

# THE BOUNDED $L^2$ CURVATURE CONJECTURE

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ABSTRACT. This is the main paper in a sequence in which we give a complete proof of the bounded  $L^2$  curvature conjecture. More precisely we show that the time of existence of a classical solution to the Einstein-vacuum equations depends only on the  $L^2$ -norm of the curvature and a lower bound on the volume radius of the corresponding initial data set. We note that though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to another, more subtle, scaling tied to its causal geometry. Indeed,  $L^2$  bounds on the curvature is the minimum requirement necessary to obtain lower bounds on the radius of injectivity of causal boundaries. We note also that, while the first nontrivial improvements for well posedness for quasilinear hyperbolic systems in spacetime dimensions greater than  $1+1$  (based on Strichartz estimates) were obtained in [2], [3], [48], [49], [19] and optimized in [20], [36], the result we present here is the first in which the full structure of the quasilinear hyperbolic system, not just its principal part, plays a crucial role.

To achieve our goals we recast the Einstein vacuum equations as a quasilinear  $so(3, 1)$ -valued Yang-Mills theory and introduce a Coulomb type gauge condition in which the equations exhibit a specific new type of *null structure* compatible with the quasilinear, covariant nature of the equations. To prove the conjecture we formulate and establish bilinear and trilinear estimates on rough backgrounds which allow us to make use of that crucial structure. These require a careful construction and control of parametrices including  $L^2$  error bounds which is carried out in [41]-[44], as well as a proof of sharp Strichartz estimates for the wave equation on a rough background which is carried out in [45]. It is at this level that the *null scaling* mentioned above makes our problem critical. Indeed, any known notion of a parametrix relies in an essential way on the eikonal equation, and our space-time possesses, barely, the minimal regularity needed to make sense of its solutions.

## 1. INTRODUCTION

This is the main in a sequence of papers in which we give a complete proof of the bounded  $L^2$  curvature conjecture. According to the conjecture the time of existence of a classical solution to the Einstein-vacuum equations depends only on the  $L^2$ -norm of the curvature and a lower bound on the volume radius of the corresponding initial data set. At a deep level the  $L^2$  curvature conjecture concerns the relationship between the curvature tensor and the causal geometry of an Einstein vacuum space-time. Thus, though

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the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to a different scaling, which we call *null scaling*, tied to its causal properties. More precisely,  $L^2$  curvature bounds are strictly necessary to obtain lower bounds on the radius of injectivity of causal boundaries. These lower bounds turn out to be crucial for the construction of parametrices and derivation of bilinear and trilinear spacetime estimates for solutions to scalar wave equations. We note also that, while the first nontrivial improvements for well posedness for quasilinear hyperbolic systems in spacetime dimensions greater than  $1 + 1$  (based on Strichartz estimates) were obtained in [2], [3], [48], [49], [19] and optimized in [20], [36], the result we present here is the first in which the full structure of the quasilinear hyperbolic system, not just its principal part, plays a crucial role.

**1.1. Initial value problem.** We consider the Einstein vacuum equations (EVE),

$$\mathbf{Ric}_{\alpha\beta} = 0 \tag{1.1}$$

where  $\mathbf{Ric}_{\alpha\beta}$  denotes the Ricci curvature tensor of a four dimensional Lorentzian space time  $(\mathcal{M}, \mathbf{g})$ . An initial data set for (1.1) consists of a three dimensional 3-surface  $\Sigma_0$  together with a Riemannian metric  $g$  and a symmetric 2-tensor  $k$  verifying the constraint equations,

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \text{tr}k = 0, \\ R_{scal} - |k|^2 + (\text{tr}k)^2 = 0, \end{cases} \tag{1.2}$$

where the covariant derivative  $\nabla$  is defined with respect to the metric  $g$ ,  $R_{scal}$  is the scalar curvature of  $g$ , and  $\text{tr}k$  is the trace of  $k$  with respect to the metric  $g$ . In this work we restrict ourselves to asymptotically flat initial data sets with one end. For a given initial data set the Cauchy problem consists in finding a metric  $\mathbf{g}$  satisfying (1.1) and an embedding of  $\Sigma_0$  in  $\mathcal{M}$  such that the metric induced by  $\mathbf{g}$  on  $\Sigma_0$  coincides with  $g$  and the 2-tensor  $k$  is the second fundamental form of the hypersurface  $\Sigma_0 \subset \mathcal{M}$ . The first local existence and uniqueness result for (EVE) was established by Y.C. Bruhat, see [5], with the help of wave coordinates which allowed her to cast the Einstein vacuum equations in the form of a system of nonlinear wave equations to which one can apply<sup>1</sup> the standard theory of nonlinear hyperbolic systems. The optimal, classical<sup>2</sup> result states the following,

**Theorem 1.1** (Classical local existence [12] [14]). *Let  $(\Sigma_0, g, k)$  be an initial data set for the Einstein vacuum equations (1.1). Assume that  $\Sigma_0$  can be covered by a locally finite system of coordinate charts, related to each other by  $C^1$  diffeomorphisms, such that*

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<sup>1</sup>The original proof in [5] relied on representation formulas, following an approach pioneered by Sobolev, see [37].

<sup>2</sup>Based only on energy estimates and classical Sobolev inequalities.

$(g, k) \in H_{loc}^s(\Sigma_0) \times H_{loc}^{s-1}(\Sigma_0)$  with  $s > \frac{5}{2}$ . Then there exists a unique<sup>3</sup> (up to an isometry) globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$ , verifying (1.1), for which  $\Sigma_0$  is a Cauchy hypersurface<sup>4</sup>.

**1.2. Bounded  $L^2$  curvature conjecture.** The classical exponents  $s > 5/2$  are clearly not optimal. By straightforward scaling considerations one might expect to make sense of the initial value problem for  $s \geq s_c = 3/2$ , with  $s_c$  the natural scaling exponent for  $L^2$  based Sobolev norms. Note that for  $s = s_c = 3/2$  a local in time existence result, for sufficiently small data, would be equivalent to a global result. More precisely any smooth initial data, small in the corresponding critical norm, would be globally smooth. Such a well-posedness (WP) result would be thus comparable with the so called  $\epsilon$ -regularity results for nonlinear elliptic and parabolic problems, which play such a fundamental role in the global regularity properties of general solutions. For quasilinear hyperbolic problems critical WP results have only been established in the case of 1 + 1 dimensional systems, or spherically symmetric solutions of higher dimensional problems, in which case the  $L^2$ -Sobolev norms can be replaced by bounded variation (BV) type norms<sup>5</sup>. A particularly important example of this type is the critical BV well-posedness result established by Christodoulou for spherically symmetric solutions of the Einstein equations coupled with a scalar field, see [7]. The result played a crucial role in his celebrated work on the Weak Cosmic Censorship for the same model, see [8]. As well known, unfortunately, the BV-norms are completely inadequate in higher dimensions; the only norms which can propagate the regularity properties of the data are necessarily  $L^2$  based.

The quest for optimal well-posedness in higher dimensions has been one of the major themes in non-linear hyperbolic PDE's in the last twenty years. Major advances have been made in the particular case of semi-linear wave equations. In the case of geometric wave equations such as Wave Maps and Yang-Mills, which possess a well understood null structure, well-posedness holds true for all exponents larger than the corresponding critical exponent. For example, in the case of Wave Maps defined from the Minkowski space  $\mathbb{R}^{n+1}$  to a complete Riemannian manifold, the critical scaling exponents is  $s_c = n/2$  and well-posedness is known to hold all the way down to  $s_c$  for all dimensions  $n \geq 2$ . This critical well-posedness result, for  $s = n/2$ , plays a fundamental role in the recent, large data, global results of [46], [39], [40] and [28] for 2 + 1 dimensional wave maps.

The role played by critical exponents for quasi-linear equations is much less understood. The first well posedness results, on any (higher dimensional) quasilinear hyperbolic system, which go beyond the classical Sobolev exponents, obtained in [2], [3], and [48], [49]

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<sup>3</sup>The original proof in [12], [14] actually requires one more derivative for the uniqueness. The fact that uniqueness holds at the same level of regularity than the existence has been obtained in [33]

<sup>4</sup>That is any past directed, in-extendable causal curve in  $\mathcal{M}$  intersects  $\Sigma_0$ .

<sup>5</sup>Recall that the entire theory of shock waves for 1+1 systems of conservation laws is based on BV norms, which are *critical* with respect to the scaling of the equations. Note also that these BV norms are not, typically, conserved and that Glimm's famous existence result [13] can be interpreted as a global well posedness result for initial data with small BV norms.

and [19], do not take into account the specific (null) structure of the equations. Yet the presence of such structure was crucial in the derivation of the optimal results mentioned above, for geometric semilinear equations. In the case of the Einstein equations it is not at all clear what such structure should be, if there is one at all. Indeed, the only specific structural condition, known for (EVE), discovered in [30] under the name of the *weak null condition*, is not at all adequate for improved well posedness results, see remark 1.3. It is known however, see [29], that without such a structure one cannot have well posedness for exponents<sup>6</sup>  $s \leq 2$ . Yet (EVE) are of fundamental importance and as such it is not unreasonable to expect that such a structure must exist.

Even assuming such a structure, a result of well-posedness for the Einstein equations at, or near, the critical regularity  $s_c = 3/2$  is not only completely out of reach but may in fact be wrong. This is due to the presence of a different scaling connected to the geometry of boundaries of causal domains. It is because of this more subtle scaling that we need at least  $L^2$ -bounds for the curvature to derive a lower bound on the radius of injectivity of null hypersurfaces and thus control their local regularity properties. This imposes a crucial obstacle to well posedness below  $s = 2$ . Indeed, as we will show in the next subsection, any such result would require, crucially, bilinear and even trilinear estimates for solutions to wave equations of the form  $\square_{\mathbf{g}}\phi = F$ . Such estimates, however, depend on Fourier integral representations, with a phase function  $u$  which solves the eikonal equation  $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ . Thus the much needed bilinear estimates depend, ultimately, on the regularity properties of the level hypersurfaces of the phase  $u$  which are, of course, null. The catastrophic breakdown of the regularity of these null hypersurfaces, in the absence of a lower bound for the injectivity radius, would make these Fourier integral representations entirely useless.

These considerations lead one to conclude that, the following conjecture, proposed in [18], is most probably sharp in so far as the minimal number of derivatives in  $L^2$  is concerned:

**Conjecture**[Bounded  $L^2$  Curvature Conjecture (BCC)] *The Einstein- vacuum equations admit local Cauchy developments for initial data sets  $(\Sigma_0, g, k)$  with locally finite  $L^2$  curvature and locally finite  $L^2$  norm of the first covariant derivatives of  $k$ <sup>7</sup>.*

**Remark 1.2.** *It is important to emphasize here that the conjecture should be primarily interpreted as a continuation argument for the Einstein equations; that is the space-time constructed by evolution from smooth data can be smoothly continued, together with a time foliation, as long as the curvature of the foliation and the first covariant derivatives of its second fundamental form remain  $L^2$ - bounded on the leaves of the foliation. In fact the conjecture implies the break-down criterion previously obtained in [26] and improved in [31], [51]. According to that criterion a vacuum space-time, endowed with a constant*

<sup>6</sup>Note that the dimension here is  $n = 3$ .

<sup>7</sup>As we shall see, from the precise theorem stated below, other weaker conditions, such as a lower bound on the volume radius, are needed.

mean curvature (CMC) foliation  $\Sigma_t$ , can be extended, together with the foliation, as long as the  $L_t^1 L^\infty(\Sigma_t)$  norm of the deformation tensor of the future unit normal to the foliation remains bounded. It is straightforward to see, by standard energy estimates, that this condition implies bounds for the  $L_t^\infty L^2(\Sigma_t)$  norm of the space-time curvature from which one can derive bounds for the induced curvature tensor  $R$  and the first derivatives of the second fundamental form  $k$ . Thus, if we can ensure that the time of existence of a space-time foliated by  $\Sigma_t$  depends only on the  $L^2$  norms of  $R$  and first covariant derivatives of  $k$ , we can extend the space-time indefinitely.

In this paper we provide the framework and the main ideas of the proof of the conjecture. We rely on the results of [41], [42], [43], [44], [45], which we use here as a black box. A summary of the entire proof is given in [27].

**1.3. Brief history.** The conjecture has its roots in the remarkable developments of the last twenty years centered around the issue of optimal well-posedness for semilinear wave equations. The case of the Einstein equations turns out to be a lot more complicated due to the quasilinear character of the equations. To make the discussion more tangible it is worthwhile to recall the form of the Einstein vacuum equations in the wave gauge. Assuming given coordinates  $x^\alpha$ , verifying  $\square_{\mathbf{g}} x^\alpha = 0$ , the metric coefficients  $g_{\alpha\beta} = \mathbf{g}(\partial_\alpha, \partial_\beta)$ , with respect to these coordinates, verify the system of quasilinear wave equations,

$$g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g) \quad (1.3)$$

where  $F_{\alpha\beta}$  are quadratic functions of  $\partial g$ , i.e. the derivatives of  $g$  with respect to the coordinates  $x^\alpha$ . In a first approximation we may compare (1.3) with the semilinear wave equation,

$$\square \phi = F(\phi, \partial \phi) \quad (1.4)$$

with  $F$  quadratic in  $\partial \phi$ . Using standard energy estimates, one can prove an estimate, roughly, of the form:

$$\|\phi(t)\|_s \lesssim \|\phi(0)\|_s \exp \left( C_s \int_0^t \|\partial \phi(\tau)\|_{L^\infty} d\tau \right).$$

The classical exponent  $s > 3/2 + 1$  arises simply from the Sobolev embedding of  $H^r$ ,  $r > 3/2$  into  $L^\infty$ . To go beyond the classical exponent, see [34], one has to replace Sobolev inequalities with Strichartz estimates of, roughly, the following type,

$$\left( \int_0^t \|\partial \phi(\tau)\|_{L^\infty}^2 d\tau \right)^{1/2} \lesssim C \left( \|\partial \phi(0)\|_{H^{1+\epsilon}} + \int_0^t \|\square \phi(\tau)\|_{H^{1+\epsilon}} \right)$$

where  $\epsilon > 0$  can be chosen arbitrarily small. This leads to a gain of  $1/2$  derivatives, i.e. we can prove well-posedness for equations of type (1.4) for any exponent  $s > 2$ .

The same type of improvement in the case of quasilinear equations requires a highly non-trivial extension of such estimates for wave operators with non-smooth coefficients.

The first improved regularity results for quasilinear wave equations of the type,

$$g^{\mu\nu}(\phi)\partial_\mu\partial_\nu\phi = F(\phi, \partial\phi) \quad (1.5)$$

with  $g^{\mu\nu}(\phi)$  a non-linear perturbation of the Minkowski metric  $m^{\mu\nu}$ , are due to [2], [3], and [48], [49] and [19]. The best known results for equations of type (1.3) were obtained in [20] and [36]. According to them one can lower the Sobolev exponent  $s > 5/2$  in Theorem 1.1 to  $s > 2$ . It turns out, see [29], that these results are sharp in the general class of quasilinear wave equations of type (1.3). To do better one needs to take into account the special structure of the Einstein equations and rely on a class of estimates which go beyond Strichartz, namely the so called bilinear estimates<sup>8</sup>.

In the case of semilinear wave equations, such as Wave Maps, Maxwell-Klein-Gordon and Yang-Mills, the first results which make use of bilinear estimates go back to [15], [16], [17]. In the particular case of the Maxwell-Klein-Gordon and Yang-Mills equation the main observation was that, after the choice of a special gauge (Coulomb gauge), the most dangerous nonlinear terms exhibit a special, null structure for which one can apply the bilinear estimates derived in [15]. With the help of these estimates one was able to derive a well posedness result, in the flat Minkowski space  $\mathbb{R}^{1+3}$ , for the exponent  $s = s_c + 1/2 = 1$ , where  $s_c = 1/2$  is the critical Sobolev exponent in that case<sup>9</sup>.

To carry out a similar program in the case of the Einstein equations one would need, at the very least, the following crucial ingredients:

- A. *Provide a coordinate condition, relative to which the Einstein vacuum equations verifies an appropriate version of the null condition.*
- B. *Provide an appropriate geometric framework for deriving bilinear estimates for the null quadratic terms appearing in the previous step.*
- C. *Construct an effective progressive wave representation  $\Phi_F$  (parametrix) for solutions to the scalar linear wave equation  $\square_{\mathbf{g}}\phi = F$ , derive appropriate bounds for both the parametrix and the corresponding error term  $E = F - \square_{\mathbf{g}}\Phi_F$  and use them to derive the desired bilinear estimates.*

As it turns out, the proof of several bilinear estimates of Step B reduces to the proof of sharp  $L^4(\mathcal{M})$  Strichartz estimates for a localized version of the parametrix of step C. Thus we will also need the following fourth ingredient.

- D. *Prove sharp  $L^4(\mathcal{M})$  Strichartz estimates for a localized version of the parametrix of step C.*

Note that the last three steps are to be implemented using only hypothetical  $L^2$  bounds for the space-time curvature tensor, consistent with the conjectured result. To start with,

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<sup>8</sup>Note that no such result, i.e. well-posedness for  $s = 2$ , is presently known for either scalar equations of the form (1.5) or systems of the form (1.3).

<sup>9</sup>This corresponds precisely to the  $s = 2$  exponent in the case of the Einstein-vacuum equations

it is not at all clear what should be the correct coordinate condition, or even if there is one for that matter.

**Remark 1.3.** *As mentioned above, the only known structural condition related to the classical null condition, called the weak null condition [30], tied to wave coordinates, fails the regularity test. Indeed, the following simple system in Minkowski space verifies the weak null condition and yet, according to [29], it is ill posed for  $s = 2$ .*

$$\square\phi = 0, \quad \square\psi = \phi \cdot \Delta\phi.$$

*Other coordinate conditions, such as spatial harmonic<sup>10</sup>, also do not seem to work.*

We rely instead on a Coulomb type condition, for orthonormal frames, adapted to a maximal foliation. Such a gauge condition appears naturally if we adopt a Yang-Mills description of the Einstein field equations using Cartan's formalism of moving frames<sup>11</sup>, see [6]. It is important to note that it is not at all a priori clear that such a choice would do the job. Indeed, the null form nature of the Yang-Mills equations in the Coulomb gauge is only revealed once we commute the resulting equations with the projection operator  $\mathcal{P}$  on the divergence free vectorfields. Such an operation is natural in that case, since  $\mathcal{P}$  commutes with the flat d'Alembertian. In the case of the Einstein equations, however, the corresponding commutator term  $[\square_{\mathbf{g}}, \mathcal{P}]$  generates<sup>12</sup> a whole host of new terms and it is quite a miracle that they can all be treated by an extended version of bilinear estimates. At an even more fundamental level, the flat Yang-Mills equations possess natural energy estimates based on the time symmetry of the Minkowski space. There are no such timelike Killing vectorfield in curved space. We have to rely instead on the future unit normal to the maximal foliation  $\Sigma_t$  whose deformation tensor is non-trivial. This leads to another class of nonlinear terms which have to be treated by a novel trilinear estimate.

We will make more comments concerning the implementations of all four ingredients later on, in the section 2.4.

**Remark 1.4.** *In addition to the ingredients mentioned above, we also need a mechanism of reducing the proof of the conjecture to small data, in an appropriate sense. Indeed, even in the flat case, the Coulomb gauge condition cannot be globally imposed for large data. In fact [17] relied on a cumbersome technical device based on local Coulomb gauges, defined on domain of dependence of small balls. Here we rely instead on a variant of the gluing construction of [10], [11], see section 2.3.*

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<sup>10</sup>Maximal foliation together with spatial harmonic coordinates on the leaves of the foliation would be the coordinate condition closest in spirit to the Coulomb gauge.

<sup>11</sup>We would like to thank L. Anderson for pointing out to us the possibility of using such a formalism as a potential bridge to [16].

<sup>12</sup>Note also that additional error terms are generated by projecting the equations on the components of the frame.

## 2. STATEMENT OF THE MAIN RESULTS

**2.1. Maximal foliations.** In this section, we recall some well-known facts about maximal foliations (see for example the introduction in [9]). We assume the space-time  $(\mathcal{M}, \mathbf{g})$  to be foliated by the level surfaces  $\Sigma_t$  of a time function  $t$ . Let  $T$  denote the unit normal to  $\Sigma_t$ , and let  $k$  the the second fundamental form of  $\Sigma_t$ , i.e.  $k_{ab} = -\mathbf{g}(\mathbf{D}_a T, e_b)$ , where  $e_a, a = 1, 2, 3$  denotes an arbitrary frame on  $\Sigma_t$  and  $\mathbf{D}_a T = \mathbf{D}_{e_a} T$ . We assume that the  $\Sigma_t$  foliation is maximal, i.e. we have:

$$\mathrm{tr}_g k = 0 \tag{2.1}$$

where  $g$  is the induced metric on  $\Sigma_t$ . The constraint equations on  $\Sigma_t$  for a maximal foliation are given by:

$$\nabla^a k_{ab} = 0, \tag{2.2}$$

where  $\nabla$  denotes the induced covariant derivative on  $\Sigma_t$ , and

$$R_{scal} = |k|^2. \tag{2.3}$$

Also, we denote by  $n$  the lapse of the  $t$ -foliation, i.e.  $n^{-2} = -\mathbf{g}(\mathbf{D}t, \mathbf{D}t)$ .  $n$  satisfies the following elliptic equation on  $\Sigma_t$ :

$$\Delta n = n|k|^2. \tag{2.4}$$

Finally, we recall the structure equations of the maximal foliation:

$$\nabla_0 k_{ab} = \mathbf{R}_{a0b0} - n^{-1} \nabla_a \nabla_b n - k_{ac} k_b{}^c, \tag{2.5}$$

$$\nabla_a k_{bc} - \nabla_b k_{ac} = \mathbf{R}_{c0ab} \tag{2.6}$$

and:

$$R_{ab} - k_{ac} k_b{}^c = \mathbf{R}_{a0b0}. \tag{2.7}$$

**2.2. Main Theorem.** We recall below the definition of the volume radius on a general Riemannian manifold  $M$ .

**Definition 2.1.** Let  $B_r(p)$  denote the geodesic ball of center  $p$  and radius  $r$ . The volume radius  $r_{vol}(p, r)$  at a point  $p \in M$  and scales  $\leq r$  is defined by

$$r_{vol}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r'^3},$$

with  $|B_r|$  the volume of  $B_r$  relative to the metric on  $M$ . The volume radius  $r_{vol}(M, r)$  of  $M$  on scales  $\leq r$  is the infimum of  $r_{vol}(p, r)$  over all points  $p \in M$ .

Our main result is the following:



**Theorem 2.2** (Main theorem). *Let  $(\mathcal{M}, \mathbf{g})$  an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation by space-like hypersurfaces  $\Sigma_t$  defined as level hypersurfaces of a time function  $t$ . Assume that the initial slice  $(\Sigma_0, g, k)$  is such that the Ricci curvature  $Ric \in L^2(\Sigma_0)$ ,  $\nabla k \in L^2(\Sigma_0)$ , and  $\Sigma_0$  has a strictly positive volume radius on scales  $\leq 1$ , i.e.  $r_{vol}(\Sigma_0, 1) > 0$ . Then,*

(1)  **$L^2$  regularity.** *There exists a time*

$$T = T(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

and a constant

$$C = C(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

such that the following control holds on  $0 \leq t \leq T$ :

$$\|\mathbf{R}\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \leq C, \|\nabla k\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \leq C \text{ and } \inf_{0 \leq t \leq T} r_{vol}(\Sigma_t, 1) \geq \frac{1}{C}.$$

(2) **Higher regularity.** *Within the same time interval as in part (1) we also have the higher derivative estimates<sup>13</sup>,*

$$\sum_{|\alpha| \leq m} \|\mathbf{D}^{(\alpha)} \mathbf{R}\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \leq C_m \sum_{|i| \leq m} \left[ \|\nabla^{(i)} Ric\|_{L^2(\Sigma_0)} + \|\nabla^{(i)} \nabla k\|_{L^2(\Sigma_0)} \right], \quad (2.8)$$

where  $C_m$  depends only on the previous  $C$  and  $m$ .

**Remark 2.3.** *Since the core of the main theorem is local in nature we do not need to be very precise here with our asymptotic flatness assumption. We may thus assume the existence of a coordinate system at infinity, relative to which the metric has two derivatives bounded in  $L^2$ , with appropriate asymptotic decay. Note that such bounds could be deduced from weighted  $L^2$  bounds assumptions for  $Ric$  and  $\nabla k$ .*

**Remark 2.4.** *Note that the dependence on  $\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}$  in the main theorem can be replaced by dependence on  $\|\mathbf{R}\|_{L^2(\Sigma_0)}$  where  $\mathbf{R}$  denotes the space-time curvature tensor<sup>14</sup>. Indeed this follows from the following well known  $L^2$  estimate (see section 8 in [26]).*

$$\int_{\Sigma_0} |\nabla k|^2 + \frac{1}{4}|k|^4 \leq \int_{\Sigma_0} |\mathbf{R}|^2. \quad (2.9)$$

and the Gauss equation relating  $Ric$  to  $\mathbf{R}$ .

<sup>13</sup>Assuming that the initial has more regularity so that the right-hand side of (2.8) makes sense.

<sup>14</sup>Here and in what follows the notations  $R, \mathbf{R}$  will stand for the Riemann curvature tensors of  $\Sigma_t$  and  $\mathcal{M}$ , while  $Ric, \mathbf{Ric}$  and  $R_{scal}, \mathbf{R}_{scal}$  will denote the corresponding Ricci and scalar curvatures.

**2.3. Reduction to small initial data.** We first need an appropriate covering of  $\Sigma_0$  by harmonic coordinates. This is obtained using the following general result based on Cheeger-Gromov convergence of Riemannian manifolds.

**Theorem 2.5** ([1] or Theorem 5.4 in [32]). *Given  $c_1 > 0, c_2 > 0, c_3 > 0$ , there exists  $r_0 > 0$  such that any 3-dimensional, complete, Riemannian manifold  $(M, g)$  with  $\|\text{Ric}\|_{L^2(M)} \leq c_1$  and volume radius at scales  $\leq 1$  bounded from below by  $c_2$ , i.e.  $r_{\text{vol}}(M, 1) \geq c_2$ , verifies the following property:*

*Every geodesic ball  $B_r(p)$  with  $p \in M$  and  $r \leq r_0$  admits a system of harmonic coordinates  $x = (x_1, x_2, x_3)$  relative to which we have*

$$(1 + c_3)^{-1}\delta_{ij} \leq g_{ij} \leq (1 + c_3)\delta_{ij}, \quad (2.10)$$

and

$$r \int_{B_r(p)} |\partial^2 g_{ij}|^2 \sqrt{|g|} dx \leq c_3. \quad (2.11)$$

We consider  $\epsilon > 0$  which will be chosen as a small universal constant. We apply theorem 2.5 to the Riemannian manifold  $\Sigma_0$ . Then, there exists a constant:

$$r_0 = r_0(\|\text{Ric}\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{\text{vol}}(\Sigma_0, 1), \epsilon) > 0$$

such that every geodesic ball  $B_r(p)$  with  $p \in \Sigma_0$  and  $r \leq r_0$  admits a system of harmonic coordinates  $x = (x_1, x_2, x_3)$  relative to which we have:

$$(1 + \epsilon)^{-1}\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij},$$

and

$$r \int_{B_r(p)} |\partial^2 g_{ij}|^2 \sqrt{|g|} dx \leq \epsilon.$$

Now, by the asymptotic flatness of  $\Sigma_0$ , the complement of its end can be covered by the union of a finite number of geodesic balls of radius  $r_0$ , where the number  $N_0$  of geodesic balls required only depends on  $r_0$ . In particular, it is therefore enough to obtain the control of  $\mathbf{R}$ ,  $k$  and  $r_{\text{vol}}(\Sigma_t, 1)$  of Theorem 2.2 when one restricts to the domain of dependence of one such ball. Let us denote this ball by  $B_{r_0}$ . Next, we rescale the metric of this geodesic ball by:

$$g_\lambda(t, x) = g(\lambda t, \lambda x), \quad \lambda = \min \left( \frac{\epsilon^2}{\|R\|_{L^2(B_{r_0})}^2}, \frac{\epsilon^2}{\|\nabla k\|_{L^2(B_{r_0})}^2}, r_0 \epsilon \right) > 0.$$

Let<sup>15</sup>  $R_\lambda, k_\lambda$  and  $B_{r_0}^\lambda$  be the rescaled versions of  $R, k$  and  $B_{r_0}$ . Then, in view of our choice for  $\lambda$ , we have:

$$\begin{aligned} \|R_\lambda\|_{L^2(B_{r_0}^\lambda)} &= \sqrt{\lambda} \|R\|_{L^2(B_{r_0})} \leq \epsilon, \\ \|\nabla k_\lambda\|_{L^2(B_{r_0}^\lambda)} &= \sqrt{\lambda} \|\nabla k\|_{L^2(B_{r_0})} \leq \epsilon, \end{aligned}$$

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<sup>15</sup>Since in what follows there is no danger to confuse the Ricci curvature  $\text{Ric}$  with the scalar curvature  $R$  we use the short hand  $R$  to denote the full curvature tensor  $\text{Ric}$ .

and

$$\|\partial^2 g_\lambda\|_{L^2(B_{r_0}^\lambda)} = \sqrt{\lambda} \|\partial^2 g\|_{L^2(B_{r_0})} \leq \sqrt{\frac{\lambda \epsilon}{r_0}} \leq \epsilon.$$

Note that  $B_{r_0}^\lambda$  is the rescaled version of  $B_{r_0}$ . Thus, it is a geodesic ball for  $g_\lambda$  of radius  $\frac{r_0}{\lambda} \geq \frac{1}{\epsilon} \geq 1$ . Now, considering  $g_\lambda$  on  $0 \leq t \leq 1$  is equivalent to considering  $g$  on  $0 \leq t \leq \lambda$ . Thus, since  $r_0$ ,  $N_0$  and  $\lambda$  depend only on  $\|R\|_{L^2(\Sigma_0)}$ ,  $\|\nabla k\|_{L^2(\Sigma_0)}$ ,  $r_{vol}(\Sigma_0, 1)$  and  $\epsilon$ , Theorem 2.2 is equivalent to the following theorem:

**Theorem 2.6** (Main theorem, version 2). *Let  $(\mathcal{M}, \mathbf{g})$  an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation by space-like hypersurfaces  $\Sigma_t$  defined as level hypersurfaces of a time function  $t$ . Let  $B$  a geodesic ball of radius one in  $\Sigma_0$ , and let  $D$  its domain of dependence. Assume that the initial slice  $(\Sigma_0, g, k)$  is such that:*

$$\|R\|_{L^2(B)} \leq \epsilon, \|\nabla k\|_{L^2(B)} \leq \epsilon \text{ and } r_{vol}(B, 1) \geq \frac{1}{2}.$$

Let  $B_t = D \cap \Sigma_t$  the slice of  $D$  at time  $t$ . Then:

- (1)  **$L^2$  regularity.** *There exists a small universal constant  $\epsilon_0 > 0$  such that if  $0 < \epsilon < \epsilon_0$ , then the following control holds on  $0 \leq t \leq 1$ :*

$$\|\mathbf{R}\|_{L_{[0,1]}^\infty L^2(B_t)} \lesssim \epsilon, \|\nabla k\|_{L_{[0,1]}^\infty L^2(B_t)} \lesssim \epsilon \text{ and } \inf_{0 \leq t \leq 1} r_{vol}(B_t, 1) \geq \frac{1}{4}.$$

- (2) **Higher regularity.** *The following control holds on  $0 \leq t \leq 1$ :*

$$\sum_{|\alpha| \leq m} \|\mathbf{D}^{(\alpha)} \mathbf{R}\|_{L_{[0,1]}^\infty L^2(B_t)} \lesssim \|\nabla^{(i)} Ric\|_{L^2(B)} + \|\nabla^{(i)} \nabla k\|_{L^2(B)}. \quad (2.12)$$

**Notation:** In the statement of Theorem 2.6, and in the rest of the paper, the notation  $f_1 \lesssim f_2$  for two real positive scalars  $f_1, f_2$  means that there exists a universal constant  $C > 0$  such that:

$$f_1 \leq C f_2.$$

Theorem 2.6 is not yet in suitable form for our proof since some of our constructions will be global in space and may not be carried out on a subregion  $B$  of  $\Sigma_0$ . Thus, we glue a smooth asymptotically flat solution of the constraint equations (1.2) outside of  $B$ , where the gluing takes place in an annulus just outside  $B$ . This can be achieved using the construction in [10], [11]. We finally get an asymptotically flat solution to the constraint equations, defined everywhere on  $\Sigma_0$ , which agrees with our original data set  $(\Sigma_0, g, k)$  inside  $B$ . We still denote this data set by  $(\Sigma_0, g, k)$ . It satisfies the bounds:

$$\|R\|_{L^2(\Sigma_0)} \leq 2\epsilon, \|\nabla k\|_{L^2(\Sigma_0)} \leq 2\epsilon \text{ and } r_{vol}(\Sigma_0, 1) \geq \frac{1}{4}.$$

**Remark 2.7.** Notice that the gluing process in [10]–[11] requires the kernel of a certain linearized operator to be trivial. This is achieved by conveniently choosing the asymptotically flat solution to (1.2) that is glued outside of  $B$  to our original data set. This choice is always possible since the metrics for which the kernel is nontrivial are non generic (see [4]).

**Remark 2.8.** Assuming only  $L^2$  bounds on  $R$  and  $\nabla k$  is not enough to carry out the construction in the above mentioned results. However, the problem solved there remains subcritical at our desired level of regularity and thus we believe that a closer look at the construction in [10]–[11], or an alternative construction, should be able to provide the desired result. This is an open problem.

**Remark 2.9.** Since  $\|k\|_{L^4(\Sigma_0)}^2 \leq \|Ric\|_{L^2}$  we deduce that  $\|k\|_{L^2(B)} \lesssim \epsilon^{1/2}$  on the geodesic ball  $B$  of radius one. Furthermore, asymptotic flatness is compatible with a decay of  $|x|^{-2}$  at infinity, and in particular with  $k$  in  $L^2(\Sigma_0)$ . So we may assume that the gluing process is such that the resulting  $k$  satisfies:

$$\|k\|_{L^2(\Sigma_0)} \lesssim \epsilon.$$

Finally, we have reduced Theorem 2.2 to the case of a small initial data set:

**Theorem 2.10** (Main theorem, version 3). *Let  $(\mathcal{M}, \mathbf{g})$  an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation by space-like hypersurfaces  $\Sigma_t$  defined as level hypersurfaces of a time function  $t$ . Assume that the initial slice  $(\Sigma_0, g, k)$  is such that:*

$$\|R\|_{L^2(\Sigma_0)} \leq \epsilon, \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \epsilon \text{ and } r_{vol}(\Sigma_0, 1) \geq \frac{1}{2}.$$

Then:

- (1)  **$L^2$  regularity.** *There exists a small universal constant  $\epsilon_0 > 0$  such that if  $0 < \epsilon < \epsilon_0$ , the following control holds on  $0 \leq t \leq 1$ :*

$$\|\mathbf{R}\|_{L_{[0,1]}^\infty L^2(\Sigma_t)} \lesssim \epsilon, \|k\|_{L_{[0,1]}^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_{[0,1]}^\infty L^2(\Sigma_t)} \lesssim \epsilon \text{ and } \inf_{0 \leq t \leq 1} r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}.$$

- (2) **Higher regularity.** *The following estimates hold on  $0 \leq t \leq 1$  and any  $m > 0$ :*

$$\sum_{|\alpha| \leq m} \|\mathbf{D}^{(\alpha)} \mathbf{R}\|_{L_{[0,1]}^\infty L^2(\Sigma_t)} \lesssim \sum_{|i| \leq m} \|\nabla^{(i)} Ric\|_{L^2(\Sigma_0)} + \|\nabla^{(i)} \nabla k\|_{L^2(\Sigma_0)}. \quad (2.13)$$

The rest of this paper is devoted to the proof of Theorem 2.10. Note that we will concentrate mainly on part (1) of the theorem. The proof of part (2) - which concerns the propagation of higher regularity - follows exactly the same steps as the proof of part (1) and is sketched in section 13.

2.4. **Strategy of the proof.** The proof of Theorem 2.10 consists in four steps.

**Step A (Yang-Mills formalism)** We first cast the Einstein-vacuum equations within a Yang-Mills formalism. This relies on the Cartan formalism of moving frames. The idea is to give up on a choice of coordinates and express instead the Einstein vacuum equations in terms of the connection 1-forms associated to moving orthonormal frames, i.e. vectorfields  $e_\alpha$ , which verify,

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \text{diag}(-1, 1, 1, 1).$$

The connection 1-forms (they are to be interpreted as 1-forms with respect to the external index  $\mu$  with values in the Lie algebra of  $so(3, 1)$ ), defined by the formulas,

$$(\mathbf{A}_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha) \quad (2.14)$$

verify the equations,

$$\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0 \quad (2.15)$$

where, denoting  $(\mathbf{F}_{\mu\nu})_{\alpha\beta} := \mathbf{R}_{\alpha\beta\mu\nu}$ ,

$$(\mathbf{F}_{\mu\nu})_{\alpha\beta} = (\mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu])_{\alpha\beta}. \quad (2.16)$$

In other words we can interpret the curvature tensor as the curvature of the  $so(3, 1)$ -valued connection 1-form  $\mathbf{A}$ . Note also that the covariant derivatives are taken only with respect to the *external indices*  $\mu, \nu$  and do not affect the *internal indices*  $\alpha, \beta$ . We can rewrite (2.15) in the form,

$$\square_{\mathbf{g}} \mathbf{A}_\nu - \mathbf{D}_\nu (\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{J}_\nu(\mathbf{A}, \mathbf{D}\mathbf{A}) \quad (2.17)$$

where,

$$\mathbf{J}_\nu = \mathbf{D}^\mu ([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}_\mu, \mathbf{F}_{\mu\nu}].$$

Observe that the equations (2.15)-(2.16) look just like the Yang-Mills equations on a fixed Lorentzian manifold  $(\mathcal{M}, \mathbf{g})$  except, of course, that in our case  $\mathbf{A}$  and  $\mathbf{g}$  are not independent but connected rather by (2.14), reflecting the quasilinear structure of the Einstein equations. Just as in the case of [15], which establishes the well-posedness of the Yang-Mills equation in Minkowski space in the energy norm (i.e.  $s = 1$ ), we rely in an essential manner on a Coulomb type gauge condition. More precisely, we take  $e_0$  to be the future unit normal to the  $\Sigma_t$  foliation and choose  $e_1, e_2, e_3$  an orthonormal basis to  $\Sigma_t$ , in such a way that we have, essentially (see precise discussion in section 3.2),  $\text{div} A = \nabla^i A_i = 0$ , where  $A$  is the spatial component of  $\mathbf{A}$ . It turns out that  $A_0$  satisfies an elliptic equation while each component  $A_i = \mathbf{g}(\mathbf{A}, e_i)$ ,  $i = 1, 2, 3$  verifies an equation of the form,

$$\square_{\mathbf{g}} A_i = -\partial_i (\partial_0 A_0) + A^j \partial_j A_i + A^j \partial_i A_j + \text{l.o.t.} \quad (2.18)$$

with l.o.t. denoting nonlinear terms which can be treated by more elementary techniques (including non sharp Strichartz estimates).

