1+\epsilon)d_e(q_1,q_2), \qquad \forall\, q_1,q_2 \in \Sigma_t \cap \mathcal{O},\ t \in I_p \tag{22}$$

Observe also that, since σ is the metric induced by g on S_t,

$$d_g(q_1,q_2) \le d_\sigma(q_1,q_2), \qquad \forall\, q_1,q_2 \in S_t,\ \ t(p) - \ell_*(p) < t \le t(p) \tag{23}$$

Remark 4. In what follows we will assume, without loss of generality, that $t(p) = 0$ and the $x = (x^1, x^2, x^3)$ coordinates of p are $x = 0$. Without loss of generality we may also assume that $n(p) = 1$. Indeed if $n(p) \neq 0$ we can rescale the time variable $t = \tau/n(p)$ such that relative to the new time we have $\mathbf{g} = -\frac{n^2}{n(p)^2}d\tau^2 + g_{ij}dx^i dx^j$. Once we find a convenient value for δ'_* such that no distinct past null geodesics from p intersect for values of $|\tau| \le \delta'_*$ we find the desired $\delta_* = \delta'_* \cdot n(p) \ge \delta'_* N_0^{-1}$.

According to Remark 1 we can find a largest value of time $t_* < t(p)$ where two distinct null geodesics from p intersect, say at point q with $t(q) = t_*$. Our next result will imply that at q the angle between projections[8] of $\dot\gamma_1(t_*)$ and $\dot\gamma_2(t_*)$ onto $T_q\Sigma_{t_*}$ is precisely π.

Lemma 3.1. *Let $t_* = t_*(p) < 0$ be the largest[9] value of t such that that there exist two distinct past directed null geodesics γ_1, γ_2 from p intersecting at q with $t(q) = t_*$. Assume also that the exponential map $\mathcal{G} = (t,\omega) \to \gamma_\omega(t)$ is a global diffeomorphism from $(t_*(p), t(p)] \times \mathbb{S}^2$ to its image on $\mathcal{N}^-(p)$ and a local diffeomorphism in a neighborhood of q. Then, at q, the projections of $\dot\gamma_1(t_*)$ and $\dot\gamma_2(t_*)$ onto $T_q\Sigma_{t_*}$ belong to the same line and point in the opposite directions.*

Remark 5. Similar statement also holds for future directed null geodesics.

Proof : The distinct null geodesics γ_1, γ_2 can be identified with the null geodesics $\gamma_{\omega_1}, \gamma_{\omega_2}$ with $\omega_1 \neq \omega_2 \in \mathbb{S}^2$, two distinct directions in the tangent space $T_p\mathcal{M}$.

By assumptions $\gamma_{\omega_1}(t_*) = \gamma_{\omega_2}(t_*) = q$ and there exist disjoint neighborhoods $\mathcal{V}_1$ of (t_*, ω_1) and $\mathcal{V}_2$ of (t_*, ω_2) in $\mathbb{R} \times \mathbb{S}^2$ such that the restrictions of $\mathcal{G}$ to $\mathcal{V}_1$, $\mathcal{V}_2$ are both diffeomorphisms. We can choose both neighborhoods to be of the form $\mathcal{V}_i = (t_* - \epsilon, t_* + \epsilon) \times W_i$ with $\omega_i \in W_i$ for $i = 1, 2$. Let $\mathcal{G}_t(\omega) = \mathcal{G}(t,\omega)$ and define $S_{t,i} = \mathcal{G}_t(W_i), i = 1, 2$. They are both pieces of embedded 2-surfaces in Σ_t, $t \in (t_* - \epsilon_0, t_* + \epsilon_0)$ for some $\epsilon_0 > 0$ and, as the exponential map $\mathcal{G}(t,\cdot)$ is assumed to be a global diffeomorphism for any $t > t_*$, they are disjoint for all $t > t_*$.

[8]defined relative to the decomposition $X = -X^0\mathbf{T} + \underline{X}$, where $\underline{X} \in T_q\Sigma_{t_*}$.

[9]We assume that such a point exists.

For $t = t_*$ the surfaces $S_{t_*,i}$ intersect at the point q. We claim that the tangent spaces of $T_q(S_{t_*,1})$ and $T_q(S_{t_*,2})$ must coincide in $T_q(\Sigma_*)$. Otherwise, since $T_q(S_{t_*,1})$ and $T_q(S_{t_*,2})$ are two dimensional hyperplanes in a three dimensional space $T_q\Sigma_{t_*}$, they intersect transversally and by an implicit function we conclude that the surfaces $S_{t_*,i}$ also intersect transversally at q. The latter is impossible, as $S_{t,1}, S_{t,2}$ are continuous families of 2-surfaces *disjoint* for all $t > t_*$.

■

The following lemma is a consequence of **A3** and the normalization made in Remark 4. Recall that $\mathcal{I}^-(p)$ denotes the causal past of point p, i.e., it consists of all points which can be reached by continuous, past time-like curves from p, and $\mathcal{N}^-(p)$ is the boundary of $\mathcal{I}^-(p)$.

Lemma 3.2. *Let $t \in [-r_0/3, 0]$ and and let p_t be the point on Σ_t which has the same coordinates $x = (x^1, x^2, x^3) = 0$ as p. Let $B_{t,r} = B(p_t, r) \subset \Sigma_t$ be the euclidean ball centered at p_t of radius r and B_r^c its complement in Σ_t. Then,*

$$B_{t,(1-3\epsilon)|t|} \subset \mathcal{I}^-(p) \cap \Sigma_t, \quad B^c_{t,(1+3\epsilon)|t|} \cap \big(\mathcal{I}^-(p) \cup \mathcal{N}^-(p)\big) = \emptyset. \tag{24}$$

Proof . According to Remark 4 p has coordinates $t = 0, x = 0$ and $n(p) = 1$. Hence, according to (19), $|n-1| < \epsilon$ and $|g_{ij} - \delta_{ij}| < \epsilon$. The point p_t has coordinates $(t, 0)$, $t > -r_0/3$. Let $q \in B_{t,(1-2\epsilon)|t|}$ of coordinate (t, y) and $\ell(\tau) = (\tau, y\frac{\tau}{t})$ be the straight segment connecting p with q. Thus, in view of (19), for sufficiently small ϵ,

$$\begin{aligned}
\mathbf{g}(\dot\ell(\tau), \dot\ell(\tau)) &= \mathbf{m}(\dot\ell(\tau), \dot\ell(\tau)) + (\mathbf{g} - \mathbf{m})(\dot\ell(\tau), \dot\ell(\tau)) \\
&\le (-1 + \frac{|y|^2}{t^2}) + \epsilon(1 + \frac{|y|^2}{t^2}) = -1 + \epsilon + (1+\epsilon)\frac{|y|^2}{t^2} \\
&< -1 + \epsilon + (1+\epsilon)(1-2\epsilon)^2 = -2\epsilon + O(\epsilon)^2 < 0.
\end{aligned}$$

Thus q can be reached by a time-like curve from p, therefore $q \in \mathcal{J}^-(p)$.

On the other hand, if $\ell(\tau) = (\tau, x(\tau))$ is an arbitrary causal curve from p then,

$$\begin{aligned}
0 \ge \mathbf{g}(\dot\ell(\tau), \dot\ell(\tau)) &= \mathbf{m}(\dot\ell(\tau), \dot\ell(\tau)) + (\mathbf{g} - \mathbf{m})(\dot\ell(\tau), \dot\ell(\tau)) \\
&\ge (-1 + |\dot x|^2) - \epsilon(1 + |\dot x|^2) = -(1+\epsilon) + (1-\epsilon)|\dot x|^2.
\end{aligned}$$

Therefore $|\dot x| \le \frac{1+\epsilon}{1-\epsilon} < 1 + 2\epsilon + O(\epsilon^2)$ and thus, for sufficiently small $\epsilon > 0$, $|x(\tau)| \le (1 + 3\epsilon)\tau$. Consequently points q in the complement of the ball $B_{t,(1+3\epsilon)t}$ cannot be reached from p by a causal curve. ■

Corollary 3.3. *Any continuous curve $x(\tau) \subset \Sigma_t$ between two points $q_1 \in B_{t,(1-3\epsilon)|t|}$ and $q_2 \in B^c_{t,r(1+3\epsilon)|t|}$ has to intersect $\mathcal{N}^-(p) \cap \Sigma_t$.*

Proof : This is an immediate consequence of proposition 3.2 and the fact that both $\mathcal{I}^-(p)$ and $(\mathcal{I}^-(p) \cup \mathcal{N}^-(p))^c$ are connected open, disjoint, sets. ■

Remark 6. Observe that the argument used in the proof of proposition 3.2 also shows the inclusion,

$$\mathcal{N}^-(p) \cap \Sigma_t \subset B_{t,(1+3\epsilon)|t|} \cap B^c_{t,(1-3\epsilon)|t|}, \qquad \forall t: \; -r_0/3 \le t \le 0. \tag{25}$$

We are now ready prove the following,

Proposition 3.4. *Assume* **A1**, **A2**, **A3** *satisfied and* $\ell_*(p,t) < \delta_* << r_0$. *Then, no two null geodesics originating at* p *and also opposite*[10] *at* p *can intersect in the slab* $[t(p) - \ell_*(p,t), t(p))$.

