Overview of the proof of the Bounded L^2 Curvature Conjecture

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Summary

This memoir contains an overview of the proof of the bounded L^2 curvature conjecture. More precisely we show that the time of existence of a classical solution to the Einstein-vacuum equations depends only on the L^2-norm of the curvature and a lower bound on the volume radius of the corresponding initial data set. We note that though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to another, more subtle, scaling tied to its causal geometry. Indeed, L^2 bounds on the curvature is the minimum requirement necessary to obtain lower bounds on the radius of injectivity of causal boundaries. We note also that, while the first nontrivial improvements for well posedness for quasilinear hyperbolic systems in spacetime dimensions greater than $1+1$ (based on Strichartz estimates) were obtained in [**2**], [**3**], [**49**], [**50**], [**19**] and optimized in [**20**], [**36**], the result we present here is the first in which the full structure of the quasilinear hyperbolic system, not just its principal part, plays a crucial role.

The entire proof is obtained in the following sequence of papers

S. Klainerman, I. Rodnianski, J. Szeftel, The bounded L^2 curvature conjecture. arXiv:1204.1767, 91 pp. This is the main part of the series in which the proof is completed based on the results of the papers below.

J. Szeftel, Parametrix for wave equations on a rough background I: regularity of the phase at initial time. arXiv:1204.1768, 145 pp.

J. Szeftel, Parametrix for wave equations on a rough background II: control of the parametrix at initial time. arXiv:1204.1769, 84 pp.

J. Szeftel, Parametrix for wave equations on a rough background III: space-time regularity of the phase. arXiv:1204.1770, 276 pp.

J. Szeftel, Parametrix for wave equations on a rough background IV: Control of the error term. arXiv:1204.1771, 284 pp.

J. Szeftel, Sharp Strichartz estimates for the wave equation on a rough background. arXiv:1301.0112, 30 pp.

Contents

CHAPTER 1

Introduction

1.1. General Introduction

We present a summary of our proof of the bounded L^2-curvature conjecture in General Relativity. According to the conjecture the time of existence of a classical solution to the Einstein-vacuum equations depends only on the L^2-norm of the curvature and a lower bound on the volume radius of the corresponding initial data set. At a deep level the L^2 curvature conjecture concerns the relationship between the curvature tensor and the causal geometry of an Einstein vacuum space-time. Thus, though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to a different scaling, which we call *null scaling*, tied to its causal properties. More precisely, L^2 curvature bounds are strictly necessary to obtain lower bounds on the radius of injectivity of causal boundaries. These lower bounds turn out to be crucial for the construction of parametrices and derivation of bilinear and trilinear spacetime estimates for solutions to scalar wave equations. We note also that, while the first nontrivial improvements for well posedness for quasilinear hyperbolic systems in spacetime dimensions greater than $1+1$ (based on Strichartz estimates) were obtained in [**2**], [**3**], [**49**], [**50**], [**19**] and optimized in [**20**], [**36**], the result we present here is the first in which the full structure of the quasilinear hyperbolic system, not just its principal part, plays a crucial role.

1.1.1. Initial value problem. We consider the Einstein vacuum equations (EVE),

$$\mathbf{Ric}_{\alpha\beta} = 0 \tag{1.1}$$

where $\mathbf{Ric}_{\alpha\beta}$ denotes the Ricci curvature tensor of a four dimensional Lorentzian space time $(\mathcal{M}, \mathbf{g})$. An initial data set for (1.1) consists of a three dimensional 3-surface Σ_0 together with a Riemannian metric g and a symmetric 2-tensor k verifying the constraint equations,

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \mathrm{tr} k = 0, \\ R_{scal} - |k|^2 + (\mathrm{tr} k)^2 = 0, \end{cases} \tag{1.2}$$

where the covariant derivative ∇ is defined with respect to the metric g, R_{scal} is the scalar curvature of g, and $\mathrm{tr}k$ is the trace of k with respect to the metric g. In this work we restrict ourselves to asymptotically flat initial data sets with one end. For a given initial data set the Cauchy problem consists in finding a metric $\mathbf{g}$ satisfying (1.1) and an embedding of Σ_0 in $\mathcal{M}$ such that the metric induced by $\mathbf{g}$ on Σ_0 coincides with g and the 2-tensor k is the second fundamental form of the hypersurface $\Sigma_0 \subset \mathcal{M}$. The first local existence and uniqueness result for (EVE) was established by Y.C. Bruhat, see [**5**], with the help of wave coordinates which allowed her to cast the Einstein vacuum equations in the form of a system of nonlinear wave equations to which one can apply[1] the standard theory of nonlinear hyperbolic systems. The optimal, classical[2] result states the following,

THEOREM 1.1 (Classical local existence [**12**] [**14**]). *Let* (Σ_0, g, k) *be an initial data set for the Einstein vacuum equations* (1.1). *Assume that* Σ_0 *can be covered by a locally finite system of coordinate charts, related to each other by* C^1 *diffeomorphisms, such that* $(g, k) \in H^s_{loc}(\Sigma_0) \times H^{s-1}_{loc}(\Sigma_0)$ *with* $s > \frac{5}{2}$. *Then there exists a unique*[3] *(up to an isometry) globally hyperbolic development* $(\mathcal{M}, \mathbf{g})$, *verifying* (1.1), *for which* Σ_0 *is a Cauchy hypersurface*[4].

1.1.2. Bounded L^2 curvature conjecture. The classical exponents $s > 5/2$ are clearly not optimal. By straightforward scaling considerations one might expect to make sense of the initial value problem for $s \geq s_c = 3/2$, with s_c the natural scaling exponent for L^2 based Sobolev norms. Note that for $s = s_c = 3/2$ a local in time existence result, for sufficiently small data, would be equivalent to a global result. More precisely any smooth initial data, small in the corresponding critical norm, would be globally smooth. Such a well-posedness (WP) result would be thus comparable with the so called ϵ- regularity results for nonlinear elliptic and parabolic problems, which play such a fundamental role in the global regularity properties of general solutions. For quasilinear hyperbolic problems critical WP results have only been established in the case of $1 + 1$ dimensional systems, or spherically symmetric solutions of higher dimensional problems, in which case the L^2-Sobolev norms can be replaced by bounded variation (BV) type norms[5]. A particularly important example of this type is the critical BV well-posedness result established by Christodoulou for spherically symmetric solutions of the Einstein equations coupled with a scalar field, see [**7**]. The result played a crucial role in his famous Cosmic Censorship results for the same model, see [**8**]. As well known, unfortunately, the BV-norms are completely inadequate in higher dimensions; the only norms which can propagate the regularity properties of the data are necessarily L^2 based.

The quest for optimal well-posedness in higher dimensions has been one of the major themes in non-linear hyperbolic PDE's in the last twenty years. Major advances have been made in the particular case of semi-linear wave equations. In the case of geometric

[1]The original proof in [**5**] relied on representation formulas, following an approach pioneered by Sobolev, see [**37**].

[2]Based only on energy estimates and classical Sobolev inequalities.

[3]The original proof in [**12**], [**14**] actually requires one more derivative for the uniqueness. The fact that uniqueness holds at the same level of regularity than the existence has been obtained in [**33**]

[4]That is any past directed, in-extendable causal curve in $\mathcal{M}$ intersects Σ_0.

[5]Recall that the entire theory of shock waves for 1+1 systems of conservation laws is based on BV norms, which are *critical* with respect to the scaling of the equations. Note also that these BV norms are not, typically, conserved and that Glimm's famous existence result [**13**] can be interpreted as a global well posedness result for initial data with small BV norms.

wave equations such as Wave Maps and Yang-Mills, which possess a well understood null structure, well-posedness holds true for all exponents larger than the corresponding critical exponent. For example, in the case of Wave Maps defined from the Minkowski space $\mathbb{R}^{n+1}$ to a complete Riemannian manifold, the critical scaling exponents is $s_c = n/2$ and well-posedness is known to hold all the way down to s_c for all dimensions $n \geq 2$. This critical well-posedness result, for $s = n/2$, plays a fundamental role in the recent, large data, global results of [**47**], [**40**], [**41**] and [**28**] for $2+1$ dimensional wave maps.

The role played by critical exponents for quasi-linear equations is much less understood. The first well posedness results, on any (higher dimensional) quasilinear hyperbolic system, which go beyond the classical Sobolev exponents, obtained in [**2**], [**3**], and [**49**], [**50**] and [**19**], do not take into account the specific (null) structure of the equations. Yet the presence of such structure was crucial in the derivation of the optimal results mentioned above, for geometric semilinear equations. In the case of the Einstein equations it is not at all clear what such structure should be, if there is one at all. Indeed, the only specific structural condition, known for (EVE), discovered in [**30**] under the name of the *weak null condition*, is not at all adequate for improved well posedness results, see remark 1.3. It is known however, see [**29**], that without such a structure one cannot have well posedeness for exponents[6] $s \leq 2$. Yet (EVE) are of fundamental importance and as such it is not unreasonable to expect that such a structure must exist.

Even assuming such a structure, a result of well-posedness for the Einstein equations at, or near, the critical regularity $s_c = 3/2$ is not only completely out of reach but may in fact be wrong. This is due to the presence of a different scaling connected to the geometry of boundaries of causal domains. It is because of this more subtle scaling that we need at least L^2-bounds for the curvature to derive a lower bound on the radius of injectivity of null hypersurfaces and thus control their local regularity properties. This imposes a crucial obstacle to well posedness below $s = 2$. Indeed, as we will show in the next subsection, any such result would require, crucially, bilinear and even trillinear estimates for solutions to wave equations of the form $\square_{\mathbf{g}}\phi = F$. Such estimates, however, depend on Fourier integral representations, with a phase function u which solves the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$. Thus the much needed bilinear estimates depend, ultimately, on the regularity properties of the level hypersurfaces of the phase u which are, of course, null. The catastrophic breakdown of the regularity of these null hypersurfaces, in the absence of a lower bound for the injectivity radius, would make these Fourier integral representations entirely useless.

These considerations lead one to conclude that, the following conjecture, proposed in [**18**], is most probably sharp in so far as the minimal number of derivatives in L^2 is concerned:

Conjecture[Bounded L^2 Curvature Conjecture (BCC)] *The Einstein- vacuum equations admit local Cauchy developments for initial data sets* (Σ_0, g, k) *with locally finite* L^2

[6]Note that the dimension here is $n = 3$.

curvature and locally finite L^2 norm of the first covariant derivatives of k[7].

REMARK 1.2. *It is important to emphasize here that the conjecture should be primarily interpreted as a continuation argument for the Einstein equations; that is the space-time constructed by evolution from smooth data can be smoothly continued, together with a time foliation, as long as the curvature of the foliation and the first covariant derivatives of its second fundamental form remain L^2- bounded on the leaves of the foliation. In fact the conjecture implies the break-down criterion previously obtained in* [**26**] *and improved in* [**31**], [**52**]. *According to that criterion a vacuum space-time, endowed with a constant mean curvature (CMC) foliation Σ_t, can be extended, together with the foliation, as long as the $L^1_t L^\infty(\Sigma_t)$ norm of the deformation tensor of the future unit normal to the foliation remains bounded. It is straightforward to see, by standard energy estimates, that this condition implies bounds for the $L^\infty_t L^2(\Sigma_t)$ norm of the space-time curvature from which one can derive bounds for the induced curvature tensor R and the first derivatives of the second fundamental form k. Thus, if we can ensure that the time of existence of a space-time foliated by Σ_t depends only on the L^2 norms of R and first covariant derivatives of k, we can extend the space-time indefinitely.*

1.1.3. Brief history. The conjecture has its roots in the remarkable developments of the last twenty years centered around the issue of optimal well-posedness for semilinear wave equations. The case of the Einstein equations turns out to be a lot more complicated due to the quasilinear character of the equations. To make the discussion more tangible it is worthwhile to recall the form of the Einstein vacuum equations in the wave gauge. Assuming given coordinates x^α, verifying $\square_{\mathbf{g}} x^\alpha = 0$, the metric coefficients $g_{\alpha\beta} = \mathbf{g}(\partial_\alpha, \partial_\beta)$, with respect to these coordinates, verify the system of quasilinear wave equations,

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g) \tag{1.3}$$

where $F_{\alpha\beta}$ are quadratic functions of ∂g, i.e. the derivatives of g with respect to the coordinates x^α. In a first approximation we may compare (1.3) with the semilinear wave equation,

$$\square\phi = F(\phi, \partial\phi) \tag{1.4}$$

with F quadratic in $\partial\phi$. Using standard energy estimates, one can prove an estimate, roughly, of the form:

$$\|\phi(t)\|_s \lesssim \|\phi(0)\|_s \exp\left(C_s \int_0^t \|\partial\phi(\tau)\|_{L^\infty} d\tau\right).$$

The classical exponent $s > 3/2 + 1$ arises simply from the Sobolev embedding of H^r, $r > 3/2$ into L^∞. To go beyond the classical exponent, see [**34**], one has to replace

[7]As we shall see, from the precise theorem stated below, other weaker conditions, such as a lower bound on the volume radius, are needed.

Sobolev inequalities with Strichartz estimates of, roughly, the following type,

$$\left(\int_0^t \|\partial\phi(\tau)\|_{L^\infty}^2 d\tau\right)^{1/2} \lesssim C\left(\|\partial\phi(0)\|_{H^{1+\varepsilon}} + \int_0^t \|\Box\phi(\tau)\|_{H^{1+\varepsilon}}\right)$$

where $\varepsilon > 0$ can be chosen arbitrarily small. This leads to a gain of $1/2$ derivatives, i.e. we can prove well-posedness for equations of type (1.4) for any exponent $s > 2$.

The same type of improvement in the case of quasilinear equations requires a highly non-trivial extension of such estimates for wave operators with non-smooth coefficients. The first improved regularity results for quasilinear wave equations of the type,

$$g^{\mu\nu}(\phi)\partial_\mu\partial_\nu\phi = F(\phi, \partial\phi) \tag{1.5}$$

with $g^{\mu\nu}(\phi)$ a non-linear perturbation of the Minkowski metric $m^{\mu\nu}$, are due to [**2**], [**3**], and [**49**], [**50**] and [**19**]. The best known results for equations of type (1.3) were obtained in [**20**] and [**36**]. According to them one can lower the Sobolev exponent $s > 5/2$ in Theorem 1.1 to $s > 2$. It turns out, see [**29**], that these results are sharp in the general class of quasilinear wave equations of type (1.3). To do better one needs to take into account the special structure of the Einstein equations and rely on a class of estimates which go beyond Strichartz, namely the so called bilinear estimates[8].

In the case of semilinear wave equations, such as Wave Maps, Maxwell-Klein-Gordon and Yang-Mills, the first results which make use of bilinear estimates go back to [**15**], [**16**], [**17**]. In the particular case of the Maxwell-Klein-Gordon and Yang-Mills equation the main observation was that, after the choice of a special gauge (Coulomb gauge), the most dangerous nonlinear terms exhibit a special, null structure for which one can apply the bilinear estimates derived in [**15**]. With the help of these estimates one was able to derive a well posedness result, in the flat Minkowski space $\mathbb{R}^{1+3}$, for the exponent $s = s_c + 1/2 = 1$, where $s_c = 1/2$ is the critical Sobolev exponent in that case[9].

To carry out a similar program in the case of the Einstein equations one would need, at the very least, the following crucial ingredients:

- **A**. *Provide a coordinate condition, relative to which the Einstein vacuum equations verifies an appropriate version of the null condition.*
- **B**. *Provide an appropriate geometric framework for deriving bilinear estimates for the null quadratic terms appearing in the previous step.*
- **C**. *Construct an effective progressive wave representation Φ_F (parametrix) for solutions to the scalar linear wave equation $\Box_{\mathbf{g}}\phi = F$, derive appropriate bounds for both the parametrix and the corresponding error term $E = F - \Box_{\mathbf{g}}\Phi_F$ and use them to derive the desired bilinear estimates.*

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[8]Note that no such result, i.e. well-posedness for $s = 2$, is presently known for either scalar equations of the form (1.5) or systems of the form (1.3).

[9]This corresponds precisely to the $s = 2$ exponent in the case of the Einstein-vacuum equations

As it turns out, the proof of several bilinear estimates of Step B reduces to the proof of sharp $L^4(\mathcal{M})$ Strichartz estimates for a localized version of the parametrix of step C. Thus we will also need the following fourth ingredient.

D. *Prove sharp $L^4(\mathcal{M})$ Strichartz estimates for a localized version of the parametrix of step C.*

Note that the last three steps are to be implemented using only hypothetical L^2 bounds for the space-time curvature tensor, consistent with the conjectured result. To start with, it is not at all clear what should be the correct coordinate condition, or even if there is one for that matter.

REMARK 1.3. *As mentioned above, the only known structural condition related to the classical null condition, called the weak null condition, tied to wave coordinates, fails the test. Indeed, the following simple system in Minkowski space verifies the weak null condition and yet, according to* [**29**], *it is ill posed for $s = 2$.*

$$\Box\phi = 0, \qquad \Box\psi = \phi \cdot \Delta\phi.$$

Coordinate conditions, such as spatial harmonic[10], *also do not seem to work.*

We rely instead on a Coulomb type condition, for orthonormal frames, adapted to a maximal foliation. Such a gauge condition appears naturally if we adopt a Yang-Mills description of the Einstein field equations using Cartan's formalism of moving frames[11], see [**6**]. It is important to note nevertheless that it is not at all a priori clear that such a choice would do the job. Indeed, the null form nature of the Yang-Mills equations in the Coulomb gauge is only revealed once we commute the resulting equations with the projection operator $\mathcal{P}$ on the divergence free vectorfields. Such an operation is natural in that case, since $\mathcal{P}$ commutes with the flat d'Alembertian. In the case of the Einstein equations, however, the corresponding commutator term $[\Box_{\mathbf{g}}, \mathcal{P}]$ generates[12] a whole host of new terms and it is quite a miracle that they can all be treated by an extended version of bilinear estimates. At an even more fundamental level, the flat Yang-Mills equations possess natural energy estimates based on the time symmetry of the Minkowski space. There are no such timelike Killing vectorfield in curved space. We have to rely instead on the future unit normal to the maximal foliation Σ_t whose deformation tensor is non-trivial. This leads to another class of nonlinear terms which have to be treated by a novel trilinear estimate.

We will make more comments concerning the implementations of all four ingredients later on, in the section 1.2.4.

[10]Maximal foliation together with spatial harmonic coordinates on the leaves of the foliation would be the coordinate condition closest in spirit to the Coulomb gauge.

[11]We would like to thank L. Anderson for pointing out to us the possibility of using such a formalism as a potential bridge to [**16**] .

[12]Note also that additional error terms are generated by projecting the equations on the components of the frame.

REMARK 1.4. *In addition to the ingredients mentioned above, we also need a mechanism of reducing the proof of the conjecture to small data, in an appropriate sense. Indeed, even in the flat case, the Coulomb gauge condition cannot be globally imposed for large data. In fact* [**17**] *relied on a cumbersome technical device based on local Coulomb gauges, defined on domain of dependence of small balls. Here we rely instead on a variant of the gluing construction of* [**10**], [**11**], *see section 1.2.3.*

1.2. Statement of the main results

1.2.1. Maximal foliations. In this section, we recall some well-known facts about maximal foliations (see for example the introduction in [**9**]). We assume the space-time $(\mathcal{M}, \mathbf{g})$ to be foliated by the level surfaces Σ_t of a time function t. Let T denote the unit normal to Σ_t, and let k the the second fundamental form of Σ_t, i.e. $k_{ab} = -\mathbf{g}(\mathbf{D}_a T, e_b)$, where $e_a, a = 1, 2, 3$ denotes an arbitrary frame on Σ_t and $\mathbf{D}_a T = \mathbf{D}_{e_a} T$. We assume that the Σ_t foliation is maximal, i.e. we have:

$$\mathrm{tr}_g k = 0 \tag{1.6}$$

where g is the induced metric on Σ_t. The constraint equations on Σ_t for a maximal foliation are given by:

$$\nabla^a k_{ab} = 0, \tag{1.7}$$

where ∇ denotes the induced covariant derivative on Σ_t, and

$$R_{scal} = |k|^2. \tag{1.8}$$

Also, we denote by n the lapse of the t-foliation, i.e. $n^{-2} = -\mathbf{g}(\mathbf{D}t, \mathbf{D}t)$. n satisfies the following elliptic equation on Σ_t:

$$\Delta n = n|k|^2. \tag{1.9}$$

Finally, we recall the structure equations of the maximal foliation:

$$\nabla_0 k_{ab} = \mathbf{R}_{a\,0\,b\,0} - n^{-1}\nabla_a \nabla_b n - k_{ac} k_b{}^c, \tag{1.10}$$

$$\nabla_a k_{bc} - \nabla_b k_{ac} = \mathbf{R}_{c0ab} \tag{1.11}$$

and:

$$R_{ab} - k_{ac} k^c{}_b = \mathbf{R}_{a0b0}. \tag{1.12}$$

1.2.2. Main Theorem. We recall below the definition of the volume radius on a general Riemannian manifold M.

DEFINITION 1.5. *Let $B_r(p)$ denote the geodesic ball of center p and radius r. The volume radius $r_{vol}(p, r)$ at a point $p \in M$ and scales $\leq r$ is defined by*

$$r_{vol}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r^3},$$

with $|B_r|$ the volume of B_r relative to the metric on M. The volume radius $r_{vol}(M,r)$ of M on scales $\le r$ is the infimum of $r_{vol}(p,r)$ over all points $p \in M$.

Our main result is the following:

THEOREM 1.6 (Main theorem). *Let $(\mathcal{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum equations (1.1) together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Assume that the initial slice (Σ_0, g, k) is such that the Ricci curvature $Ric \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$, and Σ_0 has a strictly positive volume radius on scales ≤ 1, i.e. $r_{vol}(\Sigma_0, 1) > 0$. Then,*

(1) L^2 ***regularity.*** *There exists a time*

$$T = T(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

and a constant

$$C = C(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

such that the following control holds on $0 \le t \le T$:

$$\|\mathbf{R}\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \le C,\ \|\nabla k\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \le C \text{ and } \inf_{0 \le t \le T} r_{vol}(\Sigma_t, 1) \ge \frac{1}{C}.$$

(2) ***Higher regularity.*** *Within the same time interval as in part (1) we also have the higher derivative estimates*[13]*,*

$$\sum_{|\alpha| \le m} \|\mathbf{D}^{(\alpha)} \mathbf{R}\|_{L^\infty_{[0,T]} L^2(\Sigma_t)} \le C_m \sum_{|i| \le m} \Big[\|\nabla^{(i)} Ric\|_{L^2(\Sigma_0)} + \|\nabla^{(i)} \nabla k\|_{L^2(\Sigma_0)} \Big], \qquad (1.13)$$

where C_m depends only on the previous C and m.

REMARK 1.7. *Since the core of the main theorem is local in nature we do not need to be very precise here with our asymptotic flatness assumption. We may thus assume the existence of a coordinate system at infinity, relative to which the metric has two derivatives bounded in L^2, with appropriate asymptotic decay. Note that such bounds could be deduced from weighted L^2 bounds assumptions for Ric and ∇k.*

REMARK 1.8. *Note that the dependence on $\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}$ in the main theorem can be replaced by dependence on $\|\mathbf{R}\|_{L^2(\Sigma_0)}$ where $\mathbf{R}$ denotes the space-time curvature tensor*[14]*. Indeed this follows from the following well known L^2 estimate (see section 8 in* [**26**]*).*

$$\int_{\Sigma_0} |\nabla k|^2 + \frac{1}{4}|k|^4 \le \int_{\Sigma_0} |\mathbf{R}|^2. \qquad (1.14)$$

and the Gauss equation relating Ric to $\mathbf{R}$.

[13] Assuming that the initial has more regularity so that the right-hand side of (1.13) makes sense.

[14] Here and in what follows the notations $R, \mathbf{R}$ will stand for the Riemann curvature tensors of Σ_t and $\mathcal{M}$, while Ric, $\mathbf{Ric}$ and $R_{scal}, \mathbf{R}_{scal}$ will denote the corresponding Ricci and scalar curvatures.

1.2.3. Reduction to small initial data. We first need an appropriate covering of Σ_0 by harmonic coordinates. This is obtained using the following general result based on Cheeger-Gromov convergence of Riemannian manifolds.

THEOREM 1.9 ([**1**] or Theorem 5.4 in [**32**]). *Given $c_1 > 0, c_2 > 0, c_3 > 0$, there exists $r_0 > 0$ such that any 3-dimensional, complete, Riemannian manifold (M, g) with $\|Ric\|_{L^2(M)} \leq c_1$ and volume radius at scales ≤ 1 bounded from below by c_2, i.e. $r_{vol}(M, 1) \geq c_2$, verifies the following property:*

Every geodesic ball $B_r(p)$ with $p \in M$ and $r \leq r_0$ admits a system of harmonic coordinates $x = (x_1, x_2, x_3)$ relative to which we have

$$(1 + c_3)^{-1}\delta_{ij} \leq g_{ij} \leq (1 + c_3)\delta_{ij}, \tag{1.15}$$

and

$$r \int_{B_r(p)} |\partial^2 g_{ij}|^2 \sqrt{|g|} dx \leq c_3. \tag{1.16}$$

We consider $\varepsilon > 0$ which will be chosen as a small universal constant. We apply theorem 1.9 to the Riemannian manifold Σ_0. Then, there exists a constant:

$$r_0 = r_0(\|\text{Ric}\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1), \varepsilon) > 0$$

such that every geodesic ball $B_r(p)$ with $p \in \Sigma_0$ and $r \leq r_0$ admits a system of harmonic coordinates $x = (x_1, x_2, x_3)$ relative to which we have:

$$(1 + \varepsilon)^{-1}\delta_{ij} \leq g_{ij} \leq (1 + \varepsilon)\delta_{ij},$$

and

$$r \int_{B_r(p)} |\partial^2 g_{ij}|^2 \sqrt{|g|} dx \leq \varepsilon.$$

Now, by the asymptotic flatness of Σ_0, the complement of its end can be covered by the union of a finite number of geodesic balls of radius r_0, where the number N_0 of geodesic balls required only depends on r_0. In particular, it is therefore enough to obtain the control of $\mathbf{R}$, k and $r_{vol}(\Sigma_t, 1)$ of Theorem 1.6 when one restricts to the domain of dependence of one such ball. Let us denote this ball by B_{r_0}. Next, we rescale the metric of this geodesic ball by:

$$g_\lambda(t, x) = g(\lambda t, \lambda x),\ \lambda = \min\left(\frac{\varepsilon^2}{\|R\|^2_{L^2(B_{r_0})}}, \frac{\varepsilon^2}{\|\nabla k\|^2_{L^2(B_{r_0})}}, r_0\varepsilon\right) > 0.$$

Let[15] R_λ, k_λ and $B^\lambda_{r_0}$ be the rescaled versions of R, k and B_{r_0}. Then, in view of our choice for λ, we have:

$$\|R_\lambda\|_{L^2(B^\lambda_{r_0})} = \sqrt{\lambda}\|R\|_{L^2(B_{r_0})} \leq \varepsilon,$$
$$\|\nabla k_\lambda\|_{L^2(B^\lambda_{r_0})} = \sqrt{\lambda}\|\nabla k\|_{L^2(B_{r_0})} \leq \varepsilon,$$

[15]Since in what follows there is no danger to confuse the Ricci curvature Ric with the scalar curvature R we use the short hand R to denote the full curvature tensor Ric.

and

$$\|\partial^2 g_\lambda\|_{L^2(B^\lambda_{r_0})} = \sqrt{\lambda}\|\partial^2 g\|_{L^2(B_{r_0})} \leq \sqrt{\frac{\lambda\varepsilon}{r_0}} \leq \varepsilon.$$

Note that $B^\lambda_{r_0}$ is the rescaled version of B_{r_0}. Thus, it is a geodesic ball for g_λ of radius $\frac{r_0}{\lambda} \geq \frac{1}{\varepsilon} \geq 1$. Now, considering g_λ on $0 \leq t \leq 1$ is equivalent to considering g on $0 \leq t \leq \lambda$. Thus, since r_0, N_0 and λ depend only on $\|R\|_{L^2(\Sigma_0)}$, $\|\nabla k\|_{L^2(\Sigma_0)}$, $r_{vol}(\Sigma_0, 1)$ and ε, Theorem 1.6 is equivalent to the following theorem:

Theorem 1.10 (Main theorem, version 2). *Let $(\mathcal{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum equations* (1.1) *together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Let B a geodesic ball of radius one in Σ_0, and let D its domain of dependence. Assume that the initial slice (Σ_0, g, k) is such that:*

$$\|R\|_{L^2(B)} \leq \varepsilon,\ \|\nabla k\|_{L^2(B)} \leq \varepsilon \text{ and } r_{vol}(B, 1) \geq \frac{1}{2}.$$

Let $B_t = D \cap \Sigma_t$ the slice of D at time t. Then:

(1) L^2 ***regularity.*** *There exists a small universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the following control holds on $0 \leq t \leq 1$:*

$$\|\mathbf{R}\|_{L^\infty_{[0,1]}L^2(B_t)} \lesssim \varepsilon,\ \|\nabla k\|_{L^\infty_{[0,1]}L^2(B_t)} \lesssim \varepsilon \text{ and } \inf_{0 \leq t \leq 1} r_{vol}(B_t, 1) \geq \frac{1}{4}.$$

(2) ***Higher regularity.*** *The following control holds on $0 \leq t \leq 1$:*

$$\sum_{|\alpha| \leq m} \|\mathbf{D}^{(\alpha)}\mathbf{R}\|_{L^\infty_{[0,1]}L^2(B_t)} \lesssim \|\nabla^{(i)} Ric\|_{L^2(B)} + \|\nabla^{(i)}\nabla k\|_{L^2(B)}. \tag{1.17}$$

Notation: In the statement of Theorem 1.10, and in the rest of the paper, the notation $f_1 \lesssim f_2$ for two real positive scalars f_1, f_2 means that there exists a universal constant $C > 0$ such that:

$$f_1 \leq C f_2.$$

Theorem 1.10 is not yet in suitable form for our proof since some of our constructions will be global in space and may not be carried out on a subregion B of Σ_0. Thus, we glue a smooth asymptotically flat solution of the constraint equations (1.2) outside of B, where the gluing takes place in an annulus just outside B. This can be achieved using the construction in [**10**], [**11**]. We finally get an asymptotically flat solution to the constraint equations, defined everywhere on Σ_0, which agrees with our original data set (Σ_0, g, k) inside B. We still denote this data set by (Σ_0, g, k). It satisfies the bounds:

$$\|R\|_{L^2(\Sigma_0)} \leq 2\varepsilon,\ \|\nabla k\|_{L^2(\Sigma_0)} \leq 2\varepsilon \text{ and } r_{vol}(\Sigma_0, 1) \geq \frac{1}{4}.$$

REMARK 1.11. *Notice that the gluing process in* [**10**]–[**11**] *requires the kernel of a certain linearized operator to be trivial. This is achieved by conveniently choosing the asymptotically flat solution to* (1.2) *that is glued outside of B to our original data set. This choice is always possible since the metrics for which the kernel is nontrivial are non generic (see* [**4**]*).*

REMARK 1.12. *Assuming only L^2 bounds on R and ∇k is not enough to carry out the construction in the above mentioned results. However, the problem solved there remains subcritical at our desired level of regularity and thus we believe that a closer look at the construction in* [**10**]–[**11**]*, or an alternative construction, should be able to provide the desired result. This is an open problem.*

REMARK 1.13. *Since $\|k\|^2_{L^4(\Sigma_0)} \leq \|Ric\|_{L^2}$ we deduce that $\|k\|_{L^2(B)} \lesssim \varepsilon^{1/2}$ on the geodesic ball B of radius one. Furthermore, asymptotic flatness is compatible with a decay of $|x|^{-2}$ at infinity, and in particular with k in $L^2(\Sigma_0)$. So we may assume that the gluing process is such that the resulting k satisfies:*

$$\|k\|_{L^2(\Sigma_0)} \lesssim \varepsilon.$$

Finally, we have reduced Theorem 1.6 to the case of a small initial data set:

THEOREM 1.14 (Main theorem, version 3). *Let $(\mathcal{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum equations* (1.1) *together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Assume that the initial slice (Σ_0, g, k) is such that:*

$$\|R\|_{L^2(\Sigma_0)} \leq \varepsilon,\ \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \varepsilon \text{ and } r_{vol}(\Sigma_0, 1) \geq \frac{1}{2}.$$

Then:

(1) L^2 ***regularity.*** *There exists a small universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, the following control holds on $0 \leq t \leq 1$:*

$$\|\mathbf{R}\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \varepsilon,\ \|k\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \varepsilon \text{ and } \inf_{0\leq t\leq 1} r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}.$$

(2) ***Higher regularity.*** *The following control holds on $0 \leq t \leq 1$:*

$$\sum_{|\alpha|\leq m} \|\mathbf{D}^{(\alpha)}\mathbf{R}\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \|\nabla^{(i)}Ric\|_{L^2(\Sigma_t)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma_t)}. \tag{1.18}$$

The rest of this paper is devoted to the proof of Theorem 1.14.

1.2.4. Strategy of the proof. The proof of Theorem 1.14 consists in four steps.

Step A (Yang-Mills formalism) We first cast the Einstein-vacuum equations within a Yang-Mills formalism. This relies on the Cartan formalism of moving frames. The idea is to give up on a choice of coordinates and express instead the Einstein vacuum equations in terms of the connection 1-forms associated to moving orthonormal frames, i.e. vectorfields e_α, which verify,

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \text{diag}(-1, 1, 1, 1).$$

The connection 1-forms (they are to be interpreted as 1-forms with respect to the external index μ with values in the Lie algebra of $so(3,1)$), defined by the formulas,

$$(\mathbf{A}_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha) \tag{1.19}$$

verify the equations,

$$\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0 \tag{1.20}$$

where, denoting $(\mathbf{F}_{\mu\nu})_{\alpha\beta} := \mathbf{R}_{\alpha\beta\mu\nu}$,

$$(\mathbf{F}_{\mu\nu})_{\alpha\beta} = \big(\mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu]\big)_{\alpha\beta}. \tag{1.21}$$

In other words we can interpret the curvature tensor as the curvature of the $so(3,1)$-valued connection 1-form $\mathbf{A}$. Note also that the covariant derivatives are taken only with respect to the *external indices* μ, ν and do not affect the *internal indices* α, β. We can rewrite (1.20) in the form,

$$\square_{\mathbf{g}} \mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{J}_\nu(\mathbf{A}, \mathbf{D}\mathbf{A}) \tag{1.22}$$

where,

$$\mathbf{J}_\nu = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}_\mu, \mathbf{F}_{\mu\nu}].$$

Observe that the equations (1.20)-(1.21) look just like the Yang-Mills equations on a fixed Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ except, of course, that in our case $\mathbf{A}$ and $\mathbf{g}$ are not independent but connected rather by (1.19), reflecting the quasilinear structure of the Einstein equations. Just as in the case of [**15**], which establishes the well-posedness of the Yang-Mills equation in Minkowski space in the energy norm (i.e. $s = 1$), we rely in an essential manner on a Coulomb type gauge condition. More precisely, we take e_0 to be the future unit normal to the Σ_t foliation and choose e_1, e_2, e_3 an orthonormal basis to Σ_t, in such a way that we have, essentially (see precise discussion in section 2.1.2), $div\, A = \nabla^i A_i = 0$, where A is the spatial component of $\mathbf{A}$. It turns out that A_0 satisfies an elliptic equation while each component $A_i = \mathbf{g}(\mathbf{A}, e_i)$, $i = 1, 2, 3$ verifies an equation of the form,

$$\square_{\mathbf{g}} A_i \;\; = \;\; -\partial_i(\partial_0 A_0) + A^j \partial_j A_i + A^j \partial_i A_j + \text{ l.o.t.} \tag{1.23}$$

with l.o.t. denoting nonlinear terms which can be treated by more elementary techniques (including non sharp Strichartz estimates).

Step B (Bilinear and trilinear estimates) To eliminate $\partial_i(\partial_0 A_0)$ in (1.23), we need to project (1.23) onto divergence free vectorfields with the help of a non-local operator which we denote by $\mathcal{P}$. In the case of the flat Yang-Mills equations, treated in [**15**], this leads to an equation of the form,

$$\Box A_i \;=\; \mathcal{P}(A^j\partial_j A_i) + \mathcal{P}(A^j\partial_i A_j) + \text{ l.o.t.}$$

where both terms on the right can be handled by bilinear estimates. In our case we encounter however three fundamental differences with the flat situation of [**15**].

- To start with the operator $\mathcal{P}$ does not commute with $\Box_{\mathbf{g}}$. It turns out, fortunately, that the terms generated by commutation can still be estimated by an extended class of bilinear estimates which includes contractions with the curvature tensor, see section 2.2.5.
- All energy estimates used in [**15**] are based on the standard timelike Killing vectorfield ∂_t. In our case the corresponding vectorfield $e_0 = T$ (the future unit normal to Σ_t) is not Killing. This leads to another class of trilinear error terms which we discuss in sections 2.5 and 2.2.5.
- The main difference with [**15**] is that we now need bilinear and trilinear estimates for solutions of wave equations on background metrics which possess only limited regularity.

This last item is a major problem, both conceptually and technically. On the conceptual side we need to rely on a more geometric proof of bilinear estimates based on a plane wave representation formula[16] for solutions of scalar wave equations,

$$\Box_{\mathbf{g}}\phi = 0.$$

The proof of the bilinear estimates rests on the representation formula[17]

$$\phi_f(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u(t,x)}\, f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{1.24}$$

where f represents schematically the initial data[18], and where ${}^\omega u$ is a solution of the eikonal equation[19],

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\, {}^\omega u\, \partial_\beta\, {}^\omega u = 0, \tag{1.25}$$

with appropriate initial conditions on Σ_0 and $d\omega$ the area element of the standard sphere in $\mathbb{R}^3$.

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[16]We follow the proof of the bilinear estimates outlined in [**21**] which differs substantially from that of [**15**] and is reminiscent of the null frame space strategy used by Tataru in his fundamental paper [**48**].

[17](1.24) actually corresponds to the representation formula for a half-wave. The full representation formula corresponds to the sum of two half-waves (see section 2.7)

[18]Here f is in fact at the level of the Fourier transform of the initial data and the norm $\|\lambda f\|_{L^2(\mathbb{R}^3)}$ corresponds, roughly, to the H^1 norm of the data .

[19]In the flat Minkowski space ${}^\omega u(t,x) = t \pm x\cdot\omega$.

REMARK 1.15. *Note that* (1.24) *is a parametrix for a scalar wave equation. The lack of a good parametrix for a covariant wave equation forces us to develop a strategy based on writing the main equation in components relative to a frame, i.e. instead of dealing with the tensorial wave equation* (1.22) *directly, we consider the system of scalar wave equations* (1.23). *Unlike in the flat case, this scalarization procedure produces several terms which are potentially dangerous, and it is fortunate that they can still be controlled by the use of an extended*[20] *class of bilinear estimates.*

Step C (Control of the parametrix) To prove the bilinear and trilinear estimates of Step B, we need in particular to control the parametrix at initial time (i.e. restricted to the initial slice Σ_0)

$$\phi_f(0,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(0,x)}\, f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{1.26}$$

and the error term corresponding to (1.24)

$$Ef(t,x) = \square_{\mathbf{g}}\phi_f(t,x) = i\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(t,x)}\, (\square_{\mathbf{g}}\ {}^\omega u) f(\lambda\omega)\lambda^3 d\lambda d\omega \tag{1.27}$$

i.e. ϕ_f is an exact solution of $\square_{\mathbf{g}}\phi = 0$ only in flat space in which case $\square_{\mathbf{g}}\ {}^\omega u = 0$. This requires the following four sub steps

- **C1** *Make an appropriate choice for the equation satisfied by* ${}^\omega u(0,x)$ *on* Σ_0*, and control the geometry of the foliation of* Σ_0 *by the level surfaces of* ${}^\omega u(0,x)$.
- **C2** *Prove that the parametrix at* $t=0$ *given by* (1.26) *is bounded in* $\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma_0))$ *using the estimates for* ${}^\omega u(0,x)$ *obtained in* **C1**.
- **C3** *Control the geometry of the foliation of* $\mathcal{M}$ *given by the level hypersurfaces of* ${}^\omega u$.
- **C4** *Prove that the error term* (1.27) *satisfies the estimate* $\|Ef\|_{L^2(\mathcal{M})} \leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}$ *using the estimates for* ${}^\omega u$ *and* $\square_{\mathbf{g}}\ {}^\omega u$ *proved in* **C3**.

To achieve Step C3 and Step C4, we need, at the very least, to control $\square_{\mathbf{g}}\ {}^\omega u$ in L^∞. This issue was first addressed in the sequence of papers [**22**]–[**24**] where an L^∞ bound for $\square_{\mathbf{g}}\ {}^\omega u$ was established, depending only on the L^2 norm of the curvature flux along null hypersurfaces. The proof required an interplay between both geometric and analytic techniques and had all the appearances of being sharp, i.e. we don't expect an L^∞ bound for $\square_{\mathbf{g}}\ {}^\omega u$ which requires bounds on less than two derivatives in L^2 for the metric[21].

To obtain the L^2 bound for the Fourier integral operator E defined in (1.27), we need, of course, to go beyond uniform estimates for $\square_{\mathbf{g}}\ {}^\omega u$. The classical L^2 bounds for Fourier integral operators of the form (1.27) are not at all economical in terms of the number of integration by parts which are needed. In our case the total number of such integration by parts is limited by the regularity properties of the function $\square_{\mathbf{g}}\ {}^\omega u$. To get an L^2 bound

[20]such as contractions between the Riemann curvature tensor and derivatives of solutions of scalar wave equations.

[21]classically, this requires, at the very least, the control of $\mathbf{R}$ in L^∞

for the parametrix at initial time (1.26) and the error term (1.27) within such restrictive regularity properties we need, in particular:

- In Step C1 and Step C3, a precise control of derivatives of ${}^{\omega}u$ and $\square_{\mathbf{g}}\, {}^{\omega}u$ with respect to both ω as well as with respect to various directional derivatives[22]. To get optimal control we need, in particular, a very careful construction of the initial condition for ${}^{\omega}u$ on Σ_0 and then sharp space-time estimates of Ricci coefficients, and their derivatives, associated to the foliation induced by ${}^{\omega}u$.
- In Step C2 and Step C4, a careful decompositions of the Fourier integral operators (1.26) and (1.27) in both λ and ω, similar to the first and second dyadic decomposition in harmonic analysis, see [**39**], as well as a third decomposition, which in the case of (1.27) is done with respect to the space-time variables relying on the geometric Littlewood-Paley theory developed in [**24**].

Below, we make further comments on Steps C1-C4:

(1) *The choice of $u(0,x,\omega)$ on Σ_0 in Step C1.* Let us note that the typical choice $u(0,x,\omega) = x\cdot\omega$ in a given coordinate system would not work for us, since we don't have enough control on the regularity of a given coordinate system within our framework. Instead, we need to find a geometric definition of $u(0,x,\omega)$. A natural choice would be

$$\square_{\mathbf{g}} u = 0 \text{ on } \Sigma_0$$

which by a simple computation turns out to be the following simple variant of the minimal surface equation[23]

$$\mathrm{d}iv\left(\frac{\nabla u}{|\nabla u|}\right) = k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$

Unfortunately, this choice does not allow us to have enough control of the derivatives of u in the normal direction to the level surfaces of u. This forces us to look for an alternate equation for u:

$$\mathrm{d}iv\left(\frac{\nabla u}{|\nabla u|}\right) = 1 - \frac{1}{|\nabla u|} + k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$

This equation turns out to be parabolic in the normal direction to the level surfaces of u, and allows us to obtain the desired regularity in Step C1. On closer inspection it is related with the well known mean curvature flow on Σ_0.

(2) *How to achieve Step C3.* The regularity obtained in Step C1, together with null transport equations tied to the eikonal equation, elliptic systems of Hodge type, the geometric Littlewood-Paley theory of [**24**], sharp trace theorems, and an extensive use of the structure of the Einstein equations, allows us to propagate the regularity on Σ_0 to the space-time, thus achieving Step C3.

[22]Taking into account the different behavior in tangential and transversal directions with respect to the level surfaces of ${}^{\omega}u$.

[23]In the time symmetric case $k = 0$, this is exactly the minimal surface equation

(3) *The regularity with respect to ω in Steps C1 and C3.* The regularity with respect to x for u is clearly limited as a consequence of the fact that we only assume L^2 bounds on $\mathbf{R}$. On the other hand, $\mathbf{R}$ is independent of the parameter ω, and one might infer that u is smooth with respect to ω. Surprisingly, this is not at all the case. Indeed, the regularity in x obtained for u in Steps C1 and C3 is better in directions tangent to the level hypersurfaces of u. Now, the ω derivatives of the tangential directions have non zero normal components. Thus, when differentiating the structure equations with respect to ω, tangential derivatives to the level surfaces of u are transformed in non tangential derivatives which in turn severely limits the regularity in ω obtained in Steps C1 and C3.

(4) *How to achieve Steps C2 and C4.* Let us note that the classical arguments for proving L^2 bounds for Fourier operators are based either on a TT^* argument, or a T^*T argument, which requires several integration by parts either with respect to x for T^*T, of with respect to (λ, ω) for TT^*. Both methods would fail by far within the regularity for u obtained in Step C1 and Step C3. This forces us to design a method which allows to take advantage both of the regularity in x and ω. This is achieved using in particular the following ingredients:
   - geometric integrations by parts taking full advantage of the better regularity properties in directions tangent to the level hypersurfaces of u,
   - the standard first and second dyadic decomposition in frequency space, with respect to both size and angle (see [**39**]), an additional decomposition in physical space relying on the geometric Littlewood-Paley projections of [**24**] for Step C4, as well as another decomposition involving frequency and angle for Step C2.

   Even with these precautions, at several places in the proof, one encounters log-divergences which have to be tackled by ad-hoc techniques, taking full advantage of the structure of the Einstein equations.

Step D (Sharp $L^4(\mathcal{M})$ Strichartz estimates) Recall that the parametrix constructed in Step C needs also to be used to prove sharp $L^4(\mathcal{M})$ Strichartz estimates. Indeed the proof of several bilinear estimates of Step B reduces to the proof of sharp $L^4(\mathcal{M})$ Strichartz estimates for the parametrix (1.24) with λ localized in a dyadic shell.

More precisely, let $j \geq 0$, and let ψ a smooth function on $\mathbb{R}^3$ supported in

$$\frac{1}{2} \leq |\xi| \leq 2.$$

Let $\phi_{f,j}$ the parametrix (1.24) with a additional frequency localization $\lambda \sim 2^j$

$$\phi_{f,j}(t,x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda\ {}^\omega u(t,x)} \psi(2^{-j}\lambda) f(\lambda\omega) \lambda^2 d\lambda d\omega. \tag{1.28}$$

We will need the sharp[24] $L^4(\mathcal{M})$ Strichartz estimate

$$\|\phi_{f,j}\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{1.29}$$

The standard procedure for proving[25] (1.29) is based on a TT^* argument which reduces it to an L^∞ estimate for an oscillatory integral with a phase involving ${}^\omega u$. This is then achieved by the method of stationary phase which requires quite a few integrations by parts. In fact the standard argument would require, at the least[26], that the phase function $u = {}^\omega u$ verifies,

$$\partial_{t,x}u \in L^\infty,\ \partial_{t,x}\partial_\omega^2 u \in L^\infty. \tag{1.30}$$

This level of regularity is, unfortunately, incompatible with the regularity properties of solutions to our eikonal equation (1.25). In fact, based on the estimates for ${}^\omega u$ derived in step C3, we are only allowed to assume

$$\partial_{t,x}u \in L^\infty,\ \partial_{t,x}\partial_\omega u \in L^\infty. \tag{1.31}$$

We are thus forced to follow an alternative approach[27] to the stationary phase method inspired by [**35**] and [**36**] .

Remark 1.16. *Note that apart from the results of Chapter 2 which require the projection of various tensors on a frame, the computations and estimates in all the other chapters are covariant.*

1.2.5. Structure of the paper. In Chapter 2, we perform Step A and Step B, i.e. we recast the Einstein equations as a quasilinear Yang-Mills type system, we prove bilinear estimates, and we reduce the proof of Theorem 1.14 to Step C and Step D. Next, we perform Step C on the control of the plane wave parametrix (1.24). More precisely, in Chapter 3, we perform Step C4 on the control of the error term (1.27). Next, in Chapter 4, we perform Step C3 on the space-time control of the optical function ${}^\omega u$. Then, in Chapter 5, we perform Step C2 on the control of the parametrix at initial time (1.26). In Chapter 6, we perform Step C1 on the control of the optical function ${}^\omega u$ on the initial slice Σ_0. Finally, in Chapter 7, we prove sharp $L^4(\mathcal{M})$ Strichartz estimates localized in frequency which corresponds to Step D.

Remark 1.17. *Chapter 2 summarizes the results obtained in* [**27**]*. Chapter 3 summarizes the results obtained in* [**45**]*. Chapter 4 summarizes the results obtained in* [**44**]*. Chapter 5 summarizes the results obtained in* [**43**]*. Chapter 6 summarizes the results obtained in* [**42**]*. Finally, Chapter 7, summarizes the results obtained in* [**46**]*.*

[24]Note in particular that the corresponding estimate in the flat case is sharp.

[25]Note that the procedure we describe would prove not only (1.29) but the full range of mixed Strichartz estimates.

[26]The regularity (1.30) is necessary to make sense of the change of variables involved in the stationary phase method.

[27]We refer to the approach based on the overlap estimates for wave packets derived in [**35**] and [**36**] in the context of Strichartz estimates respectively for $C^{1,1}$ and $H^{2+\varepsilon}$ metrics. Note however that our approach does not require a wave packet decomposition.