**Step B (Bilinear and trilinear estimates)** To eliminate  $\partial_i(\partial_0 A_0)$  in (2.18), we need to project (2.18) onto divergence free vectorfields with the help of a non-local operator which we denote by  $\mathcal{P}$ . In the case of the flat Yang-Mills equations, treated in [15], this leads to an equation of the form,

$$\square A_i = \mathcal{P}(A^j \partial_j A_i) + \mathcal{P}(A^j \partial_i A_j) + \text{l.o.t.}$$

where both terms on the right can be handled by bilinear estimates. In our case we encounter however three fundamental differences with the flat situation of [15].

- To start with the operator  $\mathcal{P}$  does not commute with  $\square_{\mathbf{g}}$ . It turns out, fortunately, that the terms generated by commutation can still be estimated by an extended class of bilinear estimates which includes contractions with the curvature tensor, see section 5.4.
- All energy estimates used in [15] are based on the standard timelike Killing vectorfield  $\partial_t$ . In our case the corresponding vectorfield  $e_0 = T$  (the future unit normal to  $\Sigma_t$ ) is not Killing. This leads to another class of trilinear error terms which we discuss in sections 8 and 5.4.
- The main difference with [15] is that we now need bilinear and trilinear estimates for solutions of wave equations on background metrics which possess only limited regularity.

This last item is a major problem, both conceptually and technically. On the conceptual side we need to rely on a more geometric proof of bilinear estimates based on a plane wave representation formula<sup>16</sup> for solutions of scalar wave equations,

$$\square_{\mathbf{g}} \phi = 0.$$

The proof of the bilinear estimates rests on the representation formula<sup>17</sup>

$$\phi_f(t, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t, x)} f(\lambda \omega) \lambda^2 d\lambda d\omega \quad (2.19)$$

where  $f$  represents schematically the initial data<sup>18</sup>, and where  $\omega u$  is a solution of the eikonal equation<sup>19</sup>,

$$\mathbf{g}^{\alpha\beta} \partial_\alpha \omega u \partial_\beta \omega u = 0, \quad (2.20)$$

<sup>16</sup>We follow the proof of the bilinear estimates outlined in [21] which differs substantially from that of [15] and is reminiscent of the null frame space strategy used by Tataru in his fundamental paper [47].

<sup>17</sup>(2.19) actually corresponds to the representation formula for a half-wave. The full representation formula corresponds to the sum of two half-waves (see section 10)

<sup>18</sup>Here  $f$  is in fact at the level of the Fourier transform of the initial data and the norm  $\|\lambda f\|_{L^2(\mathbb{R}^3)}$  corresponds, roughly, to the  $H^1$  norm of the data .

<sup>19</sup>In the flat Minkowski space  $\omega u(t, x) = t \pm x \cdot \omega$ .

with appropriate initial conditions on  $\Sigma_0$  and  $d\omega$  the area element of the standard sphere in  $\mathbb{R}^3$ .

**Remark 2.11.** *Note that (2.19) is a parametrix for a scalar wave equation. The lack of a good parametrix for a covariant wave equation forces us to develop a strategy based on writing the main equation in components relative to a frame, i.e. instead of dealing with the tensorial wave equation (2.17) directly, we consider the system of scalar wave equations (2.18). Unlike in the flat case, this scalarization procedure produces several terms which are potentially dangerous, and it is fortunate that they can still be controlled by the use of an extended<sup>20</sup> class of bilinear estimates.*

**Step C (Control of the parametrix)** To prove the bilinear and trilinear estimates of Step B, we need in particular to control the parametrix at initial time (i.e. restricted to the initial slice  $\Sigma_0$ )

$$\phi_f(0, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(0, x)} f(\lambda \omega) \lambda^2 d\lambda d\omega \quad (2.21)$$

and the error term corresponding to (2.19)

$$Ef(t, x) = \square_{\mathbf{g}} \phi_f(t, x) = i \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t, x)} (\square_{\mathbf{g}} \omega u) f(\lambda \omega) \lambda^3 d\lambda d\omega \quad (2.22)$$

i.e.  $\phi_f$  is an exact solution of  $\square_{\mathbf{g}} \phi = 0$  only in flat space in which case  $\square_{\mathbf{g}} \omega u = 0$ . This requires the following four sub steps

- C1** *Make an appropriate choice for the equation satisfied by  $\omega u(0, x)$  on  $\Sigma_0$ , and control the geometry of the foliation of  $\Sigma_0$  by the level surfaces of  $\omega u(0, x)$ .*
- C2** *Prove that the parametrix at  $t = 0$  given by (2.21) is bounded in  $\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma_0))$  using the estimates for  $\omega u(0, x)$  obtained in **C1**.*
- C3** *Control the geometry of the foliation of  $\mathcal{M}$  given by the level hypersurfaces of  $\omega u$ .*
- C4** *Prove that the error term (2.22) satisfies the estimate  $\|Ef\|_{L^2(\mathcal{M})} \leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}$  using the estimates for  $\omega u$  and  $\square_{\mathbf{g}} \omega u$  proved in **C3**.*

To achieve Step C3 and Step C4, we need, at the very least, to control  $\square_{\mathbf{g}} \omega u$  in  $L^\infty$ . This issue was first addressed in the sequence of papers [22]–[24] where an  $L^\infty$  bound for  $\square_{\mathbf{g}} \omega u$  was established, depending only on the  $L^2$  norm of the curvature flux along null hypersurfaces. The proof required an interplay between both geometric and analytic techniques and had all the appearances of being sharp, i.e. we don't expect an  $L^\infty$  bound for  $\square_{\mathbf{g}} \omega u$  which requires bounds on less than two derivatives in  $L^2$  for the metric<sup>21</sup>.

To obtain the  $L^2$  bound for the Fourier integral operator  $E$  defined in (2.22), we need, of course, to go beyond uniform estimates for  $\square_{\mathbf{g}} \omega u$ . The classical  $L^2$  bounds for Fourier

<sup>20</sup>such as contractions between the Riemann curvature tensor and derivatives of solutions of scalar wave equations.

<sup>21</sup>classically, this requires, at the very least, the control of  $\mathbf{R}$  in  $L^\infty$

integral operators of the form (2.22) are not at all economical in terms of the number of integration by parts which are needed. In our case the total number of such integration by parts is limited by the regularity properties of the function  $\square_{\mathbf{g}} \omega u$ . To get an  $L^2$  bound for the parametrix at initial time (2.21) and the error term (2.22) within such restrictive regularity properties we need, in particular:

- In Step C1 and Step C3, a precise control of derivatives of  $\omega u$  and  $\square_{\mathbf{g}} \omega u$  with respect to both  $\omega$  as well as with respect to various directional derivatives<sup>22</sup>. To get optimal control we need, in particular, a very careful construction of the initial condition for  $\omega u$  on  $\Sigma_0$  and then sharp space-time estimates of Ricci coefficients, and their derivatives, associated to the foliation induced by  $\omega u$ .
- In Step C2 and Step C4, a careful decompositions of the Fourier integral operators (2.21) and (2.22) in both  $\lambda$  and  $\omega$ , similar to the first and second dyadic decomposition in harmonic analysis, see [38], as well as a third decomposition, which in the case of (2.22) is done with respect to the space-time variables relying on the geometric Littlewood-Paley theory developed in [24].

Below, we make further comments on Steps C1-C4:

- (1) *The choice of  $u(0, x, \omega)$  on  $\Sigma_0$  in Step C1.* Let us note that the typical choice  $u(0, x, \omega) = x \cdot \omega$  in a given coordinate system would not work for us, since we don't have enough control on the regularity of a given coordinate system within our framework. Instead, we need to find a geometric definition of  $u(0, x, \omega)$ . A natural choice would be

$$\square_{\mathbf{g}} u = 0 \text{ on } \Sigma_0$$

which by a simple computation turns out to be the following simple variant of the minimal surface equation<sup>23</sup>

$$\operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) = k \left( \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right) \text{ on } \Sigma_0.$$

Unfortunately, this choice does not allow us to have enough control of the derivatives of  $u$  in the normal direction to the level surfaces of  $u$ . This forces us to look for an alternate equation for  $u$ :

$$\operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right) = 1 - \frac{1}{|\nabla u|} + k \left( \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right) \text{ on } \Sigma_0.$$

This equation turns out to be parabolic in the normal direction to the level surfaces of  $u$ , and allows us to obtain the desired regularity in Step C1. On closer inspection it is related with the well known mean curvature flow on  $\Sigma_0$ .

- (2) *How to achieve Step C3.* The regularity obtained in Step C1, together with null transport equations tied to the eikonal equation, elliptic systems of Hodge type, the

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<sup>22</sup>Taking into account the different behavior in tangential and transversal directions with respect to the level surfaces of  $\omega u$ .

<sup>23</sup>In the time symmetric case  $k = 0$ , this is exactly the minimal surface equation



geometric Littlewood-Paley theory of [24], sharp trace theorems, and an extensive use of the structure of the Einstein equations, allows us to propagate the regularity on  $\Sigma_0$  to the space-time, thus achieving Step C3.

- (3) *The regularity with respect to  $\omega$  in Steps C1 and C3.* The regularity with respect to  $x$  for  $u$  is clearly limited as a consequence of the fact that we only assume  $L^2$  bounds on  $\mathbf{R}$ . On the other hand,  $\mathbf{R}$  is independent of the parameter  $\omega$ , and one might infer that  $u$  is smooth with respect to  $\omega$ . Surprisingly, this is not at all the case. Indeed, the regularity in  $x$  obtained for  $u$  in Steps C1 and C3 is better in directions tangent to the level hypersurfaces of  $u$ . Now, the  $\omega$  derivatives of the tangential directions have non zero normal components. Thus, when differentiating the structure equations with respect to  $\omega$ , tangential derivatives to the level surfaces of  $u$  are transformed in non tangential derivatives which in turn severely limits the regularity in  $\omega$  obtained in Steps C1 and C3.
- (4) *How to achieve Steps C2 and C4.* Let us note that the classical arguments for proving  $L^2$  bounds for Fourier operators are based either on a  $TT^*$  argument, or a  $T^*T$  argument, which requires several integration by parts either with respect to  $x$  for  $T^*T$ , or with respect to  $(\lambda, \omega)$  for  $TT^*$ . Both methods would fail by far within the regularity for  $u$  obtained in Step C1 and Step C3. This forces us to design a method which allows to take advantage both of the regularity in  $x$  and  $\omega$ . This is achieved using in particular the following ingredients:
- geometric integrations by parts taking full advantage of the better regularity properties in directions tangent to the level hypersurfaces of  $u$ ,
  - the standard first and second dyadic decomposition in frequency space, with respect to both size and angle (see [38]), an additional decomposition in physical space relying on the geometric Littlewood-Paley projections of [24] for Step C4, as well as another decomposition involving frequency and angle for Step C2.

Even with these precautions, at several places in the proof, one encounters log-divergences which have to be tackled by ad-hoc techniques, taking full advantage of the structure of the Einstein equations.

**Step D (Sharp  $L^4(\mathcal{M})$  Strichartz estimates)** Recall that the parametrix constructed in Step C needs also to be used to prove sharp  $L^4(\mathcal{M})$  Strichartz estimates. Indeed the proof of several bilinear estimates of Step B reduces to the proof of sharp  $L^4(\mathcal{M})$  Strichartz estimates for the parametrix (2.19) with  $\lambda$  localized in a dyadic shell.

More precisely, let  $j \geq 0$ , and let  $\psi$  a smooth function on  $\mathbb{R}^3$  supported in

$$\frac{1}{2} \leq |\xi| \leq 2.$$

Let  $\phi_{f,j}$  the parametrix (2.19) with a additional frequency localization  $\lambda \sim 2^j$

$$\phi_{f,j}(t, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega. \quad (2.23)$$

We will need the sharp<sup>24</sup>  $L^4(\mathcal{M})$  Strichartz estimate

$$\|\phi_{f,j}\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{j}{2}} \|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \quad (2.24)$$

The standard procedure for proving<sup>25</sup> (2.24) is based on a  $TT^*$  argument which reduces it to an  $L^\infty$  estimate for an oscillatory integral with a phase involving  ${}^\omega u$ . This is then achieved by the method of stationary phase which requires quite a few integrations by parts. In fact the standard argument would require, at the least<sup>26</sup>, that the phase function  $u = {}^\omega u$  verifies,

$$\partial_{t,x}u \in L^\infty, \partial_{t,x}\partial_\omega^2 u \in L^\infty. \quad (2.25)$$

This level of regularity is, unfortunately, incompatible with the regularity properties of solutions to our eikonal equation (2.20). In fact, based on the estimates for  ${}^\omega u$  derived in step C3, we are only allowed to assume

$$\partial_{t,x}u \in L^\infty, \partial_{t,x}\partial_\omega u \in L^\infty. \quad (2.26)$$

We are thus forced to follow an alternative approach<sup>27</sup> to the stationary phase method inspired by [35] and [36].

**2.5. Structure of the paper.** The rest of this paper is devoted to the proof of Theorem 2.10. Here are the main steps.

- In section 3, we start by describing the Cartan formalism and introduce compatible frames, i.e. frames  $e_0, e_1, e_2, e_3$  with  $e_0$  the future unit normal to the foliation  $\Sigma_t$  and  $(e_1, e_2, e_3)$  an orthonormal basis on  $\Sigma_t$ . We choose  $e_1, e_2, e_3$  such that the spatial components  $A = (A_1, A_2, A_3)$  verify the Coulomb condition  $\nabla^i A_i = 0$ . We then decompose the equations (3.10)-(3.11) relative to the frame. This leads to scalar equations for  $A_0 = \mathbf{g}(\mathbf{A}, e_0)$  and  $A_i = \mathbf{g}(\mathbf{A}, e_i)$  of the form (see Proposition 3.5),

$$\begin{aligned} \Delta A_0 &= \text{l.o.t.} \\ \square A_i &= -\partial_i(\partial_0 A_0) + A^j \partial_j A_i + A^j \partial_j A_i + \text{l.o.t.} \end{aligned}$$

where l.o.t. denote nonlinear terms for which the specific structure is irrelevant, i.e. no bilinear estimates are needed. The entire proof of the bounded  $L^2$  conjecture is designed to treat the difficult terms  $A^j \partial_j A_i$  and  $A^j \partial_j A_i$ .

<sup>24</sup>Note in particular that the corresponding estimate in the flat case is sharp.

<sup>25</sup>Note that the procedure we describe would prove not only (2.24) but the full range of mixed Strichartz estimates.

<sup>26</sup>The regularity (2.25) is necessary to make sense of the change of variables involved in the stationary phase method.

<sup>27</sup>We refer to the approach based on the overlap estimates for wave packets derived in [35] and [36] in the context of Strichartz estimates respectively for  $C^{1,1}$  and  $H^{2+\epsilon}$  metrics. Note however that our approach does not require a wave packet decomposition.

- To eliminate  $\partial_i(\partial_0 A_0)$  and exhibit the null structure of the term  $A^j \partial_j A_i$  we need to project the second equation onto divergence free vectorfields. Unlike the flat case of the Yang-Mills equation (see [17]), the projection does not commute with  $\square$  and we have to be very careful with the commutator terms which it generates. We effectively achieve the desired effects of the projection by introducing the quantity  $B = (-\Delta)^{-1} \text{curl} A$ . The main commutation formulas are discussed in section 6 and proved in the appendix.
- In section 4, we start by deriving various preliminary estimates on the initial slice  $\Sigma_0$ , discuss an appropriate version of Uhlenbeck's lemma and show how to control  $A_0$ ,  $A$  as well as  $B = (-\Delta)^{-1} \text{curl} A$ , from our initial assumptions on  $\Sigma_0$ .
- In section 5, we introduce our bootstrap assumptions and describe the principal steps in the proof of version 3 our main theorem, i.e. Theorem 2.10. Note that we are by no means economical in our choice of bootstrap assumptions. We have decided to give a longer list, than strictly necessary, in the hope that it will make the proof more transparent. We make, in particular, a list of bilinear, and even trilinear and Strichartz bootstrap assumptions which take advantage of the special structure of the Einstein equations. The trilinear bootstrap assumption is needed in order to derive the crucial  $L^2$  estimates for the curvature tensor. The entire proof of Theorem 2.10 is summarized in Propositions 5.7 and 5.8 in which all the bootstrap assumptions are improved by estimates which depend only on the initial data, as well as Proposition 5.9 in which we prove the propagation of higher regularity.
- In section 6, we discuss various elliptic estimates on the slices  $\Sigma_t$ , derive estimates for  $B$  from the bootstrap assumptions on  $A$ , and we show how to derive estimates for  $A$  from those of  $B$ .
- In section 7, we use the bootstrap assumptions to derive  $L^2$ -spacetime estimates for  $\square B$  and  $\partial \square B$ , estimates which are crucial in order to provide a parametrix representation for  $B$  and prove the bilinear estimates stated in proposition 5.8. It is crucial here that all the commutator terms generated in the process continue to have the crucial bilinear structure discussed above and thus can be all estimated by our bilinear bootstrap assumptions.
- In section 8, we derive energy estimates for the wave equations  $\square_{\mathbf{g}} \phi = F$ , relying again on the bootstrap assumptions, in particular the trilinear ones.
- In section 9, we improve on our basic bootstrap assumption, i.e. all bootstrap assumptions except the bilinear, trilinear and Strichartz bootstrap assumptions. This corresponds to proving Proposition 5.7.
- In section 10, we show how to construct parametric representation formulas for solutions to the scalar wave equation  $\square_{\mathbf{g}} \phi = F$ . The main result of the section, Theorem 10.7, depends heavily on Theorem 10.3 whose proof requires, essentially, all the constructions and proofs of the papers [41]-[44]. Theorem 10.3 is in fact the main black box of this paper.

- In sections 11 and 12, we improve on our bilinear, trilinear and Strichartz bootstrap assumptions. This corresponds to proving Proposition 5.8. Note that we rely on sharp  $L^4(\mathcal{M})$  Strichartz estimates for a parametrix localized in frequency (see Proposition 12.1) which are proved in [45].
- Finally, we prove the propagation of higher regularity in section 13. This corresponds to proving Proposition 5.9.

The rest of this paper is devoted to the proof of Theorem 2.10. Note that we will concentrate mainly on part (1) of the theorem. The proof of part (2), which follows exactly the same steps as the proof (with some obvious simplifications) of part (1), is sketched in section 13.

**Remark 2.12.** *To re-emphasize that the special structure of the Einstein equations is of fundamental importance in deriving our result we would like to stress that the bilinear estimates are needed not only to treat the terms of the form  $A^j \partial_j A_i$  and  $A^j \partial_j A_i$  mentioned above (which are also present in flat space) but also to derive energy estimates for solutions to  $\square_{\mathbf{g}} \phi = F$ . Moreover, a trilinear estimate is required to get  $L^2$  bounds for  $\mathbf{R}$ . In addition to these, a result such as Theorem 10.3 cannot possibly hold true, for metrics  $\mathbf{g}$  with our limited degree of regularity, unless the Einstein equations are satisfied, i.e.  $\text{Ric}(\mathbf{g}) = 0$ . Indeed a crucial element of a construction of a parametrix representation for solutions to  $\square_{\mathbf{g}} \phi = F$ , guaranteed in Theorem 10.3, is the control and regularity of a family of phase functions with level hypersurfaces which are null with respect to  $\mathbf{g}$ . As mention a few times in this introduction, such controls are intimately tied to the null geometry of a space-time, e.g. lower lower bounds for the radius of injectivity of null hypersurfaces, and would fail, by a lot, for a general Lorentzian metric  $\mathbf{g}$ .*

**Conclusion.** Though this result falls short of the crucial goal of finding a scale invariant well-posedness criterion in GR, it is clearly optimal in terms of all currently available ideas and techniques. Indeed, within our current understanding, a better result would require enhanced bilinear estimates, which in turn would rely heavily on parametrices. On the other hand, parametrices are based on solutions to the eikonal equation whose control requires, at least,  $L^2$  bounds for the curvature tensor, as can be seen in many instances in our work. Thus, if we are to ultimately find a scale invariant well-posedness criterion, it is clear that an entirely new circle of ideas is needed. Such a goal is clearly of fundamental importance not just to GR, but also to any physically relevant quasilinear hyperbolic system.

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program, starting with a series of papers between the first author and M. Machedon, in which spacetime bilinear estimates were first introduced and used to take advantage of the null structure of geometric semilinear equations such as Wave Maps and Yang-Mills. The works of Bahouri- Chemin [2]-[3] and D.Tataru [49] were the first to go below the classical Sobolev exponent  $s = 5/2$ , for any quasilinear system in higher dimensions. This was, at the time, a major psychological and technical breakthrough which opened the way for future developments. Another major breakthrough of the period, with direct influence on our approach to bilinear estimates in curved spacetimes, is D. Tataru's work [47] on critical well posedness for Wave Maps, in which null frame spaces were first introduced. His joint work with H. Smith [36] which, together with [20], is the first to reach optimal well-posedness without bilinear estimates, has also influenced our approach on parametrices and Strichartz estimates. The authors would also like to acknowledge fruitful conversations with L. Anderson, and J. Sterbenz.

### 3. EINSTEIN VACUUM EQUATIONS AS YANG-MILLS GAUGE THEORY

**3.1. Cartan formalism.** Consider an Einstein vacuum spacetime  $(\mathcal{M}, \mathbf{g})$ . We denote the covariant differentiation by  $\mathbf{D}$ . Let  $e_\alpha$  be an orthonormal frame on  $\mathcal{M}$ , i.e.

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \text{diag}(-1, 1, \dots, 1).$$

Consistent with the Cartan formalism we define the connection 1 form,

$$(\mathbf{A})_{\alpha\beta}(X) = \mathbf{g}(\mathbf{D}_X e_\beta, e_\alpha) \tag{3.1}$$

where  $X$  is an arbitrary vectorfield in  $T(\mathcal{M})$ . Observe that,

$$(\mathbf{A})_{\alpha\beta}(X) = -(\mathbf{A})_{\beta\alpha}(X)$$

i.e. the 1-form  $\mathbf{A}_\mu dx^\mu$  takes values in the Lie algebra of  $so(3, 1)$ . We separate the internal indices  $\alpha, \beta$  from the external indices  $\mu$  according to the following notation.

$$(\mathbf{A}_\mu)_{\alpha\beta} := (\mathbf{A})_{\alpha\beta}(\partial_\mu) = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha) \tag{3.2}$$

Recall that the Riemann curvature tensor is defined by

$$\mathbf{R}(X, Y, U, V) = \mathbf{g}(X, [\mathbf{D}_U \mathbf{D}_V - \mathbf{D}_V \mathbf{D}_U - \mathbf{D}_{[U, V]} Y])$$

with  $X, Y, U, V$  arbitrary vectorfields in  $T(\mathcal{M})$ . Thus, taking  $U = \partial_\mu, V = \partial_\nu$ , coordinate vector-fields,

$$\mathbf{R}(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = \mathbf{g}(e_\alpha, \mathbf{D}_\mu \mathbf{D}_\nu e_\beta - \mathbf{D}_\nu \mathbf{D}_\mu e_\beta).$$

We write,

$$\mathbf{D}_\nu e_\beta = (D_\nu e_\beta, e_\lambda) e_\lambda = (\mathbf{A}_\nu)^\lambda{}_\beta e_\lambda$$

and,

$$\begin{aligned} \mathbf{D}_\mu \mathbf{D}_\nu e_\beta &= \mathbf{D}_\mu((\mathbf{A}_\nu)^\lambda{}_\beta e_\lambda) = \partial_\mu (\mathbf{A}_\nu)^\lambda{}_\beta e_\lambda + (\mathbf{A}_\nu)^\lambda{}_\beta \mathbf{D}_\mu e_\lambda \\ &= \partial_\mu (\mathbf{A}_\nu)^\lambda{}_\beta e_\lambda + (\mathbf{A}_\nu)^\lambda{}_\beta (\mathbf{A}_\mu)^\sigma{}_\lambda e_\sigma. \end{aligned}$$

Hence,

$$\mathbf{R}(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) = \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} + (\mathbf{A}_\nu)_\alpha{}^\lambda(\mathbf{A}_\mu)_{\lambda\beta} - (\mathbf{A}_\mu)_\alpha{}^\lambda(\mathbf{A}_\nu)_{\lambda\beta}. \quad (3.3)$$

Thus defining the Lie bracket,

$$([\mathbf{A}_\mu, \mathbf{A}_\nu])_{\alpha\beta} = (\mathbf{A}_\mu)_\alpha{}^\gamma(\mathbf{A}_\nu)_{\gamma\beta} - (\mathbf{A}_\nu)_\alpha{}^\gamma(\mathbf{A}_\mu)_{\gamma\beta} \quad (3.4)$$

we obtain:

$$\mathbf{R}_{\alpha\beta\mu\nu} = \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} - ([\mathbf{A}_\mu, \mathbf{A}_\nu])_{\alpha\beta},$$

or, since  $\partial_\mu(\mathbf{A}_\nu) - \partial_\nu(\mathbf{A}_\mu) = \mathbf{D}_\mu\mathbf{A}_\nu - \mathbf{D}_\nu\mathbf{A}_\mu$

$$(\mathbf{F}_{\mu\nu})_{\alpha\beta} = \mathbf{R}_{\alpha\beta\mu\nu} = (\mathbf{D}_\mu\mathbf{A}_\nu - \mathbf{D}_\nu\mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu])_{\alpha\beta}. \quad (3.5)$$

Therefore we can interpret  $\mathbf{F}$  as the curvature of the connection  $\mathbf{A}$ .

Consider now the covariant derivative of the Riemann curvature tensor,

$$\begin{aligned} \mathbf{D}_\sigma\mathbf{R}_{\alpha\beta\mu\nu} &= (\mathbf{D}_\sigma\mathbf{F}_{\mu\nu})_{\alpha\beta} - \mathbf{R}_{\mathbf{D}_\sigma\alpha\beta\mu\nu} - \mathbf{R}_{\alpha\mathbf{D}_\sigma\beta\mu\nu} \\ &= (\mathbf{D}_\sigma\mathbf{F}_{\mu\nu})_{\alpha\beta} - \mathbf{R}^\delta{}_{\beta\mu\nu}\mathbf{g}(\mathbf{D}_\sigma f_\alpha, f_\delta) - \mathbf{R}_{\alpha\mu\nu}{}^\delta\mathbf{g}(\mathbf{D}_\sigma f_\beta, f_\delta) \\ &= (\mathbf{D}_\sigma\mathbf{F}_{\mu\nu})_{\alpha\beta} - (\mathbf{A}_\sigma)_\alpha{}^\delta(\mathbf{F}_{\mu\nu})_{\delta\beta} - (\mathbf{A}_\sigma)_\beta{}^\delta(\mathbf{F}_{\mu\nu})_{\alpha\delta} \\ &= (\mathbf{D}_\sigma\mathbf{F}_{\mu\nu})_{\alpha\beta} + (\mathbf{A}_\sigma)_\alpha{}^\delta(\mathbf{F}_{\mu\nu})_{\delta\beta} - (\mathbf{F}_{\mu\nu})_\alpha{}^\delta(\mathbf{A}_\sigma)_{\delta\beta} \\ &= (\mathbf{D}_\sigma\mathbf{F}_{\mu\nu} + [\mathbf{A}_\sigma, \mathbf{F}_{\mu\nu}])_{\alpha\beta}. \end{aligned}$$

Hence,

$$\mathbf{D}_\sigma\mathbf{R}_{\alpha\beta\mu\nu} = {}^{(\mathbf{A})}\mathbf{D}_\sigma\mathbf{F}_{\mu\nu} := \mathbf{D}_\sigma\mathbf{F}_{\mu\nu} + [\mathbf{A}_\sigma, \mathbf{F}_{\mu\nu}] \quad (3.6)$$

where we denote by  ${}^{(\mathbf{A})}\mathbf{D}$  the covariant derivative on the corresponding vector bundle. More precisely if  $\mathbf{U} = \mathbf{U}_{\mu_1\mu_2\dots\mu_k}$  is any  $k$ -tensor on  $\mathcal{M}$  with values on the Lie algebra of  $so(3,1)$ ,

$${}^{(\mathbf{A})}\mathbf{D}_\sigma\mathbf{U} = \mathbf{D}_\sigma\mathbf{U} + [\mathbf{A}_\sigma, \mathbf{U}]. \quad (3.7)$$

**Remark 3.1.** Recall that in  $(\mathbf{A}_\mu)_{\alpha\beta}$ ,  $\alpha, \beta$  are called the internal indices, while  $\mu$  are called the external indices. Now, the internal indices will be irrelevant for most of the paper. Thus, from now on, we will drop these internal indices, except for rare instances where we will need to distinguish between internal indices of the type  $ij$  and internal indices of the type  $0i$ .

The Bianchi identities for  $\mathbf{R}_{\alpha\beta\mu\nu}$  take the form

$${}^{(\mathbf{A})}\mathbf{D}_\sigma\mathbf{F}_{\mu\nu} + {}^{(\mathbf{A})}\mathbf{D}_\mu\mathbf{F}_{\nu\sigma} + {}^{(\mathbf{A})}\mathbf{D}_\nu\mathbf{F}_{\sigma\mu} = 0. \quad (3.8)$$

As it is well known the Einstein vacuum equations  $\mathbf{R}_{\alpha\beta} = 0$  imply  $\mathbf{D}^\mu\mathbf{R}_{\alpha\beta\mu\nu} = 0$ . Thus, in view of equation (3.6),

$$0 = {}^{(\mathbf{A})}\mathbf{D}^\mu\mathbf{F}_{\mu\nu} = \mathbf{D}^\mu\mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] \quad (3.9)$$

or, in view of (3.5) and the vanishing of the Ricci curvature of  $\mathbf{g}$ ,

$$\square\mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu\mathbf{A}_\mu) = \mathbf{J}_\nu \quad (3.10)$$

where

$$\mathbf{J}_\nu = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}_\mu, \mathbf{F}_{\mu\nu}]. \quad (3.11)$$

Using again the vanishing of the Ricci curvature it is easy to check,

$$\mathbf{D}^\nu \mathbf{J}_\nu = 0. \quad (3.12)$$

Finally we recall the general formula of transition between two different orthonormal frames  $e_\alpha$  and  $\tilde{e}_\alpha$  on  $\mathcal{M}$ , related by,

$$\tilde{e}_\alpha = \mathbf{O}_\alpha^\gamma e_\gamma$$

where  $\mathbf{m}_{\alpha\beta} = \mathbf{O}_\alpha^\gamma \mathbf{O}_\beta^\delta \mathbf{m}_{\gamma\delta}$ , i.e.  $\mathbf{O}$  is a smooth map from  $\mathcal{M}$  to the Lorentz group  $O(3,1)$ . In other words, raising and lowering indices with respect to  $\mathbf{m}$ ,

$$\mathbf{O}_{\alpha\lambda} \mathbf{O}^{\beta\lambda} = \delta_\alpha^\beta \quad (3.13)$$

Now,  $(\tilde{\mathbf{A}}_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu \tilde{e}_\beta, \tilde{e}_\alpha)$ . Therefore,

$$(\tilde{\mathbf{A}}_\mu)_{\alpha\beta} = \mathbf{O}_\alpha^\gamma \mathbf{O}_\beta^\delta (\mathbf{A}_\mu)_{\gamma\delta} + \partial_\mu (\mathbf{O}_\alpha^\gamma) \mathbf{O}_\beta^\delta \mathbf{m}_{\gamma\delta} \quad (3.14)$$

**3.2. Compatible frames.** Recall that our spacetime is assumed to be foliated by the level surfaces  $\Sigma_t$  of a time function  $t$ , which are maximal, i.e. denoting by  $k$  the second fundamental form of  $\Sigma_t$  we have,

$$\text{tr}_g k = 0 \quad (3.15)$$

where  $g$  is the induced metric on  $\Sigma_t$ . Let us choose  $e_{(0)} = T$ , the future unit normal to the  $\Sigma_t$  foliation, and  $e_{(i)}$ ,  $i = 1, 2, 3$  an orthonormal frame tangent to  $\Sigma_t$ . We call this a frame compatible with our  $\Sigma_t$  foliation. We consider the connection coefficients (3.2) with respect to this frame. Thus, in particular, denoting by  $A_0$ , respectively  $A_i$ , the temporal and spatial components of  $\mathbf{A}_\mu$

$$(A_i)_{0j} = (A_j)_{0i} = -k_{ij}, \quad i, j = 1, 2, 3 \quad (3.16)$$

$$(A_0)_{0i} = -n^{-1} \nabla_i n \quad i = 1, 2, 3 \quad (3.17)$$

where  $n$  denotes the lapse of the  $t$ -foliation, i.e.  $n^{-2} = -\mathbf{g}(\mathbf{D}t, \mathbf{D}t)$ . With this notation we note that,

$$\nabla_l k_{ij} = \nabla_l (k_i)_j + k_{in} (A_l)_j{}^n = \nabla^l (A_i)_{0j} + k_{in} (A_l)_j{}^n$$

where  $\nabla$  is the induced covariant derivative on  $\Sigma_t$  and, as before, the notation  $\nabla_l (k_i)_j$  or  $\nabla^l (A_i)_{0j}$ , is meant to suggest that the covariant differentiation affects only the external index  $i$ . Recalling from (2.2) that  $k$  verifies the constraint equations,

$$\nabla^i k_{ij} = 0,$$

we derive,

$$\nabla^i (A_i)_{0j} = k_i{}^m (A_i)_{mj}. \quad (3.18)$$

Besides the choice of  $e_0$  we are still free to make a choice for the spatial elements of the frame  $e_1, e_2, e_3$ . In other words we consider frame transformations which keep  $e_0$  fixed, i.e transformations of the type,

$$\tilde{e}_i = O_i^j e_j$$

with  $O$  in the orthogonal group  $O(3)$ . We now have, according to (3.14),

$$(\tilde{A}_m)_{ij} = O_i^k O_j^l (A_m)_{kl} + \partial_m (O_i^k) O_j^l \delta_{kl}$$

or, schematically,

$$\tilde{A}_m = O A_m O^{-1} + (\partial_m O) O^{-1} \quad (3.19)$$

formula in which we understand that only the spatial internal indices are involved. We shall use this freedom later to exhibit a frame  $e_1, e_2, e_3$  such that the corresponding connection  $A$  satisfies the Coulomb gauge condition  $\nabla^l (A_l)_{ij} = 0$  (see Lemma 4.2).

**3.3. Notations.** We introduce notations used throughout the paper. From now on, we use greek indices to denote general indices on  $\mathcal{M}$  which do not refer to the particular frame  $(e_0, e_1, e_2, e_3)$ . The letters  $a, b, c, d$  will be used to denote general indices on  $\Sigma_t$  which do not refer to the particular frame  $(e_1, e_2, e_3)$ . Finally, the letters  $i, j, l, m, n$  will only denote indices relative to the frame  $(e_1, e_2, e_3)$ . Also, recall that  $\mathbf{D}$  denotes the covariant derivative on  $\mathcal{M}$ , while  $\nabla$  denotes the induced covariant derivative on  $\Sigma_t$ . Furthermore,  $\boldsymbol{\partial}$  will always refer to the derivative of a scalar quantity relative to one component of the frame  $(e_0, e_1, e_2, e_3)$ , while  $\partial$  will always refer to the derivative of a scalar quantity relative to one component of the the frame  $(e_1, e_2, e_3)$ , so that  $\boldsymbol{\partial} = (\partial_0, \partial)$ . For example,  $\partial A$  may be any term of the form  $\partial_i(A_j)$ ,  $\partial_0 A$  may be any term of the form  $\partial_0(A_j)$ ,  $\partial A_0$  may be any term of the form  $\partial_j(A_0)$ , and  $\boldsymbol{\partial} \mathbf{A} = (\boldsymbol{\partial} A, \boldsymbol{\partial} A_0) = (\partial_0 A_0, \partial A_0, \partial_0 A, \partial A)$ .

We introduce the curl operator  $curl$  defined for any  $so(3, 1)$ -valued triplet  $(\omega_1, \omega_2, \omega_3)$  of functions on  $\Sigma_t$  as follows:

$$(curl \omega)_i = \epsilon_i^{jl} \partial_j (\omega_l), \quad (3.20)$$

where  $\epsilon_{ijl}$  is fully antisymmetric and such that  $\epsilon_{123} = 1$ . We also introduce the divergence operator  $div$  defined for any  $so(3, 1)$ -valued tensor  $A$  on  $\Sigma_t$  as follows:

$$div A = \nabla^l (A_l) = \partial^l (A_l) + A^2. \quad (3.21)$$

**Remark 3.2.** *Since  $\partial_0$  and  $\partial_j$  are not coordinate derivatives, note that the commutators  $[\partial_j, \partial_0]$  and  $[\partial_j, \partial_l]$  do not vanish. Indeed, we have for any scalar function  $\phi$  on  $\mathcal{M}$ :*

$$\begin{aligned} [\partial_i, \partial_j] \phi &= [e_i, e_j] \phi = (\mathbf{D}_i e_j - \mathbf{D}_j e_i) \phi \\ &= -((\mathbf{D}_i e_j, e_0) - (\mathbf{D}_j e_i, e_0)) e_0(\phi) + ((\mathbf{D}_i e_j, e_l) - (\mathbf{D}_j e_i, e_l)) e_l(\phi) \\ &= -((A_i)_{0j} - (A_j)_{0i}) \partial_0 \phi + ((A_i)_{lj} - (A_j)_{li}) \partial_l \phi, \end{aligned}$$



and:

$$\begin{aligned}
[\partial_i, \partial_0]\phi &= [e_i, e_0]\phi = (\mathbf{D}_i e_0 - \mathbf{D}_0 e_i)\phi \\
&= -((\mathbf{D}_i e_0, e_0) - (\mathbf{D}_0 e_i, e_0))e_0(\phi) + ((\mathbf{D}_i e_0, e_l) - (\mathbf{D}_0 e_i, e_l))e_l(\phi) \\
&= (A_0)_{0i}\partial_0\phi + ((A_i)_{l0} - (A_0)_{li})\partial_l\phi.
\end{aligned}$$

This can be written schematically as:

$$[\partial_i, \partial_j]\phi = A\partial\phi \text{ and } [\partial_j, \partial_0]\phi = \mathbf{A}\boldsymbol{\partial}\phi, \quad (3.22)$$

for any scalar function  $\phi$  on  $\mathcal{M}$ .

**Remark 3.3.** The term  $A^2$  in (3.21) corresponds to a quadratic expression in components of  $A$ , where the particular indices do not matter. In the rest of the paper, we will adopt this schematic notation for lower order terms (e.g. terms of the type  $A^2$  and  $A^3$ ) where the particular indices do not matter.

Finally,  $\square A_0$  and  $\square A_i$  will always be understood as  $\square$  applied to the scalar functions  $A_0$  and  $A_i$ , while  $(\square A)_\alpha$  will refer to  $\square$  acting on the differential form  $A_\alpha$ . Also,  $\Delta A_0$  will always refer to  $\Delta(A_0)$ .

**3.4. Main equations for  $(A_0, A)$ .** In what follows we rewrite equations (3.10)–(3.11) with respect to the components  $A_0$  and  $A = (A_1, A_2, A_3)$ . To do this we need the following simple lemma.