Remark. Modulo the assumption that the intersecting geodesics have to be opposite at p, Proposition 3.4 gives the desired contradiction and implies the Main Theorem. Indeed, Remark 1. implies that if $\ell_*(p,t) < \delta_*$ then there exist two distinct null geodesics from p necessarily intersecting at time $t(p) - \ell_*(p,t)$, which contradicts the above proposition. The extra assumption that the geodesics are opposite at p will be settled below by showing existence of a point $p \in \mathcal{M}_I$ with the property that there exist two null geodesics from p intersecting precisely at time $t(p) - \ell_*(p,t)$, which are also opposite at p.

Proof : Once again we set $t(p) = x(p) = 0$ and $n(p) = 1$. We now argue by contradiction. Assume that there exist two opposite null geodesics $\gamma_1 \neq \gamma_2$ from p such that $\gamma_1(t_*, \omega_1) = \gamma_2(t_*, \omega_2)$, where t_* is the first time of intersection of all such geodesics, with $t_* \ge -\ell_*(p,t)$. We choose the time $t_0 > t_*$ such that the distance $d_g(\gamma_1(t_0,\omega_1), \gamma_2(t_0,\omega_2)) < t_0\,\epsilon/2$. Let $q_1 = \gamma_1(t_0,\omega_1)$ and $q_2 = \gamma_2(t_0,\omega_2)$. Note that by our assumptions the exponential map $\mathcal{G}(t,\cdot)$ is a global diffeomorphism for all $0 < t \le t_0$. Our assumption also implies that ω_1 and ω_2 represent antipodal points on $\mathbb{S}^2$.

We consider the set,

$$\Omega = \{\omega \in \mathbb{S}^2 : \; d_{\mathbb{S}^2}(\omega, \omega_2) < \frac{\pi}{4}\} = \{\omega \in \mathbb{S}^2 : \; d_{\mathbb{S}^2}(\omega, \omega_1) \ge \frac{3\pi}{4}\}.$$

Then, in view of (20) and (23), the set $\tilde{\Omega}_{t_0} = \mathcal{G}(t_0, \Omega)$ has the property that[11],

$$d_g(q_2, q) \le d_\sigma(q_2, q) < (1+\epsilon) d_0(q_2, q) = (1+\epsilon)|t_0|\frac{\pi}{4}, \quad \forall q \in \tilde{\Omega}_{t_0}.$$

Thus, in view of $d_g(q_1, q_2) < t_0\,\epsilon/2$ and the triangle inequality,

$$d_g(q_1, q) \le (1+\epsilon)|t_0|\frac{\pi}{4} + \frac{\epsilon}{2} t_0 \le \frac{\pi}{4}|t_0|\left(1 + O(\epsilon)\right), \quad \forall q \in \tilde{\Omega}_{t_0}. \tag{26}$$

Thus, taking into account (22),

$$d_e(q_1, q) \le \frac{1}{1-\epsilon} d_g(q_1, q) \le \frac{\pi}{4}|t_0|\left(1 + O(\epsilon)\right), \quad \forall q \in \tilde{\Omega}_{t_0}. \tag{27}$$

---

[10] i.e., two null geodesics γ_1, γ_2 with the property that $\gamma_1(0) = \gamma_2(0) = p$ and the projections of the tangent vectors $\dot\gamma_1(0), \dot\gamma_2(0)$ to $T_p\Sigma_{t(p)}$ belong to the same line and point in the opposite directions.

[11] Recall that d_0 denotes the distance function on S_t defined with respect to the metric σ_0 obtained by pushing forward the standard metric on $\mathbb{S}^2$ by the exponential map $\mathcal{G}(t,\cdot)$.

On the other hand, from (20), since any point in the complement of $\tilde{\Omega}_{t_0}$ in $\mathcal{N}^-(p)\cap\Sigma_{t_0}$ lies in the image by $\mathcal{G}$ of the complement of $\Omega\subset\mathbb{S}^2$,

$$d_\sigma(q_1,q)\le(1+\epsilon)d_0(q_1,q)\le\frac{3\pi}{4}(1+\epsilon)|t_0|,\quad\forall q\in\big(\mathcal{N}^-(p)\cap\Sigma_{t_0}\big)\setminus\tilde{\Omega}\tag{28}$$

Observe that, since $q_1\in\mathcal{N}^-(p)\cap\Sigma_{t_0}$ and using (25), we have $q_1\in B_{t_0,(1+3\epsilon)|t_0|}\cap B^c_{t_0,(1-3\epsilon)|t_0|}$. Thus if (t_0,y) are the coordinates of q_1 we must have

$$1-3\epsilon\le\frac{|y|}{|t_0|}\le 1+3\epsilon.$$

Thus, for sufficiently small $\epsilon>0$, the point $(t_0,-(1-7\epsilon\,|t_0|)y)\in B_{t_0,(1-3\epsilon)|t_0|}$ while $(t_0,-(1+7\epsilon\,|t_0|)y)\in B^c_{t_0,(1+3\epsilon)|t_0|}$. Let $y(\tau)=-\big(14\epsilon\tau+(1-7\epsilon\,|t_0|)\big)\cdot y$, with $\tau\in[0,1]$ and I the segment $I(\tau)=\big(t_0,y(\tau)\big)$. Observe that all points of I are within euclidean distance $O(\epsilon\,|t_0|)$ from the point $q^{opp}=(t_0,-y)$. Clearly, the extremities of I verify, $I(0)\in B_{t_0,(1-3\epsilon)|t_0|}$ and $I(1)\in B^c_{t_0,(1+3\epsilon)|t_0|}$. To reach a contradiction with Corollary 3.3 we will show that in I does not intersect $\mathcal{N}^-(p)\cap\Sigma_{t_0}$.

We first show that $I\cap\tilde{\Omega}_{t_0}=\emptyset$. Indeed, if $q\in\tilde{\Omega}_{t_0}$, we have, from(27),

$$d_e(q_1,q)\le\frac{\pi}{4}|t_0|\,(1+O(\epsilon))<|t_0|$$

while, if $q\in I$,

$$d_e(q_1,q)\ge|y|(2-7\epsilon\,|t_0|)\ge|t_0|(1-3\epsilon)(2-7\epsilon\,|t_0|)>|t_0|$$

Now, assume by contradiction, the existence of $q\in\Big((\mathcal{N}^-(p)\cap\Sigma_{t_0})\setminus\tilde{\Omega}_{t_0}\Big)\cap I$. From (28) we must infer that,

$$d_\sigma(q_1,q)\le\frac{3\pi}{4}(1+\epsilon)|t_0|.$$

On the other hand let $x(\tau)$, $\tau\in[0,1]$, be the σ-geodesic connecting q_1 and q in S_{t_0}. Since $S_{t_0}=\mathcal{N}^-(p)\cap\Sigma_{t_0}$ is contained in the set $B_{t_0,(1+3\epsilon)|t_0|}\setminus B^c_{t_0,(1-3\epsilon)|t_0|}$ so is the entire curve $x(\tau)$ for all $0\le\tau\le1$. Now observe that the euclidean distance of any curve which connects q and q^{opp} while staying outside $B_{t_0,(1-3\epsilon)|t_0|}$ must be greater than $\pi(1-3\epsilon)|t_0|$. Since all points in I are within euclidean distance $O(\epsilon\,|t_0|)$ from q^{opp} we infer that

$$\int_0^1|\dot{x}(\tau)|_e d\tau\ge\pi|t_0|\,(1-O(\epsilon))\,.$$

This implies that,

$$d_\sigma(q_1,q)=\int_0^1|\dot{x}(\tau)|_g d\tau\ge(1-\epsilon)\int_0^1|\dot{x}(\tau)|_e d\tau\ge\pi|t_0|\,(1-O(\epsilon))\,.$$

which is a contradiction. Thus I does not intersect $\mathcal{N}^-(p)\cap\Sigma_{t_0}$.