Remark 1.18. *The structure of this overview is such that each part motivates the next one. In particular, Chapter 2 relies on the control of the parametrix* (1.24) *(see Theorem 2.27 in Chapter 2), and thus motivates Chapters 3, 4, 5 and 6 which precisely deal with the control of that parametrix. Next, in order to control the error term* (1.27) *in Chapter 3, we rely on estimates for the optical function* ${}^{\omega}u$*, which motivates Chapter 4 where these estimates are proved. In turn, the space-time estimates for* ${}^{\omega}u$ *in Chapter 4 are obtained in particular using transport equations, and we need the corresponding control for* ${}^{\omega}u$ *on the initial slice* Σ_0*, which motivates Chapter 6. Finally, in order to control the parametrix at initial time* (1.26) *in Chapter 5, we rely on estimates for the function* ${}^{\omega}u$ *on* Σ_0*, which motivates again Chapter 6. Finally, Chapter 2 also relies on sharp* $L^4(\mathcal{M})$ *Strichartz estimates localized in frequency (see Proposition 2.32 in Chapter 2), and thus motivates Chapter 7.*

1.2.6. Conclusion. Though this result falls short of the crucial goal of finding a scale invariant well-posedness criterion in GR, it is clearly optimal in terms of all currently available ideas and techniques. Indeed, within our current understanding, a better result would require enhanced bilinear estimates, which in turn would rely heavily on parametrices. On the other hand, parametrices are based on solutions to the eikonal equation whose control requires, at least, L^2 bounds for the curvature tensor, as can be seen in many instances in our work. Thus, if we are to ultimately find a scale invariant well-posedness criterion, it is clear that an entirely new circle of ideas is needed. Such a goal is clearly of fundamental importance not just to GR, but also to any physically relevant quasilinear hyperbolic system.

1.2.7. Acknowledgements. This work would be inconceivable without the extraordinary advancements made on nonlinear wave equations in the last twenty years in which so many have participated. We would like to single out the contributions of those who have affected this work in a more direct fashion, either through their papers or through relevant discussions, in various stages of its long gestation. D. Christodoulou's seminal work [**8**] on the weak cosmic censorship conjecture had a direct motivating role on our program, starting with a series of papers between one of the authors and M. Machedon, in which spacetime bilinear estimates were first introduced and used to take advantage of the null structure of geometric semilinear equations such as Wave Maps and Yang-Mills. The works of Bahouri- Chemin [**2**]-[**3**] and D.Tataru [**50**] were the first to go below the classical Sobolev exponent $s = 5/2$, for any quasilinear system in higher dimensions. This was, at the time, a major psychological and technical breakthrough which opened the way for future developments. Another major breakthrough of the period, with direct influence on our approach to bilinear estimates in curved spacetimes, is D. Tataru's work [**48**] on critical well posedness for Wave Maps, in which null frame spaces were first introduced. His joint work with H. Smith [**36**] which, together with [**20**] is the first to reach optimal well-posedness without bilinear estimates, has also influenced our approach on parametrices and Strichartz estimates. The authors would also like to acknowledge fruitful conversations with L. Anderson, and J. Sterbenz.

CHAPTER 2

Einstein vacuum equations as a Yang-Mills gauge theory

Recall Steps A, B, C and D introduced in section 1.2.4. In this chapter, we perform Step A and Step B, i.e. we recast the Einstein equations as a quasilinear Yang-Mills type system and we prove bilinear estimates. This allows us to reduce the proof of Theorem 1.14 to Step C and Step D. Here, we only outline the main ideas, and we refer to [**27**] for the details.

2.1. Yang-Mills formalism

2.1.1. Cartan formalism. Consider an Einstein vacuum space-time $(\mathcal{M}, \mathbf{g})$. We denote the covariant differentiation by $\mathbf{D}$. Let e_α be an orthonormal frame on $\mathcal{M}$, i.e.

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \mathrm{diag}(-1, 1, \dots, 1).$$

Consistent with the Cartan formalism we define the connection 1 form,

$$(\mathbf{A})_{\alpha\beta}(X) = \mathbf{g}(\mathbf{D}_X e_\beta, e_\alpha) \tag{2.1}$$

where X is an arbitrary vectorfield in $T(\mathcal{M})$. Observe that,

$$(\mathbf{A})_{\alpha\beta}(X) = -(\mathbf{A})_{\beta\alpha}(X)$$

i.e. the 1 -form $\mathbf{A}_\mu dx^\mu$ takes values in the Lie algebra of $SO(1,3)$. We separate the internal indices α, β from the external indices μ according to the following notation.

$$(\mathbf{A}_\mu)_{\alpha\beta} := (\mathbf{A})_{\alpha\beta}(\partial_\mu) = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha) \tag{2.2}$$

The Riemann curvature tensor is defined by $\mathbf{R}(X, Y, U, V) = \mathbf{g}\big(X, \big[\mathbf{D}_U\mathbf{D}_V - \mathbf{D}_V\mathbf{D}_U - \mathbf{D}_{[U,V]}Y\big]\big)$ with X, Y, U, V arbitrary vectorfields in $T(\mathcal{M})$. Thus, taking $U = \partial_\mu, V = \partial_\nu$, coordinate vector-fields,

$$\mathbf{R}(e_\alpha, e_\beta, \partial_\mu, \partial_\nu) \;=\; \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} + (\mathbf{A}_\nu)_\alpha{}^\lambda(\mathbf{A}_\mu)_{\lambda\beta} - (\mathbf{A}_\mu)_\alpha{}^\lambda(\mathbf{A}_\nu)_{\lambda\beta}. \tag{2.3}$$

Defining the Lie bracket,

$$([\mathbf{A}_\mu, \mathbf{A}_\nu])_{\alpha\beta} = (\mathbf{A}_\mu)_\alpha{}^\gamma(\mathbf{A}_\nu)_{\gamma\beta} - (\mathbf{A}_\nu)_\alpha{}^\gamma(\mathbf{A}_\mu)_{\gamma\beta} \tag{2.4}$$

we obtain:

$$\mathbf{R}_{\alpha\beta\mu\nu} = \partial_\mu(\mathbf{A}_\nu)_{\alpha\beta} - \partial_\nu(\mathbf{A}_\mu)_{\alpha\beta} - ([\mathbf{A}_\mu, \mathbf{A}_\nu])_{\alpha\beta},$$

or, since $\partial_\mu(\mathbf{A}_\nu) - \partial_\nu(\mathbf{A}_\mu) = \mathbf{D}_\mu\mathbf{A}_\nu - \mathbf{D}_\nu\mathbf{A}_\mu$

$$(\mathbf{F}_{\mu\nu})_{\alpha\beta} = \mathbf{R}_{\alpha\beta\mu\nu} = \big(\mathbf{D}_\mu\mathbf{A}_\nu - \mathbf{D}_\nu\mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu]\big)_{\alpha\beta}. \tag{2.5}$$

where interpret $\mathbf{F}$ is the curvature of the connection $\mathbf{A}$.

The usual covariant derivative of the Riemann curvature tensor can be expressed as follows:

$$\mathbf{D}_\sigma \mathbf{R}_{\alpha\beta\mu\nu} = {}^{(\mathbf{A})}\mathbf{D}_\sigma F_{\mu\nu} := \mathbf{D}_\sigma \mathbf{F}_{\mu\nu} + [\mathbf{A}_\sigma, \mathbf{F}_{\mu\nu}] \tag{2.6}$$

where we denote by ${}^{(\mathbf{A})}\mathbf{D}$ the covariant derivative on the corresponding vector bundle. More precisely if $\mathbf{U} = \mathbf{U}_{\mu_1\mu_2\ldots\mu_k}$ is any k-tensor on $\mathcal{M}$ with values on the Lie algebra of $SO(3,1)$,

$$ {}^{(\mathbf{A})}\mathbf{D}_\sigma \mathbf{U} = \mathbf{D}_\sigma \mathbf{U} + [\mathbf{A}_\sigma, \mathbf{U}]. \tag{2.7}$$

REMARK 2.1. *Recall that in $(\mathbf{A}_\mu)_{\alpha\beta}$, α, β are called the internal indices, while μ are called the external indices. Now, the internal indices are mostly irrelevant in our work. Thus, from now on, we will drop them, except for rare instances where we will need to distinguish between internal indices of the type ij and internal indices of the type $0i$.*

The Bianchi identities for $\mathbf{R}_{\alpha\beta\mu\nu}$ take the form

$$ {}^{(\mathbf{A})}\mathbf{D}_\sigma \mathbf{F}_{\mu\nu} + {}^{(\mathbf{A})}\mathbf{D}_\mu F_{\nu\sigma} + {}^{(\mathbf{A})}\mathbf{D}_\nu \mathbf{F}_{\sigma\mu} = 0. \tag{2.8}$$

As it is well known the Einstein vacuum equations $\mathbf{R}_{\alpha\beta} = 0$ imply $\mathbf{D}^\mu \mathbf{R}_{\alpha\beta\mu\nu} = 0$. Thus, in view of equation (2.6),

$$0 = {}^{(\mathbf{A})}\mathbf{D}^\mu \, \mathbf{F}_{\mu\nu} = \mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] \tag{2.9}$$

or, in view of (2.5) and the vanishing of the Ricci curvature of $\mathbf{g}$,.

$$\square \mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{J}_\nu \tag{2.10}$$

where

$$\mathbf{J}_\nu = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}_\mu, \mathbf{F}_{\mu\nu}]. \tag{2.11}$$

Using again the vanishing of the Ricci curvature it is easy to check,

$$\mathbf{D}^\nu \mathbf{J}_\nu = 0. \tag{2.12}$$

Finally we recall the general formula of transition between two different orthonormal frames e_α and $\widetilde{e}_\alpha$ on $\mathcal{M}$, related by,

$$\widetilde{e}_\alpha = \mathbf{O}_\alpha^\gamma e_\gamma$$

where $\mathbf{m}_{\alpha\beta} = \mathbf{O}_\alpha^\gamma \mathbf{O}_\beta^\delta \mathbf{m}_{\gamma\delta}$, i.e. $\mathbf{O}$ is a smooth map from $\mathcal{M}$ to the Lorentz group $O(3,1)$. In other words, raising and lowering indices with respect to $\mathbf{m}$,

$$\mathbf{O}_{\alpha\lambda} \mathbf{O}^{\beta\lambda} = \delta_\alpha^\beta \tag{2.13}$$

Now, $(\widetilde{\mathbf{A}}_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu \widetilde{e}_\beta, \widetilde{e}_\alpha)$. Therefore,

$$(\widetilde{\mathbf{A}}_\mu)_{\alpha\beta} = \mathbf{O}_\alpha^\gamma \mathbf{O}_\beta^\delta (\mathbf{A}_\mu)_{\gamma\delta} + \partial_\mu(\mathbf{O}_\alpha^\gamma) \, \mathbf{O}_\beta^\delta \, \mathbf{m}_{\gamma\delta} \tag{2.14}$$

2.1.2. Compatible frames. Recall that our space-time is assumed to be foliated by the level surfaces Σ_t of a time function t, which are maximal, i.e. denoting by k the second fundamental form of Σ_t we have,

$$\mathrm{tr}_g k = 0 \tag{2.15}$$

where g is the induced metric on Σ_t. Let us choose $e_{(0)} = T$, the future unit normal to the Σ_t foliation, and $e_{(i)}$, $i = 1, 2, 3$ an orthonormal frame tangent to Σ_t. We call this a frame compatible with our Σ_t foliation. We consider the connection coefficients (2.2) with respect to this frame. Thus, in particular, denoting by A_0, respectively A_i, the temporal and spatial components of $\mathbf{A}_\mu$

$$(A_i)_{0j} = (A_j)_{0i} = -k_{ij}, \qquad i, j = 1, 2, 3 \tag{2.16}$$

$$(A_0)_{0i} = -n^{-1}\nabla_i n \qquad i = 1, 2, 3 \tag{2.17}$$

where n denotes the lapse of the t-foliation, i.e. $n^{-2} = -\mathbf{g}(\mathbf{D}t, \mathbf{D}t)$. With this notation we note that,

$$\nabla_l k_{ij} = \nabla_l (k_i)_j + k_{in}(A_l)_j{}^n = \nabla^l (A_i)_{0j} + k_{in}(A_l)_j{}^n$$

where, as before, the notation $\nabla_l (k_i)_j$ or $\nabla^l (A_i)_{0j}$, is meant to suggest that the covariant differentiation affects only the external index i. Recalling from (1.7) that k verifies the constraint equations,

$$\nabla^i k_{ij} = 0,$$

we derive,

$$\nabla^i (A_i)_{0j} = k_i{}^m (A_i)_{mj}. \tag{2.18}$$

Besides the choice of e_0 we are still free to make a choice for the spatial elements of the frame e_1, e_2, e_3. In other words we consider frame transformations which keep e_0 fixed, i.e transformations of the type,

$$\tilde{e}_i = O_i^j e_j$$

with O in the orthogonal group $O(3)$. We now have, according to (2.14),

$$(\tilde{A}_m)_{ij} = O_i^k O_j^l (A_m)_{kl} + \partial_m (O_i^k) O_j^l \delta_{kl}$$

or, schematically,

$$\tilde{A}_m = O A_m O^{-1} + (\partial_m O) O^{-1} \tag{2.19}$$

formula in which we understand that only the spatial internal indices are involved. We shall use this freedom later to exhibit a frame e_1, e_2, e_3 such that the corresponding connection A satisfies the coulomb gauge condition $\nabla^l (A_l)_{ij} = 0$ (see Lemma 2.6).

2.1.3. Notations. We use greek indices to denote general indices on $\mathcal{M}$ which do not refer to the particular frame (e_0, e_1, e_2, e_3). The letters a, b, c, d will be used to denote general indices on Σ_t which do not refer to the particular frame (e_1, e_2, e_3). Finally, the letters i, j, l, m, n will only denote indices relative to the frame (e_1, e_2, e_3). Also, recall that $\mathbf{D}$ denotes the covariant derivative on $\mathcal{M}$, while ∇ denotes the induced covariant derivative on Σ_t. Furthermore, $\boldsymbol{\partial}$ will always refer to the derivative of a scalar quantity relative to one component of the frame (e_0, e_1, e_2, e_3), while ∂ will always refer to the derivative of a scalar quantity relative to one component of the the frame (e_1, e_2, e_3), so that $\boldsymbol{\partial} = (\partial_0, \partial)$. For example, ∂A may be any term of the form $\partial_i(A_j)$, $\partial_0(A)$ may be any term of the form $\partial_0(A_j)$, $\partial(A_0)$ may be any term of the form $\partial_j(A_0)$, and $\boldsymbol{\partial}\mathbf{A} = (\boldsymbol{\partial} A, \boldsymbol{\partial}(A_0)) = \big(\partial_0(A_0), \partial(A_0), \partial_0(A), \partial(A)\big)$. Note that we use brackets such as (A_j) to emphasize that we are dealing with $su(3,1)$ objects. Often, however, we will simply drop them.

We introduce the curl operator $curl$ defined for any $su(3,1)$-valued triplet $(\omega_1, \omega_2, \omega_3)$ of functions on Σ_t as follows:

$$(curl\, \omega)_i = \in_i{}^{jl} \partial_j(\omega_l), \tag{2.20}$$

where $\in_{ijl}$ is fully antisymmetric and such that $\in_{123} = 1$. We also introduce the divergence operator $\mathrm{d}iv$ defined for any $su(3,1)$-valued tensor A on Σ_t as follows:

$$\mathrm{d}iv\, A = \nabla^l(A_l) = \partial^l(A_l) + A^2. \tag{2.21}$$

REMARK 2.2. *The term A^2 in (2.21) corresponds to a quadratic expression in components of A, where the particular indices do not matter. In the rest of this part, we will adopt this schematic notation for lower order terms (e.g. terms of the type A^2 and A^3) where the particular indices do not matter.*

Finally, $\Box A_0$ and $\Box A_i$ will always be understood as $\Box(A_0)$ and $\Box(A_i)$, while $(\Box A)_\alpha$ refers to the tensorial wave equation. Also, ΔA_0 will always refer to $\Delta(A_0)$.

REMARK 2.3. *Since ∂_0 and ∂_j are not coordinate derivatives, note that the commutators $[\partial_j, \partial_0]$ and $[\partial_j, \partial_l]$ do not vanish. In fart we have, schematically,*

$$[\partial_i, \partial_j]\phi = A\boldsymbol{\partial}\phi \text{ and } [\partial_j, \partial_0]\phi = \mathbf{A}\boldsymbol{\partial}\phi, \tag{2.22}$$

for any scalar function ϕ on $\mathcal{M}$.

2.1.4. Main equations for (A_0, A). Using the conventions above one can prove the following proposition.

PROPOSITION 2.4. *Consider an orthonormal frame e_α compatible with a maximal Σ_t foliation of the space-time $\mathcal{M}$ with connection coefficients $\mathbf{A}_\mu$ defined by (2.2), their decomposition $\mathbf{A} = (A_0, A)$ relative to the same frame e_α, and Coulomb- like condition on the frame,*

$$div\, A = A^2.$$

In such a frame the Einstein-vacuum equations take the form,

$$\Delta A_0 = \mathbf{A}\boldsymbol{\partial} A + \mathbf{A}\partial(A_0) + \mathbf{A}^3, \tag{2.23}$$

$$\Box A_i + \partial_i(\partial_0 A_0) = A^j\partial_j A_i + A^j\partial_i A_j + A_0\boldsymbol{\partial}\mathbf{A} + A\boldsymbol{\partial}(A_0) + \mathbf{A}^3. \tag{2.24}$$

REMARK 2.5. *It is extremely important to our strategy that we have reduced the covariant wave equation* (2.10) *to the system of scalar equations* (2.23) (2.24) *(see remark 1.15).*

2.2. Strategy of the proof of the main Theorem

In this section, we discuss the strategy of the proof of the main theorem after reduction to small initial data, i.e. Theorem 1.14.

2.2.1. The Uhlenbeck type lemma. In order to exhibit a frame e_1, e_2, e_3 such that together with $e_0 = T$ we obtain a connection $\mathbf{A}$ satisfying our Coulomb type gauge on the slice Σ_t, we will need the following result in the spirit of the Uhlenbeck lemma [**51**].

LEMMA 2.6. *Let (M, g) a 3 dimensional Riemannian asymptotically flat manifold. Let R denote its curvature tensor and $r_{vol}(M,1)$ its volume radius on scales ≤ 1. Let $\tilde{A}$ a connection on M corresponding to an orthonormal frame. Assume the following bounds:*

$$\|\tilde{A}\|_{L^2(M)} + \|\nabla\tilde{A}\|_{L^2(M)} + \|R\|_{L^2(M)} \leq \delta \qquad \textit{and } r_{vol}(M,1) \geq \frac{1}{4},$$

where $\delta > 0$ is a small enough constant. Assume also that $\tilde{A}$ and $\nabla\tilde{A}$ belong to $L^2(M)$. Then, there is another connection A on M satisfying he Coulomb like gauge condition, and such that

$$\|\tilde{A}\|_{L^2(M)} + \|\nabla\tilde{A}\|_{L^2(M)} \leq \delta$$

Furthermore, if $\nabla^2\tilde{A}$ belongs to $L^2(M)$, then $\nabla^2 A$ belongs to $L^2(M)$.

The proof of Lemma 2.6 is a straightforward adaptation, in a simpler situation, of [**51**].

2.2.2. Classical local existence. We rely on the following standard well-posedness result for the Cauchy problem for the Einstein equations (1.1) in the maximal foliation.

THEOREM 2.7 (Well-posedness for the Einstein equation in the maximal foliation). *Let (Σ_0, g, k) be asymptotically flat and satisfying the constraint equations* (1.2), *with Ric, ∇Ric, k, ∇k and $\nabla^2 k$ in $L^2(\Sigma_0)$, and $r_{vol}(\Sigma_0, 1) > 0$. Then, there exists a unique asymptotically flat solution $(\mathcal{M}, \mathbf{g})$ to the Einstein vacuum equations* (1.1) *corresponding to this initial data set, together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Furthermore, there exists a time*

$$T_* = T_*(\|\nabla^{(l)} Ric\|_{L^2(\Sigma_0)}, 0 \leq l \leq 1, \|\nabla^{(j)} k\|_{L^2(\Sigma_0)}, 0 \leq j \leq 2, r_{vol}(\Sigma_0, 1)) > 0$$

such that the maximal foliation exists for on $0 \le t \le T_*$ *with a corresponding control in* $L^\infty_{[0,T_*]} L^2(\Sigma_t)$ *for Ric,* ∇Ric*,* k*,* ∇k *and* $\nabla^2 k$*.*

Theorem 2.7 requires two more derivatives both for R and k with respect to the main Theorem 1.6. Its proof is standard and relies solely on energy estimates (as opposed to Strichartz or bilinear estimates). We refer the reader to [**9**] chapter 10 for a related statement.

REMARK 2.8. *In the proof of our main theorem the result above will be used only as an extension tool (see steps 1 and 3 below), only for very tiny values of the time interval.*

2.2.3. Weakly regular null hypersurfaces. We shall be working with null hyper surfaces in $\mathcal{M}$ verifying a set of reasonable assumptions, described below. These assumptions will be easily verified by the level hyper surfaces $\mathcal{H}_u$ solutions u of the eikonal equation $\mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu = 0$ discussed in section 2.7. The regularity of the eikonal equation is studied in detail in [**44**] (see also Chapter 4).

DEFINITION 2.9. *Let* $\mathcal{H}$ *be a null hypersurface with future null normal* L *verifying* $\mathbf{g}(L,T) = -1$*. Let also* $N = L - T$*. We denote by* $\nabla\!\!\!\!/$ *the induced connection along the* 2*-surfaces* $\mathcal{H} \cap \Sigma_t$*. We say that* $\mathcal{H}$ *is weakly regular provided that,*

$$\|\mathbf{D}L\|_{L^3(\mathcal{H})} + \|\mathbf{D}N\|_{L^3(\mathcal{H})} \lesssim 1, \tag{2.25}$$

and the following Sobolev embedding holds for any scalar function f *on* $\mathcal{H}$*:*

$$\|f\|_{L^6(\mathcal{H})} \lesssim \|\nabla\!\!\!\!/ f\|_{L^2(\mathcal{H})} + \|L(f)\|_{L^2(\mathcal{H})} + \|f\|_{L^2(\mathcal{H})}. \tag{2.26}$$

2.2.4. Main bootstrap assumptions. Let $M \ge 1$ a large enough constant to be chosen later in terms only of universal constants. By choosing $\varepsilon > 0$ sufficiently small, we can also ensure $M\varepsilon$ is small enough. From now on, we assume the following bootstrap assumptions hold true on a fixed interval $[0, T^*]$, for some $0 < T^* \le 1$. Note that $\mathcal{H}$ denotes an arbitrary weakly regular null hypersurface with future normal L, normalized by the condition $\mathbf{g}(L,T) = -1$.

- *Bootstrap curvature assumptions*

$$\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \le M\varepsilon. \tag{2.27}$$

  Also,

$$\|\mathbf{R} \cdot L\|_{L^2(\mathcal{H})} \le M\varepsilon, \tag{2.28}$$

  where $\mathbf{R}\cdot L$ denotes any component of $\mathbf{R}$ such that at least one index is contracted with L.
- *Bootstrap assumptions for the connection* We also assume that there exist $\mathbf{A} = (A_0, A)$ verifying our Coulomb type condition on $[0, T^*]$, such that,

$$\|A\|_{L^\infty_t L^2(\Sigma_t)} + \|\boldsymbol{\partial} A\|_{L^\infty_t L^2(\Sigma_t)} + \|A\|_{L^2_t L^7(\Sigma_t)} \le M\varepsilon, \tag{2.29}$$

and:

$$\|A_0\|_{L_t^\infty L^2(\Sigma_t)}+\|\boldsymbol{\partial} A_0\|_{L_t^\infty L^2(\Sigma_t)}+\|A_0\|_{L_t^2 L^\infty(\Sigma_t)}+\|\boldsymbol{\partial} A_0\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial\boldsymbol{\partial} A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \leq M\varepsilon. \quad (2.30)$$

REMARK 2.10. *Together with the estimates in* [**44**] *(see section 4.4 in that paper, and also section 4.2.3), the bootstrap assumption* (2.27) *yields:*

$$\|k\|_{L_t^\infty L^2(\Sigma_t)}+\|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim M\varepsilon. \quad (2.31)$$

Furthermore, the bootstrap assumption (2.28) *together with the estimates in* [**44**] *(see section 4.2 in that paper, and also section 4.2.2) yields:*

$$\inf_t r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}. \quad (2.32)$$

In addition we make the following bilinear estimates assumptions for $\mathbf{A}$ and $\mathbf{R}$.:

- *Bilinear assumptions I.* Assume,

$$\|A^j\partial_j A\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2. \quad (2.33)$$

Also, let $B=(-\Delta)^{-1}curl\, A$ (see (2.61) and the accompanying explanations). Then, we have:

$$\|A^j\partial_j(\boldsymbol{\partial} B)\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2, \quad (2.34)$$

and:

$$\|\mathbf{R}_{\cdot\,\cdot\, j\, 0}\partial^j B\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2. \quad (2.35)$$

Finally, for any weakly regular null hypersurface $\mathcal{H}$ and any smooth scalar function ϕ on $\mathcal{M}$,

$$\|k_{j\,.}\partial^j\phi\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon \sup_{\mathcal{H}}\|\nabla\!\!\!/\phi\|_{L^2(\mathcal{H})}, \quad (2.36)$$

and

$$\|A^j\partial_j\phi\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon \sup_{\mathcal{H}}\|\nabla\!\!\!/\phi\|_{L^2(\mathcal{H})}, \quad (2.37)$$

where the supremum is taken over all null hypersurfaces $\mathcal{H}$.

- *Bilinear assumptions II.* We assume,

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(A,A))\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2, \quad (2.38)$$

where the bilinear form Q_{ij} is given by $Q_{ij}(\phi,\psi)=\partial_i\phi\partial_j\psi-\partial_j\phi\partial_i\psi$. Furthermore, we also have:

$$\|(-\Delta)^{-\frac{1}{2}}(\partial(A^l)\partial_l A)\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2. \quad (2.39)$$

- *Non-sharp Strichartz assumption*

$$\|A\|_{L_t^2 L^7(\Sigma_t)} \lesssim M^2 \varepsilon. \tag{2.40}$$

and, for $B = (-\Delta)^{-1} curl\, A$, (see (2.61) and the accompanying explanations).

$$\|\partial B\|_{L_t^2 L^7(\Sigma_t)} \lesssim M^2 \varepsilon. \tag{2.41}$$

REMARK 2.11. *Note that the Strichartz estimate for $\|A\|_{L_t^2 L^7(\Sigma_t)}$ is far from being sharp. Nevertheless, this estimate will be sufficient for the proof as it will only be used to deal with lower order terms.*

Finally we also need a trilinear bootstrap assumption. For this we need to introduce the Bell Robinson tensor,

$$Q_{\alpha\beta\gamma\delta} = \mathbf{R}_{\alpha}{}^{\lambda}{}_{\gamma}{}^{\sigma}\mathbf{R}_{\beta\,\lambda\,\delta\,\sigma} + {}^*\mathbf{R}_{\alpha}{}^{\lambda}{}_{\gamma}{}^{\sigma}\,{}^*\mathbf{R}_{\beta\,\lambda\,\delta\,\sigma} \tag{2.42}$$

- *Trilinear bootstrap assumption.* We assume the following,

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta} k^{ij} e_0^{\gamma} e_0^{\delta}\right| \lesssim M^4 \varepsilon^3. \tag{2.43}$$

Let us conclude this section by remarking that the bootstrap assumptions are verified for a sufficiently small final value T^*.

PROPOSITION 2.12. *The above bootstrap assumptions are verified on $0 \le t \le T^*$ for a sufficiently small $T^* > 0$.*

The only challenge in the proof of Proposition 2.12 is to show the existence of the desired connection $\mathbf{A}$ using in particular the Uhlenbeck type Lemma 2.6. All other estimates follow trivially from our initial bounds and the local existence theorem above, for sufficiently small T^*. We refer to Proposition 4.6 in [**27**].

2.2.5. Proof of the bounded L^2 curvature conjecture. In the following two propositions, we state the improvement of our bootstrap assumptions.

PROPOSITION 2.13. *Let us assume that all bootstrap assumptions of the previous section hold for $0 \le t \le T^*$. If $\varepsilon > 0$ is sufficiently small, then the following improved estimates hold true on $0 \le t \le T^*$:*

$$\|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon + M^2 \varepsilon^{\frac{3}{2}} + M^3 \varepsilon^2, \tag{2.44}$$

$$\|\mathbf{R} \cdot L\|_{L^2(\mathcal{H})} \lesssim \varepsilon + M^2 \varepsilon^{\frac{3}{2}} + M^3 \varepsilon^2, \tag{2.45}$$

$$\|A\|_{L_t^\infty L^2(\Sigma_t)} + \|\boldsymbol{\partial} A_i\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon + M^2 \varepsilon^{\frac{3}{2}} + M^3 \varepsilon^2, \tag{2.46}$$

$$\begin{aligned} &\|A_0\|_{L_t^\infty L^2(\Sigma_t)} + \|\boldsymbol{\partial} A_0\|_{L_t^\infty L^2(\Sigma_t)} + \|A_0\|_{L_t^2 L^\infty(\Sigma_t)} \\ &\quad + \|\boldsymbol{\partial} A_0\|_{L_t^\infty L^3(\Sigma_t)} + \|\partial \boldsymbol{\partial} A_0\|_{L_t^\infty L^{\frac{3}{2}}(\Sigma_t)} \lesssim \varepsilon + M^2 \varepsilon^{\frac{3}{2}} + M^3 \varepsilon^2, \end{aligned} \tag{2.47}$$

PROPOSITION 2.14. *Let us assume that all bootstrap assumptions of the previous section hold for $0 \leq t \leq T^*$. If $\varepsilon > 0$ is sufficiently small, then the following improved estimates hold true on $0 \leq t \leq T^*$:*

$$\|A^j\partial_j A\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2, \tag{2.48}$$

$$\|A^j\partial_j(\boldsymbol{\partial} B)\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2, \tag{2.49}$$

and

$$\|\mathbf{R}_{\cdot\,\cdot\,j\,0}\partial^j B\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2. \tag{2.50}$$

Also, for any scalar function ϕ on $\mathcal{M}$, we have:

$$\|k_j.\partial^j\phi\|_{L^2(\mathcal{M})} \lesssim M\varepsilon \sup_{\mathcal{H}}\|\nabla\!\!\!/\phi\|_{L^2(\mathcal{H})}, \tag{2.51}$$

and

$$\|A^j\partial_j\phi\|_{L^2(\mathcal{M})} \lesssim M\varepsilon \sup_{\mathcal{H}}\|\nabla\!\!\!/\phi\|_{L^2(\mathcal{H})}, \tag{2.52}$$

where the supremum is taken over all null hypersurfaces $\mathcal{H}$. Finally, we have:

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(A,A))\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2, \tag{2.53}$$

$$\|(-\Delta)^{-\frac{1}{2}}(\partial A^l\partial_l A)\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2. \tag{2.54}$$

$$\|A\|_{L^2_tL^7(\Sigma_t)} \lesssim M\varepsilon, \tag{2.55}$$

$$\|\partial B\|_{L^2_tL^7(\Sigma_t)} \lesssim M\varepsilon. \tag{2.56}$$

and

$$\left|\int_{\mathcal{M}} Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right| \lesssim M^3\varepsilon^3. \tag{2.57}$$

The proof of Proposition 2.13 is postponed to section 2.6, while the proof of Proposition 2.14 is postponed to section 2.8. We also need a proposition on the propagation of higher regularity.

PROPOSITION 2.15. *Let us assume that the estimates corresponding to all bootstrap assumptions of the previous section hold for $0 \leq t \leq T^*$ with a universal constant M. Then for any $t \in [0, T^*)$ and for $\varepsilon > 0$ sufficiently small, the following propagation of higher regularity holds:*

$$\|\mathbf{DR}\|_{L^\infty_tL^2(\Sigma_t)} \leq 2\left(\|Ric\|_{L^2(\Sigma_0)} + \|\nabla Ric\|_{L^2(\Sigma_0)} + \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} + \|\nabla^2 k\|_{L^2(\Sigma_0)}\right).$$

The proof of Proposition 2.15 follows along the same lines as the proof Proposition 2.13 and Proposition 2.14, and we refer to [**27**] for its proof. Next, let us show how Propositions 2.12, 2.13, 2.14 and 2.15 imply our main theorem 1.14. We proceed, by the standard bootstrap method , along the following steps:

Step 1. We show that all bootstrap assumptions are verified for a sufficiently small final value T^*.

Step2. Assuming that all bootstrap assumptions hold for fixed values of $0 < T^* \leq 1$ and M sufficiently large we show that, for $\varepsilon > 0$ sufficiently small, we may improve on the constant M in our bootstrap assumptions.

Step 3. Using the estimates derived in step 2 we can extend the time of existence T^* to $T^* + \delta$ such that all the bootstrap assumptions remain true.

Now, *Step 1* follows from Proposition 2.12. *Step 2* follows from Proposition 2.13 and Proposition 2.14. In view of *Step 2*, the estimates corresponding to all bootstrap assumptions of the previous section hold for $0 \leq t \leq T^*$ with a universal constant M. Thus the conclusion of Proposition 2.15 holds, and arguing as in the proof of Proposition 2.12, we obtain *Step 3*. Thus, the bootstrap assumptions hold on $0 \leq t \leq 1$ for a universal constant M. In particular, this yields together with (2.31):

$$\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon \text{ and } \|k\|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon \text{ for all } 0 \leq t \leq 1. \tag{2.58}$$

In view of (2.32), we also obtain the following control on the volume radius:

$$\inf_{0 \leq t \leq 1} r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}. \tag{2.59}$$

Furthermore, Proposition 2.15 yields the following propagation of higher regularity

$$\sum_{|\alpha| \leq m} \|\mathbf{D}^{(\alpha)}\mathbf{R}\|_{L^\infty_{[0,1]} L^2(\Sigma_t)} \leq C_m \left[\|\nabla^{(i)}\mathrm{Ric}\|_{L^2(\Sigma_t)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma_t)} \right] \tag{2.60}$$

where C_m only depends on m.

Remark 2.16. *Note that Proposition 2.15 only yields the case $m = 1$ in* (2.60). *The fact that* (2.60) *also holds for higher derivatives $m \geq 2$ follows from the standard propagation of regularity for the classical local existence result of Theorem 2.7 and the bound* (2.60) *with $m = 1$ coming from Proposition 2.15.*

Finally, (2.58), the control on the volume radius (2.59) and the propagation of higher regularity (2.60) yield the conclusion of Theorem 1.14. Together with the reduction to small initial data performed in section 1.2.3, this concludes the proof of the main Theorem 1.6.

The rest of the chapter deals with the proofs of propositions 2.13 and 2.14. The core of the proofs is to control A, the spatial part of the connection $\mathbf{A}$. As explained in the introduction we need to project our equation for the spatial components A onto divergence free vectorfields. This is needed for two reasons, to eliminate the term $\partial_i(\partial_0 A_0)$ on the left hand side of (2.24) and to obtain, on the right hand side, terms which exhibit the crucial null structure we need to implement our proof. Rather than work with the projection $\mathcal{P}$, which is too complicated, we introduce instead the new variable,

$$B = (-\Delta)^{-1} curl\,(A) \tag{2.61}$$

for which we derive a suitable wave equation. Since we have (see Lemma 2.20) $A = curl\,(B) + l.o.t$ it suffices to obtain estimates for B which lead us to an improvement of the bootstrap assumption (2.29) on A. In section 2.4, we derive space-time estimates for $\square B$ and its derivatives. Proposition 2.13, which does not require a parametrix representation, is proved in 2.6. Proposition 2.14 is proved in sections 2.8 and 2.9 based on the representation formula of theorem 2.30 derived in section 2.7.

2.3. Simple consequences of the bootstrap assumptions

2.3.1. Sobolev embeddings and elliptic estimates on Σ_t. The bootstrap assumption (2.27) on R and the estimate for k (2.31) together with the estimates in [**44**] (see section 4.4 in that paper) yield the following lapse estimates:

$$\begin{aligned}&\|n-1\|_{L^\infty(\mathcal{M})}+\|\nabla n\|_{L^\infty(\mathcal{M})}+\|\nabla^2 n\|_{L^\infty_tL^2(\Sigma_t)}+\|\nabla^2 n\|_{L^\infty_tL^3(\Sigma_t)}\\ &\quad+\|\nabla(\partial_0 n)\|_{L^\infty_tL^3(\Sigma_t)}+\|\nabla^3 n\|_{L^\infty_tL^{\frac{3}{2}}(\Sigma_t)}+\|\nabla^2(\partial_0(n))\|_{L^\infty_tL^{\frac{3}{2}}(\Sigma_t)}\ \lesssim\ M\varepsilon,\end{aligned}\tag{2.62}$$

where ∇ denotes the induced covariant derivative on Σ_t.

Remark 2.17. *Recall from* (2.17) *that* $(A_0)_{0i} = -n^{-1}\nabla_i n$. *Thus, the estimates* (2.62) *for* n *could in principle be deduced from the bootstrap assumptions* (2.30) *for* A_0. *However, notice that* $\nabla n \in L^\infty(\mathcal{M})$ *in view of* (2.62), *while* A_0 *is only in* $L^2_tL^\infty(\Sigma_t)$ *according to* (2.30). *This improvement for the components* $(A_0)_{0i}$ *of* A_0 *will turn out to be crucial (see remark 2.24). Its proof is given in section 4.4 of* [**44**] *(see also the discussion in section 4.2.3).*

Next, we record the following Sobolev embeddings and elliptic estimates on Σ_t derived under the assumptions (2.28) and (2.27) in [**44**] (see sections 3.5 and 4.2 in that paper).

Lemma 2.18 (Calculus inequalities on Σ_t). *Assume that the assumptions* (2.28) *and* (2.27) *hold, and assume that the volume radius at scales* ≤ 1 *on* Σ_0 *is bounded from below by a universal constant. Then, the Sobolev embedding on* Σ_t *holds for any tensor* F

$$\|F\|_{L^6(\Sigma_t)} \lesssim \|\nabla F\|_{L^2(\Sigma_t)},\tag{2.63}$$

Also, we define the operator $(-\Delta)^{-\frac{1}{2}}$ *acting on tensors on* Σ_t *as:*

$$(-\Delta)^{-\frac{1}{2}}F = \frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_0^{+\infty}\tau^{-\frac{3}{4}}U(\tau)Fd\tau,$$

where Γ *is the Gamma function, and where* $U(\tau)F$ *is defined using the heat flow on* Σ_t*:*

$$(\partial_\tau - \Delta)U(\tau)F = 0,\ U(0)F = F.$$

We have the following Bochner estimates:

$$\|\nabla(-\Delta)^{-\frac{1}{2}}\|_{\mathcal{L}(L^2(\Sigma_t))} \lesssim 1 \text{ and } \|\nabla^2(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Sigma_t))} \lesssim 1,\tag{2.64}$$

where $\mathcal{L}(L^2(\Sigma_t))$ denotes the set of bounded linear operators on $L^2(\Sigma_t)$. (2.64) *together with the Sobolev embedding* (2.63) *yields:*

$$\|(-\Delta)^{-\frac{1}{2}}F\|_{L^2(\Sigma_t)} \lesssim \|F\|_{L^{\frac{6}{5}}(\Sigma_t)}. \tag{2.65}$$

2.3.2. Elliptic estimates for B. Here we record simple estimates for B, based the bootstrap assumptions (2.29) (2.30) for A and A_0 and standard elliptic estimates such as the Bochner and Sobolev inequalities on Σ_t, see (2.64) and (2.65).

PROPOSITION 2.19. *Let $B_i = (-\Delta)^{-1}(curl\,(A)_i)$. Then, we have, for each component of B:*

$$\|\boldsymbol{\partial} B\|_{L^\infty_t L^2(\Sigma_t)} + \|\partial^2 B\|_{L^\infty_t L^2(\Sigma_t)} + \|\partial(\partial_0 B)\|_{L^\infty_t L^2(\Sigma_t)} \lesssim M\varepsilon. \tag{2.66}$$

2.3.3. Decomposition for A. Recall that $B = (-\Delta)^{-1}(curl\,(A))$. We indicate below how to recover A from B:

LEMMA 2.20. *We have the following estimate:*

$$A = curl\,(B) + E$$

where E satisfies:

$$\|\partial E\|_{L^\infty_t L^3(\Sigma_t)} + \|\partial^2 E\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} + \|E\|_{L^2_t L^\infty(\Sigma_t)} \lesssim M^2\varepsilon^2.$$

PROOF. We have, symbolically,

$$A = (-\Delta)^{-1}curl\,(curl\,(A) + (-\Delta)^{-1}(A\partial A + A^3).$$

from which,

$$A \;=\; curl\,(B) - (-\Delta)^{-1}[\Delta, curl\,]B + (-\Delta)^{-1}(A\partial A + A^3)$$

The rest of the proof uses elliptic estimates on Σ_t, the bootstrap assumptions for $\mathbf{A}$ and $\mathbf{R}$, and the bootstrap assumption (2.40). We refer to [**27**] for the details. □

2.4. Estimates for $\Box B$

We outline the proof of two important propositions concerning estimates for $\Box curl\, A$ and $\Box B$, with $B = \Delta^{-1}curl\,(A)$. The proofs makes use of the special structure of various bilinear expressions and thus is based not only on the bootstrap assumptions for A_0, A, k and $\mathbf{R}$ but also some of our bilinear bootstrap assumptions.

We record first the following straightforward commutation lemma, see [**27**].

LEMMA 2.21. *Let ϕ a $so(3,1)$ scalar function on $\mathcal{M}$. We have, schematically,*

$$\partial_j(\Box\phi) - \Box(\partial_j(\phi)) = 2(A^\lambda)_j{}^\mu\, \partial_\lambda\partial_\mu\phi + \partial_0(A_0)\boldsymbol{\partial}\phi + \mathbf{A}^2\boldsymbol{\partial}\phi. \tag{2.67}$$

We also have:

$$
\begin{aligned}
[\Box, \Delta]\phi &= -4k^{ab}\nabla_a\nabla_b(\partial_0\phi) + 4n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0\phi)) - 2\nabla_0 k^{ab}\nabla_a\nabla_b\phi \\
&+ F^{(1)}\boldsymbol{\partial}^2\phi + F^{(2)}\boldsymbol{\partial}\phi, \\
F^{(1)} &= \boldsymbol{\partial}A_0 + \mathbf{A}^2, \\
F^{(2)} &= \partial\boldsymbol{\partial}A_0 + \mathbf{A}\boldsymbol{\partial}\mathbf{A} + \mathbf{A}^3,
\end{aligned}
\tag{2.68}
$$

where ∇_a and ∇_b denote induced covariant derivatives on Σ_t applied to the scalars ϕ, $\partial_0\phi$ and $\partial_0(\partial_0\phi)$.

The estimates for $\Box curl\, A$ and $\Box B$ are given by the following propositions.

Proposition 2.22. *We have*

$$\sum_{i=1}^{3}\|(-\Delta)^{-\frac{1}{2}}\Box(curl\, A_i)\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2. \tag{2.69}$$

Proposition 2.23 (Estimates for $\Box B$). *The components $B_i = (-\Delta)^{-1}(curl\,(A)_i)$ verify the following estimate,*

$$\sum_{i=1}^{3}\left(\|\Box B_i\|_{L^2(\mathcal{M})} + \|\partial\,\Box B_i\|_{L^2(\mathcal{M})}\right) \lesssim M^2\varepsilon^2. \tag{2.70}$$

We also have,

$$\sum_{i=1}^{3}\|\partial_0\partial_0 B_i\|_{L^2(\mathcal{M})} \lesssim M\varepsilon. \tag{2.71}$$

The proof of Proposition 2.22 and Proposition 2.23 are similar in spirit. We give below a short outline of the proof of Proposition 2.23 which is slightly more difficult.

Proof. In what follows we outline the main steps in the proof of space-time estimates (2.69), (2.70) for $\Box B$ and $\partial_0^2 B$. We have:

$$
\begin{aligned}
\Box(B_i) &= [\Box, (-\Delta)^{-1}](curl\,(A)_i) + (-\Delta)^{-1}(\Box(curl\,(A)_i)) \\
&= -(-\Delta)^{-1}[\Box, \Delta](-\Delta)^{-1}(curl\,(A)_i) + (-\Delta)^{-1}(\Box(curl\,(A)_i)) \\
&= -(-\Delta)^{-1}[\Box, \Delta](B_i) + (-\Delta)^{-1}(\Box(curl\,(A)_i)).
\end{aligned}
$$

Thus, using the L^2 boundedness of $\partial(-\Delta)^{1/2}$ and result of proposition 2.22, we obtain:

$$\|\partial\Box(B_i)\|_{L^2(\mathcal{M})} \lesssim \|(-\Delta)^{-\frac{1}{2}}[\Box, \Delta](B_i)\|_{L^2(\mathcal{M})} + M^3\varepsilon^2, \tag{2.72}$$

It remains to estimate $\|(-\Delta)^{-\frac{1}{2}}[\Box,\Delta](B_i)\|_{L^2(\mathcal{M})}$. We rely on the commutator formula (2.68) to write,

$$
\begin{aligned}
[\Box,\Delta]B &= -4k^{ab}\nabla_a\nabla_b(\partial_0 B) + 4n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0\ B)) - 2\nabla_0 k^{ab}\nabla_a\nabla_b B \qquad (2.73)\\
&+ F^{(1)}\boldsymbol{\partial}^2 B + F^{(2)}\boldsymbol{\partial} B\\
F^{(1)} &= \boldsymbol{\partial}(A_0) + \mathbf{A}^2,\\
F^{(2)} &= \partial\boldsymbol{\partial}(A_0) + \mathbf{A}\boldsymbol{\partial}\mathbf{A} + \mathbf{A}^3.
\end{aligned}
$$

with B any component (B_l), $l=1,2,3$. It is easy to check that,

$$\|F^{(1)}\|_{L^\infty_t L^3(\Sigma_t)} + \|F^{(2)}\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} \lesssim M\varepsilon,$$

We write,

$$\|(-\Delta)^{-\frac{1}{2}}[\Box,\Delta](B_l)\|_{L^2(\mathcal{M})} \lesssim N_1 + N_2 + N_3 + N_4 \tag{2.74}$$

$$
\begin{aligned}
N_1 &= \|(-\Delta)^{-\frac{1}{2}}[k^{ab}\nabla_a\nabla_b(\partial_0(B_l))]\|_{L^2(\mathcal{M})}\\
N_2 &= \|(-\Delta)^{-\frac{1}{2}}[n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0(B_l)))]\|_{L^2(\mathcal{M})}\\
N_3 &= \|(-\Delta)^{-\frac{1}{2}}[\nabla_0 k^{ab}\nabla_a\nabla_b(B_l)]\|_{L^2(\mathcal{M})}\\
N_4 &= \|F^{(1)}\|_{L^\infty_t L^3(\Sigma_t)}\|\boldsymbol{\partial}^2(B_l)\|_{L^2(\mathcal{M})} + \|F^{(2)}\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)}\|\boldsymbol{\partial}(B_l)\|_{L^\infty_t L^6(\Sigma_t)}
\end{aligned}
$$

Using the estimates (2.66) we easily infer that

$$N_4 \lesssim M\varepsilon^2. \tag{2.75}$$

To estimate N_1 we proceed as follows, using the constraint equations (1.7) for k,

$$k^{ab}\nabla_a\nabla_b(\partial_0(B_l)) \quad = \quad \nabla_a[k^{ab}\nabla_b(\partial_0(B_l))]$$

Together with the Bochner inequality on Σ_t (2.64) and the bilinear assumption (2.34) , we obtain:

$$N_1 \lesssim \|k^{ab}\partial_b(\partial_0(B_l))]\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2. \tag{2.76}$$

To estimate N_2 we write,

$$n^{-1}\nabla_b n\nabla_b(\partial_0(\partial_0(B_l))) = \nabla^b[n^{-1}\nabla_b n\partial_0(\partial_0(B_l))] - (n^{-1}\Delta n - n^{-2}|\nabla n|^2)\partial_0(\partial_0(B_l)).$$

Together with the estimates (2.62) for the lapse n and the Sobolev embedding on Σ_t (2.65), this yields:

$$
\begin{aligned}
N_2 &\lesssim \|n^{-1}\nabla_b n\partial_0(\partial_0(B_l))\|_{L^2(\mathcal{M})} + \|(n^{-1}\Delta n - n^{-2}|\nabla n|^2)\partial_0(\partial_0(B_l))\|_{L^2_t L^{\frac{6}{5}}(\Sigma_t)}\\
&\lesssim (\|\nabla n\|_{L^\infty} + \|n^{-1}\Delta n - n^{-2}|\nabla n|^2\|_{L^\infty_t L^3(\Sigma_t)})\|\partial_0(\partial_0(B_l))\|_{L^2(\mathcal{M})}\\
&\lesssim M\varepsilon\|\partial_0(\partial_0(B_l))\|_{L^2(\mathcal{M})}. \qquad (2.77)
\end{aligned}
$$

Remark 2.24. *Note that there is no room in the estimate* (2.77)*. Indeed the sharp estimate* $\|\nabla n\|_{L^\infty(\mathcal{M})} \lesssim M\varepsilon$ *given by* (2.62) *is crucial as emphasized in remark 2.17.*

Finally, we consider N_3. Recall from (1.10) that the second fundamental form satisfies the following equation:

$$\nabla_0 k_{ab} = E_{ab} - n^{-1}\nabla_a\nabla_b n - k_{ac}k_b{}^c = E_{ab} + \text{ l.o.t.} \tag{2.78}$$

where E is the 2-tensor on Σ_t defined as $E_{ab} = \mathbf{R}_{a\,0\,b\,0}$. In view of the estimates (2.31) for k and (2.62) for n,

$$\|\text{ l.o.t.}\|_{L^\infty_t L^3(\Sigma_t)} \lesssim \|\nabla^2 n\|_{L^\infty_t L^3(\Sigma_t)} + \|k\|^2_{L^\infty_t L^6(\Sigma_t)} \lesssim M\varepsilon.$$

Thus instead of estimating $\nabla_0 k^{ab}\nabla_a\nabla_b B$ in the definition of N_3 it suffices to estimate the term

$$E^{ab}\nabla_a\nabla_b B \;\; = \;\; \nabla_a(E^{ab}\nabla_b B) - \nabla_a E^{ab}\nabla_b B$$

Using the maximal foliation assumption, the Bianchi identities and the symmetries of $\mathbf{R}$, we can write, schematically, $\nabla^a E_{ab} = \mathbf{A}\mathbf{R}$ and therefore, together with the bootstrap assumptions (2.29) for A and (2.30) for A_0, and the bootstrap assumption (2.27) for $\mathbf{R}$ yields:

$$\|\nabla^a E_{ab}\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)} \lesssim \|\mathbf{A}\|_{L^\infty_t L^6(\Sigma_t)}\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \lesssim M^2\varepsilon^2. \tag{2.79}$$

We thus have,

$$\nabla_0 k_{ab} = \nabla_a(E^{ab}\nabla_b B) + \text{ l.o.t.}$$

Now, in view of the bilinear assumption (2.35),

$$\|(-\Delta)^{1/2}\nabla_a(E^{ab}\nabla_b B)\|_{L^2(\mathcal{M})} \;\; \lesssim \;\; \|\mathbf{R}_{0a0b}\nabla_b B\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2$$

Hence, putting all the above together we infer that,

$$N_3 \lesssim M^3\varepsilon^2.$$

Together with (2.74), (2.76), (2.77) and (2.75), we derive,

$$\|\partial\,\square B\|_{L^2(\mathcal{M})} \lesssim M^3\varepsilon^2 + M\varepsilon\|\partial_0(\partial_0 B)\|_{L^2(\mathcal{M})}. \tag{2.80}$$

To close the estimate for $\|\partial\,\square B\|_{L^2(\mathcal{M})}$ it remains to control the right-hand side of (2.80). This is achieved relying in particular on the following formula

$$\partial_0(\partial_0 B) = -\square B + \Delta B + n^{-1}\nabla n\cdot\nabla B.$$

□

2.5. Energy estimate for the wave equation on a curved background

Recall that $e_0 = T$, the future unit normal to the Σ_t foliation. Let π be the deformation tensor of e_0, that is the symmetric 2-tensor on $\mathcal{M}$ defined as:

$$\pi_{\alpha\beta} = \mathbf{D}_\alpha T_\beta + \mathbf{D}_\beta T_\alpha.$$

In view of the definition of the second fundamental form k and the lapse n, we have:

$$\pi_{ab} = -2k_{ab},\; \pi_{a0} = \pi_{0a} = n^{-1}\nabla_a n,\; \pi_{00} = 0. \tag{2.81}$$

In what follows $\mathcal{H}$ denotes an arbitrary weakly regular null hypersurface[1] with future normal L verifying $\mathbf{g}(L,T) = -1$. We denote by $\nabla\!\!\!/$ the induced connection on the 2-surfaces $\mathcal{H}\cap\Sigma_t$.