**Lemma 3.4.** For any vectorfield  $X$ , we have:

$$X^\alpha(\square \mathbf{A})_\alpha = \square(X \cdot \mathbf{A}) - 2\mathbf{D}^\lambda X \cdot \mathbf{D}_\lambda \mathbf{A} - (\square X) \cdot \mathbf{A}. \quad (3.23)$$

Taking  $X = e_0$  in the lemma and noting that,

$$\square e_0 = \mathbf{D}^\lambda \mathbf{D}_\lambda e_0 = -\mathbf{D}^\lambda (\mathbf{A}_\lambda)_0^\gamma e_\gamma - (\mathbf{A}_\lambda)_0^\gamma (\mathbf{A}^\lambda)_\gamma^\mu e_\mu$$

as well as<sup>28</sup>

$$\mathbf{D}^\mu (\mathbf{A}_\mu) = -\mathbf{D}_0 (\mathbf{A}_0) + \mathbf{D}^i (\mathbf{A}_i) = -[\partial_0 A_0 + (A_0)_0^i (A_i)] + [\nabla^i (A_i)]. \quad (3.24)$$

we derive, keeping track of the term in  $\partial_0 A_0$ ,

$$\begin{aligned}
(\square \mathbf{A})_0 &= \square A_0 + 2(\mathbf{A}^\lambda)_0^\gamma \mathbf{D}_\lambda (\mathbf{A}_\gamma) - (\square e_0) \cdot \mathbf{A} \\
&= \square A_0 + 2(\mathbf{A}^\lambda)_0^\gamma \mathbf{D}_\lambda (\mathbf{A}_\gamma) + \mathbf{D}^\lambda (\mathbf{A}_\lambda)_0^\gamma \mathbf{A}_\gamma + (\mathbf{A}_\lambda)_0^\gamma (\mathbf{A}^\lambda)_\gamma^\mu \mathbf{A}_\mu \\
&= \square A_0 - \partial_0 (A_0)_0^i (A_i) + \mathbf{A}\boldsymbol{\partial}A + \mathbf{A}\partial A_0 + \mathbf{A}^3.
\end{aligned}$$

On the other hand,

$$\partial_0 (\mathbf{D}^\mu (\mathbf{A}_\mu)) = -\partial_0^2 A_0 - \partial_0 (A_0)_0^i (A_i) + \partial_0 (\nabla^i (A_i))$$

<sup>28</sup>Recall that  $\text{tr}k = 0$ .

Hence,

$$\begin{aligned} (\square \mathbf{A})_0 - \partial_0(\mathbf{D}^\mu(\mathbf{A}_\mu)) &= \square A_0 - \partial_0(A_0)_0{}^i(A_i) + \partial_0^2 A_0 + \partial_0(A_0)_0{}^i(A_i) - \partial_0(\nabla^i(A_i)) \\ &\quad + \mathbf{A}\partial A + \mathbf{A}\partial(A_0) + \mathbf{A}^3 \\ &= \square A_0 + \partial_0^2 A_0 - \partial_0(\nabla^i(A_i)) + \mathbf{A}\partial A + \mathbf{A}\partial(A_0) + \mathbf{A}^3. \end{aligned}$$

On the other hand we have, by a straightforward computation, for any scalar  $\phi$ ,

$$\square \phi = -\partial_0(\partial_0 \phi) + \Delta \phi + n^{-1} \nabla n \cdot \nabla \phi, \quad (3.25)$$

with  $\Delta$  denoting the standard Laplace-Beltrami operator on  $\Sigma_t$ . Therefore,

$$(\square \mathbf{A})_0 - \partial_0(\mathbf{D}^\mu(\mathbf{A}_\mu)) = \Delta A_0 - \partial_0(\nabla^i(A_i)) + \mathbf{A}\partial A + \mathbf{A}\partial A_0 + \mathbf{A}^3.$$

Finally, recalling (3.11), we have,

$$\mathbf{J}_0 = -\mathbf{D}^\mu[\mathbf{A}_\mu, \mathbf{A}_0] + [\mathbf{A}_\mu, \mathbf{F}_{\mu 0}] = \mathbf{A}\partial A + \mathbf{A}\partial A_0 + \mathbf{A}^3.$$

Hence the  $e_0$  component of (3.10) takes the form,

$$\Delta A_0 - \partial_0(\nabla^i A_i) = \mathbf{A}\partial A + \mathbf{A}\partial A_0 + \mathbf{A}^3. \quad (3.26)$$

According to (3.18) we have,

$$\nabla^i(A_i)_{0j} = -k_i{}^m(A_i)_{mj}.$$

We are thus free to impose the *Coulomb like* gauge condition,

$$\nabla^i(A_i)_{jk} = 0. \quad (3.27)$$

In fact we write both (3.18) and (3.27) in the form,

$$\nabla^i(A_i) = A^2. \quad (3.28)$$

With this choice of gauge equation (3.26) takes the form,

$$\Delta A_0 = \mathbf{A}\partial A + \mathbf{A}\partial A_0 + \mathbf{A}^3. \quad (3.29)$$

It remains to derive equations for the scalar components  $A_i$ ,  $i = 1, 2, 3$ . First we observe, in view of (3.24) and (3.28),

$$\mathbf{D}^\lambda \mathbf{A}_\lambda = -\mathbf{D}_0 A_0 + \mathbf{D}^i A_i = -\partial_0 A_0 + \nabla^i A_i + A^2 = -\partial_0 A_0 + A^2. \quad (3.30)$$

Using lemma 3.4 with  $X = e_{(i)}$ ,  $i = 1, 2, 3$  we derive,

$$\square A_i = (\square \mathbf{A})_i - 2(\mathbf{A}^\lambda)_i{}^\gamma \mathbf{D}_\lambda(\mathbf{A}_\gamma) - \mathbf{D}^\lambda(\mathbf{A}_\lambda)_i{}^\gamma \mathbf{A}_\gamma - (\mathbf{A}_\lambda)_i{}^\gamma (\mathbf{A}_\lambda)_\gamma{}^\mu \mathbf{A}_\mu$$

or, schematically, ignoring signs or numerical constants in front of the quadratic and cubic terms:

$$\square A_i = (\square \mathbf{A})_i + A^j \partial_j A_i + A_0 \partial \mathbf{A} + A \partial A_0 + \mathbf{A}^3.$$

Recalling (3.12) we have,

$$(\square \mathbf{A})_i - \partial_i(\mathbf{D}^\mu(\mathbf{A}_\mu)) = J_i.$$

where  $J_i$  is the  $e_{(i)}$  component of  $\mathbf{J}$ . Therefore,

$$\square A_i + \partial_i(\partial_0 A_0) = A^j \cdot \partial_j A_i + J_i + A_0 \boldsymbol{\partial} \mathbf{A} + A \boldsymbol{\partial} A_0 + \mathbf{A}^3.$$

On the other hand, recalling the definition of  $\mathbf{J}$  in (3.11), we easily find,

$$J_i = A^i \cdot \partial_i A + [A^j, F_{ji}] + A_0 \boldsymbol{\partial} \mathbf{A} + A \boldsymbol{\partial} A_0 + \mathbf{A}^3.$$

Therefore, schematically,

$$\square A_i + \partial_i(\partial_0 A_0) = A^j \cdot \partial_j A_i + A^j \cdot \partial_i A_j + A_0 \boldsymbol{\partial} \mathbf{A} + A \boldsymbol{\partial} A_0 + \mathbf{A}^3.$$

We summarize the results of this subsection in the following proposition.

**Proposition 3.5.** *Consider an orthonormal frame  $e_\alpha$  compatible with a maximal  $\Sigma_t$  foliation of the space-time  $\mathcal{M}$  with connection coefficients  $\mathbf{A}_\mu$  defined by (3.2), their decomposition  $\mathbf{A} = (A_0, A)$  relative to the same frame  $e_\alpha$ , and Coulomb-like condition on the frame,*

$$\operatorname{div} A = A^2.$$

In such a frame the Einstein-vacuum equations take the form,

$$\Delta A_0 = \mathbf{A} \boldsymbol{\partial} A + \mathbf{A} \boldsymbol{\partial} A_0 + \mathbf{A}^3, \quad (3.31)$$

$$\square A_i + \partial_i(\partial_0 A_0) = A^j \partial_j A_i + A^j \partial_i A_j + A_0 \boldsymbol{\partial} \mathbf{A} + A \boldsymbol{\partial} A_0 + \mathbf{A}^3. \quad (3.32)$$

**Remark 3.6.** *It is extremely important to our strategy that we have reduced the covariant wave equation (3.10) to the system of scalar equations (3.31) (3.32) (see remark 2.11).*

We also record below the following useful computation.

**Lemma 3.7.** *We have the following symbolic identity:*

$$\operatorname{curl}(\operatorname{curl}(A))_j = \partial_j(\operatorname{div} A) - \Delta(A_j) + A \boldsymbol{\partial} A. \quad (3.33)$$

*Proof.* To prove (3.33) we write, using the fact that  $[\partial_i, \partial_j] = A \boldsymbol{\partial}$  in view of (3.22), and the definition (3.21) of  $\operatorname{div}$ :

$$\begin{aligned} \operatorname{curl}(\operatorname{curl}(A))_j &= \epsilon_{jli} \partial_l(\epsilon_{imn} \partial_m(A_n)) \\ &= \epsilon_{jli} \epsilon_{imn} \partial_l(\partial_m(A_n)) \\ &= (\delta_{jm} \delta_{ln} - \delta_{jn} \delta_{lm}) \partial_l(\partial_m(A_n)) \\ &= \partial_l(\partial_j(A_l)) - \partial_l(\partial_l(A_j)) \\ &= \partial_j(\operatorname{div} A) - \Delta(A_j) + A \boldsymbol{\partial} A. \end{aligned}$$

which is (3.33). This concludes the proof of the lemma.  $\square$

## 4. PRELIMINARIES

4.1. **The initial slice.** By the assumptions of Theorem 2.10, we have:

$$\|R\|_{L^2(\Sigma_0)} \leq \epsilon, \quad (4.1)$$

$$\|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \epsilon, \quad (4.2)$$

and:

$$r_{vol}(\Sigma_0, 1) \geq \frac{1}{2}. \quad (4.3)$$

(4.2), (4.1) and (4.3) together with the estimates in [43] (see section 4.4 in that paper) yields:

$$\|n - 1\|_{L^\infty(\Sigma_0)} + \|\nabla^2 n\|_{L^2(\Sigma_0)} \lesssim \epsilon. \quad (4.4)$$

Also, we record the following Sobolev embeddings and elliptic estimates on  $\Sigma_0$  that were derived under the assumptions (4.1) and (4.3) in [43] (see section 3.5 in that paper).

**Lemma 4.1** (Calculus inequalities on  $\Sigma_0$  [43]). *Assume that (4.1) and (4.3) hold. We have on  $\Sigma_0$  the following Sobolev embedding for any tensor  $F$ :*

$$\|F\|_{L^6(\Sigma_0)} \lesssim \|\nabla F\|_{L^2(\Sigma_0)}. \quad (4.5)$$

Also, we define the operator  $(-\Delta)^{-\frac{1}{2}}$  acting on tensors on  $\Sigma_0$  as:

$$(-\Delta)^{-\frac{1}{2}} F = \frac{1}{\Gamma\left(\frac{1}{4}\right)} \int_0^{+\infty} \tau^{-\frac{3}{4}} U(\tau) F d\tau,$$

where  $\Gamma$  is the Gamma function, and where  $U(\tau)F$  is defined using the heat flow on  $\Sigma_0$ :

$$(\partial_\tau - \Delta)U(\tau)F = 0, \quad U(0)F = F.$$

We have the following Bochner estimates:

$$\|\nabla(-\Delta)^{-\frac{1}{2}}\|_{\mathcal{L}(L^2(\Sigma_0))} \lesssim 1 \quad \text{and} \quad \|\nabla^2(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Sigma_t))} \lesssim 1, \quad (4.6)$$

where  $\mathcal{L}(L^2(\Sigma_0))$  denotes the set of bounded linear operators on  $L^2(\Sigma_0)$ . (4.6) together with the Sobolev embedding (4.5) yields:

$$\|(-\Delta)^{-\frac{1}{2}} F\|_{L^2(\Sigma_0)} \lesssim \|F\|_{L^{\frac{6}{5}}(\Sigma_0)}. \quad (4.7)$$

4.1.1. *The Uhlenbeck type lemma.* In order to exhibit a frame  $e_1, e_2, e_3$  such that together with  $e_0 = T$  we obtain a connection  $\mathbf{A}$  satisfying our Coulomb type gauge on the initial slice  $\Sigma_0$ , we will need the following result in the spirit of the Uhlenbeck lemma<sup>29</sup> [50].

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<sup>29</sup>Note that our smallness assumptions on  $\tilde{A}$  make the proof of the Lemma much simpler than the original result of Uhlenbeck.

**Lemma 4.2.** *Let  $(M, g)$  a 3 dimensional Riemannian asymptotically flat manifold. Let  $R$  denote its curvature tensor and  $r_{vol}(M, 1)$  its volume radius on scales  $\leq 1$ . Let  $\tilde{A}$  be a connection on  $M$  corresponding to an orthonormal frame  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ . Assume the following bounds:*

$$\|\tilde{A}\|_{L^2(M)} + \|\nabla\tilde{A}\|_{L^2(M)} + \|R\|_{L^2(M)} \leq \delta \quad \text{and } r_{vol}(M, 1) \geq \frac{1}{4}, \quad (4.8)$$

where  $\delta > 0$  is a small enough constant. Assume also that  $\tilde{A}$  and  $\nabla\tilde{A}$  belong to  $L^2(M)$ . Then, there is another connection  $A$  on  $M$  satisfying the Coulomb gauge condition  $\nabla^l(A_l) = 0$ , and such that

$$\|A\|_{L^2(M)} + \|\nabla A\|_{L^2(M)} \lesssim \delta \quad (4.9)$$

Furthermore, if  $\nabla^2\tilde{A}$  belongs to  $L^2(M)$ , then  $\nabla^2 A$  belongs to  $L^2(M)$ .

*Proof.* This is a straightforward adaptation, in a simpler situation, of [50]. Note that in the new frame  $e_1, e_2, e_3$ , defined by  $e_i = O_i^j \tilde{e}_j$ , with  $O$  in the orthogonal group  $O(3)$ , we have,

$$A_m = O\tilde{A}_m O^{-1} + (\partial_m O)O^{-1}.$$

Our Coulomb gauge condition leads to the elliptic equation for  $O$ ,

$$\nabla^m((\partial_m O)O^{-1} + O\tilde{A}_m O^{-1}) = 0, \quad O \cdot O^t = I, \quad (4.10)$$

which, in view of the smallness assumptions and the boundary condition  $O \rightarrow 1$  at infinity along  $M$ , admits the unique solution. We leave the remaining details to the reader.  $\square$

4.1.2. *Control of  $A$ ,  $A_0$  and  $B = \Delta^{-1}curl(A)$  on the initial slice.* Let us first deduce from the Uhlenbeck type Lemma 4.2 the existence of a connection  $A$  on  $\Sigma_0$  satisfying the Coulomb gauge condition (3.27). In view of Theorem 2.5, the bound on  $R$  in  $L^2(\Sigma_0)$  and on  $r_{vol}(\Sigma_0, 1)$  assumed in Theorem 2.10 yields the existence of a system of harmonic coordinates. Furthermore, let  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$  an orthonormal frame obtained from  $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}$  by a standard orthonormalisation procedure, and let  $\tilde{A}$  the corresponding connection. Then, the estimates of Theorem 2.5 yield the fact that  $\tilde{A}$  and  $\nabla\tilde{A}$  belong to  $L^2(M)$ . Together with the estimates (4.1) on  $R$  and (4.3) on  $r_{vol}(\Sigma_0, 1)$ , and the Uhlenbeck type Lemma 4.2, we obtain the existence of a connection  $A$  on  $\Sigma_0$  satisfying the Coulomb gauge condition (3.27).

Next, using the fact that  $A$  satisfies the Coulomb gauge (3.27), and using also the estimates (4.1) (4.2) and the estimates of Lemma 4.1 on the initial slice  $\Sigma_0$ , we may estimate  $A, A_0$  and  $B = \Delta^{-1}curl(A)$ . We will make use of the following computation,

**Proposition 4.3.** *We have the following estimate for  $A, A_0$  and  $B = \Delta^{-1}curl(A)$  on the initial slice  $\Sigma_0$ :*

$$\|\mathbf{A}\|_{L^2(\Sigma_0)} + \|\partial A\|_{L^2(\Sigma_0)} + \|\partial A_0\|_{L^2(\Sigma_0)} + \|\partial\partial B\|_{L^2(\Sigma_0)} \lesssim \epsilon.$$

*Proof.* We estimate separately the components  $(A_i)_{j0}, (A_i)_{jl}, (A_0)_{i0}$  and  $(A_0)_{ij}$ . We start with  $(A_i)_{j0}$ . Recall that  $(A_i)_{0j} = k_{ij}$ . Together with (4.2), we obtain:

$$\|(\partial A)_{0j}\|_{L^2(\Sigma_0)} \lesssim \|\nabla k\|_{L^2(\Sigma_0)} + \|A^2\|_{L^2(\Sigma_0)} \lesssim \epsilon + \|A\|_{L^4(\Sigma_0)}^2. \quad (4.11)$$

Also,  $(A_i)_{jl} = g(\mathbf{D}_i e_j, e_l) = g(\nabla_i e_j, e_l)$ . A computation similar to (3.3) yields:

$$R(e_i, e_j, e_l, e_m) = \partial_l(\mathbf{A}_m)_{ij} - \partial_m(\mathbf{A}_l)_{ij} + (\mathbf{A}_m)_i{}^n(\mathbf{A}_l)_{nj} - (\mathbf{A}_l)_i{}^n(\mathbf{A}_m)_{nj}.$$

Thus, we have schematically:

$$(\text{curl } A)_{ij} = R + A^2.$$

On the other hand, we have from the Coulomb gauge condition:

$$\text{div } A = A^2.$$

Using (3.33), we obtain, writing again schematically:

$$(\Delta A)_{ij} = \nabla R + A\partial A + A^3, \quad (4.12)$$

which after multiplication by  $A_{ij}$  and integration by parts yields:

$$\begin{aligned} \|(\partial A)_{ij}\|_{L^2(\Sigma_0)}^2 &\lesssim (\|R\|_{L^2(\Sigma_0)} + \|A\|_{L^4(\Sigma_0)}^2)\|\partial A\|_{L^2(\Sigma_0)} + \|A\|_{L^4(\Sigma_0)}^4 \\ &\lesssim (\epsilon + \|A\|_{L^4(\Sigma_0)}^2)\|\partial A\|_{L^2(\Sigma_0)} + \|A\|_{L^4(\Sigma_0)}^4, \end{aligned} \quad (4.13)$$

where we used (4.1) in the last inequality. Now, recall  $(A_i)_{00} = 0$ , which together with (4.11) and (4.13) yields:

$$\|\partial A\|_{L^2(\Sigma_0)} \lesssim \epsilon + \|A\|_{L^4(\Sigma_0)}^2.$$

Together with the Sobolev embedding (4.5), this implies:

$$\|\partial A\|_{L^2(\Sigma_0)} \lesssim \epsilon. \quad (4.14)$$

Next, we estimate  $\nabla_0 k$ . Recall (2.5):

$$\nabla_0 k_{ab} = \mathbf{R}_{a0b0} - n^{-1}\nabla_a \nabla_b n - k_{ac}k_b{}^c.$$

Also recall Gauss equation (2.7):

$$\mathbf{R}_{a0b0} = R_{ab} - k_a{}^c k_{cb}.$$

Thus, we have:

$$\nabla_0 k = R - n^{-1}\nabla^2 n + A^2. \quad (4.15)$$

(4.1), (4.4), (4.14), (4.15) and the Sobolev embedding (4.5) imply:

$$\|\nabla_0 k\|_{L^2(\Sigma_0)} \lesssim \epsilon. \quad (4.16)$$

Now,  $(A_j)_{0i} = k_{ij}$ , and thus:

$$(\partial_0 A)_{0i} = \nabla_0 k + AA_0,$$

which together with (4.16), (4.14) and the Sobolev embedding (4.5) yields:

$$\begin{aligned} \|(\partial_0 A)_{0i}\|_{L^2(\Sigma_0)} &\lesssim \|\nabla_0 k\|_{L^2(\Sigma_0)} + \|A\|_{L^4(\Sigma_0)}\|A_0\|_{L^4(\Sigma_0)} \\ &\lesssim \epsilon + \epsilon\|\partial(A_0)\|_{L^2(\Sigma_0)}. \end{aligned} \quad (4.17)$$

Next, we estimate  $(\partial_0 A)_{ij}$ . In view of (3.3), we have:

$$\mathbf{R}(e_i, e_j, e_0, e_l) = (\partial_0 A_l)_{ij} - (\partial_l A_0)_{ij} + A_0 A.$$

Furthermore, we have:

$$\mathbf{R}_{0lij} = (\partial_i A_j)_{0l} - (\partial_j A_i)_{0l} + A^2 = \partial A + A^2.$$

Using the symmetry of the curvature tensor  $\mathbf{R}_{ij0l} = \mathbf{R}_{0lij}$ , we obtain:

$$(\partial_0 A_l)_{ij} = \partial A_0 + \partial A + \mathbf{A}\mathbf{A},$$

which together with (4.14) and the Sobolev embedding (4.5) yields:

$$\begin{aligned} \|(\partial_0 A)_{ij}\|_{L^2(\Sigma_0)} &\lesssim \|\partial A_0\|_{L^2(\Sigma_0)} + \|\partial A\|_{L^2(\Sigma_0)} + \|A\|_{L^4(\Sigma_0)} \|\mathbf{A}\|_{L^4(\Sigma_0)} \\ &\lesssim \epsilon + \|\partial A_0\|_{L^2(\Sigma_0)}. \end{aligned} \quad (4.18)$$

Since  $A_{00} = 0$ , (4.17) and (4.18) yield:

$$\|\partial_0 A\|_{L^2(\Sigma_0)} \lesssim \epsilon + \|\partial A_0\|_{L^2(\Sigma_0)}. \quad (4.19)$$

Next, we estimate  $\partial(A_0)$ . Recall (3.31):

$$\Delta A_0 = \mathbf{A}\partial A + \mathbf{A}\partial A_0 + \mathbf{A}^3.$$

After multiplication by  $A_0$  and integration by parts, and together with (4.14), (4.19) and and the Sobolev embedding (4.5), this yields:

$$\begin{aligned} \|\partial A_0\|_{L^2(\Sigma_0)}^2 &\lesssim (\|\mathbf{A}\|_{L^4(\Sigma_0)} \|\partial A\|_{L^2(\Sigma_0)} + \|\mathbf{A}\|_{L^4(\Sigma_0)} \|\partial A_0\|_{L^2(\Sigma_0)} + \|\mathbf{A}\|_{L^4(\Sigma_0)}^3) \|A_0\|_{L^4(\Sigma_0)} \\ &\lesssim \epsilon^2 \|\partial A_0\|_{L^2(\Sigma_0)} + \|\partial A_0\|_{L^2(\Sigma_0)}^3, \end{aligned}$$

which implies:

$$\|\partial A_0\|_{L^2(\Sigma_0)} \lesssim \epsilon. \quad (4.20)$$

Together with (4.19), we obtain:

$$\|\partial_0 A\|_{L^2(\Sigma_0)} \lesssim \epsilon. \quad (4.21)$$

Finally, we estimate  $B$  on the initial slice  $\Sigma_0$  using the estimates for  $\mathbf{A}$  (4.14), (4.20) and (4.21). This will be done on  $\Sigma_t$  in Proposition 6.4. Arguing as in Proposition 6.4 for  $t = 0$  together with (4.14), (4.20), (4.21), the Sobolev embeddings (4.5) and (4.7) on  $\Sigma_0$ , the Bochner inequality on  $\Sigma_0$  (4.6), we immediately obtain:

$$\|\partial\partial B\|_{L^2(\Sigma_0)} \lesssim \epsilon.$$

This concludes the proof of the proposition.  $\square$

## 5. STRATEGY OF THE PROOF OF THEOREM 2.10

**5.1. Classical local existence.** We will need the following well-posedness result for the Cauchy problem for the Einstein equations (1.1) in the maximal foliation.

**Theorem 5.1** (Well-posedness for the Einstein equation in the maximal foliation). *Let  $(\Sigma_0, g, k)$  be asymptotically flat and satisfying the constraint equations (1.2), with  $Ric, \nabla Ric, k, \nabla k$  and  $\nabla^2 k$  in  $L^2(\Sigma_0)$ , and  $r_{vol}(\Sigma_0, 1) > 0$ . Then, there exists a unique asymptotically flat solution  $(\mathcal{M}, \mathbf{g})$  to the Einstein vacuum equations (1.1) corresponding to this initial data set, together with a maximal foliation by space-like hypersurfaces  $\Sigma_t$  defined as level hypersurfaces of a time function  $t$ . Furthermore, there exists a time*

$$T_* = T_*(\|\nabla^{(l)} Ric\|_{L^2(\Sigma_0)}, 0 \leq l \leq 1, \|\nabla^{(j)} k\|_{L^2(\Sigma_0)}, 0 \leq j \leq 2, r_{vol}(\Sigma_0, 1)) > 0$$

*such that the maximal foliation exists for on  $0 \leq t \leq T_*$  with a corresponding control in  $L^\infty_{[0, T_*]} L^2(\Sigma_t)$  for  $Ric, \nabla Ric, k, \nabla k$  and  $\nabla^2 k$ .*

Theorem 5.1 requires two more derivatives both for  $R$  and  $k$  with respect to the main Theorem 2.2. Its proof is standard and relies solely on energy estimates (as opposed to Strichartz estimates of bilinear estimates). We refer the reader to [9] chapter 10 for a related statement.

**Remark 5.2.** *In the proof of our main theorem, the result above will be used only in the context of an extension and continuity arguments (see Step 1 and Step 3 in section 5.4).*

**5.2. Weakly regular null hypersurfaces.** We shall be working with null hyper surfaces in  $\mathcal{M}$  verifying a set of assumptions, described below. These assumptions will be easily verified by the level hyper surfaces  $\mathcal{H}_u$  solutions  $u$  of the eikonal equation  $\mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu u = 0$  discussed in section (10). The regularity of the eikonal equation is studied in detail in [43].

**Definition 5.3** (Weakly regular null hypersurfaces). *Let  $\mathcal{H}$  be a null hypersurface with future null normal  $L$  verifying  $\mathbf{g}(L, T) = -1$ . Let also  $N = L - T$ . We denote by  $\nabla$  the induced connection along the 2-surfaces  $\mathcal{H} \cap \Sigma_t$ . We say that  $\mathcal{H}$  is weakly regular provided that,*

$$\|\mathbf{D}L\|_{L^3(\mathcal{H})} + \|\mathbf{D}N\|_{L^3(\mathcal{H})} \lesssim 1, \tag{5.1}$$

*and the following Sobolev embedding holds for any scalar function  $f$  on  $\mathcal{H}$ :*

$$\|f\|_{L^6(\mathcal{H})} \lesssim \|\nabla f\|_{L^2(\mathcal{H})} + \|L(f)\|_{L^2(\mathcal{H})} + \|f\|_{L^2(\mathcal{H})}. \tag{5.2}$$

**5.3. Main bootstrap assumptions.** Let  $M \geq 1$  a large enough constant to be chosen later in terms only of universal constants. By choosing  $\epsilon > 0$  sufficiently small, we can also ensure  $M\epsilon$  is small enough. From now on, we assume the following bootstrap assumptions hold true on a fixed interval  $[0, T^*]$ , for some  $0 < T^* \leq 1$ . Note that  $\mathcal{H}$  denotes an arbitrary weakly regular null hypersurface, with future directed normal  $L$ , normalized by the condition  $\mathbf{g}(L, T) = -1$ .



- *Bootstrap curvature assumptions*

$$\|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \leq M\epsilon. \quad (5.3)$$

Also,

$$\|\mathbf{R} \cdot L\|_{L^2(\mathcal{H})} \leq M\epsilon, \quad (5.4)$$

where  $\mathbf{R} \cdot L$  denotes any component of  $\mathbf{R}$  such that at least one index is contracted with  $L$ .

- *Bootstrap assumptions for the connection  $\mathbf{A}$ .* We also assume that there exist  $\mathbf{A} = (A_0, A)$  verifying our Coulomb type condition on  $[0, T^*]$ , such that,

$$\|A\|_{L_t^\infty L^2(\Sigma_t)} + \|\boldsymbol{\partial}A\|_{L_t^\infty L^2(\Sigma_t)} \leq M\epsilon, \quad (5.5)$$

and:

$$\begin{aligned} \|A_0\|_{L_t^\infty L^2(\Sigma_t)} + \|\boldsymbol{\partial}A_0\|_{L_t^\infty L^2(\Sigma_t)} + \|A_0\|_{L_t^2 L^\infty(\Sigma_t)} + \|\boldsymbol{\partial}A_0\|_{L_t^\infty L^3(\Sigma_t)} \\ + \|\partial\boldsymbol{\partial}A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \leq M\epsilon. \end{aligned} \quad (5.6)$$

**Remark 5.4.** *Together with the estimates in [43] (see section 4.4 in that paper), the bootstrap assumption (5.3) yields:*

$$\|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim M\epsilon. \quad (5.7)$$

*Furthermore, the bootstrap assumption (5.4) together with the estimates in [43] (see section 4.2 in that paper) yields:*

$$\inf_t r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}. \quad (5.8)$$

In addition we make the following bilinear estimates assumptions for  $\mathbf{A}$  and  $\mathbf{R}$ :

- *Bilinear assumptions I.* Assume,

$$\|A^j \partial_j A\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2. \quad (5.9)$$

Also, for  $B = (-\Delta)^{-1} \text{curl}(A)$  (see (5.37) and the accompanying explanations):

$$\|A^j \partial_j (\boldsymbol{\partial}B)\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2, \quad (5.10)$$

and:

$$\|\mathbf{R} \cdot \cdot j_0 \partial^j B\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2. \quad (5.11)$$

Finally, for any weakly regular null hypersurface  $\mathcal{H}$  and any smooth scalar function  $\phi$  on  $\mathcal{M}$ ,

$$\|k_j \cdot \partial^j \phi\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon \sup_{\mathcal{H}} \|\nabla \phi\|_{L^2(\mathcal{H})}, \quad (5.12)$$

and

$$\|A^j \partial_j \phi\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon \sup_{\mathcal{H}} \|\nabla \phi\|_{L^2(\mathcal{H})}, \quad (5.13)$$

where the supremum is taken over all null hypersurfaces  $\mathcal{H}$ .

- *Bilinear assumptions II.* We assume,

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(A, A))\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2, \quad (5.14)$$

where the bilinear form  $Q_{ij}$  is given by  $Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_j \phi \partial_i \psi$ . Furthermore, we also have:

$$\|(-\Delta)^{-\frac{1}{2}}(\partial(A^l) \partial_l(A))\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2. \quad (5.15)$$

- *Non-sharp Strichartz assumptions*

$$\|A\|_{L_t^2 L^7(\Sigma_t)} \lesssim M^2 \epsilon. \quad (5.16)$$

and, for  $B = (-\Delta)^{-1} \text{curl } A$ , (see (5.37) and the accompanying explanations):

$$\|\partial B\|_{L_t^2 L^7(\Sigma_t)} \lesssim M^2 \epsilon. \quad (5.17)$$

**Remark 5.5.** *Note that the Strichartz estimate for  $\|A\|_{L_t^2 L^7(\Sigma_t)}$  is far from being sharp. Nevertheless, this estimate will be sufficient for the proof as it will only be used to deal with lower order terms.*

Finally we also need a trilinear bootstrap assumption. For this we need to introduce the Bel-Robinson tensor,

$$Q_{\alpha\beta\gamma\delta} = \mathbf{R}_\alpha{}^\lambda \gamma^\sigma \mathbf{R}_{\beta\lambda\delta\sigma} + {}^* \mathbf{R}_\alpha{}^\lambda \gamma^\sigma {}^* \mathbf{R}_{\beta\lambda\delta\sigma} \quad (5.18)$$

- *Trilinear bootstrap assumption.* We assume the following,

$$\left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right| \lesssim M^4 \epsilon^3. \quad (5.19)$$

We conclude this section by showing that the bootstrap assumptions are verified for some positive  $T^*$ .

**Proposition 5.6.** *The above bootstrap assumptions are verified on  $0 \leq t \leq T^*$  for a sufficiently small  $T^* > 0$ .*

*Proof.* The only challenge here is to prove the existence of the desired connection  $\mathbf{A}$ , all other estimates follow trivially from our initial bounds and the local existence theorem above, for sufficiently small  $T^*$ . More precisely we need to exhibit a frame  $e_1, e_2, e_3$  such that, together with  $e_0 = T$ , we obtain a connection  $\mathbf{A}$  satisfying our Coulomb type gauge on the slice  $\Sigma_t$ . To achieve this we start on  $\Sigma_0$  with the orthonormal frame  $e_1, e_2, e_3$ , discussed in section 4.1<sup>30</sup> and transport it to an orthonormal frame on  $\Sigma_t$ ,  $0 \leq t \leq T^*$ , according to the equation,

$$\mathbf{D}_T(\tilde{e}_j) = 0, \quad \tilde{e}_j(0) = e_j, \quad j = 1, 2, 3.$$

Differentiating, we obtain schematically the following transport equation for  $\tilde{A}$ :

$$\mathbf{D}_T(\tilde{A}) = \mathbf{R}, \quad \tilde{A}(0) = A.$$

---

<sup>30</sup>such that the corresponding connection  $\mathbf{A}$  verify the Coulomb gauge condition (3.27) and the estimates of proposition 4.3

We can then rely on the estimates of the local existence theorem, for sufficiently small  $T^*$ , to derive  $L^\infty_{[0, T^*]} L^2(\Sigma_t)$  bounds for  $\tilde{A}$ ,  $\partial\tilde{A}$  and  $\partial^2\tilde{A}$ . Since all the bounds for  $\tilde{A}$  and  $R$  are controlled from the initial data, for small  $T^*$  (thus proportional to  $\epsilon$ ), we are in a position to apply Uhlenbeck's lemma 4.2 on  $\Sigma_t$  to produce the desired connection  $A$ . Furthermore, differentiating (4.10) twice with respect to  $\mathbf{D}_T$ , and applying standard elliptic estimates, we finally obtain the fact that  $A$ ,  $\partial A$  and  $\partial^2 A$  are also controlled in  $L^\infty_{[0, T^*]} L^2(\Sigma_t)$  in conformity with our bootstrap assumptions.  $\square$

**5.4. Proof of the bounded  $L^2$  curvature conjecture.** In the following two propositions, we state the improvement of our bootstrap assumptions.

**Proposition 5.7.** *Let us assume that all bootstrap assumptions of the previous section hold for  $0 \leq t \leq T^*$ . If  $\epsilon > 0$  is sufficiently small, then the following improved estimates hold true on  $0 \leq t \leq T^*$ :*

$$\|\mathbf{R}\|_{L^\infty L^2(\Sigma_t)} \lesssim \epsilon + M^2 \epsilon^{\frac{3}{2}} + M^3 \epsilon^2, \quad (5.20)$$

$$\|\mathbf{R} \cdot L\|_{L^2(\mathcal{H})} \lesssim \epsilon + M^2 \epsilon^{\frac{3}{2}} + M^3 \epsilon^2, \quad (5.21)$$

$$\|A\|_{L^\infty L^2(\Sigma_t)} + \|\partial A_i\|_{L^\infty L^2(\Sigma_t)} \lesssim \epsilon + M^2 \epsilon^{\frac{3}{2}} + M^3 \epsilon^2, \quad (5.22)$$

$$\begin{aligned} & \|A_0\|_{L^\infty L^2(\Sigma_t)} + \|\partial A_0\|_{L^\infty L^2(\Sigma_t)} + \|A_0\|_{L^2_t L^\infty(\Sigma_t)} \\ & + \|\partial A_0\|_{L^\infty L^3(\Sigma_t)} + \|\partial\partial A_0\|_{L^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim \epsilon + M^2 \epsilon^{\frac{3}{2}} + M^3 \epsilon^2, \end{aligned} \quad (5.23)$$

**Proposition 5.8.** *Let us assume that all bootstrap assumptions of the previous section hold for  $0 \leq t \leq T^*$ . If  $\epsilon > 0$  is sufficiently small, then the following improved estimates hold true on  $0 \leq t \leq T^*$ :*

$$\|A^j \partial_j A\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2, \quad (5.24)$$

$$\|A^j \partial_j (\partial B)\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2, \quad (5.25)$$

and

$$\|\mathbf{R} \cdot \cdot_j \partial^j B\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2. \quad (5.26)$$

Also, for any scalar function  $\phi$  on  $\mathcal{M}$ , we have:

$$\|k_j \cdot \partial^j \phi\|_{L^2(\mathcal{M})} \lesssim M \epsilon \left( \sup_{\mathcal{H}} \|\nabla \phi\|_{L^2(\mathcal{H})} + \|\partial \phi\|_{L^\infty L^2(\Sigma_t)} \right), \quad (5.27)$$

and

$$\|A^j \partial_j \phi\|_{L^2(\mathcal{M})} \lesssim M \epsilon \left( \sup_{\mathcal{H}} \|\nabla \phi\|_{L^2(\mathcal{H})} + \|\partial \phi\|_{L^\infty L^2(\Sigma_t)} \right), \quad (5.28)$$

where the supremum is taken over all (weakly regular) null hypersurfaces  $\mathcal{H}$ . Finally, we have:

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(A, A))\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2, \quad (5.29)$$

$$\|(-\Delta)^{-\frac{1}{2}}(\partial(A^l)\partial_l(A))\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2. \quad (5.30)$$

Also,

$$\|A\|_{L_t^2 L^\gamma(\Sigma_t)} \lesssim M\epsilon \quad (5.31)$$

$$\|\partial B\|_{L_t^2 L^\gamma(\Sigma_t)} \lesssim M\epsilon. \quad (5.32)$$

and

$$\left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right| \lesssim M^3 \epsilon^3. \quad (5.33)$$

The proof of Proposition 5.7 is postponed to section 9, while the proof of Proposition 5.8 is postponed to sections 11 and 12. We also need a proposition on the propagation of higher regularity.

**Proposition 5.9.** *Let us assume that the estimates corresponding to all bootstrap assumptions of the previous section hold for  $0 \leq t \leq T^*$  with a universal constant  $M$ . Then for any  $t \in [0, T^*)$  and for  $\epsilon > 0$  sufficiently small, the following propagation of higher regularity holds:*

$$\|\mathbf{DR}\|_{L_t^\infty L^2(\Sigma_t)} \leq 2 \left( \|Ric\|_{L^2(\Sigma_0)} + \|\nabla Ric\|_{L^2(\Sigma_0)} + \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} + \|\nabla^2 k\|_{L^2(\Sigma_0)} \right).$$

The proof of Proposition 5.9 is postponed to section 13. Next, let us show how Propositions 5.6, 5.7, 5.8 and 5.9 imply our main theorem 2.10. We proceed, by the standard bootstrap method, along the following steps:

*Step 1.* We show that all bootstrap assumptions are verified for a sufficiently small final value  $T^*$ .

*Step 2.* Assuming that all bootstrap assumptions hold for fixed values of  $0 < T^* \leq 1$  and  $M$  sufficiently large we show that, for  $\epsilon > 0$  sufficiently small, we may improve on the constant  $M$  in our bootstrap assumptions.

*Step 3.* Using the estimates derived in step 2 we can extend the time of existence  $T^*$  to  $T^* + \delta$  such that all the bootstrap assumptions remain true.

Now, *Step 1* follows from Proposition 5.6. *Step 2* follows from Proposition 5.7 and Proposition 5.8. In view of *Step 2*, the estimates corresponding to all bootstrap assumptions of the previous section hold for  $0 \leq t \leq T^*$  with a universal constant  $M$ . Thus the conclusion of Proposition 5.9 holds, and arguing as in the proof of Proposition 5.6, we obtain *Step 3*. Thus, the bootstrap assumptions hold on  $0 \leq t \leq 1$  for a universal constant  $M$ . In particular, this yields together with (5.7):

$$\|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \epsilon \text{ and } \|k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \epsilon \text{ for all } 0 \leq t \leq 1. \quad (5.34)$$

In view of (5.8), we also obtain the following control on the volume radius:

$$\inf_{0 \leq t \leq 1} r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}. \quad (5.35)$$

Furthermore, Proposition 5.9 yields the following propagation of higher regularity

$$\sum_{|\alpha| \leq m} \|\mathbf{D}^{(\alpha)} \mathbf{R}\|_{L^\infty_{[0,1]} L^2(\Sigma_t)} \leq C_m \left[ \|\nabla^{(i)} \text{Ric}\|_{L^2(\Sigma_t)} + \|\nabla^{(i)} \nabla k\|_{L^2(\Sigma_t)} \right] \quad (5.36)$$

where  $C_m$  only depends on  $m$ .