■

The proof of proposition 3.4 depends on the fact that the intersecting null geodesics from p are opposite to each other in the tangent space $T_p(\Sigma_{t(p)})$. According to lemma 3.1 we know that, at the first time t when two past directed null geodesics

γ, λ from p intersect at a point q, they must intersect opposite to each other. However the situation is not entirely symmetric, as there may exist another pair of future directed null geodesics γ', λ' from q which intersect at a point p_1 with $t(p_1)$ strictly smaller than $t(p)$. We can then repeat the procedure with p replaced by p_1 and with a new pair of null geodesics γ_1, λ_2 from p_1 intersecting at q_1 with $t(q_1)$ the smallest value of t such that any two null geodesics from p_1 intersect on Σ_t. Proceeding by induction we can construct a sequence of points p_k, q_k with $t(p_k)$ monotonically decreasing and $t(q_k)$ monotonically increasing, and sequence of pairs of distinct null geodesics γ_k, λ_k passing through both p_k and q_k. Our construction also insures that at q_k the geodesics γ_k, λ_k are opposite to each other. We would like to pass to limit and thus obtain two null geodesics which intersect each other at two distinct points. This procedure is behind the proof of the following

Proposition 3.5. *Assume that the region $\mathcal{M}_I$ verifies* **A4**. *Then, if there exist two distinct null geodesics λ_0, γ_0 intersecting at two points p_0, q_0 such that $0 < t(p_0) - t(q_0) < \delta_*$, then there must exist a pair of null geodesics λ, γ intersecting at points p, q with $t(q_0) \le t(q) < t(p) \le t(p_0)$ which are opposite at both p and q*

Proof : Let

$$\Delta t := \min_{p,q \in \mathcal{M}_I} t(p) - t(q) \tag{29}$$

such that there exists a pair of distinct past directed null geodesics originating at p and intersecting at q. By the assumption of the proposition $\Delta t < \delta_*$. On the other hand, for all points $p \in \mathcal{D}^c$, where the set $\mathcal{D}$ is that of the condition **A4**, we have that $\ell_*(p,t) > \delta_0$. Assuming, without loss of generality that $\delta_* < \delta_0$, we see that it suffices to impose the restriction $p \in \mathcal{D}$ in (29). Since $\mathcal{D}$ is compact and the manifold $\mathcal{M}_I$ is smooth we can conclude that $\Delta t > 0$.

Let $p_n \in \mathcal{D}$ be a sequence of points such that $\ell_*(p_n, t) \to \Delta t$. Since for each p_n with sufficiently large n we have $\ell_*(p_n, t) < \delta_*$ we may assume, with the help of Theorem 2.12, that **A1**–**A4** are satisfied for p_n.

Choosing a subsequence, if necessary, we can assume that $p_n \to p$. We claim that $\ell_*(p,t) = \Delta t$, i.e., there exists a pair of distinct past null geodesics from p intersecting at time $t(p) - \Delta t$, and that these geodesics are opposite to each other at p. First, to show existence of such geodesics we assume, by contradiction, that there exists an $\epsilon_0 > 0$ such that no two distinct geodesics from p intersect at $t \ge t(p) - \Delta t - \epsilon_0$. Since by assumption $\Delta t < \delta_*$ we may assume that $\mathcal{N}^-(p)$ does not contain points conjugate to p in the slab $(t(p) - \Delta t - \epsilon_0, t(p))$. This implies that the exponential map $\mathcal{G}_p(t, \cdot)$ is a global diffeomorphism for all $t \in (t(p) - \Delta t - \epsilon_0, t(p))$. Smooth dependence of the exponential map $\mathcal{G}_q$ on the base point q implies that there exists a small neighborhood $\mathcal{U}$ of p such that for any $q \in \mathcal{U}$ the exponential map $\mathcal{G}_q(t, \cdot)$ is a global diffeomorphism for any $t \in (t(p) - \Delta t - \epsilon_0/2, t(p))$. This however contradicts the existence of our sequence $p_n \to p$ since by construction $\ell_*(p_n, t) \to \Delta t$.

Therefore we may assume that there exists a pair of null geodesics γ_1, γ_2, originating at p and intersecting at a point q with $t(q) = t(p) - \Delta t = t(p) - \ell_*(p,t)$. By Lemma 3.1 the geodesics γ_1 and γ_2 are opposite at q. We need to show that they are also opposite at p. Consider the boundary of the causal future of $q - \mathcal{N}^+(q)$. It contains a pair of null geodesics, the same γ_1 and γ_2, intersecting at p. Thus, either $t(p)$ is the first time of intersection among all distinct future directed null geodesics from q, in which case Remarks after Theorem 2.12 and Lemma 3.1 imply that γ_1 and γ_2 are opposite at p, or there exists a pair of null geodesics from q intersecting at a point p' such that $t(p') < t(p)$. But then $t(p') - t(q) < \Delta t$ contradicting the definition of Δt. ■

4. Proof of Main Theorem II

We start with the following proposition.

Proposition 4.1. *Assume* (5), (6) *verified. Then, if the initial metric g on Σ_0 verifies* (7)*, there exists a large constant $C = C(N_0, \mathcal{K}_0)$ such that, relative to the induced transported coordinates in $I \times U$ we have,*

$$C^{-1}|\xi|^2 \le g_{ij}(t,x)\xi^i\xi^j \le C|\xi|^2, \qquad \forall x \in U \tag{30}$$

Proof : We fix a coordinate chart U and consider the transported coordinates t, x^1, x^2, x^3 on $I \times U$. Thus $\partial_t g_{ij} = -\frac{1}{2} n\, k_{ij}$. Let $X = X$ be a time-independent vector on $\mathbf{M}$ tangent to Σ_t. Then,

$$\partial_t g(X,X) = -\frac{1}{2} n\, k(X,X).$$

Clearly,

$$|nk(X,X)| \le |nk|_g |X|_g^2 \le \|nk(t)\|_{L^\infty} |X|_g^2$$

with $|k|_g^2 = g^{ac}g^{bd}k_{ab}k_{cd}$ and $|X|_g^2 = X^iX^jg_{ij} = g(X,X)$. Therefore, since $\partial_t|X|_g^2 = \partial_t g(X,X)$,

$$-\frac{1}{2}\|nk(t)\|_{L^\infty}|X|_g^2 \le \partial_t|X|_g^2 \le \frac{1}{2}\|nk(t)\|_{L^\infty}|X|_g^2.$$

Thus,

$$|X|_{g_0} e^{-\int_{t_0}^t \|nk(\tau)\| d\tau} \le |X|_{g_t}^2 \le |X|_{g_0} e^{\int_{t_0}^t \|nk(\tau)\| d\tau}$$

from which (30) immediately follows. ■

Corollary 4.2. *Let $p \in \Sigma_t$ in a coordinate chart $U_t = \Sigma_t \cap (I \times U)$ with transported coordinates (t, x^1, x^2, x^3). Denote by e the euclidean metric on U_t relative to the coordinates $x = (x^1, x^2, x^3)$. Let $B_r^{(e)}(p) \subset U_t$ be an euclidean ball of radius r centered at p. Then, for all $\rho \ge Cr$, with $C = C(N_0, \mathcal{K}_0)$ the constant of proposition 4.1, the euclidean ball $B_r^{(e)}(p)$ is included in the geodesic balls $B_\rho(p)$, relative to the metric g_t,*

$$B_r^{(e)}(p) \subset B_\rho(p), \qquad \rho \ge Cr.$$

Proof : Let $q \in B_r^{(e)}(p)$ and $\gamma : [0,1] \to B_r^{(e)}(p)$ be the line segment between p and q. Clearly, in view of (30),

$$\mathrm{d}ist_e(p,q) = \int_0^1 \big(e(\dot\gamma,\dot\gamma)\big)^{\frac12}\, d\tau \ge C^{-1}\int_0^1 \big(g_t(\gamma(\tau))(\dot\gamma,\dot\gamma)\big)^{\frac12} d\tau \ge C^{-1}\mathrm{d}ist_{g_t}(p,q).$$

Thus for any $q \in B_r^{(e)}(p)$ we have $\mathrm{d}ist_{g_t}(p,q) \le C\mathrm{d}ist_e(p,q) \le Cr$. Therefore q belongs to the geodesic ball $B_\rho(p)$ for any $\rho \ge Cr$, as desired. ■

The Corollary allows us to get a lower bound for the volume radius. We recall below the definition of volume radius on a general Riemannian manifold M. The Corollary allows us to get a lower bound for the volume radius. We recall below the definition of volume radius on a general Riemannian manifold M.