LEMMA 2.25. *Let F a scalar function on $\mathcal{M}$, and let ϕ_0 and ϕ_1 two scalar functions on Σ_0. Let ϕ the solution of the following wave equation on $\mathcal{M}$:*

$$\begin{cases} \Box\phi = F, \\ \phi|_{\Sigma_0} = \phi_0,\ \partial_0(\phi)|_{\Sigma_0} = \phi_1. \end{cases} \tag{2.82}$$

Let $\mathcal{E}_0, \mathcal{E}_1$ denote the energy quantities,

$$\begin{aligned} \mathcal{E}_0[\phi] &:= \|\boldsymbol{\partial}\phi\|_{L^\infty_t L^2(\Sigma_t)} + \sup_{\mathcal{H}}(\|\nabla\!\!\!/\phi\|_{L^2(\mathcal{H})} + \|L(\phi)\|_{L^2(\mathcal{H})}) \\ \mathcal{E}_1[\phi] &:= \|\partial(\boldsymbol{\partial}\phi)\|_{L^\infty_t L^2(\Sigma_t)} + \|\partial_0(\partial_0\phi)\|_{L^2(\mathcal{M})} + \sup_{\mathcal{H}}\left(\|\nabla\!\!\!/(\partial\phi)\|_{L^2(\mathcal{H})} + \|L(\partial\phi)\|_{L^2(\mathcal{H})}\right) \end{aligned}$$

where the supremum is taken over all weakly regular null hypersurfaces $\mathcal{H}$ (satisfying assumptions (2.25) *and* (2.26)*). The following estimates hold true, provided that εM^2 is sufficiently small,*

$$\mathcal{E}_0 \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}, \tag{2.83}$$

$$\mathcal{E}_1 \lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\nabla F\|_{L^2(\mathcal{M})}. \tag{2.84}$$

PROOF. We introduce the energy momentum tensor $Q_{\alpha\beta}$ on $\mathcal{M}$ given by:

$$Q_{\alpha\beta} = Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\mathbf{g}_{\alpha\beta}\left(\mathbf{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right).$$

In view of the equation (2.82) satisfied by ϕ, we have:

$$\mathbf{D}^\alpha Q_{\alpha\beta} = F\partial_\beta\phi.$$

Now, consider the divergence of the 1-tensor $P_\alpha = \mathbf{Q}_{\alpha\beta}e_0^\beta = Q_{\alpha 0}$,

$$\mathbf{D}^\alpha P_\alpha = F\partial_0\phi + \frac{1}{2}Q_{\alpha\beta}\pi^{\alpha\beta},$$

where π is the deformation tensor of e_0. Integrating over well-chosen regions of $\mathcal{M}$, we easily obtain:

$$\begin{aligned} \mathcal{E}_0 &\lesssim \|\nabla\phi_0\|^2_{L^2(\Sigma_0)} + \|\phi_1\|^2_{L^2(\Sigma_0)} + \left|\int_{\mathcal{M}} F\partial_0\phi d\mathcal{M}\right| + \left|\int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta}d\mathcal{M}\right| \\ &\lesssim \|\nabla\phi_0\|^2_{L^2(\Sigma_0)} + \|\phi_1\|^2_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}\|\partial_0\phi\|_{L^2(\mathcal{M})} + \left|\int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta}d\mathcal{M}\right|. \end{aligned} \tag{2.85}$$

Next, we deal with the last term in the right-hand side of (2.85). In view of (2.81) and our maximal foliation assumption, we have:

$$\int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta}d\mathcal{M} = -2\int_{\mathcal{M}}\partial_a\phi\partial_b\phi k^{ab}d\mathcal{M} + \int n^{-1}\nabla^a n\partial_a\phi\partial_0\phi d\mathcal{M}.$$

[1]i.e. it satisfies assumptions (2.25) and (2.26)

Together with the bilinear bootstrap assumptions (2.36) and the estimates (2.62) for the lapse n, this yields:

$$\begin{aligned}\left|\int_{\mathcal{M}} Q_{\alpha\beta}\pi^{\alpha\beta}d\mathcal{M}\right| &\lesssim \|k_a.\partial^a\phi\|_{L^2(\mathcal{M})}\|\partial\phi\|_{L^2(\mathcal{M})} + \|\nabla n\|_{L^\infty(\mathcal{M})}\|\boldsymbol{\partial}\phi\|^2_{L^2(\mathcal{M})}\\ &\lesssim M^2\varepsilon\left(\sup_{\mathcal{H}}\|\nabla\!\!\!/\phi\|_{L^2(\mathcal{H})}\right)\|\boldsymbol{\partial}\phi\|_{L^2(\mathcal{M})} + M\varepsilon\|\boldsymbol{\partial}\phi\|^2_{L^2(\mathcal{M})},\end{aligned}$$

which together with (2.85) concludes the proof of the (2.83). Though more technical the proof of (2.84) follows the same ideas, and we refer to [**27**] for the details. □

2.6. Improvement of the bootstrap assumptions (part 1)

In this section, we discuss the proof of Proposition 2.13. More precisely, we derive estimates for $\mathbf{R}, A_0$ and A which allow us to improve the basic bootstrap assumptions (2.27), (2.28), (2.29) and (2.30).

2.6.1. Curvature estimates. We derive the curvature estimates using the Bell-Robinson tensor,

$$Q_{\alpha\beta\gamma\delta} = \mathbf{R}_{\alpha}{}^{\lambda}{}_{\gamma}{}^{\sigma}\mathbf{R}_{\beta\,\lambda\,\delta\,\sigma} + {}^*\mathbf{R}_{\alpha}{}^{\lambda}{}_{\gamma}{}^{\sigma}\,{}^*\mathbf{R}_{\beta\,\lambda\,\delta\,\sigma}$$

Let

$$P_\alpha = Q_{\alpha\beta\gamma\delta}e_0^\beta e_0^\gamma e_0^\delta.$$

Then, we have:

$$D^\alpha P_\alpha = 3Q_{\alpha\beta\gamma\delta}\pi^{\alpha\beta}e_0^\gamma e_0^\delta, \tag{2.86}$$

where π is the deformation tensor of e_0. We introduce the Riemannian metric,

$$h_{\alpha\beta} = g_{\alpha\beta} + 2(e_0)_\alpha(e_0)_\beta \tag{2.87}$$

and use it to define the following space-time norm for tensors U:

$$|U|^2 = U_{\alpha_1\cdots\alpha_k}U_{\alpha'_1\cdots\alpha'_k}h^{\alpha_1\alpha'_1}\cdots h^{\alpha_k\alpha'_k}.$$

Given two space-time tensors U, V we denote by $U\cdot V$ a given contraction between the two tensors and by $|U\cdot V|$ the norm of the contraction according to the above definition.

Let $\mathcal{H}$ be a weakly regular null hypersurface with future normal L, $\mathbf{g}(L,T) = -1$. Integrating (2.86) on a well-chosen, causal, space-time region, we have:

$$\int_{\Sigma_t}|\mathbf{R}|^2 + \int_{\mathcal{H}}|\mathbf{R}\cdot L|^2 \lesssim \|\mathbf{R}\|^2_{L^2(\Sigma_0)} + \left|\int_{\mathcal{M}} Q_{\alpha\beta\gamma\delta}\pi^{\alpha\beta}e_0^\gamma e_0^\delta\right| \lesssim \varepsilon^2 + \left|\int_{\mathcal{M}} Q_{\alpha\beta\gamma\delta}\pi^{\alpha\beta}e_0^\gamma e_0^\delta\right|.$$

We need to estimate the term in the right-hand side of the previous inequality. Note that since $\pi_{00} = 0$, $\pi_{0j} = n^{-1}\nabla_j n$, and $\pi_{ij} = k_{ij}$, the bootstrap assumption (2.27) for $\mathbf{R}$, and the estimates (2.62) for n yield:

$$\begin{aligned}\int_{\Sigma_t}|\mathbf{R}|^2+\int_{\mathcal{H}}|\mathbf{R}\cdot L|^2 &\lesssim \varepsilon^2+\|\nabla n\|_{L^\infty}\|\mathbf{R}\|^2_{L^\infty_t L^2(\Sigma_t)}+\left|\int_{\mathcal{M}}Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right|\\ &\lesssim \varepsilon^2+(M\varepsilon)^3+\left|\int_{\mathcal{M}}Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right|.\end{aligned}$$

The term in the right-hand side of the previous inequality is dangerous. Schematically it has the form $\left|\int_{\mathcal{M}}k\mathbf{R}^2\right|$. Typically this term is estimated by:

$$\left|\int_{\mathcal{M}}k\mathbf{R}^2\right| \lesssim \|k\|_{L^2_t L^\infty(\Sigma_t)}\|\mathbf{R}\|^2_{L^\infty_t L^2(\Sigma_t)},$$

which requires a Strichartz estimate for k which is false even in flat space. It is for this reason that we need the trilinear bootstrap assumption (2.43). Using it we derive,

$$\int_{\Sigma_t}|\mathbf{R}|^2+\int_{\mathcal{H}}|\mathbf{R}\cdot L|^2 \lesssim \varepsilon^2+M^4\varepsilon^3. \tag{2.88}$$

which, for small ε, improves the bootstrap assumptions (2.27) and (2.28).

2.6.2. Improvement of the bootstrap assumption for A_0. Recall (2.23)

$$\Delta A_0=\mathbf{A}\partial A+\mathbf{A}\partial A_0+\mathbf{A}^3. \tag{2.89}$$

Then, using (2.89), elliptic estimates on Σ_t, and commuting (2.89) with ∂_t in order to control $\partial_t A_0$, we are able to obtain the improved estimate (2.47) (see [**27**] for the details).

2.6.3. Improvement of the bootstrap assumption for A. Using the estimates for $\Box B_i$ derived in Lemma 2.23, the estimates for B on the initial slice Σ_0, and the energy estimate (2.84) derived in Lemma 2.25, we have:

$$\|\partial^2 B\|_{L^\infty_t L^2(\Sigma_t)}\lesssim \varepsilon+M^2\varepsilon^2. \tag{2.90}$$

Using then (2.90) with Lemma 2.20, we obtain:

$$\|\partial A\|_{L^\infty_t L^2(\Sigma_t)}\lesssim\|\partial^2 B\|_{L^\infty_t L^2(\Sigma_t)}+\|\partial E\|_{L^\infty_t L^2(\Sigma_t)}\lesssim\varepsilon+M^2\varepsilon^2. \tag{2.91}$$

which proves corresponding estimate in (2.46).

To estimate $\partial_0 A$. we recall that, $\partial_0(A_j)=\partial_j(A_0)+\mathbf{R}_{0j\cdots}$. Thus, we have:

$$\|\partial_0 A\|_{L^\infty_t L^2(\Sigma_t)}\lesssim\|\partial A_0\|_{L^\infty_t L^2(\Sigma_t)}+\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)},$$

which together with the improved estimates for $\mathbf{R}$ and A_0 yields:

$$\|\partial_0 A\|_{L^\infty_t L^2(\Sigma_t)}\lesssim\varepsilon+(M\varepsilon)^{\frac{3}{2}}. \tag{2.92}$$

2.7. Parametrix for the wave equation

Let $u_\pm$ two families, indexed by $\omega \in \mathbb{S}^2$, of scalar functions on the space-time $\mathcal{M}$ satisfying the Eikonal equation for each $\omega \in \mathbb{S}^2$. We also denote ${}^\omega u_\pm(t,x) = u_\pm(t,x,\omega)$. We have the freedom of choosing ${}^\omega u_\pm$ on the initial slice Σ_0, and in order for the results in [**43**], [**45**] to apply, we need to initialize ${}^\omega u_\pm$ on Σ_0 as in [**42**] (see also Chapter 6).

Let $\mathcal{H}_{{}^\omega u_\pm}$ denote the corresponding null level hypersurfaces. Let ${}^\omega L_\pm$ its normal. ${}^\omega L_\pm$ is null, and we fix it by imposing $\mathbf{g}({}^\omega L_\pm, T) = -1$. Let the vectorfield tangent to Σ_t ${}^\omega N_\pm$ be defined such as to satisfy:

$$ {}^\omega L_\pm = \pm T + {}^\omega N_\pm. $$

We pick $({}^\omega e_\pm)_A$, $A = 1, 2$ vectorfields in Σ_t such that together with ${}^\omega N_\pm$ we obtain an orthonormal basis of Σ_t. Finally, we denote by $\nabla\!\!\!/_\pm$ derivatives in the directions $({}^\omega e_\pm)_A$, $A = 1, 2$.

REMARK 2.26. *Note that $\mathcal{H}_{{}^\omega u_\pm}$ satisfy assumptions* (2.25) *and* (2.26) *from the results in* [**44**] *(see Theorem 2.15 and section 3.4 in that paper).*

For any pair of functions $f_\pm$ on $\mathbb{R}^3$, we define the following scalar function on $\mathcal{M}$:

$$ \psi[f_+, f_-](t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u_+(t,x)} f_+(\lambda\omega)\lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u_-(t,x)} f_-(\lambda\omega)\lambda^2 d\lambda d\omega. $$

We appeal to the following result from [**43**] [**45**] (see also Chapters 3 and 5):

THEOREM 2.27. *Let ϕ_0 and ϕ_1 two scalar functions on Σ_0. Then, there is a unique pair of functions (f_+, f_-) such that:*

$$ \psi[f_+, f_-]|_{\Sigma_0} = \phi_0 \text{ and } \partial_0(\psi[f_+, f_-])|_{\Sigma_0} = \phi_1. $$

Furthermore, $f_\pm$ satisfy the following estimates:

$$ \|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)}, $$

and:

$$ \|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)}. $$

Finally, $\Box\psi[f_+, f_-]$ satisfies the following estimates:

$$ \|\Box\psi[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)}), $$

and:

$$ \|\partial\Box\psi[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)}). $$

REMARK 2.28. *The existence of $f_\pm$ and the first two estimates of Theorem 2.27 are proved in* [**43**] *(see also Chapter 5), while the last two estimates in Theorem 2.27 are proved in* [**45**] *(see also Chapter 3).*

We associate to any pair of functions ϕ_0, ϕ_1 on Σ_0 the function $\Psi_{om}[\phi_0, \phi_1]$ defined for $(t, x) \in \mathcal{M}$ as:

$$\Psi_{om}[\phi_0, \phi_1] = \psi[f_+, f_-]$$

where (f_+, f_-) is defined in view of Theorem 2.27 as the unique pair of functions associated to (ϕ_0, ϕ_1). In particular, we obtain:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla \phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)},$$

$$\|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla^2 \phi_0\|_{L^2(\Sigma_0)} + \|\nabla \phi_1\|_{L^2(\Sigma_0)},$$

$$\|\Box \Psi_{om}[\phi_0, \phi_1]\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla \phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)}), \tag{2.93}$$

and:

$$\|\partial \Box \Psi_{om}[\phi_0, \phi_1]\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla^2 \phi_0\|_{L^2(\Sigma_0)} + \|\nabla \phi_1\|_{L^2(\Sigma_0)}). \tag{2.94}$$

Next, let ${}^{\omega,s}u_\pm$ two families, indexed by $\omega \in \mathbb{S}^2$ and $s \in \mathbb{R}$, of scalar functions on the space-time $\mathcal{M}$ satisfying the Eikonal equation for each $\omega \in \mathbb{S}^2$ and $s \in \mathbb{R}$. We have the freedom of choosing ${}^{\omega,s}u_\pm$ on the slice Σ_s, and in order for the results in [**43**] [**45**] to apply, we need to initialize ${}^{\omega,s}u_\pm$ on Σ_s as in [**42**]. Note that the families ${}^{\omega}u_\pm$ correspond to ${}^{\omega,s}u$ with the choice $s = 0$. For any pair of functions $f_\pm$ on $\mathbb{R}^3$, and for any $s \in \mathbb{R}$, we define the following scalar function on $\mathcal{M}$:

$$\psi_s[f_+, f_-](t, x, s) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda\, {}^{\omega,s}u_+(t,x)} f_+(\lambda\omega) \lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda\, {}^{\omega,s}u_-(t,x)} f_-(\lambda\omega) \lambda^2 d\lambda d\omega.$$

We have the following straightforward corollary of Theorem 2.27:

Corollary 2.29. *Let $s \in \mathbb{R}$. Let ϕ_0 and ϕ_1 two scalar functions on Σ_s. Then, there is a unique pair of functions (f_+, f_-) such that:*

$$\psi_s[f_+, f_-]|_{\Sigma_s} = \phi_0 \text{ and } \partial_0(\psi_s[f_+, f_-])|_{\Sigma_s} = \phi_1.$$

Furthermore, $f_\pm$ satisfy the following estimates:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla \phi_0\|_{L^2(\Sigma_s)} + \|\phi_1\|_{L^2(\Sigma_s)},$$

and:

$$\|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla^2 \phi_0\|_{L^2(\Sigma_s)} + \|\nabla \phi_1\|_{L^2(\Sigma_s)}.$$

Finally, $\Box \psi_s[f_+, f_-]$ satisfies the following estimates:

$$\|\Box \psi_s[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla \phi_0\|_{L^2(\Sigma_s)} + \|\phi_1\|_{L^2(\Sigma_s)}),$$

and:

$$\|\partial \Box \psi_s[f_+, f_-]\|_{L^2(\mathcal{M})} \lesssim M\varepsilon(\|\nabla^2 \phi_0\|_{L^2(\Sigma_s)} + \|\nabla \phi_1\|_{L^2(\Sigma_s)}).$$

Next, for any $s \in \mathbb{R}$, we associate to any function F on Σ_s the function $\Psi(t,s)F$ defined for $(t,x) \in \mathcal{M}$ as:

$$\Psi(t,s)F = \psi_s[f_+, f_-](t)$$

where (f_+, f_-) is defined in view of Corollary 2.29 as the unique pair of functions associated to the choice $(\phi_0, \phi_1) = (0, F)$. In particular, we obtain:

$$\begin{aligned} \|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} &\lesssim \|F\|_{L^2(\Sigma_s)}, \\ \|\lambda^2 f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda^2 f_-\|_{L^2(\mathbb{R}^3)} &\lesssim \|\nabla F\|_{L^2(\Sigma_s)}, \\ \|\square\Psi(t,s)F\|_{L^2(\mathcal{M})} &\lesssim M\varepsilon\|F\|_{L^2(\Sigma_s)}, \end{aligned} \tag{2.95}$$

and:

$$\|\partial\square\Psi(t,s)F\|_{L^2(\mathcal{M})} \lesssim M\varepsilon\|\nabla F\|_{L^2(\Sigma_s)}. \tag{2.96}$$

Now, we are in position to construct an exact parametrix for the wave equation (2.82):

THEOREM 2.30 (Representation formula). *Let F a scalar function on $\mathcal{M}$, and let ϕ_0 and ϕ_1 two scalar functions on Σ_0. Let ϕ the solution of the wave equation (2.82) on $\mathcal{M}$. Then, there is a sequence $\phi^{(j)}$, $j \geq 0$, of scalar functions approximations of ϕ and a sequence $F^{(j)}$, $j \geq 0$, of scalar functions on $\mathcal{M}$, with of the form:*

$$\phi^{(0)} = \Psi_{om}[\phi_0, \phi_1] + \int_0^t \Psi(t,s)F^{(0)}(s,.)ds, \qquad F^{(0)} = F$$

and for all $j \geq 1$:

$$\phi^{(j)} = \int_0^t \Psi(t,s)F^{(j)}(s,.)ds,$$

such that,

$$\phi = \sum_{j=0}^{+\infty} \phi^{(j)},$$

and such that $\phi^{(j)}$ and $F^{(j)}$ satisfy the following estimates:

$$\|\boldsymbol{\partial}\phi^{(j)}\|_{L^\infty_t L^2(\Sigma_t)} + \|F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\varepsilon)^j(\|\nabla\phi_0\|_{L^2(\Sigma_0)} + \|\phi_1\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\mathcal{M})}),$$

and:

$$\|\partial\boldsymbol{\partial}\phi^{(j)}\|_{L^\infty_t L^2(\Sigma_t)} + \|\partial F^{(j)}\|_{L^2(\mathcal{M})} \lesssim (M\varepsilon)^j(\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})}),$$

PROOF. Let us define:

$$F^{(0)} = F \text{ and } \phi^{(0)} = \Psi_{om}[\phi_0, \phi_1] + \int_0^t \Psi(t,s)F^{(0)}(s,.)ds.$$

Then, we define iteratively for $j \geq 1$:

$$F^{(j)} = -\square\phi^{(j-1)} + F^{(j-1)} \text{ and } \phi^{(j)} = \int_0^t \Psi(t,s)F^{(j)}(s,.)ds.$$

The proof follows from the estimates (2.93), (2.94), (2.95) and (2.96), together with the energy estimates for the wave equation of Lemma 2.25. We refer to [**27**] for the details. □

2.8. Improvement of the bootstrap assumptions (part 2)

The goal of this section and next section is to prove Proposition 2.14. This requires in particular to write B using the representation formula of Theorem 2.30. In this section we derive the improved bilinear estimate (2.48), (2.49), (2.50), (2.51) and (2.52) of Proposition 2.14. We also derive the improved trilinear estimate (2.57).

2.8.1. Improvement of the bilinear bootstrap assumptions I. In this section, we give the main ideas on the how we derive the improved bilinear estimate (2.48), (2.49), (2.50), (2.51) and (2.52) of Proposition 2.14. These bilinear estimates all involve the norm in $L^2(\mathcal{M})$ of quantities of the type:

$$\mathcal{C}(U, \partial\phi),$$

where $\mathcal{C}(U, \partial\phi)$ denotes a contraction with respect to one index between a tensor U and $\partial\phi$, with ϕ being a scalar function which is solution to the wave equation (2.82) with F, ϕ_0 and ϕ_1 satisfying the estimate:

$$\|\nabla^2\phi_0\|_{L^2(\Sigma_0)} + \|\nabla\phi_1\|_{L^2(\Sigma_0)} + \|\partial F\|_{L^2(\mathcal{M})} \lesssim M\varepsilon.$$

In particular, we may use the parametrix constructed in Lemma 2.30 for ϕ:

$$\phi = \sum_{j=0}^{+\infty} \phi^{(j)},$$

with:

$$\phi^{(0)} = \Psi_{om}[\phi_0, \phi_1] + \int_0^t \Psi(t,s)F(s,.)ds,$$

and for all $j \geq 1$:

$$\phi^{(j)} = \int_0^t \Psi(t,s)F^{(j)}(s,.)ds.$$

Thus, we need to estimate the norm in $L^2(\mathcal{M})$ of contractions of quantities of the type:

$$\mathcal{C}(U, \partial(\Psi_{om}[\phi_0, \phi_1])) + \sum_{j=0}^{+\infty} \int_0^t \mathcal{C}(U, \partial(\Psi(t,s)F^{(j)}(s,.)))ds.$$

After using the definition of Ψ_{om} and $\Psi(t,s)$, and the estimates for $F^{(j)}$ provided by Lemma 2.30, this reduces to estimating:

$$\int_{\mathbb{S}^2}\int_0^\infty \mathcal{C}(U, \partial(e^{i\lambda\ ^\omega u_+(t,x)}))f_+(\lambda\omega)\lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2}\int_0^\infty \mathcal{C}(U, \partial(e^{i\lambda\ ^\omega u_-(t,x)}))f_-(\lambda\omega)\lambda^2 d\lambda d\omega,$$

where $f_\pm$ in view of Theorem 2.27 and the estimates for F, ϕ_0 and ϕ_1 satisfies:

$$\|\lambda^2 f_\pm\|_{L^2(\mathbb{R}^3)} \lesssim M\varepsilon.$$

Since both half waves parametrices are estimated in the same way, the bilinear estimates (2.33), (2.34), (2.35), (2.36) and (2.37) all estimate the norm in $L^2(\mathcal{M})$ of contractions of quantities of the type:

$$\int_{\mathbb{S}^2}\int_0^\infty \mathcal{C}(U, \partial(e^{i\lambda\ {}^\omega u(t,x)}))f(\lambda\omega)\lambda^2 d\lambda d\omega,$$

where f satisfies:

$$\|\lambda^2 f\|_{L^2(\mathbb{R}^3)} \lesssim M\varepsilon. \tag{2.97}$$

Furthermore we observe that $\partial_j(e^{i\lambda\ {}^\omega u}) = i\lambda e^{i\lambda\ {}^\omega u}\partial_j({}^\omega u)$, and that the gradient of ${}^\omega u$ on Σ_t is given by: $\nabla({}^\omega u) = {}^\omega b^{-1}\ {}^\omega N$, with ${}^\omega b = |\nabla({}^\omega u)|^{-1}$ is the null lapse, and ${}^\omega N = \frac{\nabla\ {}^\omega u}{|\nabla\ {}^\omega u|}$ is the unit normal to $\mathcal{H}_{{}^\omega u} \cap \Sigma_t$ along Σ_t. Thus, the bilinear estimates (2.33), (2.34), (2.35), (2.36) and (2.37) all reduce to $L^2(\mathcal{M})$-estimates of expressions of the form:

$$\mathfrak{C}[U,f] := \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(t,x)}\ {}^\omega b^{-1}\mathcal{C}(U, {}^\omega N)f(\lambda\omega)\lambda^3 d\lambda d\omega, \tag{2.98}$$

where f satisfies (2.97). To estimate $\mathfrak{C}[U,f]$ we follow the strategy of [**21**].

$$\begin{aligned}
&\|\mathfrak{C}[U,f]\|_{L^2(\mathcal{M})} &(2.99)\\
\lesssim\ & \int_{\mathbb{S}^2}\left\| {}^\omega b^{-1}\mathcal{C}(U, {}^\omega N)\left(\int_0^{+\infty} e^{i\lambda\ {}^\omega u(t,x)}f(\lambda\omega)\lambda^3 d\lambda\right)\right\|_{L^2(\mathcal{M})} d\omega\\
\lesssim\ & \int_{\mathbb{S}^2}\|{}^\omega b^{-1}\|_{L^\infty(\mathcal{M})}\|\mathcal{C}(U, {}^\omega N)\|_{L^\infty_{{}^\omega u}L^2(\mathcal{H}_{{}^\omega u})}\left\|\int_0^{+\infty} e^{i\lambda\ {}^\omega u(t,x)}f(\lambda\omega)\lambda^3 d\lambda\right\|_{L^2_{{}^\omega u}} d\omega\\
\lesssim\ & \left(\sup_{\omega\in\mathbb{S}^2}\|{}^\omega b^{-1}\|_{L^\infty(\mathcal{M})}\right)\left(\sup_{\omega\in\mathbb{S}^2}\|\mathcal{C}(U, {}^\omega N)\|_{L^\infty_{{}^\omega u}L^2(\mathcal{H}_{{}^\omega u})}\right)\left(\int_{\mathbb{S}^2}\|\lambda^3 f(\lambda\omega)\|_{L^2_\lambda} d\omega\right)\\
\lesssim\ & \left(\sup_{\omega\in\mathbb{S}^2}\|{}^\omega b^{-1}\|_{L^\infty(\mathcal{M})}\right)\left(\sup_{\omega\in\mathbb{S}^2}\|\mathcal{C}(U, {}^\omega N)\|_{L^\infty_{{}^\omega u}L^2(\mathcal{H}_{{}^\omega u})}\right)\|\lambda^2 f\|_{L^2(\mathbb{R}^3)},
\end{aligned}$$

Now, since ${}^\omega u$ has been initialized on Σ_0 as in [**42**], and satisfies the Eikonal equation on $\mathcal{M}$, the results in [**44**] (see Theorem 2.15 in that paper, and also (4.42)) under the assumption of Theorem 1.14 imply:

$$\sup_{\omega\in\mathbb{S}^2}\|{}^\omega b^{-1}\|_{L^\infty(\mathcal{M})} \lesssim 1.$$

Together with the fact that f satisfies (2.97), and with (2.99), we finally obtain:

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(t,x)}\ {}^\omega b^{-1}\mathcal{C}(U, {}^\omega N)f(\lambda\omega)\lambda^3 d\lambda d\omega\right\|_{L^2(\mathcal{M})} &(2.100)\\
\lesssim\ & M\varepsilon\left(\sup_{\omega\in\mathbb{S}^2}\|\mathcal{C}(U, {}^\omega N)\|_{L^\infty_{{}^\omega u}L^2(\mathcal{H}_{{}^\omega u})}\right).
\end{aligned}$$

It remains to estimate the right-hand side of (2.100) for the contractions appearing in the bilinear estimates (2.33), (2.34), (2.35), (2.36) and (2.37). Since all the estimates in the proof will be uniform in ω, we drop the index ω to ease the notations.

REMARK 2.31. *In the proof of bilinear estimates* (2.48)*,* (2.49)*,* (2.50)*,* (2.51) *and* (2.52)*, the tensor U appearing in the expression $\mathcal{C}(U,N)$ is either $\mathbf{R}$ or derivatives of solutions ϕ of a a scalar wave equation. In view of the bootstrap assumption* (2.28) *for the curvature flux, as well as the first energy estimate for the wave equation in Lemma 2.25, we can control $\|\mathcal{C}(U,N)\|_{L^\infty_u L^2(\mathcal{H}_u)}$ as long as we can show that $\mathcal{C}(U,N)$ can be expressed*[2] *in terms of, $\mathbf{R}\cdot L$, $\nabla\!\!\!/\phi$ and $L(\phi)$.*

2.8.1.1. *Proof of* (2.48). Since $A = curl\,(B)+E$ in view of Lemma 2.20 and bootstrap assumption (2.29), we have:

$$\begin{aligned}\|A^j\partial_j(A)\|_{L^2(\mathcal{M})} &\lesssim \|(curl\,(B))^j\partial_j(A)\|_{L^2(\mathcal{M})} + \|E\|_{L^2_tL^\infty(\Sigma_t)}\|\partial A\|_{L^\infty_tL^2(\Sigma_t)} \quad (2.101)\\ &\lesssim \|(curl\,(B))^j\partial_j(A)\|_{L^2(\mathcal{M})} + M^2\varepsilon^2,\end{aligned}$$

To estimate $\|(curl\,(B))^j\partial_j(A)\|_{L^2(\mathcal{M})}$ we write, $(curl\,(B))^j\partial_j(A) = \in_{jmn}\partial_m(B_n)\partial_j(A)$. We are now ready to apply the representation theorem 2.30 to B. Indeed, according to Lemma 2.23, and proposition 2.19, we have

$$\begin{aligned}\Box B = F, \qquad \|\partial F\|_{L^2(\mathcal{M})} &\lesssim M^2\varepsilon^2 \quad (2.102)\\ \|\boldsymbol{\partial} B(0)\|_{L^2(\Sigma_0)} + \|\partial^2 B(0)\|_{L^2(\Sigma_0)} + \|\partial(\partial_0 B(0))\|_{L^2(\Sigma_0)} &\lesssim M\varepsilon.\end{aligned}$$

We are thus in a position to apply the reduction discussed in the subsection above and reduce our desired bilinear estimate to an estimate for,

$$\mathcal{C}(U,N) \;=\; \in_{jm\cdot} N_m\partial_j(A)$$

Now, we decompose ∂_j on the orthonormal frame $N, f_A, A=1,2$ of Σ_t, where we recall that $f_A, A=1,2$ denotes an orthonormal basis of $\mathcal{H}_u\cap\Sigma_t$. We have schematically:

$$\partial_j = N_jN + \nabla\!\!\!/, \qquad (2.103)$$

where $\nabla\!\!\!/$ denotes derivatives which are tangent to $\mathcal{H}_u\cap\Sigma_t$. Thus, we have:

$$\in_{jm\cdot} N_m\partial_j(A) = \in_{jm\cdot} N_mN_j\partial_N(A) + \nabla\!\!\!/(A) = \nabla\!\!\!/(A),$$

where we used the antisymmetry of $\in_{jm\cdot}$ in the last equality. Therefore, we obtain in this case:

$$\|\mathcal{C}(U,N)\|_{L^\infty_uL^2(\mathcal{H}_u)} \lesssim \|\nabla\!\!\!/(A)\|_{L^\infty_uL^2(\mathcal{H}_u)}.$$

---

[2] In other words, our goal is to check that the term $\mathcal{C}(U,N)$ does not involve the dangerous terms of the type $\underline{\alpha}$ and $\underline{L}\phi$, where $\underline{L}$ is the vectorfield defined as $\underline{L} = 2T - L$, and $\underline{\alpha}$ is the two tensor on $\Sigma_t\cap\mathcal{H}_u$ defined by $\underline{\alpha}_{AB} = \mathbf{R}_{\underline{L}\,A\,\underline{L}\,B}$.

It remains to estimate $\|\nabla\!\!\!/(A)\|_{L^\infty_u L^2(\mathcal{H}_u)}$. Since $A = curl\,(B) + E$ we have, using Lemma 2.20 again, followed by Proposition 2.23 and Lemma 2.25:

$$\begin{aligned}\|\nabla\!\!\!/(A)\|_{L^\infty_u L^2(\mathcal{H}_u)} &\lesssim \|\nabla\!\!\!/(\partial B)\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/(E)\|_{L^\infty_u L^2(\mathcal{H}_u)}\\ &\lesssim \|\nabla\!\!\!/(\partial B)\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\partial E\|_{L^\infty_t L^3(\Sigma_t)} + \|\partial^2 E\|_{L^\infty_t L^{\frac{3}{2}}(\Sigma_t)}\\ &\lesssim \|\nabla\!\!\!/(\partial B)\|_{L^\infty_u L^2(\mathcal{H}_u)} + M\varepsilon,\\ &\lesssim M\varepsilon.\end{aligned}$$

Therefore,

$$\|A^j\partial_j(A)\|_{L^2(\mathcal{M})} \lesssim \|(curl\,(B))^j\partial_j(A)\|_{L^2(\mathcal{M})} + M^2\varepsilon^2 \lesssim M^2\varepsilon^2,$$

as desired.

2.8.1.2. *Proof of* (2.49). The proof of (2.49) is similar to the one of (2.48) in view of Lemma 2.20.

2.8.1.3. *Proof of* (2.50). Since B satisfies a wave equation in view of Lemma 2.23, the quantity $\mathcal{C}(U,N)$ is in this case[3],

$$\mathcal{C}(U,N) = N_j\mathbf{R}_{0\,j\,\cdot\,\cdot} = \mathbf{R}_{0\,N\,\cdot\,\cdot} = \frac{1}{2}\mathbf{R}_{L\,\underline{L}\,\cdot\,\cdot}$$

which together with the bootstrap assumption for the curvature flux (2.28) improves the bilinear estimate (2.35).

2.8.1.4. *Proof of* (2.51). We have $k_{j\,\cdot} = A^j$ and $A = curl\,(B) + E$ in view of Lemma 2.20. Arguing as in (2.101), we reduce the proof to the estimate of:

$$\|(curl\,B)^j\partial_j\phi\|_{L^2(\mathcal{M})}.$$

Then, the proof proceeds as the one of (2.48).

2.8.1.5. *Proof of* (2.52). The proof of (2.52) proceeds as in (2.51).

2.8.2. Improvement of the trilinear estimate. In this section, we shall derive the improved trilinear estimate (2.57). To estimate the trilinear quantity $\left|\int_{\mathcal{M}} Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right|$. we first write, according to Lemma 2.20, $A = curl\,(B) + E$ by. Arguing as in (2.101), we reduce the proof of (2.57) to an estimate for:

$$\left|\int_{\mathcal{M}} Q_{\cdot\,j\gamma\delta}(curl\,(B))_j e_0^\gamma e_0^\delta\right|.$$

Making use of the wave equation (2.102) for B we argue as in the beginning of section 2.8.1 to reduce the proof to an estimate of the following:

$$\left|\int_{\mathcal{M}}\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\,{}^\omega u(t,x)}\,{}^\omega b^{-1}\left(\in_{jm\cdot}\,{}^\omega N_m Q_{j\dots}\right)f(\lambda\omega)\lambda^3 d\lambda d\omega d\mathcal{M}\right|$$

where f satisfies:

$$\|\lambda^2 f\|_{L^2(\mathbb{R}^3)} \lesssim M\varepsilon.$$

[3]Use also $L = T + N$, $\underline{L} = T - N$ and the symmetries of $\mathbf{R}$

Arguing exactly as in (2.99) (2.100), we can estimate the latter integral by the quantity $\sup_{\omega\in\mathbb{S}^2}\|\in_{jm\cdot}N_mQ_{j\dots}\|_{L^2_{\omega_u}L^1(\mathcal{H}_{\omega_u})}M\varepsilon$. In other words,

$$\left|\int_{\mathcal{M}}Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right|\lesssim\sup_{\omega\in\mathbb{S}^2}\|\in_{jm\cdot}N_mQ_{j\dots}\|_{L^2_{\omega_u}L^1(\mathcal{H}_{\omega_u})}M\varepsilon+M^3\varepsilon^3. \tag{2.104}$$

Next, we estimate the right-hand side of (2.104). Since all the estimates in the proof will be uniform in ω, we drop the index ω to ease the notations. The formula (2.42) for the Bell-Robinson tensor Q yields:

$$\begin{aligned}Q_{j\cdots} &= \mathbf{R}_j{}^{\lambda}\cdot\cdot\mathbf{R}_{\cdot\lambda\cdot\cdot}+dual\\ &= -\frac{1}{2}\mathbf{R}_{jL\cdot\cdot}\mathbf{R}_{\cdot\underline{L}\cdot\cdot}-\frac{1}{2}\mathbf{R}_{j\underline{L}\cdot\cdot}\mathbf{R}_{\cdot L\cdot\cdot}+\mathbf{R}_{jA\cdot\cdot}\mathbf{R}_{\cdot A\cdot\cdot}+dual,\end{aligned}$$

where we used the frame L, $\underline{L}$, $f_A, A=1,2$ in the last equality. Thus, we have schematically:

$$\in_{jm\cdot}N_mQ_{j\dots}=\mathbf{R}(\mathbf{R}\cdot L+\in_{jm\cdot}N_m\mathbf{R}_{jA\cdot\cdot})$$

Decomposing e_j with respect to the orthonormal frame $N, f_B, B=1,2$, we note that:

$$\in_{jm\cdot}N_m\mathbf{R}_{jA\cdot\cdot}=\in_{jm\cdot}N_jN_m\mathbf{R}_{NA\cdot\cdot}+\in_{jm\cdot}(f_B)_jN_m\mathbf{R}_{BA\cdot\cdot}=\mathbf{R}_{BA\cdot\cdot}.$$

On the other hand, decomposing $\mathbf{R}_{BA\cdot\cdot}$ further and using the symmetries of $\mathbf{R}$, one easily checks that $\mathbf{R}_{BA\cdot\cdot}$ must contain at least one L so that it is of the type $\mathbf{R}\cdot\mathbf{L}$. Thus, we have schematically:

$$\in_{jm\cdot}N_mQ_{j\dots}=\mathbf{R}(\mathbf{R}\cdot L). \tag{2.105}$$

Thus, in view of (2.104), making use of the bootstrap assumptions (2.27) on R and (2.28) on the curvature flux, we deduce,

$$\begin{aligned}\left|\int_{\mathcal{M}}Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right| &\lesssim (M\varepsilon)^3+M\varepsilon\|\mathbf{R}\mathbf{R}_L\|_{L^2_uL^1(\mathcal{H}_u)}\\ &\lesssim (M\varepsilon)^3+M\varepsilon\|\mathbf{R}\|_{L^2(\mathcal{M})}\|\mathbf{R}_L\|_{L^\infty_uL^2(\mathcal{H}_u)}\\ &\lesssim M^3\varepsilon^3\end{aligned}$$

In other words,

$$\left|\int_{\mathcal{M}}Q_{ij\gamma\delta}k^{ij}e_0^\gamma e_0^\delta\right|\lesssim(M\varepsilon)^3. \tag{2.106}$$

which yields the desired improvement of the trilinear estimate (2.43).

2.9. Improvement of the bootstrap assumptions (part 3)

In this section, we conclude the proof of Proposition 2.14. More precisely, we give the main ideas in the improvement of the bilinear bootstrap assumptions II. We start with a discussion of the sharp $L^4(\mathcal{M})$- Strichartz estimate.

2.9.1. The sharp Strichartz $L^4(\mathcal{M})$ estimate. To a function f on $\mathbb{R}^3$ and a family ${}^\omega u$ indexed by $\omega \in \mathbb{S}^2$ of scalar functions on the space-time $\mathcal{M}$ satisfying the Eikonal equation for each $\omega \in \mathbb{S}^2$, we associate a half wave parametrix:

$$\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u(t,x)} f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

Let an integer p and a smooth cut-off function ψ on $(0,+\infty)$ supported in a shell. We call a half wave parametrix localized at frequencies of size $\lambda \sim 2^p$ the following Fourier integral operator:

$$\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u(t,x)} \psi(2^{-p}\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

We have the following $L^4(\mathcal{M})$ Strichartz estimates localized in frequency for a half wave parametrix which are proved in [**46**] (see also Chapter 7):

Proposition 2.32 (Corollary 2.8 in [**46**]). *Let f a function on $\mathbb{R}^3$, let $p \in \mathbb{N}$, and let ψ a smooth function on $(0,+\infty)$ compactly supported in the shell $1/2 \leq \lambda \leq 2$. Let ${}^\omega u$ a family indexed by $\omega \in \mathbb{S}^2$ of scalar functions on the space-time $\mathcal{M}$ satisfying the Eikonal equation for each $\omega \in \mathbb{S}^2$ and initialized on the initial slice Σ_0 as in* [**42**]. *Let ϕ_p the scalar function on $\mathcal{M}$ defined by the following oscillatory integral:*

$$\phi_p(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u(t,x)} \psi(2^{-p}\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

Then, we have the following $L^4(\mathcal{M})$ Strichartz estimates for ϕ_p:

$$\|\phi_p\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{p}{2}}\|\psi(2^{-p}\lambda)f\|_{L^2(\mathbb{R}^3)}, \tag{2.107}$$

$$\|\partial\phi_p\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{3p}{2}}\|\psi(2^{-p}\lambda)f\|_{L^2(\mathbb{R}^3)}, \tag{2.108}$$

$$\|\partial^2\phi_p\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{5p}{2}}\|\psi(2^{-p}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{2.109}$$

Note that this Strichartz estimate is sharp.

2.9.2. Improvement of the non sharp Strichartz estimates. Here, we derive the improved non sharp Strichartz estimates (2.55) and (2.56). In view of Lemma 2.20, (2.55) easily follows from (2.56), so we focus on the later improved estimate.

Corollary 2.33. *B satisfies the following Strichartz estimate:*

$$\|\partial B\|_{L^2_t L^7(\Sigma_t)} \lesssim M\varepsilon.$$

Proof. Recall (2.102) which allows us to apply the representation formula of Theorem 2.30 to B. By a straightforward reduction the proof then reduces to the following non-sharp Strichartz estimate for a half wave parametrix:

$$\left\|\partial\left(\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\, {}^\omega u(t,x)} f(\lambda\omega)\lambda^2 d\lambda d\omega\right)\right\|_{L^2_t L^7(\Sigma_t)} \lesssim \|\lambda^2 f\|_{L^2(\mathbb{R}^3)}. \tag{2.110}$$

Then, the proof of Corollary 2.33 follows in particular from the sharp Strichartz estimate of Proposition 2.32. We refer to [**27**] for the details. □

2.9.3. Improvement of the bilinear bootstrap assumptions II. In this section, we sketch the proofs of the improved bilinear estimates (2.53) and (2.54) of Proposition 2.14. Based on the decomposition $A = curl\,(B) + E$ of Lemma 2.20 it is easy to show that that the proof of the bilinear estimates (2.38) and (2.39) reduces to:

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} \lesssim M^2\varepsilon^2. \tag{2.111}$$

Decomposing B according to Theorem 2.30,

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\partial B, \partial B))\|_{L^2(\mathcal{M})} \leq \sum_{m,n=0}^{+\infty} \|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\phi^{(m)}, \phi^{(n)}))\|_{L^2(\mathcal{M})}. \tag{2.112}$$

Thus it suffices to prove for all $m, n \geq 0$:

$$\|(-\Delta)^{-\frac{1}{2}}(Q_{ij}(\phi^{(m)}, \phi^{(n)}))\|_{L^2(\mathcal{M})} \lesssim (M\varepsilon)^{m+1}(M\varepsilon)^{n+1}. \tag{2.113}$$

The estimates in (2.113) are analogous for all m, n, so it suffices to prove (2.113) in the case $(m, n) = (0, 0)$. In view of the definition of $\phi^{(0)}$, the estimates for B on the initial slice Σ_0, estimate (2.70) for $\partial\Box B$, and the definition of Ψ_{om} and $\Psi(t, s)$, (2.113) reduces to the following bilinear estimate for half wave parametrices:

$$\left\|(-\Delta)^{-\frac{1}{2}}\left(Q_{ij}(\phi_{f_1}, \phi_{f_2}\right)\right\|_{L^2(\mathcal{M})} \lesssim \|\lambda f_1\|_{L^2(\mathbb{R}^3)}\|\lambda f_2\|_{L^2(\mathbb{R}^3)} \tag{2.114}$$

with,

$$\phi_f = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(t,x)} f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

We then decompose f_1, f_2 with respect to frequency and reduce the desired estimate to $L^4(\mathcal{M})$ Strichartz estimate localized in frequency of Proposition 2.32, see details in [**27**].

This concludes the proof of Proposition 2.14.

CHAPTER 3

Control of the error term

In this chapter, we consider the Fourier integral operator E given by (1.27) in which corresponds to the error term of a plane wave type parametrix. Recall that E is given by:

$$Ef(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)}\square_{\mathbf{g}}u(t,x,\omega)f(\lambda\omega)\lambda^3 d\lambda d\omega,\ (t,x)\in\mathcal{M},$$

where $u(.,.,\omega)$ is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that $u(0,x,\omega)\sim x\cdot\omega$ when $|x|\to+\infty$ on Σ_0 (see section 3.1.1). The goal of this chapter is to outline the main ideas allowing us to obtain the control for the error term E in [**45**].

3.1. Geometric set-up and main results

3.1.1. Geometry of the foliation of $\mathcal{M}$ by u. Recall that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ depending on a extra parameter $\omega\in\mathbb{S}^2$. The level hypersufaces $u(t,x,\omega) = u$ of the optical function u are denoted by $\mathcal{H}_u$. Let L' denote the space-time gradient of u, i.e.:

$$L' = -\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha. \tag{3.1}$$

Using the fact that u satisfies the eikonal equation, we obtain:

$$\mathbf{D}_{L'}L' = 0, \tag{3.2}$$

which implies that L' is the geodesic null generator of $\mathcal{H}_u$.

We foliate the space-time $\mathcal{M}$ by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t and we denote by T the unit, future oriented, normal to Σ_t. We have:

$$T(u) = \pm|\nabla u|$$

where $|\nabla u|^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t. Since the sign of $T(u)$ is irrelevant, we choose by convention:

$$T(u) = |\nabla u|. \tag{3.3}$$

We denote by $P_{t,u}$ the surfaces of intersection between Σ_t and $\mathcal{H}_u$.