**Remark 5.10.** *Note that Proposition 5.9 only yields the case  $m = 1$  in (5.36). The fact that (5.36) also holds for higher derivatives  $m \geq 2$  follows from the standard propagation of regularity for the classical local existence result of Theorem 5.1 and the bound (5.36) with  $m = 1$  coming from Proposition 5.9.*

Finally, (5.34), the control on the volume radius (5.35) and the propagation of higher regularity (5.36) yield the conclusion of Theorem 2.10. Together with the reduction to small initial data performed in section 2.3, this concludes the proof of the main Theorem 2.2.

The rest of the paper deals with the proofs of propositions 5.7, 5.8 and 5.9. The core of the proof is to control  $A$ , the spatial part of the connection  $\mathbf{A}$ . As explained in the introduction we need to project our equation for the spatial components  $A$  onto divergence free vectorfields. This is needed for two reasons, to eliminate the term  $\partial_i(\partial_0 A_0)$  on the left hand side of (3.32) and to obtain, on the right hand side, terms which exhibit the crucial null structure we need to implement our proof. Rather than work with the projection  $\mathcal{P}$ , which is too complicated, we rely instead on the new variable,

$$B = (-\Delta)^{-1} \text{curl}(A) \quad (5.37)$$

for which we derive a wave equation. Since we have (see Lemma 6.5):

$$A = \text{curl}(B) + l.o.t$$

it suffices to obtain estimates for  $B$  which lead us to an improvement of the bootstrap assumption (5.5) on  $A$ . In section 7, we derive space-time estimates for  $\square B$  and its derivatives. Proposition 5.7, which does not require a parametrized representation, is proved in 9. Proposition 5.8 is proved in sections 11 and 12 based on the representation formula of theorem 10.7 derived in section 10. Finally, Proposition 5.9 is proved in section 13.

## 6. SIMPLE CONSEQUENCES OF THE BOOTSTRAP ASSUMPTIONS

In this section, we discuss elliptic estimates on  $\Sigma_t$ , we derive estimates for  $B$  from the bootstrap assumptions on  $A$ , and we show how to recover  $A$  from  $B$ .

**6.1. Sobolev embeddings and elliptic estimates on  $\Sigma_t$ .** First, we derive estimates for the lapse  $n$  on  $\Sigma_t$ . The bootstrap assumption on  $R$  (5.3) and the estimate for  $k$  (5.7) together with the estimates in [43] (see section 4.4 in that paper) yield:

$$\begin{aligned} & \|n - 1\|_{L^\infty(\mathcal{M})} + \|\nabla n\|_{L^\infty(\mathcal{M})} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^3(\Sigma_t)} \\ & + \|\nabla(\partial_0 n)\|_{L_t^\infty L^3(\Sigma_t)} + \|\nabla^3 n\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|\nabla^2(\partial_0 n)\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim M\epsilon. \end{aligned} \quad (6.1)$$

**Remark 6.1.** Recall from (3.17) that:

$$(A_0)_{0i} = -n^{-1}\nabla_i n.$$

Thus, the estimates (6.1) for  $n$  could in principle be deduced from the bootstrap assumptions (5.6) for  $A_0$ . However, notice that  $\nabla n \in L^\infty(\mathcal{M})$  in view of (6.1), while  $A_0$  is only in  $L_t^2 L^\infty(\Sigma_t)$  according to (5.6). This improvement for the components  $(A_0)_{0i}$  of  $A_0$  will turn out to be crucial and subtle<sup>31</sup> (see remark 7.5).

Next, we record the following Sobolev embeddings and elliptic estimates on  $\Sigma_t$  that were derived under the assumptions (5.4) and (5.3) in [43] (see sections 3.5 and 4.2 in that paper).

**Lemma 6.2** (Calculus inequalities on  $\Sigma_t$  [43]). *Assume that the assumptions (5.4) and (5.3) hold, and assume that the volume radius at scales  $\leq 1$  on  $\Sigma_0$  is bounded from below by a universal constant. Let  $\delta > 0$ . Then, there exists  $r_0(\delta) > 0$  and a finite covering of  $\Sigma_t$  by geodesic balls of radius  $r_0(\delta)$  such that each geodesic ball in the covering admits a system of harmonic coordinates  $x = (x_1, x_2, x_3)$  relative to which we have*

$$(1 + \delta)^{-1}\delta_{ij} \leq g_{ij} \leq (1 + \delta)\delta_{ij}, \quad (6.2)$$

and

$$r_0(\delta) \int_{B_{r_0}(p)} |\partial^2 g_{ij}|^2 \sqrt{|g|} dx \leq \delta. \quad (6.3)$$

Furthermore, we have on  $\Sigma_t$  the following estimates for any tensor  $F$ :

$$\|F\|_{L^3(\Sigma_t)} \lesssim \|\nabla F\|_{L^{\frac{3}{2}}(\Sigma_t)}, \quad (6.4)$$

$$\|F\|_{L^6(\Sigma_t)} \lesssim \|\nabla F\|_{L^2(\Sigma_t)}, \quad (6.5)$$

$$\|F\|_{L^\infty(\Sigma_t)} \lesssim \|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)} \quad \forall p > 3, \quad (6.6)$$

and:

$$\|\nabla^2 F\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|\Delta F\|_{L^{\frac{3}{2}}(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)}. \quad (6.7)$$

<sup>31</sup>Using the lapse equation  $\Delta n = n|k|^2$  and  $k, \nabla k \in L_t^\infty L^2(\Sigma_t)$ , see (5.7), together with the Sobolev embedding (6.5) we only deduce  $k \in L_t^\infty L^6(\Sigma_t)$  from which  $\Delta n \in L_t^\infty L^3(\Sigma_t)$ . This would yield  $\nabla^2 n \in L_t^\infty L^3(\Sigma_t)$ , and thus  $\nabla n$  misses to be in  $L^\infty(\mathcal{M})$  by a log divergence. However, one can overcome this loss by exploiting the Besov improvement with respect to the Sobolev embedding (6.5). We refer the reader to section 4.4 in [43] for the details.

Finally, we define the operator  $(-\Delta)^{-\frac{1}{2}}$  acting on tensors on  $\Sigma_t$  as:

$$(-\Delta)^{-\frac{1}{2}}F = \frac{1}{\Gamma(\frac{1}{4})} \int_0^{+\infty} \tau^{-\frac{3}{4}} U(\tau) F d\tau,$$

where  $\Gamma$  is the Gamma function, and where  $U(\tau)F$  is defined using the heat flow on  $\Sigma_t$ :

$$(\partial_\tau - \Delta)U(\tau)F = 0, U(0)F = F.$$

We have the following Bochner estimates:

$$\|\nabla(-\Delta)^{-\frac{1}{2}}\|_{\mathcal{L}(L^2(\Sigma_t))} \lesssim 1 \text{ and } \|\nabla^2(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Sigma_t))} \lesssim 1, \quad (6.8)$$

where  $\mathcal{L}(L^2(\Sigma_t))$  denotes the set of bounded linear operators on  $L^2(\Sigma_t)$ . (6.8) together with the Sobolev embedding (6.5) yields:

$$\|(-\Delta)^{-\frac{1}{2}}F\|_{L^2(\Sigma_t)} \lesssim \|F\|_{L^{\frac{6}{5}}(\Sigma_t)}. \quad (6.9)$$

**Remark 6.3.** Note that  $\partial^2 f = \nabla^2 f + A\partial f$  for any scalar function  $f$  on  $\Sigma_t$ . Thus, in view of the bootstrap assumption (5.5) for  $A$ , we may replace  $\nabla^2$  with  $\partial^2$  in the Bochner inequality (6.8) when applied to a scalar function.

**6.2. Elliptic estimates for  $B$ .** Here we derive estimates for  $B$  using the bootstrap assumptions (5.5) (5.6) for  $A$  and  $A_0$ .

**Proposition 6.4.** Let  $B_i = (-\Delta)^{-1}(\text{curl}(A)_i)$ . Then, we have:

$$\|\partial(B_i)\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial^2(B_i)\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial(\partial_0(B_i))\|_{L_t^\infty L^2(\Sigma_t)} \lesssim M\epsilon.$$

*Proof.* Using the Böchner inequality on  $\Sigma_t$  (6.8) together with Remark 6.3, and from the bootstrap assumption (5.5) on  $A$ , we have:

$$\|\partial(B_i)\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial^2(B_i)\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|A\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim M\epsilon. \quad (6.10)$$

Next, we estimate  $\partial(\partial_0(B_i))$ . In view of the definition of  $B$ , we have:

$$\begin{aligned} \partial_0(B_i) &= (-\Delta)^{-1}(\text{curl}(\partial_0(A))) + [\partial_0, (-\Delta)^{-1}]\text{curl}(A) + (-\Delta)^{-1}([\partial_0, \text{curl}]A) \\ &= (-\Delta)^{-1}(\text{curl}(\partial_0(A))) - (-\Delta)^{-1}[\partial_0, \Delta](-\Delta)^{-1}\text{curl}(A) + (-\Delta)^{-1}([\partial_0, \text{curl}]A) \\ &= (-\Delta)^{-1}(\text{curl}(\partial_0(A))) - (-\Delta)^{-1}[\partial_0, \Delta]B + (-\Delta)^{-1}([\partial_0, \text{curl}]A). \end{aligned}$$

Thus, in view of the bootstrap assumption (5.5) for  $A$ , the Bochner inequality on  $\Sigma_t$  (6.8) and the Sobolev embedding on  $\Sigma_t$  (6.9), we have:

$$\begin{aligned} &\|\partial\partial_0(B_i)\|_{L_t^\infty L^2(\Sigma_t)} \quad (6.11) \\ &\lesssim \|\partial(-\Delta)^{-1}(\text{curl}(\partial_0(A)))\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial(-\Delta)^{-1}[\partial_0, \Delta]B\|_{L_t^\infty L^2(\Sigma_t)} \\ &\quad + \|\partial(-\Delta)^{-1}([\partial_0, \text{curl}]A)\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim \|\partial_0 A\|_{L_t^\infty L^2(\Sigma_t)} + \|(-\Delta)^{-\frac{1}{2}}[\partial_0, \Delta]B\|_{L_t^\infty L^2(\Sigma_t)} + \|(-\Delta)^{-\frac{1}{2}}([\partial_0, \text{curl}]A)\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim M\epsilon + \|(-\Delta)^{-\frac{1}{2}}[\partial_0, \Delta]B\|_{L_t^\infty L^2(\Sigma_t)} + \|(-\Delta)^{-\frac{1}{2}}([\partial_0, \text{curl}]A)\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim M\epsilon + \|[\partial_0, \Delta]B\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|[\partial_0, \text{curl}]A\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)}. \end{aligned}$$

Next, we estimate the right-hand side of (6.11). Recall the commutator formula (C.4)  $[\partial_0, \Delta](B_l) = -2k^{ab}\nabla_a\nabla_b(B_l) + 2n^{-1}\nabla_b n\nabla_b(\partial_0(B_l)) + n^{-1}\Delta n\partial_0(B_l) - 2n^{-1}\nabla_a n k^{ab}\nabla_b(B_l)$ . Together with the Sobolev embedding on  $\Sigma_t$  (6.5), the bootstrap assumption (5.5) for  $A$ , and the estimate (6.10) for  $B_i$ , this yields:

$$\begin{aligned} & \|[\partial_0, \Delta](B_i)\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} & (6.12) \\ & \lesssim \|k\|_{L_t^\infty L^3(\Sigma_t)} \|\partial^2(B_i)\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^3(\Sigma_t)} \|\partial(\partial_0(B_i))\|_{L_t^\infty L^2(\Sigma_t)} \\ & \quad + \|\Delta n\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \|\partial_0(B_i)\|_{L_t^\infty L^6(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} \|k\|_{L_t^\infty L^6(\Sigma_t)} \|\partial(B_i)\|_{L_t^\infty L^6(\Sigma_t)} \\ & \lesssim M^2\epsilon^2 + M\epsilon \|\partial(\partial_0(B_i))\|_{L_t^\infty L^2(\Sigma_t)}. \end{aligned}$$

Next, we estimate the last term in the right-hand side of (6.11). In view of the commutator formulas (C.3) and (3.22), and in view of the definition of  $curl$ , we have schematically:

$$[\partial_0, curl]A = k\nabla A + n^{-1}\nabla n\partial_0 A + A\partial A = \mathbf{A}\partial\mathbf{A},$$

which together with the bootstrap assumption (5.5) for  $A$  yields:

$$\|[\partial_0, curl]A\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \lesssim \|\mathbf{A}\|_{L_t^\infty L^3(\Sigma_t)} \|\partial\mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} \lesssim M^2\epsilon^2. \quad (6.13)$$

Finally, (6.11)-(6.13) imply:

$$\begin{aligned} \|\partial\partial_0(B_i)\|_{L_t^\infty L^2(\Sigma_t)} & \lesssim M\epsilon + \|[\partial_0, \Delta]B\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|[\partial_0, curl]A\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\ & \lesssim M\epsilon + M\epsilon \|\partial(\partial_0(B_i))\|_{L_t^\infty L^2(\Sigma_t)} \end{aligned}$$

which yields:

$$\|\partial\partial_0(B_i)\|_{L_t^\infty L^2(\Sigma_t)} \lesssim M\epsilon.$$

Together with (6.10), this concludes the proof of the proposition.  $\square$

**6.3. A decomposition for  $A$ .** Recall that  $B = (-\Delta)^{-1}(curl(A))$ . We show how to recover  $A$  from  $B$ :

**Lemma 6.5.** *We have the following estimate:*

$$A = curl(B) + E$$

where  $E$  satisfies:

$$\|\partial E\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial^2 E\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|E\|_{L_t^2 L^\infty(\Sigma_t)} \lesssim M^2\epsilon^2.$$

*Proof.* In view of Lemma 3.7, we have:

$$A = (-\Delta)^{-1}curl(curl(A) + (-\Delta)^{-1}(A\partial A + A^3)).$$

This yields:

$$\begin{aligned} A & = curl(-\Delta)^{-1}(curl(A)) + [(-\Delta)^{-1}, curl]curl(A) + (-\Delta)^{-1}(A\partial A + A^3) \\ & = curl(B) - (-\Delta)^{-1}[\Delta, curl](-\Delta)^{-1}curl(A) + (-\Delta)^{-1}(A\partial A + A^3) \\ & = curl(B) - (-\Delta)^{-1}[\Delta, curl]B + (-\Delta)^{-1}(A\partial A + A^3), \end{aligned}$$



which implies:

$$E = -(-\Delta)^{-1}[\Delta, \text{curl}]B + (-\Delta)^{-1}(A\partial A + A^3).$$

Now, we have

$$[\Delta, \partial]\phi = R\partial\phi + \partial A\partial\phi + A\partial^2\phi$$

for any scalar function  $\phi$  in  $\Sigma_t$  where the curvature tensor  $R$  on  $\Sigma_t$  is related to  $\mathbf{R}$  through the Gauss equation which can be written schematically:

$$R = \mathbf{R} + A^2.$$

Thus, we obtain:

$$[\Delta, \text{curl}]B = \mathbf{R}\partial B + \partial A\partial B + A\partial^2 B + A^2\partial B.$$

This yields:

$$E = -(-\Delta)^{-1}(\mathbf{R}\partial B + \partial A\partial B + A\partial^2 B + A^2\partial B) + (-\Delta)^{-1}(A\partial A + A^3). \quad (6.14)$$

Using the elliptic estimate (6.7) on  $\Sigma_t$ , we have:

$$\begin{aligned} \|\partial^2 E\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} &\lesssim \|\Delta E\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ &\lesssim \|\mathbf{R}\partial B\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|\partial B\partial A\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|A\partial^2 B\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ &\quad + \|A^2\partial B\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|A\partial A\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|A^3\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ &\lesssim \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \|\partial B\|_{L_t^\infty L^6(\Sigma_t)} + \|A\|_{L_t^\infty L^6(\Sigma_t)} \|\partial^2 B\|_{L_t^\infty L^2(\Sigma_t)} \\ &\quad + (\|A\|_{L_t^\infty L^6(\Sigma_t)} + \|\partial B\|_{L_t^\infty L^6(\Sigma_t)}) \|\partial A\|_{L_t^\infty L^2(\Sigma_t)} \\ &\quad + \|A\|_{L_t^\infty L^6(\Sigma_t)}^2 (\|\partial B\|_{L_t^\infty L^6(\Sigma_t)} + \|A\|_{L_t^\infty L^6(\Sigma_t)}) \\ &\lesssim M^2 \epsilon^2, \end{aligned}$$

where we used in the last inequality the Sobolev embedding (6.5) on  $\Sigma_t$ , the bootstrap estimates (5.5) for  $A$ , the bootstrap estimate (5.3) for  $\mathbf{R}$  and the estimates (6.10) for  $B$ . Together with the Sobolev embedding (6.4) on  $\Sigma_t$ , we finally obtain:

$$\|\partial E\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial^2 E\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim M^2 \epsilon^2.$$

Next, we estimate  $\|E\|_{L_t^2 L^\infty(\Sigma_t)}$ . We first claim the following non sharp embedding on  $\Sigma_t$ . For any scalar function  $v$  on  $\Sigma_t$ , we have:

$$\|(-\Delta)^{-1}v\|_{L^\infty(\Sigma_t)} \lesssim \|v\|_{L^{\frac{14}{9}}(\Sigma_t)} + \|v\|_{L^{\frac{13}{9}}(\Sigma_t)}. \quad (6.15)$$

The proof of (6.15) requires the use of Littlewood-Paley projections on  $\Sigma_t$  and is postponed to Appendix A. We now come back to the estimate of  $\|E\|_{L_t^2 L^\infty(\Sigma_t)}$ . Using (6.14) and

(6.15), we have:

$$\begin{aligned}
& \|E\|_{L_t^2 L^\infty(\Sigma_t)} \\
\lesssim & \|\mathbf{R}\partial B\|_{L_t^2 L^{\frac{14}{9}}(\Sigma_t)} + \|A\partial A\|_{L_t^2 L^{\frac{14}{9}}(\Sigma_t)} + \|A^3\|_{L_t^2 L^{\frac{14}{9}}(\Sigma_t)} \\
& + \|\mathbf{R}\partial B\|_{L_t^2 L^{\frac{13}{9}}(\Sigma_t)} + \|A\partial A\|_{L_t^2 L^{\frac{13}{9}}(\Sigma_t)} + \|A^3\|_{L_t^2 L^{\frac{13}{9}}(\Sigma_t)} \\
\lesssim & \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} (\|\partial B\|_{L_t^2 L^7(\Sigma_t)} + \|\partial B\|_{L_t^\infty L^{\frac{26}{5}}(\Sigma_t)}) \\
& + (\|A\|_{L_t^2 L^7(\Sigma_t)} + \|A\|_{L_t^\infty L^{\frac{26}{5}}(\Sigma_t)}) \|\partial A\|_{L_t^\infty L^2(\Sigma_t)} + \|A\|_{L_t^\infty L^{\frac{14}{3}}(\Sigma_t)}^3 + \|A\|_{L_t^\infty L^{\frac{13}{3}}(\Sigma_t)}^3 \\
\lesssim & M^2 \epsilon^2,
\end{aligned}$$

where we used in the last inequality the bootstrap assumptions (5.5) for  $A$ , the bootstrap assumption (5.3) for  $\mathbf{R}$ , the bootstrap Strichartz estimate for  $B$ , see (5.16), and Proposition 6.4. This concludes the proof of the lemma.  $\square$

## 7. ESTIMATES FOR $\square B$

The goal of this section is to derive estimates for  $\square B$ , with  $B = \Delta^{-1} \text{curl}(A)$  using the wave equation (3.32) satisfied by each component of  $A_i$ . We provide the proof of two important propositions concerning estimates for  $\square \text{curl} A$  and  $\square B$ , with  $B = \Delta^{-1} \text{curl}(A)$ . The proofs makes use of the special structure of various bilinear expressions and thus is based not only on the bootstrap assumptions for  $A_0, A, k$  and  $\mathbf{R}$  but also some of our bilinear bootstrap assumptions.

We will need the following straightforward commutation lemma.

**Lemma 7.1.** *Let  $\phi$  be a  $so(3,1)$  scalar function on  $\mathcal{M}$ . We have, schematically,*

$$\partial_j(\square\phi) - \square(\partial_j\phi) = 2(\mathbf{A}^\lambda)_j{}^\mu \partial_\lambda \partial_\mu \phi + \partial_0 A_0 \partial\phi + \mathbf{A}^2 \partial\phi. \quad (7.1)$$

We also have:

$$\begin{aligned}
[\square, \Delta]\phi &= -4k^{ab} \nabla_a \nabla_b (\partial_0 \phi) + 4n^{-1} \nabla_b n \nabla_b (\partial_0 (\partial_0 \phi)) - 2\nabla_0 k^{ab} \nabla_a \nabla_b \phi \\
&+ F^{(1)} \partial^2 \phi + F^{(2)} \partial\phi, \\
F^{(1)} &= \partial A_0 + \mathbf{A}^2, \\
F^{(2)} &= \partial \partial A_0 + \mathbf{A} \partial \mathbf{A} + \mathbf{A}^3,
\end{aligned} \quad (7.2)$$

where  $\nabla_a$  and  $\nabla_b$  denote induced covariant derivatives on  $\Sigma_t$  applied to the scalars  $\phi$ ,  $\partial_0 \phi$  and  $\partial_0(\partial_0 \phi)$ .

**Remark 7.2.** *The derivation of (7.2) involves the full use of the Einstein equations. The resulting structure of the terms on the right hand side of (7.2) is crucial to the strategy of the proof of the main result and reflects “hidden” null cancellations. We refer the reader to (7.27)-(7.34), where this structure allows us to use bilinear estimates.*

*Proof.* We start with the following general covariant calculation for any scalar function  $\phi$  on  $\mathcal{M}$ :

$$[\mathbf{D}_\mu, \square]\phi = 0. \quad (7.3)$$

This follows trivially from the vanishing of the spacetime Ricci curvature, i.e.

$$[\mathbf{D}_\mu, \square]\phi = \mathbf{R}_{\nu\mu}{}^{\nu\lambda}\mathbf{D}_\lambda\phi = 0.$$

On the other hand, Lemma 3.4 yields:

$$(e_j)^\mu \square(\mathbf{D}\phi)_\mu = \square(\partial_j\phi) - 2(\mathbf{A}^\lambda)_j{}^\mu \mathbf{D}_\lambda\partial_\mu\phi - \mathbf{D}^\lambda(\mathbf{A}_\lambda)_j{}^\gamma \partial_\gamma\phi - (\mathbf{A}_\lambda)_j{}^\gamma (\mathbf{A}^\lambda)_{\gamma}{}^\sigma \partial_\sigma\phi.$$

Together with our Coulomb like gauge condition, we obtain for  $\partial_j\phi$ ,  $j = 1, 2, 3$ :

$$\partial_j(\square\phi) - \square(\partial_j\phi) = 2(\mathbf{A}^\lambda)_j{}^\mu \partial_\lambda\partial_\mu\phi + \partial_0(A_0)^\gamma{}_j \partial_\gamma\phi + \mathbf{A}^2\partial\phi,$$

which proves the first part of the lemma. The proof of the second part of Lemma 7.1 is postponed to Appendix C.  $\square$

### 7.1. Estimates for $\square \operatorname{curl}(A)$ .

**Proposition 7.3.** *The following estimate holds true,*

$$\sum_{i=1}^3 \|(-\Delta)^{-\frac{1}{2}}\square(\operatorname{curl}(A)_i)\|_{L^2(\mathcal{M})} \lesssim M^2\epsilon^2.$$

*Proof.* We have:

$$\square(\partial_j(A_i) - \partial_i(A_j)) = \partial_j(\square(A_i)) - \partial_i(\square(A_j)) + [\square, \partial_j](A_i) - [\square, \partial_i](A_j). \quad (7.4)$$

We evaluate the first term on the right-hand side of (7.4) by differentiating (3.32). We obtain:

$$\partial_j(\square(A_i)) = -\partial_j(\partial_i(\partial_0 A_0)) - \partial_j(A^l \partial_l A_i) + \partial_j(h_i^{(1)}), \quad (7.5)$$

where  $h_i^{(1)}$  is given by:

$$h_i^{(1)} = A^l \partial_l A_i + A_0 \partial \mathbf{A} + A \partial A_0 + \mathbf{A}^3.$$

We estimate  $h_i^{(1)}$  using the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$ , the Sobolev embedding on  $\Sigma_t$  (6.9), and the Bochner inequality (6.8) on  $\Sigma_t$ :

$$\begin{aligned} \|(-\Delta)^{-\frac{1}{2}}\partial_j(h_i^{(1)})\|_{L^2(\mathcal{M})} &\lesssim \|h_i^{(1)}\|_{L^2(\mathcal{M})} + \|Ah_i^{(1)}\|_{L_t^2 L_x^{\frac{6}{5}}(\Sigma_t)} \\ &\lesssim \|A^l \partial_l(A_i)\|_{L^2(\mathcal{M})} + \|A_0\|_{L_t^2 L^\infty(\Sigma_t)} \|\partial \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} \\ &\quad + \|A\|_{L_t^\infty L^6(\Sigma_t)} \|\partial A_0\|_{L_t^\infty L^3(\Sigma_t)} + \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)}^3 \\ &\lesssim M^2\epsilon^2. \end{aligned} \quad (7.6)$$

In view of (7.5), we have:

$$\begin{aligned} \partial_j(\square(A_i)) - \partial_i(\square(A_j)) &= -\partial_j(\partial_i(\partial_0 A_0)) + \partial_i(\partial_j(\partial_0 A_0)) \\ &\quad -\partial_j(A^l \partial_i A_l) + \partial_i(A^l \partial_j A_l) + \partial_j(h_i^{(1)}) - \partial_i(h_j^{(1)}) \\ &= h_{ij}^{(2)}, \end{aligned} \tag{7.7}$$

where  $h_{ij}^{(2)}$  is given by:

$$h_{ij}^{(2)} = Q_{ij}(A^l, A_l) + A\partial(\partial_0 A_0) + A^2 \boldsymbol{\partial} \mathbf{A} + \partial_j(h_i^{(1)}) - \partial_i(h_j^{(1)}),$$

and where the quadratic form  $Q_{ij}$  is defined as  $Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_j \phi \partial_i \psi$ . Note that the most dangerous term in  $h_{ij}^{(2)}$  is  $Q_{ij}(A^l, A_l)$ . Using the bilinear assumption (5.14), the Sobolev embeddings on  $\Sigma_t$  (6.9) and (6.5), the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$ , and the estimate (7.6), we have:

$$\begin{aligned} &\|(-\Delta)^{-\frac{1}{2}}(h_{ij}^{(2)})\|_{L^2(\mathcal{M})} \\ &\lesssim \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(A^l, A_l))\|_{L^2(\mathcal{M})} + \|A\partial(\partial_0 A_0)\|_{L_t^2 L^{\frac{6}{5}}(\Sigma_t)} + \|A^2 \boldsymbol{\partial} \mathbf{A}\|_{L_t^2 L^{\frac{6}{5}}(\Sigma_t)} \\ &\quad + \|(-\Delta)^{-\frac{1}{2}} \partial_j(h_i^{(1)})\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}} \partial_i(h_j^{(1)})\|_{L^2(\mathcal{M})} \\ &\lesssim M^3 \epsilon^2 + \|A\|_{L_t^\infty L^6(\Sigma_t)} \|\partial(\partial_0 A_0)\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|A\|_{L_t^\infty L^6(\Sigma_t)}^2 \|\boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} + M^2 \epsilon^2 \\ &\lesssim M^3 \epsilon^2. \end{aligned} \tag{7.8}$$

Next, we consider the commutator terms in the right-hand side of (7.4). In view of (7.1), we have:

$$[\square, \partial_j](A_i) = 2(A^\lambda)_j{}^\mu \partial_\lambda \partial_\mu(A_i) + h_{ij}^{(3)}, \tag{7.9}$$

where  $h_{ij}^{(3)}$  is given by:

$$h_{ij}^{(3)} = \partial_0 A_0 \partial(A_i) + \mathbf{A}^2 \partial(A_i).$$

Using the Sobolev embeddings on  $\Sigma_t$  (6.9) and (6.5), and the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$ , we have:

$$\begin{aligned} &\|(-\Delta)^{-\frac{1}{2}}(h_{ij}^{(3)})\|_{L^2(\mathcal{M})} \\ &\lesssim \|(-\Delta)^{-\frac{1}{2}}(h_{ij}^{(3)})\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim \|\partial_0 A_0 \partial(A_i)\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|A^2 \boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\ &\lesssim \|\boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} \|\partial_0 A_0\|_{L_t^\infty L^3(\Sigma_t)} + \|A\|_{L_t^\infty L^6(\Sigma_t)}^2 \|\boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim M^2 \epsilon^2. \end{aligned} \tag{7.10}$$

Next, we consider the term  $(A^\lambda)_j{}^\mu \partial_\lambda \partial_\mu(A_i)$ . We have:

$$\begin{aligned} (A^\lambda)_j{}^\mu \partial_\lambda \partial_\mu(A_i) &= -(A_0)_j{}^l \partial_0 \partial_l(A_i) + (A_0)_{j0} \partial_0 \partial_0(A_i) \\ &\quad + (A^l)_j{}^m \partial_l \partial_m(A_i) - (A^l)_{j0} \partial_l \partial_0(A_i) + A^2 \boldsymbol{\partial} \mathbf{A}. \end{aligned} \tag{7.11}$$

Note that the most dangerous terms in (7.11) are the third and the fourth one. They will both require the use of bilinear estimates.

We deal with each term in the right-hand side of (7.11), starting with the first one. We have:

$$\begin{aligned} (A_0)_j{}^l \partial_0 \partial_l (A_i) &= (A_0)_j{}^l \partial_l (\partial_0 (A_i)) + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A} \\ &= \partial_l (A_0 \boldsymbol{\partial} \mathbf{A}) + \partial_l (A_0) \boldsymbol{\partial} \mathbf{A} + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A}, \end{aligned}$$

which together with the Sobolev embeddings on  $\Sigma_t$  (6.9) and (6.5), and the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$  yields:

$$\begin{aligned} & \|(-\Delta)^{-\frac{1}{2}}((A_0)_j{}^l \partial_0 \partial_l (A_i))\|_{L^2(\mathcal{M})} \\ & \lesssim \|A_0 \boldsymbol{\partial} \mathbf{A}\|_{L^2(\mathcal{M})} + \|\partial_l (A_0) \boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|\mathbf{A}^2 \boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\ & \lesssim \|A_0\|_{L_t^2 L^\infty(\Sigma_t)} \|\boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial_l (A_0)\|_{L_t^\infty L^3(\Sigma_t)} \|\boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} \\ & \quad + \|A\|_{L_t^\infty L^6(\Sigma_t)}^2 \|\boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} \\ & \lesssim M^2 \epsilon^2. \end{aligned} \tag{7.12}$$

Next, we consider the second term in the right-hand side of (7.11). For that term, we would like to factorize the  $\partial_0$  derivative in order to get two terms of the type  $\partial_0(A_0 \partial_0(A))$  and  $\partial_0(A_0) \partial_0(A)$ , and then conclude using elliptic estimates and Sobolev embeddings on  $\Sigma_t$ . A similar strategy worked for the first term in the right-hand side of (7.11). But it does not work directly for this term since  $(-\Delta)^{-\frac{1}{2}} \partial_0$  is not necessarily bounded on  $L^2(\Sigma_t)$ . Thus, we first start by showing how one may replace one  $\partial_0$  with  $\partial$ . Using the identity (3.3) relating  $\mathbf{A}$  and  $\mathbf{R}$ , we have:

$$\begin{aligned} (A_0)_{j0} \partial_0 \partial_0 (A_i) &= (A_0)_{j0} \partial_0 (\partial_0 (A_i)) + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A} \\ &= (A_0)_{j0} \partial_0 (\partial_i (A_0)) + A_0 \partial_0 (\mathbf{R}_{0i\dots}) + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A} \\ &= A_0 \partial_i (\partial_0 (A_0)) + A_0 \mathbf{D}_0 \mathbf{R}_{0i\dots} + \mathbf{A}^2 \mathbf{R} + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A}. \end{aligned}$$

Using the Bianchi identities for  $\mathbf{R}$ , we have:

$$\mathbf{D}_0 \mathbf{R}_{0i\dots} = \mathbf{D}_l \mathbf{R}_{li\dots}$$

Thus we obtain:

$$\begin{aligned} (A_0)_{j0} \partial_0 \partial_0 (A_i) &= A_0 \partial_i (\partial_0 (A_0)) + A_0 \mathbf{D}_l \mathbf{R}_{li\dots} + \mathbf{A}^2 \mathbf{R} + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A} \\ &= A_0 \partial_i (\partial_0 (A_0)) + \partial_l (A_0 \mathbf{R}) + \partial_l (A_0) \mathbf{R} + \mathbf{A}^2 \mathbf{R} + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A}. \end{aligned}$$

Using the Sobolev embeddings on  $\Sigma_t$  (6.9) and (6.5), the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$ , and the bootstrap assumption (5.3) for  $\mathbf{R}$ , we obtain:

$$\begin{aligned}
& \|(-\Delta)^{-\frac{1}{2}}((A_0)_{j0} \partial_0 \partial_0(A_i))\|_{L^2(\mathcal{M})} \\
\lesssim & \|A_0 \partial_i(\partial_0(A_0))\|_{L_t^2 L^{\frac{6}{5}}(\Sigma_t)} + \|A_0 \mathbf{R}\|_{L^2(\mathcal{M})} + \|\partial_t(A_0) \mathbf{R}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\
& + \|\mathbf{A}^2 \mathbf{R}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|\mathbf{A}^2 \partial \mathbf{A}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\
\lesssim & \|A_0\|_{L_t^\infty L^6(\Sigma_t)} \|\partial_i(\partial_0(A_0))\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|A_0\|_{L_t^2 L^\infty(\Sigma_t)} \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \\
& + \|\partial_t(A_0)\|_{L_t^\infty L^3(\Sigma_t)} \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} + \|A\|_{L_t^\infty L^6(\Sigma_t)}^2 (\|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)}) \\
\lesssim & M^2 \epsilon^2.
\end{aligned} \tag{7.13}$$

Next, we consider the third term in the right-hand side of (7.11). We have:

$$\begin{aligned}
(A^l)_j{}^m \partial_l \partial_m(A_i) &= A^l \partial_m(\partial_l(A)) + \mathbf{A}^2 \partial \mathbf{A} \\
&= \partial(A^l \partial_l(A)) + \partial(A^l) \partial_l(A) + \mathbf{A}^2 \partial \mathbf{A}.
\end{aligned}$$

Together with the bilinear assumptions (5.9) and (5.15), the Sobolev embeddings on  $\Sigma_t$  (6.9) and (6.5), and the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$ , we obtain:

$$\begin{aligned}
& \|(-\Delta)^{-\frac{1}{2}}((A^l)_j{}^m \partial_l \partial_m(A_i))\|_{L^2(\mathcal{M})} \\
\lesssim & \|A^l \partial_l(A)\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}(\partial(A^l) \partial_l(A))\|_{L^2(\mathcal{M})} + \|\mathbf{A}^2 \partial \mathbf{A}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\
\lesssim & M^3 \epsilon^2.
\end{aligned} \tag{7.14}$$

Finally, we consider the fourth term in the right-hand side of (7.11). We would like to factorize the  $\partial_0$  derivative in order to get two terms of the type  $\partial_0(A^l \partial_l(A))$  and  $\partial_0(A^l) \partial_l(A)$ , and then conclude using the bilinear assumptions (5.9) and (5.15). A similar strategy worked for the third term in the right-hand side of (7.11). But it does not work directly for this term since  $(-\Delta)^{-\frac{1}{2}} \partial_0$  is not necessarily bounded on  $L^2(\Sigma_t)$ . Thus, as for the second term, we first start by showing how one may replace  $\partial_0$  with  $\partial$ . Using the identity (3.3) relating  $\mathbf{A}$  and  $\mathbf{R}$ , we have schematically:

$$\partial_0(A_i) - \partial_i(A_0) + \mathbf{A}^2 = \mathbf{R}_{0i..}$$

which yields:

$$\begin{aligned}
(A^l)_{j0} \partial_l \partial_0(A_i) &= (A^l)_{j0} \partial_l(\partial_0(A_i)) + \mathbf{A}^2 \partial \mathbf{A} \\
&= (A^l)_{j0} \partial_l(\partial_i(A_0)) + (A^l)_{j0} \partial_l(\mathbf{R}_{0i..}) + \mathbf{A}^2 \partial \mathbf{A} \\
&= \partial_l((A^l)_{j0} \mathbf{R}_{0i..}) + \partial_l(A^l) \mathbf{R} + A \partial^2 A_0 + \mathbf{A}^2 \partial \mathbf{A} \\
&= \partial_l((A^l)_{j0} \mathbf{R}_{0i..}) + \mathbf{A}^2 \mathbf{R} + \mathbf{A}^2 \partial \mathbf{A} + A \partial^2 A_0,
\end{aligned}$$

where we used in the last inequality our Coulomb like gauge choice which yields  $\partial_l(A^l) = \nabla_l(A^l) = A^2$ . Thus, we have:

$$(A^l)_{j0} \partial_l \partial_0(A_i) = \partial_l((A^l)_{j0} \mathbf{R}_{0i..}) + h_{ij}^{(4)}, \tag{7.15}$$

where  $h_{ij}^{(4)}$  is given by:

$$h_{ij}^{(4)} = \mathbf{A}^2 \mathbf{R} + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A} + A \partial^2 A_0.$$

Using the Sobolev embeddings on  $\Sigma_t$  (6.9) and (6.5), the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$ , and the bootstrap assumption (5.3) for  $\mathbf{R}$ , we obtain:

$$\begin{aligned} & \|(-\Delta)^{-\frac{1}{2}}(h_{ij}^{(4)})\|_{L^2(\mathcal{M})} \\ \lesssim & \|\mathbf{A}^2 \mathbf{R}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|\mathbf{A}^2 \boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|\mathbf{A} \partial^2 A_0\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\ \lesssim & \|A\|_{L_t^\infty L^6(\Sigma_t)}^2 (\|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} + \|\boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)}) + \|A\|_{L_t^\infty L^6(\Sigma_t)} \|\partial^2 A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ \lesssim & M^2 \epsilon^2. \end{aligned} \tag{7.16}$$

Next, we estimate the first term on the right-hand side of (7.15). Since  $\mathbf{R}_{0i00} = 0$ , the terms  $\mathbf{R}_{0i..}$  are of two types:  $\mathbf{R}_{0imn}$  or  $\mathbf{R}_{0i0m}$ . Now, from the symmetries of  $\mathbf{R}$  and the Einstein equations, we have:

$$\mathbf{R}_{0imn} = \mathbf{R}_{mno i} \text{ and } \mathbf{R}_{0i0m} = -\mathbf{R}_{ninm}.$$

Also, in view of the link between  $\mathbf{R}$  and  $\mathbf{A}$  (3.3), we have schematically:

$$\mathbf{R}_{mno i} = \partial_m(A_n) - \partial_n(A_m) + \mathbf{A}^2 \text{ and } \mathbf{R}_{ninm} = \partial_n(A_i) - \partial_i(A_n) + \mathbf{A}^2.$$

Thus, we obtain schematically:

$$\mathbf{R}_{0i..} = \partial A + \mathbf{A}^2.$$

which, using the Coulomb gauge, yields:

$$\begin{aligned} \partial_l((A^l)_{j0} \mathbf{R}_{0i..}) &= A^l \partial_l \partial(A) + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A} \\ &= \partial(A^l \partial_l(A)) + \partial(A^l) \partial_l(A) + \mathbf{A}^2 \boldsymbol{\partial} \mathbf{A}. \end{aligned}$$

Together with the bilinear assumptions (5.9) and (5.15), the Sobolev embeddings on  $\Sigma_t$  (6.9) and (6.5), and the bootstrap assumptions (5.5) and (5.6) for  $A$  and  $A_0$ , we obtain:

$$\begin{aligned} & \|(-\Delta)^{-\frac{1}{2}}(\partial_l((A^l)_{j0} \mathbf{R}_{0i..}))\|_{L^2(\mathcal{M})} \\ \lesssim & \|A^l \partial_l(A)\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}(\partial(A^l) \partial_l(A))\|_{L^2(\mathcal{M})} + \|\mathbf{A}^2 \boldsymbol{\partial} \mathbf{A}\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\ \lesssim & M^3 \epsilon^2. \end{aligned} \tag{7.17}$$

Now, (7.15)-(7.17) imply:

$$\|(-\Delta)^{-\frac{1}{2}}((A^l)_{j0} \partial_l \partial_0(A_i))\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2. \tag{7.18}$$

Finally, (7.11)-(7.18) imply:

$$\|(-\Delta)^{-\frac{1}{2}}((A^\lambda)_j{}^\mu \partial_\lambda \partial_\mu(A_i))\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2. \tag{7.19}$$

In the end, (7.4), (7.7)-(7.10), and (7.19) yield:

$$\|(-\Delta)^{-\frac{1}{2}} \square(\partial_j(A_i))\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2.$$

This implies:

$$\|(-\Delta)^{-\frac{1}{2}}[\square(\partial_j(A_i) - \partial_i(A_j))]\|_{L^2(\mathcal{M})} \lesssim M^3 \epsilon^2,$$

which concludes the proof of the proposition.  $\square$

**7.2. Estimates for  $\square B$ .** Here we derive a wave equation for each component of  $B = \Delta^{-1} \text{curl}(A)$  and prove the following,

**Proposition 7.4** (Estimates for  $\square B$ ). *The components  $B_i = (-\Delta)^{-1}(\text{curl}(A)_i)$  verify the following estimate,*

$$\sum_{i=1}^3 \|\partial \square B_i\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2. \quad (7.20)$$

We also have,

$$\sum_{i=1}^3 \|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})} \lesssim M \epsilon. \quad (7.21)$$

*Proof.* The estimates for  $\square B$  are simpler than those for  $\partial \square B$  and  $\square \partial B$ . We prove first the estimates for  $\partial \square B$  and derive those for  $\square \partial B$  using the commutation formula (7.1).