Definition 4.3. The volume radius $r_v(p,\rho)$ at point $p \in M$ and scales $\le \rho$ is defined by,

$$r_{vol}(p,\rho) = \inf_{r\le\rho}\frac{|B_r(p)|}{r^3}$$

with $|B_r|$ the volume of B_r relative to the metric g. The volume radius $r_{vol}(M,\rho)$ of M on scales $\le \rho$ is the infimum of $r_{vol}(p,\rho)$ over all points $p \in M$.

Let ρ_0 be the positive number of the initial assumption **I1**. Thus every point $p \in \Sigma_t$ belongs to an euclidean ball $B_{\rho_0}^{(e)}(p)$, relative to local transported coordinates. Let $B_r(p)$ be a geodesic ball around p. According to Corollary 4.2 for any $a \le \min\{\rho, r/C\}$ we must have $B_a^{(e)}(p) \subset B_r(p)$. Therefore, according to Proposition 4.1,

$$|B_r(p)|_{g_t} \ge |B_a^{(e)}(p)|_{g_t} = \int_{B_a^{(e)}(p)} \sqrt{|g_t|}\; dx \ge C^{-3/2}|B_a^{(e)}(p)|_e \ge C^{-3/2}a^3$$

This means that, for all $r \le C\rho$,

$$|B_r(p)| \ge C^{-3/2}(r/C)^3$$

Thus, on scales $\rho' \le C\rho$, $\rho \le \rho_0$ we must have, $r_{vol}(p,\rho') \ge C^{-9/2}$. Choosing $\rho \le \rho_0$ such that $C\rho = 1$ we deduce the following,

Proposition 4.4. *Under the assumptions* **I1** *as well as* (5), (6) *there exists a sufficiently small constant $v = v(I_0,\rho_0,N_0,\mathcal{K}_0) > 0$, depending only on I_0,ρ_0, N_0, $\mathcal{K}_0$, such that the volume radius of each Σ_t, for scales ≤ 1, is bounded from below,*

$$r_{vol}(\Sigma_t, 1) \ge v.$$

We rely on proposition 4.4 to prove the existence of good local space-time coordinates on $\mathcal{M}_I$. The key to our construction is the following general result, based on Cheeger -Gromov convergence of Riemannian manifolds, see [A2] or Theorem 5.4. in [Pe].

Theorem 4.5. *Given*[12] $\Lambda > 0$, $v > 0$ *and* $\epsilon > 0$ *there exists an* $r_0 > 0$ *such that on any 3- dimensional, complete, Riemannian maniflod* (M,g) *with* $\|R\|_{L^2} \le \Lambda$ *and*

[12] An appropriate version of the theorem holds in every dimension N with an L^p bound of the Riemann curvature tensor and $p > N/2$.

volume radius, at scales ≤ 1 *bounded from below by* v*, i.e.,* $r_{vol}(M,1) \ge v$*, verifies the following property:*

Every geodesic ball $B_r(p)$*, with* $p \in M$ *and* $r \le r_0$ *admits a system of harmonic coordinates* $x = (x^1, x^2, x^3)$ *relative to which we have,*

$$(1+\epsilon)^{-1}\delta_{ij} \le g_{ij} \quad \le \quad (1+\epsilon)\delta_{ij} \tag{31}$$

$$r\int_{B_r(p)} |\partial^2 g_{ij}|^2 dv_g \quad \le \quad \epsilon \tag{32}$$

We apply this theorem for the family of complete Riemannian manifolds $(\Sigma_t, g_t)_{t\in I}$, for $p = 2$. According to proposition 4.4 we have a uniform lower bound for the volume radius $r_{vol}(\Sigma_t, 1)$. On the other hand we also have a uniform bound on the L^2 norm of the Ricci curvature tensor[13]. Indeed, according to proposition 5.3 of the next section, there exists a constant $C = C(N_0, \mathcal{K}_0)$ such that, for any $t \in I$,

$$\|\mathbf{R}(t)\|_{L^2} \le C(N_0, \mathcal{R}_0)\|\mathbf{R}(t_0)\|_{L^2} = C\mathcal{R}_0.$$

Therefore, for any $\epsilon > 0$, there exists r_0 depending only on $\epsilon, I_0, \rho_0, N_0, \mathcal{K}_0, R_0$ such that on any geodesic ball, $B_r \subset \Sigma_t$, $r \le r_0$, centered at a point $p_t \in \Sigma_t$, there exist local coordinates relative to which the metric g_t verify conditions (31)-(32). Starting with any such coordinate system $x = (x^1, x^2, x^3)$ we consider a cylinder $J \times B_r$, with $J = \big(t-\delta, t+\delta\big) \cap I$ and the associated transported coordinates (t, x) for which (2) holds, i.e.

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j,$$

Integrating equation (3) and using assumptions (5), (6) we derive, for all $t' \in J$ and δ sufficiently small,

$$\begin{aligned} |g_{ij}(t', x) - g_{ij}(t, x)| \quad &\le 2 \quad \int_J \|nk(s)\|_{L^\infty} ds \le 2N_0|J| \sup_{t\in J} \|k(t)\|_{L^\infty} \\ &\le \quad 2N_0 \frac{|J|}{|I|}\mathcal{K}_0 \le \epsilon \end{aligned}$$

provided that $4\delta|I|^{-1}N_0\mathcal{K}_0 < \epsilon$. On the other hand, according to (31) we have for all $x \in B_r$,

$$|g_{ij}(t,x) - \delta_{ij}| \le \epsilon$$

Therefore, for sufficiently small interval J, whose size 2δ depends only on N_0, $\mathcal{K}_0$ and $\epsilon > 0$, we have, for all $(t', x) \in J \times B_r$,

$$|g_{ij}(t', x) - \delta_{ij}| \le 2\epsilon \tag{33}$$

On the other hand assumption (6) also provides us with a bound for $\partial_t \log n$, i.e. $|I| \cdot \sup_{t\in I} \|\partial_t \log n(t)\|_{L^\infty} \le \mathcal{K}_0$. Hence also,

$$|J| \sup_{t\in J} \|\partial_t n(t)\|_{L^\infty} \le N_0^{-1} \frac{|I|}{|J|}\mathcal{K}_0$$

[13] which coincides with the full Riemann curvature tensor in three dimensions.

Therefore, with a similar choice of $|J| = 2\delta$ we have,

$$|n(t',x) - n(t,x)| \le 2\delta N_0^{-1}\frac{|I|}{|J|}\mathcal{K}_0 < \epsilon.$$

Now, let $n(p)$ be the value of the lapse n the center p of $B_r \subset \Sigma_t$. Clearly, for all $x \in B_r$,

$$|n(t,x) - n(p)| \le r\|\nabla n\|_{L^\infty(B_r)} \le rN_0^{-1}\|\nabla \log n\|_{L^\infty(B_r)} \le rN_0^{-1}|I|^{-1}\cdot\mathcal{K}_0 \le \epsilon$$

provided that $rN_0^{-1}|I|^{-1}\cdot\mathcal{K}_0 < \epsilon$. Thus, for all $(t',x) \in J \times B_r$,

$$|n(t',x) - n(p)| \le 2\epsilon \tag{34}$$

This concludes the proof of the following.

Proposition 4.6. *Under assumptions* **I1**, **I2** *as well as* (5) *and* (6) *the globally hyperbolic region of space-time $\mathcal{M}_I$ verifies assumption* **C**. *More precisely, for every $\epsilon > 0$ there exists a constant r_0, depending only on the fundamental constants $\rho_0, I_0, N_0, \mathcal{K}_0, \mathcal{R}_0$, such that every point $p \in \mathcal{M}_I$ admits a coordinate neighborhood $I_p \times U_p$, with each U_p containing a geodesic ball $B_{r_0}(p)$ of radius r_0, and a system of transported coordinates (t,x) such that,* (12), (13) *and* (14) *hold true.*

The proof of theorem 2.8 is now an immediate consequence of Theorem 2.7 and proposition 4.6.

5. Radius of conjugacy

The remaining part of the paper will be devoted to the proof of Theorems 2.9 and 2.12. As was mentioned before the key results on the radius of conjugacy were obtained[14] in [KR1]-[KR3] and here we will show how to deduce Theorems 2.9 and 2.12 from these results.