Definition 3.1 (*Canonical null pair*).

$$L = bL' = T + N, \qquad \underline{L} = 2T - L = T - N \tag{3.4}$$

where L' is the space-time gradient of u (3.1), b is the lapse of the null foliation (or shortly null lapse)

$$b^{-1} = - < L', T >= T(u), \tag{3.5}$$

and N is a unit normal, along Σ_t, to the surfaces $P_{t,u}$. Since u satisfies the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$, this yields $L'(u) = 0$ and thus $L(u) = 0$. In view of the definition of L and (3.3), we obtain:

$$N = -\frac{\nabla u}{|\nabla u|}. \tag{3.6}$$

DEFINITION 3.2. *A null frame e_1, e_2, e_3, e_4 at a point $p \in P_{t,u}$ consists, in addition to the null pair $e_3 = \underline{L}, e_4 = L$, of* arbitrary orthonormal vectors *e_1, e_2 tangent to $P_{t,u}$.*

DEFINITION 3.3 (*Second fundamental form*). *Let e_1, e_2, e_3, e_4 be a null frame on $P_{t,u}$ as above. The second fundamental form on $P_{t,u}$ associated to our canonical null pair is given by*

$$\chi_{AB} =< \mathbf{D}_A e_4, e_B > .$$

We decompose χ into its trace and traceless component.

$$tr\chi = \mathbf{g}^{AB}\chi_{AB},\ \widehat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} tr\chi \mathbf{g}_{AB}.$$

Recall that $\mathrm{tr}\chi$ satisfies a transport equation called the Raychaudhuri equation:

$$L(\mathrm{tr}\chi) + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\widehat{\chi}|^2 + \cdots \tag{3.7}$$

(see precise equation in (4.19)).

We conclude this section with the identification of the symbol $\Box_{\mathbf{g}} u$ of the error term. We have (see for example [**44**] for a proof):

$$\Box_{\mathbf{g}} u = b^{-1}\mathrm{tr}\chi. \tag{3.8}$$

Thus, we may rewrite the error term E as:

$$Ef(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)} b^{-1}(t,x,\omega)\mathrm{tr}\chi(t,x,\omega) f(\lambda\omega)\lambda^3 d\lambda d\omega. \tag{3.9}$$

3.1.2. Some norms. We define some norms on $\mathcal{H}$. For any $1 \leq p \leq +\infty$ and for any tensor F on $\mathcal{H}_u$, we have:

$$\|F\|_{L^p(\mathcal{H}_u)} = \left(\int_0^1 dt \int_{P_{t,u}} |F|^p d\mu_{t,u}\right)^{\frac{1}{p}},$$

where $d\mu_{t,u}$ denotes the area element of $P_{t,u}$.

Let x' a coordinate system on $P_{0,u}$. By transporting this coordinate system along the null geodesics generated by L, we obtain a coordinate system (t,x') of $\mathcal{H}$. We define the following norms:

$$\|F\|_{L^\infty_{x'}L^2_t} = \sup_{x'\in P_{0,u}}\left(\int_0^1 |F(t,x')|^2dt\right)^{\frac{1}{2}},$$
$$\|F\|_{L^2_{x'},L^\infty_t} = \left\|\sup_{0\leq t\leq 1}|F(t,x'))|\right\|_{L^2(P_{0,u})}.$$

3.1.3. Estimates for the space-time foliation. In this section, we collect the estimates that are needed to follow the discussion of the control of the error term contained in this chapter. An outline of the proof of these estimates will be given in Chapter 4 (see [**44**] for the complete proof).

We start with the regularity in (t,x) of the lapse b and the second fundamental for χ. We need:

$$\|\mathrm{tr}\chi\|_{L^\infty(\mathcal{M})}+\|\nabla\mathrm{tr}\chi\|_{L^\infty_{x'}L^2_t}+\|b-1\|_{L^\infty(\mathcal{M})}+\|\nabla b\|_{L^\infty_u L^2(\mathcal{H}_u)}+\|\widehat{\chi}\|_{L^\infty_{x'}L^2_t}+\|\nabla\!\!\!/\widehat{\chi}\|_{L^\infty_u L^2(\mathcal{H}_u)}\lesssim\varepsilon. \tag{3.10}$$

Remark 3.4. *In this section, all estimates hold for any $\omega\in\mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.*

We also need an estimate for two derivatives of $\mathrm{tr}\chi$ with respect to ∇_N

$$\|\nabla_N P_j(\nabla_N\mathrm{tr}\chi)\|_{L^2(\mathcal{H}_u)}\lesssim\varepsilon 2^j+2^{\frac{j}{2}}\mu(u), \tag{3.11}$$

where μ in a function satisfying:

$$\|\mu\|_{L^2(\mathbb{R})}\lesssim\varepsilon.$$

Next, we consider the regularity with respect to ω. We have:

$$\|\partial_\omega b\|_{L^\infty(\mathcal{M})}\lesssim\varepsilon, \tag{3.12}$$

$$|N(t,x,\omega)-N(t,x,\omega')|\simeq|\omega-\omega'|,\ \forall(t,x)\in\mathcal{M},\omega,\omega'\in\mathbb{S}^2, \tag{3.13}$$

and

$$\|\partial_\omega N\|_{L^\infty(\mathcal{M})}\lesssim 1. \tag{3.14}$$

Furthermore, we have the following decomposition for $\widehat{\chi}$:

$$\widehat{\chi}=\chi_1+\chi_2, \tag{3.15}$$

where χ_1 and χ_2 are two symmetric traceless $P_{t,u}$-tangent 2-tensors satisfying in particular, for any $2\leq p<+\infty$:

$$\|\chi_1\|_{L^p_t L^\infty_{x'}}+\|\partial_\omega\chi_2\|_{L^{6_-}(\mathcal{H}_u)}\lesssim\varepsilon. \tag{3.16}$$

Remark 3.5. *The point of decomposition* (3.15) *is that χ_1 has a better regularity with respect to (t,x) than $\widehat{\chi}$, while χ_2 has a better regularity with respect to ω than $\widehat{\chi}$ (see explanation in section 4.4.1).*

Finally, we need to compare quantities evaluated at two angles ω and ν in $\mathbb{S}^2$ satisfying $|\omega-\nu|\lesssim 2^{-\frac{j}{2}}$. We have the following decomposition for $N(t,x,\omega)-N(t,x,\nu)$:

$$2^{\frac{j}{2}}(N(t,x,\omega)-N(t,x,\nu))=F_1^j(t,x,\nu)+F_2^j(t,x,\omega,\nu) \tag{3.17}$$

where the tensor F_1^j does not depend on ω and satisfies:

$$\|F_1^j\|_{L^\infty}\lesssim 1,$$

and the tensor F_2^j satisfies:

$$\|F_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)}\lesssim 2^{-\frac{j}{2}}.$$

Here $L_u^\infty L^2(\mathcal{H}_u)$ is defined with respect to $u=u(t,x,\omega)$. We also have the following decomposition for $\mathrm{tr}\chi$:

$$\mathrm{tr}\chi(t,x,\omega)=f_1^j(t,x,\nu)+f_2^j(t,x,\omega,\nu) \tag{3.18}$$

where the scalar f_1^j does not depend on ω and satisfies:

$$\|f_1^j\|_{L^\infty}\lesssim\varepsilon,$$

and where the scalar f_2^j satisfies:

$$\|f_2^j\|_{L_u^\infty L^2(\mathcal{H}_u)}\lesssim \varepsilon 2^{-\frac{j}{2}}.$$

3.1.4. Main result. The main result of this chapter is the following.

THEOREM 3.6. *Let u be a function on $\mathcal{M}\times\mathbb{S}^2$ satisfying suitable assumptions (we refer to* [**45**] *for the complete set of assumptions, and to section 3.1.3 for some typical assumptions). Let E the Fourier integral operator with phase $u(t,x,\omega)$ and symbol $\square_{\mathbf{g}}u$:*

$$Ef(t,x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u(t,x,\omega)}b^{-1}(t,x,\omega)tr\chi(t,x,\omega)f(\lambda\omega)\lambda^3d\lambda d\omega. \tag{3.19}$$

Then, E satisfies the estimate:

$$\|Ef\|_{L^2(\mathcal{M})}\lesssim\varepsilon\|\lambda f\|_{L^2(\mathbb{R}^3)}. \tag{3.20}$$

3.1.5. Geometric Littlewood-Paley projections on the 2-surfaces $P_{t,u}$. Throughout the paper, we will use the geometric Littlewood-Paley projections on 2-surfaces ($P_{t,u}$ in our case) constructed in [**24**]. In that paper, the following properties are proved

THEOREM 3.7. *The LP-projections P_j verify the following properties:*
i) L^p-boundedness For any $1\le p\le\infty$, and any interval $I\subset\mathbb{Z}$,

$$\|P_IF\|_{L^p(P_{t,u})}\lesssim\|F\|_{L^p(P_{t,u})} \tag{3.21}$$

ii) Bessel inequality

$$\sum_j\|P_jF\|^2_{L^2(P_{t,u})}\lesssim\|F\|^2_{L^2(P_{t,u})}$$

iii) Finite band property For any $1 \leq p \leq \infty$.

$$\begin{array}{lll}\|\not{\Delta} P_j F\|_{L^p(P_{t,u})} & \lesssim & 2^{2j}\|F\|_{L^p(P_{t,u})}\\ \|P_j F\|_{L^p(P_{t,u})} & \lesssim & 2^{-2j}\|\not{\Delta} F\|_{L^p(P_{t,u})}.\end{array} \tag{3.22}$$

In addition, the L^2 *estimates*

$$\begin{array}{lll}\|\nabla\!\!\!/ P_j F\|_{L^2(P_{t,u})} & \lesssim & 2^{j}\|F\|_{L^2(P_{t,u})}\\ \|P_j F\|_{L^2(P_{t,u})} & \lesssim & 2^{-j}\|\nabla\!\!\!/ F\|_{L^2(P_{t,u})}\end{array} \tag{3.23}$$

hold together with the dual estimate

$$\|P_j \nabla\!\!\!/ F\|_{L^2(P_{t,u})} \lesssim 2^j\|F\|_{L^2(P_{t,u})}$$

iv) Weak Bernstein inequality For any $2 \leq p < \infty$

$$\|P_j F\|_{L^p(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j}+1)\|F\|_{L^2(P_{t,u})},$$
$$\|P_{<0} F\|_{L^p(P_{t,u})} \lesssim \|F\|_{L^2(P_{t,u})}$$

together with the dual estimates

$$\|P_j F\|_{L^2(P_{t,u})} \lesssim (2^{(1-\frac{2}{p})j}+1)\|F\|_{L^{p'}(P_{t,u})},$$
$$\|P_{<0} F\|_{L^2(P_{t,u})} \lesssim \|F\|_{L^{p'}(P_{t,u})}$$

3.2. Control of the error term

3.2.1. The basic computation. We start the proof of Theorem 3.6 with the following instructive computation:

$$\begin{aligned}\|Ef\|_{L^2(\mathcal{M})} \leq & \int_{\mathbb{S}^2}\left\|b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)\int_0^{+\infty}e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2(\mathcal{M})}d\omega\\ & \leq \int_{\mathbb{S}^2}\|b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)\|_{L^\infty_u L^2(\mathcal{H}_u)}\left\|\int_0^{+\infty}e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2_u}d\omega\\ & \leq \varepsilon\|\lambda^2 f\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{3.24}$$

where we have used Plancherel with respect to λ, Cauchy-Schwarz with respect to ω, the estimates (3.10) for b and $\mathrm{tr}\chi$. (3.24) misses the conclusion (3.20) of Theorem 3.6 by a power of λ. Now, assume for a moment that we may replace a power of λ by a derivative on $b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)$. Then, the same computation yields:

$$\begin{aligned}&\left\|\int_{\mathbb{S}^2}\int_0^{+\infty}\nabla(b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega))e^{i\lambda u}f(\lambda\omega)\lambda d\lambda d\omega\right\|_{L^2(\mathcal{M})}\\ \leq\ & \int_{\mathbb{S}^2}\|\nabla(b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega))\|_{L^\infty_u L^2(\mathcal{H}_u)}\left\|\int_0^{+\infty}e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2_u}d\omega\\ \leq\ & \varepsilon\|\lambda f\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{3.25}$$

where we used the fact that

$$\|\nabla(b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega))\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon \tag{3.26}$$

in view of (3.10). Note that the estimate provided by (3.25) is consistent with the control of the error term (3.20). This suggests a strategy which consists in making integrations by parts to trade powers of λ against derivatives of the symbol $b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)$.

3.2.2. Structure of the proof of Theorem 3.6. The proof of Theorem 3.6 proceeds in three steps. We first localize in frequencies of size $\lambda \sim 2^j$. We then localize the angle ω in patches on the sphere $\mathbb{S}^2$ of diameter $2^{-j/2}$. Finally, we estimate the diagonal terms.

3.2.2.1. *Step 1: decomposition in frequency.* For the first step, we introduce φ and ψ two smooth compactly supported functions on $\mathbb{R}$ such that:

$$\varphi(\lambda) + \sum_{j\geq 0} \psi(2^{-j}\lambda) = 1 \text{ for all } \lambda \in \mathbb{R}. \tag{3.27}$$

We use (3.27) to decompose Ef as follows:

$$Ef(t,x) = \sum_{j\geq -1} E_j f(t,x), \tag{3.28}$$

where for $j \geq 0$:

$$E_j f(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u} b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)\psi(2^{-j}\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{3.29}$$

and

$$E_{-1} f(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u} b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)\varphi(\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{3.30}$$

This decomposition is classical and is known as the first dyadic decomposition (see [**39**]). The goal of this first step is to prove the following proposition:

PROPOSITION 3.8. *The decomposition* (3.28) *satisfies an almost orthogonality property, from which it follows that:*

$$\|Ef\|^2_{L^2(\mathcal{M})} \lesssim \sum_{j\geq -1} \|E_j f\|^2_{L^2(\mathcal{M})} + \varepsilon^2 \|f\|^2_{L^2(\mathbb{R}^3)}. \tag{3.31}$$

A discussion of the proof of Proposition 3.8 is postponed to section 3.3.

3.2.2.2. *Step 2: decomposition in angle.* Proposition 3.8 enables us to estimate $\|E_j f\|_{L^2(\mathcal{M})}$ instead of $\|Ef\|_{L^2(\mathcal{M})}$. The analog of computation (3.24) for $\|E_j f\|_{L^2(\mathcal{M})}$ yields:

$$\|E_j f\|_{L^2(\mathcal{M})} \leq \varepsilon \|\lambda\psi(2^{-j}\lambda) f\|_{L^2(\sigma)} \lesssim \varepsilon 2^j \|\psi(2^{-j}\lambda) f\|_{L^2(\mathbb{R}^3)}, \tag{3.32}$$

which misses the wanted estimate by a power of 2^j. We thus need to perform a second dyadic decomposition (see [**39**]). We introduce a smooth partition of unity on the sphere $\mathbb{S}^2$:

$$\sum_{\nu\in\Gamma} \eta_j^\nu(\omega) = 1 \text{ for all } \omega \in \mathbb{S}^2, \tag{3.33}$$

where Γ is a lattice on $\mathbb{S}^2$ of size $2^{-\frac{j}{2}}$, where the support of η_j^ν is a patch on $\mathbb{S}^2$ of diameter $\sim 2^{-j/2}$. We use (3.33) to decompose E_jf as follows:

$$E_jf(t,x) = \sum_{\nu\in\Gamma} E_j^\nu f(t,x), \tag{3.34}$$

where:

$$E_j^\nu f(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u} b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{3.35}$$

We also define:

$$\begin{array}{l}\gamma_{-1} = \|\varphi(\lambda)f\|_{L^2(\mathbb{R}^3)},\ \gamma_j = \|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\\ \gamma_j^\nu = \|\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\ \nu\in\Gamma,\end{array} \tag{3.36}$$

which satisfy:

$$\|f\|^2_{L^2(\mathbb{R}^3)} = \sum_{j\geq -1}\gamma_j^2 = \sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2. \tag{3.37}$$

The goal of this second step is to prove the following proposition:

Proposition 3.9. *The decomposition* (3.34) *satisfies an almost orthogonality property, from which it follows that*

$$\|E_jf\|^2_{L^2(\mathcal{M})} \lesssim \sum_{\nu\in\Gamma}\|E_j^\nu f\|^2_{L^2(\mathcal{M})} + \varepsilon^2\gamma_j^2. \tag{3.38}$$

A discussion of the proof of Proposition 3.9 is postponed to section 3.5.

3.2.2.3. *Step 3: control of the diagonal term.* Proposition 3.9 allows us to estimate $\|E_j^\nu f\|_{L^2(\mathcal{M})}$ instead of $\|E_jf\|_{L^2(\mathcal{M})}$. The analog of computation (3.24) for $\|E_j^\nu f\|_{L^2(\mathcal{M})}$ yields:

$$\begin{aligned} & \|E_j^\nu f\|_{L^2(\mathcal{M})} \\ \leq\ & \int_{\mathbb{S}^2}\|b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)\|_{L^\infty_u L^2(\mathcal{H}_u)}\left\|\int_0^{+\infty} e^{i\lambda u}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2_u} d\omega \\ \leq\ & 2^j\varepsilon\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))}\|\lambda\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f\|_{L^2(\mathbb{R}^3)} \\ \lesssim\ & \varepsilon 2^{j/2}\gamma_j^\nu, \end{aligned} \tag{3.39}$$

where the term $\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))}$ comes from the fact that we apply Cauchy-Schwarz in ω. Note that we have used in (3.39) the fact that the support of η_j^ν is 2 dimensional and has diameter $2^{-j/2}$ so that:

$$\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))} \lesssim 2^{-j/2}. \tag{3.40}$$

Now, (3.39) still misses the wanted estimate by a power of $2^{j/2}$. Nevertheless, using more refined techniques, we are able to estimate the diagonal term:

PROPOSITION 3.10. *The diagonal term $E_j^\nu f$ satisfies the following estimate:*

$$\|E_j^\nu f\|_{L^2(\mathcal{M})} \lesssim \varepsilon\gamma_j^\nu. \tag{3.41}$$

A discussion of the proof of Proposition 3.10 is postponed to section 3.4.

REMARK 3.11. *Note that Proposition 3.9 together with Proposition 3.10 yields the estimate:*

$$\|E_j f\|_{L^2(\mathcal{M})} \lesssim \varepsilon\gamma_j. \tag{3.42}$$

Now, since the proof of Proposition 3.9 and the proof of Proposition 3.10 do not depend on the proof of Proposition 3.8, we are allowed to use the conclusion of Proposition 3.9 and Proposition 3.10 in the proof of Proposition 3.8. In particular, the estimate (3.42) *will be used for the proof of Proposition 3.8. In the same spirit, since the proof of Proposition 3.10 does not depend on the proof of Proposition 3.9, we are allowed to use the conclusion of Proposition 3.10 in the proof of Proposition 3.9.*

Convention. In the rest of this chapter, we will use several integration by parts. In turn, these integration by parts will each generate a large number of terms. For the sake of simplicity, we will only discuss few typical terms. We will constantly use the notation "$+\cdots$" in various identities and estimates in order to refer to the additional terms. That is not to say that these additional terms are lower order or estimated in the same way, but simply that the typical terms that we exhibit allow for a simple exposition of the main ideas of the proof. We refer the reader to [**45**] for a complete proof which contains the control of the typical terms discussed here as well as the numerous additional terms.

3.2.2.4. *Proof of Theorem 3.6.* Proposition 3.8, 3.9 and 3.10 immediately yield the proof of Theorem 3.6. Indeed, (3.31), (3.37), (3.38) and (3.41) imply:

$$\begin{aligned}
\|Ef\|^2_{L^2(\mathcal{M})} &\lesssim \sum_{j\geq -1}\|E_jf\|^2_{L^2(\mathcal{M})} + \varepsilon^2\|f\|^2_{L^2(\mathbb{R}^3)} \\
&\lesssim \sum_{j\geq -1}\sum_{\nu\in\Gamma}\|E_j^\nu f\|^2_{L^2(\mathcal{M})} + \varepsilon^2\sum_{j\geq -1}\gamma_j^2 + \varepsilon^2\|f\|^2_{L^2(\mathbb{R}^3)} \\
&\lesssim \varepsilon^2\sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2 + \varepsilon^2\sum_{j\geq -1}\gamma_j^2 + \varepsilon^2\|f\|^2_{L^2(\mathbb{R}^3)} \\
&\lesssim \varepsilon^2\|f\|^2_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{3.43}$$

which is the conclusion of Theorem 3.6.

The rest of this chapter is dedicated to a discussion of the proof of Propositions 3.8, 3.9 and 3.10. The details of the proofs being very involved, we only give a very sketchy summary of the main ideas. We refer the reader to [**45**] for the details.

3.3. Almost orthogonality in frequency

We have to prove (3.31):

$$\|Ef\|^2_{L^2(\mathcal{M})} \lesssim \sum_{j\geq -1} \|E_j f\|^2_{L^2(\mathcal{M})} + \varepsilon^2 \|f\|^2_{L^2(\mathbb{R}^3)}. \tag{3.44}$$

This will result from the following inequality using Shur's Lemma:

$$\left|\int_{\mathcal{M}} E_j f(t,x)\overline{E_k f(t,x)} d\mathcal{M}\right| \lesssim \varepsilon^2 2^{-\frac{|j-k|}{4}} \gamma_j \gamma_k \text{ for } |j-k| > 2. \tag{3.45}$$

In turn, (3.45) will follow from integrations by parts in u.

3.3.1. A first integration by parts. From now on, we focus on proving (3.45). We may assume $j \geq k+3$. We have:

$$\begin{aligned} &\int_{\mathcal{M}} E_j f(t,x)\overline{E_k f(t,x)} d\mathcal{M} \quad (3.46)\\ = &\int_{\mathbb{S}^2}\int_0^{+\infty}\int_{\mathbb{S}^2}\int_0^{+\infty} \left(\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b(t,x,\omega)^{-1}\mathrm{tr}\chi(t,x,\omega)\overline{b(x,\omega')^{-1}\mathrm{tr}\chi(t,x,\omega')} d\mathcal{M}\right)\\ &\times \psi(2^{-j}\lambda) f(\lambda\omega)\lambda^2 \psi(2^{-k}\lambda')\overline{f(\lambda'\omega')}(\lambda')^2 d\lambda d\omega d\lambda' d\omega'. \end{aligned}$$

We consider the coordinate system (t,u,x') on $\mathcal{M}$, and we would like to integrate by parts with respect to ∂_u in (3.46). Since $\nabla u = b^{-1}N$ and $\nabla u' = b'^{-1}N'$, we have:

$$e^{i\lambda u - i\lambda' u'} = -\frac{i}{\lambda - \lambda' \frac{b}{b'} g(N,N')} \partial_u (e^{i\lambda u - i\lambda' u'}), \tag{3.47}$$

where we use the notation u for $u(t,x,\omega)$, b for $b(t,x,\omega)$, N for $N(t,x,\omega)$, u' for $u(t,x,\omega')$, b' for $b(t,x,\omega')$ and N' for $N(x,\omega')$. We will also use the notation $\mathrm{tr}\chi$ for $\mathrm{tr}\chi(t,x,\omega)$ and $\mathrm{tr}\chi'$ for $\mathrm{tr}\chi(t,x,\omega')$. Using (3.47), we obtain:

$$\begin{aligned} \int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} b\overline{b'} d\mathcal{M} = \; & i\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1}\partial_u \mathrm{tr}\chi \overline{b'^{-1}\mathrm{tr}\chi'}}{\lambda - \lambda' \frac{b}{b'} g(N,N')} d\mathcal{M} \\ & + i\int_{\mathcal{M}} e^{i\lambda u - i\lambda' u'} \frac{b^{-1}\mathrm{tr}\chi \partial_u(\overline{b'^{-1}\mathrm{tr}\chi'})}{\lambda - \lambda' \frac{b}{b'} g(N,N')} d\mathcal{M} + \cdots, \end{aligned} \tag{3.48}$$

where the additional terms in (3.48) arise when ∂_u falls on the volume element of $\mathcal{M}$ or on the denominator in the right-hand side of (3.47). Note that:

$$\left|\frac{\lambda'}{\lambda}\frac{b}{b'} g(N,N')\right| \leq \frac{\lambda'}{\lambda}\left|\frac{b}{b'}\right| \leq \frac{1}{2} + O(\varepsilon) < 1,$$

where we used the estimate (3.10) satisfied by b and b' and the fact that $j\geq k+3$ so that $\lambda'\leq\lambda/2$. Thus, we may expand the fraction in (3.48):

$$\frac{1}{\lambda-\lambda'\frac{b}{b'}g(N,N')}=\sum_{p\geq 0}\frac{1}{\lambda}\left(\frac{\lambda'\frac{b}{b'}g(N,N')}{\lambda}\right)^p. \tag{3.49}$$

REMARK 3.12. *The expansion* (3.49) *generates quantities of the type*

$$\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)f(\lambda\omega)(2^{-j}\lambda)^p\lambda^2d\lambda.$$

where $p\in\mathbb{Z}$. For simplicity, we omit the index p and denote them by

$$F_j(u)=\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2d\lambda. \tag{3.50}$$

since they are essentially equivalent. Note that Plancherel yields:

$$\|F_j\|_{L^2_{\omega,u}}\leq\|\psi(2^{-j}\lambda)f(\lambda\omega)\lambda\|_{L^2(\mathbb{R}^3)}\lesssim 2^j\gamma_j. \tag{3.51}$$

Also, using Cauchy-Schwarz in λ, we have

$$\|F_j\|_{L^2_\omega L^\infty_u}\leq 2^{\frac{j}{2}}\|\psi(2^{-j}\lambda)f(\lambda\omega)\lambda\|_{L^2(\mathbb{R}^3)}\lesssim 2^{\frac{3j}{2}}\gamma_j. \tag{3.52}$$

(3.46), (3.48) and (3.49) imply:

$$\begin{aligned}&\int_{\mathcal{M}}E_jf(t,x)\overline{E_kf(t,x)}d\mathcal{M} \qquad (3.53)\\ =\;& 2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}\nabla_N\mathrm{tr}\chi F_j(u)d\omega\right)\overline{\left(\int_{\mathbb{S}^2}b'^{-1}\mathrm{tr}\chi' F_k(u')d\omega'\right)}d\mathcal{M}\\ &+2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}\mathrm{tr}\chi NF_j(u)d\omega\right)\cdot\overline{\left(\int_{\mathbb{S}^2}\nabla(b'^{-1}\mathrm{tr}\chi')F_k(u')d\omega'\right)}d\mathcal{M}+\cdots,\end{aligned}$$

where we only kept the first term in the expansion (3.49) in order to simplify the exposition[1].

REMARK 3.13. *The second term in the right-hand side of* (3.53) *is easier because the derivative falls on the low frequency term. This is why we estimate this term directly while the other term requires a more elaborate treatment which is explained in section 3.3.2.*

We estimate the second term in the right-hand side of (3.53). We have:

$$\begin{aligned}&\left|2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}\mathrm{tr}\chi NF_j(u)d\omega\right)\cdot\overline{\left(\int_{\mathbb{S}^2}\nabla(b'^{-1}\mathrm{tr}\chi')F_k(u')d\omega'\right)}d\mathcal{M}\right| \qquad (3.54)\\ \lesssim\;& 2^{-j}\left\|\int_{\mathbb{S}^2}\mathrm{tr}\chi NF_j(u)d\omega\right\|_{L^2(\mathcal{M})}\left\|\int_{\mathbb{S}^2}\nabla(b'^{-1}\mathrm{tr}\chi')F_k(u')d\omega'\right\|_{L^2(\mathcal{M})}.\end{aligned}$$

[1]note that in the last term in the right-hand side of (3.53), we wrote $\nabla_N(b'^{-1}\mathrm{tr}\chi')$ as $N\cdot\nabla(b'^{-1}\mathrm{tr}\chi')$

We have the following analog of (3.42):

$$\left\|\int_{\mathbb{S}^2}\textrm{tr}\chi NF_j(u)d\omega\right\|_{L^2(\mathcal{M})}\lesssim\varepsilon\gamma_j. \tag{3.55}$$

Indeed, one can show that the symbol $\textrm{tr}\chi$ satisfies regularity assumptions which are at least as good as $b^{-1}\textrm{tr}\chi$ (see [**44**], and also section 6.1.5), so that the proof of (3.42) may adapted in a straightforward manner to obtain (3.55).

Next, we consider the second term in the right-hand side of (3.54). Then proceeding as in the basic computation (3.24), and using the estimate (3.26), we obtain

$$\left\|\int_{\mathbb{S}^2}\nabla(b'^{-1}\textrm{tr}\chi')F_k(u')d\omega'\right\|_{L^2(\mathcal{M})}\lesssim\varepsilon\|\lambda\psi(2^{-k}\lambda)f\|_{L^2(\mathbb{R}^3)}\lesssim\varepsilon 2^k\gamma_k. \tag{3.56}$$

Together with (3.54) and (3.55), we finally obtain:

$$\left|2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}\textrm{tr}\chi NF_j(u)d\omega\right)\cdot\overline{\left(\int_{\mathbb{S}^2}\nabla(b'^{-1}\textrm{tr}\chi')F_k(u')d\omega'\right)}d\mathcal{M}\right|\lesssim 2^{-j+k}\gamma_k\gamma_j, \tag{3.57}$$

which is consistent with (3.45).

REMARK 3.14. *Estimating the first term in the right-hand side of* (3.53) *in the same way would only yield:*

$$\left|2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}\nabla_N tr\chi F_j(u)d\omega\right)\overline{\left(\int_{\mathbb{S}^2}b'^{-1}tr\chi' F_k(u')d\omega'\right)}d\mathcal{M}\right|\lesssim\varepsilon^2\gamma_j\gamma_k, \tag{3.58}$$

which is not sufficient to obtain (3.45).

3.3.2. A more precise estimate. In this section, we estimate the first term the right-hand side of (3.53). Using the geometric Littlewood-Paley projections on the 2-surfaces $P_{t,u}$, we decompose $\nabla_N\textrm{tr}\chi$ as:

$$\nabla_N\textrm{tr}\chi=P_{\leq\frac{j+k}{2}}(\nabla_N\textrm{tr}\chi)+P_{>\frac{j+k}{2}}(\nabla_N\textrm{tr}\chi).$$

In turn, this yields a decomposition for the first term in the right-hand side of (3.53):

$$2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}\nabla_N\textrm{tr}\chi F_j(u)d\omega\right)\overline{\left(\int_{\mathbb{S}^2}b'^{-1}\textrm{tr}\chi' F_k(u')d\omega'\right)}d\mathcal{M}=A_1+A_2, \tag{3.59}$$

where:

$$\begin{aligned}A_1&=2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}P_{>\frac{j+k}{2}}(\nabla_N\textrm{tr}\chi)F_j(u)d\omega\right)\overline{E_kf(t,x)}d\mathcal{M},\\ A_2&=2^{-j}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}P_{\leq\frac{j+k}{2}}(\nabla_N\textrm{tr}\chi)F_j(u)d\omega\right)\overline{E_kf(t,x)}d\mathcal{M}.\end{aligned} \tag{3.60}$$

We first estimate the easier term A_1. The definition of P_l implies $P_l = 2^{-2l}\Delta\!\!\!/ P_l$, and thus

$$P_{>\frac{j+k}{2}}(\nabla_N \mathrm{tr}\chi) = \sum_{l>\frac{j+k}{2}} P_l(\nabla_N \mathrm{tr}\chi) = \sum_{l>\frac{j+k}{2}} 2^{-2l}\Delta\!\!\!/ P_l(\nabla_N \mathrm{tr}\chi),$$

which yields the following decomposition for A_1:

$$A_1 = \sum_{l>\frac{j+k}{2}} A_{1,l} \tag{3.61}$$

where $A_{1,l}$ is given by:

$$A_{1,l} = 2^{-j-2l}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2}\Delta\!\!\!/ P_l(\nabla_N \mathrm{tr}\chi)F_j(u)d\omega\right)\overline{E_k f(t,x)}d\mathcal{M}.$$

Integrating by parts $\Delta\!\!\!/$ on $P_{t,u}$ and using the fact that $\nabla\!\!\!/ F_{j,-1}(u) = 0$, we obtain:

$$A_{1,l} = -2^{-j-2l}\int_{\mathbb{S}^2}\int_{t,u}\left(\int_{P_{t,u}}\nabla\!\!\!/ P_l(\nabla_N \mathrm{tr}\chi)\nabla\!\!\!/(\overline{E_k f(t,x)}b)d\mu_{t,u}\right)F_j(u)du\,dt\,d\omega + \cdots,$$

where the additional term corresponds to the case where the derivative falls on the volume element of $\mathcal{M}$. Next, we apply Cauchy-Schwartz to the integral on $\mathcal{M}$ and obtain:

$$\begin{aligned}
|A_{1,l}| &\leq 2^{-j-2l}\int_{\mathbb{S}^2}\|\nabla\!\!\!/ P_l(\nabla_N \mathrm{tr}\chi)F_j(u)\|_{L^2(\mathcal{M})}\|\nabla\!\!\!/ E_k\|_{L^2(\mathcal{M})}d\omega \\
&\lesssim 2^{-j-2l}\int_{\mathbb{S}^2}\|\nabla\!\!\!/ P_l(\nabla_N \mathrm{tr}\chi)\|_{L^\infty_u L^2(\mathcal{H}_u)}\|F_j(u)\|_{L^2_u}\|\nabla\!\!\!/ E_k\|_{L^2(\mathcal{M})}d\omega \\
&\lesssim 2^{-j-l}\int_{\mathbb{S}^2}\|\nabla_N \mathrm{tr}\chi\|_{L^\infty_u L^2(\mathcal{H}_u)}\|F_j(u)\|_{L^2_u}\|\nabla\!\!\!/ E_k\|_{L^2(\mathcal{M})}d\omega \\
&\lesssim \varepsilon 2^{-j-l}\int_{\mathbb{S}^2}\|F_j(u)\|_{L^2_u}\|\nabla E_k\|_{L^2(\mathcal{M})}d\omega,
\end{aligned} \tag{3.62}$$

where we used the finite band property for P_l and the estimates (3.10) for $\mathrm{tr}\chi$. In view of (3.62), we also need to estimate $\|\nabla E_k\|_{L^2(\mathcal{M})}$. We have:

$$\begin{aligned}
\nabla E_k f(t,x) = &\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u}\nabla(b^{-1}\mathrm{tr}\chi)\psi(2^{-k}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega \\
&+i2^k\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u}b^{-1}\mathrm{tr}\chi\nabla u\psi(2^{-k}\lambda)(2^{-k}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega.
\end{aligned} \tag{3.63}$$

Using the basic computation (3.24) for the first term together with the estimate (3.26), and (3.42) for the second term together with the fact that $\mathrm{tr}\chi L$ satisfies the same regularity assumptions than $b^{-1}\mathrm{tr}\chi$, we obtain:

$$\|\nabla E_k\|_{L^2(\mathcal{M})} \lesssim \varepsilon 2^k \gamma_k. \tag{3.64}$$

(3.62), (3.51), and (3.64) yield:

$$|A_{1,l}| \lesssim \varepsilon^2 2^{-l+k} \varepsilon^2 \gamma_j \gamma_k.$$

Together with (3.61), this yields:

$$|A_1| \lesssim \left(\sum_{l > \frac{j+k}{2}} 2^{-l} \right) \varepsilon 2^k \varepsilon^2 \gamma_j \gamma_k \lesssim \varepsilon^2 2^{-\frac{j-k}{2}} \gamma_j \gamma_k, \tag{3.65}$$

which is consistent with (3.45).

3.3.3. A second integration by parts in u. To estimate A_2, we perform a second integration by parts relying again on (3.47). This leads to:

$$A_2 \;=\; 2^{-2j} \int_{\mathcal{M}} \int_{\mathbb{S}^2} \nabla_N P_{\leq \frac{j+k}{2}} (\nabla_N \mathrm{tr}\chi) F_j(u) \overline{E_k f(t,x)} d\mathcal{M} + \cdots , \tag{3.66}$$

where we only keep the worst term, which is the one containing two derivatives of $\mathrm{tr}\chi$. It is at this stage that we need the estimate (3.11) for $\nabla_N P_l(\nabla_N \mathrm{tr}\chi)$ which we recall now. We have:

$$\|\nabla_N P_l(\nabla_N \mathrm{tr}\chi)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^l + 2^{\frac{l}{2}} \mu(u), \tag{3.67}$$

where μ in a function satisfying:

$$\|\mu\|_{L^2(\mathbb{R})} \lesssim \varepsilon.$$

In view of the estimate (3.67), we have:

$$\begin{aligned}
\left\| \nabla_N P_{\leq \frac{j+k}{2}} (\nabla_N \mathrm{tr}\chi) \right\|_{L^2(\mathcal{H}_u)} &\lesssim \sum_{l \leq \frac{j+k}{2}} \|\nabla_N P_l(\nabla_N \mathrm{tr}\chi)\|_{L^2(\mathcal{H}_u)} \\
&\lesssim \sum_{l \leq \frac{j+k}{2}} (2^l \varepsilon + 2^{\frac{l}{2}} \mu(u)) \\
&\lesssim \varepsilon 2^{\frac{j+k}{2}} + 2^{\frac{j+k}{4}} \mu(u).
\end{aligned}$$

In view of (3.66), this yields after applying Cauchy-Schwartz:

$$\begin{aligned}
|A_2| &\lesssim 2^{-2j} \|E_k f\|_{L^2(\mathcal{M})} \int_{\mathbb{S}^2} \left\| \left\| \nabla_N P_{\leq \frac{j+k}{2}} (\nabla_N \mathrm{tr}\chi) \right\|_{L^2(\mathcal{H}_u)} F_j(u) \right\|_{L^2_u} d\omega + \cdots \qquad (3.68) \\
&\lesssim 2^{-2j} \varepsilon \gamma_k \int_{\mathbb{S}^2} \left\| (\varepsilon 2^{\frac{j+k}{2}} + \varepsilon 2^{\frac{j+k}{4}} \mu(u)) F_j(u) \right\|_{L^2_u} d\omega + \cdots \\
&\lesssim 2^{-2j} \varepsilon \gamma_k \left(\varepsilon 2^{\frac{j+k}{2}} \int_{\mathbb{S}^2} \|F_j(u)\|_{L^2_u} d\omega + 2^{\frac{j+k}{4}} \int_{\mathbb{S}^2} \|\mu\|_{L^2(\mathbb{R})} \|F_j(u)\|_{L^\infty_u} d\omega \right) + \cdots \\
&\lesssim 2^{-\frac{j-k}{4}} \varepsilon^2 \gamma_k \gamma_j + \cdots ,
\end{aligned}$$

where we used (3.42) for E_kf, Cauchy-Schwarz in ω, and the estimates (3.51) and (3.52) for $F_j(u)$.

3.3.4. End of the proof of Proposition 3.8. In view of (3.53), (3.57), (3.59), (3.65) and (3.68), we obtain:

$$\left|\int_{\mathcal{M}} E_jf(t,x)\overline{E_kf(t,x)}d\mathcal{M}\right| \lesssim \varepsilon^2 2^{-\frac{|j-k|}{4}}\gamma_j\gamma_k \text{ for } |j-k|>2. \tag{3.69}$$

Finally, (3.69) together with Shur's Lemma yields:

$$\|Ef\|^2_{L^2(\mathcal{M})} \lesssim \sum_{j\geq -1}\|E_jf\|^2_{L^2(\mathcal{M})}+\varepsilon^2\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{3.70}$$

This concludes the proof of Proposition 3.8.

3.4. Control of the diagonal term

Since the orthogonality argument in angle is the core of this chapter, we choose to deal first with the control of the diagonal term in this section. We will then proceed with the orthogonality argument in angle in the rest of the chapter.

In order to control the diagonal term, we have to prove (3.41):

$$\|E^\nu_jf\|_{L^2(\mathcal{M})} \lesssim \varepsilon\gamma^\nu_j. \tag{3.71}$$

Recall that E^ν_j is given by:

$$E^\nu_jf(t,x) = \int_{\mathbb{S}^2} b^{-1}(t,x,\omega)\mathrm{tr}\chi(t,x,\omega)F_j(u)\eta^\nu_j(\omega)d\omega, \tag{3.72}$$

where $F_j(u)$ is defined by:

$$F_j(u) = \int_0^{+\infty} e^{i\lambda u}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2d\lambda. \tag{3.73}$$

The proof of the estimate (3.71) will proceed in four steps:

Step 1. We first consider a decomposition roughly of the type:

$$E^\nu_jf(t,x) = b^{-1}(t,x,\nu)\mathrm{tr}\chi(t,x,\nu)\left(\int_{\mathbb{S}^2}F_j(u)\eta^\nu_j(\omega)d\omega\right)+\cdots,$$

so that we have to prove estimate (3.71) with $b^{-1}\mathrm{tr}\chi$ replaced by 1.

Step 2. That estimate is obtained by considering the transport equation along L_ν:

$$L_\nu\left(\int_{\mathbb{S}^2}F_j(u)\eta^\nu_j(\omega)d\omega\right) = \cdots.$$

Step 3. A certain term in the transport equation of Step 2 needs to be estimated using an energy estimate for the wave equation.

Step 4. We conclude the proof using the estimates obtained in Step 2 and Step 3.

3.4.1. Step 1: freezing the ω dependance in b^{-1}trχ. In view of the estimate (3.12) for $\partial_\omega b$, the estimate (3.10) for b, and the decomposition (3.18) for trχ, we have:

$$b^{-1}(t,x,\omega)\text{tr}\chi(t,x,\omega) = f_1^j(t,x,\nu) + f_2^j(t,x,\nu,\omega), \tag{3.74}$$

where f_1^j only depends on (t,x,ν) and satisfies:

$$\|f_1^j\|_{L^\infty} \lesssim \varepsilon, \tag{3.75}$$

and where f_2^j satisfies:

$$\|f_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon, \tag{3.76}$$

with $u = u(.,\omega)$. (3.74) yields the following decomposition for the diagonal term:

$$E_j^\nu f(t,x) = f_1^j(t,x,\nu)\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega + \int_{\mathbb{S}^2} F_j(u)f_2^j(t,x,\omega,\nu)\eta_j^\nu(\omega)d\omega,$$

which implies:

$$\begin{aligned}
& \left\|E_j^\nu f(t,x)\right\|_{L^2(\mathcal{M})} \qquad (3.77)\\
\lesssim\ & \|f_1^j\|_{L^\infty(\mathcal{M})}\left\|\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} + \int_{\mathbb{S}^2}\|F_j(u)\|_{L^2_u}\|f_2^j\|_{L^\infty_u L^2(\mathcal{H}_u)}\eta_j^\nu(\omega)d\omega\\
\lesssim\ & \varepsilon\left\|\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} + \varepsilon\gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality the estimates (3.75) and (3.76), Cauchy-Schwarz in ω, the size of the patch, and the estimate (3.51) for $F_j(u)$.

Remark 3.15. *The point of the decomposition* (3.74) *is to allow us to replace in the diagonal term* (3.72) *the symbol $b^{-1}tr\chi$ with 1. An obvious way to achieve this is to write the following decomposition:*

$$b^{-1}tr\chi(t,x,\omega) = b^{-1}tr\chi(t,x,\nu) + (b^{-1}tr\chi(t,x,\omega) - b^{-1}tr\chi(t,x,\nu)). \tag{3.78}$$

The first term clearly satisfies (3.75) *in view of the estimate* (3.10) *for b and $tr\chi$. On the other hand, we obtain in* [**44**] *(see also* (4.49)*) the estimate $\partial_\omega tr\chi \in L^\infty_u L^2(\mathcal{H}_u)$ which together with the estimate* (3.12) *for $\partial_\omega b$ yields:*

$$\|\partial_\omega(b^{-1}tr\chi)\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{3.79}$$

Now, we have:

$$b^{-1}tr\chi(t,x,\omega) - b^{-1}tr\chi(t,x,\nu) = (\omega-\nu)\int_0^1 \partial_\omega(b^{-1}tr\chi)(t,x,\omega_\sigma)d\sigma$$

which together with (3.79) *is not enough to conclude since $L^\infty_u L^2(\mathcal{H}_u)$ and $L^\infty_{u_\sigma} L^2(\mathcal{H}_{u_\sigma})$ are not comparable. We refer the reader to* [**44**] *where the decomposition* (3.74) *as well as several others are proved (see also the discussion in section 4.5).*

The following proposition allows us to estimate the right-hand side of (3.77).

PROPOSITION 3.16. *We have the following bound:*

$$\left\|\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2_{u_\nu,x_\nu}L^\infty_t} \lesssim \gamma_j^\nu. \tag{3.80}$$

REMARK 3.17. *In order to control the diagonal term, it suffices to have a bound of the $L^2(\mathcal{M})$ norm for the left-hand side of* (3.80). *The improvement to a bound for the $L^2_{u_\nu,x_\nu}L^\infty_t$ norm will be crucial when proving the almost orthogonality in angle.*

Assuming the result of the proposition, estimates (3.80) and (3.77) yield:

$$\left\|E_j^\nu f(t,x)\right\|_{L^2(\mathcal{M})} \lesssim \varepsilon\gamma_j^\nu,$$

which together with (3.72) and (3.77) implies:

$$\|E_j^\nu f\|_{L^2(\mathcal{M})} \lesssim \varepsilon\gamma_j^\nu.$$

which is the wanted estimate (3.71). This concludes the proof of Proposition 3.10.

3.4.2. Step 2: A transport equation in the L_ν direction. We still need to prove Proposition 3.16. Note that it suffices to show:

$$\left\|L_\nu\left(\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right)\right\|_{L^2(\mathcal{M})} \lesssim \gamma_j^\nu. \tag{3.81}$$

Now, since the space-time gradient of u is given by $b^{-1}L$, we have:

$$L_\nu\left(\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right) = 2^j\int_{\mathbb{S}^2} b^{-1}\mathbf{g}\Big(L(t,x,\omega),L(t,x,\nu)\Big)F_j(u)\eta_j^\nu(\omega)d\omega,$$

where F_j has been defined in (3.50). In view of (3.82), we have:

$$\begin{aligned} & L_\nu\left(\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right) \\ = \; & 2^j b^{-1}(t,x,\nu)\int_{\mathbb{S}^2}\mathbf{g}(L(t,x,\omega),L(t,x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega \\ & +2^j\int_{\mathbb{S}^2}(b^{-1}(t,x,\omega)-b^{-1}(t,x,\nu))\mathbf{g}(L(t,x,\omega),L(t,x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega. \end{aligned} \tag{3.82}$$

Next, we estimate the second term in the right-hand side of (3.82). We have:

$$\mathbf{g}\Big(L(t,x,\omega),L(t,x,\nu)\Big) = \mathbf{g}(N(t,x,\omega)-N(t,x,\nu),N(t,x,\omega)-N(t,x,\nu)). \tag{3.83}$$

Thus, the estimate (3.14) for $\partial_\omega N$ and the size of the patch yields:

$$\|\mathbf{g}(L(t,x,\omega),L(t,x,\nu))\|_{L^\infty(\mathcal{H}_u)} \lesssim 2^{-j}, \tag{3.84}$$

which implies:

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^2}(b^{-1}(t,x,\omega)-b^{-1}(t,x,\nu))\mathbf{g}(L(t,x,\omega),L(t,x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} \quad (3.85)\\
\lesssim\ & \int_{\mathbb{S}^2}\|b^{-1}(t,x,\omega)-b^{-1}(t,x,\nu)\|_{L^\infty_u L^2(\mathcal{H}_u)}\|\mathbf{g}(L(t,x,\omega),L(t,x,\nu))\|_{L^\infty}\|F_j(u)\|_{L^2_u}\eta_j^\nu(\omega)d\omega\\
\lesssim\ & 2^{-j}\varepsilon\gamma_j^\nu,
\end{aligned}$$

where we used in the last inequality (3.84), the estimate (3.12) for $\partial_\omega b$, Cauchy-Schwarz in ω, the size of the patch, and (3.51) for $F_j(u)$. (3.82) together with (3.85) and the estimate (3.12) for $\partial_\omega b$ yields:

$$\begin{aligned}
&\left\|L_\nu\left(\int_{\mathbb{S}^2}F_j(u)\eta_j^\nu(\omega)d\omega\right)\right\|_{L^2(\mathcal{M})} \quad (3.86)\\
\lesssim\ & 2^j\left\|\int_{\mathbb{S}^2}\mathbf{g}(L(t,x,\omega),L(t,x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}+\varepsilon\gamma_j^\nu.
\end{aligned}$$

Next, we estimate the right-hand side of (3.86). Using (3.83), the decomposition (3.17) for $N-N'$, and arguing as in (3.77), we obtain:

$$\left\|\int_{\mathbb{S}^2}\mathbf{g}(L(t,x,\omega),L(t,x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}\lesssim\left\|\int_{\mathbb{S}^2}(\omega-\nu)^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}+2^{-j}\gamma_j^\nu.$$

Together with (3.86), this implies:

$$\left\|L_\nu\left(\int_{\mathbb{S}^2}F_j(u)\eta_j^\nu(\omega)d\omega\right)\right\|_{L^2(\mathcal{M})}\lesssim\left\|\int_{\mathbb{S}^2}(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}+\gamma_j^\nu. \quad (3.87)$$

Finally, we need to estimate the first term in the right-hand side of (3.87). We will rely on the energy estimate for the wave equation.