We have:

$$\begin{aligned} \square(B_i) &= [\square, (-\Delta)^{-1}](\text{curl}(A)_i) + (-\Delta)^{-1}(\square(\text{curl}(A)_i)) \\ &= -(-\Delta)^{-1}[\square, \Delta](-\Delta)^{-1}(\text{curl}(A)_i) + (-\Delta)^{-1}(\square(\text{curl}(A)_i)) \\ &= -(-\Delta)^{-1}[\square, \Delta](B_i) + (-\Delta)^{-1}(\square(\text{curl}(A)_i)). \end{aligned} \quad (7.22)$$

Thus, we obtain:

$$\begin{aligned} &\|\partial \square(B_i)\|_{L^2(\mathcal{M})} \\ &\lesssim \|\partial(-\Delta)^{-1}[\square, \Delta](B_i)\|_{L^2(\mathcal{M})} + \|\partial(-\Delta)^{-1}(\square(\text{curl}(A)_i))\|_{L^2(\mathcal{M})} \\ &\lesssim \|(-\Delta)^{-\frac{1}{2}}[\square, \Delta](B_i)\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}(\square(\text{curl}(A)_i))\|_{L^2(\mathcal{M})} \\ &\lesssim \|(-\Delta)^{-\frac{1}{2}}[\square, \Delta](B_i)\|_{L^2(\mathcal{M})} + M^3 \epsilon^2, \end{aligned} \quad (7.23)$$

where we used Proposition 7.3 in the last inequality.

In view of (7.23), we need to estimate  $\|(-\Delta)^{-\frac{1}{2}}[\square, \Delta](B_i)\|_{L^2(\mathcal{M})}$ . Recall the commutator formula (7.2):

$$\begin{aligned} [\square, \Delta]\phi &= -4k^{ab}\nabla_a\nabla_b(\partial_0\phi) + 4n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0\phi)) - 2\nabla_0 k^{ab}\nabla_a\nabla_b\phi \\ &\quad + F^{(1)}\partial^2\phi + F^{(2)}\partial\phi, \\ F^{(1)} &= \partial A_0 + \mathbf{A}^2, \\ F^{(2)} &= \partial\partial A_0 + \mathbf{A}\partial\mathbf{A} + \mathbf{A}^3. \end{aligned} \quad (7.24)$$

Using the bootstrap assumptions (5.5) for  $A$  and (5.6) for  $A_0$ , we have:

$$\|F^{(1)}\|_{L_t^\infty L^3(\Sigma_t)} \lesssim \|\partial A_0\|_{L_t^\infty L^3(\Sigma_t)} + \|A\|_{L_t^\infty L^6(\Sigma_t)}^2 \lesssim M \epsilon, \quad (7.25)$$



and:

$$\begin{aligned}
& \|F^{(2)}\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} & (7.26) \\
\lesssim & \|\partial\partial A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)} \|\partial\mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)}^3 \\
\lesssim & M\epsilon.
\end{aligned}$$

Using (7.24)-(7.26) together with the estimates of Proposition 6.4, we obtain:

$$\begin{aligned}
& \|(-\Delta)^{-\frac{1}{2}}[\square, \Delta](B_l)\|_{L^2(\mathcal{M})} & (7.27) \\
\lesssim & \|(-\Delta)^{-\frac{1}{2}}[k^{ab}\nabla_a\nabla_b(\partial_0(B_l))]\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}[n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0(B_l)))]\|_{L^2(\mathcal{M})} \\
& + \|(-\Delta)^{-\frac{1}{2}}[\nabla_0 k^{ab}\nabla_a\nabla_b(B_l)]\|_{L^2(\mathcal{M})} + \|F^{(1)}\|_{L_t^\infty L^3(\Sigma_t)} \|\partial^2(B_l)\|_{L^2(\mathcal{M})} \\
& + \|F^{(2)}\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \|\partial(B_l)\|_{L_t^\infty L^6(\Sigma_t)} \\
\lesssim & \|(-\Delta)^{-\frac{1}{2}}[k^{ab}\nabla_a\nabla_b(\partial_0(B_l))]\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}[n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0(B_l)))]\|_{L^2(\mathcal{M})} \\
& + \|(-\Delta)^{-\frac{1}{2}}[\nabla_0 k^{ab}\nabla_a\nabla_b(B_l)]\|_{L^2(\mathcal{M})} + M^2\epsilon^2 + M\epsilon\|\partial_0(\partial_0(B_l))\|_{L^2(\mathcal{M})}.
\end{aligned}$$

Next, we estimate the various terms in the right-hand side of (7.27). The first and the third will require bilinear estimates, while the second will require the estimate  $\nabla n \in L^\infty(\mathcal{M})$ . We start with the first one. We have:

$$\begin{aligned}
k^{ab}\nabla_a\nabla_b(\partial_0(B_l)) &= \nabla_a[k^{ab}\nabla_b(\partial_0(B_l))] - \nabla_a k^{ab}\nabla_b(\partial_0(B_l)) \\
&= \nabla_a[k^{ab}\nabla_b(\partial_0(B_l))],
\end{aligned}$$

where we used the constraint equations (2.2) for  $k$  in the last equality. Together with the Bochner inequality on  $\Sigma_t$  (6.8) and the bilinear assumption (5.10), we obtain:

$$\|(-\Delta)^{-\frac{1}{2}}[k^{ab}\nabla_a\nabla_b(\partial_0(B_l))]\|_{L^2(\mathcal{M})} \lesssim \|k^{ab}\partial_b(\partial_0(B_l))\|_{L^2(\mathcal{M})} \lesssim M^3\epsilon^2. \quad (7.28)$$

Next, we estimate the second term in the right-hand side of (7.27). We have:

$$n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0(B_l))) = \nabla^b[n^{-1}\nabla_b n\partial_0(\partial_0(B_l))] - (n^{-1}\Delta n - n^{-2}|\nabla n|^2)\partial_0(\partial_0(B_l)).$$

Together with the estimates (6.1) for the lapse  $n$  and the Sobolev embedding on  $\Sigma_t$  (6.9), this yields:

$$\begin{aligned}
& \|(-\Delta)^{-\frac{1}{2}}[n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0(B_l)))]\|_{L^2(\mathcal{M})} & (7.29) \\
\lesssim & \|n^{-1}\nabla_b n\partial_0(\partial_0(B_l))\|_{L^2(\mathcal{M})} + \|(n^{-1}\Delta n - n^{-2}|\nabla n|^2)\partial_0(\partial_0(B_l))\|_{L_t^2 L^{\frac{6}{5}}(\Sigma_t)} \\
\lesssim & (\|\nabla n\|_{L^\infty} + \|n^{-1}\Delta n - n^{-2}|\nabla n|^2\|_{L_t^\infty L^3(\Sigma_t)}) \|\partial_0(\partial_0(B_l))\|_{L^2(\mathcal{M})} \\
\lesssim & M\epsilon\|\partial_0(\partial_0(B_l))\|_{L^2(\mathcal{M})}.
\end{aligned}$$

**Remark 7.5.** Note that there is no room in the estimate (7.29). In particular, the estimate  $\|\nabla n\|_{L^\infty(\mathcal{M})} \lesssim M\epsilon$  given by (6.1) is crucial as emphasized in remark 6.1.

Finally, we consider the third term in the right-hand side of (7.27). Recall from (2.5) that the second fundamental form satisfies the following equation:

$$\nabla_0 k_{ab} = E_{ab} + F_{ab}^{(3)}, \quad (7.30)$$

where  $E$  is the 2-tensor on  $\Sigma_t$  defined as:

$$E_{ab} = \mathbf{R}_{a0b0},$$

and where  $F_{ab}^{(3)}$  is given by:

$$F_{ab}^{(3)} = -n^{-1} \nabla_a \nabla_b n - k_{ac} k_b{}^c.$$

In view of the estimates (5.7) for  $k$  and (6.1) for  $n$ ,  $F_{ab}^{(3)}$  satisfies the estimate:

$$\|F_{ab}^{(3)}\|_{L_t^\infty L^3(\Sigma_t)} \lesssim \|\nabla^2 n\|_{L_t^\infty L^3(\Sigma_t)} + \|k\|_{L_t^\infty L^6(\Sigma_t)}^2 \lesssim M\epsilon. \quad (7.31)$$

Next, we consider the term involving  $E$  in the right-hand side of (7.30). Using the maximal foliation assumption, the Bianchi identities and the symmetries of  $\mathbf{R}$ , we obtain:

$$\nabla^a E_{ab} = \mathbf{D}^a \mathbf{R}_{a0b0} + \mathbf{A} \mathbf{R} = -\mathbf{D}^0 \mathbf{R}_{00b0} + \mathbf{A} \mathbf{R} = -\partial_0(\mathbf{R}_{00b0}) + \mathbf{A} \mathbf{R} = \mathbf{A} \mathbf{R}$$

which together with the bootstrap assumptions (5.5) for  $A$  and (5.6) for  $A_0$ , and the bootstrap assumption (5.3) for  $\mathbf{R}$  yields:

$$\|\nabla^a E_{ab}\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)} \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \lesssim M^2 \epsilon^2. \quad (7.32)$$

Now, we have:

$$E_{ab} \nabla_a \nabla_b (B_l) = \nabla^a [E_{ab} \nabla_b (B_l)] - \nabla^a E_{ab} \nabla_b (B_l)$$

which together with the bilinear estimate (5.11), the estimates of Lemma 6.4 for  $B$  and (7.32) yields:

$$\begin{aligned} & \|(-\Delta)^{-\frac{1}{2}} [E_{ab} \nabla_a \nabla_b (B_l)]\|_{L^2(\mathcal{M})} \\ & \lesssim \|(-\Delta)^{-\frac{1}{2}} \nabla^a [E_{ab} \nabla_b (B_l)]\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}} [\nabla^a E_{ab} \partial_b (B_l)]\|_{L^2(\mathcal{M})} \\ & \lesssim \|\mathbf{R}_{a0b0} \partial_b (B_l)\|_{L^2(\mathcal{M})} + \|\nabla^a E_{ab}\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \|\partial (B_l)\|_{L_t^\infty L^6(\Sigma_t)} \\ & \lesssim M^2 \epsilon^2. \end{aligned} \quad (7.33)$$

(7.30), (7.31), (7.33) and the estimates of Lemma 6.4 for  $B$  yield:

$$\begin{aligned} & \|(-\Delta)^{-\frac{1}{2}} [\nabla_0 k^{ab} \nabla_a \nabla_b (B_l)]\|_{L^2(\mathcal{M})} \\ & \lesssim \|(-\Delta)^{-\frac{1}{2}} [E_{ab} \nabla_a \nabla_b (B_l)]\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}} [F_{ab}^{(3)} \nabla_a \nabla_b (B_l)]\|_{L^2(\mathcal{M})} \\ & \lesssim M^2 \epsilon^2 + \|F_{ab}^{(3)}\|_{L_t^\infty L^3(\Sigma_t)} \|\partial^2 (B)\|_{L_t^\infty L^2(\Sigma_t)} \\ & \lesssim M^2 \epsilon^2. \end{aligned} \quad (7.34)$$

Finally, (7.27)-(7.29) and (7.34) yield:

$$\|(-\Delta)^{-\frac{1}{2}} [\square, \Delta](B_i)\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2 + M\epsilon \|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})},$$

which together with (7.23) implies:

$$\|\partial\Box(B_i)\|_{L^2(\mathcal{M})} \lesssim M^2\epsilon^2 + M\epsilon\|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})}. \quad (7.35)$$

Recalling (3.25), we have:

$$\partial_0(\partial_0(B_i)) = -\Box(B_i) + \Delta(B_i) + n^{-1}\nabla n \cdot \nabla(B_i),$$

which together with the estimates of Lemma 6.4 for  $B$ , the estimates (6.1) for  $n$ , and (7.35) yields:

$$\begin{aligned} & \|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})} & (7.36) \\ \lesssim & \|\Box(B_i)\|_{L^2(\mathcal{M})} + \|\Delta(B_i)\|_{L^2(\mathcal{M})} + \|\nabla n \cdot \nabla(B_i)\|_{L^2(\mathcal{M})} \\ \lesssim & M^2\epsilon^2 + M\epsilon\|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})} + \|\partial^2(B_i)\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla n\|_{L^\infty}\|\partial(B_i)\|_{L_t^\infty L^2(\Sigma_t)} \\ \lesssim & M^2\epsilon^2 + M\epsilon\|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})}. \end{aligned}$$

Choosing  $\epsilon > 0$  such that  $M\epsilon$  is small enough to absorb the term  $\|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})}$  in the right-hand side, (7.35) and (7.36) gives the desired estimate for both  $\|\partial\Box B\|_{L^2(\mathcal{M})}$  and  $\|\partial_0(\partial_0(B_i))\|_{L^2(\mathcal{M})}$  of the lemma.  $\square$

## 8. ENERGY ESTIMATE FOR THE WAVE EQUATION ON A CURVED BACKGROUND WITH BOUNDED $L^2$ CURVATURE

Recall that  $e_0 = T$ , the future unit normal to the  $\Sigma_t$  foliation. Let  $\pi$  be the deformation tensor of  $e_0$ , that is the symmetric 2-tensor on  $\mathcal{M}$  defined as:

$$\pi_{\alpha\beta} = \mathbf{D}_\alpha T_\beta + \mathbf{D}_\beta T_\alpha.$$

In view of the definition of the second fundamental form  $k$  and the lapse  $n$ , we have:

$$\pi_{ab} = -2k_{ab}, \pi_{a0} = \pi_{0a} = n^{-1}\nabla_a n, \pi_{00} = 0. \quad (8.1)$$

In what follows  $\mathcal{H}$  denotes an arbitrary weakly regular null hypersurface<sup>32</sup> with future normal  $L$  verifying  $\mathbf{g}(L, T) = -1$ . We denote by  $\nabla$  the induced connection on the 2-surfaces  $\mathcal{H} \cap \Sigma_t$ .

We have the following energy estimate for the scalar wave equation:

**Lemma 8.1.** *Let  $F$  a scalar function on  $\mathcal{M}$ , and let  $\phi_0$  and  $\phi_1$  two scalar functions on  $\Sigma_0$ . Let  $\phi$  the solution of the following wave equation on  $\mathcal{M}$ :*

$$\begin{cases} \Box\phi = F, \\ \phi|_{\Sigma_0} = \phi_0, \partial_0\phi|_{\Sigma_0} = \phi_1. \end{cases} \quad (8.2)$$

*Then,  $\phi$  satisfies the following energy estimate:*

$$\begin{aligned} & \|\partial\phi\|_{L_t^\infty L^2(\Sigma_t)} + \sup_{\mathcal{H}}(\|\nabla\phi\|_{L^2(\mathcal{H})} + \|L(\phi)\|_{L^2(\mathcal{H})}) \\ \lesssim & \|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}, \end{aligned} \quad (8.3)$$

<sup>32</sup>i.e. it satisfies assumptions (5.1) and (5.2)

where the supremum is taken over all null hypersurfaces  $\mathcal{H}$  satisfying assumptions (5.1) and (5.2).

*Proof.* We introduce the energy momentum tensor  $Q_{\alpha\beta}$  on  $\mathcal{M}$  given by:

$$Q_{\alpha\beta} = Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\mathbf{g}_{\alpha\beta}(\mathbf{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi).$$

In view of the equation (8.2) satisfied by  $\phi$ , we have:

$$\mathbf{D}^\alpha Q_{\alpha\beta} = F\partial_\beta\phi.$$

Now, we form the 1-tensor  $P$ :

$$P_\alpha = Q_{\alpha 0},$$

and we obtain:

$$\mathbf{D}^\alpha P_\alpha = \mathbf{D}^\alpha Q_{\alpha 0} + Q_{\alpha\beta}\mathbf{D}^\alpha T^\beta = F\partial_0\phi + \frac{1}{2}Q_{\alpha\beta}\pi^{\alpha\beta},$$

where  $\pi$  is the deformation tensor of  $e_0$ . Integrating a specifically chosen region of  $\mathcal{M}$ , bounded by  $\Sigma_0, \Sigma_t$  and  $\mathcal{H}$ , we obtain:

$$\begin{aligned} & \|\partial\phi\|_{L_t^\infty L^2(\Sigma_t)}^2 + \sup_{\mathcal{H}} \|\nabla\phi\|_{L^2(\mathcal{H})}^2 \\ & \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)}^2 + \|\phi_1\|_{L^2(\Sigma_0)}^2 + \left| \int_{\mathcal{M}} F\partial_0\phi d\mathcal{M} \right| + \left| \int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta} d\mathcal{M} \right| \\ & \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)}^2 + \|\phi_1\|_{L^2(\Sigma_0)}^2 + \|F\|_{L^2(\mathcal{M})} \|\partial_0\phi\|_{L^2(\mathcal{M})} + \left| \int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta} d\mathcal{M} \right|. \end{aligned} \quad (8.4)$$

Next, we deal with the last term in the right-hand side of (8.4). In view of (8.1), we have:

$$\begin{aligned} & \int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta} d\mathcal{M} \\ & = -2 \int_{\mathcal{M}} Q_{ab}k^{ab} d\mathcal{M} + \int n^{-1}\nabla^i n Q_{0i} d\mathcal{M} \\ & = -2 \int_{\mathcal{M}} \partial_a\phi\partial_b\phi k^{ab} d\mathcal{M} + \int_{\mathcal{M}} \text{tr}_g k(\mathbf{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) d\mathcal{M} + \int n^{-1}\nabla^a n \partial_a\phi\partial_0\phi d\mathcal{M} \\ & = -2 \int_{\mathcal{M}} \partial_a\phi\partial_b\phi k^{ab} d\mathcal{M} + \int n^{-1}\nabla^a n \partial_a\phi\partial_0\phi d\mathcal{M}, \end{aligned}$$

where we used in the last inequality the maximal foliation assumption. Together with the bilinear bootstrap assumption(5.12) and the estimates (6.1) for the lapse  $n$ , this yields:

$$\begin{aligned} \left| \int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta} d\mathcal{M} \right| & \lesssim \|k_a \cdot \partial^a\phi\|_{L^2(\mathcal{M})} \|\partial\phi\|_{L^2(\mathcal{M})} + \|\nabla n\|_{L^\infty(\mathcal{M})} \|\partial\phi\|_{L^2(\mathcal{M})}^2 \\ & \lesssim M^2\epsilon \left( \sup_{\mathcal{H}} \|\nabla\phi\|_{L^2(\mathcal{H})} \right) \|\partial\phi\|_{L^2(\mathcal{M})} + M\epsilon \|\partial\phi\|_{L^2(\mathcal{M})}^2, \end{aligned}$$

which together with (8.4) concludes the proof of the lemma.  $\square$

**Remark 8.2.** *The most dangerous term in the right-hand side of the previous inequality is  $\|k_a \cdot \partial^a \phi\|_{L^2(\mathcal{M})}$ . Usually, when deriving energy estimates for the wave equation, this term is typically estimated by:*

$$\|k_a \cdot \partial^a \phi\|_{L^2(\mathcal{M})} \lesssim \|k\|_{L_t^2 L^\infty(\Sigma_t)} \|\partial \phi\|_{L_t^\infty L^2(\Sigma_t)}$$

which requires a Strichartz estimate for  $k$ . This Strichartz estimate fails under the assumptions of Theorem 2.2, and we need to rely instead on the bilinear estimate (5.12).

We have the following higher order energy estimate for the scalar wave equation:

**Lemma 8.3.** *Let  $F$  a scalar function on  $\mathcal{M}$ , and let  $\phi_0$  and  $\phi_1$  two scalar functions on  $\Sigma_0$ . Let  $\phi$  the solution of the wave equation (8.2) on  $\mathcal{M}$ . Then,  $\phi$  satisfies the following energy estimate:*

$$\begin{aligned} & \|\partial(\boldsymbol{\partial}\phi)\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial_0(\partial_0\phi)\|_{L^2(\mathcal{M})} + \sup_{\mathcal{H}} (\|\nabla(\partial\phi)\|_{L^2(\mathcal{H})} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}) \\ & \lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\nabla F\|_{L^2(\mathcal{M})}, \end{aligned} \quad (8.5)$$

where the supremum is taken over all null hypersurfaces  $\mathcal{H}$  satisfying assumption (5.1) and (5.2). Furthermore,  $\square(\partial_j\phi)$  satisfies the following estimate:

$$\|\square(\partial_j\phi)\|_{L^2(\mathcal{M})} \lesssim M\epsilon(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}).$$

*Proof.* We derive an equation for  $\partial_j\phi$ . Differentiating (8.2), we obtain:

$$\begin{cases} \square(\partial_j\phi) = \partial_j F + [\square, \partial_j](\phi), \\ \partial_j\phi|_{\Sigma_0} = \partial_j\phi_0, \quad \partial_0(\partial_j\phi)|_{\Sigma_0} = (\partial_j(\partial_0\phi) + [\partial_0, \partial_j]\phi)|_{\Sigma_0} = \partial_j\phi_1 + \mathbf{A}\phi_1 + \mathbf{A}\nabla\phi_0. \end{cases} \quad (8.6)$$

Applying the energy estimate of Lemma 8.1 to (8.6), we obtain:

$$\begin{aligned} & \|\boldsymbol{\partial}(\partial_j\phi)\|_{L_t^\infty L^2(\Sigma_t)} + \sup_{\mathcal{H}} (\|\nabla(\partial_j\phi)\|_{L^2(\mathcal{H})} + \|L(\partial_j\phi)\|_{L^2(\mathcal{H})}) \\ & \lesssim \|\nabla(\partial_j\phi_0)\|_{L^2(\Sigma_0)} + \|\partial_j\phi_1 + \mathbf{A}\phi_1 + \mathbf{A}\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\partial_j F + [\square, \partial_j]\phi\|_{L^2(\mathcal{M})}. \end{aligned}$$

which after taking the supremum over  $j = 1, 2, 3$  yields:

$$\begin{aligned} & \|\partial(\boldsymbol{\partial}\phi)\|_{L_t^\infty L^2(\Sigma_t)} + \sup_{\mathcal{H}} (\|\nabla(\partial\phi)\|_{L^2(\mathcal{H})} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}) \\ & \lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\nabla F\|_{L^2(\mathcal{M})} + \|\mathbf{A}\phi_1\|_{L^2(\Sigma_0)} + \|\mathbf{A}\nabla\phi_0\|_{L^2(\Sigma_0)} \\ & \quad + \|\mathbf{A}\boldsymbol{\partial}\phi\|_{L_t^\infty L^2(\Sigma_t)} + \sup_j \|[\square, \partial_j]\phi\|_{L^2(\mathcal{M})}, \end{aligned} \quad (8.7)$$

where the term  $\mathbf{A}\boldsymbol{\partial}\phi$  in the last inequality comes from the commutator formula (3.22) applied to  $[\partial_0, \partial_j]$ .

Next, we estimate the last term in the right-hand side of (8.7). In view of the commutator formula (7.1), we have:

$$\begin{aligned} [\square, \partial_j]\phi &= 2(\mathbf{A}^\lambda)_j{}^\mu \partial_\lambda \partial_\mu \phi + \partial_0 A_0 \boldsymbol{\partial}\phi + \mathbf{A}^2 \boldsymbol{\partial}\phi \\ &= A^i \partial_i (\partial_l \phi) + A^i \partial_i (\partial_0 \phi) + A^0 \partial_0 (\partial_l \phi) + (A^0)_{j0} \partial_0 (\partial_0 \phi) + \partial_0 A_0 \boldsymbol{\partial}\phi + \mathbf{A}^2 \boldsymbol{\partial}\phi \\ &= A^i \partial_i (\partial_l \phi) + A^i \partial_i (\partial_0 \phi) + h_j, \end{aligned} \quad (8.8)$$

where  $h_j$  is defined in view of the identity (3.17) as:

$$h_j = A^0 \partial_0(\partial_t \phi) + n^{-1} \nabla_j n \partial_0(\partial_0 \phi) + \partial_0 A_0 \boldsymbol{\partial} \phi + \mathbf{A}^2 \boldsymbol{\partial} \phi.$$

We estimate the various terms in the right-hand of (8.8) starting with  $h_j$ . The Sobolev embedding on  $\Sigma_t$  (6.5), the bootstrap estimates (5.6) for  $A_0$  and (5.5) for  $A$ , and the estimate (6.1) for the lapse  $n$  yield:

$$\begin{aligned} & \|h_j\|_{L^2(\mathcal{M})} \\ \lesssim & \|A^0\|_{L_t^\infty L^\infty(\Sigma_t)} \|\partial_0(\partial_t \phi)\|_{L_t^\infty L^2(\Sigma_t)} \\ & + \|\nabla n\|_{L^\infty(\mathcal{M})} \|\partial_0(\partial_0 \phi)\|_{L^2(\mathcal{M})} + \|\partial_0 A_0\|_{L_t^\infty L^3(\Sigma_t)} \|\boldsymbol{\partial} \phi\|_{L_t^\infty L^6(\Sigma_t)} + \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)}^2 \|\boldsymbol{\partial} \phi\|_{L_t^\infty L^6(\Sigma_t)} \\ \lesssim & M\epsilon (\|\partial_0(\partial_0 \phi)\|_{L^2(\mathcal{M})} + \|\partial(\boldsymbol{\partial} \phi)\|_{L_t^\infty L^2(\Sigma_t)}). \end{aligned} \tag{8.9}$$

Note again in view of the previous inequality that the estimate  $\nabla n \in L^\infty(\mathcal{M})$  is crucial as emphasized by Remarks 6.1 and 7.5. Next, we deal with the first and the second term in the right-hand of (8.8). Using the bilinear estimate (5.13), we have:

$$\|A^i \partial_i(\partial_t \phi)\|_{L^2(\mathcal{M})} + \|A^i \partial_i(\partial_0 \phi)\|_{L^2(\mathcal{M})} \lesssim M\epsilon \left( \sup_{\mathcal{H}} (\|\nabla(\partial_t \phi)\|_{L^2(\mathcal{H})} + \|\nabla(\partial_0 \phi)\|_{L^2(\mathcal{H})}) \right),$$

which together with (8.8) and (8.9) yields:

$$\begin{aligned} & \|[\square, \partial_j] \phi\|_{L^2(\mathcal{M})} \\ \lesssim & M\epsilon \left( \sup_{\mathcal{H}} (\|\nabla(\partial_t \phi)\|_{L^2(\mathcal{H})} + \|\nabla(\partial_0 \phi)\|_{L^2(\mathcal{H})}) \right) + M\epsilon (\|\partial_0(\partial_0 \phi)\|_{L^2(\mathcal{M})} + \|\partial(\boldsymbol{\partial} \phi)\|_{L_t^\infty L^2(\Sigma_t)}). \end{aligned} \tag{8.10}$$

It remains to estimate the term  $\|\nabla(\partial_0 \phi)\|_{L^2(\mathcal{H})}$ . Let us define the vectorfield  $N = L - e_0$ . Since  $\mathbf{g}(L, e_0) = -1$ , and since  $L$  is null,  $N$  is tangent to  $\Sigma_t$ . Decomposing  $e_0 = L - N$ , we obtain schematically:

$$\begin{aligned} |\nabla(\partial_0 \phi)| & \leq |\nabla(\nabla_N \phi)| + |\nabla(L(\phi))| \\ & \lesssim |\nabla(N_j \partial_j \phi)| + |\partial(L(\phi))| \\ & \lesssim |\nabla(\partial \phi)| + |L(\partial \phi)| + |(\mathbf{D}N)(\partial \phi)| + |(\mathbf{D}L)(\partial \phi)| + |A \partial \phi| \end{aligned} \tag{8.11}$$

which together with the assumptions (5.1) and (5.2) for  $\mathcal{H}$ , and the embedding (10.2) on  $\mathcal{H}$  yields:

$$\begin{aligned} & \|\nabla(\partial_0 \phi)\|_{L^2(\mathcal{H})} \\ \lesssim & \|\nabla(\partial \phi)\|_{L^2(\mathcal{H})} + \|L(\partial \phi)\|_{L^2(\mathcal{H})} + \|(\mathbf{D}N)(\partial \phi)\|_{L^2(\mathcal{H})} + \|(\mathbf{D}L)(\partial \phi)\|_{L^2(\mathcal{H})} + \|A \partial \phi\|_{L^2(\mathcal{H})} \\ \lesssim & \|\nabla(\partial \phi)\|_{L^2(\mathcal{H})} + \|L(\partial \phi)\|_{L^2(\mathcal{H})} + (\|\mathbf{D}N\|_{L^3(\mathcal{H})} + \|\mathbf{D}L\|_{L^3(\mathcal{H})} + \|A\|_{L^3(\mathcal{H})}) \|\partial \phi\|_{L^6(\mathcal{H})} \\ \lesssim & (\|\nabla(\partial \phi)\|_{L^2(\mathcal{H})} + \|L(\partial \phi)\|_{L^2(\mathcal{H})}) (1 + \|\nabla A\|_{L_t^\infty L^2(\Sigma_t)}) \\ \lesssim & \|\nabla(\partial \phi)\|_{L^2(\mathcal{H})} + \|L(\partial \phi)\|_{L^2(\mathcal{H})}, \end{aligned} \tag{8.12}$$

where we used the bootstrap assumptions (5.5) for  $A$  in the last inequality. Finally, (8.10)-(8.12) yield:

$$\begin{aligned} \sup_j \|\llbracket \square, \partial_j \rrbracket(\phi)\|_{L^2(\mathcal{M})} &\lesssim M\epsilon \left( \sup_{\mathcal{H}} (\|\nabla(\partial\phi)\|_{\mathcal{H}} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}) \right) \\ &\quad + M\epsilon (\|\partial_0(\partial_0\phi)\|_{L^2(\mathcal{M})} + \|\partial(\boldsymbol{\partial}\phi)\|_{L_t^\infty L^2(\Sigma_t)}). \end{aligned} \quad (8.13)$$

Now, (8.7) and (8.13) imply:

$$\begin{aligned} &\|\partial(\boldsymbol{\partial}\phi)\|_{L_t^\infty L^2(\Sigma_t)} + \sup_{\mathcal{H}} (\|\nabla(\partial\phi)\|_{L^2(\mathcal{H})} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}) \\ &\lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\nabla F\|_{L^2(\mathcal{M})} + \|\mathbf{A}\phi_1\|_{L^2(\Sigma_0)} + \|\mathbf{A}\nabla\phi_0\|_{L^2(\Sigma_0)} \\ &\quad + \|\mathbf{A}\boldsymbol{\partial}\phi\|_{L_t^\infty L^2(\Sigma_t)} + M\epsilon \left( \sup_{\mathcal{H}} (\|\nabla(\partial\phi)\|_{L^2(\mathcal{H})} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}) \right) \\ &\quad + M\epsilon (\|\partial_0(\partial_0\phi)\|_{L^2(\mathcal{M})} + \|\partial(\boldsymbol{\partial}\phi)\|_{L_t^\infty L^2(\Sigma_t)}), \end{aligned}$$

which together with the Sobolev embedding on  $\Sigma_t$ , the bootstrap estimates (5.6) for  $A_0$  and (5.5) for  $A$ , and the fact that we may choose  $\epsilon$  such that  $M\epsilon$  is small enough, yields:

$$\begin{aligned} &\|\partial(\boldsymbol{\partial}\phi)\|_{L_t^\infty L^2(\Sigma_t)} + \sup_{\mathcal{H}} (\|\nabla(\partial\phi)\|_{L^2(\mathcal{H})} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}) \\ &\lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})} + M\epsilon \|\partial_0(\partial_0\phi)\|_{L^2(\mathcal{M})}. \end{aligned} \quad (8.14)$$

In view of (8.14), we need an estimate for  $\partial_0(\partial_0\phi)$ . Proceeding as in (7.36), we have:

$$\begin{aligned} \|\partial_0(\partial_0(\phi))\|_{L^2(\mathcal{M})} &\lesssim \|\square(\phi)\|_{L^2(\mathcal{M})} + \|\Delta\phi\|_{L^2(\mathcal{M})} + \|\nabla n \cdot \nabla(\phi)\|_{L^2(\mathcal{M})} \\ &\lesssim \|F\|_{L^2(\mathcal{M})} + \|\nabla^2\phi\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla n\|_{L^\infty} \|\nabla\phi\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim \|F\|_{L^2(\mathcal{M})} + \|\partial^2\phi\|_{L_t^\infty L^2(\Sigma_t)}. \end{aligned} \quad (8.15)$$

Finally, (8.6) and (8.13)-(8.15) yields:

$$\begin{aligned} \|\square(\partial_j\phi)\|_{L^2(\mathcal{M})} &\lesssim \|\partial_j F\|_{L^2(\mathcal{M})} + \|\llbracket \square, \partial_j \rrbracket\phi\|_{L^2(\mathcal{M})} \\ &\lesssim \|\partial F\|_{L^2(\mathcal{M})} + M\epsilon \left( \sup_{\mathcal{H}} (\|\nabla(\partial\phi)\|_{\mathcal{H}} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}) \right) \\ &\quad + M\epsilon (\|\partial_0(\partial_0\phi)\|_{L^2(\mathcal{M})} + \|\partial(\boldsymbol{\partial}\phi)\|_{L_t^\infty L^2(\Sigma_t)}) \\ &\lesssim M\epsilon (\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}) \end{aligned}$$

which together with (8.14) and (8.15) concludes the proof of the lemma.  $\square$

## 9. PROOF OF PROPOSITION 5.7

Here we derive estimates for  $\mathbf{R}$ ,  $A_0$  and  $A$  and thus improve the basic bootstrap assumptions (5.3), (5.4), (5.5) and (5.6).

**9.1. Curvature estimates.** We derive the curvature estimates using the Bel-Robinson tensor,

$$Q_{\alpha\beta\gamma\delta} = \mathbf{R}_\alpha{}^\lambda \gamma^\sigma \mathbf{R}_{\beta\lambda\delta\sigma} + {}^* \mathbf{R}_\alpha{}^\lambda \gamma^\sigma {}^* \mathbf{R}_{\beta\lambda\delta\sigma}$$

Let

$$P_\alpha = Q_{\alpha\beta\gamma\delta} e_0^\beta e_0^\gamma e_0^\delta.$$

Then, we have:

$$D^\alpha P_\alpha = 3Q_{\alpha\beta\gamma\delta} \pi^{\alpha\beta} e_0^\gamma e_0^\delta, \quad (9.1)$$

where  $\pi$  is the deformation tensor of  $e_0$ . We introduce the Riemannian metric,

$$h_{\alpha\beta} = g_{\alpha\beta} + 2(e_0)_\alpha (e_0)_\beta \quad (9.2)$$

and use it to define the following space-time norm for tensors  $U$ :

$$|U|^2 = U_{\alpha_1 \dots \alpha_k} U_{\alpha'_1 \dots \alpha'_k} h^{\alpha_1 \alpha'_1} \dots h^{\alpha_k \alpha'_k}.$$

Given two space-time tensors  $U, V$  we denote by  $U \cdot V$  a given contraction between the two tensors and by  $|U \cdot V|$  the norm of the contraction according to the above definition.

Let  $\mathcal{H}$  be a weakly regular null hypersurface with future normal  $L$  such that  $\mathbf{g}(L, T) = -1$ . Integrating (9.1) over a space-time region, bounded by  $\Sigma_0, \Sigma_t$  and  $\mathcal{H}$ , and using well-known properties of the Bel-Robinson tensor, we have:

$$\int_{\Sigma_t} |\mathbf{R}|^2 + \int_{\mathcal{H}} |\mathbf{R} \cdot L|^2 \lesssim \|\mathbf{R}\|_{L^2(\Sigma_0)}^2 + \left| \int_{\mathcal{M}} Q_{\alpha\beta\gamma\delta} \pi^{\alpha\beta} e_0^\gamma e_0^\delta \right| \lesssim \epsilon^2 + \left| \int_{\mathcal{M}} Q_{\alpha\beta\gamma\delta} \pi^{\alpha\beta} e_0^\gamma e_0^\delta \right|.$$

We need to estimate the term in the right-hand side of the previous inequality. Note that since  $\pi_{00} = 0$ ,  $\pi_{0j} = n^{-1} \nabla_j n$ , and  $\pi_{ij} = k_{ij}$ , the bootstrap assumption (5.3) for  $\mathbf{R}$ , and the estimates (6.1) for  $n$  yield:

$$\begin{aligned} \int_{\Sigma_t} |\mathbf{R}|^2 + \int_{\mathcal{H}} |\mathbf{R} \cdot L|^2 &\lesssim \epsilon^2 + \|\nabla n\|_{L^\infty} \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)}^2 + \left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right| \\ &\lesssim \epsilon^2 + (M\epsilon)^3 + \left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right|. \end{aligned}$$

The last term on the right-hand side of the previous inequality is dangerous. Schematically it has the form  $\left| \int_{\mathcal{M}} k \mathbf{R}^2 \right|$ . Typically this term is estimated by:

$$\left| \int_{\mathcal{M}} k \mathbf{R}^2 \right| \lesssim \|k\|_{L_t^2 L^\infty(\Sigma_t)} \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)}^2,$$

requiring a Strichartz estimate for  $k$  which is false even in flat space. It is for this reason that we need the trilinear bootstrap assumption (5.19). Using it we derive,

$$\int_{\Sigma_t} |\mathbf{R}|^2 + \int_{\mathcal{H}} |\mathbf{R} \cdot L|^2 \lesssim \epsilon^2 + M^4 \epsilon^3. \quad (9.3)$$

which, for small  $\epsilon$ , improves the bootstrap assumptions (5.3) and (5.4).