A lower bound on the radius of conjugacy in [KR1]-[KR3] is given by the following theorem. Let $\mathcal{L}^-(p)$ denote the union of all past directed null geodesics from p. Clearly $\mathcal{N}^-(p) \subset \mathcal{L}^-(p)$. We can extend the null geodesic (potentially non-smooth) vectorfield $\mathbf{L}$ to $\mathcal{L}^-(p)$ and define $S_{s_0} = \mathcal{L}^-(p) \cap \{s = s_0\}$ a two dimensional foliation of $\mathcal{L}^-(p)$ by the level surfaces of the affine parameter s ($\mathbf{L}(s) = 1$). The conjugacy radii of $\mathcal{N}^-(p)$ and $\mathcal{L}^-(p)$ coincide and

$$\mathcal{N}^-(p) \cap \big(\cup_{s \le i_*(p)} S_s\big) = \cup_{s \le i_*(p)} S_s$$

Theorem 5.1. *Let $\varpi > 0$ be a sufficiently small universal constant and let $\mathcal{R}(p,s)$ denote the reduced curvature flux, associated with $\cup_{s' \le s} S_s$, to be defined below. Then there exists a large constant C_ϖ such that if the radius of conjugacy $s_*(p) \le \varpi$ then $\mathcal{R}(p, s_*(p)) \ge C_\varpi$.*

To deduce Theorems 2.9 and 2.12 from Theorem 5.1 it suffices to show that the reduced curvature flux $\mathcal{R}(p,s) \le C$ for all values of $s \le \min(\ell_*(p), \delta_*)$, where δ_* is

[14] An extension of these results to null hypersurfaces with a vertex is part of Q.Wang's thesis, Princeton University, 2006.

allowed to depend on N_0, $\mathcal{R}_0, \mathcal{K}_0$. As we shall see below the reduced curvature flux itself is only well defined for the values of $s < i_*(p)$. For $s_*(p) \le \min(\ell_*(p), \delta_*)$ we will then show that for all $s < s_*$ we have the bound $\mathcal{R}(p,s) \le C(N_0, \mathcal{R}_0, \mathcal{K}_0)$ and thus by Theorem 5.1, in fact, $s_*(p) > \min(\ell_*(p), \delta_*)$. In the latter case, we will also show that $\mathcal{R}(p,s) \le C(N_0, \mathcal{R}_0, \mathcal{K}_0)$ for all $s < \min(\ell_*(p), \delta_*)$.

5.2. **Basic definitions and inequalities.** We start with a quick review of the Bel-Robinson tensor and the corresponding energy inequalities induced by $\mathbf{T}$. The fully symmetric, traceless and divergence free Bel-Robinson tensor is given by

$$\mathbf{Q}[\mathbf{R}]_{\alpha\beta\gamma\delta} = \mathbf{R}_{\alpha\lambda\gamma\mu}\mathbf{R}_{\beta\ \delta}^{\ \lambda\ \mu} + {}^\star\mathbf{R}_{\alpha\lambda\gamma\mu}\,{}^\star\mathbf{R}_{\beta\ \delta}^{\ \lambda\ \mu} \tag{35}$$

The curvature tensor $\mathbf{R}$ can be decomposed into its electric and magnetic parts E, H as follows,

$$E(X,Y) =< \mathbf{g}(\mathbf{R}(X,\mathbf{T})\mathbf{T}, Y), \qquad H(X,Y) = \mathbf{g}({}^\star\mathbf{R}(X,\mathbf{T})\mathbf{T}, Y) \tag{36}$$

with ${}^\star\mathbf{R}$ the Hodge dual of $\mathbf{R}$. One can easily check that E and H are tangent, traceless 2-tensors, to Σ_t and that $|\mathbf{R}|^2 = |E|^2 + |H|^2$. We easily check the formulas relative to an orthonormal frame $e_0 = T, e_1, e_2, e_3$,

$$\begin{aligned} \mathbf{R}_{abc0} &= -\in_{abs} H_{sc}, \qquad {}^\star\mathbf{R}_{abc0} = \in_{abs} E_{sc} \\ \mathbf{R}_{abcd} &= \in_{abs}\in_{cdt} E_{st}, \qquad {}^\star\mathbf{R}_{abcd} = -\in_{abs}\in_{cdt} H_{st} \end{aligned} \tag{37}$$

Observe that,

$$|\mathbf{Q}| \le 4(|E|^2 + |H|^2) \tag{38}$$

and,

$$\mathbf{Q}_{0000} = |E|^2 + |H|^2 \tag{39}$$

Let $\mathbf{P}_\alpha = \mathbf{Q}[\mathbf{R}]_{\alpha\beta\gamma\delta}\mathbf{T}^\beta\mathbf{T}^\gamma\mathbf{T}^\delta$. By a straightforward calculation,

$$\mathbf{D}^\alpha \mathbf{P}_\alpha = \frac{3}{2}\,{}^{(\mathbf{T})}\pi^{\alpha\beta}\mathbf{Q}_{\alpha\beta\gamma\delta}\mathbf{T}^\gamma\mathbf{T}^\delta \tag{40}$$

Therefore, integrating in a slab $\mathcal{M}_J = \cup_{t\in J}\Sigma_t$, $J = [t_0, t] \subset I$, we derive the following.

$$\int_{\Sigma_t}\mathbf{Q}_{0000} = \int_{\Sigma_0}\mathbf{Q}_{0000} + \frac{3}{2}\int_{t_0}^t\int_{\Sigma_{t'}} n\,{}^{(\mathbf{T})}\pi^{\alpha\beta}\mathbf{Q}_{\alpha\beta 00}\,dv_g \tag{41}$$

with dv_g denoting the volume element on Σ_t. Now,

$$\begin{aligned} \Big|\int_{t_0}^t\int_{\Sigma_{t'}} n\,{}^{(\mathbf{T})}\pi^{\alpha\beta}\mathbf{Q}_{\alpha\beta 00}dv_g\Big| &\lesssim N_0\int_{t_0}^t\int_{\Sigma_{t'}} |{}^{(\mathbf{T})}\pi|(|E|^2 + |H|^2)dv_g \\ &\lesssim N_0\int_{t_0}^t \|{}^{(\mathbf{T})}\pi(t')\|_{L^\infty}\big(\|E(t')\|_{L^2} + \|H(t')\|_{L^2}\big)dt' \end{aligned}$$

Thus, if we denote

$$\mathcal{Q}(t) = \int_{\Sigma_t}\mathbf{Q}_{0000} = \int_{\Sigma_t}(|E|^2 + |H|^2)dv_g,$$

we deduce,

$$\mathcal{Q}(t) - \mathcal{Q}(t_0) \lesssim N_0\int_{t_0}^t \|\pi(t')\|_{L^\infty}\mathcal{Q}(t')dt'$$

and by Gronwall,

$$\mathcal{Q}(t) \lesssim \mathcal{Q}(t_0)\mathrm{exp}\Big(\int_{t_0}^{t} N_0\| \,^{(\mathbf{T})}\pi(t')\|_{L^\infty}dt'\Big)$$

Thus, in view of (6),

$$\mathcal{Q}(t) \lesssim \mathcal{Q}(t_0)\,\mathrm{exp}\big(N_0\mathcal{K}_0\big)$$

We have just proved the following,

Proposition 5.3. *Assume that the assumptions* (5) *and* (6) *are true. There exists a constant* $C = C(N_0, \mathcal{K}_0)$ *such that, for any* $t \in I$,

$$\|\mathbf{R}(t)\|_{L^2} \le C\|\mathbf{R}(t_0)\|_{L^2} = C\mathcal{R}_0. \tag{42}$$

Instead of integrating (40) in the slab $\mathcal{M}_J$ we will now integrate it in the region $\mathcal{D}^-_J(P) = \mathcal{J}^-(p)\cap\mathcal{M}_J$ whose boundary consists of the null part $\mathcal{N}^-(p)$ and space-like part $D_0(p) = \mathcal{J}^-(p)\cap\Sigma_0$. We recall that $\mathcal{N}^-(p)$ is a Lipschitz manifold and the set of its terminal points $\mathcal{T}^-(p)$ has measure zero relative to $dA_{\mathcal{N}^-(p)}$.