3.4.3. Step3: The energy estimate for the wave equation.

Recall from (3.8) that:

$$\square_{\mathbf{g}}u=b^{-1}\mathrm{tr}\chi.$$

Thus, we have:

$$\square_{\mathbf{g}}\left(\int_{\mathbb{S}^2}(\omega-\nu)^2F_j(u)\eta_j^\nu(\omega)d\omega\right)=\int_{\mathbb{S}^2}b^{-1}(t,x,\omega)\mathrm{tr}\chi(t,x,\omega)(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega. \quad (3.88)$$

Arguing as in (3.77), we may replace $b^{-1}\mathrm{tr}\chi$ by 1:

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^2}b^{-1}(t,x,\omega)\mathrm{tr}\chi(t,x,\omega)(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}\\
\lesssim\ & \varepsilon\left\|\int_{\mathbb{S}^2}(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}+\varepsilon\gamma_j^\nu
\end{aligned}$$

which together with (3.88) implies:

$$\begin{aligned}&\left\|\Box_{\mathbf{g}}\left(\int_{\mathbb{S}^2}(\omega-\nu)^2F_j(u)\eta_j^\nu(\omega)d\omega\right)\right\|_{L^2(\mathcal{M})} \\ \lesssim\ & \varepsilon\left\|\int_{\mathbb{S}^2}(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}+\varepsilon\gamma_j^\nu.\end{aligned} \tag{3.89}$$

Let ϕ be the scalar function in the left-hand side of (3.88), i.e.:

$$\phi=\int_{\mathbb{S}^2}(\omega-\nu)^2F_j(u)\eta_j^\nu(\omega)d\omega.$$

Then, using the energy estimate for the wave equation (2.85) we obtain:

$$\begin{aligned}\|\mathbf{D}\phi\|^2_{L^\infty_tL^2(\Sigma_t)} \lesssim\ & \|\nabla\phi(0,.)\|^2_{L^2(\Sigma_0)}+\|T\phi(0,.)\|^2_{L^2(\Sigma_0)}+\|\Box_{\mathbf{g}}\phi\|_{L^2(\mathcal{M})}\|\mathbf{D}\phi\|_{L^2(\mathcal{M})} \\ & +\left|\int_{\mathcal{M}}Q_{\alpha\beta}\pi^{\alpha\beta}d\mathcal{M}\right|,\end{aligned} \tag{3.90}$$

where π is the deformation tensor of T

$$\pi_{\alpha\beta}=\mathbf{D}_\alpha T_\beta+\mathbf{D}_\beta T_\alpha.$$

and $Q_{\alpha\beta}$ on $\mathcal{M}$ is the energy momentum tensor associated to ϕ

$$Q_{\alpha\beta}=Q_{\alpha\beta}[\phi]=\partial_\alpha\phi\partial_\beta\phi-\frac{1}{2}\mathbf{g}_{\alpha\beta}\left(\mathbf{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right).$$

The control of the parametrix at initial time in [**43**] (see also Chapter 5) yields

$$\|\nabla\phi(0,.)\|_{L^2(\Sigma_0)}+\|T\phi(0,.)\|_{L^2(\Sigma_0)}\lesssim\gamma_j^\nu. \tag{3.91}$$

Next, we consider the last term in the right-hand side of (3.90). From the maximal foliation assumption, π is traceless, so that

$$\begin{aligned}Q_{\alpha\beta}\pi^{\alpha\beta} &= \pi^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\ &= 2^{2j}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\pi_{NN'}(\omega-\nu)^2(\omega'-\nu)^2F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^\nu(\omega')d\omega d\omega'\end{aligned}$$

Using (3.14), one obtains

$$Q_{\alpha\beta}\pi^{\alpha\beta}=2^{2j}\pi_{N_\nu N_\nu}\phi^2+\cdots,$$

and thus

$$Q_{\alpha\beta}\pi^{\alpha\beta}\simeq\pi_{N_\nu N_\nu}(\mathbf{D}\phi)^2+\cdots. \tag{3.92}$$

It turns out that we have a trace estimate for $\pi_{N_\nu N_\nu}$ (see details in [**45**]):

$$\|\pi_{N_\nu N_\nu}\|_{L^\infty_{u_\nu,x'_\nu}L^2_t}\lesssim\varepsilon$$

which together with (3.92) implies

$$\begin{aligned}\|Q_{\alpha\beta}\pi^{\alpha\beta}\|_{L^2(\mathcal{M})} &\lesssim \|\pi_{N_\nu N_\nu}\|_{L^\infty_{u_\nu,x'_\nu}L^2_t}\|\mathbf{D}\phi\|_{L^2_{u_\nu,x_\nu}L^\infty_t}\|\mathbf{D}\phi\|_{L^2(\mathcal{M})} \\ &\lesssim \varepsilon\|\mathbf{D}\phi\|_{L^2_{u_\nu,x_\nu}L^\infty_t}\|\mathbf{D}\phi\|_{L^2(\mathcal{M})}.\end{aligned} \tag{3.93}$$

Finally, (3.89)-(3.93) implies

$$\|\mathbf{D}\phi\|_{L^2(\mathcal{M})} \lesssim \varepsilon\|\mathbf{D}\phi\|_{L^2_{u_\nu,x_\nu}L^\infty_t} + \gamma_j^\nu,$$

which is equivalent to

$$\left\|\int_{\mathbb{S}^2}(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})} \lesssim \varepsilon\left\|\int_{\mathbb{S}^2}(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2_{u_\nu,x_\nu}L^\infty_t} + \gamma_j^\nu. \tag{3.94}$$

In view of (3.94) and (3.87), we obtain:

$$\left\|L_\nu\left(\int_{\mathbb{S}^2}F_j(u)\eta_j^\nu(\omega)d\omega\right)\right\|_{L^2(\mathcal{M})} \lesssim \varepsilon\left\|\int_{\mathbb{S}^2}(2^{\frac{j}{2}}(\omega-\nu))^2F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2_{u_\nu,x_\nu}L^\infty_t} + \gamma_j^\nu,$$

and thus:

$$\left\|\int_{\mathbb{S}^2}F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2_{u_\nu,x_\nu}L^\infty_t} \lesssim \gamma_j^\nu,$$

which is the desired estimate (3.80).

3.5. Almost orthogonality in angle

We have to prove (3.38):

$$\|E_jf\|^2_{L^2(\mathcal{M})} \lesssim \sum_{\nu\in\Gamma}\|E_j^\nu f\|^2_{L^2(\mathcal{M})} + \varepsilon^2\gamma_j^2. \tag{3.95}$$

This will result from an estimate for:

$$\left|\int_{\mathcal{M}}E_j^\nu f(t,x)\overline{E_j^{\nu'}f(t,x)}d\mathcal{M}\right|. \tag{3.96}$$

Let us introduce integration by parts first with respect to tangential directions, and then with respect to L.

3.5.1. Integration by parts.

3.5.1.1. *Integration by parts in tangential directions.* By definition of $\nabla\!\!\!/$, we have $\nabla\!\!\!/ h = \nabla h - (\nabla_N h)N$ for any function h on Σ_t. In particular, we have $\nabla\!\!\!/(u) = 0$ and $\nabla\!\!\!/(u') = b'^{-1}N' - b'^{-1}\mathbf{g}(N',N)N$. Now, since $\mathbf{g}(N' - \mathbf{g}(N,N')N, N') = 1 - \mathbf{g}(N',N)^2$ and $\nabla u' = b'^{-1}N'$, we deduce:

$$\begin{aligned} e^{i\lambda u - i\lambda' u'} &= \frac{i}{\lambda'\mathbf{g}(N' - \mathbf{g}(N,N')N, \nabla u')}\nabla_{N'-\mathbf{g}(N,N')N}(e^{i\lambda u - i\lambda' u'}) \\ &= \frac{ib'}{\lambda'(1-\mathbf{g}(N',N)^2)}\nabla_{N'-\mathbf{g}(N,N')N}(e^{i\lambda u - i\lambda' u'}), \end{aligned} \tag{3.97}$$

where we have used the fact that $N' - \mathbf{g}(N,N')N$ is a tangent vector with respect of the level surfaces of u. We consider an oscillatory integral of the following form:

$$\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2}h(t,x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M},$$

where h is a scalar function on $\mathcal{M}$. Integrating by parts once using (3.97) yields:

$$\begin{aligned}&\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2} h(t,x)F_j(u)F_j(u')\eta_j^{\nu}(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M}\\ = \ & -i2^{-j}\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2}\frac{b'}{1-\mathbf{g}(N,N')^2}((N'-\mathbf{g}(N,N')N)(h)+\cdots)\\ &\times F_j(u)F_j(u')\eta_j^{\nu}(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M},\end{aligned}$$

where we only kept the term where the derivative falls on h, and neglected for the simplicity of the exposition the terms when the derivative falls on the denominator of the right-hand side of (3.97) or on the volume element of $\mathcal{M}$. In view of (3.13), we have:

$$N'-\mathbf{g}(N,N')N\sim N'-N\sim|\omega-\omega'|\sim|\nu-\nu'|, \tag{3.98}$$

and:

$$1-\mathbf{g}(N,N')=\frac{\mathbf{g}(N-N',N-N')}{2}\sim|\omega-\omega'|^2\sim|\nu-\nu'|^2, \tag{3.99}$$

and we thus obtain:

$$\begin{aligned}&\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2} h(t,x)F_j(u)F_j(u')\eta_j^{\nu}(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M} \qquad (3.100)\\ = \ & i\frac{1}{2^j|\nu-\nu'|}\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2} b'\nabla\!\!\!/(h)F_j(u)F_j(u')\eta_j^{\nu}(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M}+\cdots.\end{aligned}$$

Remark 3.18. *In the formula* (3.100)*, we neglect two types of terms for the simplicity of the exposition. First, we neglect the term when the derivative falls on the denominator of the right-hand side of* (3.97) *or on the volume element of* $\mathcal{M}$*. Next, make the following approximation:*

$$\frac{N'-\mathbf{g}(N,N')N}{1-\mathbf{g}(N,N')^2}\sim\frac{1}{2^j|\nu-\nu'|}.$$

In the actual proof, we use (3.99) *to derive the following expansion:*

$$\frac{1}{1-\mathbf{g}(N,N')^2}=\frac{1}{|N_\nu-N_{\nu'}|^2}\left(\sum_{p,q\geq 0}c_{pq}\left(\frac{N-N_\nu}{|N_\nu-N_{\nu'}|}\right)^p\left(\frac{N'-N_{\nu'}}{|N_\nu-N_{\nu'}|}\right)^q\right), \tag{3.101}$$

for some explicit real coefficients c_{pq} *such that the series*

$$\sum_{p,q\geq 0}c_{pq}x^py^q$$

has radius of convergence 1. Then, (3.100) *corresponds to the first term in the expansion* (3.101) *with the additional simplification which consists in replacing* $|N_\nu-N_{\nu'}|$ *with* $|\nu-\nu'|$ *again in view of* (3.99)*. While these approximations greatly simplify the exposition, they still allow us to exhibit typical terms in the proof of the almost orthogonality in angle.*

3.5.1.2. *Integration by parts in L.* Next, we also introduce integrations by parts with respect to L. Since $L(u)=0$ and $L(u')={b'}^{-1}\mathbf{g}(L,L')$, we have:

$$e^{i\lambda u-i\lambda' u'}=\frac{ib'}{\lambda'\mathbf{g}(L,L')}L(e^{i\lambda u-i\lambda' u'}). \tag{3.102}$$

We consider an oscillatory integral of the following form:

$$\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2}h(t,x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M},$$

where h is a scalar function on $\mathcal{M}$. Integrating by parts once using (3.97) yields:

$$\begin{aligned}&\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2}h(t,x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M}\\ =\ & -i2^{-j}\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2}\frac{b'}{\mathbf{g}(L,L')}(L(h)+\cdots)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M},\end{aligned}$$

where we only kept the term where the derivative falls on h, and neglected for the simplicity of the exposition the term when the derivative falls on the denominator of the right-hand side of (3.102) or on the volume element. Using the fact that:

$$\mathbf{g}(L,L')=-1+\mathbf{g}(N,N') \tag{3.103}$$

together with (3.99), and keeping only the first term in the expansion (3.101), with the additional simplification which consists in replacing $|N_\nu-N_{\nu'}|$ with $|\nu-\nu'|$, we obtain:

$$\begin{aligned}&\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2}h(t,x)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M} \qquad (3.104)\\ =\ & i\frac{1}{2^j|\nu-\nu'|^2}\int_{\mathcal{M}}\int_{\mathbb{S}^2\times\mathbb{S}^2}b'L(h)F_j(u)F_j(u')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\mathcal{M}+\cdots.\end{aligned}$$

3.5.2. Presence of a log-loss. Let us explain why proceeding directly by integration by parts in (3.96) results in a log-loss. Let us define $\mathcal{E}_{j,\nu,\nu'}$ as:

$$\mathcal{E}_{j,\nu,\nu'}=\int_{\mathcal{M}}E_j^\nu f(t,x)\overline{E_j^{\nu'}f(t,x)}d\mathcal{M}.$$

We have:

$$\begin{aligned}\mathcal{E}_{j,\nu,\nu'}\ =\ &\int_{\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\left(\int_{\mathcal{M}}e^{i\lambda u-i\lambda' u'}b^{-1}\mathrm{tr}\chi {b'}^{-1}\mathrm{tr}\chi' d\mathcal{M}\right)\\ &\times\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega'.\end{aligned}$$

We integrate by parts tangentially using (3.100). Consider the term where the tangential derivative falls on $\mathrm{tr}\chi$, which is of the form:

$$\begin{aligned}\frac{1}{2^j|\nu-\nu'|}\quad&\int_{\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\left(\int_{\mathcal{M}}e^{i\lambda u-i\lambda' u'}b^{-1}\nabla\mathrm{tr}\chi {b'}^{-1}\mathrm{tr}\chi' d\mathcal{M}\right)\\ &\times\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega'.\end{aligned}$$

Since $L\nabla\!\!\!\!/\,\mathrm{tr}\chi$ is the only derivative of $\nabla\!\!\!\!/\,\mathrm{tr}\chi$ for which we have an estimate, our next integration by parts must be with respect to L, that is we use (3.104). Consider the term where the L derivative falls on $\mathrm{tr}\chi'$, which is of the form:

$$\begin{aligned}\frac{1}{2^{2j}|\nu-\nu'|^3} \quad & \int_{\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\left(\int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'}b^{-1}\nabla\!\!\!\!/\,\mathrm{tr}\chi b'^{-1}L(\mathrm{tr}\chi')d\mathcal{M}\right) \\ & \times\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega'.\end{aligned} \tag{3.105}$$

Now, note in view of (3.103), (3.99) and the estimate (3.14) for $\partial_\omega N$, that:

$\mathbf{g}(L,L')\sim|\nu-\nu'|^2$, $\mathbf{g}(L,e'_A)=\mathbf{g}(L-L',e'_A)\sim|\nu-\nu'|$ and $\mathbf{g}(L,\underline{L}')=-2+\mathbf{g}(L,L')\sim-2$.

Thus, decomposing L on the frame L', N', e'_A, we obtain:

$$L\sim L'+|\nu-\nu'|\nabla\!\!\!\!/' +|\nu-\nu'|^2N'. \tag{3.106}$$

Together with (3.105), we finally obtain the sum of three terms:

$$\begin{aligned} & \mathcal{E}_{j,\nu,\nu'} \\ = \ & \mathcal{E}_{j,\nu,\nu'}[1]+\mathcal{E}_{j,\nu,\nu'}[2]+\mathcal{E}_{j,\nu,\nu'}[3] \\ = \ & \frac{1}{2^{2j}|\nu-\nu'|^3}\int_{\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\left(\int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'}b^{-1}\nabla\!\!\!\!/\,\mathrm{tr}\chi b'^{-1}L'(\mathrm{tr}\chi')d\mathcal{M}\right) \\ & \times\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega' \\ & +\frac{1}{2^{2j}|\nu-\nu'|^2}\int_{\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\left(\int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'}b^{-1}\nabla\!\!\!\!/\,\mathrm{tr}\chi b'^{-1}\nabla\!\!\!\!/'(\mathrm{tr}\chi')d\mathcal{M}\right) \\ & \times\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega' \\ & +\frac{1}{2^{2j}|\nu-\nu'|}\int_{\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\left(\int_{\mathcal{M}} e^{i\lambda u-i\lambda' u'}b^{-1}\nabla\!\!\!\!/\,\mathrm{tr}\chi b'^{-1}N'(\mathrm{tr}\chi')d\mathcal{M}\right) \\ & \times\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega'.\end{aligned} \tag{3.107}$$

We consider the second term in the right-hand side of (3.107) which is of the form:

$$\begin{aligned}\mathcal{E}_{j,\nu,\nu'}[2] \ = \ & \frac{1}{2^{2j}|\nu-\nu'|^2}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2} b^{-1}\nabla\!\!\!\!/\,\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right) \\ & \times\left(\int_{\mathbb{S}^2} b'^{-1}\nabla\!\!\!\!/'\mathrm{tr}\chi' F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right)d\mathcal{M}.\end{aligned}$$

We claim that such a term leads to a log-loss. Indeed, we have:

$$
\begin{aligned}
& |\mathcal{E}_{j,\nu,\nu'}[2]| \\
\lesssim\ & \frac{1}{2^{2j}|\nu-\nu'|^2}\left\|\int_{\mathbb{S}^2} b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}\left\|\int_{\mathbb{S}^2} b'^{-1}\nabla\!\!\!/'\mathrm{tr}\chi' F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})} \\
\lesssim\ & \frac{1}{2^{2j}|\nu-\nu'|^2}\left(\int_{\mathbb{S}^2}\|b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\|_{L^2(\mathcal{M})}\eta_j^\nu(\omega)d\omega\right) \\
& \times\left(\int_{\mathbb{S}^2}\|b'^{-1}\nabla\!\!\!/'\mathrm{tr}\chi' F_j(u')\|_{L^2(\mathcal{M})}\eta_j^{\nu'}(\omega')d\omega'\right) \\
\lesssim\ & \frac{1}{2^{2j}|\nu-\nu'|^2}\left(\int_{\mathbb{S}^2}\|b^{-1}\|_{L^\infty}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_u L^2(\mathcal{H}_u)}\|F_j(u)\|_{L^2_u}\eta_j^\nu(\omega)d\omega\right) \\
& \times\left(\int_{\mathbb{S}^2}\|b'^{-1}\|_{L^\infty}\|\nabla\!\!\!/'\mathrm{tr}\chi'\|_{L^\infty_u L^2(\mathcal{H}_u)}\|F_j(u')\|_{L^2_{u'}}\eta_j^{\nu'}(\omega')d\omega'\right) \\
\lesssim\ & \frac{\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu-\nu'|)^2},
\end{aligned}
\tag{3.108}
$$

where we used in the last inequality Cauchy-Schwartz in ω and ω' which gains the square root of the volume of the patch, the estimates (3.10) for b and $\mathrm{tr}\chi$, and the estimate (3.51) for $F_j(u)$ and $F_j(u')$. This leads to a log-loss since we have:

$$
\sup_\nu\sum_{\nu'}\frac{1}{(2^{j/2}|\nu-\nu'|)^2}\sim j. \tag{3.109}
$$

Indeed, note that ν' runs on a lattice on $\mathbb{S}^2$ of basic size $2^{-j/2}$ so that (3.109) corresponds to the sum

$$
\sum_{l\in\mathbb{Z}^2,\,1\leq|l|\leq 2^{j/2}}\frac{1}{|l|^2}\sim j.
$$

3.5.3. Strategy of the proof of Proposition 3.9. Let us explain informally the strategy of the proof. As we noticed in the previous section, the second term in (3.107) contains a log-loss. Let us start by showing that the first and the third term in the right-hand side of (3.107) do not contain a log-loss.

3.5.3.1. *Control of the first term in the right-hand side of* (3.107). We have

$$
\begin{aligned}
\mathcal{E}_{j,\nu,\nu'}[1] \;=\; & \frac{1}{2^{2j}|\nu-\nu'|^3}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2} b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right) \\
& \times\left(\int_{\mathbb{S}^2} b'^{-1}L'(\mathrm{tr}\chi')F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right)d\mathcal{M}.
\end{aligned}
$$

In view of the Raychaudhuri equation (3.7), we have:

$$
L'(\mathrm{tr}\chi') = -|\widehat{\chi}'|^2+\cdots,
$$

where we keep only the worst term. Thus, we obtain

$$\mathcal{E}_{j,\nu,\nu'}[1] = \frac{1}{2^{2j}|\nu-\nu'|^3}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2} b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right)\left(\int_{\mathbb{S}^2} b'^{-1}|\widehat{\chi}'|^2F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right)d\mathcal{M}.$$

Let us decompose:

$$b'^{-1}|\widehat{\chi}'|^2 = b_{\nu'}^{-1}\widehat{\chi}_{\nu'}\widehat{\chi} + (b'^{-1}|\widehat{\chi}'|^2 - b_{\nu'}^{-1}\widehat{\chi}_{\nu'}\widehat{\chi}), \tag{3.110}$$

and let us assume for the moment that we can control the second term in (3.110). Then, we are led to control:

$$\frac{1}{2^{2j}|\nu-\nu'|^3}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2} b^{-1}\widehat{\chi}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right)b_{\nu'}^{-1}\widehat{\chi}_{\nu'}\left(\int_{\mathbb{S}^2}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right)d\mathcal{M}.$$

We have:

$$\begin{aligned}
&\left|\frac{1}{2^{2j}|\nu-\nu'|^3}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2} b^{-1}\widehat{\chi}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right)b_{\nu'}^{-1}\widehat{\chi}_{\nu'}\left(\int_{\mathbb{S}^2}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right)d\mathcal{M}\right|\\
\lesssim\;& \frac{1}{2^{2j}|\nu-\nu'|^3}\left\|\int_{\mathbb{S}^2} b^{-1}\widehat{\chi}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}\left\|b_{\nu'}^{-1}\widehat{\chi}_{\nu'}\int_{\mathbb{S}^2}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})}\\
\lesssim\;& \frac{1}{2^{2j}|\nu-\nu'|^3}\left(\int_{\mathbb{S}^2}\|b^{-1}\widehat{\chi}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\|_{L^2(\mathcal{M})}\eta_j^\nu(\omega)d\omega\right)\\
&\times\left\|b_{\nu'}^{-1}\widehat{\chi}_{\nu'}\right\|_{L^\infty_{u_{\nu'},x'_{\nu'}}L^2_t}\left\|\int_{\mathbb{S}^2}F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2_{u_{\nu'},x'_{\nu'}}L^\infty_t}\\
\lesssim\;& \frac{\varepsilon\gamma_j^{\nu'}}{2^{2j}|\nu-\nu'|^3}\int_{\mathbb{S}^2}\|b^{-1}\|_{L^\infty(\mathcal{M})}\|\widehat{\chi}\|_{L^\infty_{x'}L^2_t}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_{x'}L^2_t}\|F_j(u)\|_{L^2_u}\eta_j^\nu(\omega)d\omega\\
\lesssim\;& \frac{\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'}}{(2^{\frac{j}{2}}|\nu-\nu'|)^3},
\end{aligned}\tag{3.111}$$

where we used the estimate (3.10) for $\mathrm{tr}\chi$, $\widehat{\chi}$ and b, the estimate (3.80), Cauchy-Schwartz in ω, the size of the patch, and the estimate (3.51) for $F_j(u)$. Note that the right-hand side of (3.111) does not contain a log-loss since:

$$\sup_\nu\sum_{\nu'}\frac{1}{(2^{j/2}|\nu-\nu'|)^3}\lesssim 1. \tag{3.112}$$

Remark 3.19. *While the estimate obtained in* (3.111) *is correct, one has to modify slightly the method leading to it. Indeed,* $\widehat{\chi}$ *does not have enough regularity with respect to* ω *to be able to handle the second term in the decomposition* (3.110)*. The way to overcome this is to make use of the decomposition* (3.15) *for* $\widehat{\chi}$*:*

$$\widehat{\chi} = \chi_1 + \chi_2.$$

Then, we exploit the fact that, in view of the estimate (3.16)*,* χ_1 *has better regularity than* $\widehat{\chi}$ *with respect to* (t,x)*, while* χ_2 *has better regularity than* $\widehat{\chi}$ *with respect to* ω*. We refer to* [**45**] *for more details.*

3.5.3.2. *Control of the third term in the right-hand side of* (3.107). We have

$$\mathcal{E}_{j,\nu,\nu'}[3] = \frac{1}{2^{2j}|\nu-\nu'|}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2} b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right) \times\left(\int_{\mathbb{S}^2} b'^{-1}N'(\mathrm{tr}\chi')F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right)d\mathcal{M}.$$

Now, we have:

$$\begin{aligned}
&\left|\frac{1}{2^{2j}|\nu-\nu'|}\int_{\mathcal{M}}\left(\int_{\mathbb{S}^2} b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right)\left(\int_{\mathbb{S}^2} b'^{-1}N'\mathrm{tr}\chi' F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right)d\mathcal{M}\right|\\
\lesssim\ & \frac{1}{2^{2j}|\nu-\nu'|}\left\|\int_{\mathbb{S}^2} b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathcal{M})}\left\|\int_{\mathbb{S}^2} b'^{-1}N'\mathrm{tr}\chi' F_j(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathcal{M})}\\
\lesssim\ & \frac{1}{2^{2j}|\nu-\nu'|}\left(\int_{\mathbb{S}^2}\|b^{-1}\nabla\!\!\!/\mathrm{tr}\chi F_j(u)\|_{L^2(\mathcal{M})}\eta_j^\nu(\omega)d\omega\right)\\
&\times\left(\int_{\mathbb{S}^2}\|b'^{-1}N'\mathrm{tr}\chi' F_j(u')\|_{L^2(\mathcal{M})}\eta_j^{\nu'}(\omega')d\omega'\right)\\
\lesssim\ & \frac{1}{2^{2j}|\nu-\nu'|}\left(\int_{\mathbb{S}^2}\|b^{-1}\|_{L^\infty}\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^\infty_u L^2(\mathcal{H}_u)}\|F_j(u)\|_{L^2_u}\eta_j^\nu(\omega)d\omega\right)\\
&\times\left(\int_{\mathbb{S}^2}\|b'^{-1}\|_{L^\infty}\|N'\mathrm{tr}\chi'\|_{L^\infty_u L^2(\mathcal{H}_u)}\|F_j(u')\|_{L^2_{u'}}\eta_j^{\nu'}(\omega')d\omega'\right)\\
\lesssim\ & \frac{\varepsilon^2\gamma_j^\nu\gamma_j^{\nu'}}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu-\nu'|)},
\end{aligned} \tag{3.113}$$

where we used in the last inequality Cauchy-Schwartz in ω and ω' which gains the square root of the volume of the patch, the estimates (3.10) for b and $\mathrm{tr}\chi$, and the estimate (3.51) for $F_j(u)$ and $F_j(u')$. Note that the right-hand side of (3.113) does not contain a log-loss since we have:

$$\sup_\nu\sum_{\nu'}\frac{1}{2^{\frac{j}{2}}(2^{\frac{j}{2}}|\nu-\nu'|)}\lesssim 1. \tag{3.114}$$

3.5.3.3. *A decomposition for $E_j^\nu f$.* To remove the log-loss exhibited in (3.108) (3.109), we rely on a decomposition of $\mathrm{tr}\chi$ using the geometric Littlewood-Paley projections P_j. We have:

$$\mathrm{tr}\chi = P_{\leq j/2}(\mathrm{tr}\chi)+\sum_{l>j/2}P_l\mathrm{tr}\chi$$

which in turn yields the following decomposition for $E_j^\nu f$:

$$E_j^\nu f(t,x)=\sum_{l\geq j/2}E_j^{\nu,l}f(t,x), \tag{3.115}$$

where:

$$E_j^{\nu,l}f(t,x) = \int_{\mathbb{S}^2} b(t,x,\omega)^{-1}P_l\mathrm{tr}\chi(t,x,\omega)F_j(u)\eta_j^\nu(\omega)d\omega,\ \ \forall l > \frac{j}{2} \tag{3.116}$$

and:

$$E_j^{\nu,j/2}f(t,x) = \int_{\mathbb{S}^2} b(t,x,\omega)^{-1}P_{\leq j/2}\mathrm{tr}\chi(t,x,\omega)F_j(u)\eta_j^\nu(\omega)d\omega. \tag{3.117}$$

In order to prove almost orthogonality in angle, i.e. (3.95), we will estimate:

$$\left|\sum_{l,m}\int_{\mathcal{M}} E_j^{\nu,l}f(t,x)\overline{E_j^{\nu',m}f(t,x)}d\mathcal{M}\right|. \tag{3.118}$$

3.5.3.4. *The mechanism to remove the log-loss.* In order to explain the mechanism which allows us to remove the log-loss, let us assume for convenience that $m \leq l$ in (3.118). Then, notice first from (3.107), (3.109), (3.112) and (3.114) that the only term in the right-hand side of (3.107) which contains a log-loss is the second one, i.e the term which contains only tangential derivatives. In order to remove the log-loss, our goal will be to always put more tangential derivatives on the lowest frequency, i.e. $P_m\mathrm{tr}\chi'$ (as opposed to the higher frequency $P_l\mathrm{tr}\chi$). This is achieved as follows (see [**45**] for the details):

(1) Integrate by parts with respect to L using (3.104).
(2) One term corresponds to the case where the L derivative falls on the largest frequency $P_l\mathrm{tr}\chi$, while the other term corresponds to the case where L falls on the lowest frequency $P_m\mathrm{tr}\chi'$. For the second term, decompose the L derivative on the frame L', N', e'_A as in (3.106).
(3) Notice that the terms involving L, L' or N' are estimated in the spirit of (3.111) and (3.113), and should in principle contain no log-loss in view of (3.112) and (3.114).
(4) Finally, the last term is the one containing the $\nabla\!\!\!/'$ derivative. This term is the only one which contains the log-loss exhibited in (3.109). Now, we have achieved our goal since after integration by parts, the tangential derivative fell on $P_m\mathrm{tr}\chi'$ which is the lowest frequency.

Remark 3.20. *Due to the decomposition* (3.115)*, we now not only need to obtain summability in* (ν,ν')*, but also in* (l,m)*. This creates additional difficulties, in particular when estimating the terms* $\mathcal{E}_{j,\nu,\nu'}[1]$ *and* $\mathcal{E}_{j,\nu,\nu'}[3]$ *in* (3.107)*. We refer to* [**45**] *for more details.*

CHAPTER 4

Control of the space-time foliation

The goal of this chapter is to prove the estimates on the control of the space-time foliation by the optical function u which are needed for the proof of Theorem 3.6 (see section 3.1.3), i.e. for the control of the error term. Here, we outline the main ideas and we refer to [**44**] for the details.

4.1. Geometric set-up and main results

4.1.1. Geometry of the foliation of $\mathcal{M}$ by u. Recall from section 1.2.1 that the space-time $\mathcal{M}$ is foliated by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t, where T denotes the unit, future oriented, normal to Σ_t and k its second fundamental form. Recall also that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ depending on a extra parameter $\omega \in \mathbb{S}^2$. The level hypersufaces $u(t,x,\omega) = u$ of the optical function u are denoted by $\mathcal{H}_u$. Let L' denote the space-time gradient of u, i.e.:

$$L' = -\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha. \tag{4.1}$$

Using the fact that u satisfies the eikonal equation, we obtain:

$$\mathbf{D}_{L'}L' = 0, \tag{4.2}$$

which implies that L' is the geodesic null generator of $\mathcal{H}_u$.

We have:

$$T(u) = \pm|\nabla u|$$

where $|\nabla u|^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t. Since the sign of $T(u)$ is irrelevant, we choose by convention:

$$T(u) = |\nabla u|. \tag{4.3}$$

We denote by $P_{t,u}$ the surfaces of intersection between Σ_t and $\mathcal{H}_u$. They play a fundamental role in our discussion.

Definition 4.1 (*Canonical null pair*).

$$L = bL' = T + N, \qquad \underline{L} = 2T - L = T - N \tag{4.4}$$

where L' is the space-time gradient of u (4.1), *b is the lapse of the null foliation (or shortly null lapse)*

$$b^{-1} = - < L', T >= T(u), \tag{4.5}$$

and N *is a unit normal, along* Σ_t*, to the surfaces* $P_{t,u}$*. Since* u *satisfies the eikonal equation* $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ *on* $\mathcal{M}$*, this yields* $L'(u) = 0$ *and thus* $L(u) = 0$*. In view of the definition of* L *and (4.3), we obtain:*

$$N = -\frac{\nabla u}{|\nabla u|}. \tag{4.6}$$

Remark 4.2. u *is prescribed on* Σ_0 *as in* [**42**]*. For any* $(0,x)$ *on* Σ_0*,* L *is defined as* $L = T + N$ *where* T *is the unit normal to* Σ_0 *at* $(0,x)$ *and* $N = -\nabla u/|\nabla u|$ *at* $(0,x)$*, and* b *is defined as* $b^{-1} = |\nabla u|$*. Let* $\kappa_x(t)$ *denote the null geodesic parametrized by* t *and such that* $\kappa_x(0) = (0,x)$ *and* $\kappa_x'(0) = b^{-1}L$*. Then, we claim that*

$$\kappa_x'(t) = b(\kappa_x(t))^{-1} L_{\kappa_x(t)} \text{ for all } t. \tag{4.7}$$

Indeed, $L' = b^{-1}L$ *is the geodesic null generator of* $\mathcal{H}_u$ *(see (4.2)).*

Definition 4.3. *A null frame* e_1, e_2, e_3, e_4 *at a point* $p \in P_{t,u}$ *consists, in addition to the null pair* $e_3 = \underline{L}, e_4 = L$*, of* arbitrary orthonormal *vectors* e_1, e_2 *tangent to* $P_{t,u}$*.*

Definition 4.4 (*Ricci coefficients*). *Let* e_1, e_2, e_3, e_4 *be a null frame on* $P_{t,u}$ *as above. The following tensors on* $S_{t,u}$

$$\begin{aligned} \chi_{AB} &= <\mathbf{D}_A e_4, e_B>, \qquad \underline{\chi}_{AB} = <\mathbf{D}_A e_3, e_B>, \\ \zeta_A &= \frac{1}{2} <\mathbf{D}_3 e_4, e_A>, \qquad \underline{\zeta}_A = \frac{1}{2} <\mathbf{D}_4 e_3, e_A>, \\ \underline{\xi}_A &= \frac{1}{2} <\mathbf{D}_3 e_3, e_A>. \end{aligned} \tag{4.8}$$

are called the Ricci coefficients associated to our canonical null pair.

We decompose χ *and* $\underline{\chi}$ *into their trace and traceless components.*

$$tr\chi = \mathbf{g}^{AB}\chi_{AB}, \qquad tr\underline{\chi} = \mathbf{g}^{AB}\underline{\chi}_{AB}, \tag{4.9}$$

$$\widehat{\chi}_{AB} = \chi_{AB} - \frac{1}{2}tr\chi\mathbf{g}_{AB}, \quad \widehat{\underline{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2}tr\underline{\chi}\mathbf{g}_{AB}, \tag{4.10}$$

Definition 4.5. *The null components of the curvature tensor* $\mathbf{R}$ *of the space-time metric* $\mathbf{g}$ *are given by:*

$$\alpha_{ab} = \mathbf{R}(L, e_a, L, e_b), \qquad \beta_a = \frac{1}{2}\mathbf{R}(e_a, L, \underline{L}, L), \tag{4.11}$$

$$\rho = \frac{1}{4}\mathbf{R}(\underline{L}, L, \underline{L}, L), \qquad \sigma = \frac{1}{4}{}^\star\mathbf{R}(\underline{L}, L, \underline{L}, L) \tag{4.12}$$

$$\underline{\beta}_a = \frac{1}{2}\mathbf{R}(e_a, \underline{L}, \underline{L}, L), \qquad \underline{\alpha}_{ab} = \mathbf{R}(\underline{L}, e_a, \underline{L}, e_b) \tag{4.13}$$

where ${}^\star\mathbf{R}$ *denotes the Hodge dual of* $\mathbf{R}$*.*

Observe that all tensors defined above are $P_{t,u}$-tangent.

REMARK 4.6. *Note that $\underline{\alpha}$ is the only null component which does not contain a contraction of $\mathbf{R}$ with L. With the notation of Chapter 2 (see for instance* (2.28)*), we have:*

$$\mathbf{R}\cdot L = (\alpha, \beta, \rho, \sigma, \underline{\beta}).$$

DEFINITION 4.7. *We decompose the symmetric traceless 2 tensor k into the scalar δ, the $P_{t,u}$-tangent 1-form ϵ, and the $P_{t,u}$-tangent symmetric 2-tensor η as follows:*

$$\left\{\begin{array}{l} k_{NN} = \delta \\ k_{AN} = \epsilon_A \\ k_{AB} = \eta_{AB}. \end{array}\right. \tag{4.14}$$

The following *Ricci equations* can be easily derived from the definition of T, the fact that L' is geodesic (4.2), and the definition (4.8) of the Ricci coefficients (see [**9**] p. 171):

$$\begin{array}{ll} \mathbf{D}_A e_4 = \chi_{AB} e_B - \epsilon_A e_4, & \mathbf{D}_A e_3 = \underline{\chi}_{AB} e_B + \epsilon_A e_3, \\ \mathbf{D}_4 e_4 = -\overline{\delta} e_4, & \mathbf{D}_4 e_3 = 2\underline{\zeta}_A e_A + \overline{\delta} e_3, \\ \mathbf{D}_3 e_4 = 2\zeta_A e_A + (\delta + n^{-1}\nabla_N n) e_4, & \mathbf{D}_3 e_3 = 2\underline{\xi}_A e_A - (\delta + n^{-1}\nabla_N n) e_3, \\ \mathbf{D}_4 e_A = \nabla\!\!\!/_4 e_A + \underline{\zeta}_A e_4, & \mathbf{D}_3 e_A = \nabla\!\!\!/_3 e_A + \zeta_A e_3 + \underline{\xi}_A e_4, \\ \mathbf{D}_B e_A = \nabla\!\!\!/_B e_A + \frac{1}{2}\chi_{AB}\, e_3 + \frac{1}{2}\underline{\chi}_{AB}\, e_4 & \end{array} \tag{4.15}$$

where, $\nabla\!\!\!/_3$, $\nabla\!\!\!/_4$ denote the projection on $P_{t,u}$ of $\mathbf{D}_3$ and $\mathbf{D}_4$, $\nabla\!\!\!/$ denotes the induced covariant derivative on $P_{t,u}$ and $\overline{\delta}, \overline{\epsilon}$ are defined by:

$$\overline{\delta} = \delta - n^{-1}N(n),\ \overline{\epsilon}_A = \epsilon_A - n^{-1}\nabla_A n. \tag{4.16}$$

Also,

$$\begin{array}{l} \underline{\chi}_{AB} = -\chi_{AB} - 2k_{AB}, \\ \underline{\zeta}_A = -\overline{\epsilon}_A, \\ \underline{\xi}_A = \epsilon_A + n^{-1}\nabla\!\!\!/_A n - \zeta_A. \end{array} \tag{4.17}$$

4.1.2. Null structure equations. Below we write down our main structure equations (see [**9**] chapter 7 or [**44**] for a proof).

PROPOSITION 4.8. *The components* $tr\chi$, $\widehat{\chi}$, ζ *and the lapse* b *verify the following equations*[1]*:*

$$L(b) = -b\,\overline{\delta}, \tag{4.18}$$

$$L(tr\chi) + \frac{1}{2}(tr\chi)^2 = -|\widehat{\chi}|^2 - \overline{\delta}\,tr\chi, \tag{4.19}$$

$$\nabla\!\!\!/_4\widehat{\chi} + tr\chi\widehat{\chi} = -\overline{\delta}\widehat{\chi} - \alpha, \tag{4.20}$$

$$\nabla\!\!\!/_4\zeta_A + \frac{1}{2}(tr\chi)\zeta_A = -(\overline{\epsilon}_B + \zeta_B)\widehat{\chi}_{AB} - \frac{1}{2}tr\chi\overline{\epsilon}_A - \beta_A,. \tag{4.21}$$

REMARK 4.9. *Equation* (4.19) *is known as the Raychaudhuri equation in the relativity literature.*

To obtain estimates for χ, we may use the transport equations (4.19) (4.20). However, this does not allow us to get enough regularity. Instead, we follow [**9**] [**20**] [**22**] and consider (4.19) for trχ together with an elliptic system of Hodge type for $\widehat{\chi}$.

PROPOSITION 4.10. *The expression* $(d\!\!\!/iv\widehat{\chi})_A = \nabla\!\!\!/^B\widehat{\chi}_{AB}$ *verifies the following equation:*

$$(d\!\!\!/iv\widehat{\chi})_A + \widehat{\chi}_{AB}\epsilon_B = \frac{1}{2}(\nabla\!\!\!/_A tr\chi + \epsilon_A tr\chi) - \beta_A. \tag{4.22}$$

See [**9**] chapter 7 or [**44**] for a proof.

Finally, we consider the control of ζ and $\underline{L}$trχ. To this end, we follow again [**20**] [**22**]: we derive an elliptic system of Hodge type for ζ and a transport equation for $\underline{L}$trχ.

PROPOSITION 4.11. *We have:*

$$\underline{L}(tr\chi) + \frac{1}{2}tr\underline{\chi}tr\chi = 2d\!\!\!/iv\zeta + (\delta + n^{-1}\nabla_N n)tr\chi - \widehat{\chi}\cdot\widehat{\underline{\chi}} + 2\zeta\cdot\zeta + 2\rho. \tag{4.23}$$

Also, the expressions $d\!\!\!/iv\zeta = \nabla\!\!\!/^B\zeta_B$ *and* $cu\!\!\!/rl\zeta = \in^{AB}\nabla\!\!\!/_A\zeta_B$ *verify the following equations:*

$$d\!\!\!/iv\zeta = \frac{1}{2}\left(\mu + \frac{1}{2}tr\chi tr\underline{\chi} + \widehat{\chi}\cdot\widehat{\underline{\chi}} - 2|\zeta|^2\right) - \rho, \tag{4.24}$$

$$cu\!\!\!/rl\zeta = -\frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}} + \sigma, \tag{4.25}$$

where for F, G *symmetric traceless* $P_{t,u}$*-tangent 2-tensors, we denote by* $F\wedge G$ *the tensor* $F\wedge G_{AB} = \in_{AB} F_{AC}G_{BC}$*. Finally, setting,*

$$\mu = \underline{L}(tr\chi) - \big(\delta + n^{-1}\nabla_N n\big)tr\chi \tag{4.26}$$

[1] which can be interpreted as transport equations along the null geodesics generated by L. Indeed observe that if a $P_{t,u}$-tangent tensor Π satisfies the homogeneous equation $\nabla\!\!\!/_4\Pi = 0$ then Π is parallel transported along null geodesics.

we find

$$
\begin{aligned}
L(\mu) + tr\chi\mu = 2(\underline{\zeta} - \zeta)\cdot \nabla\!\!\!/ tr\chi - 2\widehat{\chi}\cdot\Big(\nabla\!\!\!/\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta - \delta\widehat{\chi}\Big) \\
- tr\chi\Bigg(2 d\!\!\!/ iv\zeta + 2\zeta\cdot\zeta + 4(\epsilon - \zeta)\cdot n^{-1}\nabla\!\!\!/ n - 2\overline{\delta}(\delta + n^{-1}\nabla_N n) + 4\rho \\
- \frac{1}{2}tr\chi tr\underline{\chi} + 2|\epsilon|^2 + 3|\widehat{\chi}|^2 + 4\widehat{\chi}\cdot\widehat{\eta} - 2|n^{-1}N(n)|^2\Bigg).
\end{aligned}
\tag{4.27}
$$

See [**20**] or [**44**] for a proof.

4.1.3. Commutation formulas. We have the following useful commutation formulas (see [**9**] p. 159):

LEMMA 4.12. *Let $U_{\underline{A}}$ be an m-covariant tensor tangent to the surfaces $P_{t,u}$. Then,*

$$
\begin{aligned}
\nabla\!\!\!/_B\nabla\!\!\!/_4 U_{\underline{A}} - \nabla\!\!\!/_4\nabla\!\!\!/_B U_{\underline{A}} &= \chi_{BC}\nabla\!\!\!/_C U_{\underline{A}} - n^{-1}\nabla\!\!\!/_B n\nabla\!\!\!/_4 U_{\underline{A}} \qquad (4.28)\\
&+ \sum_i(\chi_{A_iB}\overline{\epsilon}_C - \chi_{BC}\overline{\epsilon}_{A_i} - \in_{A_iC}{}^*\beta_B)U_{A_1..\check{C}..A_m},
\end{aligned}
$$

$$
\begin{aligned}
\nabla\!\!\!/_B\nabla\!\!\!/_3 U_{\underline{A}} - \nabla\!\!\!/_3\nabla\!\!\!/_B U_{\underline{A}} &= \underline{\chi}_{BC}\nabla\!\!\!/_C U_{\underline{A}} - \underline{\xi}_B\nabla\!\!\!/_4 U_{\underline{A}} - b^{-1}\nabla_B b\nabla\!\!\!/_3 U_{\underline{A}} + \sum_i(-\chi_{A_iB}\underline{\xi}_C \\
&+ \chi_{BC}\underline{\xi}_{A_i} - \underline{\chi}_{A_iB}\zeta_C + \underline{\chi}_{BC}\zeta_{A_i} + \in_{A_iC}{}^*\underline{\beta}_B)U_{A_1..\check{C}..A_m}, \qquad (4.29)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\!\!\!/_3\nabla\!\!\!/_4 U_{\underline{A}} - \nabla\!\!\!/_4\nabla\!\!\!/_3 U_{\underline{A}} &= -\overline{\delta}\nabla\!\!\!/_3 U_{\underline{A}} + (\delta + n^{-1}\nabla_N n)\nabla\!\!\!/_4 U_{\underline{A}} + 2(\zeta_B - \underline{\zeta}_B)\nabla\!\!\!/_B U_{\underline{A}} \qquad (4.30)\\
&+ 2\sum_i(\underline{\zeta}_{A_i}\zeta_C - \underline{\zeta}_C\zeta_{A_i} + \in_{A_iC}{}^*\sigma)U_{A_1..\check{C}..A_m}.
\end{aligned}
$$

4.1.4. Bianchi identities. In view of the formulas on p. 161 of [**9**], the Bianchi equations for $\alpha, \beta, \rho, \sigma, \underline{\beta}$ are:

$$\nabla\!\!\!/_L(\beta) = d\!\!\!/iv\alpha - \overline{\delta}\beta + (2\epsilon - \overline{\epsilon})\cdot\alpha \tag{4.31}$$

$$\nabla\!\!\!/_{\underline{L}}(\beta) = \nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^* + 2\widehat{\chi}\cdot\underline{\beta} + (\delta + n^{-1}\nabla_N n)\beta + \underline{\xi}\cdot\alpha + 3(\zeta\rho + {}^*\zeta\sigma) \tag{4.32}$$

$$L(\rho) = d\!\!\!/iv\beta - \frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha + (\epsilon - 2\overline{\epsilon})\cdot\beta \tag{4.33}$$

$$\underline{L}(\rho) = -d\!\!\!/iv\underline{\beta} - \frac{1}{2}\widehat{\chi}\cdot\underline{\alpha} + 2\underline{\xi}\cdot\beta + (\epsilon - 2\zeta)\cdot\underline{\beta} \tag{4.34}$$

$$L(\sigma) = -cu\!\!\!/rl\beta + \frac{1}{2}\widehat{\underline{\chi}}{}^*\alpha + (-\epsilon + 2\overline{\epsilon}){}^*\beta \tag{4.35}$$

$$\underline{L}(\sigma) = -cu\!\!\!/rl\underline{\beta} - \frac{1}{2}\widehat{\chi}{}^*\underline{\alpha} - 2\underline{\xi}{}^*\beta + (\epsilon - 2\zeta){}^*\underline{\beta} \tag{4.36}$$

$$\nabla\!\!\!/_L(\underline{\beta}) = -\nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^* + 2\widehat{\underline{\chi}}\cdot\beta + \overline{\delta}\underline{\beta} - 3(\underline{\zeta}\rho - {}^*\underline{\zeta}\sigma) \tag{4.37}$$

4.1.5. Main results. We introduce the L^2 curvature flux $\mathcal{R}$ relative to the time foliation:

$$\mathcal{R} = \left(\|\alpha\|^2_{L^2(\mathcal{H}_u)} + \|\beta\|^2_{L^2(\mathcal{H}_u)} + \|\rho\|^2_{L^2(\mathcal{H}_u)} + \|\sigma\|^2_{L^2(\mathcal{H}_u)} + \|\underline{\beta}\|^2_{L^2(\mathcal{H}_u)}\right)^{\frac{1}{2}}. \tag{4.38}$$

In view of Remark 4.6, we have $\mathcal{R} = \|\mathbf{R}\cdot L\|_{L^2(\mathcal{H}_u)}$. Thus, the we may rewrite the bootstrap assumptions of Chapter 2 on $\mathbf{R}$ as:

$$\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \leq M\varepsilon, \ \sup_u \mathcal{R} \leq M\varepsilon.$$

To ease the notations, we drop the bootstrap constant M:

$$\|\mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \leq \varepsilon, \ \sup_u \mathcal{R} \leq \varepsilon. \tag{4.39}$$

The goal of this part is to control the geometry of the null hypersurfaces $\mathcal{H}_u$ of u up to time $t = 1$ when only assuming the smallness assumption (4.39).

REMARK 4.13. *In the rest of the chapter, all inequalities, except the ones of Theorem 4.16 below, hold for any u with the constant in the right-hand side being independent of u. Thus, one may take the supremum in u in these inequalities. To ease the notations, we do not explicitly write down the supremum in u in these estimates.*

u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ depending on a extra parameter $\omega \in \mathbb{S}^2$. Now, for u to be uniquely defined, we need to prescribe it on Σ_0 (i.e. at $t = 0$). This issue has been settled in [**42**] (see also Chapter 6). From now on, we assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ which is prescribed on Σ_0 as in [**42**].