9.2. **Improvement of the bootstrap assumption for  $A_0$ .** Recall (3.31):

$$\Delta A_0 = \mathbf{A}\partial\mathbf{A} + \mathbf{A}^3. \quad (9.4)$$

Using the elliptic estimate (6.7) and the Sobolev embedding (6.4) together with (9.4), we have:

$$\begin{aligned} \|\partial A_0\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial^2 A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} &\lesssim \|\Delta A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ &\lesssim \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)} \|\partial\mathbf{A}\|_{L_t^\infty L^3(\Sigma_t)} + \|A\|_{L_t^\infty L^2(\Sigma_t)}^3 \\ &\lesssim M^2 \epsilon^2 \end{aligned} \quad (9.5)$$

where we used the bootstrap assumptions (5.5) on  $A$  and the Sobolev embedding (6.5) in the last inequality.

Next, using the Sobolev embedding (6.15) together with (9.4), we have:

$$\begin{aligned} \|A_0\|_{L_t^2 L^\infty(\Sigma_t)} &= \|(-\Delta)^{-1}(\mathbf{A}\partial\mathbf{A} + \mathbf{A}^3)\|_{L_t^2 L^\infty(\Sigma_t)} \\ &\lesssim \|\mathbf{A}\partial\mathbf{A}\|_{L_t^2 L^{\frac{14}{9}}(\Sigma_t)} + \|\mathbf{A}^3\|_{L_t^2 L^{\frac{14}{9}}(\Sigma_t)} \\ &\lesssim \|A\|_{L_t^2 L^7(\Sigma_t)} \|\partial A\|_{L_t^\infty L^2(\Sigma_t)} + \|A\|_{L_t^\infty L^6(\Sigma_t)}^3 \\ &\lesssim M^2 \epsilon^2, \end{aligned} \quad (9.6)$$

where we used the bootstrap assumptions (5.5) on  $A$  and the Sobolev embedding (6.5) in the last inequality.

Next, we consider  $\partial_0 A_0$ . In view of (9.4), we have:

$$\begin{aligned} \Delta(\partial_0 A_0) &= \partial_0(\mathbf{A}\partial\mathbf{A}) + \partial_0(\mathbf{A}^3) + [\partial_0, \Delta](A_0) \\ &= \partial_0\mathbf{A}\partial\mathbf{A} + \partial(\mathbf{A}\partial\mathbf{A}) + \mathbf{A}^2\partial\mathbf{A} + [\partial_0, \Delta](A_0). \end{aligned}$$

Together with (C.4), we obtain:

$$\begin{aligned} \Delta(\partial_0 A_0) &= \partial_0\mathbf{A}\partial\mathbf{A} + \partial(\mathbf{A}\partial\mathbf{A}) + \mathbf{A}^2\partial\mathbf{A} + [\partial_0, \Delta](A_0) \\ &= f_1 + \partial f_2, \end{aligned} \quad (9.7)$$

where  $f_1$  is given by:

$$f_1 = \partial_0\mathbf{A}\partial\mathbf{A} + \mathbf{A}^2\partial\mathbf{A},$$

and where  $f_2$  is given by:

$$f_2 = \mathbf{A}\partial\mathbf{A}.$$

In view of the bootstrap assumptions (5.5) for  $A$  and (5.6) for  $A_0$ , we have:

$$\begin{aligned} &\|f_1\|_{L_t^\infty L^1(\Sigma_t)} + \|f_2\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ &\lesssim \|\partial\mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)}^2 + \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)}^3 + \|\mathbf{A}\|_{L_t^\infty L^6(\Sigma_t)} \|\partial\mathbf{A}\|_{L_t^\infty L^2(\Sigma_t)} \\ &\lesssim M^2 \epsilon^2. \end{aligned} \quad (9.8)$$

We will use the following elliptic estimate on  $\Sigma_t$ :

**Lemma 9.1.** *Let  $v$  a scalar function on  $\Sigma_t$  satisfying the following Laplace equation:*

$$\Delta v = f_1 + \partial f_2.$$

*Then, we have the following estimate:*

$$\|v\|_{L^3(\Sigma_t)} + \|\partial v\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|f_1\|_{L^1(\Sigma_t)} + \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}.$$

The proof of Lemma 9.1 requires the use of Littlewood-Paley projections on  $\Sigma_t$  and is postponed to Appendix B. We now come back to the estimate of  $\partial_0 A_0$ . In view of (9.7), Lemma 9.1 and the estimate (9.8), we have:

$$\|\partial_0 A_0\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial \partial_0 A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim M^2 \epsilon^2. \quad (9.9)$$

Finally, (9.5), (9.6) and (9.9) lead to an improvement of the bootstrap assumption (5.6) for  $A_0$ .

**9.3. Improvement of the bootstrap assumption for  $A$ .** Using the estimates for  $\square B_i$  derived in Lemma 7.4, the estimates for  $B$  on the initial slice  $\Sigma_0$  obtained in Lemma 4.3, and the energy estimate (8.5) derived in Lemma 8.3, we have:

$$\|\partial^2 B\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \epsilon + M^2 \epsilon^2. \quad (9.10)$$

Using (9.10) with Lemma 6.5, we obtain:

$$\|\partial A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\partial^2 B\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial E\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \epsilon + M^2 \epsilon^2. \quad (9.11)$$

Next, we estimate  $\partial_0(A)$ . Recall that:

$$\partial_0(A_j) = \partial_j(A_0) + \mathbf{R}_{0j\dots}$$

Thus, we have:

$$\|\partial_0 A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\partial A_0\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)},$$

which together with the improved estimates for  $\mathbf{R}$  and  $A_0$  yields:

$$\|\partial_0 A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \epsilon + (M\epsilon)^{\frac{3}{2}}. \quad (9.12)$$

Finally, (9.11) and (9.12) lead to an improvement of the bootstrap assumption (5.5) for  $A$ .

Finally, (9.3), (9.5), (9.6), (9.9), (9.11) and (9.12) yield the improved estimates (5.20), (5.21), (5.23) and (5.22). This concludes the proof of Proposition 5.7.

## 10. PARAMETRIX FOR THE WAVE EQUATION

Let  $u_{\pm}$  be two families of scalar functions defined on the space-time  $\mathcal{M}$  and indexed by  $\omega \in \mathbb{S}^2$ , satisfying the eikonal equation  $\mathbf{g}^{\alpha\beta} \partial_{\alpha} u_{\pm} \partial_{\beta} u_{\pm} = 0$  for each  $\omega \in \mathbb{S}^2$ . We also denote  ${}^{\omega}u_{\pm}(t, x) = u_{\pm}(t, x, \omega)$ . We have the freedom of choosing  ${}^{\omega}u_{\pm}$  on the initial slice  $\Sigma_0$ , and in order for the results in [42], [44] to apply, we need to initialize  ${}^{\omega}u_{\pm}$  on  $\Sigma_0$  as in [41]. The dependence of  ${}^{\omega}u_{\pm}$  on  $\omega$  is manifested in particular through the requirement that on  $\Sigma_0$  the behavior of  $u_{\pm}$  asymptotically approaches that of  $x \cdot \omega$ .

Let  $\mathcal{H}_{{}^{\omega}u_{\pm}}$  denote the null level hypersurfaces of  ${}^{\omega}u_{\pm}$ . Let  ${}^{\omega}L_{\pm}$  be their null normals, fixed by the condition  $\mathbf{g}({}^{\omega}L_{\pm}, T) = \mp 1$ . Let the vectorfield tangent to  $\Sigma_t$   ${}^{\omega}N_{\pm}$  be defined such as to satisfy:

$${}^{\omega}L_{\pm} = \pm e_0 + {}^{\omega}N_{\pm}.$$

We pick  $({}^{\omega}e_{\pm})_A$ ,  $A = 1, 2$  vectorfields in  $\Sigma_t$  such that together with  ${}^{\omega}N_{\pm}$  we obtain an orthonormal basis of  $\Sigma_t$ . Finally, we denote by  $\nabla_{\pm}$  derivatives in the directions  $({}^{\omega}e_{\pm})_A$ ,  $A = 1, 2$ .

**Remark 10.1.** *Note that from the results in [43] (see Theorem 2.15 and section 3.4 in that paper)  $\mathcal{H}_{{}^{\omega}u_{\pm}}$  satisfy assumptions (5.1) and (5.2).*

We record the following Sobolev embedding/trace type inequality on  $\mathcal{H}_u$  for functions defined on  $\mathcal{M}$ , derived in [43] (see sections 3.5 in that paper).

**Lemma 10.2** (An embedding on  $\mathcal{H}$  [43]). *For any null hypersurface  $\mathcal{H}_u$ , defined as above, and for any  $\Sigma_t$ -tangent tensor  $F$ , we have:*

$$\|F\|_{L^2(\mathcal{H})} \lesssim \|\nabla F\|_{L_t^{\infty} L^{\frac{3}{2}}(\Sigma_t)} + \|F\|_{L_t^{\infty} L^3(\Sigma_t)}, \quad (10.1)$$

and for any  $2 \leq p \leq 4$ :

$$\|F\|_{L^p(\mathcal{H})} \lesssim \|\nabla F\|_{L_t^{\infty} L^2(\Sigma_t)} + \|F\|_{L_t^{\infty} L^2(\Sigma_t)}. \quad (10.2)$$

For any pair of functions  $f_{\pm}$  on  $\mathbb{R}^3$ , we define the following scalar function on  $\mathcal{M}$ :

$$\psi[f_+, f_-](t, x) = \int_{\mathbb{S}^2} \int_0^{\infty} e^{i\lambda {}^{\omega}u_+(t, x)} f_+(\lambda\omega) \lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2} \int_0^{\infty} e^{i\lambda {}^{\omega}u_-(t, x)} f_-(\lambda\omega) \lambda^2 d\lambda d\omega.$$

We appeal to the following result from [42] [44]:

**Theorem 10.3** (Theorem 2.11 in [42] and Theorem 2.17 in [44]). *Let  $\phi_0$  and  $\phi_1$  two scalar functions on  $\Sigma_0$ . Then, there is a unique pair of functions  $(f_+, f_-)$  such that:*

$$\psi[f_+, f_-]|_{\Sigma_0} = \phi_0 \text{ and } \partial_0(\psi[f_+, f_-])|_{\Sigma_0} = \phi_1.$$

Furthermore,  $f_{\pm}$  satisfy the following estimates:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla \phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)},$$

and:

$$\|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla^2 \phi_0\|_{L^2(\Sigma_0)} + \|\nabla \phi_1\|_{L^2(\Sigma_0)}.$$

Finally,  $\square\psi[f_+, f_-]$  satisfies the following estimates:

$$\|\square\psi[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\epsilon(\|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)}),$$

and:

$$\|\partial\square\psi[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\epsilon(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)}).$$

**Remark 10.4.** *The content of Theorem 10.3 is a deep statement about existence of a generalized Fourier transform and its inverse on  $\Sigma_0$ , and existence and accuracy of a parametrix for the scalar wave equation on  $\mathcal{M}$ , with merely  $L^2$  curvature bounds assumptions on the ambient geometry. The existence of  $f_\pm$  and the first two estimates of Theorem 10.3 are proved in [42], while the last two estimates in Theorem 10.3 are proved in [44].*

We associate to any pair of functions  $\phi_0, \phi_1$  on  $\Sigma_0$  the function  $\Psi_{om}[\phi_0, \phi_1]$  defined for  $(t, x) \in \mathcal{M}$  as:

$$\Psi_{om}[\phi_0, \phi_1] = \psi[f_+, f_-]$$

where  $(f_+, f_-)$  is defined in view of Theorem 10.3 as the unique pair of functions associated to  $(\phi_0, \phi_1)$ . In particular, we obtain:

$$\begin{aligned} \|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} &\lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)}, \\ \|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} &\lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)}, \\ \|\square\Psi_{om}[\phi_0, \phi_1]\|_{L^2(\mathcal{M})} &\lesssim M\epsilon(\|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)}), \end{aligned} \quad (10.3)$$

and:

$$\|\partial\square\Psi_{om}[\phi_0, \phi_1]\|_{L^2(\mathcal{M})} \lesssim M\epsilon(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)}). \quad (10.4)$$

Next, let  ${}^{\omega,s}u_\pm$  two families, indexed by  $\omega \in \mathbb{S}^2$  and  $s \in \mathbb{R}^{33}$ , of scalar functions on the space-time  $\mathcal{M}$  satisfying the eikonal equation for each  $\omega \in \mathbb{S}^2$  and  $s \in \mathbb{R}$ . We have the freedom of choosing  ${}^{\omega,s}u_\pm$  on the slice  $\Sigma_s$ , and in order for the results in [42] [44] to apply, we need to initialize  ${}^{\omega,s}u_\pm$  on  $\Sigma_s$  as in [41]. Note that the families  ${}^{\omega}u_\pm$  correspond to  ${}^{\omega,s}u$  with the choice  $s = 0$ . For any pair of functions  $f_\pm$  on  $\mathbb{R}^3$ , and for any  $s \in \mathbb{R}$ , we define the following scalar function on  $\mathcal{M}$ :

$$\psi_s[f_+, f_-](t, x, s) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^{\omega,s}u_+(t,x)} f_+(\lambda\omega) \lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^{\omega,s}u_-(t,x)} f_-(\lambda\omega) \lambda^2 d\lambda d\omega.$$

We have the following straightforward corollary of Theorem 10.3:

**Corollary 10.5.** *Let  $s \in \mathbb{R}$ . Let  $\phi_0$  and  $\phi_1$  two scalar functions on  $\Sigma_s$ . Then, there is a unique pair of functions  $(f_+, f_-)$  such that:*

$$\psi_s[f_+, f_-]|_{\Sigma_s} = \phi_0 \text{ and } \partial_0(\psi_s[f_+, f_-])|_{\Sigma_s} = \phi_1.$$

Furthermore,  $f_\pm$  satisfy the following estimates:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_s)} + \|\phi_1\|_{L^2(\Sigma_s)},$$

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<sup>33</sup>In fact, we need to restrict  $s$  to an open interval of  $[0, 1]$  for which the bootstrap assumptions (5.3) (5.4) on  $\mathbf{R}$  hold

and:

$$\|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla^2 \phi_0\|_{L^2(\Sigma_s)} + \|\nabla \phi_1\|_{L^2(\Sigma_s)}.$$

Finally,  $\square \psi_s[f_+, f_-]$  satisfies the following estimates:

$$\|\square \psi_s[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\epsilon(\|\nabla \phi_0\|_{L^2(\Sigma_s)} + \|\phi_1\|_{L^2(\Sigma_s)}),$$

and:

$$\|\partial \square \psi_s[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\epsilon(\|\nabla^2 \phi_0\|_{L^2(\Sigma_s)} + \|\nabla \phi_1\|_{L^2(\Sigma_s)}).$$

Next, for any  $s \in \mathbb{R}$ , we associate to any function  $F$  on  $\Sigma_s$  the function  $\Psi(t, s)F$  defined for  $(t, x) \in \mathcal{M}$  as:

$$\Psi(t, s)F = \psi_s[f_+, f_-](t)$$

where  $(f_+, f_-)$  is defined in view of Corollary 10.5 as the unique pair of functions associated to the choice  $(\phi_0, \phi_1) = (0, -nF)$ . In particular, we obtain in view of Corollary 10.5 and the control of the lapse  $n$  given by (6.1):

$$\begin{aligned} \|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} &\lesssim \|F\|_{L^2(\Sigma_s)}, \\ \|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} &\lesssim \|\nabla F\|_{L^2(\Sigma_s)}, \\ \|\square \Psi(t, s)F\|_{L^2(\mathcal{M})} &\lesssim M\epsilon\|F\|_{L^2(\Sigma_s)}, \end{aligned} \tag{10.5}$$

and:

$$\|\partial \square \Psi(t, s)F\|_{L^2(\mathcal{M})} \lesssim M\epsilon\|\nabla F\|_{L^2(\Sigma_s)}. \tag{10.6}$$

**Remark 10.6.** Note that we have

$$\square \left( \int_0^t \Psi(t, s)F(s)ds \right) = F(t) + \int_0^t \square \Psi(t, s)F(s)ds.$$

Now, we are in position to construct a parametrix for the wave equation (8.2).

**Theorem 10.7** (Representation formula). *Let  $F$  a scalar function on  $\mathcal{M}$ , and let  $\phi_0$  and  $\phi_1$  two scalar functions on  $\Sigma_0$ . Let  $\phi$  the solution of the wave equation (8.2) on  $\mathcal{M}$ . Then, there is a sequence of scalar functions  $(\phi^{(j)}, F^{(j)})$ ,  $j \geq 0$  on  $\mathcal{M}$ , defined according to:*

$$\phi^{(0)} = \Psi_{om}[\phi_0, \phi_1] + \int_0^t \Psi(t, s)F^{(0)}(s, \cdot)ds, \quad F^{(0)} = F$$

and for all  $j \geq 1$ :

$$\phi^{(j)} = \int_0^t \Psi(t, s)F^{(j)}(s, \cdot)ds, \quad F^{(j)} = -\square \phi^{(j-1)} + F^{(j-1)}$$

such that,

$$\phi = \sum_{j=0}^{+\infty} \phi^{(j)},$$

and such that  $\phi^{(j)}$  and  $F^{(j)}$  satisfy the following estimates:

$$\|\partial \phi^{(j)}\|_{L_t^\infty L^2(\Sigma_t)} + \|F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\epsilon)^j(\|\nabla \phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}),$$

and:

$$\|\partial\boldsymbol{\theta}\phi^{(j)}\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\epsilon)^j (\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}).$$

*Proof.* Let us define:

$$F^{(0)} = F \text{ and } \phi^{(0)} = \Psi_{om}[\phi_0, \phi_1] + \int_0^t \Psi(t, s) F^{(0)}(s, \cdot) ds.$$

Then, we define iteratively for  $j \geq 1$ :

$$F^{(j)} = -\square\phi^{(j-1)} + F^{(j-1)} \text{ and } \phi^{(j)} = \int_0^t \Psi(t, s) F^{(j)}(s, \cdot) ds.$$

In view of Remark 10.6, note that for  $j \geq 1$ :

$$\square\phi^{(j)} = \square \int_0^t \Psi(t, s) F^{(j)}(s, \cdot) ds = F^{(j)} + \int_0^t \square\Psi(t, s) F^{(j)}(s, \cdot) ds,$$

which yields:

$$F^{(j+1)} = -\int_0^t \square\Psi(t, s) F^{(j)}(s, \cdot) ds.$$

Thus, we obtain in view of (10.5) and (10.6):

$$\|F^{(j+1)}\|_{L^2(\mathcal{M})} \lesssim M\epsilon \|F^{(j)}\|_{L^2(\mathcal{M})},$$

and:

$$\|\partial F^{(j+1)}\|_{L^2(\mathcal{M})} \lesssim M\epsilon \|\partial F^{(j)}\|_{L^2(\mathcal{M})}.$$

Therefore, we obtain for all  $j \geq 2$ :

$$\|F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\epsilon)^{j-1} \|F^{(1)}\|_{L^2(\mathcal{M})}, \quad (10.7)$$

and:

$$\|\partial F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\epsilon)^{j-1} \|\partial F^{(1)}\|_{L^2(\mathcal{M})}. \quad (10.8)$$

Also, we have:

$$\square\phi^{(0)} = F^{(0)} + \square\Psi_{om}[\phi_0, \phi_1] + \int_0^t \square\Psi(t, s) F(s, \cdot) ds,$$

This yields:

$$F^{(1)} = -\square\Psi_{om}[\phi_0, \phi_1] - \int_0^t \square\Psi(t, s) F(s, \cdot) ds$$

which together with (10.3), (10.4), (10.5) and (10.6) implies:

$$\|F^{(1)}\|_{L^2(\mathcal{M})} \lesssim M\epsilon (\|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}),$$

and:

$$\|\partial F^{(1)}\|_{L^2(\mathcal{M})} \lesssim M\epsilon (\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}).$$

Together with (10.7) and (10.8), we obtain for any  $j \geq 1$ :

$$\|F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\epsilon)^j (\|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}), \quad (10.9)$$

and:

$$\|\partial F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\epsilon)^j (\|\nabla^2 \phi_0\|_{L^2(\Sigma_0)} + \|\nabla \phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}). \quad (10.10)$$

We now estimate  $\phi^{(j)}$ ,  $j \geq 1$ . For  $j \geq 1$ ,  $\phi^{(j)}$  satisfies the following wave equation:

$$\begin{cases} \square \phi^{(j)} = F^{(j)} - F^{(j+1)}, \\ \phi^{(j)}|_{\Sigma_0} = 0, \partial_0(\phi^{(j)})|_{\Sigma_0} = 0. \end{cases}$$

which together with Lemma 8.3, (10.9) and (10.10) yields:

$$\|\partial \phi^{(j)}\|_{L_t^\infty L^2(\Sigma_t)} \lesssim (M\epsilon)^j (\|\nabla \phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}), \quad (10.11)$$

and:

$$\|\partial(\partial \phi^{(j)})\|_{L_t^\infty L^2(\Sigma_t)} \lesssim (M\epsilon)^j (\|\nabla^2 \phi_0\|_{L^2(\Sigma_0)} + \|\nabla \phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}). \quad (10.12)$$

Now, we have:

$$\square \left( \sum_{j=0}^J \phi^{(j)} \right) = \sum_{j=0}^J (F^{(j)} - F^{(j+1)}) = F - F^{(J+1)},$$

which together with (10.10) and (10.12) yields in the limit  $j \rightarrow +\infty$ :

$$\square \left( \sum_{j=0}^{+\infty} \phi^{(j)} \right) = F.$$

Note also that

$$\phi^{(0)}|_{\Sigma_0} = \phi_0 \text{ and } \partial_0 \phi^{(0)}|_{\Sigma_0} = \phi_1,$$

while for all  $j \geq 1$ , we have:

$$\phi^{(j)}|_{\Sigma_0} = 0 \text{ and } \partial_0 \phi^{(j)}|_{\Sigma_0} = 0.$$

Thus,  $\sum_{j=0}^{+\infty} \phi^{(j)}$  satisfies the wave equation (8.2), and by uniqueness, we have:

$$\phi = \sum_{j=0}^{+\infty} \phi^{(j)}.$$

This concludes the proof of the theorem.  $\square$

## 11. PROOF OF PROPOSITION 5.8 (PART 1)

The goal of this and next sections is to prove Proposition 5.8. This requires the use of the representation formula of Theorem 10.7. In this section we derive the improved bilinear estimate (5.24), (5.25), (5.26), (5.27) and (5.28) of Proposition 5.8. We also derive the improved trilinear estimate (5.33).

**11.1. Improvement of the bilinear bootstrap assumptions I.** We prove the bilinear estimates (5.24), (5.25), (5.26), (5.27), (5.28). These bilinear estimates all involve the  $L^2(\mathcal{M})$  norm of quantities of the type:

$$\mathcal{C}(U, \partial\phi),$$

where  $\mathcal{C}(U, \partial\phi)$  denotes a contraction with respect to one index between a tensor  $U$  and  $\partial\phi$ , for  $\phi$  a solution of the scalar wave equation (8.2) with  $F, \phi_0$  and  $\phi_1$  satisfying the estimate:

$$\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})} \lesssim M\epsilon.$$

In particular, we may use the parametrix constructed in Lemma 10.7 for  $\phi$ :

$$\phi = \sum_{j=0}^{+\infty} \phi^{(j)},$$

with:

$$\phi^{(0)} = \Psi_{om}[\phi_0, \phi_1] + \int_0^t \Psi(t, s)F(s, \cdot)ds,$$

and for all  $j \geq 1$ :

$$\phi^{(j)} = \int_0^t \Psi(t, s)F^{(j)}(s, \cdot)ds.$$

Thus, we need to estimate the norm in  $L^2(\mathcal{M})$  of contractions of quantities of the type:

$$\mathcal{C}(U, \partial(\Psi_{om}[\phi_0, \phi_1])) + \sum_{j=0}^{+\infty} \int_0^t \mathcal{C}(U, \partial(\Psi(t, s)F^{(j)}(s, \cdot)))ds.$$

After using the definition of  $\Psi_{om}$  and  $\Psi(t, s)$ , and the estimates for  $F^{(j)}$  provided by Lemma 10.7, this reduces to estimating:

$$\int_{\mathbb{S}^2} \int_0^\infty \mathcal{C}(U, \partial(e^{i\lambda \omega u_+(t,x)}))f_+(\lambda\omega)\lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2} \int_0^\infty \mathcal{C}(U, \partial(e^{i\lambda \omega u_-(t,x)}))f_-(\lambda\omega)\lambda^2 d\lambda d\omega,$$

where  $f_\pm$  in view of Theorem 10.3 and the estimates for  $F, \phi_0$  and  $\phi_1$  satisfies:

$$\|\lambda^2 f_\pm\|_{L^2(\mathbb{R}^3)} \lesssim M\epsilon.$$

Since both half-wave parametrices are estimated in the same way, the bilinear estimates (5.9), (5.10), (5.11), (5.12) and (5.13) all estimate the norm in  $L^2(\mathcal{M})$  of contractions of quantities of the type:

$$\int_{\mathbb{S}^2} \int_0^\infty \mathcal{C}(U, \partial(e^{i\lambda \omega u(t,x)}))f(\lambda\omega)\lambda^2 d\lambda d\omega,$$

where  $f$  satisfies:

$$\|\lambda^2 f\|_{L^2(\mathbb{R}^3)} \lesssim M\epsilon. \tag{11.1}$$

Now, we have:

$$\partial_j(e^{i\lambda \omega u}) = i\lambda e^{i\lambda \omega u} \partial_j(\omega u),$$



and the gradient of  ${}^\omega u$  on  $\Sigma_t$  is given by:

$$\nabla({}^\omega u) = {}^\omega b^{-1} {}^\omega N,$$

where  ${}^\omega b = |\nabla({}^\omega u)|^{-1}$  is the null lapse, and

$${}^\omega N = \frac{\nabla {}^\omega u}{|\nabla {}^\omega u|}$$

is the unit normal to  $\mathcal{H}_{\omega_u} \cap \Sigma_t$  along  $\Sigma_t$ . Thus, the bilinear estimates (5.9), (5.10), (5.11), (5.12) and (5.13) all reduce to  $L^2(\mathcal{M})$ -estimates of expressions of the form:

$$\mathfrak{C}[U, f] := \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^\omega u(t,x)} {}^\omega b^{-1} \mathcal{C}(U, {}^\omega N) f(\lambda\omega) \lambda^3 d\lambda d\omega, \quad (11.2)$$

where  $f$  satisfies (11.1).

To estimate  $\mathfrak{C}[U, f]$  we follow the strategy of [21].

$$\begin{aligned} \|\mathfrak{C}[U, f]\|_{L^2(\mathcal{M})} &\lesssim \int_{\mathbb{S}^2} \left\| {}^\omega b^{-1} \mathcal{C}(U, {}^\omega N) \left( \int_0^{+\infty} e^{i\lambda {}^\omega u(t,x)} f(\lambda\omega) \lambda^3 d\lambda \right) \right\|_{L^2(\mathcal{M})} d\omega & (11.3) \\ &\lesssim \int_{\mathbb{S}^2} \| {}^\omega b^{-1} \|_{L^\infty(\mathcal{M})} \| \mathcal{C}(U, {}^\omega N) \|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})} \left\| \int_0^{+\infty} e^{i\lambda {}^\omega u(t,x)} f(\lambda\omega) \lambda^3 d\lambda \right\|_{L^2_{\omega_u}} d\omega \\ &\lesssim \left( \sup_{\omega \in \mathbb{S}^2} \| {}^\omega b^{-1} \|_{L^\infty(\mathcal{M})} \right) \left( \sup_{\omega \in \mathbb{S}^2} \| \mathcal{C}(U, {}^\omega N) \|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})} \right) \left( \int_{\mathbb{S}^2} \| \lambda^3 f(\lambda\omega) \|_{L^2_\lambda} d\omega \right) \\ &\lesssim \left( \sup_{\omega \in \mathbb{S}^2} \| {}^\omega b^{-1} \|_{L^\infty(\mathcal{M})} \right) \left( \sup_{\omega \in \mathbb{S}^2} \| \mathcal{C}(U, {}^\omega N) \|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})} \right) \| \lambda^2 f \|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

where we used Plancherel in  $\lambda$  and Cauchy Schwarz in  $\omega$ . Now, since  ${}^\omega u$  has been initialized on  $\Sigma_0$  as in [41], and satisfies the eikonal equation on  $\mathcal{M}$ , the results in [43] (see section 4.8 in that paper) under the assumption of Theorem 2.10 imply:

$$\sup_{\omega \in \mathbb{S}^2} \| {}^\omega b^{-1} \|_{L^\infty(\mathcal{M})} \lesssim 1.$$

Together with the fact that  $f$  satisfies (11.1), and with (11.3), we finally obtain:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^\omega u(t,x)} {}^\omega b^{-1} \mathcal{C}(U, {}^\omega N) f(\lambda\omega) \lambda^3 d\lambda d\omega \right\|_{L^2(\mathcal{M})} & (11.4) \\ &\lesssim M \epsilon \left( \sup_{\omega \in \mathbb{S}^2} \| \mathcal{C}(U, {}^\omega N) \|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})} \right). \end{aligned}$$

It remains to estimate the right-hand side of (11.4) for the contractions appearing in the bilinear estimates (5.24), (5.25), (5.26), (5.27) and (5.28). Since all the estimates in the proof are uniform in  $\omega$ , we drop the index  $\omega$  to ease the notations.

**Remark 11.1.** *In the proof of bilinear estimates (5.24), (5.25), (5.26), (5.27) and (5.28), the tensor  $U$  appearing in the expression  $\mathcal{C}(U, N)$  is either  $\mathbf{R}$  or derivatives of solutions  $\phi$  of a scalar wave equation. In view of the bootstrap assumption (5.4) for the curvature flux, as well as the energy estimate for the wave equation in Lemma 8.1, we can control  $\|\mathcal{C}(U, N)\|_{L_u^\infty L^2(\mathcal{H}_u)}$  as long as we can show that  $\mathcal{C}(U, N)$  can be expressed in terms of,*

$$\mathbf{R} \cdot L, \nabla \phi \text{ and } L(\phi).$$

*In other words, our goal is to check that the term  $\mathcal{C}(U, N)$  does not involve the dangerous terms of the type:*

$$\underline{\alpha} \text{ and } \underline{L}\phi$$

*where  $\underline{L}$  is the vectorfield defined as  $\underline{L} = 2T - L$ , and  $\underline{\alpha}$  is the two tensor on  $\Sigma_t \cap \mathcal{H}_u$  defined as:*

$$\underline{\alpha}_{AB} = \mathbf{R}_{\underline{L}A \underline{L}B}.$$

11.1.1. *Proof of (5.24).* Since  $A = \text{curl}(B) + E$  in view of Lemma 6.5, we have:

$$\begin{aligned} \|A^j \partial_j(A)\|_{L^2(\mathcal{M})} &\lesssim \|(\text{curl}(B))^j \partial_j(A)\|_{L^2(\mathcal{M})} + \|E\|_{L_t^2 L^\infty(\Sigma_t)} \|\partial A\|_{L_t^\infty L^2(\Sigma_t)} \quad (11.5) \\ &\lesssim \|(\text{curl}(B))^j \partial_j(A)\|_{L^2(\mathcal{M})} + M^2 \epsilon^2, \end{aligned}$$

where we used in the last inequality Lemma 6.5 for  $E$ , and the bootstrap assumption (5.5) for  $A$ . Next, we estimate  $\|(\text{curl}(B))^j \partial_j(A)\|_{L^2(\mathcal{M})}$ . Recall that we have:

$$(\text{curl}(B))^j \partial_j(A) = \epsilon_{jmn} \partial_m(B_n) \partial_j(A).$$

We are now ready to apply the representation theorem 10.7 to  $B$ . Indeed, according to Lemma 7.4, and proposition 6.4, we have

$$\begin{aligned} \square B = F, \quad \|\partial F\|_{L^2(\mathcal{M})} &\lesssim M^2 \epsilon^2 \quad (11.6) \\ \|\partial B(0)\|_{L^2(\Sigma_0)} + \|\partial^2 B(0)\|_{L^2(\Sigma_0)} + \|\partial(\partial_0 B(0))\|_{L^2(\Sigma_0)} &\lesssim M\epsilon. \end{aligned}$$

We are thus in a position to apply the reduction discussed in the subsection above and reduce our desired bilinear estimate to an estimate for,

$$\mathcal{C}(U, N) = \epsilon_{jm} N_m \partial_j(A)$$

Now, we decompose  $\partial_j$  on the orthonormal frame  $N, f_A, A = 1, 2$  of  $\Sigma_t$ , where we recall that  $f_A, A = 1, 2$  denotes an orthonormal basis of  $\mathcal{H}_u \cap \Sigma_t$ . We have schematically:

$$\partial_j = N_j N + \nabla, \quad (11.7)$$

where  $\nabla$  denotes derivatives which are tangent to  $\mathcal{H}_u \cap \Sigma_t$ . Thus, we have:

$$\mathcal{C}(U, N) = \epsilon_{jm} N_m N_j \partial_N(A) + \nabla(A) = \nabla(A),$$

where we have used the antisymmetry of  $\epsilon_{jm}$  in the last equality. Therefore, we obtain in this case:

$$\|\mathcal{C}(U, N)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim \|\nabla(A)\|_{L_u^\infty L^2(\mathcal{H}_u)}.$$

It remains to estimate  $\|\nabla(A)\|_{L_u^\infty L^2(\mathcal{H}_u)}$ . Since we have  $A = \text{curl}(B) + E$  in view of Lemma 6.5, we obtain:

$$\begin{aligned} \|\nabla(A)\|_{L_u^\infty L^2(\mathcal{H}_u)} &\lesssim \|\nabla(\partial B)\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\nabla(E)\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim \|\nabla(\partial B)\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|\partial E\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial^2 E\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \\ &\lesssim \|\nabla(\partial B)\|_{L_u^\infty L^2(\mathcal{H}_u)} + M\epsilon, \\ &\lesssim M\epsilon \end{aligned}$$

where we used the embedding (10.1) and the estimates for  $E$  given by Lemma 6.5. Furthermore, we have in view of Proposition 7.4 and Lemma 8.3 the following estimate for  $B$ :

$$\|\nabla(\partial B)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim M\epsilon.$$

We finally obtain:

$$\|\nabla(A)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim M\epsilon.$$

This improves the bilinear estimate (5.9).

11.1.2. *Proof of (5.25).* In view of Lemma 6.5  $A = \text{curl}(B) + E$ . Arguing as in (11.5), we reduce the proof to the estimate of:

$$\|(\text{curl } B)^j \partial_j(\partial B)\|_{L^2(\mathcal{M})}.$$

Since  $B$  satisfies the wave equation (11.6), the quantity  $\mathcal{C}(U, N)$  is in this case,

$$\mathcal{C}(U, N) = \epsilon_{jm} \cdot N_m \partial_j(\partial B).$$

Using the decomposition of  $\partial_j$  (11.7) and the antisymmetry of  $\epsilon_{jm}$ , we have schematically:

$$\begin{aligned} \epsilon_{jm} \cdot N_m \partial_j(\partial B) &= \epsilon_{jm} \cdot N_m N_j \partial_N(\partial B) + \nabla(\partial B) \\ &= \nabla(\partial B) \\ &= \nabla(\partial B) + \nabla(\partial_0 B) \\ &= \nabla(\partial B) + L(\partial B) + (\mathbf{D}L + \mathbf{D}N + A)\partial(B), \end{aligned} \tag{11.8}$$

where in the last equality we used the decomposition (8.11) for  $\nabla(\partial_0(B))$ . Together with the assumptions (5.1) and (5.2) on  $\mathcal{H}_u$ , and the Sobolev embedding (10.2) on  $\mathcal{H}_u$ , we obtain:

$$\begin{aligned} &\|\epsilon_{jm} \cdot N_m \partial_j(\partial B)\|_{L^2(\mathcal{H}_u)} \\ &\lesssim \|\nabla(\partial B)\|_{L^2(\mathcal{H}_u)} + \|L(\partial B)\|_{L^2(\mathcal{H}_u)} \\ &\quad + (\|\mathbf{D}N\|_{L^3(\mathcal{H}_u)} + \|\mathbf{D}L\|_{L^3(\mathcal{H}_u)} + \|A\|_{L^3(\mathcal{H}_u)}) \|\partial B\|_{L^6(\mathcal{H}_u)} \\ &\lesssim (1 + \|A\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial A\|_{L_t^\infty L^2(\Sigma_t)}) (\|\nabla(\partial B)\|_{L^2(\mathcal{H}_u)} + \|L(\partial B)\|_{L^2(\mathcal{H}_u)}) \\ &\lesssim \|\nabla(\partial B)\|_{L^2(\mathcal{H}_u)} + \|L(\partial B)\|_{L^2(\mathcal{H}_u)}, \end{aligned}$$

where we used the bootstrap assumption (5.5) for  $A$  in the last inequality. Now, we have in view of Lemma 7.4 and Lemma 8.3 the following estimate for  $B$ :

$$\|\nabla(\partial B)\|_{L_u^\infty L^2(\mathcal{H}_u)} + \|L(\partial B)\|_{L_u^\infty L^2(\mathcal{H}_u)} \lesssim M\epsilon,$$

which improves the bilinear estimate (5.10).

11.1.3. *Proof of (5.26).* Since  $B$  satisfies a wave equation in view of Lemma 7.4, the quantity  $\mathcal{C}(U, N)$  is in this case:

$$N_j \mathbf{R}_{0j\dots} = \mathbf{R}_{0N\dots}$$

Thus, using the fact that  $L = T + N$ ,  $\underline{L} = T - N$  and the symmetries of  $\mathbf{R}$ , we deduce:

$$N_j \mathbf{R}_{0j\dots} = \frac{1}{2} \mathbf{R}_{L\underline{L}\dots}$$

which together with the bootstrap assumption for the curvature flux (5.4) improves the bilinear estimate (5.11).

11.1.4. *Proof of (5.27).* We have  $k_{j\dots} = A^j$  and  $A = \text{curl}(B) + E$  in view of Lemma 6.5. Arguing as in (11.5), we reduce the proof to the estimate of:

$$\|(\text{curl } B)^j \partial_j \phi\|_{L^2(\mathcal{M})}.$$

Since  $B$  satisfies a wave equation in view of Lemma 7.4, the quantity  $\mathcal{C}(U, N)$  is in this case:

$$\in_{jm\dots} N_m \partial_j \phi.$$

Using the decomposition (11.7) for  $\partial_j$  and the antisymmetry of  $\in_{jm\dots}$ , we obtain schematically:

$$\in_{jm\dots} N_m \partial_j \phi = \in_{jm\dots} N_m N_j \partial_N \phi + \nabla \phi = \nabla \phi,$$

which improves the bilinear estimate (5.12).