Let $(\mathbf{P}^*)_{a\beta\gamma} =\in_{\alpha\beta\gamma\mu} \mathbf{P}^\mu$ and the associated differential form, ${}^*\mathbf{P} = ({}^*\mathbf{P})_{\alpha\beta\gamma}dx^\beta dx^\gamma dx^\delta$. We can rewrite equation (40) in the form, $d^*\mathbf{P} = -{}^*\mathbf{F}$, with $({}^*\mathbf{F})_{\alpha\beta\gamma\delta} =\in_{\alpha\beta\gamma\delta} \mathbf{F}$, and,

$$\mathbf{F} = \frac{3}{2}\,^{(\mathbf{T})}\pi^{\alpha\beta}\mathbf{Q}_{\alpha\beta\gamma\delta}\mathbf{T}^\gamma\mathbf{T}^\delta.$$

Integrating the last expression in the space-time region $\mathcal{D}^-_I(p) = \mathcal{J}^-(p)\cap\mathcal{M}_J$, with $J = [t_0, t]$, $p \in \mathcal{M}_J$, and applying Stokes theorem we derive,

$$\int_{\mathcal{D}^-_J(p)} {}^\star\mathbf{F} \quad = \quad -\int_{\mathcal{N}^-(p)\cap\mathcal{M}_J} {}^\star\mathbf{P} = \mathcal{F}_p(\mathcal{N}^-(p)\cap\mathcal{M}_J) - \mathrm{En}(D_0(p)) \tag{43}$$

where

$$\mathrm{En}(D_0(p)) \quad = \quad -\int_{D_0(p)} {}^\star\mathbf{P} = \int_{D_0(p)} \mathbf{Q}(\mathbf{T},\mathbf{T},\mathbf{T},\mathbf{T})dv_g \tag{44}$$

$$\mathcal{F}_p(\mathcal{N}^-(p)\cap\mathcal{M}_J) \quad = \quad -\int_{\mathcal{N}^-(p)\cap\mathcal{M}_J} {}^\star\mathbf{P} \tag{45}$$

The energy integral (44) through $D_0(p)\subset\Sigma_0$ can clearly be bounded by $\|\mathbf{R}(t_0)\|_{L^2}$. Moreover, in view of proposition 5.3 the integral $\int_{\mathcal{D}^-_J(p)} {}^\star\mathbf{F}$ can be bounded by $C(\mathcal{K}_0)\cdot\mathcal{R}_0$. Therefore,

$$\mathcal{F}_p(\mathcal{N}^-(p)\cap\mathcal{M}_J) \lesssim C(\mathcal{K}_0)\cdot\mathcal{R}_0 \tag{46}$$

We recall that the null boundary $\mathcal{N}^-(p)$ is a Lipschitz manifold. This means that every point $p \in \mathcal{N}^{-1}(p)$ has a local coordinate chart U_p together with local coordinate $x^\alpha = x^\alpha(\tau, \omega^1, \omega^2)$ which are Lipschitz continuous. The coordinates are such that for all fixed[15] $\omega = (\omega^1, \omega^2)$ the curves $\tau \to x^\alpha(\tau, \omega)$ are null and for any fixed value τ the 2 dimensional surfaces S_τ, given by $x^\alpha = x^a(\tau, \omega)$, are space-like. In particular there is a well defined null normal $\frac{dx^\alpha}{d\tau} = l^\alpha$ at all points of U_p with the

[15] The statements here are understood to be true with the possible exception of a set of measure zero relative to the measure $dA_{\mathcal{N}^-(p)}$ along $\mathcal{N}^-(p)$ introduced just before definition 2.4.

possible exception of a set of measure zero. Moreover we can choose our coordinate charts such that at each point where the normal l is defined we have $\mathbf{g}(l, \mathbf{T}) > 0$, i.e. l is past oriented. Observe that on such coordinate chart U we have,

$$\int_U {}^\star\mathbf{P} \quad = \quad \int_U {}^\star\mathbf{P}_{\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma = \int_U \mathbf{g}(\mathbf{P}, l) d\tau dA_\tau = \int_U \mathbf{Q}(\mathbf{T}, \mathbf{T}, \mathbf{T}, l) d\tau dA_\tau$$

with dA_τ the volume element of the space-like surfaces S_τ. Since $\mathbf{T}$ is future time-like and l is null past directed we have $\mathbf{Q}(\mathbf{T}, \mathbf{T}, \mathbf{T}, l) < 0$. Consequently, for every coordinate chart $U \subset \mathcal{N}^-(p)$, $\mathcal{F}_p^-(U) \geq 0$, where

$$\mathcal{F}_p^-(U) = -\int_U {}^\star\mathbf{P} \tag{47}$$

Using a partition of unity it follows that $\mathcal{F}_p(U) \geq 0$ for any $U \subset \mathcal{N}^-(p)$ and therefore $\mathcal{F}_p^-(U_1) \leq \mathcal{F}_p^-(U_2)$ whenever $U_1 \subset U_2 \subset \mathcal{N}^-(p)$. We can thus identify $\mathcal{F}_p(U)$ as the flux of curvature through $U \subset \mathcal{N}^-(p)$.

Therefore we have the following:

Proposition 5.4. *Under assumptions* (5),(6) *and* (8) *the flux of curvature in* $\mathcal{M}_I \cap \mathcal{N}^-(p)$, $\mathcal{F}_p^-(\mathcal{M}_I) = \mathcal{F}_p^-(\mathcal{M}_I \cap \mathcal{N}^-(p))$, *can be bounded by a uniform constant independent of* p. *More precisely,*

$$\mathcal{F}_p(\mathcal{M}_I) \leq C(N_0, \mathcal{K}_0) \cdot \mathcal{R}_0.$$

5.5. **Reduced curvature flux.** Let S_s be the 2 dimensional space-like surface of a constant affine parameter s, defined by the condition $\mathbf{L}(s) = 1$ and $s(p) = 0$. Clearly for $s \leq \delta < i_*(p)$ the union of S_s defines a regular foliation of

$$\mathcal{N}^-(p, \delta) = \cup_{s<\delta} S_s$$

At any point of $\mathcal{N}^-(p, \delta) \setminus \{p\}$ we can define a conjugate null vector $\underline{\mathbf{L}}$ with $\mathbf{g}(\mathbf{L}, \underline{\mathbf{L}}) = -2$ and such that $\underline{\mathbf{L}}$ is orthogonal to the leafs S_s. In addition we can choose $(e_a)_{a=1,2}$ tangent S_s such that together with $\mathbf{L}$ and $\underline{\mathbf{L}}$ we obtain a null frame,

$$\begin{aligned} &g(\mathbf{L}, \underline{\mathbf{L}}) = -2, \qquad \mathbf{g}(\mathbf{L}, \mathbf{L}) = \mathbf{g}(\underline{\mathbf{L}}, \underline{\mathbf{L}}) = 0, \\ &\mathbf{g}(\mathbf{L}, e_a) = \mathbf{g}(\underline{\mathbf{L}}, e_a) = 0, \qquad \mathbf{g}(e_a, e_b) = \delta_{ab}. \end{aligned} \tag{48}$$

We denote by σ the restriction of $\mathbf{g}$ to S_s. Endowed with this metric S_s is a 2 dimensional compact riemannian manifold with $\gamma(e_a, e_b) = \delta_{ab}$. Let $|S_s|$ denotes the area of S_s and define $r = r(s)$ by the formula

$$4\pi r^2 = |S_s|. \tag{49}$$

We say that a tensor π along $\mathcal{N}^-(p)$ is S_s tangent, or simply S-tangent, if, at every point of $\mathcal{N}^-(p)$ it is orthogonal to both null vectors $\mathbf{L}$ and $\underline{\mathbf{L}}$. Given such a tensor, say π_{ab}, we denote by $|\pi|$ its length relative to the metric γ, i.e. $|\pi|^2 = \sum_{a,b=1}^2 |\pi_{ab}|^2$.

We denote by ∇ the restriction of $\mathbf{D}$ to S_s, Clearly, for all $X, Y \in T(S_s)$,

$$\nabla_X Y = \mathbf{D}_X Y + \frac{1}{2} < \mathbf{D}_X Y, \underline{\mathbf{L}} > \mathbf{L} + \frac{1}{2} < \mathbf{D}_X Y, \mathbf{L} > \underline{\mathbf{L}} \tag{50}$$

Given an $S-$ tangent tensor π we define $(\nabla_L \pi)$ to be the projection to S_s of $\mathbf{D}_\mathbf{L}\pi$. We write $\overline{\nabla}\pi = (\nabla\pi, \nabla_L\pi)$ and

$$|\overline{\nabla}\pi|^2 = |\nabla_L \pi|^2 + |\nabla\pi|^2.$$

We recall the definition of the null second fundamental form $\chi, \underline{\chi}$, and torsion ζ associated to the S_s foliation.

$$\chi_{ab} = \mathbf{g}(\mathbf{D}_{e_a}\mathbf{L}, e_b), \qquad \underline{\chi}_{ab} = g(\mathbf{D}_{e_a}\underline{\mathbf{L}}, e_b) \qquad \zeta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_a\mathbf{L}, \underline{\mathbf{L}}) \tag{51}$$

We also introduce,

$$\varphi^{-1} = g(\mathbf{T}, \mathbf{L}), \qquad \psi_a = \mathbf{g}(e_a, \mathbf{T}) \tag{52}$$