REMARK 4.14. *In the rest of the chapter, all inequalities hold for any $\omega \in \mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.*

We define some norms on $\mathcal{H}_u$. For any $1 \leq p \leq +\infty$ and for any tensor F on $\mathcal{H}_u$, we have:

$$\|F\|_{L^p(\mathcal{H}_u)} = \left(\int_0^1 dt \int_{P_{t,u}} |F|^p d\mu_{t,u}\right)^{\frac{1}{p}},$$

where $d\mu_{t,u}$ denotes the area element of $P_{t,u}$. We also introduce the following norms:

$$\mathcal{N}_1(F) = \|F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_L F\|_{L^2(\mathcal{H}_u)},$$
$$\mathcal{N}_2(F) = \mathcal{N}_1(F) + \|\nabla\!\!\!/^2 F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/ \nabla\!\!\!/_L F\|_{L^2(\mathcal{H}_u)}.$$

Let x' a coordinate system on $P_{0,u}$. By transporting this coordinate system along the null geodesics generated by L, we obtain a coordinate system (t, x') of $\mathcal{H}$. We define the following norms:

$$\|F\|_{L^\infty_{x'} L^2_t} = \sup_{x' \in P_{0,u}} \left(\int_0^1 |F(t,x')|^2 dt\right)^{\frac{1}{2}},$$

$$\|F\|_{L^2_{x'}L^\infty_t} = \left\| \sup_{0\le t\le 1} |F(t,x'))| \right\|_{L^2(P_{0,u})}.$$

The following theorem investigates the regularity of u with respect to (t,x):

THEOREM 4.15. *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in [**42**]. Assume also that the estimate (4.39) is satisfied. Then, null geodesics generating $\mathcal{H}_u$ do not have conjugate points and distinct null geodesics do not intersect. Furthermore, the following estimates are satisfied:*

$$\|n-1\|_{L^\infty} + \|\nabla n\|_{L^\infty_t L^2_{x'}} + \|\nabla^2 n\|_{L^\infty_t L^2_{x'}} + \|\nabla \mathbf{D}_T n\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon, \tag{4.40}$$

$$\mathcal{N}_1(k) + \|\nabla\!\!\!/_{\underline{L}}\epsilon\|_{L^2(\mathcal{H}_u)} + \|\underline{L}(\delta)\|_{L^2(\mathcal{H}_u)} + \|\bar{\epsilon}\|_{L^\infty_{x'}L^2_t} + \|\bar{\delta}\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon, \tag{4.41}$$

$$\|b-1\|_{L^\infty} + \mathcal{N}_2(b) + \|\underline{L}(b)\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon, \tag{4.42}$$

$$\|tr\chi\|_{L^\infty} + \|\nabla\!\!\!/ tr\chi\|_{L^2_{x'}L^\infty_t} + \|\underline{L}tr\chi\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon, \tag{4.43}$$

$$\|\widehat{\chi}\|_{L^2_{x'}L^\infty_t} + \mathcal{N}_1(\widehat{\chi}) + \|\nabla\!\!\!/_{\underline{L}}\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon, \tag{4.44}$$

$$\|\zeta\|_{L^2_{x'}L^\infty_t} + \mathcal{N}_1(\zeta) \lesssim \varepsilon. \tag{4.45}$$

We introduce the family of intrinsic Littlewood-Paley projections P_j which have been constructed in [**24**] using the heat flow on the surfaces $P_{t,u}$ (see also section 3.1.5). This allows us to state our second theorem which investigates the regularity of $\underline{L}\,\underline{L}\mathrm{tr}\chi$ and $\nabla\!\!\!/_{\underline{L}}\zeta$.

THEOREM 4.16. *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in [**42**]. Assume also that the assumption (4.39) is satisfied. Then, there exists a function λ in $L^2(\mathbb{R})$ such that for all $j\ge 0$, we have:*

$$\|P_j\,\underline{L}\,\underline{L}tr\chi\|_{L^2(\mathcal{H}_u)} \lesssim 2^j\varepsilon + 2^{\frac{j}{2}}\lambda(u), \tag{4.46}$$

and

$$\|P_j\nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon + 2^{-\frac{j}{2}}\lambda(u). \tag{4.47}$$

The following theorem investigates the regularity with respect to the parameter $\omega\in\mathbb{S}^2$.

THEOREM 4.17. *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in [**42**]. Assume also that the estimate (4.39) is satisfied. Then, we have the following estimates:*

$$\|\partial_\omega N\|_{L^\infty} \lesssim 1, \tag{4.48}$$

$$\|\mathbf{D}\partial_\omega N\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega b\|_{L^\infty} + \|\nabla\!\!\!/\partial_\omega b\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega\chi\|_{L^2_{x'}L^\infty_t} + \|\partial_\omega\zeta\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon. \tag{4.49}$$

Furthermore, we have the following decomposition for $\widehat{\chi}$:

$$\widehat{\chi} = \chi_1 + \chi_2, \tag{4.50}$$

where χ_1 and χ_2 are two symmetric traceless $P_{t,u}$-tangent 2-tensors satisfying:

$$\mathcal{N}_1(\chi_1) + \|\nabla\!\!\!/_{\underline{L}}\chi_1\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega\chi_1\|_{L^\infty_t L^2_{x'}} + \mathcal{N}_1(\chi_2) + \|\nabla\!\!\!/_{\underline{L}}\chi_2\|_{L^2(\mathcal{H}_u)} + \|\partial_\omega\chi_2\|_{L^\infty_t L^2_{x'}} \lesssim \varepsilon \tag{4.51}$$

and for any $2 \leq p < +\infty$, we have:

$$\|\chi_1\|_{L^p_t L^\infty_{x'}} + \|\partial_\omega \chi_2\|_{L^p_t L^{4_-}_{x'}} + \|\partial_\omega \chi_2\|_{L^{6_-}(\mathcal{H}_u)} + \|\nabla \partial_\omega \chi_2\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon, \tag{4.52}$$

where for any real number a, $a_- = a - \delta$ for any $\delta > 0$.

REMARK 4.18. *Notice from (4.51) that χ_1 and χ_2 have at least the same regularity as $\widehat{\chi}$. Now, the point of the decomposition (4.50) is that both χ_1 and χ_2 have better regularity properties than $\widehat{\chi}$. Indeed, in view of (4.52), χ_1 has better regularity with respect to (t, x) while χ_2 has better regularity with respect to ω.*

Next, the following theorem contains estimates for second order derivatives with respect to ω.

THEOREM 4.19. *Assume that u is the solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ such that u is prescribed on Σ_0 as in* [**42**]. *Assume also that the estimate (4.39) is satisfied. Then, we have the following estimates:*

$$\|\partial^2_\omega N\|_{L^2_{x'}, L^\infty_t} \lesssim 1, \tag{4.53}$$

$$\|\nabla_L \Pi(\partial^2_\omega N)\|_{L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{4.54}$$

$$\|P_j \nabla_{\underline{L}} \Pi(\partial^2_\omega N)\|_{L^p_t L^2_{x'}} + \|P_j \Pi(\partial^2_\omega \chi)\|_{L^\infty_t L^2_{x'}} + \|P_j \Pi(\partial^2_\omega \zeta)\|_{L^p_t L^2_{x'}} \lesssim 2^j \varepsilon, \tag{4.55}$$

where p is any real number such that $2 \leq p < +\infty$, and where Π denotes the projection on $P_{t,u}$-tangent tensors.

Finally, we need to compare quantities evaluated at two angles ω and ν. The following decompositions are used in sections 3.4 and 3.5

THEOREM 4.20. *Let ω and ν in $\mathbb{S}^2$ such that $|\omega - \nu| \lesssim 2^{-\frac{j}{2}}$. Let $u = u(., \omega)$, $N = N(., \omega)$ and $N_\nu = N(., \nu)$. For any $j \geq 0$, we have the following decomposition for $N - N_\nu$:*

$$2^{\frac{j}{2}}(N - N_\nu) = F^j_1 + F^j_2 \tag{4.56}$$

where the tensor F^j_1 does not depend on ω and satisfies:

$$\|F^j_1\|_{L^\infty} \lesssim 1,$$

and where the tensor F^j_2 satisfies:

$$\|F^j_2\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim 2^{-\frac{j}{2}}.$$

We also have following decomposition for $tr\chi$:

$$tr\chi = f^j_1 + f^j_2 \tag{4.57}$$

where the scalar f^j_1 does not depend on ω and satisfies:

$$\|f^j_1\|_{L^\infty} \lesssim \varepsilon,$$

and where the scalar f^j_2 satisfies:

$$\|f^j_2\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon 2^{-\frac{j}{2}}.$$

Let us conclude this section by mentioning several ingredients of [**44**] that have been omitted here for the sake of simplicity:

- estimates for the transport equations along L, and the elliptic systems of Hodge type on $P_{t,u}$ involved in the null structure equations
- embeddings on $\mathcal{H}_u$, Σ_t and $P_{t,u}$
- geometric Littlewood-Paley projections and Besov spaces on Σ_t
- control of the Gauss curvature of $P_{t,u}$
- Bochner inequalities on Σ_t and $P_{t,u}$
- estimates for various commutator terms of the type: $[\mathbf{D}_L, \nabla\!\!\!/]$, $[\mathbf{D}_{\underline{L}}, \nabla\!\!\!/]$, $[\mathbf{D}_L, P_j]$, $[\mathbf{D}_{\underline{L}}, P_j]$, ...

4.2. Regularity of the foliation with respect to (t,x)

In this section, we outline the main ideas of the proof of Theorem 4.15. We assume the following bootstrap assumptions:

$$\|n-1\|_{L^\infty(\mathcal{H}_u)} + \|b-1\|_{L^\infty(\mathcal{H}_u)} \le \frac{1}{10}, \tag{4.58}$$

$$\|\nabla n\|_{L^\infty_t L^2_{x'}} + \|\nabla^2 n\|_{L^\infty_t L^2_{x'}} + \|\nabla \mathbf{D}_T n\|_{L^\infty_t L^2_{x'}} + \mathcal{N}_2(b) + \|\underline{L}(b)\|_{L^2_{x'} L^\infty_t} \le D\varepsilon, \tag{4.59}$$

$$\mathcal{N}_1(k) + \|\nabla\!\!\!/_{\underline{L}}\epsilon\|_{L^2(\mathcal{H}_u)} + \|\mathbf{D}_{\underline{L}}\delta\|_{L^2(\mathcal{H}_u)} + \|\bar{\epsilon}\|_{L^\infty_{x'} L^2_t} + \|\bar{\delta}\|_{L^\infty_{x'} L^2_t} \le D\varepsilon, \tag{4.60}$$

$$\|\mathrm{tr}\chi\|_{L^\infty(\mathcal{H}_u)} + \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2_{x'} L^\infty_t} + \|\underline{L}\mathrm{tr}\chi\|_{L^2_{x'} L^\infty_t} \le D\varepsilon, \tag{4.61}$$

$$\|\widehat{\chi}\|_{L^2_{x'} L^\infty_t} + \mathcal{N}_1(\widehat{\chi}) + \|\nabla\!\!\!/_{\underline{L}}\widehat{\chi}\|_{L^2(\mathcal{H}_u)} \le D\varepsilon, \tag{4.62}$$

$$\|\zeta\|_{L^2_{x'} L^\infty_t} + \mathcal{N}_1(\zeta) \le D\varepsilon, \tag{4.63}$$

where $D > 0$ is a large enough constant. We will improve on these estimates.

4.2.1. Non intersection of null geodesics on $\mathcal{H}_u$. The control we obtain on the geometric quantities associated to our foliation is only valid as long as there are no conjugate points and null geodesics do not intersect. The goal of this section is to prove that this holds at least until $t = 1$. In addition to the bound (4.39) on the curvature tensor $\mathbf{R}$ of $\mathbf{g}$, we make the following regularity assumption on $\mathbf{g}$. There exists a coordinate chart such that

$$\|\mathbf{g}\|_{C^2(\mathcal{M})} \le M, \tag{4.64}$$

where M is a very large constant.

REMARK 4.21. *The assumption* (4.64) *is only used to prove the absence of caustic and that null geodesics do not intersect at least until* $t = 1$*, which is a qualitative property. On the other hand, we only rely on the bound* (4.39) *on* $\mathbf{R}$ *to prove the various quantitative bounds of Theorems 4.15, 4.16, 4.17 and 4.19.*

For $(0,x)$ in Σ_0, recall the definition in Remark 4.2 of the null geodesic $\kappa_x(t)$. For all $0 \le t \le 1$, let $\Phi_t : \Sigma \to \Sigma_t$ defined by $\Phi_t(0,x) = \kappa_x(t)$. We have $\Phi_0(0,x) = (0,x)$ on Σ_0. We define $t_0 \ge 0$ as the supremum of $0 \le t \le 1$ such that Φ_t is bijective from Σ_0 to Σ_t.

Remark 4.22. *As long as $0 \leq t < t_0$, there are no conjugate points and no distinct null geodesic intersections. Thus, we may assume that the u-foliation exists and satisfies the bounds* (4.58)-(4.63) *given by the bootstrap assumptions. Furthermore, we may assume the identity* (4.7) *for the null geodesics $\kappa_x(t)$.*

Our goal is to show that we have in fact $t_0 = 1$. We proceed in three steps (see [**44**] for the details):

Step 1. As noticed in Remark 4.22, the L^∞ bound for $\mathrm{tr}\chi$ given by (4.61) holds for $0 \leq t < t_0$. Furthermore, using the Raychaudhuri equation (4.19) and the bound (4.64), we obtain the existence of a constant $\delta > 0$ depending on M such that the L^∞ bound for $\mathrm{tr}\chi$ given by (4.61) holds for $0 \leq t < t_0 + \delta$. This control for $\mathrm{tr}\chi$ allows us to prove that there are no conjugate points on $0 \leq t < t_0 + \delta$.

Step 2. Next, we prove that $\Sigma_t = \cup_u P_{t,u}$ for $0 \leq t \leq t_0 + \delta$ where $\delta > 0$ is a constant depending on M. This requires the bound (4.64), and the control it induces on forward and backward light cones for small time intervals with a size depending on M.

Step 3. Assume now that $0 < t_0 < 1$. In view of *Step 1* and *Step 2*, the only thing that can go wrong at $t = t_0$ is that two distinct null geodesics intersect in Σ_{t_0}. Assume by contradiction that this is indeed the case so that there exists $(0, x_1) \neq (0, x_2)$ two points in Σ_0 such that $\kappa_{x_1}(t_0) = \kappa_{x_2}(t_0) = (t_0, x_0)$. Since

$$\kappa'_{x_j}(t) = b(\kappa_{x_j(t)})^{-1} L_{\kappa_{x_j(t)}},\ j = 1, 2,$$

in view of Remark 4.2, the regularity of b and L yields $\kappa'_{x_1}(t_0) = \kappa'_{x_2}(t_0)$. From the classical uniqueness result for ODEs, we deduce that $\kappa_{x_1}(t) = \kappa_{x_2}(t)$ for all t. In particular, taking $t = 0$, we obtain $(0, x_1) = (0, x_2)$ which yields a contradiction.

Finally, Steps 1, 2 and 3, yield $t_0 \geq 1$. In particular, we have:

On $0 \leq t \leq 1,$ there are no conjugate points and no intersection of distinct null geodesics. In particular, u exists on $0 \leq t \leq 1$ and the bootstrap assumptions (4.58)-(4.63) hold. Furthermore, $\Sigma_t = \cup_u P_{t,u}$ for all $0 \leq t \leq 1$. (4.65)

4.2.2. Lower bound on the volume radius of Σ_t. In this section, we prove the lower bound on the volume radius of Σ_t given by the estimate (2.32). We use the global coordinate system $x' = (x^1, x^2)$ on $P_{0,u}$ which has been constructed in [**42**] (see also Proposition 6.13). Transporting this coordinate system along the null geodesics generated by L, we obtain a coordinate system x' of $P_{t,u}$, which in particular satisfies

$$(1 - O(\varepsilon))|\xi|^2 \leq \gamma_{AB}(p)\xi^A\xi^B \leq (1 + O(\varepsilon))|\xi|^2, \qquad \text{uniformly for all } \ p \in P_{t,u}, \quad (4.66)$$

where γ is the metric induces by $\mathbf{g}$ on $P_{t,u}$. We denote by x' this global coordinate system on $P_{t,u}$.

Next, we obtain a global coordinate system on Σ_t as follows. First, recall from (4.65) that $\Sigma_t = \cup P_{t,u}$ so that u is defined on Σ_t. To any $p \in \Sigma_t$, we associate the coordinates $(u(p), x'(p))$ where $u(p)$ is the value of the optical function u at p, and $x'(p)$ are the coordinate of p in the coordinate system of $P_{t,u}$. In this coordinate system, the metric g_t on Σ_t (i.e. the restriction of $\mathbf{g}$ on Σ_t) takes the following form:

$$g_t = \begin{pmatrix} b^{-2} & 0 \\ 0 & \gamma \end{pmatrix}, \tag{4.67}$$

where γ is the induced metric on $P_{t,u}$. Together with the estimate (4.58) for b and (4.66) for γ, we obtain the following lower bound on the volume radius of Σ_t at scales ≤ 1:

$$r_{vol}(\Sigma_t, 1) \geq \frac{1}{4}, \tag{4.68}$$

which is the estimate (2.32).

4.2.3. Estimates for the second fundamental form k and the lapse n. We first estimate k on Σ_t. k satisfies the following symmetric Hodge system on Σ_t:

$$\begin{cases} \text{curl}k_{ij} = {}^*\mathbf{R}_{\mu i \nu j} T^\mu T^\nu, \\ \nabla^j k_{ij} = 0, \\ \text{tr}k = 0, \end{cases} \tag{4.69}$$

where $\text{curl}k_{ij} = \frac{1}{2}(\in_i^{lm} \nabla_l k_{mj} + \in_j^{lm} \nabla_l k_{mi})$ and $\text{tr}k = g^{ij}k_{ij}$. Using an elliptic estimate for the Hodge system (4.69), we easily obtain:

$$\|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon. \tag{4.70}$$

Recall from (1.9) that the lapse n satisfies the following elliptic equation on Σ_t:

$$\Delta n = |k|^2 n. \tag{4.71}$$

Using (4.71) and (4.70), together with elliptic estimates on Σ_t, we improve the estimate for n in the bootstrap assumptions (4.58) (4.59). We also prove the following estimate which is needed for the estimate (2.62)

$$\|\nabla n\|_{L^\infty(\mathcal{M})} \lesssim \varepsilon. \tag{4.72}$$

Using (4.71) and (4.70) together with the Sobolev embedding on the three dimensional riemannian manifold Σ_t yields $\Delta n \in L_t^\infty L^3(\Sigma_t)$. Together with elliptic estimates, this implies $\nabla^2 n \in L_t^\infty L^3(\Sigma_t)$, and thus ∇n misses to be in $L^\infty(\mathcal{M})$ by a log divergence. However, one can overcome this loss by exploiting the Besov improvement with respect to the Sobolev embedding on Σ_t. This requires to introduce a geometric Littlewood-Paley theory on Σ_t[2]. We refer the reader to section 4.4 in [**44**] for the details.

Finally, we estimate k on $\mathcal{H}_u$. To this end, we use the decomposition of k (4.14) in δ, ϵ and η, and obtain a Hodge system for δ, ϵ and η on $\mathcal{H}_u$. This allows us to derive the following estimate

$$\mathcal{N}_1(k) \lesssim \mathcal{N}_1(\eta) + \mathcal{N}_1(\epsilon) + \mathcal{N}_1(\delta) \lesssim \varepsilon. \tag{4.73}$$

[2]Note that we use a geometric construction based on the heat flow on Σ_t since we don't have enough regularity for the metric in order to use a coordinate dependent Littlewood-Paley decomposition

Deriving an estimate for $T(\delta)$ and $\nabla\!\!\!/_T\epsilon$, together with (4.73), then yields

$$\|\mathbf{D}_{\underline{L}}\delta\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{\underline{L}}\epsilon\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \varepsilon. \tag{4.74}$$

4.2.4. Time foliation versus geodesic foliation. While we work with a time foliation, we recall that the estimates corresponding to the bootstrap assumptions on χ and ζ have already been proved in the context of a geodesic foliation in [**22**] [**24**] [**23**]. One may reprove these estimates by adapting the proofs to the context of a time foliation. However, this would be rather lengthy and we suggest a more elegant solution which consists in translating certain estimates from the geodesic foliation to the time foliation, and in obtaining directly the rest of the estimates. More precisely, we wish to obtain the L^∞ bound from $\mathrm{tr}\chi$, and the trace bounds for $\widehat{\chi}$ and ζ by exploiting the corresponding estimates in the geodesic foliation. We will obtain the trace bounds for δ and ϵ by reducing to estimates in the geodesic foliation in section 4.2.5. Finally, these trace bounds and the null structure equations will allow us to get all the remaining estimates in section 4.2.6. We start by recalling some of the results obtained in the context of the geodesic foliation in [**22**] [**24**] [**23**].

4.2.4.1. *The case of the geodesic foliation.* Recall that $L' = -\mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha$ is the geodesic null generator of $\mathcal{H}_u$. Let s denote its affine parameter, i.e. $L'(s) = 1$. We denote by $P'_{s,u}$ the level surfaces of s in $\mathcal{H}_u$.

DEFINITION 4.23. *A null frame e'_1, e'_2, e'_3, e'_4 at a point $p \in P'_{s,u}$ consists, in addition to $e'_4 = L'$, of arbitrary orthonormal vectors e'_1, e'_2 tangent to $P'_{s,u}$ and the unique vectorfield $e'_3 = \underline{L}'$ satisfying the relations:*

$$\mathbf{g}(e'_3, e'_4) = -2,\ \mathbf{g}(e'_3, e'_3) = 0,\ \mathbf{g}(e'_3, e'_1) = 0,\ \mathbf{g}(e'_3, e'_2) = 0.$$

DEFINITION 4.24 (*Ricci coefficients in the geodesic foliation*). *Let e'_1, e'_2, e'_3, e'_4 be a null frame on $P'_{s,u}$ as above. The following tensors on $P'_{s,u}$*

$$\begin{array}{ll} \chi'_{AB} = <\mathbf{D}_{e'_A} e'_4, e'_B>, & \underline{\chi}'_{AB} = <\mathbf{D}_{e'_A} e'_3, e'_B>, \\ \zeta'_A = \frac{1}{2} <\mathbf{D}_{e'_A} e'_4, e'_3> & \end{array} \tag{4.75}$$

are called the Ricci coefficients associated to the geodesic foliation.

We decompose χ' and $\underline{\chi}'$ into their trace and traceless components.

$$tr\chi' = \mathbf{g}^{AB}\chi'_{AB}, \qquad tr\underline{\chi}' = \mathbf{g}^{AB}\underline{\chi}'_{AB}, \tag{4.76}$$

$$\widehat{\chi}'_{AB} = \chi'_{AB} - \frac{1}{2}tr\chi'\mathbf{g}_{AB}, \quad \widehat{\underline{\chi}}'_{AB} = \underline{\chi}'_{AB} - \frac{1}{2}tr\underline{\chi}'\mathbf{g}_{AB}. \tag{4.77}$$

DEFINITION 4.25. *The null components of the curvature tensor* $\mathbf{R}$ *of the space-time metric* $\mathbf{g}$ *in the geodesic foliation are given by:*

$$\alpha'_{ab} = \mathbf{R}(L', e'_a, L', e'_b), \qquad \beta'_a = \frac{1}{2}\mathbf{R}(e'_a, L', \underline{L}', L'), \tag{4.78}$$

$$\rho' = \frac{1}{4}\mathbf{R}(\underline{L}', L', \underline{L}', L'), \qquad \sigma' = \frac{1}{4}{}^\star\mathbf{R}(\underline{L}', L', \underline{L}', L') \tag{4.79}$$

$$\underline{\beta}'_a = \frac{1}{2}\mathbf{R}(e'_a, \underline{L}', \underline{L}', L'), \qquad \underline{\alpha}'_{ab} = \mathbf{R}(\underline{L}', e'_a, \underline{L}', e'_b) \tag{4.80}$$

where ${}^\star\mathbf{R}$ *denotes the Hodge dual of* $\mathbf{R}$.

We now recall the main estimates obtained in [**22**] [**24**] [**23**]. We have:

$$\|\mathrm{tr}\chi'\|_{L^\infty(\mathcal{H}_u)} + \|\widehat{\chi}'\|_{L^2_{x'}L^\infty_s} + \|\zeta'\|_{L^2_{x'}L^\infty_s} \lesssim \varepsilon \tag{4.81}$$

and

$$\|\underline{\chi}'\|_{L^2_{x'}L^\infty_s} + \mathcal{N}'_1(\chi') + \mathcal{N}'_1(\zeta') \lesssim \varepsilon, \tag{4.82}$$

where the norm $\mathcal{N}'_1$ is given by

$$\mathcal{N}'_1(F) = \|F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/'F\|_{L^2(\mathcal{H}_u)} + \|\nabla\!\!\!/_{L'}F\|_{L^2(\mathcal{H}_u)}.$$

REMARK 4.26. *Note that the norm* $L^\infty(\mathcal{H}_u)$ *does not depend on the particular foliation. Now, this is also the case for the trace norm* $L^2_{x'}L^\infty_s$. *Indeed, recall the definition of the null geodesic* κ_x *in Remark 4.2. Then, we have:*

$$\|F\|^2_{L^\infty_{x'}L^2_t} = \sup_{(0,x)\in\Sigma_0}\int_0^1 |F(\kappa_x(t))|^2 dt = \sup_{(0,x)\in\Sigma_0}\int_0^1 |F(\kappa_x(s))|^2 n^{-1}b^{-1} ds \sim \|F\|^2_{L^\infty_{x'}L^2_s}$$

where we used the fact that $\frac{dt}{ds} = n^{-1}b^{-1}$ *and the fact that* $nb \sim 1$ *by the bootstrap assumption* (4.58).

In the next section, we will obtain the estimates corresponding to (4.81) in the time foliation. For now, we conclude this section by recalling the definition and some properties of the Besov spaces constructed in [**22**] [**24**] [**23**]. For $P'_{s,u}$-tangent tensors F on $\mathcal{H}_u$, $0 \leq a \leq 1$, we introduce the Besov norms:

$$\|F\|_{\mathcal{B}'^a} = \sum_{j\geq 0} 2^{ja}\sup_{0\leq s\leq 1}\|P'_jF\|_{L^2(P'_{s,u})} + \sup_{0\leq s\leq 1}\|P'_{<0}F\|_{L^2(P'_{s,u})}, \tag{4.83}$$

$$\|F\|_{\mathcal{P}'^a} = \sum_{j\geq 0} 2^{ja}\|P'_jF\|_{L^2(\mathcal{H}_u)} + \|P'_{<0}F\|_{L^2(\mathcal{H}_u)} \tag{4.84}$$

where P'_j are the geometric Littlewood-Paley projections on the 2-surfaces $P'_{s,u}$. Using the definition of these Besov spaces, we have (see [**22**] [**24**] [**23**])

$$\|\underline{\chi}'\|_{\mathcal{B}'^0} \lesssim \varepsilon. \tag{4.85}$$

Furthermore, we have for scalar functions on $\mathcal{H}_u$ (see [**22**] section 5):

$$\|f\|_{L^\infty(\mathcal{H}_u)} \lesssim \|f\|_{\mathcal{B}'^1} \lesssim \|f\|_{L^\infty_s L^2_{x'}} + \|\nabla\!\!\!/'f\|_{\mathcal{B}'^0}. \tag{4.86}$$

Finally, we have the following version of the sharp classical trace theorem (see Corollary 4.21 in [**44**] for a proof).

PROPOSITION 4.27. *Assume F is an $P'_{s,u}$-tangent tensor which admits a decomposition of the form, $\nabla\!\!\!/' F = A\nabla\!\!\!/_{L'}P + E$. Then,*

$$\|F\|_{L^\infty_{x'}L^2_s} \lesssim \mathcal{N}'_1(F) + \mathcal{N}'_1(P)(\|A\|_{L^\infty} + \|\nabla\!\!\!/' A\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'} A\|_{L^2_{x'}L^\infty_s}) + \|E\|_{\mathcal{P}'^0}. \tag{4.87}$$

4.2.4.2. *Estimates in the time foliation.* In this section, we obtain the L^∞ bound for $\mathrm{tr}\chi$, and the trace bounds for $\widehat{\chi}$ and ζ by relying on the corresponding estimates in the geodesic foliation (4.81). We start by establishing the relation between the Ricci coefficients in the time and in the geodesic foliation. Recall from (4.4) that $L = bL'$. Since (e_1, e_2) and (e'_1, e'_2) are both orthonormal vectors in the tangent space of $\mathcal{H}_u$ which are both orthogonal to L, we may chose these vectors such that there is a tensor F' on $P'_{s,u}$ satisfying:

$$e_A = e'_A + F'_A L',\ A = 1, 2.$$

We then easily express $\underline{L}$ in the frame $(L', \underline{L}', e'_A)$. Finally, we have the following relations:

$$\begin{array}{l} L = bL', \\ e_A = e'_A + F'_A L',\ A = 1, 2, \\ \underline{L} = b^{-1}\,\underline{L}' + 2b^{-1}F'_A e'_A + b^{-1}|F'|^2 L'. \end{array} \tag{4.88}$$

Next, using the definition (4.8) and (4.75) of the Ricci coefficients respectively in the time and geodesic foliation, and the identities (4.88), we easily obtain

$$\chi = b\chi',\ \mathrm{tr}\chi = b\mathrm{tr}\chi',\ \widehat{\chi} = b\widehat{\chi}',\ \zeta_A = \zeta'_A + \chi'_{AC}F'_C. \tag{4.89}$$

(4.89) together with the bootstrap assumption (4.58) and the estimate (4.81) yields:

$$\begin{array}{l} \|\mathrm{tr}\chi\|_{L^\infty(\mathcal{H}_u)} \leq \|b\|_{L^\infty(\mathcal{H}_u)}\|\mathrm{tr}\chi'\|_{L^\infty(\mathcal{H}_u)} \lesssim \varepsilon, \\ \|\widehat{\chi}\|_{L^\infty_{x'}L^2_t} \leq \|b\|_{L^\infty(\mathcal{H}_u)}\|\widehat{\chi}'\|_{L^2_{x'}L^\infty_s} \lesssim \varepsilon, \\ \|\zeta\|_{L^\infty_{x'}L^2_t} \lesssim \|\zeta'\|_{L^\infty_{x'}L^2_s} + \|\chi'\|_{L^\infty_{x'}L^2_s}\|F'\|_{L^\infty} \lesssim \varepsilon + \varepsilon\|F'\|_{L^\infty}, \end{array} \tag{4.90}$$

where we have used the fact that the trace norms $L^2_{x'}L^\infty_t$ and $L^2_{x'}L^\infty_s$ are equivalent by Remark 4.26.

In view of the trace estimate for ζ given by (4.90), we need to estimate $\|F'\|_{L^\infty}$. To this end, we estimate $\nabla\!\!\!/' F'$. Using the definition (4.8) of $\underline{\chi}$ and (4.75) of $\underline{\chi}'$, and the identities (4.88), we obtain:

$$\mathbf{g}(\mathbf{D}_{e'_A}F', e'_B) = -\frac{1}{2}\underline{\chi}'_{AB} + \cdots,$$

where we only kept the main term. Together with the estimate for $\underline{\chi}'$ (4.85), this yields

$$\|\nabla\!\!\!/' F'\|_{\mathcal{B}'^0} \lesssim D\varepsilon$$

which together with (4.86) implies:

$$\|F'\|_{L^\infty} \lesssim D\varepsilon. \tag{4.91}$$

In particular, (4.90) and (4.91) imply:

$$\|\zeta\|_{L^\infty_{x'}L^2_t} \lesssim \varepsilon. \tag{4.92}$$

Note that (4.90) and (4.92) are improvements of the corresponding estimates in the bootstrap assumptions (4.61)-(4.63).

4.2.5. Trace norm bounds for $\overline{\delta}$ and $\overline{\epsilon}$. The goal of this section is to improve the estimate for $\|\overline{\delta}\|_{L^\infty_{x'}L^2_t}$ and $\|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t}$ given by the bootstrap assumption (4.60), where $\overline{\delta}$ and $\overline{\epsilon}$ are defined in (4.16). Let us first define k_{LL} and k_{LA}:

$$k_{LL} = -\mathbf{g}(\mathbf{D}_L T, L),\ k_{LA} = -\mathbf{g}(\mathbf{D}_L T, e_A),\ A = 1, 2. \tag{4.93}$$

Then, using in particular the definition (4.16), we have:

$$\overline{\delta} = k_{LL} \text{ and } \overline{\epsilon}_A = k_{LA}. \tag{4.94}$$

We also define $k_{L'L'}$ and $k_{L'A}$:

$$k_{L'L'} = -\mathbf{g}(\mathbf{D}_{L'} T, L'),\ k_{L'A} = -\mathbf{g}(\mathbf{D}_{L'} T, e'_A),\ A = 1, 2. \tag{4.95}$$

Then, the relations (4.88) between L, e_1, e_2 and L', e'_1, e'_2 together with the definitions (4.93) and (4.95) yield:

$$k_{LL} = b^2 k_{L'L'} \text{ and } k_{LA} = b k_{L'A} + b F'_A k_{L'L'}. \tag{4.96}$$

Thus, (4.94) and (4.96) imply:

$$\begin{aligned} \|\overline{\delta}\|_{L^\infty_{x'}L^2_t} &\lesssim \|b k_{L'L'}\|_{L^\infty_{x'}L^2_s} \lesssim \|k_{L'L'}\|_{L^\infty_{x'}L^2_s} \\ \|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t} &\lesssim \|b k_{L'A}\|_{L^\infty_{x'}L^2_s} + \|b F'_A k_{L'L'}\|_{L^\infty_{x'}L^2_s} \lesssim \|k_{L'L'}\|_{L^\infty_{x'}L^2_s} + \|k_{L'A}\|_{L^\infty_{x'}L^2_s} \end{aligned} \tag{4.97}$$

where we used the bootstrap assumption (4.58), the L^∞ bound for F' (4.91) and Remark 4.26.

In view of (4.97), it is enough to bound the trace norms $\|k_{L'L'}\|_{L^\infty_{x'}L^2_s}$ and $\|k_{L'A}\|_{L^\infty_{x'}L^2_s}$. To this end, we would like to apply the trace estimate (4.87), which requires to show that $\nabla\!\!\!/' k_{L'L'}$ and $\nabla\!\!\!/' k_{L'A}$ admit a decomposition of the form, $A\nabla\!\!\!/_{L'} P + E$. We only discuss the estimate for $k_{L'L'}$, and we refer the reader to [**44**] for $k_{L'A}$. We have:

$$\nabla\!\!\!/'_{e'_A} k_{L'L'} = -\mathbf{D}_{e'_A} \mathbf{g}(\mathbf{D}_{L'} T, L') = -\mathbf{g}(\mathbf{D}_{e'_A} \mathbf{D}_{L'} T, L') - \mathbf{g}(\mathbf{D}_{L'} T, D_{e'_A} L').$$

Introducing the commutator term $[\mathbf{D}_{e'_A}, \mathbf{D}_{L'}]$, and decomposing the corresponding component of $\mathbf{R}$, we obtain

$$\nabla\!\!\!/'_{e'_A} k_{L'L'} = -b^{-1} F'_B \alpha'_{AB} + \cdots, \tag{4.98}$$

where we only kept a typical term for simplicity. Relying on the Bianchi identities, the following decomposition for α' was obtained in [**22**]:

$$\alpha' = \nabla\!\!\!/_{L'}(P) + E, \tag{4.99}$$

where $P = \mathcal{D}'^{-1}_2 \beta'$, and

$$\mathcal{N}'_1(P) + \|E\|_{\mathcal{P}'^0} \lesssim \varepsilon. \tag{4.100}$$

Together with (4.98), we obtain a decomposition of the following form

$$\nabla\!\!\!/'k_{L'L'} = A_1\nabla\!\!\!/_{L'}P_1 + E_1, \tag{4.101}$$

where

$$\|A_1\|_{L^\infty} + \|\nabla\!\!\!/' A_1\|_{L^2_{x'}L^\infty_s} + \|\nabla\!\!\!/_{L'}A_1\|_{L^2_{x'}L^\infty_s} + \mathcal{N}_1'(P_1) + \|E_1\|_{\mathcal{P}'^0} \lesssim \varepsilon. \tag{4.102}$$

Using (4.101), (4.102) and the trace estimate (4.87), we deduce

$$\|k_{L'L'}\|_{L^\infty_{x'}L^2_s} \lesssim \varepsilon,$$

which together with (4.97) allows us to improve the estimate for $\|\overline{\delta}\|_{L^\infty_{x'}L^2_t}$ and $\|\overline{\epsilon}\|_{L^\infty_{x'}L^2_t}$ given by the bootstrap assumption (4.60).

4.2.6. Remaining estimates for trχ, $\widehat{\chi}$, ζ and b. In order to improve the remaining estimates in the bootstrap assumptions (4.58)-(4.63), we use the null structure equation of section 4.1.2, which consists of transport equations along L and Hodge systems on $P_{t,u}$. We refer the reader to section 4.8 in [**44**], where using the L^∞ bound of trχ, the trace estimates for $\widehat{\chi}, \overline{\delta}$ and $\overline{\epsilon}$, and the estimates for the lapse n, we easily obtain the remaining estimates. Thus, there exists a universal constant $D > 0$ such that (4.58)-(4.63) hold. This yields (4.40)-(4.45) which concludes the proof of Theorem 4.15.

4.3. An estimate for $\underline{\mathbf{L}}\mathbf{L}tr\chi$

In this section, we outline the main ideas of the proof of Theorem 4.16. Let $\mu_1 = b\,\underline{L}(\mu)$. Then, we first derive a transport equation for μ_1, and a Hodge system for $\nabla\!\!\!/_{\underline{L}}\zeta$. For simplicity, we only discuss the transport equation for μ_1. We differentiate the transport equation (4.27) satisfied by μ with respect to $\underline{L}$ and multiply it by nb. We also use commutator formulas of section 4.1.3, the Bianchi identity (4.34) for ρ, the curvature bound (4.39) and the estimates (4.40)-(4.45) obtained in Theorem 4.15. We obtain

$$\begin{aligned} nL(\mu_1) + n\mathrm{tr}\chi\mu_1 \;\; = \;\; & -2bn\nabla\!\!\!/_{\underline{L}}(\zeta)\cdot\nabla\!\!\!/\mathrm{tr}\chi - 2bn\widehat{\chi}\cdot\left(\nabla\!\!\!/\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta) + b^{-1}\nabla\!\!\!/ b\nabla\!\!\!/_{\underline{L}}(\zeta) + 2\nabla\!\!\!/_{\underline{L}}\zeta\widehat{\otimes}\zeta\right) \\ & +2n\mathrm{tr}\chi bn^{-1}\nabla\!\!\!/ n\cdot\nabla\!\!\!/_{\underline{L}}(\zeta) + \mathrm{d}\!\!\!/\mathrm{iv}(F_1) + f_2, \end{aligned}$$

where the tensor F_1 and the scalar f_2 satisfy

$$\|F_1\|_{L^2(\mathcal{H}_u)} + \|f_2\|_{L^1(\mathcal{H}_u)} \lesssim \varepsilon.$$

This yields:

$$\|P_j(\mu_1)\|_{L^2(\mathcal{H}_u)} \lesssim 2^{\frac{j}{2}}\lambda(u)\varepsilon + 2^j\varepsilon + \left\|P_j\left(\int_0^t(bn\widehat{\chi}\cdot(\nabla\!\!\!/\widehat{\otimes}\nabla\!\!\!/_{\underline{L}}(\zeta))d\tau\right)\right\|_{L^2(\mathcal{H}_u)} + \cdots. \tag{4.103}$$

Here, the term $2^j\varepsilon$ comes from the estimate for F_1 and f_2 together with Bernstein and the finite band property for P_j, and the term $2^{\frac{j}{2}}\lambda(u)\varepsilon$ comes from the initial data term for the transport equation - i.e. μ_1 at $t = 0$ which is estimated in [**42**] - together with

Bernstein for P_j. We have only kept one typical term in the right-hand side of (4.103) for the sake of simplicity.

In view of the desired estimate (4.47), we are bootstrapping an estimate of the type

$$\|P_j \nabla\!\!\!/_{\underline{L}}(\zeta)\|_{L^2(\mathcal{H}_u)} \lesssim D\varepsilon + D2^{-\frac{j}{2}}\lambda(u)$$

for some large enough bootstrap constant D, and where λ is a function in $L^2(\mathbb{R})$. Thus, estimating directly the term in the right-hand side of (4.103) would yield an upper bound of the type

$$\sum_{l,q} 2^j 2^{-\frac{|q-l|}{2}} \gamma_q^{(1)} \gamma_l^{(2)}, \text{ where } \gamma_q^{(1)} \in \ell^2(\mathbb{N}) \text{ and } \gamma_l^{(2)} \in \ell^\infty(\mathbb{N}) \tag{4.104}$$

which is not summable. Instead, we rely on the following decomposition for $bn\widehat{\chi}$:

$$\nabla\!\!\!/(bn\widehat{\chi}) = \nabla\!\!\!/_{nL} P + E \tag{4.105}$$

where P, E are $P_{t,u}$-tangent tensors, and P, E satisfy:

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \varepsilon.$$

REMARK 4.28. *A similar decomposition has been proved in the geodesic foliation in* [**22**]*, and adapted to the time foliation in the spirit of section 4.2.4. In order to obtain* (4.105)*, we use the fact that the proof in the geodesic foliation relies on a specific structure of certain commutators and of the Bianchi identities, which can be recovered in the time foliation. We refer to* [**44**] *for the details.*

Using the decomposition(4.105), we decompose the term in the right-hand side of (4.103) in a sum of two terms which are estimated as follows (see [**44**] for the details):

- For the term involving $\nabla\!\!\!/_{nL}P$, we integrate by parts in $\nabla\!\!\!/_{nL}$, and consider the term where the L derivative falls on $\nabla\!\!\!/_{\underline{L}}\zeta$. Differentiating the transport equation (4.21) satisfied by ζ with respect to $\nabla\!\!\!/_{\underline{L}}$, commutators formula, and the Bianchi identity (4.32), we obtain
$$\nabla\!\!\!/_{nL}\nabla\!\!\!/_{\underline{L}}\zeta = \nabla\!\!\!/_{nL}\mathcal{D}^0(\underline{\beta}) + \cdots$$
for some elliptic operator of order 0 $\mathcal{D}^0$ on $P_{t,u}$. We then integrate by parts the L derivatives, and obtain for this term an upper bound of the type
$$2^j \mathcal{N}_1(P)\|\underline{\beta}\|_{L^2(\mathcal{H}_u)} + \cdots .$$
Then, using the estimate for P and the the curvature bound (4.39) for $\underline{\beta}$, this is enough to bound the term involving $\nabla\!\!\!/_{nL}P$ in the right-hand side of (4.103).
- For the term involving E, we we rely on the Besov improvement for E, and we derive an upper bound of the form
$$\sum_{l,q} 2^j 2^{-\frac{|q-l|}{2}} \gamma_q^{(1)} \gamma_l^{(2)}, \text{ where } \gamma_q^{(1)} \in \ell^1(\mathbb{N}) \text{ and } \gamma_l^{(2)} \in \ell^\infty(\mathbb{N}),$$
which is summable unlike (4.104). This is enough to bound the term involving E in the right-hand side of (4.103).

REMARK 4.29. *The reader may wonder why the estimate for $\underline{L}L\mathrm{tr}\chi$ is not better than L^2 with respect to the variable t - instead of L^∞ as one should expect since we rely on a transport equation for the corresponding quantity μ_1. The reason is that boundary terms arise from several integration by parts in $\nabla\!\!\!/_L$ in the course of the proof. The point is that we do not have better estimates than L^2 with respect to the variable t for these terms.*

4.4. Regularity of the foliation with respect to ω

4.4.1. First order derivatives with respect to ω. In this section, we outline the main ideas of the proof of Theorem 4.17. Let us first explain how to control $\partial_\omega\mathrm{tr}\chi$. Differentiating the Raychaudhuri equation (4.19) with respect to ω, we obtain

$$L(\partial_\omega\mathrm{tr}\chi) = [L, \partial_\omega]\mathrm{tr}\chi + \cdots .$$

Now, we have

$$[L, \partial_\omega]\mathrm{tr}\chi = -\partial_\omega N(\mathrm{tr}\chi) = -\nabla\!\!\!/_{\partial_\omega N}\mathrm{tr}\chi$$

where we used the fact that $\mathbf{g}(N, N) = 1$, which differentiated with respect to ω implies that $\partial_\omega N$ is a vectorfield tangent to $P_{t,u}$. Thus, we obtain

$$L(\partial_\omega\mathrm{tr}\chi) = -\nabla\!\!\!/_{\partial_\omega N}\mathrm{tr}\chi + \cdots$$

which together with the estimate (4.43) for $\nabla\!\!\!/\mathrm{tr}\chi$ immediately yields

$$\|\partial_\omega\mathrm{tr}\chi\|_{L^2_{x'}L^\infty_t} \lesssim \varepsilon.$$

REMARK 4.30. *In view of the commutator*

$$[L, \partial_\omega] = -\nabla\!\!\!/_{\partial_\omega N},$$

a derivative with respect to ω has essentially the same regularity as a $\nabla\!\!\!/$-derivative.

The estimates in (4.48) and (4.49) are obtained in the same way, i.e. by differentiating the ricci equations (4.15) and the transport equations (4.18) (4.19) (4.20) and (4.21) with respect to ω, computing the commutators $[\partial_\omega, \mathbf{D}_L]$ and $[\partial_\omega, \nabla\!\!\!/_L]$, and estimating the corresponding transport equations (see [**44**] for the details).

Next, let us explain how to derive the decomposition (4.50) for $\widehat{\chi}$:

$$\widehat{\chi} = \chi_1 + \chi_2,$$

where χ_1 and χ_2 are two symmetric traceless $P_{t,u}$-tangent 2-tensors satisfying the estimates (4.51) and (4.52). Recall the Codazzi type equation (4.22) satisfied by $\widehat{\chi}$:

$$\mathrm{d}\!\!/\mathrm{iv}\,\widehat{\chi} = \frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi - \beta + \cdots .$$

This is an elliptic system on $P_{t,u}$, and we may write formally

$$\widehat{\chi} = \frac{1}{2}\mathcal{D}^{-1}\nabla\!\!\!/\mathrm{tr}\chi - \mathcal{D}^{-1}\beta + \cdots ,$$

where $\mathcal{D}^{-1}$ is a pseudodifferential operator of order -1 on $P_{t,u}$. This allows us to define χ_1 and χ_2 as

$$\chi_1 = \frac{1}{2}\mathcal{D}^{-1}\nabla\!\!\!/ \mathrm{tr}\chi + \cdots \text{ and } \chi_2 = -\mathcal{D}^{-1}\beta.$$

The estimate (4.51) corresponds to the estimate (4.44) for $\widehat{\chi}$ and is prove similarly, so we focus on the estimate (4.52). For the sake of clarity, we only explain why, compared to $\widehat{\chi}$, χ_1 has better regularity with respect to (t,x) while χ_2 has better regularity with respect to ω, which is the point of the decomposition (4.50) (see Remark 4.18). Indeed, since the estimate (4.43) for $\mathrm{tr}\chi$ is better than the estimate (4.44) for $\widehat{\chi}$, and since $\nabla\!\!\!/ \mathcal{D}^{-1}$ is a pseudodifferential operator of order 0 on $P_{t,u}$, we are able to obtain better regularity in (t,x) for χ_1 compared to $\widehat{\chi}$. Next, we focus on χ_2. Now, note that the curvature tensor $\mathbf{R}$ does not depend on ω. Thus, when differentiating β with respect to ω, the ω derivative falls on the frame $(L, \underline{L}, e_A)$, and we obtain schematically

$$\partial_\omega\beta = (\alpha + \rho + \sigma)\partial_\omega N.$$

In particular, we have

$$\|\partial_\omega\beta\|_{L^\infty_u L^2(\mathcal{H}_u)} \lesssim \|\partial_\omega N\|_{L^\infty}(\|\alpha\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\rho\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\sigma\|_{L^\infty_u L^2(\mathcal{H}_u)}) \lesssim \varepsilon,$$

where we used the curvature bound (4.39) for α, ρ and σ, and the estimate (4.48) for $\partial_\omega N$. Thus, $\partial_\omega\beta$ has the same regularity with respect to (t,x) than β. In view of the definition of χ_2, we obtain that $\partial_\omega\chi_2$ has essentially the same regularity as χ_2, while the estimate (4.49) for $\partial_\omega\widehat{\chi}$ looses one $\nabla\!\!\!/$-derivative with respect to the estimate (4.44) for $\widehat{\chi}$. Thus, the regularity of χ_2 with respect to ω is better than the corresponding regularity for $\widehat{\chi}$.

4.4.2. Second order derivatives with respect to ω. In this section, we outline the main ideas of the proof of Theorem 4.19. We focus on the estimate (4.55) for $\partial_\omega^2\zeta$ which is typical. Differentiating twice with respect to ω the transport equation (4.21) for ζ, and computing the commutator $[\nabla\!\!\!/_L, \partial_\omega^2]$, we obtain

$$\nabla\!\!\!/_L(\Pi(\partial_\omega^2\zeta)) = -\chi\cdot\Pi(\partial_\omega^2\zeta) + \nabla\!\!\!/(F_1) + F_2 + \cdots, \tag{4.106}$$

where the $P_{t,u}$-tangent tensors F_1 and F_2 satisfy

$$\|F_1\|_{L^2(\mathcal{H}_u)} + \|F_2\|_{L^1_{x'}L^2_t} \lesssim \varepsilon.$$

We first get rid of the first term in the right-hand side of (4.106) which is troublesome. To this end, we use the following lemma.