11.1.5. *Proof of (5.28).* We have  $A = \text{curl}(B) + E$  in view of Lemma 6.5. Arguing as in (11.5), we reduce the proof to the estimate of:

$$\|(\text{curl } B)^j \partial_j \phi\|_{L^2(\mathcal{M})}.$$

Since  $B$  satisfies a wave equation in view of Lemma 7.4, the quantity  $\mathcal{C}(U, N)$  is in this case:

$$\in_{jm\dots} N_m \partial_j \phi = \nabla \phi.$$

Using again the decomposition (11.7) for  $\partial_j$  and the antisymmetry of  $\in_{jm\dots}$ , we obtain schematically:

$$\in_{jm\dots} N_m \partial_j \phi = \in_{jm\dots} N_m N_j \partial_N \phi + \nabla \phi = \nabla \phi,$$

which improves the bilinear estimate (5.13).

**11.2. Improvement of the trilinear estimate.** In this section, we shall derive the improved trilinear estimate (5.33). Let  $Q_{\alpha\beta\gamma\delta}$  the Bell-Robinson tensor of  $\mathbf{R}$ :

$$Q_{\alpha\beta\gamma\delta} = \mathbf{R}_\alpha{}^\lambda \gamma^\sigma \mathbf{R}_{\beta\lambda\delta\sigma} + {}^* \mathbf{R}_\alpha{}^\lambda \gamma^\sigma {}^* \mathbf{R}_{\beta\lambda\delta\sigma} \quad (11.9)$$

We need an trilinear estimate for the following quantity

$$\left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right|.$$

We have  $A = \text{curl}(B) + E$  by Lemma 6.5. Arguing as in (11.5), we reduce the proof to the estimate of:

$$\left| \int_{\mathcal{M}} Q_{\cdot j\gamma\delta} (\text{curl}(B))_j e_0^\gamma e_0^\delta \right|.$$

Making use of the wave equation (11.6) for  $B$  we argue as in the beginning of section 11.1 to reduce the proof to an estimate of the following:

$$\left| \int_{\mathcal{M}} \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} \omega b^{-1} (\in_{jm} \cdot {}^\omega N_m Q_j \dots) f(\lambda\omega) \lambda^3 d\lambda d\omega d\mathcal{M} \right|$$

where  $f$  satisfies:

$$\|\lambda^2 f\|_{L^2(\mathbb{R}^3)} \lesssim M\epsilon.$$

Arguing as in (11.3) (11.4), we obtain:

$$\begin{aligned} & \left| \int_{\mathcal{M}} \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} \omega b^{-1} (\in_{jm} \cdot {}^\omega N_m Q_j \dots) f(\lambda\omega) \lambda^3 d\lambda d\omega d\mathcal{M} \right| \\ & \lesssim \int_{\mathbb{S}^2} \left\| \omega b^{-1} (\in_{jm} \cdot {}^\omega N_m Q_j \dots) \left( \int_0^{+\infty} e^{i\lambda \omega u(t,x)} f(\lambda\omega) \lambda^3 d\lambda \right) \right\|_{L^1(\mathcal{M})} d\omega \\ & \lesssim \int_{\mathbb{S}^2} \|\omega b^{-1}\|_{L^\infty(\mathcal{M})} \|\in_{jm} \cdot {}^\omega N_m Q_j \dots\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} \left\| \int_0^{+\infty} e^{i\lambda \omega u(t,x)} f(\lambda\omega) \lambda^3 d\lambda \right\|_{L^2_{\omega_u}} d\omega \\ & \lesssim \left( \sup_{\omega \in \mathbb{S}^2} \|\omega b^{-1}\|_{L^\infty(\mathcal{M})} \right) \left( \sup_{\omega \in \mathbb{S}^2} \|\in_{jm} \cdot {}^\omega N_m Q_j \dots\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} \right) \left( \int_{\mathbb{S}^2} \|\lambda^3 f(\lambda\omega)\|_{L^2_\lambda} d\omega \right) \\ & \lesssim \sup_{\omega \in \mathbb{S}^2} \|\in_{jm} \cdot {}^\omega N_m Q_j \dots\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} M\epsilon, \end{aligned}$$

where we used Plancherel in  $\lambda$  and Cauchy Schwarz in  $\omega$ . Thus, we finally obtain:

$$\left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right| \lesssim \sup_{\omega \in \mathbb{S}^2} \|\in_{jm} \cdot N_m Q_j \dots\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} M\epsilon + M^3 \epsilon^3. \quad (11.10)$$

Next, we estimate the right-hand side of (11.10). Since all the estimates in the proof will be uniform in  $\omega$ , we drop the index  $\omega$  to ease the notations. The formula for the Bell-Robinson tensor  $Q$  yields:

$$\begin{aligned} Q_{j\dots} &= \mathbf{R}_j{}^\lambda \cdot \cdot \mathbf{R}_{\cdot\lambda\dots} + \text{dual} \\ &= -\frac{1}{2} \mathbf{R}_{jL\dots} \mathbf{R}_{\cdot L\dots} - \frac{1}{2} \mathbf{R}_{jL\dots} \mathbf{R}_{\cdot L\dots} + \mathbf{R}_{jA\dots} \mathbf{R}_{\cdot A\dots} + \text{dual}, \end{aligned}$$

where we used the frame  $L, \underline{L}, f_A, A = 1, 2$  in the last equality. Thus, we have schematically:

$$\in_{jm}. N_m Q_{j\dots} = \mathbf{R}(\mathbf{R} \cdot L + \in_{jm}. N_m \mathbf{R}_{jA\dots})$$

Decomposing  $e_j$  with respect to the orthonormal frame  $N, f_B, B = 1, 2$ , we note that:

$$\in_{jm}. N_m \mathbf{R}_{jA\dots} = \in_{jm}. N_j N_m \mathbf{R}_{NA\dots} + \in_{jm}. (f_B)_j N_m \mathbf{R}_{BA\dots} = \mathbf{R}_{BA\dots}$$

On the other hand, decomposing  $\mathbf{R}_{BA\dots}$  further and using the symmetries of  $\mathbf{R}$ , one easily checks that  $\mathbf{R}_{BA\dots}$  must contain at least one  $L$  so that it is of the type  $\mathbf{R} \cdot \mathbf{L}$ . Thus, we have schematically:

$$\in_{jm}. N_m Q_{j\dots} = \mathbf{R}(\mathbf{R} \cdot L). \quad (11.11)$$

Thus, in view of (11.10), making use of the bootstrap assumptions (5.3) on  $R$  and (5.4) on the curvature flux, we deduce,

$$\begin{aligned} \left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right| &\lesssim (M\epsilon)^3 + M\epsilon \|\mathbf{R}\mathbf{R}_L\|_{L_u^2 L^1(\mathcal{H}_u)} \\ &\lesssim (M\epsilon)^3 + M\epsilon \|\mathbf{R}\|_{L^2(\mathcal{M})} \|\mathbf{R}_L\|_{L_u^\infty L^2(\mathcal{H}_u)} \\ &\lesssim M^3 \epsilon^3 \end{aligned}$$

In other words,

$$\left| \int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^\gamma e_0^\delta \right| \lesssim (M\epsilon)^3. \quad (11.12)$$

which yields the desired improvement of the trilinear estimate (5.19).

## 12. PROOF OF PROPOSITION 5.8, (PART 2)

In this section we prove the bilinear estimates II. We start with a discussion of the sharp  $L^4$  Strichartz estimate.

**12.1. The sharp  $L^4(\mathcal{M})$  Strichartz estimate.** To a function  $f$  on  $\mathbb{R}^3$  and a family  ${}^\omega u$  indexed by  $\omega \in \mathbb{S}^2$  of scalar functions on the space-time  $\mathcal{M}$  satisfying the eikonal equation for each  $\omega \in \mathbb{S}^2$ , we associate a half-wave parametrix:

$$\int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^\omega u(t,x)} f(\lambda\omega) \lambda^2 d\lambda d\omega.$$

Let  $p$  be an integer  $p$  and  $\psi$  be a smooth cut-off function on supported on the interval  $[1/2, 2]$ . We call a half-wave wave parametrix localized at frequencies of size  $\lambda \sim 2^p$  the following Fourier integral operator:

$$\int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^\omega u(t,x)} \psi(2^{-p}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega.$$

We have the following  $L^4(\mathcal{M})$  Strichartz estimates localized in frequency for a half wave parametrix which are proved in [45]:

**Proposition 12.1** (Corollary 2.8 in [45]). *Let  $f$  be a function on  $\mathbb{R}^3$ , let  $p \in \mathbb{N}$ , and let  $\psi$  be as defined above. Let  ${}^\omega u$  be a family of scalar functions on the space-time  $\mathcal{M}$  satisfying the eikonal equation for each  $\omega \in \mathbb{S}^2$  and initialized on the initial slice  $\Sigma_0$  as in [41]. Define a scalar function  $\phi_p$  on  $\mathcal{M}$  as the following oscillatory integral:*

$$\phi_p(t, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^\omega u(t, x)} \psi(2^{-p}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega.$$

Then, we have the following  $L^4(\mathcal{M})$  Strichartz estimates for  $\phi_p$ :

$$\|\phi_p\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{p}{2}} \|\psi(2^{-p}\lambda) f\|_{L^2(\mathbb{R}^3)}, \quad (12.1)$$

$$\|\partial\phi_p\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{3p}{2}} \|\psi(2^{-p}\lambda) f\|_{L^2(\mathbb{R}^3)}, \quad (12.2)$$

$$\|\partial^2\phi_p\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{5p}{2}} \|\psi(2^{-p}\lambda) f\|_{L^2(\mathbb{R}^3)}. \quad (12.3)$$

Note that these Strichartz estimates are sharp.

**12.2. Improvement of the non sharp Strichartz estimates.** In this section, we derive the improved non sharp Strichartz estimates (5.31) and (5.32). In order to do this, we first estimate the  $L_t^2 L^7(\Sigma_t)$  norm of  $\partial B$  using the  $L^4(\mathcal{M})$  Strichartz estimate together with Sobolev embeddings on  $\Sigma_t$ .

**Corollary 12.2.**  *$B$  satisfies the following Strichartz estimate:*

$$\|\partial B\|_{L_t^2 L^7(\Sigma_t)} \lesssim M\epsilon.$$

*Proof.* Decompose  $B$  as before, with the help of Theorem 10.7,

$$\|\partial B\|_{L_t^2 L^7(\Sigma_t)} \leq \sum_{j=0}^{+\infty} \|\partial\phi^{(j)}\|_{L_t^2 L^7(\Sigma_t)}. \quad (12.4)$$

Thus it suffices to prove for all  $j \geq 0$ :

$$\|\partial\phi^{(j)}\|_{L_t^2 L^7(\Sigma_t)} \lesssim (M\epsilon)^{j+1}. \quad (12.5)$$

The estimates in (12.5) are analogous for all  $j$ , so it suffices to prove (12.5) in the case  $j = 0$ . In view of the definition of  $\phi^{(0)}$ , the estimates for  $B$  on the initial slice  $\Sigma_0$  obtained in Lemma 4.3, the estimate (7.20) for  $\partial\Box B$ , and the definition of  $\Psi_{om}$  and  $\Psi(t, s)$ , (12.5) reduces to the following estimate for a half wave parametrix:

$$\left\| \partial \left( \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda {}^\omega u(t, x)} f(\lambda\omega) \lambda^2 d\lambda d\omega \right) \right\|_{L_t^2 L^7(\Sigma_t)} \lesssim \|\lambda^2 f\|_{L^2(\mathbb{R}^3)}. \quad (12.6)$$

Next, we introduce  $\varphi$  and  $\psi$  two smooth compactly supported functions on  $\mathbb{R}^+$  such that  $\psi$  is supported away from 0 and:

$$\varphi(\lambda) + \sum_{p \geq 0} \psi(2^{-p}\lambda) = 1 \text{ for all } \lambda \in \mathbb{R}. \quad (12.7)$$

We define the family of scalar functions  $\phi_p$  for  $p \geq -1$  on  $\mathcal{M}$  as:

$$\phi_{-1}(t, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} \varphi(\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega, \quad (12.8)$$

and for all  $p \geq 0$ :

$$\phi_p(t, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} \psi(2^{-p}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega. \quad (12.9)$$

In view of (12.7), we have:

$$\partial \left( \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} f(\lambda\omega) \lambda^2 d\lambda d\omega \right) = \sum_{p \geq -1} \partial \phi_p(t, x),$$

which yields:

$$\left\| \partial \left( \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} f(\lambda\omega) \lambda^2 d\lambda d\omega \right) \right\|_{L_t^2 L^7(\Sigma_t)} \lesssim \sum_{p \geq -1} \|\partial \phi_p\|_{L_t^2 L^7(\Sigma_t)}. \quad (12.10)$$

The estimate for  $\phi_{-1}$  is easier, so we focus on  $\phi_p$  for  $p \geq 0$ . Using the Sobolev embedding (6.6) on  $\Sigma_t$ , the  $L^4(\mathcal{M})$  Strichartz localized in frequency of Proposition 12.1, and the fact that  $\psi$  is supported in  $(0, +\infty)$ , we have:

$$\begin{aligned} \|\partial \phi_p\|_{L_t^2 L^7(\Sigma_t)} &\lesssim \|\partial \phi_p\|_{L^4(\mathcal{M})}^{\frac{4}{7}} \|\partial \phi_p\|_{L_t^4 L^\infty(\Sigma_t)}^{\frac{3}{7}} \\ &\lesssim \|\partial \phi_p\|_{L^4(\mathcal{M})}^{\frac{4}{7}} \|\partial^2 \phi_p\|_{L^4(\mathcal{M})}^{\frac{3}{7}} \\ &\lesssim \left( 2^{\frac{3p}{2}} \|\psi(2^{-p}\lambda) f\|_{L^2(\mathbb{R}^3)} \right)^{\frac{4}{7}} \left( 2^{\frac{5p}{2}} \|\psi(2^{-p}\lambda) f\|_{L^2(\mathbb{R}^3)} \right)^{\frac{3}{7}} \\ &\lesssim 2^{-\frac{p}{14}} \|\lambda^2 \psi(2^{-p}\lambda) f\|_{L^2(\mathbb{R}^3)} \\ &\lesssim 2^{-\frac{p}{14}} \|\lambda^2 f\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Together with (12.10), we obtain:

$$\left\| \partial \left( \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} f(\lambda\omega) \lambda^2 d\lambda d\omega \right) \right\|_{L_t^2 L^7(\Sigma_t)} \lesssim \left( \sum_{p \geq -1} 2^{-\frac{p}{14}} \right) \|\lambda^2 f\|_{L^2(\mathbb{R}^3)} \lesssim \|\lambda^2 f\|_{L^2(\mathbb{R}^3)},$$

which is (12.6). This concludes the proof of the Corollary.  $\square$

Lemma 6.5 and Corollary 12.2 yield:

$$\|A\|_{L_t^2 L^7(\Sigma_t)} \lesssim \|\partial B\|_{L_t^2 L^7(\Sigma_t)} + \|E\|_{L_t^2 L^7(\Sigma_t)} \lesssim M\epsilon + M^2\epsilon^2, \quad (12.11)$$

which is an improvement on the bootstrap assumption (5.16).



**12.3. Improvement of the bilinear bootstrap assumptions II.** In this section, we derive the improved bilinear estimate (5.29) and (5.30) of Proposition 5.8. Recall the decomposition  $A = \text{curl}(B) + E$  of Lemma 6.5. Using the bootstrap assumption 5.5 for  $A$ , the estimates for  $E$  given by Lemma 6.5 and the Sobolev embedding on  $\Sigma_t$  (6.5), we have:

$$\begin{aligned} & \|(-\Delta)^{-\frac{1}{2}}(\partial A \partial E)\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}(\partial E \partial E)\|_{L^2(\mathcal{M})} \\ \lesssim & \|\partial A \partial E\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} + \|\partial E \partial E\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\ \lesssim & \|\partial A\|_{L_t^\infty L^2(\Sigma_t)} \|\partial E\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial E\|_{L_t^\infty L^3(\Sigma_t)}^2 \\ \lesssim & M^2 \epsilon^2. \end{aligned}$$

Together with the decomposition  $A = \text{curl}(B) + E$  of Lemma 6.5, this implies that the proof of the bilinear estimates (5.14) and (5.15) reduces to:

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\text{curl}(B), \text{curl}(B)))\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}(\partial(\text{curl}(B)^l) \partial_l(\text{curl}(B)))\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2,$$

where the bilinear form  $Q_{ij}$  is given by  $Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_j \phi \partial_i \psi$ . Also, note that:

$$\partial(\text{curl}(B)^l) \partial_l(\text{curl}(B)) = Q_{ij}(\partial B, \partial B) + A \partial B \partial^2 B.$$

Together with the Sobolev embedding (4.7) on  $\Sigma_t$ , the bootstrap assumptions (5.5) on  $A$ , and the estimates of Lemma 6.4 for  $B$ , we obtain:

$$\begin{aligned} & \|(-\Delta)^{-\frac{1}{2}}(\partial(\text{curl}(B)^l) \partial_l(\text{curl}(B)))\|_{L^2(\mathcal{M})} \\ \lesssim & \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} + \|(-\Delta)^{-\frac{1}{2}}(A \partial B \partial^2 B)\|_{L^2(\mathcal{M})} \\ \lesssim & \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} + \|A \partial B \partial^2 B\|_{L_t^\infty L^{\frac{6}{5}}(\Sigma_t)} \\ \lesssim & \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} + \|A\|_{L_t^\infty L^6(\Sigma_t)} \|\partial B\|_{L_t^\infty L^6(\Sigma_t)} \|\partial^2 B\|_{L_t^\infty L^2(\Sigma_t)} \\ \lesssim & \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} + M^2 \epsilon^2. \end{aligned}$$

Finally, the proof of the bilinear estimates (5.14) and (5.15) reduces to:

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} \lesssim M^2 \epsilon^2. \quad (12.12)$$

Next, we focus on proving (12.12). Decomposing  $B$  according to Theorem 10.7, we have:

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} \leq \sum_{m,n=0}^{+\infty} \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial \phi^{(m)}, \partial \phi^{(n)}))\|_{L^2(\mathcal{M})}. \quad (12.13)$$

Thus it suffices to prove for all  $m, n \geq 0$ :

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial \phi^{(m)}, \partial \phi^{(n)}))\|_{L^2(\mathcal{M})} \lesssim (M\epsilon)^{m+1} (M\epsilon)^{n+1}. \quad (12.14)$$

The estimates in (12.14) are analogous for all  $m, n$ , so it suffices to prove (12.14) in the case  $(m, n) = (0, 0)$ . In view of the definition of  $\phi^{(0)}$ , the estimates for  $B$  on the initial

slice  $\Sigma_0$  obtained in Lemma 4.3, estimate (7.20) for  $\partial\Box B$ , (12.14) reduces to the following bilinear estimate for half-wave parametrices:

$$\left\| (-\Delta)^{-\frac{1}{2}} Q_{ij}(\partial\phi^{(1)}, \partial\phi^{(2)}) \right\|_{L^2(\mathcal{M})} \lesssim \|\lambda^2 f_1\|_{L^2(\mathbb{R}^3)} \|\lambda^2 f_2\|_{L^2(\mathbb{R}^3)}. \quad (12.15)$$

with,

$$\phi^{(k)} = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} f_k(\lambda\omega) \lambda^2 d\lambda d\omega, \quad k = 1, 2. \quad (12.16)$$

Recall the smooth cut off functions  $\varphi$  and  $\psi$  introduced in the proof of Corollary 12.2. We define two families of scalar functions  $\phi_p^j$ ,  $j = 1, 2$  for  $p \geq -1$  on  $\mathcal{M}$  as:

$$\phi_{-1}^{(j)}(t, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} \varphi(\lambda) f_j(\lambda\omega) \lambda^2 d\lambda d\omega, \quad (12.17)$$

and for all  $p \geq 0$ :

$$\phi_p^{(j)}(t, x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} \psi(2^{-p}\lambda) f_j(\lambda\omega) \lambda^2 d\lambda d\omega. \quad (12.18)$$

In view of (12.7), we have:

$$\partial\phi^{(k)} = \partial \left( \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda \omega u(t,x)} f_k(\lambda\omega) \lambda^2 d\lambda d\omega = \sum_{p \geq -1} \partial\phi_p^{(k)}(t, x) \right),$$

which yields:

$$\left\| (-\Delta)^{-\frac{1}{2}} Q_{ij}(\partial\phi^{(1)}, \partial\phi^{(2)}) \right\|_{L^2} \lesssim \sum_{p, q \geq -1} \left\| (-\Delta)^{-\frac{1}{2}} (Q_{ij} \partial\phi_p^{(1)}, \partial\phi_q^{(2)}) \right\|_{L^2}. \quad (12.19)$$

The estimates involving  $\phi_{-1}^{(k)}$  are easier, so we focus on  $\phi_p^{(1)}, \phi_q^{(2)}$  for  $p, q \geq 0$ . We may assume  $q \geq p$ . Note that the structure of  $Q_{ij}$  implies:

$$Q_{ij}(\partial\phi_p^{(1)}, \partial\phi_q^{(2)}) = \partial(\partial^2\phi_p^{(1)} \cdot \partial\phi_q^{(2)}) + A \cdot \partial^2\phi_p^{(1)} \cdot \partial\phi_q^{(2)}$$

which yields:

$$\begin{aligned} & \left\| (-\Delta)^{-\frac{1}{2}} (Q_{ij}(\partial\phi_p^{(1)}, \partial\phi_q^{(2)})) \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left\| (-\Delta)^{-\frac{1}{2}} \partial(\partial^2\phi_p^{(1)} \cdot \partial\phi_q^{(2)}) \right\|_{L^2(\mathcal{M})} + \left\| (-\Delta)^{-\frac{1}{2}} (A \cdot \partial^2\phi_p^{(1)} \cdot \partial\phi_q^{(2)}) \right\|_{L^2(\mathcal{M})} \\ & \lesssim \left\| \partial^2\phi_p^{(1)} \cdot \partial\phi_q^{(2)} \right\|_{L^2(\mathcal{M})} + \left\| A \cdot \partial^2\phi_p^{(1)} \cdot \partial\phi_q^{(2)} \right\|_{L_t^2 L_x^{\frac{6}{5}}(\Sigma_t)} \\ & \lesssim \left\| \partial^2\phi_p^{(1)} \right\|_{L^4(\mathcal{M})} \left\| \partial\phi_q^{(2)} \right\|_{L^4(\mathcal{M})} + \left\| A \right\|_{L_t^\infty L_x^6(\Sigma_t)} \left\| \partial^2\phi_p^{(1)} \right\|_{L^4(\mathcal{M})} \left\| \partial\phi_q^{(2)} \right\|_{L^4(\mathcal{M})} \\ & \lesssim \left\| \partial^2\phi_p^{(1)} \right\|_{L^4(\mathcal{M})} \left\| \partial\phi_q^{(2)} \right\|_{L^4(\mathcal{M})} \end{aligned}$$

where we used the bootstrap assumption (5.5) for  $A$  in the last inequality. Together with the  $L^4(\mathcal{M})$  frequency localized Strichartz estimate of Proposition 12.1, and the fact that  $\psi$  is supported in  $(0, +\infty)$ , we obtain:

$$\begin{aligned} \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial\phi_p^{(1)}, \partial\phi_q^{(2)}))\|_{L^2(\mathcal{M})} &\lesssim 2^{\frac{5p}{2} + \frac{3q}{2}} \|\psi(2^{-p}\lambda)f_1\|_{L^2(\mathbb{R}^3)} \|\psi(2^{-q}\lambda)f_2\|_{L^2(\mathbb{R}^3)} \\ &\lesssim 2^{\frac{p}{2} - \frac{q}{2}} \|\lambda^2\psi(2^{-p}\lambda)f_1\|_{L^2(\mathbb{R}^3)} \|\lambda^2\psi(2^{-q}\lambda)f_2\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Since we assume  $q \geq p$ , this yields:

$$\begin{aligned} &\sum_{p, q \geq -1} \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial\phi_p^{(1)}, \partial\phi_q^{(2)}))\|_{L^2(\mathcal{M})} \tag{12.20} \\ &\lesssim \sum_{p, q \geq -1} 2^{-\frac{|p-q|}{2}} \|\lambda^2\psi(2^{-p}\lambda)f_1\|_{L^2(\mathbb{R}^3)} \|\lambda^2\psi(2^{-q}\lambda)f_2\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \left( \sum_{p \geq -1} \|\lambda^2\psi(2^{-p}\lambda)f_1\|_{L^2(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}} \left( \sum_{q \geq -1} \|\lambda^2\psi(2^{-q}\lambda)f_2\|_{L^2(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\lambda^2 f_1\|_{L^2(\mathbb{R}^3)} \|\lambda^2 f_2\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Finally, (12.19) and (12.20) imply (12.15). This concludes the proof of the improved bilinear estimates (5.29) and (5.30).

Finally, (12.11), the results in section 11.1 and section 12.3, and (11.12) yield the improved estimates (5.24), (5.25), (5.26), (5.27), (5.28), (5.29), (5.30), (5.31), and (5.33). This concludes the proof of Proposition 5.8.

### 13. PROPAGATION OF REGULARITY

The goal of this section is to prove Proposition 5.9. Recall from the statement of Proposition 5.9 that we assume that the estimates corresponding to all bootstrap assumptions of the section 5.3 hold for  $0 \leq t \leq T^*$  with a universal constant  $M$ . For the convenience of the reader, we recall some of these estimates below

$$\|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R} \cdot L\|_{L^2(\mathcal{H})} \lesssim \epsilon, \tag{13.1}$$

$$\|A\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \epsilon, \tag{13.2}$$

and:

$$\begin{aligned} \|A_0\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial A_0\|_{L_t^\infty L^2(\Sigma_t)} + \|A_0\|_{L_t^2 L^\infty(\Sigma_t)} + \|\partial A_0\|_{L_t^\infty L^3(\Sigma_t)} \\ + \|\partial\partial A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim \epsilon. \end{aligned} \tag{13.3}$$

Note also that we have (6.1) with a universal constant  $M$ , i.e.

$$\|n - 1\|_{L^\infty(\mathcal{M})} + \|\nabla n\|_{L^\infty(\mathcal{M})} \lesssim \epsilon. \tag{13.4}$$

Finally, for any weakly regular null hypersurface  $\mathcal{H}$  and any smooth scalar function  $\phi$  on  $\mathcal{M}$ ,

$$\|k_j \cdot \partial^j \phi\|_{L^2(\mathcal{M})} \lesssim \epsilon \sup_{\mathcal{H}} \|\nabla \phi\|_{L^2(\mathcal{H})}, \quad (13.5)$$

and

$$\|A^j \partial_j \phi\|_{L^2(\mathcal{M})} \lesssim \epsilon \sup_{\mathcal{H}} \|\nabla \phi\|_{L^2(\mathcal{H})}, \quad (13.6)$$

where the supremum is taken over all null hypersurfaces  $\mathcal{H}$ .

We now turn to the proof of Proposition 5.9. The estimates will be similar to the one derived in Proposition 5.7 and 5.8. Thus, we will simply sketch the arguments without going into details. Note that the crucial point is to check that the null structure - which is at the core of the proof of Proposition 5.7 and 5.8 - is preserved after differentiation. This will be done in section 13.3.

**13.1. Elliptic estimates.** Recall that we need to control  $\|\mathbf{DR}\|_{L_t^\infty L^2(\Sigma_t)}$ . Note that we may restrict our attention to the control of  $\mathbf{D}_l \mathbf{R}_{ij\alpha\beta}$  and  $\mathbf{D}_0 \mathbf{R}_{ij\alpha\beta}$ . Indeed, all the other components are recovered from the Bianchi identities and the standard symmetries of  $\mathbf{R}$ . Furthermore, since the estimates are for  $\mathbf{D}_l \mathbf{R}_{ij\alpha\beta}$  and  $\mathbf{D}_0 \mathbf{R}_{ij\alpha\beta}$  similar, we will restrict only to the control of  $\mathbf{D}_l \mathbf{R}_{ij\alpha\beta}$ . These components, according to the Cartan formalism, can be expressed in the form

$$\mathbf{R}_{ij\alpha\beta} = (\partial_i A_j - \partial_j A_i + [A_i, A_j])_{\alpha\beta}. \quad (13.7)$$

Since

$$\mathbf{D}_l \mathbf{R} = \partial_l \mathbf{R} + \mathbf{A} \mathbf{R}$$

we can estimate

$$\|\mathbf{D}_l \mathbf{R}_{ij\alpha\beta}\|_{L^2(\Sigma_t)} \lesssim \|\partial_l \mathbf{R}_{ij\alpha\beta}\|_{L^2(\Sigma_t)} + \|\mathbf{R}\|_{L^6(\Sigma_t)} \|\mathbf{A}\|_{L^3(\Sigma_t)}.$$

Furthermore, from (13.7)

$$\|\partial \mathbf{R}_{ij\alpha\beta}\|_{L^2(\Sigma_t)} \lesssim \|\partial^2 A\|_{L^2(\Sigma_t)} + \|\partial A\|_{L^6(\Sigma_t)} \|\mathbf{A}\|_{L^3(\Sigma_t)} \lesssim \|\partial^2 A\|_{L^2(\Sigma_t)} + \epsilon^2.$$

Proceeding the same way with all other components of  $\mathbf{DR}$ , we derive

$$\|\partial \mathbf{R}\|_{L^2(\Sigma_t)} \lesssim \|\partial \partial \mathbf{A}\|_{L^2(\Sigma_t)} + \epsilon^2. \quad (13.8)$$

In view of (13.8), it suffices to derive  $L^2$  bounds for  $\partial \partial \mathbf{A}$ . Furthermore, since  $A_0$  satisfies an elliptic equation, and hence better estimates, we focus on the estimates for  $\partial \partial A$ . We sketch below the estimates just for  $\partial^2 A$  since the estimates  $\partial \partial_0 A$  are similar. To establish bounds on  $\|\partial^2 A\|_{L^2(\Sigma_t)}$  we use the following analog of Lemma 6.5:

**Lemma 13.1.** *The following decomposition*

$$\partial A = \text{curl}(\partial B) + E'$$

holds with  $E'$  satisfying:

$$\|\partial E'\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial^2 E'\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} + \|E'\|_{L_t^2 L^\infty(\Sigma_t)} \lesssim \epsilon \left( \|\partial^3 B\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial^3 B\|_{L_t^2 L^7(\Sigma_t)} \right) + \epsilon^2.$$

Furthermore,  $A$  satisfies

$$\|\partial^2 A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\partial^3 B\|_{L_t^\infty L^2(\Sigma_t)} + \epsilon^2.$$

*Proof.* Recall from Lemma 6.5 that we have the following decomposition

$$A = \text{curl } B + E,$$

with

$$E = -(-\Delta)^{-1}(\mathbf{R}\partial B + \partial A\partial B + A\partial^2 B + A^2\partial B) + (-\Delta)^{-1}(A\partial A + A^3) \quad (13.9)$$

given in (6.14). We now introduce the new variables  $\partial A$  and  $\partial B$ , linked by the equation

$$\partial A = \text{curl } (\partial B) + E',$$

where

$$E' = \partial E + [\partial, \text{curl}]B,$$

so that

$$E' = -\partial(-\Delta)^{-1}(\mathbf{R}\partial B + \partial A\partial B + A\partial^2 B + A^2\partial B) + A\partial A + A^3 + A\partial B.$$

It then follows that

$$\|\partial^2 A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\partial^3 B\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial E'\|_{L_t^\infty L^2(\Sigma_t)}$$

and

$$\begin{aligned} \|\partial E'\|_{L_t^\infty L^3(\Sigma_t)} &\lesssim \left( \|R\|_{L_t^\infty L^6(\Sigma_t)} + \|\partial A\|_{L_t^\infty L^6(\Sigma_t)} + \|A\|_{L_t^\infty L^{12}(\Sigma_t)}^2 \right) \|\partial B\|_{L_t^\infty L^6(\Sigma_t)} \\ &\quad + \|A\|_{L_t^\infty L^6(\Sigma_t)} \|\partial^2 B\|_{L_t^\infty L^6(\Sigma_t)} \\ &\quad + \|A\|_{L_t^\infty L^6(\Sigma_t)} \|\partial A\|_{L_t^\infty L^6(\Sigma_t)} + \|A\|_{L_t^\infty L^9(\Sigma_t)}^3 \\ &\lesssim \epsilon \left( \|\partial R\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial^2 A\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial^3 B\|_{L_t^\infty L^2(\Sigma_t)} \right) + \epsilon^2 \\ &\lesssim \epsilon \left( \|\partial E'\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial^3 B\|_{L_t^\infty L^2(\Sigma_t)} \right) + \epsilon^2 \end{aligned}$$

Using that in our, localized, setting the  $L^3(\Sigma_t)$  norm dominates the  $L^2(\Sigma_t)$  norm we obtain that

$$\|\partial^2 A\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\partial^3 B\|_{L_t^\infty L^2(\Sigma_t)} + \epsilon^2$$

and

$$\|\partial E'\|_{L_t^\infty L^3(\Sigma_t)} \lesssim \epsilon \|\partial^3 B\|_{L_t^\infty L^2(\Sigma_t)} + \epsilon^2.$$

The other estimates are proved in the same. This concludes the proof of the lemma.  $\square$

In view of Lemma 13.1, it remains to estimate  $\partial^3 B$ . This will be done following the same circle of ideas used for  $\partial^2 B$ .

13.2. **The wave equation for  $\partial B$ .** Recall from (7.22) that we have, schematically:

$$\square B = (-\Delta)^{-1}[\square, \Delta]B + (-\Delta)^{-1}\square(\operatorname{curl} A).$$

Using the commutation formula (7.1) we obtain

$$\square \partial B = F \tag{13.10}$$

where  $F$  is given by

$$F = A^\ell \partial_t \boldsymbol{\partial} B + A^0 \boldsymbol{\partial}^2 B + \mathbf{A}^2 \boldsymbol{\partial} B + \partial_0 A^0 \boldsymbol{\partial} B + \partial(-\Delta)^{-1}[\square, \Delta]B + \partial(-\Delta)^{-1}\square(\operatorname{curl} A). \tag{13.11}$$

Using the energy estimates for the wave equation derived in Lemma 8.3 and the proof of the non sharp Strichartz estimate in section 12.2, we obtain the following proposition.

**Proposition 13.2.** *We have the following estimates*

- *Energy estimate*

$$\mathcal{E} := \sup_{\mathcal{H}} (\|\nabla \partial^2 B\|_{L^2(\mathcal{H})} + \|L \partial^2 B\|_{L^2(\mathcal{H})}) + \|\boldsymbol{\partial} \partial^2 B\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \|\boldsymbol{\partial} \partial^2 B\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})} + \epsilon. \tag{13.12}$$

- *Non sharp Strichartz estimate*

$$\mathcal{S} := \|\partial^2 B\|_{L_t^2 L^7(\Sigma_t)} \lesssim \|\boldsymbol{\partial} \partial^2 B\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})} + \epsilon. \tag{13.13}$$

Furthermore, we have the following estimate for  $\partial \partial_0 \partial_0 B$

$$\|\partial \partial_0 \partial_0 B\|_{L^2(\mathcal{M})} \lesssim \|\boldsymbol{\partial} \partial^2 B\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})} + \epsilon. \tag{13.14}$$

In view of Proposition 13.2, we need to estimate  $\partial F$ . This is done in the following proposition:

**Proposition 13.3.**  *$F$  satisfies the following estimate*

$$\|\partial F\|_{L^2(\mathcal{M})} \lesssim \epsilon (\|\partial^3 B_0\|_{L^2(\Sigma_0)} + \|\partial^2 B_1\|_{L^2(\Sigma_0)} + \mathcal{E} + \mathcal{S} + \|\partial F\|_{L^2(\mathcal{M})}) + \epsilon^2.$$

The proof of Proposition 13.3 is postponed to the next section. In view of Proposition 13.2 and Proposition 13.3, we obtain the control for  $\|\boldsymbol{\partial} \partial^2 B\|_{L_t^\infty L^2(\Sigma_t)}$  by its initial data which together with the elliptic estimates of the previous section yields the desired control for  $\|\mathbf{DR}\|_{L_t^\infty L^2(\Sigma_t)}$ . This concludes the proof of Proposition 5.9.

13.3. **Proof of Proposition 13.3.** From (13.10), we have

$$\begin{aligned} \partial F &= \partial A^\ell \partial_t \boldsymbol{\partial} B + A^\ell \partial \partial_t \boldsymbol{\partial} B + \partial(A^0 \boldsymbol{\partial}^2 B + \mathbf{A}^2 \boldsymbol{\partial} B + \partial_0 A^0 \boldsymbol{\partial} B) \\ &\quad + \partial^2(-\Delta)^{-1}[\square, \Delta]B + \partial^2(-\Delta)^{-1}\square(\operatorname{curl} A). \end{aligned}$$

This yields

$$\begin{aligned}
\|\partial F\|_{L^2(\mathcal{M})} &\lesssim \|\partial A^\ell \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|[\square, \Delta]B\|_{L^2(\mathcal{M})} \\
&\quad + \|\square(\text{curl } A)\|_{L^2(\mathcal{M})} + \|\partial(A^0 \boldsymbol{\partial}^2 B + \mathbf{A}^2 \boldsymbol{\partial} B + \partial_0 A^0 \boldsymbol{\partial} B)\|_{L^2(\mathcal{M})} \\
&\lesssim \|\partial A^\ell \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|[\square, \Delta]B\|_{L^2(\mathcal{M})} \\
&\quad + \|\square(\text{curl } A)\|_{L^2(\mathcal{M})} + l.o.t.,
\end{aligned} \tag{13.15}$$

where we neglect the cubic terms and the terms involving  $A^0$  since, as in the proof of Proposition 5.7 and 5.8, they are significantly easier to treat.

Next, we isolate the terms  $\square(\text{curl } A)$  and  $[\square, \Delta]B$  on the right-hand side of (13.15). From the proof of Proposition 7.3, we have

$$\square(\text{curl } A) = \partial(A^\ell \partial_\ell A) + Q_{ij}(A^\ell, A_\ell) + \text{terms involving } A_0 + \text{cubic terms.}$$

We deduce

$$\|\square(\text{curl } A)\|_{L^2(\mathcal{M})} \lesssim \|A^\ell \partial_\ell \partial A\|_{L^2(\mathcal{M})} + \|\partial A^\ell \partial_\ell A\|_{L^2(\mathcal{M})} + \|Q_{ij}(A^\ell, A_\ell)\|_{L^2(\mathcal{M})} + l.o.t. \tag{13.16}$$

Next, we deal with  $[\square, \Delta]B$ . According to (7.24), we have schematically

$$\begin{aligned}
[\square, \Delta]B &= k^{ab} \nabla_a \nabla_b (\partial_0 B) + n^{-1} \nabla_b n \nabla_b (\partial_0 (\partial_0 B)) + \nabla_0 k^{ab} \nabla_a \nabla_b B \\
&\quad + \text{terms involving } A_0 + \text{cubic terms.}
\end{aligned}$$

We deduce

$$\begin{aligned}
\|[\square, \Delta]B\|_{L^2(\mathcal{M})} &\lesssim \|k^{ab} \nabla_a \nabla_b (\partial_0 B)\|_{L^2(\mathcal{M})} + \|\nabla_0 k^{ab} \nabla_a \nabla_b B\|_{L^2(\mathcal{M})} \\
&\quad + \|n^{-1} \nabla_b n \nabla_b (\partial_0 (\partial_0 B))\|_{L^2(\mathcal{M})} + l.o.t.
\end{aligned}$$

Together with (13.15) and (13.16), we obtain

$$\begin{aligned}
\|\partial F\|_{L^2(\mathcal{M})} &\lesssim \|\partial A^\ell \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|\partial A^\ell \partial_\ell A\|_{L^2(\mathcal{M})} \\
&\quad + \|A^\ell \partial_\ell \partial A\|_{L^2(\mathcal{M})} + \|Q_{ij}(A^\ell, A_\ell)\|_{L^2(\mathcal{M})} + \|k^{ab} \nabla_a \nabla_b (\partial_0 B)\|_{L^2(\mathcal{M})} \\
&\quad + \|\nabla_0 k^{ab} \nabla_a \nabla_b B\|_{L^2(\mathcal{M})} + \|n^{-1} \nabla_b n \nabla_b (\partial_0 (\partial_0 B))\|_{L^2(\mathcal{M})} + l.o.t.
\end{aligned} \tag{13.17}$$

We will use the following bilinear estimates.