Observe that $\varphi > 0$ with $\varphi(p) = 1$. Also

$$\frac{dt}{ds} = -n^{-1}\varphi^{-1} \tag{53}$$

with n the lapse function of the t foliation. We now recall the standard null decomposition of the Riemann curvature tensor relative to the S_s foliation:

$$\begin{aligned} \alpha_{ab} &= \mathbf{R}_{\mathbf{L}a\mathbf{L}b}\,, \quad \beta_a = \frac{1}{2}\mathbf{R}_{a\mathbf{L}\underline{\mathbf{L}}\mathbf{L}}, \quad \rho = \frac{1}{4}\mathbf{R}_{\underline{\mathbf{L}}\mathbf{L}\underline{\mathbf{L}}\mathbf{L}} \\ \sigma &= \frac{1}{4}\,{}^\star\mathbf{R}_{\underline{\mathbf{L}}\mathbf{L}\underline{\mathbf{L}}\mathbf{L}}, \quad \underline{\beta}_a = \frac{1}{2}R_{a\underline{\mathbf{L}}\underline{\mathbf{L}}\mathbf{L}}, \quad \underline{\alpha}_{ab} = \mathbf{R}_{\underline{\mathbf{L}}a\underline{\mathbf{L}}b} \end{aligned} \tag{54}$$

We can write the flux along $\mathcal{N}^-(q, \delta)$, $\delta < i_*(q)$ as follows.

$$\mathcal{F}(p, \delta) = \int_{\mathcal{N}^-(q,\delta)} \mathbf{Q}(\mathbf{T}, \mathbf{T}, \mathbf{T}, \mathbf{L}) = \int_0^\delta ds \int_{S_s} \mathbf{Q}(\mathbf{T}, \mathbf{T}, \mathbf{T}, \mathbf{L}) dA_s$$

Observe that $dsdA_s$ is precisely the measure $dA_{\mathcal{N}^-(p)}$. More generally we shall use the following notation.

Definition 5.6. Given a scalar function f on $\mathcal{N}^-(p, \delta)$, $\delta \le i_*(p)$ we denote its integral on $\mathcal{N}^-(p, \delta)$ to be,

$$\int_{\mathcal{N}^-(p,\delta)} f = \int_0^\delta ds \int_{S_s} f dA_s = \int_{\mathcal{N}^-(p,\delta)} f \; dA_{\mathcal{N}^-(p)}.$$

Or, relative to the normal coordinates (s, ω) in the tangent space to p,

$$\int_{\mathcal{N}^-(p,\delta)} f = \int_0^\delta \int_{|\omega|=1} f(s, \omega)\sqrt{|\sigma(s, \omega)|} ds d\omega$$

where $|\sigma(s, \omega)|$ is the determinant of the components of the induced metric σ on S_s relative to the coordinates s, ω.

To express the density $\mathbf{Q}(\mathbf{T}, \mathbf{T}, \mathbf{T}, \mathbf{L})$ in terms of the null components $\alpha, \beta, \rho, \sigma, \underline{\beta}$ we need to relate $\mathbf{T}$ to the null frame $\mathbf{L}, \underline{\mathbf{L}}, e_a$. To do this we first introduce another null frame attached to the t foliation. More precisely, at some point $q \in \mathcal{N}^-(p, \delta)$, we let $S_t = \Sigma_t \cap \mathcal{N}^-(p)$ for $t = t(p)$. We define $\underline{\mathbf{L}}'$ to be the null pair conjugate to $\mathbf{L}$ relative to S_t. More precisely $\mathbf{g}(\mathbf{L}, \underline{\mathbf{L}}') = -2$ and $\underline{\mathbf{L}}'$ is orthogonal to S_t. We complete $\mathbf{L}, \underline{\mathbf{L}}'$ to a full null frame on S_t by

$$e'_a = e_a - \varphi\psi_a\mathbf{L}$$

We also have,

$$\underline{\mathbf{L}}' = \underline{\mathbf{L}} - 2\varphi\psi_a e_a + 2\varphi^2|\psi|^2\mathbf{L}$$

Now,

$$\mathbf{T} \;=\; -\frac{1}{2}(\varphi\mathbf{L} + \varphi^{-1}\underline{\mathbf{L}}') = -\frac{1}{2}\varphi\mathbf{L} - \frac{1}{2}\varphi^{-1}(\underline{\mathbf{L}} - 2\varphi\psi_a e_a + 2\varphi^2|\psi|^2\mathbf{L})$$

Therefore,

$$\mathbf{T} = \varphi(-\frac{1}{2} - |\psi|^2)\mathbf{L} - \frac{1}{2}\varphi^{-1}\underline{\mathbf{L}} + \psi_a e_a \tag{55}$$

which we rewrite in the form,

$$\begin{aligned} \mathbf{T} &= T_0 + X, \qquad T_0 = -\frac{1}{2}\mathbf{L} - \frac{1}{2}\underline{\mathbf{L}} &(56)\\ X &= \big(-\frac{1}{2}(\varphi - 1) - \varphi|\psi|^2\big)\mathbf{L} - \frac{1}{2}(\varphi^{-1} - 1)\underline{\mathbf{L}} + \psi_a e_a &(57)\end{aligned}$$

Now,

$$\begin{aligned} \mathbf{Q}(\mathbf{T},\mathbf{T},\mathbf{T},\mathbf{L}) &= \mathbf{Q}(T_0 + X, T_0 + X, T_0 + X, \mathbf{L}) = \mathbf{Q}(T_0, T_0, T_0, \mathbf{L}) + \mathrm{Qr}\\ \mathrm{Qr} &= \mathbf{Q}(X, T_0, T_0, \mathbf{L}) + \mathbf{Q}(X, X, T_0, \mathbf{L}) + \mathbf{Q}(X, X, X, \mathbf{L})\end{aligned}$$

By a straightforward calculation,

$$\mathbf{Q}(T_0, T_0, T_0, \mathbf{L}) = \frac{1}{4}|\alpha|^2 + \frac{3}{2}|\beta|^2 + \frac{3}{2}(\rho^2 + \sigma^2) + \frac{1}{2}|\underline{\beta}|^2$$

For $\delta < i_*(p)$ we introduce the reduced flux, or geodesic curvature flux,

$$\mathcal{R}(p, \delta) = \Big(\int_0^\delta \int_{S_s} \big(|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\underline{\beta}|^2\big) dA_s ds\Big)^{1/2} \tag{58}$$

On the other hand the following result can be easily seen from (57) .

Lemma 5.7. *Assume that the following estimates hold on $\mathcal{N}^-(p,\delta)$, for some $\delta < i_*(p)$,*

$$|\varphi - 1| + |\psi| \le 10^{-2} \tag{59}$$

Then on $\mathcal{N}^-(p,\delta)$,

$$\mathbf{Q}(\mathbf{T},\mathbf{T},\mathbf{T},\mathbf{L}) \ge \frac{1}{2}Q(T_0, T_0, T_0, \mathbf{L}) \ge \frac{1}{8}\big(|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\underline{\beta}|^2\big)$$

Remark. We can guarantee the existence of such $\delta > 0$, as the initial conditions for φ and ψ are $\varphi(p) = 1$ and $\psi(p) = 0$. The challenge will be to extend estimate (59) to a larger region.

As an application of proposition (5.4) and lemma 5.7 above we derive,

Corollary 5.8. *Let $p \in \mathcal{M}_J$ and assume that the estimate* (59) *holds on $\mathcal{N}^-(p,\delta)$ for some $\delta < i_*(p)$. Then the reduced curvature flux $\mathcal{R}(p,\delta)$ can be bounded from above by a constant which depends only on N_0, $\mathcal{K}_0$ and the initial data $\mathcal{R}_0$.*

In view of Theorem 5.1 and Corollary 5.8 to finish the proof of Theorem 2.9 we need to show that there exists a constant $\delta_* = \delta_*(N_0, \mathcal{K}_0, \mathcal{R}_0)$ such that the bounds (59) can be extended to all values values of $s \le \min(\delta_*, i_*(p))$.

We first state a theorem which is an extension of Theorem 5.1 and another consequence of the results proved in [KR1]-[KR3]. We will then show simultaneously that for all values of $s \le \min(\delta_*, i_*(p))$ the reduced curvature flux $\mathcal{R}(p,s) \le C(\mathcal{R}_0, \mathcal{K}_0)$ and the estimates (59) hold true.