Lemma 4.31. *Let γ denotes the metric induced by $\mathbf{g}$ on $P_{t,u}$. Let M the $P_{t,u}$-tangent 2-tensor defined as the solution of the following transport equation:*

$$\nabla\!\!\!/_L M_{AB} = M_{AC}\chi_{CB},\ M_{AB} = \gamma_{AB} \text{ on } P_{0,u}, \tag{4.107}$$

Then, M_{AB} satisfies the following estimate:

$$\|M-\gamma\|_{L^\infty} + \|\nabla\!\!\!/ M\|_{\mathcal{B}^0} \lesssim \varepsilon. \tag{4.108}$$

Using the transport equation (4.106) for $\Pi(\partial_\omega^2\zeta)$ and the transport equation (4.107), for M allows us to get rid of the troublesome term $\chi\cdot\Pi(\partial_\omega^2\zeta)$:

$$\nabla\!\!\!/_L(M\cdot\Pi(\partial_\omega^2\zeta)) = \nabla\!\!\!/(M\cdot F_1) - \nabla\!\!\!/(M)\cdot F_1 + M\cdot F_2 + \cdots.$$

Together with the finite band property and the Bernstein inequality for P_j, the estimates for F_1 and F_2, and the estimate (4.108) for M, we obtain for $M\cdot\Pi(\partial_\omega^2\zeta)$ the estimate corresponding to (4.55). Then, we obtain the wanted estimate (4.55) for $\Pi(\partial_\omega^2\zeta)$ by proving that the estimate (4.108) for M is enough to ensure that the multiplication by M^{-1} preserves the estimate (4.55).

4.5. Additional decompositions

In this section, we outline the main ideas of the proof of Theorem 4.20. We need to compare N and $\mathrm{tr}\chi$ at two different angles ω and ν. The basic tool is the following lemma.

LEMMA 4.32. *Let ω and ω' in $\mathbb{S}^2$. Let $u = u(t,x,\omega)$ and $u' = u(t,x,\omega')$. Then, for any tensor F, we have:*

$$\|F\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim \|F\|_{L^\infty_u L^2(\mathcal{H}_u)} + |\omega-\omega'|^{\frac{1}{4}}\|F\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)}\left(\sup_u\left(\int_u^{u+|\omega-\omega'|}\|\mathbf{D}F\|^2_{L^2(\mathcal{H}_\tau)}d\tau\right)\right)^{\frac{1}{4}}.$$

In order to compare the norms $L^\infty_{u'}L^2(\mathcal{H}_{u'})$ and $L^\infty_u L^2(\mathcal{H}_u)$, we need coordinate systems. We define $\Phi_{t,\omega} : \Sigma_t \to \mathbb{R}^3$ defined by:

$$\Phi_{t,\omega}(t,x) := u(t,x,\omega)\omega + \partial_\omega u(t,x,\omega). \tag{4.109}$$

Then we claim that $\Phi_{t,\omega}$ is a global C^1 diffeomorphism from Σ_t to $\mathbb{R}^3$ and therefore provides a global coordinate system on Σ_t (see Proposition 6.6 for a related result on Σ_0). Next, we prove that

$$\|F\|^2_{L^2(\mathcal{H}_u)} \simeq \int_0^1\int_{\mathbb{R}^2}|F(\Phi^{-1}_{t,u,\omega}(y'))|^2dy'dt. \tag{4.110}$$

This formula allows us to compare the norms $L^\infty_{u'}L^2(\mathcal{H}_{u'})$ and $L^\infty_u L^2(\mathcal{H}_u)$. In turn, one needs to evaluate

$$|F(\Phi^{-1}_{t,u,\omega}(y'))|^2 - |F(\Phi^{-1}_{t,u',\omega'}(y'))|^2$$

In particular, we need to estimate

$$\left\|\partial_\omega\left[\Phi^{-1}_{t,u,\omega}(y')\right]\right\|_{L^\infty}.$$

We refer the reader to [**44**] for details on the proof of Lemma 4.32.

Using Lemma 4.32 as well as commutator estimates for $[\mathbf{D}, P_l]$ among others, we may prove the following corollary.

COROLLARY 4.33. *Let f a scalar function and ω, ω' in $\mathbb{S}^2$. Then, for any $l \geq 0$, we have:*

$$\|P_l f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \lesssim (2^{-l} + |\omega-\omega'|^{\frac{1}{2}}2^{-\frac{l}{2}})(\|f\|_{L^\infty_u L^2(\mathcal{H}_u)} + \|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)}),$$

and

$$
\begin{aligned}
& \|P_{\leq l}f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})} \\
\lesssim\ & (1+|\omega-\omega'|^{\frac{1}{2}}2^{\frac{l}{2}})\|f\|_{L^\infty_u L^2(\mathcal{H}_u)}+|\omega-\omega'|^{\frac{1}{4}}\|f\|^{\frac{1}{2}}_{L^\infty_u L^2(\mathcal{H}_u)} \\
& \times\left(\sup_u\sum_{q\leq l}\int_u^{u+|\omega-\omega'|}(\|P_q(nL(f))\|^2_{L^2(\mathcal{H}_\tau)}+\|P_q(bN(f))\|^2_{L^2(\mathcal{H}_\tau)})d\tau\right)^{\frac{1}{4}}.
\end{aligned}
$$

We also need the following non sharp commutator lemma.

LEMMA 4.34. *Let f a scalar function and ω,ω' in $\mathbb{S}^2$. Then, for any $l\geq 0$, we have:*

$$\|[\partial_\omega,P_{\leq l}]f\|_{L^\infty_{u'}L^2(\mathcal{H}_{u'})}\lesssim\|\mathbf{D}f\|_{L^\infty_u L^2(\mathcal{H}_u)}.$$

Using Corollary 4.33 and Lemma 4.34 together with the estimates (4.43) and (4.49) for $\mathrm{tr}\chi$ and the fact that $|\omega-\nu|\lesssim 2^{-\frac{j}{2}}$, we are able to prove the decomposition (4.57) for $\mathrm{tr}\chi$.

Next, we consider $N-N_\nu$. We have

$$2^{\frac{j}{2}}(N-N_\nu)=\int_{[\omega,\nu]}\partial_\omega N(.,\omega'')d\omega''(2^{\frac{j}{2}}(\omega-\nu)),$$

where $[\omega,\nu]$ denotes the arc of $\mathbb{S}^2$ joining ω and ν. Since $|\omega-\nu|\lesssim 2^{-\frac{j}{2}}$, we want to proceed as for the decomposition of $\mathrm{tr}\chi$. More precisely, we want to use Corollary 4.33 and Lemma 4.34 together with the estimates (4.48), (4.49), (4.53), (4.54) and (4.55) for $\partial_\omega N$, in order to prove the decomposition (4.56) for $N-N_\nu$. Now, unlike $\mathrm{tr}\chi$ which is a scalar, $\partial_\omega N$ is a tensor. Since Corollary 4.33 and Lemma 4.34 only apply to scalars, we need one last ingredient to prove the decomposition (4.56) for $N-N_\nu$ and conclude the proof of Theorem 4.20. Namely we need to scalarize $\partial_\omega N$ using a basis of the tangent space of Σ_t which does not depend on ω. We refer to [**44**] for the details.

CHAPTER 5

Construction and control of the parametrix at initial time

In this chapter, and the next one, we will only consider the leave Σ_0 of the foliation Σ_t of $\mathcal{M}$, and we denote it by Σ for simplicity. Recall the plane wave type parametrix given by (1.26)[1]

$$\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(t,x)} f(\lambda\omega)\lambda^2 d\lambda d\omega$$

where $u(.,.,\omega)$ is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$ such that $u(0,x,\omega)\sim x\cdot\omega$ when $|x|\to+\infty$ on Σ. The goal of this chapter is to outline the main ideas allowing us to obtain the control for that parametrix restricted to Σ in [**45**].

5.1. Geometric set-up and main results

5.1.1. Presentation of the parametrix. In this section, we construct a parametrix for the following homogeneous wave equation:

$$\begin{cases}\square_{\mathbf{g}}\phi = 0 \text{ on } \mathcal{M},\\ \phi_{|\Sigma}=\phi_0,\ T(\phi)_{|\Sigma}=\phi_1,\end{cases}\tag{5.1}$$

where ϕ_0 and ϕ_1 are two given functions on Σ and T is the future oriented unit normal to Σ in the space-time $\mathcal{M}$.

We recall the plane wave representation of the solution of the flat wave equation. This corresponds to the case where $\mathbf{g}$ is the Minkowski metric. (5.1) becomes:

$$\begin{cases}\square\phi = 0 \text{ on } \mathbb{R}^{1+3},\\ \phi(0,.)=\phi_0,\ \partial_t\phi(0,.)=\phi_1 \text{ on } \mathbb{R}^3.\end{cases}\tag{5.2}$$

The plane wave representation of the solution ϕ of (5.2) is given by:

$$\begin{aligned}&\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i(-t+x\cdot\omega)\lambda}\frac{1}{2}\left(\mathcal{F}\phi_0(\lambda\omega)+i\frac{\mathcal{F}\phi_1(\lambda\omega)}{\lambda}\right)d\lambda d\omega\\ &+\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i(t+x\cdot\omega)\lambda}\frac{1}{2}\left(\mathcal{F}\phi_0(\lambda\omega)-i\frac{\mathcal{F}\phi_1(\lambda\omega)}{\lambda}\right)d\lambda d\omega,\end{aligned}\tag{5.3}$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^3$.

We would like to construct a parametrix in the curved case similar to (5.3). We introduce two solutions $u_\pm$ of the eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u_\pm\partial_\beta u_\pm = 0 \text{ on } \mathcal{M},\tag{5.4}$$

[1]This is actually a half wave parametrix. See (5.6) below for the full parametrix

such that:

$$T(u_\pm) = \mp|\nabla u_\pm| = \mp a_\pm^{-1} \text{ on } \Sigma, \tag{5.5}$$

where ∇ is the covariant derivative on Σ associated to the metric g induced by $\mathbf{g}$ on Σ, $|\cdot|$ is the length associated to g for vectorfields on Σ, and $a_\pm$ is the lapse of $u_\pm$ on Σ. We look for a parametrix for (5.1) of the form:

$$S(t,x) = S_+f_+(t,x) + S_-f_-(t,x), \tag{5.6}$$

where

$$S_\pm f_\pm(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u_\pm(t,x,\omega)} f_\pm(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.7}$$

In the next two sections, we specify the parametrix (5.6) by prescribing $u_\pm$ on Σ and by making our choice for $f_\pm$ explicit.

5.1.1.1. *Prescription of u_+ and u_- on Σ.* (5.4) and (5.5) are not enough to define $u_\pm$ in a unique manner. Indeed, we still need to prescribe $u_\pm$ on Σ. To motivate our choice, we need to introduce some geometric objects connected to $u_\pm$. Let $N_\pm$ the vectorfield on Σ defined by:

$$N_\pm = \frac{\nabla u_\pm}{|\nabla u_\pm|} = a_\pm \nabla u_\pm, \tag{5.8}$$

and $L_\pm$ the vectorfield on $\mathcal{M}$ which is given on Σ by:

$$L_\pm = a_\pm \mathbf{g}^{\alpha\beta}\partial_\alpha u_\pm \partial_\beta = a_\pm(-T(u_\pm)T + \nabla u_\pm) = \pm T + N_\pm. \tag{5.9}$$

Let $P_{u_\pm} = \{x \in \Sigma / \, u_\pm(x) = u_\pm\}$ denote the level surfaces of $u_\pm$ in Σ. Since $N_\pm$ is the unit normal to $P_{u_\pm}$, the second fundamental form of $P_{u_\pm}$ in Σ is given by:

$$\theta_\pm(e_A^\pm, e_B^\pm) = g(D_{e_A^\pm} N_\pm, e_B^\pm), \; A, B = 1, 2, \tag{5.10}$$

where $(e_1^\pm, e_2^\pm)$ is an arbitrary orthonormal frame of $TP_{u_\pm}$. Let

$$\mathcal{H}_{u_\pm} = \{(t,x) \in \mathcal{M} / \, u_\pm(t,x) = u_\pm\}$$

denote the null level hypersurfaces of $u_\pm$ in $\mathcal{M}$. Since $L_\pm$ is null and orthogonal to $P_{u_\pm}$ in $\mathcal{H}_{u_\pm}$, the null second fundamental form $\chi_\pm$ is given on $P_{u_\pm}$ by:

$$\chi_\pm(e_A^\pm, e_B^\pm) = g(\mathbf{D}_{e_A^\pm} L_\pm, e_B^\pm), \; A, B = 1, 2. \tag{5.11}$$

Taking the trace in (5.10) and (5.11), and using (5.9) and the fact that k is the second fundamental form of Σ, we obtain:

$$\mathrm{tr}\chi_\pm = \pm \mathrm{tr}k + \mathrm{tr}\theta_\pm. \tag{5.12}$$

Note that $\mathrm{tr}_g k = \mathrm{tr}k + k_{NN}$, where tr_g denotes the trace for 2-tensors on Σ. In view of the maximal foliation assumption (1.6), we have $\mathrm{tr}_g k = 0$. Together with (5.12), this yields:

$$\mathrm{tr}\chi_\pm = \mp k_{N_\pm N_\pm} + \mathrm{tr}\theta_\pm. \tag{5.13}$$

Now, in [**44**] (see also Theorem 4.15), we prove that $\mathrm{tr}\chi_\pm$ belongs to $L^\infty(\mathcal{M})$ using a transport equation (the Raychaudhuri equation, see (4.19)) provided that it belongs to $L^\infty(\Sigma)$ at $t = 0$. Thus, one needs the following estimate

$$\mathrm{tr}\chi_\pm \in L^\infty(\Sigma), \tag{5.14}$$

which in view of (5.13) is equivalent to:

$$\mp k_{N_\pm N_\pm} + \mathrm{tr}\theta_\pm \in L^\infty(\Sigma). \tag{5.15}$$

We construct in [**42**] (see also Chapter 6) a function $u(x,\omega)$ on $\Sigma \times \mathbb{S}^2$ such that

$$-k_{NN} + \mathrm{tr}\theta \in L^\infty(\Sigma). \tag{5.16}$$

Note that $-u(x,-\omega)$ satisfies:

$$k_{NN} + \mathrm{tr}\theta \in L^\infty(\Sigma). \tag{5.17}$$

Thus, in view of (5.15), (5.16) and (5.17), we initialize $u_\pm$ on Σ by:

$$u_+(0,x,\omega) = u(x,\omega) \text{ and } u_-(0,x,\omega) = -u(x,-\omega) \text{ for } (x,\omega) \in \Sigma \times \mathbb{S}^2. \tag{5.18}$$

REMARK 5.1. *Note that in the particular case where $k \equiv 0$ - the so-called time symmetric case-, we may take*

$$u_+(0,x,\omega) = u_-(0,x,\omega) = u(x,\omega) \textit{ for } (x,\omega) \in \Sigma \times \mathbb{S}^2.$$

In particular, we have $u_+(0,x,\omega) = u_-(0,x,\omega) = x \cdot \omega$ in the flat case.

5.1.1.2. *The choice of f_+ and f_-.* Having defined $u_\pm$, we still need to define $f_\pm$ in the parametrix (5.6). According to (5.1), the half wave parametrix S_+ and S_- should satisfy on Σ:

$$\begin{cases} S_+f_+(0,x) + S_-f_-(0,x) = \phi_0(x), \\ T(S_+f_+)(0,x) + T(S_-f_-)(0,x) = \phi_1(x). \end{cases} \tag{5.19}$$

Let us introduce the following operators acting on functions of $\mathbb{R}^3$:

$$M_\pm f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{5.20}$$

and

$$Q_\pm f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)} a(x,\pm\omega)^{-1} f(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{5.21}$$

where $a(x,\omega) = |\nabla u(x,\omega)|^{-1}$ is the lapse of u. Using (5.5), the definition of $S_\pm$ in (5.6), (5.18), the definition (5.20) of $M_\pm$ and the definition (5.21) of $Q_\pm$, we may rewrite (5.19) as:

$$\begin{cases} M_+f_+ + M_-f_- = \phi_0, \\ Q_+(\lambda f_+) - Q_-(\lambda f_-) = i\phi_1. \end{cases} \tag{5.22}$$

The goal of this chapter will be to show that there exist a unique (f_+, f_-) satisfying (5.22), and that (f_+, f_-) satisfies the following estimate:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla\phi_0\|_{L^2(\Sigma)} + \|\phi_1\|_{L^2(\Sigma)}. \tag{5.23}$$

REMARK 5.2. *In the case of the flat wave equation* (5.2), *we have* $(\Sigma, g, k) = (\mathbb{R}^3, \delta, 0)$, $u_\pm(t,x,\omega) = \mp t + x\cdot\omega$, $u(x,\omega) = x\cdot\omega$ *and* $a(x,\omega) = 1$. *In particular, the operators* $M_\pm$ *and* $Q_\pm$ *defined respectively by* (5.20) *and* (5.21) *all coincide with the inverse Fourier transform. Then, the system* (5.22) *admits the following solutions:*

$$f_\pm(\lambda\omega) = \frac{1}{2}\left(\mathcal{F}\phi_0(\lambda\omega) \pm i\frac{\mathcal{F}\phi_1(\lambda\omega)}{\lambda}\right),$$

which clearly satisfy the estimate (5.23).

5.1.2. Geometric set-up. We define the lapse $a(x,\omega) = |\nabla u(x,\omega)|^{-1}$, and the unit vector N such that $\nabla u(x,\omega) = a(x,\omega)^{-1}N(x,\omega)$. We also define the level surfaces $P_u = \{x \,/\, u(x,\omega) = u\}$ so that N is the normal to P_u.

For $1 \le p, q \le +\infty$, we define the spaces $L^p_u L^q(P_u)$ using the norm

$$\|F\|_{L^p_u L^q(P_u)} = \left(\int_u \|F\|^p_{L^q(P_u)} du\right)^{1/p}.$$

We assume that $1/2 \le a(x) \le 2$ for all $x \in \Sigma$ (see (5.24) below) so that $L^p_u L^p(P_u)$ coincides with $L^p(\Sigma)$ for all $1 \le p \le +\infty$. We denote by γ the metric induced by g on P_u, and by $\nabla\!\!\!/$ the induced covariant derivative.

Before stating precisely the main results of this chapter, we first record the regularity obtained for the phase $u(x,\omega)$ constructed in [**42**] (see also Chapter 6).

5.1.3. Regularity assumptions on the phase $u(x,\omega)$. In this section, we collect the estimates for the phase $u(x,\omega)$ of our Fourier integral operators that are needed to follow the discussion of the control of the parametrix at initial time contained in this chapter. An outline of the proof of these estimates will be given in Chapter 6 (see [**42**] for the complete proof).

We start with the regularity in x of the lapse a. We need:

$$\|\nabla a\|_{L^\infty_u L^2(P_u)} + \|a-1\|_{L^\infty(\Sigma)} + \|\nabla\!\!\!/\nabla a\|_{L^2(\Sigma)} \lesssim \varepsilon. \tag{5.24}$$

We also need a decomposition for $\nabla_N a$. For all $j \ge 0$, there are scalar functions a^j_1 and a^j_2 such that[2]:

$$\begin{aligned}&\nabla_N a = a^j_1 + a^j_2 \text{ where } \|a^j_1\|_{L^2(\Sigma)} \lesssim 2^{-j/2}\varepsilon,\ \|a^j_2\|_{L^\infty_u L^2(P_u)} \lesssim \varepsilon\\ &\text{and } \|\nabla_N a^j_2\|_{L^2(\Sigma)} + \|a^j_2\|_{L^2_u L^\infty(P_u)} \lesssim 2^{j/2}\varepsilon.\end{aligned} \tag{5.25}$$

Next, we consider the regularity with respect to ω. We have:

$$\|\partial_\omega a\|_{L^2(\Sigma)} + \|\nabla\partial_\omega a\|_{L^2(\Sigma)} \lesssim \varepsilon, \tag{5.26}$$

$$\|\partial^\alpha_\omega a\|_{L^\infty(\Sigma)} \lesssim 1 \text{ for some } 0 < \alpha < 1, \tag{5.27}$$

where (5.27) should be understood in the Hölder sense,

$$\|\partial_\omega N\|_{L^\infty(\Sigma)} \lesssim 1, \tag{5.28}$$

[2]we choose $a^j_1 = P_{>j/2}(\nabla_N a)$ and $a^j_2 = P_{\le j/2}(\nabla_N a)$, and then obtain (5.25) using (5.24) and an estimate for $\nabla^2_N a$ (see [**42**] for the details)

and

$$\|\partial_\omega^3 u\|_{L^\infty_{\rm loc}(\Sigma)}\lesssim 1. \tag{5.29}$$

We will need the following global change of variable on Σ. Let $\omega\in\mathbb{S}^2$. Let $\phi_\omega:\Sigma\to\mathbb{R}^3$ defined by:

$$\phi_\omega(x):=u(x,\omega)\omega+\partial_\omega u(x,\omega). \tag{5.30}$$

Then ϕ_ω is a bijection, and the determinant of its Jacobian satisfies the following estimate:

$$\||\det(\text{Jac}\phi_\omega)|-1\|_{L^\infty(\Sigma)}\lesssim\varepsilon. \tag{5.31}$$

Finally, we can compare $u(x,\omega)$ with a phase linear in ω. Let $\nu\in\mathbb{S}^2$ and ϕ_ν the map defined in (5.30). Then, we have:

$$\begin{array}{l} u(x,\omega)-\phi_\nu(x)\cdot\omega=O(\varepsilon|\omega-\nu|^2),\\ \partial_\omega u(x,\omega)-\partial_\omega(\phi_\nu(x)\cdot\omega)=O(\varepsilon|\omega-\nu|),\\ \partial_\omega^2 u(x,\omega)-\partial_\omega^2(\phi_\nu(x)\cdot\omega)=O(\varepsilon).\end{array} \tag{5.32}$$

REMARK 5.3. *In* (5.24)-(5.32), *all inequalities hold for any $\omega\in\mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.*

REMARK 5.4. *In the case of the flat wave equation* (5.2), *we have $(\Sigma,g)=(\mathbb{R}^3,\delta)$, $u(x,\omega)=x\cdot\omega$, $a=1$, $N=\omega$ and $\phi_\omega=Id_{\mathbb{R}^3}$. Thus,* (5.24)-(5.32) *are clearly satisfied with $\varepsilon=0$.*

REMARK 5.5. *Recall that the lapse a is at the level of one derivative of u with respect to x. Thus, we obtain from* (5.24) *that some components of $\nabla^3 u$ are in $L^2(\Sigma)$. Note that this is not true for all components since* (5.25) *does not allow us to control $\nabla_N^2 a$ in $L^2(\Sigma)$. In fact,* (5.25) *is consistent with $3/2$ derivatives of a with respect to N in L^2.*

5.1.4. Main results. We first state a result of boundedness on L^2 for Fourier integral operators with phase $u(x,\omega)$.

THEOREM 5.6. *Let u be a function on $\Sigma\times\mathbb{S}^2$ satisfying suitable assumptions (we refer to* [**43**] *for the complete set of assumptions, and to section 5.1.3 for some typical assumptions). Let U the Fourier integral operator with phase $u(x,\omega)$ and symbol $b(x,\omega)$:*

$$Uf(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u(x,\omega)}b(x,\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.33}$$

Let $D>0$. We assume furthermore that $b(x,\omega)$ satisfies:

$$\|b\|_{L^\infty(\Sigma)}+\|\nabla b\|_{L^\infty_u L^2(P_u)}+\|\nabla\!\!\!/\nabla b\|_{L^2(\Sigma)}\lesssim D, \tag{5.34}$$

$$\|\partial_\omega b\|_{L^2(\Sigma)}+\|\nabla\partial_\omega b\|_{L^2(\Sigma)}\lesssim D, \tag{5.35}$$

and

$$\begin{array}{l}\nabla_N b=b_1^j+b_2^j \text{ where } \|b_1^j\|_{L^2(\Sigma)}\lesssim 2^{-\frac{j}{2}}D, \|b_2^j\|_{L^\infty_u L^2(P_u)}\lesssim D,\\ \text{and } \|\nabla_N b_2^j\|_{L^2(\Sigma)}+\|b_2^j\|_{L^2_u L^\infty(P_u)}\lesssim 2^{\frac{j}{2}}D.\end{array} \tag{5.36}$$

Then, U is bounded on L^2 and satisfies the estimate:

$$\|Uf\|_{L^2(\Sigma)} \lesssim D\|f\|_{L^2(\mathbb{R}^3)}. \tag{5.37}$$

REMARK 5.7. *We intend to apply Theorem 5.6 to the Fourier integral operators $M_\pm$ and $Q_\pm$ introduced in section 5.1.1.2 whose symbol are respectively 1 and a^{-1}. Thus, our assumptions on the regularity of the symbol $b(x,\omega)$ are consistent with the assumptions on the regularity of $a(x,\omega)$ given by* (5.24)-(5.29).

REMARK 5.8. *Under the additional assumption* (5.32) *on u, and under some restrictions on the constant D appearing in* (5.34), (5.35), *and* (5.36), *we may prove the opposite of* (5.37):

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|Uf\|_{L^2(\Sigma)}$$

(see Proposition 5.19). This will be a major ingredient in the proof of Theorem 5.9 below, and in particular of (5.41).

Recall the definition of the Fourier integral operators $M_\pm$ and $Q_\pm$ introduced in section 5.1.1.2:

$$M_\pm f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{5.38}$$

and

$$Q_\pm f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)} a(x,\pm\omega)^{-1} f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.39}$$

The following theorem is the main result of this chapter.

THEOREM 5.9. *Let u be a function on $\Sigma\times\mathbb{S}^2$ satisfying suitable assumptions (we refer to* [**43**] *for the complete set of assumptions, and to section 5.1.3 for some typical assumptions). Then, there exist a unique (f_+, f_-) satisfying:*

$$\left\{\begin{array}{l} M_+f_+ + M_-f_- = \phi_0, \\ Q_+(\lambda f_+) - Q_-(\lambda f_-) = i\phi_1. \end{array}\right. \tag{5.40}$$

Furthermore, (f_+, f_-) satisfies the following estimate:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla\phi_0\|_{L^2(\Sigma)} + \|\phi_1\|_{L^2(\Sigma)}. \tag{5.41}$$

Proving the estimates (5.37) and (5.41) for the Fourier integral operators U, $M_\pm$ and $Q_\pm$ will require taking several integrations by parts. The main difficulty in proving Theorem 5.6 and Theorem 5.9 will be to perform these integrations by parts within the very low level of regularity for the phase $u(x,\omega)$ given by (5.24)-(5.32) and for the symbol $b(x,\omega)$ given by (5.34) (5.35) (5.36). The proof will rely both on harmonic analysis decompositions and the geometry of the foliation of Σ by u. Theorem 5.6 will be reviewed in section 5.2 and Theorem 5.9 will be reviewed in section 5.3.

5.2. Control of Fourier integral operators

5.2.1. Structure of the proof of Theorem 5.6. The proof of Theorem 5.6 proceeds in three steps. We first localize in frequencies of size $\lambda \sim 2^j$. We then localize the angle ω in patches on the sphere $\mathbb{S}^2$ of diameter $2^{-j/2}$. Finally, we estimate the diagonal terms.

Remark 5.10. *Note that the structure of the proof is analogous to the one on the control of the error term in Chapter 3 (see section 3.2.2). However, the proof each step (almost orthogonality in frequency, almost orthogonality in angle, and control of the diagonal term) is different, more particularly the last two steps.*

5.2.1.1. *Step 1: decomposition in frequency.* For the first step, we introduce φ and ψ two smooth compactly supported functions on $\mathbb{R}$ such that:

$$\varphi(\lambda)+\sum_{j\geq 0}\psi(2^{-j}\lambda)=1 \text{ for all } \lambda\in\mathbb{R}. \tag{5.42}$$

We use (5.42) to decompose Uf as follows:

$$Uf(x)=\sum_{j\geq -1}U_jf(x), \tag{5.43}$$

where for $j\geq 0$:

$$U_jf(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u}b(x,\omega)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{5.44}$$

and

$$U_{-1}f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u}b(x,\omega)\varphi(\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.45}$$

The goal of this first step is to prove the following proposition:

Proposition 5.11. *The decomposition* (5.43) *satisfies an almost orthogonality property:*

$$\|Uf\|^2_{L^2(\Sigma)}\lesssim\sum_{j\geq -1}\|U_jf\|^2_{L^2(\Sigma)}+D^2\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{5.46}$$

The proof of Proposition 5.11 is postponed to section 5.2.2.

5.2.1.2. *Step 2: decomposition in angle.* Proposition 5.11 allows us to estimate $\|U_jf\|_{L^2(\Sigma)}$ instead of $\|Uf\|_{L^2(\Sigma)}$. We perform a second dyadic decomposition. We introduce a smooth partition of unity on the sphere $\mathbb{S}^2$:

$$\sum_{\nu\in\Gamma}\eta_j^\nu(\omega)=1 \text{ for all } \omega\in\mathbb{S}^2, \tag{5.47}$$

where Γ is a lattice on $\mathbb{S}^2$ of size $2^{-\frac{j}{2}}$, where the support of η_j^ν is a patch on $\mathbb{S}^2$ of diameter $\sim 2^{-j/2}$. We use (5.47) to decompose $U_j f$ as follows:

$$U_j f(x) = \sum_{\nu\in\Gamma} U_j^\nu f(x), \tag{5.48}$$

where:

$$U_j^\nu f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u} b(x,\omega)\psi(2^{-j}\lambda)\eta_j^\nu(\omega) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.49}$$

We also define:

$$\begin{array}{l}\gamma_{-1} = \|\varphi(\lambda) f\|_{L^2(\mathbb{R}^3)},\ \gamma_j = \|\psi(2^{-j}\lambda) f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\\ \gamma_j^\nu = \|\psi(2^{-j}\lambda)\eta_j^\nu(\omega) f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\ \nu\in\Gamma,\end{array} \tag{5.50}$$

which satisfy:

$$\|f\|^2_{L^2(\mathbb{R}^3)} = \sum_{j\geq -1}\gamma_j^2 = \sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2. \tag{5.51}$$

The goal of this second step is to prove the following proposition:

Proposition 5.12. *The decomposition* (5.48) *satisfies an almost orthogonality property:*

$$\|U_j f\|^2_{L^2(\Sigma)} \lesssim \sum_{\nu\in\Gamma}\|U_j^\nu f\|^2_{L^2(\Sigma)} + D^2\gamma_j^2. \tag{5.52}$$

The proof of Proposition 5.12 is postponed to section 5.2.3.

5.2.1.3. *Step 3: control of the diagonal term.* Proposition 5.12 allows us to estimate $\|U_j^\nu f\|_{L^2(\Sigma)}$ instead of $\|U_j f\|_{L^2(\Sigma)}$. The diagonal term is estimated as follows.

Proposition 5.13. *The diagonal term $U_j^\nu f$ satisfies the following estimate:*

$$\|U_j^\nu f\|_{L^2(\Sigma)} \lesssim D\gamma_j^\nu. \tag{5.53}$$

The proof of Proposition 5.13 is postponed to section 5.2.4.

5.2.1.4. *Proof of Theorem 5.6.* Proposition 5.11, 5.12 and 5.13 immediately yield the proof of Theorem 5.6. Indeed, (5.46), (5.51), (5.52) and (5.53) imply:

$$\begin{aligned}\|Uf\|^2_{L^2(\Sigma)} &\lesssim \sum_{j\geq -1}\|U_j f\|^2_{L^2(\Sigma)} + D^2\|f\|^2_{L^2(\mathbb{R}^3)}\\ &\lesssim \sum_{j\geq -1}\sum_{\nu\in\Gamma}\|U_j^\nu f\|^2_{L^2(\Sigma)} + D^2\sum_{j\geq -1}\gamma_j^2 + D^2\|f\|^2_{L^2(\mathbb{R}^3)}\\ &\lesssim D^2\sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2 + D^2\sum_{j\geq -1}\gamma_j^2 + D^2\|f\|^2_{L^2(\mathbb{R}^3)}\\ &\lesssim D^2\|f\|^2_{L^2(\mathbb{R}^3)},\end{aligned} \tag{5.54}$$

which is the conclusion of Theorem 5.6.

The remainder of section 5.2 is dedicated to the proof of Proposition 5.11, 5.12 and 5.13.

5.2.2. Proof of Proposition 5.11 (almost orthogonality in frequency). We have to prove (5.46):

$$\|Uf\|^2_{L^2(\Sigma)} \lesssim \sum_{j\geq -1} \|U_jf\|^2_{L^2(\Sigma)} + D^2\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{5.55}$$

This will result from the following inequality using Shur's Lemma:

$$\left|\int_\Sigma U_jf(x)\overline{U_kf(x)}d\Sigma\right| \lesssim D^2 2^{-\frac{|j-k|}{2}}\gamma_j\gamma_k \text{ for } |j-k|>2. \tag{5.56}$$

We consider a coordinate system (u, x') on Σ where x' denotes a coordinate system on P_u, and we would like to integrate by parts with respect to ∂_u. Since $\nabla u = a^{-1}N$ and $\nabla u' = a'^{-1}N'$, we have:

$$e^{i\lambda u - i\lambda' u'} = -\frac{i}{\lambda - \lambda' \frac{a}{a'} g(N, N')}\partial_u(e^{i\lambda u - i\lambda' u'}), \tag{5.57}$$

where we use the notation u for $u(x,\omega)$, a for $a(x,\omega)$, N for $N(x,\omega)$, u' for $u(x,\omega')$, a' for $a(x,\omega')$ and N' for $N(x,\omega')$. Then, the proof of (5.56) is analogous to the proof of (3.45), so we skip it for the sake of simplicity. Let us just say that, as for the proof of (3.45), most terms require one integration by parts using (5.57) and the estimate (5.34) for b (as in section 3.3.1), while one term requires a second integration by parts using (5.57) and the decomposition (5.36) for $\nabla_N b$ (as in sections 3.3.2 and 3.3.3).

5.2.3. Proof of Proposition 5.12 (almost orthogonality in angle). We have to prove (5.52):

$$\|U_jf\|^2_{L^2(\Sigma)} \lesssim \sum_{\nu\in\Gamma} \|U_j^\nu f\|^2_{L^2(\Sigma)} + D^2\gamma_j^2. \tag{5.58}$$

5.2.3.1. *Presence of a log-loss.* Integrating by parts twice in $\int_\Sigma U_j^\nu f(x)\overline{U_j^{\nu'}f(x)}d\Sigma$ would ultimately imply:

$$\left|\int_\Sigma U_j^\nu f(x)\overline{U_j^{\nu'}f(x)}d\Sigma\right| \lesssim \frac{D^2\gamma_j^\nu\gamma_j^{\nu'}}{(2^{j/2}|\nu-\nu'|)^2},\ |\nu-\nu'|\neq 0. \tag{5.59}$$

This yields to a log-loss since we have:

$$\sup_\nu \sum_{\nu'} \frac{1}{(2^{j/2}|\nu-\nu'|)^2} \sim j. \tag{5.60}$$

REMARK 5.14. *Recall that there is an analogous log-loss in the almost orthogonality argument in angle for the error term (see section 3.5.2). In section 3.5, we removed the log-loss in particular by using integration by parts with respect to the null vectorfield L. On the other hand, we work here on Σ which is Riemannian, so there is no equivalent of the null vectorfield L. Instead, we will use a second decomposition in λ (see section 5.2.3.2). Note that such a strategy can not be used to control the error term (see Remark 5.16).*

To avoid the log-loss present in (5.59), we will instead derive the following inequality:

$$\left|\int_{\Sigma} U_j^\nu f(x)\overline{U^{\nu'} f(x)} d\Sigma\right| \lesssim \frac{D^2\gamma_j^\nu\gamma_j^{\nu'}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}} + \frac{D^2\gamma_j^\nu\gamma_j^{\nu'}}{(2^{j/2}|\nu-\nu'|)^3},\ |\nu-\nu'|\neq 0, \quad (5.61)$$

where $\alpha > 0$. Indeed, since $\mathbb{S}^2$ is 2 dimensional and $1 \leq 2^{j/2}|\nu-\nu'| \leq 2^{j/2}$ for $\nu, \nu' \in \Gamma$ and $\nu \neq \nu'$, we have:

$$\sup_\nu \sum_{\nu'} \frac{1}{(2^{j/2}|\nu-\nu'|)^3} \leq C < +\infty, \quad (5.62)$$

and

$$\sup_\nu \sum_{\nu'} \frac{1}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}} \leq C_\alpha < +\infty\ \forall \alpha > 0. \quad (5.63)$$

Thus, (5.61), (5.62) and (5.63) together with Shur's Lemma imply (5.58).

5.2.3.2. *A second decomposition in frequency.* To avoid the log-loss present in (5.59), we do a second decomposition in frequency. λ belongs to the interval $[2^{j-1}, 2^{j+1}]$ which we decompose in intervals I_k:

$$[2^{j-1}, 2^{j+1}] = \bigcup_{1\leq k\leq |\nu-\nu'|^{-\alpha}} I_k \text{ where } \mathrm{diam}(I_k) \sim 2^j|\nu-\nu'|^\alpha. \quad (5.64)$$

Let ϕ_k a partition of unity of the interval $[2^{j-1}, 2^{j+1}]$ associated to the I_k's. We decompose $U_j^\nu f$ as follows:

$$U_j^\nu f(x) = \sum_{1\leq k\leq |\nu-\nu'|^{-\alpha}} U_j^{\nu,k} f(x), \quad (5.65)$$

where:

$$U_j^{\nu,k} f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u} b(x,\omega)\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\phi_k(\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega. \quad (5.66)$$

Remark 5.15. *The point of this additional decomposition is to exploit the volume in λ. Indeed, after performing Cauchy-Schwarz in λ, we obtain*

$$\sqrt{|I_k|} \sim 2^{\frac{j}{2}}|\nu-\nu'|^{\frac{\alpha}{2}}$$

which displays the crucial gain $|\nu-\nu'|^{\frac{\alpha}{2}}$.

We also define:

$$\gamma_j^{\nu,k} = \|\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\phi_k(\lambda) f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\ \nu\in\Gamma, 1\leq k\leq |\nu-\nu'|^{-\alpha}, \quad (5.67)$$

which satisfy:

$$(\gamma_j^\nu)^2 = \sum_{1\leq k\leq |\nu-\nu'|^{-\alpha}} (\gamma_j^{\nu,k})^2. \quad (5.68)$$

5.2.3.3. *The two key estimates.* We will prove the following two estimates:

$$\left|\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\right|\lesssim\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}}+\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^3}\quad\text{for } |\nu-\nu'|\neq 0,\ 1\leq k\leq|\nu-\nu'|^{-\alpha},\tag{5.69}$$

and

$$\left|\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k'}f(x)}d\Sigma\right|\lesssim\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k'}}{|k-k'|2^{j/2(1-4\alpha)}(2^{j/2}|\nu-\nu'|)^{1+4\alpha}},\quad\text{for } |\nu-\nu'|\neq 0,\ 1\leq k,k'\leq|\nu-\nu'|^{-\alpha},\ k\neq k'.\tag{5.70}$$

(5.69) and (5.70) imply:

$$\begin{aligned}&\left|\int_\Sigma U_j^{\nu}f(x)\overline{U_j^{\nu'}f(x)}d\Sigma\right| \qquad (5.71)\\
\leq\ & \sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\left|\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\right|+\sum_{1\leq k\neq k'\leq|\nu-\nu'|^{-\alpha}}\left|\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k'}f(x)}d\Sigma\right|\\
\lesssim\ & \sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}}+\sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^3}\\
&+\sum_{1\leq k\neq k'\leq|\nu-\nu'|^{-\alpha}}\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k'}}{|k-k'|2^{\frac{j}{2}(1-4\alpha)}(2^{j/2}|\nu-\nu'|)^{1+4\alpha}}\\
\lesssim\ & \frac{D^2\gamma_j^{\nu}\gamma_j^{\nu'}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}}+\frac{D^2\gamma_j^{\nu}\gamma_j^{\nu'}}{(2^{j/2}|\nu-\nu'|)^3},\end{aligned}$$

where we have used (5.68) and the fact that we may choose $0<\alpha<1/5$, together with the fact that:

$$\sup_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\sum_{1\leq k'\leq|\nu-\nu'|^{-\alpha},\,k'\neq k}\frac{1}{|k-k'|}\lesssim\alpha|\log(|\nu-\nu'|)|.\tag{5.72}$$

Since (5.71) yields the wanted estimate (5.61), we are left with proving (5.69) and (5.70). The discussion in the following section will be very informal for the sake of simplicity. We refer to [**43**] for the details.

5.2.3.4. *Proof of* (5.69). The estimate (5.69) will result of two integrations by parts with respect to tangential derivatives (in the spirit of section 3.5.1.1). Let us consider for instance the case where the two tangential derivatives fall on the symbol b of $U_j^{\nu,k}f$ defined in (5.66). This yields a term of the form

$$\int_{\mathbb{S}^2}\nabla\!\!\!/^2 b\left(\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)\phi_k(\lambda)f(\lambda\omega)\lambda^2d\lambda\right)\eta_j^\nu(\omega)d\omega.\tag{5.73}$$

Then, in view of the estimate (5.34) for b, we have in particular $\nabla\!\!\!/^2 b\in L^2(\Sigma)$ which will force us to estimate the λ integral in (5.73) in L^∞_u. To this end, we do Cauchy-Schwarz and obtain in particular the square root of the diameter of I_k. Due to (5.64), we thus gain an additional factor of $|\nu-\nu'|^\alpha$ with respect to (5.59), which yields (5.69).

Remark 5.16. *Here, the log-loss is removed by exploiting the size of the diameter of I_k. This is possible since we estimate the λ integral of* (5.73) *in L^∞_u using Cauchy Schwartz. In turn, this is a consequence of our estimate for $\nabla\!\!\!/^2 b$ in $L^2(\Sigma)$. This explains why this method cannot be used to remove the log-loss of the error term in Chapter 3. Indeed, our estimates for the space-time foliation in Chapter 4 are typically of the type $L^\infty_u L^2(\mathcal{H}_u)$, so that the integral in λ is estimated in L^2_u using Plancherel. In turn, this does not allow us to see the size of the localization in λ, so that a second decomposition in frequency of the type* (5.65) *would be useless in that case.*

5.2.3.5. *Proof of* (5.70). Note that we not only need to gain summability in (ν,ν') for this term, but also in (k,k'). This is achieved through the presence of the additional gain $k-k'$ in the right-hand side of (5.70). The estimate (5.70) will result of two integrations by parts, one with respect to the normal derivative N, and one with respect to tangential derivatives. We obtain a term analogous to (5.73)

$$\int_{\mathbb{S}^2}\nabla\!\!\!/\nabla_N b\left(\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)\phi_k(\lambda)f(\lambda\omega)\lambda^2d\lambda\right)\eta_j^\nu(\omega)d\omega. \tag{5.74}$$

In view of the estimate (5.34) for b, we have in particular $\nabla\!\!\!/\nabla_N b\in L^2(\Sigma)$. Thus, the log-loss of the summation in (ν,ν') is removed as in the previous section, in particular using the size of the diameter of I_k. Note also that the gain $k-k'$ in the right-hand side of (5.70) comes from the integration by parts in N. Indeed, we use

$$e^{i\lambda u-i\lambda' u'}=-\frac{ia}{\lambda-\lambda'\frac{a}{a'}g(N,N')}\nabla_N(e^{i\lambda u-i\lambda' u'}). \tag{5.75}$$

Now, since $\lambda\in I_k$, $\lambda\in I_{k'}$, we have in view of (5.64), the assumption (5.27) for $\partial^\alpha_\omega a$, and the assumption (5.28) for $\partial_\omega N$

$$\left|\lambda-\lambda'\frac{a}{a'}g(N,N')\right|\sim|k-k'|2^j|\nu-\nu'|^\alpha,$$

which yields the gain $k-k'$ in the right-hand side of (5.70).

5.2.4. Proof of Proposition 5.13 (control of the diagonal term). We have to prove (5.53):

$$\|U_j^\nu f\|_{L^2(\Sigma)}\lesssim D\gamma_j^\nu. \tag{5.76}$$

Recall that U_j^ν is given by:

$$U_j^\nu f(x)=\int_{\mathbb{S}^2}bF_j(u)\eta_j^\nu(\omega)d\omega, \tag{5.77}$$

where $F_j(u)$ is defined by:

$$F_j(u)=\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2d\lambda. \tag{5.78}$$

We decompose U_j^ν in the sum of two terms:

$$U_j^\nu f(x) = b(x,\nu)\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega + \int_{\mathbb{S}^2}(b(x,\omega)-b(x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega.$$

Then, using in particular the assumption (5.34) for b and the assumption (5.35) for $\partial_\omega b$, we obtain

$$\|U_j^\nu f\|_{L^2(\Sigma)} \lesssim D\left\|\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\Sigma)} + D\gamma_j^\nu. \tag{5.79}$$

In order to estimate the right-hand side of (5.79), we use the following proposition.

Proposition 5.17. *We have the following bound:*

$$\left\|\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\Sigma)} \lesssim \gamma_j^\nu. \tag{5.80}$$

The proof of Proposition 5.17 is postponed to the next section. Finally, (5.79) and (5.80) yield the wanted estimate (5.76) which concludes the proof of Proposition 5.13

5.2.4.1. *Proof of Proposition 5.17.* Recall that $\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega$ is given by:

$$\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.81}$$

Relying on the classical TT^* argument, (5.80) is equivalent to proving the boundedness on $L^2(\Sigma)$ of the operator whose kernel K is given by:

$$K(x,y) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,\ x,y\in\Sigma. \tag{5.82}$$

The decay satisfied by this kernel is stated in the following proposition.

Proposition 5.18. *The kernel K defined in* (5.82) *satisfies the following decay estimate for all x,y in Σ:*

$$\begin{aligned}|K(x,y)| \lesssim\ & \frac{2^j}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\ & \times\frac{2^j}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}.\end{aligned} \tag{5.83}$$

The proof of Proposition 5.18 is postponed to section 5.2.4.2. In the rest of this section, we show how (5.83) implies Proposition 5.17. According to Schur's Lemma, the operator whose kernel is K is bounded on $L^2(\Sigma)$ provided we can prove the following bound:

$$\sup_{x\in\Sigma}\int_\Sigma |K(x,y)|dy < +\infty,\ \sup_{y\in\Sigma}\int_\Sigma |K(x,y)|dx < +\infty. \tag{5.84}$$

Due to the symmetry of K in x,y, the two bounds in (5.84) are obtained in the same way. We focus on establishing the first bound. Using in particular (5.83) and the global change of variable on Σ given by (5.30)[3], we are able to obtain:

$$\int_\Sigma |K(x,y)|dy \lesssim \int_{\mathbb{R}^3} \frac{2^j}{(1+|2^j|\underline{y}\cdot\nu| - 2^{j/2}|\underline{y}'||)^2} \frac{2^j}{(1+2^{j/2}|\underline{y}'|)^3} d\underline{y}, \tag{5.85}$$

where $y = y\cdot\nu + y'$ and $y'\cdot\nu = 0$. Making the change of variable $y \to z$ where z is defined by $z\cdot\nu = 2^j\underline{y}\cdot\nu$ and $z' = 2^{j/2}\underline{y}'$ in the right-hand side of (5.85), and remarking that $z\cdot\nu$ is one dimensional, and z' is two dimensional, we obtain:

$$\int_\Sigma |K(x,y)|dy \lesssim \int_{\mathbb{R}^3} \frac{dz}{(1+||z\cdot\nu| - |z'||)^2(1+|z'|)^3} \lesssim 1. \tag{5.86}$$

(5.86) implies the first bound in (5.84). K being symmetric with respect to x, y, the second bound in (5.84) is also true. Thus, the operator whose kernel is K is bounded on $L^2(\Sigma)$ which concludes the proof of Proposition 5.17.

5.2.4.2. *Proof of Proposition 5.18.* Recall the definition of K:

$$K(x,y) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega) - i\lambda u(y,\omega)} \psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,\ x, y \in \Sigma. \tag{5.87}$$

We need to prove that K satisfies the following decay estimate for all x, y in Σ:

$$\begin{aligned} |K(x,y)| \lesssim & \frac{2^j}{(1+|2^j|u(x,\nu) - u(y,\nu)| - 2^{j/2}|\partial_\omega u(x,\nu) - \partial_\omega u(y,\nu)||)^2} \\ & \times \frac{2^j}{(1+2^{j/2}|\partial_\omega u(x,\nu) - \partial_\omega u(y,\nu)|)^3}. \end{aligned} \tag{5.88}$$

For the sake of simplicity, let us just describe the general strategy of the proof of Proposition 5.18. In view of the regularity for $u(x,\omega)$ with respect to ω provided by (5.24) and (5.26)-(5.29), we have

$$|u(x,\omega)| + |\partial_\omega u(x,\omega)| + |\partial_\omega^2 u(x,\omega)| + |\partial_\omega^3 u(x,\omega)| \lesssim 1 + |x|,\ \forall x \in \Sigma,\ \forall \omega \in \mathbb{S}^2. \tag{5.89}$$

This regularity allows us to integrate by part three times with respect to ω, while we may integrate as much as we want with respect to λ. The estimate (5.88) is then obtained after performing in (5.87) three integrations by parts with respect to ω and two integrations by parts with respect to λ.

5.3. Control of the parametrix at initial time

In this section, we discuss the proof of Theorem 5.9. To this end, we first show that the Fourier integral operator U of Theorem 5.6 almost preserves the L^2 norm provided we make additional assumptions on its symbol. We then use this observation to prove the estimate (5.41). Finally, we conclude the proof of Theorem 5.9 by establishing the existence and uniqueness of (f_+, f_-) solution of the system (5.40).

[3]using also the bound on the Jacobian (5.31)

5.3.1. A refinement of Theorem 5.6. In Theorem 5.6, we have proved that the Fourier integral operator U with phase u and symbol b is bounded on $L^2(\Sigma)$ provided u satisfies (5.24), (5.26)-(5.29) and (5.30)-(5.31), and the symbol b satisfies (5.34) (5.35). We now would like to prove that U satisfies the following bound from below:

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|Uf\|_{L^2(\Sigma)}, \tag{5.90}$$

provided u also satisfies (5.32) and under additional assumptions on the symbol b. This is the aim of the following proposition.