**Lemma 13.4.** *We have*

$$\begin{aligned}
&\|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|A^\ell \partial_\ell \partial A\|_{L^2(\mathcal{M})} + \|k^{ab} \nabla_a \nabla_b (\partial_0 B)\|_{L^2(\mathcal{M})} \\
&\lesssim \epsilon (\|\boldsymbol{\partial} \partial^2 B\|_{L^2(\Sigma_0)} + \mathcal{E} + \mathcal{S}) + \epsilon^2.
\end{aligned}$$

**Lemma 13.5.** *We have*

$$\begin{aligned}
&\|\partial A^\ell \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|\partial A^\ell \partial_\ell A\|_{L^2(\mathcal{M})} + \|Q_{ij}(A^\ell, A_\ell)\|_{L^2(\mathcal{M})} + \|\nabla_0 k^{ab} \nabla_a \nabla_b B\|_{L^2(\mathcal{M})} \\
&\lesssim \epsilon (\|\boldsymbol{\partial} \partial^2 B\|_{L^2(\Sigma_0)} + \mathcal{E} + \mathcal{S} + \|\partial F\|_{L^2(\mathcal{M})}) + \epsilon^2.
\end{aligned}$$

The proof of Lemma 13.4 is postponed to section 13.3.1 and the proof of Lemma 13.5 is postponed to section 13.3.2. We now conclude the proof of Proposition 13.3. In view of (13.17), Lemma 13.4 and Lemma 13.5, we have

$$\|\partial F\|_{L^2(\mathcal{M})} \lesssim \|n^{-1}\nabla_b n \nabla_b(\partial_0(\partial_0 B))\|_{L^2(\mathcal{M})} + \epsilon(\|\boldsymbol{\partial}\partial^2 B\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}) + \epsilon^2.$$

Using the estimates (13.4) and (13.14), we have

$$\begin{aligned} \|n^{-1}\nabla_b n \nabla_b(\partial_0(\partial_0 B))\|_{L^2(\mathcal{M})} &\lesssim \|n^{-1}\nabla n\|_{L^\infty(\mathcal{M})} \|\partial\partial_0\partial_0 B\|_{L^2(\mathcal{M})} \\ &\lesssim \epsilon(\|\boldsymbol{\partial}\partial^2 B\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}) + \epsilon^2 \end{aligned}$$

and we deduce

$$\|\partial F\|_{L^2(\mathcal{M})} \lesssim \epsilon(\|\boldsymbol{\partial}\partial^2 B\|_{L^2(\Sigma_0)} + \mathcal{E} + \mathcal{S} + \|\partial F\|_{L^2(\mathcal{M})}) + \epsilon^2$$

which is the desired estimate. This concludes the proof of Proposition 13.3.

13.3.1. *Proof of Lemma 13.4.* In view of the identity  $\partial A = \text{curl}(\partial B) + E'$  of Lemma 13.1, we have

$$\begin{aligned} &\|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|A^\ell \partial_\ell \partial A\|_{L^2(\mathcal{M})} + \|k^{ab} \nabla_a \nabla_b(\partial_0 B)\|_{L^2(\mathcal{M})} \\ &\lesssim \|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|k^{ab} \nabla_a \nabla_b(\partial_0 B)\|_{L^2(\mathcal{M})} + l.o.t. \end{aligned}$$

Together with the bilinear estimates (13.5) and (13.6), we deduce

$$\begin{aligned} &\|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|A^\ell \partial_\ell \partial A\|_{L^2(\mathcal{M})} + \|k^{ab} \nabla_a \nabla_b(\partial_0 B)\|_{L^2(\mathcal{M})} \\ &\lesssim \epsilon \sup_{\mathcal{H}} \|\nabla \partial \boldsymbol{\partial} B\|_{L^2(\mathcal{H})} + l.o.t. \end{aligned}$$

Arguing as in (11.8), we finally obtain

$$\begin{aligned} &\|A^\ell \partial \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|A^\ell \partial_\ell \partial A\|_{L^2(\mathcal{M})} + \|k^{ab} \nabla_a \nabla_b(\partial_0 B)\|_{L^2(\mathcal{M})} \\ &\lesssim \epsilon \sup_{\mathcal{H}} (\|\nabla \partial^2 B\|_{L^2(\mathcal{H})} + \|L \partial^2 B\|_{L^2(\mathcal{H})}) + l.o.t. \\ &\lesssim \epsilon(\|\boldsymbol{\partial}\partial^2 B\|_{L^2(\Sigma_0)} + \mathcal{E} + \mathcal{S}) + \epsilon^2 \end{aligned}$$

which is the desired estimate. This concludes the proof of Lemma 13.4.

13.3.2. *Proof of Lemma 13.5.* In view of the wave equation (13.10) satisfied by  $\partial B$ , we may use for  $\partial B$  the parametrix constructed in Lemma 10.7:

$$\partial B = \sum_{j=0}^{+\infty} \phi^{(j)},$$

with:

$$\phi^{(0)} = \Psi_{om}[\phi_0, \phi_1] + \int_0^t \Psi(t, s) F(s, \cdot) ds,$$

and for all  $j \geq 1$ :

$$\phi^{(j)} = \int_0^t \Psi(t, s) F^{(j)}(s, \cdot) ds.$$



Furthermore  $\phi^{(j)}$  and  $F^{(j)}$  satisfy the following estimate:

$$\|\partial\boldsymbol{\partial}\phi^{(j)}\|_{L_t^\infty L^2(\Sigma_t)} + \|\partial F^{(j)}\|_{L^2(\mathcal{M})} \lesssim \epsilon^j (\|\partial^3 B_0\|_{L^2(\Sigma_0)} + \|\partial^2 B_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}).$$

We will show that the proof of the bilinear estimates of Lemma 13.5 all involve the  $L^2(\mathcal{M})$  norm of quantities of the type:

$$\mathcal{C}(U, \partial(\partial B)),$$

where  $\mathcal{C}(U, \partial(\partial B))$  denotes a contraction with respect to one index between a tensor  $U$  and  $\partial(\partial B)$ . Using the parametrix for  $\partial B$  discussed above, and arguing as in section 11.1, we obtain the analog of (11.4):

$$\begin{aligned} \|\mathcal{C}(U, \partial(\partial B))\|_{L^2(\mathcal{M})} &\lesssim (\|\partial^3 B_0\|_{L^2(\Sigma_0)} + \|\partial^2 B_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}) \quad (13.18) \\ &\quad \times \left( \sup_{\mathcal{H}} \|\mathcal{C}(U, N)\|_{L^2(\mathcal{H})} \right) \end{aligned}$$

where the supremum is taken over all weakly regular null hypersurfaces, and where  $N$  is the unit normal to  $\mathcal{H} \cap \Sigma_t$  inside  $\Sigma_t$ .

We are now ready to prove the bilinear estimates of Lemma 13.5. Using the decomposition of Lemma 13.1, we have

$$\begin{aligned} &\|\partial A^\ell \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|\partial A^\ell \partial_\ell A\|_{L^2(\mathcal{M})} + \|Q_{ij}(A^\ell, A_\ell)\|_{L^2(\mathcal{M})} \quad (13.19) \\ &\lesssim \|Q_{ij}(\partial B, \boldsymbol{\partial} B)\|_{L^2(\mathcal{M})} + l.o.t. \end{aligned}$$

which is of the type  $\mathcal{C}(U, \partial(\partial B))$  with  $U = \partial\boldsymbol{\partial} B$ . Now, arguing as in section 11.1, we have in this case

$$\mathcal{C}(U, N) = \epsilon_{ij} \partial_i \boldsymbol{\partial} B N_j = \nabla \boldsymbol{\partial} B$$

and we deduce from (13.18) and (13.19)

$$\begin{aligned} &\|\partial A^\ell \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|\partial A^\ell \partial_\ell A\|_{L^2(\mathcal{M})} + \|Q_{ij}(A^\ell, A_\ell)\|_{L^2(\mathcal{M})} \quad (13.20) \\ &\lesssim \left( \sup_{\mathcal{H}} \|\nabla \boldsymbol{\partial} B\|_{L^2(\mathcal{H})} \right) (\mathcal{E} + \mathcal{S} + \|F\|_{L^2(\mathcal{M})}) + \epsilon^2 \\ &\lesssim \epsilon (\mathcal{E} + \mathcal{S} + \|F\|_{L^2(\mathcal{M})}) + \epsilon^2. \end{aligned}$$

Next, we consider the term  $\nabla_0 k^{ab} \nabla_a \nabla_b B$ . Recall from (7.30) that we have

$$\nabla_0 k_{ab} = \mathbf{R}_{a0b0} + l.o.t.$$

It follows that,

$$\|\nabla_0 k^{ab} \nabla_a \nabla_b B\|_{L^2(\mathcal{M})} \lesssim \|\mathbf{R}_{a0b0} \nabla_a \nabla_b B\|_{L^2(\mathcal{M})} + l.o.t. \quad (13.21)$$

The right-hand side of (13.21) is of the type  $\mathcal{C}(U, \partial(\partial B))$  with  $U = \mathbf{R}_{a0b0}$ . Now, arguing as in section 11.1, we have in this case

$$\mathcal{C}(U, N) = \mathbf{R}_{N0..} = \mathbf{R} \cdot L$$

and we deduce from (13.18) and (13.19)

$$\begin{aligned} & \|\partial A^\ell \partial_\ell \boldsymbol{\partial} B\|_{L^2(\mathcal{M})} + \|\partial A^\ell \partial_\ell A\|_{L^2(\mathcal{M})} + \|Q_{ij}(A^\ell, A_\ell)\|_{L^2(\mathcal{M})} \\ & \lesssim \left( \sup_{\mathcal{H}} \|\mathbf{R} \cdot L\|_{L^2(\mathcal{H})} \right) (\mathcal{E} + \mathcal{S} + \|F\|_{L^2(\mathcal{M})}) + \epsilon^2 \\ & \lesssim \epsilon (\mathcal{E} + \mathcal{S} + \|F\|_{L^2(\mathcal{M})}) + \epsilon^2 \end{aligned} \quad (13.22)$$

where we used (13.1) in the last inequality. Finally, (13.20) and (13.22) yield the desired estimate. This concludes the proof of Lemma 13.5.

#### APPENDIX A. PROOF OF (6.15)

The goal of this appendix is to prove (6.15). We first introduce Littlewood-Paley projections on  $\Sigma_t$  which will be used both for the proof of (6.15) and Lemma 9.1. These were constructed in [43] (see section 3.6 in that paper) using the heat flow on  $\Sigma_t$ . We recall below their main properties:

**Proposition A.1** (Main properties of the LP  $Q_j$  [43]). *Let  $F$  a tensor on  $\Sigma_t$ . The LP-projections  $Q_j$  on  $\Sigma_t$  verify the following properties:*

i) *Partition of unity*

$$\sum_j Q_j = I. \quad (A.1)$$

ii)  *$L^p$ -boundedness* For any  $1 \leq p \leq \infty$ , and any interval  $I \subset \mathbb{Z}$ ,

$$\|Q_I F\|_{L^p(\Sigma_t)} \lesssim \|F\|_{L^p(\Sigma_t)} \quad (A.2)$$

iii) *Finite band property* For any  $1 \leq p \leq \infty$ .

$$\begin{aligned} \|\Delta Q_j F\|_{L^p(\Sigma_t)} & \lesssim 2^{2j} \|F\|_{L^p(\Sigma_t)} \\ \|Q_j F\|_{L^p(\Sigma_t)} & \lesssim 2^{-2j} \|\Delta F\|_{L^p(\Sigma_t)}. \end{aligned} \quad (A.3)$$

*In addition, the  $L^2$  estimates*

$$\begin{aligned} \|\nabla Q_j F\|_{L^2(\Sigma_t)} & \lesssim 2^j \|F\|_{L^2(\Sigma_t)} \\ \|Q_j F\|_{L^2(\Sigma_t)} & \lesssim 2^{-j} \|\nabla F\|_{L^2(\Sigma_t)} \end{aligned} \quad (A.4)$$

*hold together with the dual estimate*

$$\|Q_j \nabla F\|_{L^2(\Sigma_t)} \lesssim 2^j \|F\|_{L^2(\Sigma_t)}$$

iv) *Bernstein inequality* For any  $2 \leq p \leq +\infty$  and  $j \in \mathbb{Z}$

$$\|Q_j F\|_{L^p(\Sigma_t)} \lesssim 2^{\frac{3}{2}(1-\frac{2}{p})j} \|F\|_{L^2(\Sigma_t)}$$

*together with the dual estimates*

$$\|Q_j F\|_{L^2(\Sigma_t)} \lesssim 2^{\frac{3}{2}(1-\frac{2}{p})j} \|F\|_{L^{p'}(\Sigma_t)}$$

We now rely on Proposition A.1 to prove (6.15). Using Proposition A.1, we have for any scalar function  $v$  on  $\Sigma_t$ :

$$\begin{aligned}
\|(-\Delta)^{-1}v\|_{L^\infty(\Sigma_t)} &\lesssim \sum_{j \in \mathbb{Z}} \|Q_j(-\Delta)^{-1}v\|_{L^\infty(\Sigma_t)} \\
&\lesssim \sum_{j \in \mathbb{Z}} 2^{\frac{3j}{2}} \|Q_j(-\Delta)^{-1}f\|_{L^2(\Sigma_t)} \\
&\lesssim \sum_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|Q_j f\|_{L^2(\Sigma_t)} \\
&\lesssim \left( \sum_{j \geq 0} 2^{-\frac{j}{14}} \right) \|f\|_{L^{\frac{14}{9}}(\Sigma_t)} + \left( \sum_{j < 0} 2^{\frac{j}{13}} \right) \|f\|_{L^{\frac{13}{9}}(\Sigma_t)} \\
&\lesssim \|f\|_{L^{\frac{14}{9}}(\Sigma_t)} + \|f\|_{L^{\frac{13}{9}}(\Sigma_t)}.
\end{aligned}$$

This concludes the proof of (6.15).

## APPENDIX B. PROOF OF LEMMA 9.1

Recall that  $v$  is the solution of

$$\Delta v = f_1 + \partial f_2.$$

Using the harmonic coordinate system on  $\Sigma_t$  of Lemma 6.2, we obtain:

$$\begin{aligned}
\dot{\Delta} v &= (g_{ij} - \delta_{ij}) \dot{\partial}_{ij}^2 v + \Gamma \dot{\partial}_j v + \Gamma f_2 + f_1 + \dot{\partial}(f_2) \\
&= \dot{\partial}_{ij}^2 [(g_{ij} - \delta_{ij})v] + \dot{\partial}_j [\Gamma v] + \dot{\partial}_j (\Gamma) v + \Gamma f_2 + f_1 + \dot{\partial}(f_2),
\end{aligned} \tag{B.1}$$

where  $\dot{\partial}$  and  $\dot{\Delta}$  denote the derivatives and the flat Laplacian relative to the coordinate system defined above, as opposed to the frame derivatives  $\partial$  and the Laplacian  $\Delta$ , and where  $\Gamma$  is the corresponding Christoffel symbol. Now, we use the following standard elliptic estimates on  $\mathbb{R}^3$ :

$$\begin{aligned}
(-\dot{\Delta})^{-1} \dot{\partial}_{ij}^2 &\in \mathcal{L}(L^3(\mathbb{R}^3)), \quad \dot{\partial}_i (-\dot{\Delta})^{-1} \dot{\partial}_{ij}^2 \in \mathcal{L}(W^{1, \frac{3}{2}}(\mathbb{R}^3), L^{\frac{3}{2}}(\mathbb{R}^3)), \\
(-\dot{\Delta})^{-1} \dot{\partial}_i &\in \mathcal{L}(L^{\frac{3}{2}}(\mathbb{R}^3), L^3(\mathbb{R}^3)), \quad (-\dot{\Delta})^{-1} \dot{\partial}_{ij}^2 \in \mathcal{L}(L^{\frac{3}{2}}(\mathbb{R}^3)) \\
(-\dot{\Delta})^{-1} &\in \mathcal{L}(L^1(\mathbb{R}^3), L^3(\mathbb{R}^3)), \quad (-\dot{\Delta})^{-1} \dot{\partial}_i \in \mathcal{L}(L^1(\mathbb{R}^3), L^{\frac{3}{2}}(\mathbb{R}^3)),
\end{aligned}$$

where the notation  $\mathcal{L}(X, Y)$  stands for the set of bounded linear operators from the space  $X$  to the space  $Y$ . Together with (B.1) and our assumptions on the harmonic coordinates

(6.2) (6.3), this yields in a coordinate Patch  $U$ :

$$\begin{aligned}
& \|v\|_{L^3(U)} + \|\partial v\|_{L^{\frac{3}{2}}(U)} \\
& \lesssim \delta(\|v\|_{L^3(U)} + \|\partial v\|_{L^{\frac{3}{2}}(U)}) + \|v_1\|_{L^2(U)}(\|\dot{\partial}\Gamma\|_{L^2(U)} + \|\Gamma\|_{L^6(U)}) \\
& \quad + \|v_2\|_{L^4(U)}(\|\dot{\partial}\Gamma\|_{L^{\frac{4}{3}}(U)} + \|\Gamma\|_{L^{\frac{12}{5}}(U)}) + \|f_1\|_{L^1(U)} + \|f_2\|_{L^{\frac{3}{2}}(U)} \\
& \lesssim \delta(\|v\|_{L^3(U)} + \|\partial v\|_{L^{\frac{3}{2}}(U)}) + C(\delta)(\|v_1\|_{L^2(U)} + \|v_2\|_{L^4(U)}) + \|f_1\|_{L^1(U)} + \|f_2\|_{L^{\frac{3}{2}}(U)},
\end{aligned}$$

where  $v = v_1 + v_2$ . We then sum the contributions of the covering of  $\Sigma_t$  by harmonic coordinate patches  $U$  satisfying (6.2) (6.3). Eventually to increasing  $C(\delta)$ , we obtain

$$\begin{aligned}
& \|v\|_{L^3(\Sigma_t)} + \|\partial v\|_{L^{\frac{3}{2}}(\Sigma_t)} \\
& \lesssim \delta(\|v\|_{L^3(\Sigma_t)} + \|\partial v\|_{L^{\frac{3}{2}}(\Sigma_t)}) + C(\delta)(\|v_1\|_{L^2(\Sigma_t)} + \|v_2\|_{L^4(\Sigma_t)}) + \|f_1\|_{L^1(\Sigma_t)} + \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}.
\end{aligned}$$

Recall from Lemma 6.2 that we have the freedom of choice for  $\delta > 0$ . By choosing  $\delta > 0$  small enough, we obtain:

$$\|v\|_{L^3(\Sigma_t)} + \|\partial v\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim C(\delta)(\|v_1\|_{L^2(\Sigma_t)} + \|v_2\|_{L^4(\Sigma_t)}) + \|f_1\|_{L^1(\Sigma_t)} + \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}. \quad (\text{B.2})$$

$\delta > 0$  is now fixed. Thus,  $C(\delta) = C$  is a constant which may not be small. We now choose

$$v_1 = Q_{\geq 0}v \text{ and } v_2 = Q_{< 0}v.$$

Together with (B.2), we obtain

$$\|v\|_{L^3(\Sigma_t)} + \|\partial v\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|Q_{\geq 0}v\|_{L^2(\Sigma_t)} + \|Q_{< 0}v\|_{L^4(\Sigma_t)} + \|f_1\|_{L^1(\Sigma_t)} + \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}. \quad (\text{B.3})$$

In view of (B.3), we still need to estimate  $\|Q_{\geq 0}v\|_{L^2(\Sigma_t)}$  and  $\|Q_{< 0}v\|_{L^4(\Sigma_t)}$ . We start with  $\|Q_{\geq 0}v\|_{L^2(\Sigma_t)}$ . We have:

$$\|Q_{\geq 0}v\|_{L^2(\Sigma_t)} \lesssim \|(-\Delta)^{-1}Q_{\geq 0}f_1\|_{L^2(\Sigma_t)} + \|(-\Delta)^{-1}Q_{\geq 0}\partial f_2\|_{L^2(\Sigma_t)}. \quad (\text{B.4})$$

Next we estimate each term in the right-hand side of (B.4) starting with the first one. Using Proposition A.1, we have for any scalar function  $f$  on  $\Sigma_t$ :

$$\begin{aligned}
\|(-\Delta)^{-1}Q_{\geq 0}f\|_{L^\infty(\Sigma_t)} & \lesssim \sum_{j \geq 0} \|Q_j(-\Delta)^{-1}f\|_{L^\infty(\Sigma_t)} \\
& \lesssim \sum_{j \geq 0} 2^{\frac{3j}{2}} \|Q_j(-\Delta)^{-1}f\|_{L^2(\Sigma_t)} \\
& \lesssim \sum_{j \geq 0} 2^{-\frac{j}{2}} \|Q_j f\|_{L^2(\Sigma_t)} \\
& \lesssim \left( \sum_{j \geq 0} 2^{-\frac{j}{2}} \right) \|f\|_{L^2(\Sigma_t)} \\
& \lesssim \|f\|_{L^2(\Sigma_t)}.
\end{aligned}$$

Taking the dual, we obtain for  $f_1$ :

$$\|(-\Delta)^{-1}Q_{\geq 0}f_1\|_{L^2(\Sigma_t)} \lesssim \|f_1\|_{L^1(\Sigma_t)}. \quad (\text{B.5})$$

Next, we consider the second term in the right-hand side of (B.4). Using property i) of Proposition A.1, we have:

$$\|(-\Delta)^{-1}Q_{\geq 0}\partial f_2\|_{L^2(\Sigma_t)} \lesssim \sum_{j \geq 0, l \in \mathbb{Z}} \|Q_j(-\Delta)^{-1}\partial Q_l f_2\|_{L^2(\Sigma_t)}. \quad (\text{B.6})$$

We now estimate the right-hand side of (B.6). We consider the two cases  $j > l$  and  $j \leq l$  separately. If  $j > l$ , we obtain using Proposition A.1:

$$\begin{aligned} \|Q_j(-\Delta)^{-1}\partial Q_l f_2\|_{L^2(\Sigma_t)} &\lesssim 2^{-2j}\|\partial Q_l f_2\|_{L^2(\Sigma_t)} \\ &\lesssim 2^{-2j+l}\|Q_l f_2\|_{L^2(\Sigma_t)} \\ &\lesssim 2^{-2j+\frac{3l}{2}}\|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}. \end{aligned} \quad (\text{B.7})$$

If  $j \leq l$ , we obtain using Proposition A.1:

$$\begin{aligned} \|Q_j(-\Delta)^{-1}\partial Q_l f_2\|_{L^2(\Sigma_t)} &\lesssim 2^{-l}\|(-\Delta)^{-1}\partial(-\Delta)^{\frac{1}{2}}Q_l f_2\|_{L^2(\Sigma_t)} \\ &\lesssim 2^{-l}\|(-\Delta)^{-1}\partial(-\Delta)^{\frac{1}{2}}\|_{\mathcal{L}(L^2(\Sigma_t))}\|Q_l f_2\|_{L^2(\Sigma_t)} \\ &\lesssim 2^{-\frac{l}{2}}\|\nabla^2(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Sigma_t))}\|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)} \\ &\lesssim 2^{-\frac{l}{2}}\|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}, \end{aligned} \quad (\text{B.8})$$

where we used the fact that

$$\|\nabla^2(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Sigma_t))} \lesssim 1$$

thanks to the Bochner inequality on  $\Sigma_t$  (6.8). Finally, (B.6)-(B.8) yields:

$$\|(-\Delta)^{-1}Q_{\leq 0}\partial f_2\|_{L^2(\Sigma_t)} \lesssim \left( \sum_{j>l, j \geq 0} 2^{-2j+\frac{3l}{2}} + \sum_{0 \leq j \leq l} 2^{-\frac{l}{2}} \right) \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}. \quad (\text{B.9})$$

(B.4), (B.5) and (B.9) yield:

$$\|Q_{\geq 0}v\|_{L^2(\Sigma_t)} \lesssim \|f_1\|_{L^1(\Sigma_t)} + \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}. \quad (\text{B.10})$$

In view of (B.3), we still need to estimate  $\|Q_{< 0}v\|_{L^4(\Sigma_t)}$ . We have:

$$\|Q_{< 0}v\|_{L^4(\Sigma_t)} \lesssim \|(-\Delta)^{-1}Q_{< 0}f_1\|_{L^4(\Sigma_t)} + \|(-\Delta)^{-1}Q_{< 0}\partial f_2\|_{L^4(\Sigma_t)}. \quad (\text{B.11})$$

Next we estimate each term in the right-hand side of (B.11) starting with the first one. Using Proposition A.1, we have for any scalar function  $f$  on  $\Sigma_t$ :

$$\begin{aligned}
\|(-\Delta)^{-1}Q_{<0}f\|_{L^\infty(\Sigma_t)} &\lesssim \sum_{j<0} \|Q_j(-\Delta)^{-1}f\|_{L^\infty(\Sigma_t)} \\
&\lesssim \sum_{j<0} 2^{\frac{3j}{2}} \|Q_j(-\Delta)^{-1}f\|_{L^2(\Sigma_t)} \\
&\lesssim \sum_{j<0} 2^{\frac{9j}{4}} \|Q_j(-\Delta)^{-1}f\|_{L^{\frac{4}{3}}(\Sigma_t)} \\
&\lesssim \sum_{j<0} 2^{\frac{j}{4}} \|Q_j f\|_{L^{\frac{4}{3}}(\Sigma_t)} \\
&\lesssim \left( \sum_{j<0} 2^{\frac{j}{4}} \right) \|f\|_{L^{\frac{4}{3}}(\Sigma_t)} \\
&\lesssim \|f\|_{L^{\frac{4}{3}}(\Sigma_t)}.
\end{aligned}$$

Taking the dual, we obtain for  $f_1$ :

$$\|(-\Delta)^{-1}Q_{<0}f_1\|_{L^{\frac{4}{3}}(\Sigma_t)} \lesssim \|f_1\|_{L^1(\Sigma_t)}. \quad (\text{B.12})$$

Next, we consider the second term in the right-hand side of (B.11). Using property i) of Proposition A.1, we have:

$$\|(-\Delta)^{-1}Q_{<0}\partial f_2\|_{L^4(\Sigma_t)} \lesssim \sum_{j<0, l \in \mathbb{Z}} \|Q_j(-\Delta)^{-1}\partial Q_l f_2\|_{L^4(\Sigma_t)}. \quad (\text{B.13})$$

We now estimate the right-hand side of (B.13). We consider the two cases  $j > l$  and  $j \leq l$  separately. If  $j > l$ , we obtain using Proposition A.1:

$$\begin{aligned}
\|Q_j(-\Delta)^{-1}\partial Q_l f_2\|_{L^4(\Sigma_t)} &\lesssim 2^{-\frac{5j}{4}} \|\partial Q_l f_2\|_{L^2(\Sigma_t)} \\
&\lesssim 2^{-\frac{5}{4}j+l} \|Q_l f_2\|_{L^2(\Sigma_t)} \\
&\lesssim 2^{-\frac{5}{4}j+\frac{3l}{2}} \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}.
\end{aligned} \quad (\text{B.14})$$

If  $j \leq l$ , we obtain using Proposition A.1:

$$\begin{aligned}
\|Q_j(-\Delta)^{-1}\partial Q_l f_2\|_{L^4(\Sigma_t)} &\lesssim 2^{\frac{3j}{4}} 2^{-l} \|(-\Delta)^{-1}\partial(-\Delta)^{\frac{1}{2}}Q_l f_2\|_{L^2(\Sigma_t)} \\
&\lesssim 2^{\frac{3j}{4}} 2^{-l} \|(-\Delta)^{-1}\partial(-\Delta)^{\frac{1}{2}}\|_{\mathcal{L}(L^2(\Sigma_t))} \|Q_l f_2\|_{L^2(\Sigma_t)} \\
&\lesssim 2^{\frac{3j}{4}-\frac{l}{2}} \|\nabla^2(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Sigma_t))} \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)} \\
&\lesssim 2^{\frac{3j}{4}-\frac{l}{2}} \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)},
\end{aligned} \quad (\text{B.15})$$

where we used the fact that

$$\|\nabla^2(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Sigma_t))} \lesssim 1$$

thanks to the Bochner inequality on  $\Sigma_t$  (6.8). Finally, (B.13)-(B.15) yields:

$$\|(-\Delta)^{-1}Q_{<0}\partial f_2\|_{L^4(\Sigma_t)} \lesssim \left( \sum_{l<j<0} 2^{-\frac{5}{4}j+\frac{3l}{2}} + \sum_{j\leq l, j<0} 2^{\frac{3j}{4}-\frac{l}{2}} \right) \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)} \quad (\text{B.16})$$

(B.11), (B.12) and (B.16) yield:

$$\|Q_{<0}v\|_{L^4(\Sigma_t)} \lesssim \|f_1\|_{L^1(\Sigma_t)} + \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}. \quad (\text{B.17})$$

Finally, (B.3), (B.10) and (B.17) imply:

$$\|v\|_{L^3(\Sigma_t)} + \|\partial v\|_{L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|f_1\|_{L^1(\Sigma_t)} + \|f_2\|_{L^{\frac{3}{2}}(\Sigma_t)}.$$

This concludes the proof of Lemma 9.1.

### APPENDIX C. PROOF OF LEMMA 7.1

The goal of this appendix is to prove Lemma 7.1. The commutation formula (7.1) has already been proved at the beginning of section 7. Thus, it only remains to prove the commutation formula (7.2). Recalling (3.25),

$$\square\phi = -\partial_0(\partial_0\phi) + \Delta\phi + n^{-1}\nabla n \cdot \nabla\phi,$$

Thus, we have:

$$\begin{aligned} [\square, \Delta]\phi &= [-\partial_0\partial_0 + n^{-1}\nabla n \cdot \nabla + \Delta, \Delta]\phi \\ &= -[\partial_0\partial_0, \Delta]\phi + [n^{-1}\nabla n \cdot \nabla, \Delta]\phi. \end{aligned} \quad (\text{C.1})$$

We thus have to calculate the commutators  $[\partial_0^2, \Delta]\phi$  and  $[n^{-1}\nabla n \cdot \nabla, \Delta]\phi$ . For any tensor  $U$  tangent to  $\Sigma_t$ , we denote by  $\nabla_0 U$  the projection of  $\mathbf{D}_0 U$  to  $\Sigma_t$ . We have the following commutator formula for any vectorfield  $U$  tangent to  $\Sigma_t$ :

$$[\nabla_b, \nabla_0]U_a = k_{bc}\nabla_c U_a - n^{-1}\nabla_b n \nabla_0 U_a + (n^{-1}k_{ab}\nabla_c n - n^{-1}k_{bc}\nabla_a n + \mathbf{R}_{0abc})U_c, \quad (\text{C.2})$$

while for a scalar  $\phi$ , the commutator formula reduces to:

$$[\nabla_b, \nabla_0]\phi = k_{bc}\nabla_c\phi - n^{-1}\nabla_b n \partial_0\phi. \quad (\text{C.3})$$

Using the commutator formulas (C.2) and (C.3) and the fact that  $[\partial_0, \Delta]\phi = [\nabla_0, \nabla^a]\nabla_a\phi + \nabla^a[\nabla_0, \nabla_a]\phi$ , we obtain:

$$[\partial_0, \Delta]\phi = -2k^{ab}\nabla_a\nabla_b\phi + 2n^{-1}\nabla_b n \nabla_b(\partial_0\phi) + n^{-1}\Delta n \partial_0\phi - 2n^{-1}\nabla_a n k^{ab}\nabla_b\phi, \quad (\text{C.4})$$

where we used the constraint equation (2.2) and the fact that, in view of the Einstein equations and the symmetries of  $\mathbf{R}$ , we have:

$$\mathbf{g}^{ab}\mathbf{R}_{0abc} = 0.$$

Differentiating the commutator formula (C.4) with respect to  $\partial_0$  and using the commutator formulas (C.2) and (C.3), we obtain:

$$\begin{aligned}
& \partial_0([\partial_0, \Delta]\phi) \\
= & -2k^{ab}\nabla_a\nabla_b(\partial_0\phi) + 2n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0\phi)) + (-2\nabla_0 k^{ab} + 4k^{ac}k_c{}^b)\nabla_a\nabla_b\phi \\
& + (2n^{-1}\nabla_b(\partial_0 n) - 10k^{ab}n^{-1}\nabla_a n)\nabla_b(\partial_0\phi) + (n^{-1}\Delta n + 2n^{-2}|\nabla n|^2)\partial_0(\partial_0\phi) \\
& + (2k^{ac}\mathbf{R}_{0acb} + 2k^{ac}\nabla_c k_{ab} - 2n^{-1}\nabla_a n\nabla_0 k^{ab} + 2k^{ab}n^{-1}\nabla_a(\partial_0 n) + 4k^{ac}k_{cb}n^{-1}\nabla_a n \\
& + 2|k|^2 n^{-1}\nabla_b n - 2k^{ab}n^{-2}\nabla_a n\partial_0 n)\nabla_b\phi \\
& + (n^{-1}\Delta(\partial_0 n) - 4k^{ab}n^{-1}\nabla_a\nabla_b n + 2n^{-2}\nabla_b n\nabla_b(\partial_0 n))\partial_0\phi.
\end{aligned}$$

Together with the commutator formula (C.4) applied to  $\partial_0\phi$ , we obtain:

$$\begin{aligned}
& [\partial_0\partial_0, \Delta]\phi \tag{C.5} \\
= & [\partial_0, \Delta]\partial_0\phi + \partial_0([\partial_0, \Delta]\phi) \\
= & -4k^{ab}\nabla_a\nabla_b(\partial_0\phi) + 4n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0\phi)) + (-2\nabla_0 k^{ab} + 4k^{ac}k_c{}^b)\nabla_a\nabla_b\phi \\
& + (2n^{-1}\nabla_b(\partial_0 n) - 12k^{ab}n^{-1}\nabla_a n)\nabla_b(\partial_0\phi) + (2n^{-1}\Delta n + 2n^{-2}|\nabla n|^2)\partial_0(\partial_0\phi) \\
& + (2k^{ac}\mathbf{R}_{0acb} + 2k^{ac}\nabla_c k_{ab} - 2n^{-1}\nabla_a n\nabla_0 k^{ab} + 2k^{ab}n^{-1}\nabla_a(\partial_0 n) + 4k^{ac}k_{cb}n^{-1}\nabla_a n \\
& + 2|k|^2 n^{-1}\nabla_b n - 2k^{ab}n^{-2}\nabla_a n\partial_0 n)\nabla_b\phi \\
& + (n^{-1}\Delta(\partial_0 n) - 4k^{ab}n^{-1}\nabla_a\nabla_b n + 2n^{-2}\nabla_b n\nabla_b(\partial_0 n))\partial_0\phi.
\end{aligned}$$

We also compute the commutator  $[n^{-1}\nabla n\nabla, \Delta]\phi$ :

$$\begin{aligned}
& [n^{-1}\nabla n\nabla, \Delta]\phi \\
= & -\Delta(n^{-1}\nabla_b n)\nabla_b\phi - \nabla_a(n^{-1}\nabla_b n)\nabla_a\nabla_b\phi + n^{-1}\nabla_b n[\nabla_b, \Delta]\phi \\
= & -n^{-1}\nabla_b(\Delta n)\nabla_b\phi - n^{-1}[\Delta, \nabla_b]n\nabla_b\phi + n^{-2}\nabla_a n\nabla_a\nabla_b n\nabla_b\phi \\
& + n^{-2}\nabla_b n\nabla_a n\nabla_a\nabla_b\phi - n^{-1}\nabla_a\nabla_b n\nabla_a\nabla_b\phi + n^{-1}\nabla_b n[\nabla_b, \Delta]\phi.
\end{aligned}$$

Now, we have the following commutator formula:

$$[\nabla_b, \Delta]\phi = R_b{}^c\nabla_c\phi = (\mathbf{R}_{b00}{}^c + k_{bd}k^{dc})\nabla_c\phi, \tag{C.6}$$

where we used the Gauss equation for  $R$ , the Einstein equations for  $\mathbf{R}$  and the maximal foliation assumption. Thus, we obtain:

$$\begin{aligned}
& [n^{-1}\nabla n\nabla, \Delta]\phi \tag{C.7} \\
= & (-n^{-1}\nabla_a\nabla_b n + n^{-2}\nabla_b n\nabla_a n)\nabla_a\nabla_b\phi + (-n^{-1}\nabla_b(\Delta n) + n^{-2}\nabla_a n\nabla_a\nabla_b n \\
& + 2(\mathbf{R}_{b00a} + k_{ba}k_a{}^c)n^{-1}\nabla_a n)\nabla_b\phi.
\end{aligned}$$



Finally, (C.1), (C.5) and (C.7) yield:

$$\begin{aligned}
& [\partial_0 \partial_0, \Delta] \phi \\
= & [\partial_0, \Delta] \partial_0 \phi + \partial_0 ([\partial_0, \Delta] \phi) \\
= & -4k^{ab} \nabla_a \nabla_b (\partial_0 \phi) + 4n^{-1} \nabla_b n \nabla_b (\partial_0 (\partial_0 \phi)) \\
& + (-2\nabla_0 k^{ab} + 4k^{ac} k_c^b - n^{-1} \nabla_a \nabla_b n + n^{-2} \nabla_b n \nabla_a n) \nabla_a \nabla_b \phi \\
& + (2n^{-1} \nabla_b (\partial_0 n) - 12k^{ab} n^{-1} \nabla_a n) \nabla_b (\partial_0 \phi) + (2n^{-1} \Delta n + 2n^{-2} |\nabla n|^2) \partial_0 (\partial_0 \phi) \\
& + (2k^{ac} \mathbf{R}_{0acb} + 2k^{ac} \nabla_c k_{ab} - 2n^{-1} \nabla_a n \nabla_0 k^{ab} + 2k^{ab} n^{-1} \nabla_a (\partial_0 n) + 4k^{ac} k_{cb} n^{-1} \nabla_a n \\
& + 2|k|^2 n^{-1} \nabla_b n - 2k^{ab} n^{-2} \nabla_a n \partial_0 n - n^{-1} \nabla_b (\Delta n) + n^{-2} \nabla_a n \nabla_a \nabla_b n \\
& + 2(\mathbf{R}_{b00a} + k_{ba} k_a^c) n^{-1} \nabla_a n) \nabla_b \phi \\
& + (n^{-1} \Delta (\partial_0 n) - 4k^{ab} n^{-1} \nabla_a \nabla_b n + 2n^{-2} \nabla_b n \nabla_b (\partial_0 n)) \partial_0 \phi,
\end{aligned}$$

from which (7.2) easily follows. This concludes the proof of Lemma 7.1.

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