Theorem 5.9. *Let $p \in \mathcal{M}_I$ fixed and assume that the reduced curvature flux verifies $\mathcal{R}(p,\delta) \le C$ for some $\delta \le i_*(p)$ and a positive constant C. Let $\varepsilon_0 > 0$ be a fixed small constant. Then for all $s \le \min(\varpi, \delta)$, where ϖ is a small constant dependent only on ϵ_0 and C, we have*

$$|tr\chi - \frac{2}{s}| \le \varepsilon_0, \qquad \int_0^s |\hat{\chi}|^2 ds' \le \varepsilon_0. \tag{60}$$

5.10. **Bounds for φ and ψ.** The proof of the bounds for the reduced curvature flux and φ and ψ depends, in addition to the results stated in Theorem 5.9 and Corollary 5.8 on the following,

Proposition 5.11. *Let δ_* be a small constant dependent only on $N_0, \mathcal{K}_0, \mathcal{R}_0$. Assume that $tr\chi$ and $\hat{\chi}$ verify (60) for all $0 \le s \le \delta < \min(\delta_*, i_*(p))$. Assume also that the condition (59) holds true for $0 \le s \le \delta$ and let $\epsilon_0 < 10^{-1}$ in Theorem 5.9. Then the following better estimate holds for all $0 \le s \le \delta$,*

$$|\varphi - 1| + |\psi| \le 10^{-3}.$$

Remark. The above proposition, Corollary 5.8 and a simple continuity argument allow us to get the desired conclusion that the reduced curvature flux $\mathcal{R}(p,\delta)$ is bounded by $C(N_0, \mathcal{K}_0, \mathcal{R}_0)$ for all $\delta < \min(i_*(p), \delta_*)$, which in turn, by Theorem 5.1, implies that $s_*(p) > \min(\ell_*(p), \delta_*)$.

Proof : We shall use the frame $\mathbf{L}, \underline{\mathbf{L}}', e'_a$ attached to the foliation S_t. Recall that,

$$e'_a = e_a - \varphi\psi_a \mathbf{L}, \qquad \underline{\mathbf{L}}' = \underline{\mathbf{L}} - 2\varphi\psi_a e_a + 2\varphi^2|\psi|^2\mathbf{L}$$

and

$$\mathbf{T} = \varphi(-\frac{1}{2} - |\psi|^2)\mathbf{L} - \frac{1}{2}\varphi^{-1}\underline{\mathbf{L}} + \psi_a e_a.$$

We denote by N the vector,

$$N = -\frac{1}{2}(\varphi\mathbf{L} - \varphi^{-1}\underline{\mathbf{L}}').$$

Observe that $\mathbf{g}(N, \mathbf{T}) = 0$ while $\mathbf{g}(N, N) = 1$. Thus N is the unit normal to S_t along the hypersurface Σ_t. We can now decompose $\mathbf{L}$ and $\underline{\mathbf{L}}'$ as follows,

$$\mathbf{L} = -\frac{1}{2}\varphi^{-1}(\mathbf{T} + N), \quad \underline{\mathbf{L}}' = -\frac{1}{2}\varphi(\mathbf{T} - N) \tag{61}$$

We shall next derive transport equations for φ and ψ_a along $\mathcal{N}^-(p)$. We start with φ and, recalling the definition of ${}^{(\mathbf{T})}\pi$, we derive,

$$\begin{aligned} \frac{d}{ds}\varphi &= \frac{d}{ds}\mathbf{g}(\mathbf{T}, \mathbf{L}) = \mathbf{g}(\mathbf{D}_{\mathbf{L}}\mathbf{T}, \mathbf{L}) = \frac{1}{2}\,{}^{(\mathbf{T})}\pi_{\mathbf{L}\mathbf{L}} \\ &= \frac{1}{4}\varphi^{-2}(\,{}^{(\mathbf{T})}\pi_{\mathbf{T}N} + \frac{1}{2}\,{}^{(\mathbf{T})}\pi_{NN}) \end{aligned}$$

On the other hand, writing

$$\mathbf{D}_{\mathbf{L}} e_a = \nabla_{\mathbf{L}} e_a - \zeta_a \mathbf{L}, \qquad \zeta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_a\mathbf{L}, \underline{\mathbf{L}}).$$

we have,

$$\nabla_{\mathbf{L}}\psi_a = \mathbf{g}(\mathbf{D}_{\mathbf{L}}\mathbf{T}, e_a) - \mathbf{g}(\mathbf{T}, \mathbf{L})\zeta_a$$

Observe that ${}^{(\mathbf{T})}\pi_{\mathbf{L}e_a} = \mathbf{g}(\mathbf{D_L T}, e_a) + \mathbf{g}(\mathbf{D}_a\mathbf{L}, \mathbf{T})$. Therefore, since $\mathbf{T} = \varphi(-\frac{1}{2} - |\varphi^2|\psi|^2)\mathbf{L} - \frac{1}{2}\varphi^{-1}\underline{\mathbf{L}} + \psi_b e_b$,

$$
\begin{aligned}
\nabla_L\psi_a &= {}^{(\mathbf{T})}\pi_{\mathbf{L}e_a} - \mathbf{g}(\mathbf{D}_a\mathbf{L}, T) - \varphi^{-1}\zeta_a \\
&= {}^{(\mathbf{T})}\pi_{\mathbf{L}e_a} - \varphi^{-1}\zeta_a - \mathbf{g}(\mathbf{D}_a\mathbf{L}, -\frac{1}{2}\varphi^{-1}\underline{\mathbf{L}} + \psi_b e_b) \\
&= \pi_{\mathbf{L}e_a} - \chi_{ab}\psi_b = {}^{(\mathbf{T})}\pi_{\mathbf{L}e'_a} - \varphi\psi_a\pi_{\mathbf{LL}} - \chi_{ab}\psi_b \\
&= -\chi_{ab}\psi_b - \frac{1}{2}\varphi^{-1}\psi_a({}^{\mathbf{T}}\pi_{TN} + \frac{1}{2}\,{}^{(\mathbf{T})}\pi_{NN}) - \frac{1}{2}\varphi^{-1}({}^{(\mathbf{T})}\pi_{0a'} + {}^{(\mathbf{T})}\pi_{Na'})
\end{aligned}
$$

Thus the scalar φ and the S_s-tangent vectorfield ψ_a satisfy the equations:

$$
\frac{d}{ds}\varphi = \frac{1}{4}\varphi^{-2}({}^{\mathbf{T}}\pi_{TN} + \frac{1}{2}\,{}^{(\mathbf{T})}\pi_{NN}),, \tag{62}
$$

$$
\nabla_{\mathbf{L}}\psi_a + \chi_{ab}\psi_b = -\frac{1}{2}\varphi^{-1}\psi_a({}^{(\mathbf{T})}\pi_{TN} + \frac{1}{2}\,{}^{(\mathbf{T})}\pi_{NN}) - \frac{1}{2}\varphi^{-1}({}^{(\mathbf{T})}\pi_{Ta'} + {}^{(\mathbf{T})}\pi_{Na'}) \tag{63}
$$

with initial conditions $\varphi(0) = 1$ and $\psi(0) = 0$. In view of our main assumptions (5), (6) we have the obvious bounds,

$$
|{}^{(\mathbf{T})}\pi_{TN}| + |{}^{(\mathbf{T})}\pi_{Ta'}| + |{}^{(\mathbf{T})}\pi_{NN}| + |{}^{(\mathbf{T})}\pi_{Na'}| \quad \lesssim \quad |I|^{-1}\mathcal{K}_0
$$

In view of these bounds we find by integrating equation (62),

$$
|\varphi(s) - 1| \lesssim \mathcal{K}_0 s/|I|
$$

To estimate ψ we first rewrite equation (63) in the form,

$$
\big|\frac{d}{ds}s^2|\psi|^2\big| \lesssim |I|^{-1}\mathcal{K}_0 s^2|\psi|(1+|\psi) + s^2(|\mathrm{tr}\chi - \frac{2}{s}| + |\hat{\chi}|)|\psi|^2,
$$

Integrating and using the bounds (60) for $\hat{\chi}$ and $\mathrm{tr}\chi$ we obtain

$$
|\psi(s)|^2 \lesssim \mathcal{K}_0 s/|I| + \varepsilon s^{1/2}
$$

for any $0 \le s \le \delta$. Therefore, for $\varepsilon \le 10^{-1}$ the desired bounds for φ and ψ of proposition 5.11 can be obtained in any interval $[0,\delta]$ as long as $\delta \cdot \mathcal{K}_0/|I| + 10^{-1}\delta^{1/2} << 10^{-3}$. ■

5.12. **Proof of Theorem 2.12.** The proof of the corresponding result for the S_t-foliation proceeds along the same lines as the one above for the geodesic foliation S_s. The connection between the two foliations is given by the relations

$$
\begin{aligned}
&\frac{dt}{ds} = -n^{-1}\varphi^{-1}, \\
&\chi_{a'b'} = \chi_{ab}, \qquad \zeta_{a'} = \zeta_a - \varphi\psi_b\chi_{ab}
\end{aligned}
$$

We leave the remaining details to the reader.

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