PROPOSITION 5.19. *Let u be a function on $\Sigma \times \mathbb{S}^2$ satisfying suitable assumptions (we refer to* [**43**] *for the complete set of assumptions, and to section 5.1.3 for some typical assumptions). Let U the Fourier integral operator with phase $u(x,\omega)$ and symbol $b(x,\omega)$:*

$$Uf(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega)} b(x,\omega) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.91}$$

We assume furthermore that $b(x,\omega)$ satisfies:

$$\|\partial_\omega b\|_{L^2(\Sigma)} + \|\nabla\partial_\omega b\|_{L^2(\Sigma)} \lesssim 1, \tag{5.92}$$

$$\|b-1\|_{L^\infty(\mathcal{M})} + \|\nabla b\|_{L^\infty_u L^2(P_u)} + \|\nabla\!\!\!/\nabla b\|_{L^2(\Sigma)} \lesssim \varepsilon, \tag{5.93}$$

and

$$\begin{aligned}&\nabla_N b = b_1^j + b_2^j \text{ where } \|b_1^j\|_{L^2(\Sigma)} \lesssim 2^{-\frac{j}{2}}\varepsilon,\ \|b_2^j\|_{L^\infty_u L^2(P_u)} \lesssim \varepsilon\\ &\text{and } \|\nabla_N b_2^j\|_{L^2(\Sigma)} + \|b_2^j\|_{L^2_u L^\infty(P_u)} \lesssim 2^{\frac{j}{2}}\varepsilon.\end{aligned} \tag{5.94}$$

Then, U is bounded on L^2 and satisfies the estimate:

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|Uf\|_{L^2(\Sigma)}. \tag{5.95}$$

REMARK 5.20. *Notice that the only difference in the assumptions with respect to Theorem 5.6 lies in the fact that u also satisfies* (5.32) *and in the constant D which has been replaced by 1 in* (5.92) *and by ε in* (5.93) (5.94)*.*

The proof of Proposition 5.19 uses the decomposition in frequency and angle of the operator U introduced in section 5.2. In order to control the diagonal term in a third step (see next section), we have to modify slightly the size of the support of our partition of unity η_j^ν on $\mathbb{S}^2$ introduced in (5.47). Let $\delta > 0$ such that:

$$0 < \sqrt{\varepsilon} \ll \delta \ll 1. \tag{5.96}$$

We now require that the support of η_j^ν is a patch on $\mathbb{S}^2$ of diameter $\sim \delta 2^{-j/2}$. With this modification, the assumptions for b in Proposition 5.19, and by carefully tracking the size of the various terms in the almost orthogonality argument in frequency and angle, and the control of the diagonal term, we obtain

$$\|Uf\|^2_{L^2(\Sigma)} = \sum_{|j-l|\leq 2}\ \sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}} \int_\Sigma S_j^\nu f(x)\overline{S_l^{\nu'} f(x)} d\Sigma + O\left(\frac{\varepsilon}{\delta^2} + \delta\right)\|f\|^2_{L^2(\mathbb{R}^3)}, \tag{5.97}$$

where the operator S_j^ν is defined on Σ by:

$$S_j^\nu f(x) = \int_\Sigma \int_0^{+\infty} e^{i\lambda u(x,\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.98}$$

Recall (5.30)-(5.31) which states that the map $\phi_\nu : \Sigma \to \mathbb{R}^3$ defined by:

$$\phi_\nu(x) := u(x,\nu)\nu + \partial_\omega u(x,\nu), \tag{5.99}$$

is a bijection, such that the determinant of its Jacobian satisfies the following estimate:

$$\||\det(\text{Jac}\phi_\nu)| - 1\|_{L^\infty(\mathcal{M})} \lesssim \varepsilon. \tag{5.100}$$

Let us note $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^3$. We introduce the operator $\widetilde{S}_j^\nu$ on Σ defined by:

$$\widetilde{S}_j^\nu f(x) = \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\eta_j^\nu f)(\phi_\nu(x)) = \int_{\mathbb{R}^3} e^{i\lambda\phi_\nu(x)\cdot\omega}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{5.101}$$

The following proposition shows that the term $\int_\Sigma S_j^\nu f(x)\overline{S_l^{\nu'}f(x)}d\Sigma$ is close to the term $\int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'}f(x)}d\Sigma$.

Proposition 5.21. *We have the following bound:*

$$\|S_j^\nu f - \widetilde{S}_j^\nu f\|_{L^2(\Sigma)} \lesssim \delta^{\frac{1}{2}}\gamma_j^\nu. \tag{5.102}$$

The proof of Proposition 5.21 relies on the classical TT^* argument, and the comparison between $u(x,\omega)$ and $\phi_\nu(x)\cdot\omega$ provided by (5.32) (see [**43**] for the details). Now, (5.97) and (5.102) yield:

$$\|Uf\|^2_{L^2(\Sigma)} = \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}} \int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'}f(x)}d\Sigma + O\left(\frac{\varepsilon}{\delta^2} + \delta^{\frac{1}{2}}\right)\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{5.103}$$

Making the change of variable $y = \phi_\nu(x)$ in $\int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'}f(x)}d\Sigma$ and using (5.100) and (5.101) implies:

$$\begin{aligned} & \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}} \int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'}f(x)}d\Sigma \\ = & \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}} \int_{\mathbb{R}^3} \mathcal{F}^{-1}(\psi(2^{-j}\cdot)\eta_j^\nu f)(y)\overline{\mathcal{F}^{-1}(\psi(2^{-l}\cdot)\eta_j^{\nu'}f)(y)}dy \\ & +O(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^3)} \\ = & \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}} \int_{\mathbb{R}^3} \psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\overline{\psi(2^{-l}\lambda)\eta_j^{\nu'}(\omega)f(\lambda\omega)}dy \\ & +O(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{5.104}$$

where we have used the fact that $\mathcal{F}^{-1}$ is an isomorphism on $L^2(\mathbb{R}^3)$ in the last equality of (5.104). Now, we have:

$$\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_{\mathbb{R}^3}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\overline{\psi(2^{-l}\lambda)\eta_j^{\nu'}(\omega)f(\lambda\omega)}dy = \|f\|^2_{L^2(\mathbb{R}^3)}, \quad (5.105)$$

which together with (5.103) and (5.104) yields:

$$\|Uf\|^2_{L^2(\Sigma)} = \|f\|^2_{L^2(\mathbb{R}^3)} + O\left(\frac{\varepsilon}{\delta^2}+\delta^{\frac{1}{2}}\right)\|f\|^2_{L^2(\mathbb{R}^3)}. \quad (5.106)$$

Choosing $\delta^{\frac{1}{2}}$ and $\varepsilon\delta^{-2}$ small enough, we deduce from (5.106):

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|Uf\|_{L^2(\Sigma)}, \quad (5.107)$$

which is the wanted estimate. This conclude the proof of Proposition 5.19.

5.3.2. Proof of Theorem 5.9. The symbol of the Fourier integral operators $M_\pm$ and $Q_\pm$ are respectively given by 1 and $a(x,\pm\omega)^{-1}$. Thus, they clearly satisfy the assumptions of Proposition 5.19. Relying on Proposition 5.19, we are then able to prove the estimate (5.41). We refer to [**43**] for the details.

The uniqueness of (f_+, f_-) solution of the system (5.40) is an immediate consequence of the estimate (5.41), so there only remains to prove the existence of (f_+, f_-) to conclude the proof of Theorem 5.9. Recall that the phase $u(x,\omega)$ of our Fourier integral operators has been constructed in [**42**] (see also Chapter 6) on $\Sigma\times\mathbb{S}^2$ under the assumption that (Σ, g, k) satisfies the following bounds consistent with the bounded L^2 curvature conjecture:

$$\|R\|_{L^2(\Sigma)} \leq \varepsilon,\ \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon, \quad (5.108)$$

where the fact that we may take $\varepsilon > 0$ small comes from a reduction to the small data case. (Σ, g, k) also satisfies the constraint equations and the maximal foliation assumption

$$\begin{cases} \nabla^j k_{ij} = 0, \\ R = |k|^2, \\ \mathrm{Tr}k = 0. \end{cases} \quad (5.109)$$

We introduce two sets V and W:

$$V = \{(\Sigma, g, k) \text{ such that (5.108) and (5.109) are satisfied}\}, \quad (5.110)$$

and

$$\begin{aligned} W = \{ \quad & (\Sigma, g, k) \in V \text{ such that } (f_+, f_-) \text{ solution of (5.40) exist for all } (\phi_0, \phi_1) \\ & \text{such that } \nabla\phi_0 \in L^2(\Sigma) \text{ and } \phi_1 \in L^2(\Sigma)\}. \end{aligned} \quad (5.111)$$

Not first that W is not empty since $(\Sigma, g, k) = (\mathbb{R}^3, \delta, 0)$ belongs to W in view of Remark 5.2. We then show that V is connected and W is both open and closed in V for a suitable topology (see [**43**] for the details). We infer $W = V$. This proves the existence of (f_+, f_-) solution of (5.40) and concludes the proof of Theorem 5.9.

CHAPTER 6

Control of the foliation at initial time

In this chapter, we will only consider $u(t, x, \omega)$ and Σ_t at $t = 0$. Thus, for simplicity, we denote in the rest of this chapter $u(0, x, \omega)$ by $u(x, \omega)$ and Σ_0 by Σ. The goal of this chapter is to prove the estimates on the control of the foliation of Σ by $u(x, \omega)$ which are needed for the proof of Theorem 5.9 (see section 5.1.3), i.e. for the control of the parametrix at initial time. The estimates obtained for $u(x, \omega)$ in this chapter must also be consistent with the control on $\mathcal{M}$ for $u(t, x, \omega)$ obtained in Chapter 4 (see section 4.1.5). Here, we outline the main ideas and we refer to [**42**] for the details.

6.1. Geometric set-up and main results

6.1.1. Reduction to small data. Recall from section 1.2.3 that we have reduced ourselves to an asymptotically flat initial data set (Σ, g, k) solution to the constraint equations which satisfies the bounds

$$\|R\|_{L^2(\Sigma)} \leq \varepsilon, \ \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon, \tag{6.1}$$

and is smooth outside of a small neighborhood U. In order to construct $u(x, \omega)$ satisfying the asymptotic behavior $u(x, \omega) \sim x \cdot \omega$ when $|x| \to +\infty$ on Σ, we need to modify (Σ, g, k) outside of U. We can glue it in a trivial way to $(\mathbb{R}^3, \delta, 0)$ so that the new initial data set is still smooth outside of U, satisfies (6.1), and coincides with $(\mathbb{R}^3, \delta, 0)$ outside of a slightly larger neighborhood. We still denote this initial data set (Σ, g, k). Of course, (Σ, g, k) does not satisfies the constraint equations in the annulus where the gluing takes place. However, for the construction of $u(x, \omega)$, we only require (Σ, g, k) to satisfy the constraint equations in U. Outside of U, (Σ, g, k) is smooth, so things are much easier.

6.1.2. Geometry of the foliation of Σ by a scalar function u. We define the lapse $a = |\nabla u|^{-1}$, and the unit vector N such that $\nabla u = a^{-1} N$. We also define the level surfaces $P_{u_0} = \{x \,/\, u = u_0\}$ so that N is the normal to P_u. The second fundamental form θ of P_u is defined by

$$\theta(X, Y) = g(\nabla_X N, Y) \tag{6.2}$$

with X, Y arbitrary vectorfields tangent to the u-foliation P_u of Σ and where ∇ denotes the covariant differentiation with respect to g. We denote by $\mathrm{tr}\theta$ the trace of θ, i.e. $\mathrm{tr}\theta = \delta^{AB}\theta_{AB}$ where θ_{AB} are the components of θ relative to an orthonormal frame $(e_A)_{A=1,2}$ on P_u.

6.1.3. Structure equations of the foliation of Σ by a scalar function u. We recall some of the structure equations of the foliation of Σ by a scalar function u which will be needed for the discussion of the main result of this chapter (see [**9**] for a proof).

PROPOSITION 6.1. *The orthonormal frame frame $N, e_A, A = 1,2$ of Σ satisfies the following system:*

$$\begin{cases} \nabla_A N = \theta_{AB} e_B, \\ \nabla_N N = -a^{-1} \nabla\!\!\!\!/ a. \end{cases} \tag{6.3}$$

Also, the lapse a satisfies the following equation

$$a^{-1} \Delta\!\!\!\!/ (a) = -\nabla_N tr\theta - |\theta|^2 + R_{NN}, \tag{6.4}$$

where $\Delta\!\!\!\!/$ is the Laplace-Beltrami for the metric γ on P_u induced by g. Finally, the second fundamental form θ satisfies the following Codazzi equation

$$\nabla\!\!\!\!/^B \theta_{AB} = \nabla\!\!\!\!/_A tr\theta + R_{NA}. \tag{6.5}$$

6.1.4. Choice of $u(x, \omega)$. We look for $u(x, \omega)$ satisfying the three following conditions:

- **(a)** *$u(x, \omega) \sim x.\omega$ when $|x| \to +\infty$ on Σ*
- **(b)** *The regularity of $u(x, \omega)$ with respect to x and ω is consistent with the regularity of $u(t, x, \omega)$ with respect to (t, x) and ω obtained in Chapter 4 (see section 4.1.5). In particular, we have $tr\theta - k_{NN} \in L^\infty(\Sigma)$ (see the discussion in section 5.1.1.1)*
- **(c)** *$u(x, \omega)$ has as enough regularity in x and ω to control the parametrix at initial time, i.e. to obtain the conclusion of Theorem 5.9*

where the initial data set (Σ, g, k) satisfies

$$\begin{cases} \nabla^j k_{ij} = 0, \\ R = |k|^2, \\ \text{tr}_g k = 0, \end{cases} \tag{6.6}$$

in U (see section 6.1.1), and where R and ∇k are in $L^2(\Sigma)$ and satisfy the smallness assumption (6.1).

In order to motivate our choice of $u(x, \omega)$, we investigate the regularity of the lapse a, which by (6.4) satisfies the following equation:

$$a^{-1} \Delta\!\!\!\!/ (a) = -\nabla_N \text{tr}\theta - |\theta|^2 - R_{NN}. \tag{6.7}$$

Since R is in $L^2(\Sigma)$, (6.7) implies that a has at most two derivatives in $L^2(\Sigma)$. Thus, $u(x, \omega)$ has at most three derivatives with respect to x in $L^2(\Sigma)$. This is not enough to satisfy **(c)**. In fact, the classical T^*T argument (see for example [**39**]) relies on integrations by parts in x and would require at least one more derivative since Σ has dimension 3.

Alternatively, we could try to use the TT^* argument which relies on integrations by parts in ω. Indeed, R being independent of ω, one would expect the regularity of $u(x, \omega)$ with respect to ω to be better. Differentiating (6.7) with respect to ω, we obtain:

$$a^{-1} \Delta\!\!\!\!/ (\partial_\omega a) = 2 \nabla\!\!\!\!/ \nabla_N a + \cdots, \tag{6.8}$$

where the term on the right-hand side comes from the commutator $[\partial_\omega, \not\Delta]$ (see section 6.4). Thus, obtaining an estimate for $\partial_\omega a$ from (6.8) requires to control $\nabla_N a$. Unfortunately, (6.7) seems to give control of tangential derivatives of a only. This is where the specific choice of $u(x,\omega)$ comes into play.

Having in mind the equation of minimal surfaces (i.e. $\mathrm{tr}\theta = 0$), condition **(b)** suggest the choice $\mathrm{tr}\theta - k_{NN} = 0$. Unfortunately, this equation together with (6.7) does not provide any control of $\nabla_N a$. We might propose as a second natural guess to take instead $\mathrm{tr}\theta - k_{NN} = \nabla_N a$. Plugging in (6.7) yields an elliptic equation for a: $\nabla_N^2 a + a^{-1}\not\Delta(a) = -|\theta|^2 - \nabla_N k_{NN} - R_{NN}$. This allows us to control $\nabla_N^2 a$ in $L^2(\Sigma)$. However, $\mathrm{tr}\theta - k_{NN} = \nabla_N a$, and $\nabla_N a$ is at most in $H^1(\Sigma)$ which does not embed in $L^\infty(\Sigma)$ - since Σ has dimension 3 - so that condition **(b)** is not satisfied. To sum up, the first guess $\mathrm{tr}\theta - k_{NN} = 0$ satisfies **(b)**, but not **(c)**, whereas the second guess $\mathrm{tr}\theta - k_{NN} = \nabla_N a$ might satisfy **(c)**, but does not satisfy **(b)**.

The correct choice is the intermediate one

$$\mathrm{tr}\theta - k_{NN} = 1 - a. \tag{6.9}$$

We will see in section 6.2 that $a - 1$ belongs to $L^\infty(\Sigma)$ so that **(b)** is satisfied. Also, plugging (6.9) in (6.7) yields the parabolic equation:

$$\nabla_N a - a^{-1}\not\Delta(a) = |\theta|^2 + \nabla_N k_{NN} + R_{NN}. \tag{6.10}$$

This will allow us to control normal derivatives of a. In turn, we will control derivatives of a with respect to ω using (6.8). Ultimately, we will prove enough regularity with respect to both x and ω, such that **(c)** is satisfied.

6.1.5. Main results. For $1 \leq p, q \leq +\infty$, we define the spaces $L^p_u L^q(P_u)$ for tensors F on Σ using the norm:

$$\|F\|_{L^p_u L^q(P_u)} = \left(\int_u \|F\|^p_{L^q(P_u)} du\right)^{1/p}.$$

REMARK 6.2. *In the rest of the paper, all inequalities hold for any $\omega \in \mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.*

We first state a result of existence and regularity with respect to x for u.

THEOREM 6.3. *Let (Σ, g, k) chosen as in section 6.1.1. There exists a scalar function u on $\Sigma \times \mathbb{S}^2$ satisfying assumption* **(a)** *and such that:*

$$\begin{aligned} &\|a-1\|_{L^\infty_u L^2(P_u)} + \|\nabla a\|_{L^\infty_u L^2(P_u)} + \|a-1\|_{L^\infty(\Sigma)} + \|\not\nabla\nabla a\|_{L^2(\Sigma)} \lesssim \varepsilon,\\ &\|tr\theta - k_{NN}\|_{L^\infty(\Sigma)} + \|\nabla\theta\|_{L^2(\Sigma)} \lesssim \varepsilon, \end{aligned} \tag{6.11}$$

where P_u, a, N and θ are associated to u as in section 6.1.2.

Notice that condition **(b)** is included in (6.11). In order to state our second result, we introduce fractional Sobolev spaces $H^b(P_u)$ on the surfaces P_u for any $b \in \mathbb{R}$ (see [**42**] for their precise definition). We have the following estimate for $\nabla_N^2 a$, and improved estimate for $\nabla_N a$.

THEOREM 6.4. *Let (Σ, g, k) chosen as in section 6.1.1. Let u the scalar function on $\Sigma\times\mathbb{S}^2$ constructed in theorem 6.3, and let P_u, a and N be associated to u as in section 6.1.2. We have:*

$$\|\nabla_N a\|_{L^\infty_u L^4(P_u)} + \|\nabla_N^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)} \lesssim \varepsilon. \tag{6.12}$$

The third theorem investigates the regularity of u with respect to ω:

THEOREM 6.5. *Let (Σ, g, k) chosen as in section 6.1.1. Let u the scalar function on $\Sigma\times\mathbb{S}^2$ constructed in theorem 6.3, and let P_u, a, N and θ be associated to u as in section 6.1.2. We have:*

$$\begin{aligned}\|\partial_\omega a\|_{L^\infty(\Sigma)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_u L^2(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(\Sigma)} + \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}\\ +\|\nabla_N^2\partial_\omega a\|_{L^2_u H^{-\frac{3}{2}}(P_u)} + \|\nabla\partial_\omega\theta\|_{L^2(\Sigma)} \lesssim \varepsilon, \quad \|\partial_\omega N\|_{L^\infty(\Sigma)} \lesssim 1,\end{aligned} \tag{6.13}$$

$$\begin{aligned}\|\partial_\omega^2 a\|_{L^2_u H^{\frac{3}{2}}(P_u)} + \|\partial_\omega^2 a\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N\partial_\omega^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)} + \|\nabla\partial_\omega^2\theta\|_{L^2(\Sigma)} \lesssim \varepsilon,\\ \|\partial_\omega^2 N\|_{L^\infty(\Sigma)} \lesssim 1,\end{aligned} \tag{6.14}$$

and

$$\|\partial_\omega^3 u\|_{L^\infty_{loc}(\Sigma)} \lesssim 1. \tag{6.15}$$

6.1.6. Additional results. The following proposition establishes the existence of a global coordinate system on Σ.

PROPOSITION 6.6. *Let $\omega\in\mathbb{S}^2$. Let $\phi_\omega:\Sigma\to\mathbb{R}^3$ defined by:*

$$\phi_\omega(x) := u(x,\omega)\omega + \partial_\omega u(x,\omega). \tag{6.16}$$

Then ϕ_ω is a bijection, and the determinant of its Jacobian satisfies the following estimate:

$$\||\det(Jac\phi_\omega)| - 1\|_{L^\infty(\Sigma)} \lesssim \varepsilon. \tag{6.17}$$

Below, we state several additional estimates. We start with a first proposition.

PROPOSITION 6.7. *Let (Σ, g, k) chosen as in section 6.1.1. Let u the scalar function on $\Sigma\times\mathbb{S}^2$ constructed in theorem 6.3. Let $\nu\in\mathbb{S}^2$ and ϕ_ν the map defined in* (6.16)*. Then, we have for all $x\in\Sigma$ and $\omega\in\mathbb{S}^2$*

$$\begin{aligned}&u(x,\omega) - \phi_\nu(x)\cdot\omega = O(\varepsilon|\omega-\nu|^2),\\ &\partial_\omega u(x,\omega) - \partial_\omega(\phi_\nu(x)\cdot\omega) = O(\varepsilon|\omega-\nu|),\\ &\partial_\omega^2 u(x,\omega) - \partial_\omega^2(\phi_\nu(x)\cdot\omega) = O(\varepsilon).\end{aligned} \tag{6.18}$$

Using the geometric Littlewood Paley projections P_j on P_u constructed in [**24**] (see section 3.1.5) together with the estimates for $\nabla_N a$ in (6.11), and the estimate for $\nabla_N^2 a$ in (6.12), we obtain the following proposition:

PROPOSITION 6.8. *Let (Σ, g, k) chosen as in section 6.1.1. Let u the scalar function on $\Sigma \times \mathbb{S}^2$ constructed in theorem 6.3, and let a and N be associated to u as in section 6.1.2. For all $j \geq 0$, there are scalar functions a_1^j and a_2^j such that:*

$$\nabla_N a = a_1^j + a_2^j \text{ where } \|a_1^j\|_{L^2(\Sigma)} \lesssim 2^{-\frac{j}{2}}\varepsilon \text{ and } \|\nabla_N a_2^j\|_{L^2(\Sigma)} \lesssim 2^{\frac{j}{4}}\varepsilon. \tag{6.19}$$

REMARK 6.9. *Recall from section 6.1.4 that we do not have enough regularity in x to apply the T^*T method. Alternatively, we could try the TT^* method which relies on integration by parts in ω. But $\partial_\omega^3 u \in L^\infty(\Sigma)$ is also not enough and we would need at least one more derivative in ω. Nevertheless, using the regularity in x and ω obtained for u in the present and the previous section, we are able to control the parametrix at initial time (see Chapter 5).*

Let us conclude this section by mentioning several ingredients of [**42**] that have been omitted here for the sake of simplicity, and that have to be proved by relying on low regularity assumptions for u which are consistent with the results stated in this section and the previous one:

- estimates for the parabolic operator $(\nabla_N - a^{-1}\Delta\!\!\!/)$
- estimates for θ and N
- product estimates in the Sobolev spaces $H^b(P_u)$
- embeddings on Σ and P_u
- a control of the Gauss curvature of P_u
- Bochner inequalities on P_u
- estimates for various commutator terms of the type: $[\nabla_N, \nabla\!\!\!/]$, $[\nabla_N, P_j]$, ...

The rest of this chapter is as follows. In section 6.2, we discuss the proof of Theorem 6.3. In section 6.3, we discuss the proof of Theorem 6.4. In section 6.4, we discuss the proof of Theorem 6.5. In section 6.5, we discuss the proof of Proposition 6.13 and Proposition 6.6. Finally, Proposition 6.7 and Proposition 6.8 are discussed in section 6.6.

6.2. Construction of the foliation and regularity with respect to x

In this section, we discuss the proof of Theorem 6.3. By section 6.1.1, we may assume that (Σ, g, k) coincides with $(\mathbb{R}^3, \delta, 0)$ outside of a compact, say $|x| \geq 1$. Notice that in $|x| \geq 1$ and for all $\omega \in \mathbb{S}^2$, the scalar function $x.\omega$ satisfies the equation (6.9) and the estimate (6.11), since $a \equiv 1, \theta \equiv 0$ and $N \equiv \omega$ in this region. Thus, we would like to construct a function u solution of (6.9) satisfying (6.11) in a region containing $|x| \leq 2$ and to glue it to $x.\omega$ in $1 \leq |x| \leq 2$. Now, (6.9) is of parabolic type - see (6.10) - where u plays the role of time. For each fixed $\omega \in \mathbb{S}^2$, we start with $u = -2$ on $x \cdot \omega = -2$. Then, we propagate with the parabolic equation (6.10), coupled with the equation for θ (6.5), to the strip $S = \{x \in \Sigma \text{ such that } -2 < u(x, \omega) < 2\}$. This strip covers the entire region $|x| \leq 1$, and we then glue u to $x \cdot \omega$ outside of $|x| \leq 1$ (see section 6.2.2). In the next section, we prove a priori estimates consistent with the estimate (6.11) and valid on

$-2 < u < 2$ for the solution u of:

$$\mathrm{tr}\theta - k_{NN} = 1 - a, \text{ on } -2 < u < 2, \tag{6.20}$$

where u is initialized on $x.\omega = -2$ by:

$$u(x, \omega) = -2 \text{ on } x.\omega = -2. \tag{6.21}$$

Note that the first equation of (6.25), (6.21) and the fact that (g, k, Σ) coincides with $(\delta, 0, \mathbb{R}^3)$ for $|x| \geq 2$ yields:

$$\nabla^p(a-1) = 0,\ \nabla^p\theta = 0,\ \nabla^p(N - \omega) = 0 \text{ for all } p \in \mathbb{N} \text{ on } u = -2. \tag{6.22}$$

6.2.1. A priori estimates for lower order derivatives. Let (Σ, g, k) chosen as in section 6.1.1. In particular, we assume:

$$\|\nabla k\|_{L^2(\Sigma)} + \|R\|_{L^2(\Sigma)} \leq \varepsilon. \tag{6.23}$$

Let u a scalar function on $\Sigma \times \mathbb{S}^2$, and let P_u, a, N and θ be associated to u as in section 6.1.2. Assume that u satisfies the additional equation (6.9), which we recall below together with (6.3) and (6.4):

$$\begin{cases} \nabla_A N = \theta_{AB} e_B, \\ \nabla_N N = -\nabla\!\!\!/ \log(a), \end{cases} \tag{6.24}$$

and

$$\begin{cases} \mathrm{tr}\theta - k_{NN} = 1 - a, \\ \nabla_N a - a^{-1}\Delta\!\!\!/(a) = |\theta|^2 + \nabla_N k_{NN} + R_{NN}. \end{cases} \tag{6.25}$$

In this section, we establish a priori estimates for a, N and θ corresponding to (6.11) in the region S of Σ between P_{-2} and P_2 (i.e. $S = \{x\, /\, -2 < u(x, \omega) < 2\}$) where u is initialized on $x.\omega = -2$ by (6.21). In particular, we have (6.22), so that the subsequent integrations by parts will not create boundary terms at $u = -2$.

For the sake of simplicity, let us just discuss the estimate (6.11) for the lapse a. We rewrite the second equation of (6.25) as:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(a - 1) = h, \tag{6.26}$$

where h is given by:

$$h = \nabla_N k_{NN} + R_{NN} + \cdots. \tag{6.27}$$

Using in particular (6.23), we obtain:

$$\|h\|_{L^2(\Sigma)} \lesssim \varepsilon.$$

Together with (6.26) and an L^2 parabolic estimate for the operator $(\nabla_N - a^{-1}\Delta\!\!\!/)$, we obtain

$$\|a - 1\|_{L^\infty_u L^2(P_u)} + \|\nabla\!\!\!/ a\|_{L^\infty_u L^2(P_u)} + \|\nabla_N a\|_{L^2(\Sigma)} + \|\nabla\!\!\!/^2 a\|_{L^2(\Sigma)} \lesssim \varepsilon. \tag{6.28}$$

In order to obtain estimates for $\nabla\!\!\!/\nabla_N a$ and $\nabla_N^2 a$, we differentiate the second equation of (6.25) by ∇_N:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N a = \nabla_N^2 k_{NN} + \nabla_N R_{NN} + \cdots, \tag{6.29}$$

where we only kept two typical terms. Note that $\nabla_N^2 k_{NN}$ and $\nabla_N R_{NN}$ are dangerous terms which cannot be estimated directly. We first need to trade a ∇_N derivative with a $\nabla\!\!\!/$ derivative using Bianchi identities and the constraints equations. We use the twice-contracted Bianchi identity on Σ

$$\nabla^j R_{ij} = \frac{1}{2}\nabla_i R. \tag{6.30}$$

In particular, using also the constraint equations (6.6), we have

$$\nabla_N R_{NN} = -\nabla_A R_{AN} + \nabla_N |k|^2.$$

Also, the constraint equations (6.6) yield

$$\nabla_N^2 k_{NN} = -\nabla_N \nabla_A k_{AN} + \cdots = -\nabla_A \nabla_N k_{AN} + \cdots .$$

Together with (6.29), we obtain

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N a = \mathrm{div}\!\!\!/(H) + h_1, \tag{6.31}$$

where

$$H = -\nabla_N k_{.N} - R_{.N}, \tag{6.32}$$

and h_1 satisfies

$$\|h_1\|_{L^2_u L^{\frac{4}{3}}(P_u)} \lesssim \varepsilon. \tag{6.33}$$

Using the smallness assumption (6.23) and the definition of H (6.32), we have

$$\|H\|_{L^2(\Sigma)} \lesssim \varepsilon \tag{6.34}$$

which together with (6.33), (6.31), and an L^2 parabolic estimate for the operator $(\nabla_N - a^{-1}\Delta\!\!\!/)$, yields

$$\|\nabla_N a\|_{L^\infty_u L^2(P_u)} + \|\nabla\!\!\!/\nabla_N a\|_{L^2(\Sigma)} \lesssim \varepsilon. \tag{6.35}$$

Finally, (6.28) and (6.35) yield the wanted estimate (6.11).

6.2.2. End of the proof of Theorem 6.3. We briefly sketch the rest of the proof of Theorem 6.3, and we refer to [**42**] for the details. In (6.28) and (6.35), we have obtained a priori estimates consistent with the estimate (6.11) and valid on $-2 < u < 2$ for the solution u of (6.20). Then, we also prove on $-2 < u < 2$ a priori estimates for higher derivatives of the solution u of (6.20). We then use the existence of u solution to[1]:

$$\begin{cases} \mathrm{tr}\theta - k_{NN} = 1 - a, \text{ on } \alpha < u < \alpha + T, \\ u = \alpha \text{ on } \underline{u} = \alpha, \end{cases} \tag{6.36}$$

where $-2 \leq \alpha \leq 2$, $\underline{u}$ is smooth, and $T > 0$ is small enough. Together with the a priori estimates, this allows us to control the solution of (6.36) on $-2+kT < u < -2+(k+1)T$ uniformly with respect to $k = 0, \dots, [4/T]$ in order to obtain a solution u of (6.20) on $-2 < u < 2$. Finally, we conclude the proof of Theorem 6.3 by showing how to glue the

[1]this local existence result could be proved either using a Nash Moser procedure or a combination of Cauchy-Kowalewska and enhanced a priori estimates for all derivatives

solution u of (6.20) to $x.\omega$ in $1 \leq |x| \leq 2$ in order to obtain a solution on Σ satisfying (6.11).

6.3. Estimates for $\nabla_N a$ and $\nabla_N^2 a$

In this section, we discuss the proof of Theorem 6.4. Recall the decomposition (6.31), (6.32), and the estimate (6.33). We introduce the scalar functions on S a_1 and a_2 solutions of:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)a_1 = h_1 \text{ on S, } a_1(-2,.) = 0, \tag{6.37}$$

and:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)a_2 = \text{di}\!\!\!/\text{v}(H) \text{ on S, } a_2(-2,.) = 0, \tag{6.38}$$

which yields, in view of (6.31) and (6.22), the decomposition:

$$\nabla_N a = a_1 + a_2. \tag{6.39}$$

The idea behind the decomposition (6.39) is to take advantage of the better regularity of h_1 for a_1 (see (6.33) compared to (6.34)), and to use the structure of $\text{di}\!\!\!/\text{v}(H)$ to obtain a useful equation for $\nabla_N a_2$. Indeed, in view of the equation (6.38) satisfied by a_2, $\nabla_N a_2$ satisfies:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(\nabla_N a_2) = \nabla_N(\text{di}\!\!\!/\text{v}(H)) + \cdots, \tag{6.40}$$

and using in particular the twice-contracted Bianchi identity on Σ, the constraint equations in the maximal foliation (6.6), and (6.40), we obtain

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N a_2 = \text{di}\!\!\!/\text{v}\,\text{di}\!\!\!/\text{v}(H_1) + \text{di}\!\!\!/\text{v}(H_2) + h_2 + \cdots, \tag{6.41}$$

where the tensors H_1, H_2 and the scalar h_2 satisfy

$$\|H_1\|_{L^2(S)} + \|H_2\|_{L^2_u L^{\frac{4}{3}}(P_u)} + \|h_2\|_{L^1(S)} \lesssim \varepsilon.$$

These ideas allow us to derive the following two propositions (see [**42**] for the detailed proof of these propositions).

Proposition 6.10. *Let a_1 be the solution of* (6.37)*, where h_1 satisfies* (6.33)*. Then, we have:*

$$\|a_1\|_{L^\infty_u L^4(P_u)} \lesssim \varepsilon, \tag{6.42}$$

and:

$$\sum_{j\geq 0} 2^{-j}\|P_j(\nabla_N a_1)\|^2_{L^2(\Sigma)} \lesssim \varepsilon^2, \tag{6.43}$$

Proposition 6.11. *Let a_2 be the solution of* (6.38)*, where H is defined in* (6.32)*. Then, we have:*

$$\|a_2\|_{L^\infty_u L^4(P_u)} \lesssim \varepsilon, \tag{6.44}$$

and:

$$\sup_{j\geq 0}\|P_j(\nabla_N a_2)\|_{L^2(\Sigma)} \lesssim \varepsilon. \tag{6.45}$$

In view of the decomposition (6.39) for $\nabla_N a$, the estimates (6.42) (6.43) for a_1, and the estimates (6.44) (6.45) for a_2, we immediately obtain the estimate (6.12) for $\nabla_N a$ and $\nabla_N^2 a$. This concludes the proof of Theorem 6.4.

6.4. Regularity of the foliation with respect to ω

Let $u(x,\omega)$ the function constructed in section 6.2. In this section, we discuss the proof of Theorem 6.5 which deals with the control of the derivatives with respect to ω of the foliation of Σ provided by $u(x,\omega)$. Recall that (Σ, g, k) coincides with $(\mathbb{R}^3, \delta, 0)$ in $|x| \geq 2$. Also, $u(x,\omega)$ coincides with $x.\omega$ in $|x| \geq 2$, and so $a \equiv 1$, $N \equiv \omega$ and $\theta \equiv 0$ in this region. Thus, u clearly satisfies the estimates of Theorem 6.5 in $|x| \geq 2$ and it is enough to control the derivatives with respect to ω of the function $u(x,\omega)$ solution to:

$$\begin{cases} \mathrm{tr}\theta - k_{NN} = 1 - a, \text{ on } -2 < u < 2, \\ u(.,\omega) = -2 \text{ on } x.\omega = -2, \end{cases} \tag{6.46}$$

in the strip $S = \{x / -2 < u < 2\}$.

6.4.1. First order derivatives with respect to ω. The goal of this section is to prove (6.13). For the sake of simplicity, we only outline of the proof of the estimate for $\partial_\omega a$. Differentiating the second equation of (6.25) with respect to ω, we obtain:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega a = 2\nabla\!\!\!/\nabla_N a + 2R_{N\partial_\omega N} + \cdots \tag{6.47}$$

where the first term on the right-hand side comes from the commutator $[\partial_\omega, \Delta\!\!\!/]$ (see [**42**]). Since $\nabla\!\!\!/\nabla_N a$ and R are in $L^2(\Sigma)$ respectively by (6.11) and (6.23), we obtain using in particular an L^2 parabolic estimate for the operator $(\nabla_N - a^{-1}\Delta\!\!\!/)$

$$\|\nabla_N\partial_\omega a\|_{L^2(\Sigma)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_u L^2(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(\Sigma)} \lesssim \varepsilon. \tag{6.48}$$

Next, we differentiate (6.47) with respect to ∇_N. We obtain:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N\partial_\omega a = 2\nabla\!\!\!/\nabla_N^2 a + 2\nabla_N R_{N\partial_\omega N} + \cdots . \tag{6.49}$$

The term $\nabla_N R_{N\partial_\omega N}$ may be treated using the contracted Bianchi identity for R - as we did for $\nabla_N R_{NN}$ in section 6.2.1 - and turns out to be in $L^2_u H^{-1}(P_u)$. On the other hand, in view of the estimate (6.12) for $\nabla_N^2 a$, $\nabla\!\!\!/\nabla_N^2 a$ belongs to $L^2_u H^{-\frac{3}{2}}(P_u)$. We obtain using in particular a refined parabolic estimate for the operator $(\nabla_N - a^{-1}\Delta\!\!\!/)$

$$\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N^2\partial_\omega a\|_{L^2_u H^{-\frac{3}{2}}(P_u)} \lesssim \varepsilon. \tag{6.50}$$

Finally, by interpolation between (6.48) and (6.50), we obtain $\partial_\omega a$ in $L^\infty_u H^{\frac{5}{4}}(P_u)$ which embeds in $L^\infty(\Sigma)$ since P_u has dimension 2. Together with (6.48) and (6.50), we obtain the estimate corresponding to $\partial_\omega a$ in (6.13).

6.4.2. Second order derivatives with respect to ω. The goal of this section is to prove (6.14). For the sake of simplicity, we only outline of the proof of the estimate for $\partial_\omega^2 a$. Differentiating the equation (6.47) for $\partial_\omega a$ with respect to ω, we obtain:

$$(\nabla_N - a^{-1}\not\Delta)\partial_\omega^2 a = 2\nabla_N^2 a + \not\nabla\nabla_N\partial_\omega a + 2R_{\partial_\omega N\partial_\omega N} + \cdots \tag{6.51}$$

where the first two terms on the right-hand side come respectively from the commutators $[\partial_\omega, \not\nabla]$ and $[\partial_\omega, \not\Delta]$. Since R is in $L^2(\Sigma)$ by (6.23), $\nabla_N^2 a$ is in $L_u^2 H^{-\frac{1}{2}}(P_u)$ by (6.12), and $\nabla_N\partial_\omega a$ is in $L_u^2 H^{\frac{1}{2}}(P_u)$ by (6.13), the right-hand side of (6.51) belongs to $L_u^2 H^{-\frac{1}{2}}(P_u)$. Using in particular estimates for the parabolic operator $(\nabla_N - a^{-1}\not\Delta)$, we deduce

$$\|\partial_\omega^2 a\|_{L_u^2 H^{\frac{3}{2}}(P_u)} + \|\partial_\omega^2 a\|_{L_u^\infty H^{\frac{1}{2}}(P_u)} + \|\nabla_N\partial_\omega^2 a\|_{L_u^2 H^{-\frac{1}{2}}(P_u)} \lesssim \varepsilon, \tag{6.52}$$

which is the estimate corresponding to $\partial_\omega^2 a$ in (6.14).

REMARK 6.12. *Note that we may not differentiate the equation* (6.51) *for $\partial_\omega^2 a$ with respect to ∇_N. Indeed, the term $\nabla_N R_{\partial_\omega N\partial_\omega N}$ has no structure: unlike R_{NN} and $R_{N\partial_\omega N}$ which were involved in the equation for a and $\partial_\omega a$, $R_{\partial_\omega N\partial_\omega N}$ does not contain any contraction with N since $\partial_\omega N$ is tangent to P_u. Thus, unlike $\nabla_N R_{NN}$ and $\nabla_N R_{N\partial_\omega N}$, we can not write $\nabla_N R_{\partial_\omega N\partial_\omega N}$ as a tangential derivative using the contracted Bianchi identities for R. Consequently, we can not obtain any estimate for $\nabla_N^2\partial_\omega^2 a$.*

6.4.3. Third order derivatives with respect to ω. The goal of this section is to prove (6.15). Recall that $\mathrm{div}(N) = \mathrm{tr}\theta$, $N = \nabla u/|\nabla u|$, $a = 1/|\nabla u|$ and $\mathrm{tr}\theta = 1 - a + k_{NN}$, so that:

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 1 - \frac{1}{|\nabla u|} + k_{NN}. \tag{6.53}$$

Differentiating (6.53) three times with respect to ω yields:

$$(\nabla_N - a^{-1}\not\Delta)\partial_\omega^3 u = \not\nabla\partial_\omega^2 \log(a) + \cdots. \tag{6.54}$$

Using in particular (6.54), the estimate (6.14) for $\partial_\omega^2 a$ and refined parabolic estimates for the operator $(\nabla_N - a^{-1}\not\Delta)$ we obtain

$$\|\partial_\omega^3 u\|_{L_u^2 H^{\frac{5}{2}}(P_u)} + \|\partial_\omega^3 u\|_{L_u^\infty H^{\frac{3}{2}}(P_u)} + \|\nabla_N\partial_\omega^3 u\|_{L_u^2 H^{\frac{1}{2}}(P_u)} \lesssim 1.$$

Now, since $\partial_\omega^3 u \in L_u^\infty H^{\frac{3}{2}}(P_u)$ and P_u is 2-dimensional, we obtain that $\partial_\omega^3 u$ belongs to $L_{loc}^\infty(\Sigma)$, which is the desired estimate (6.15).

6.5. A global coordinate system on P_u and Σ

The goal of this section is to discuss the proof of Proposition 6.6. We start by constructing a global coordinate system on P_u.

6.5.1. A global coordinate system on P_u. We have the following proposition

PROPOSITION 6.13. *Let $\omega\in\mathbb{S}^2$. Let $\Phi_u : P_u\to T_\omega\mathbb{S}^2$ defined by:*

$$\Phi_u(x) := \partial_\omega u(x,\omega), \tag{6.55}$$

where $T_\omega\mathbb{S}^2$ is the tangent space to $\mathbb{S}^2$ at ω. Then Φ_u is a global C^1 diffeomorphism from P_u to $T_\omega\mathbb{S}^2$.

For the sake of simplicity, we only briefly sketch the proof. We start by showing that Φ_u is a local C^1 diffeomorphism. We have

$$(\mathrm{Jac}\Phi_u)^T\mathrm{Jac}\Phi_u = a^{-2}\begin{pmatrix} g(\partial_\varphi N,\partial_\varphi N) & g(\partial_\psi N,\partial_\varphi N)\\ g(\partial_\psi N,\partial_\varphi N) & g(\partial_\psi N,\partial_\psi N)\end{pmatrix},$$

where (φ,ψ) denotes the usual spherical coordinates on $\mathbb{S}^2$. Using the estimates (6.11) and (6.13), we are able to derive the following estimate

$$\|(\mathrm{Jac}\Phi_u)^T\mathrm{Jac}\Phi_u - I\|_{L^\infty(\Sigma)}\lesssim\varepsilon, \tag{6.56}$$

so that Φ_u is a C^1 local diffeomorphism. In turn, this yields:

$$\||\det(\mathrm{Jac}\Phi_u)|-1\|_{L^\infty(\Sigma)}\lesssim\varepsilon. \tag{6.57}$$

It remains to show that Φ_u is onto and one-to-one. The proof relies on the estimate (6.56) for the Jacobian of Φ_u, the fact that u coincides with $x.\omega$ in the region $|x|\geq 2$ and geometric considerations on the level sets of $\partial_\varphi u$ and $\partial_\psi u$. We refer to [**42**] for the details.

6.5.2. Proof of Proposition 6.6. Let $\omega\in\mathbb{S}^2$. Recall the definition (6.16) of $\phi_\omega:\Sigma\to\mathbb{R}^3$:

$$\phi_\omega(x) := u(x,\omega)\omega+\partial_\omega u(x,\omega) = u(x,\omega)\omega+\Phi_u(x),$$

where Φ_u has been defined in Proposition 6.13. The fact that ϕ_ω is a bijection is an easy consequence of the fact that Φ_u is a bijection for all u. Then, it remains to prove (6.17). We are able to obtain

$$\begin{aligned}&(\mathrm{Jac}\phi_\omega)^T\mathrm{Jac}\phi_\omega = a^{-2}\\ &\times\begin{pmatrix} 1 & -\partial_\varphi\log(a) & -\partial_\psi\log(a)\\ -\partial_\varphi\log(a) & (\partial_\varphi\log(a))^2+g(\partial_\varphi N,\partial_\varphi N) & \partial_\varphi\log(a)\partial_\psi\log(a)+g(\partial_\psi N,\partial_\varphi N)\\ -\partial_\psi\log(a) & \partial_\varphi\log(a)\partial_\psi\log(a)+g(\partial_\psi N,\partial_\varphi N) & (\partial_\psi\log(a))^2+g(\partial_\psi N,\partial_\psi N)\end{pmatrix}.\end{aligned}$$

Taking the determinant yields:

$$\det((\mathrm{Jac}\phi_\omega)^T\mathrm{Jac}\phi_\omega) = a^{-2}\det((\mathrm{Jac}\Phi_u)^T\mathrm{Jac}\Phi_u), \tag{6.58}$$

which together with (6.56) and the estimate (6.11) for a implies:

$$\|\det((\mathrm{Jac}\phi_\omega)^T\mathrm{Jac}\phi_\omega)-1\|_{L^\infty(\Sigma)}\lesssim\varepsilon. \tag{6.59}$$

(6.59) yields (6.17). This concludes the proof of Proposition 6.6.

6.6. Additional estimates

The proof of Proposition 6.7 and Proposition 6.8 follow from the estimates of Theorem 6.3, Theorem 6.4, and Theorem 6.5 using also for some estimates the fact that u coincides with $x.\omega$ in the region $|x| \geq 2$ or the properties of the Littlewood-Paley projections P_j. For the sake of simplicity, we skip these proofs and refer the reader to [**42**] for the details.

CHAPTER 7

The Strichartz estimates

Recall Steps A, B, C and D introduced in section 1.2.4. In this chapter, we perform Step D, i.e. we prove Proposition 2.32. More precisely, let $j \geq 0$, and let ψ a smooth function on $\mathbb{R}^3$ supported in

$$\frac{1}{2} \leq |\xi| \leq 2.$$

Let φ_j the parametrix (1.24) with an additional frequency localization $\lambda \sim 2^j$

$$\varphi_j(t,x) = \int_{\mathbb{S}^2} \int_0^\infty e^{i\lambda u(t,x,\omega)} \psi(2^{-j}\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{7.1}$$

where $u(.,.,\omega)$ is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ which depends on an extra parameter $\omega \in \mathbb{S}^2$. Assume that the space-time $\mathcal{M}$ is foliated by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Let (p,q,r) such that $p,q \geq 2$, $q < +\infty$, and

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2},\ r = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}.$$

In this chapter, we outline the proof of the following sharp[1] Strichartz estimates

$$\|\varphi_j\|_{L^p_{[0,1]}L^q(\Sigma_t)} \lesssim 2^{jr}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{7.2}$$

The proof of Proposition 2.32 will then be a simple consequence of (7.2) with the choice $p = q = 4$.

Remark 7.1. *Even though we only need $L^4(\mathcal{M})$ Strichartz estimates - which corresponds to $p = q = 4$ in (7.2) - to prove Proposition 2.32, it turns out that this particular case is not easier to prove than the general case.*

7.1. Assumptions on the phase $u(t,x,\omega)$ and main results

7.1.1. Time foliation on $\mathcal{M}$. We foliate the space-time $\mathcal{M}$ by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. We assume $0 \leq t \leq 1$ so that

$$\mathcal{M} = \bigcup_{0 \leq t \leq 1} \Sigma_t. \tag{7.3}$$

[1]Note in particular that the corresponding estimates in the flat case are sharp.

We denote by T the unit, future oriented, normal to Σ_t. We also define the lapse n as

$$n^{-1} = T(t). \tag{7.4}$$

Note that we have the following identity between the volume element of $\mathcal{M}$ and the volume element corresponding to the induced metric on Σ_t

$$d\mathcal{M} = n\, d\Sigma_t\, dt. \tag{7.5}$$

We will assume the following assumption on n

$$\frac{1}{2} \le n \le 2 \tag{7.6}$$

which together with (7.5) yields

$$d\mathcal{M} \simeq d\Sigma_t\, dt. \tag{7.7}$$

REMARK 7.2. *The assumption* (7.6) *is very mild. In particular, it is compatible with the estimates for* n *derived in* [**44**] *(see also* (4.40)*).*

7.1.2. Geometry of the foliation generated by u on $\mathcal{M}$. Recall that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ depending on an extra parameter $\omega \in \mathbb{S}^2$. The level hypersufaces $u(t, x, \omega) = u$ of the optical function u are denoted by $\mathcal{H}_u$. Let L' denote the space-time gradient of u, i.e.:

$$L' = \mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha. \tag{7.8}$$

Using the fact that u satisfies the eikonal equation, we obtain:

$$\mathbf{D}_{L'}L' = 0, \tag{7.9}$$

which implies that L' is the geodesic null generator of $\mathcal{H}_u$.

We have:

$$T(u) = \pm|\nabla u|$$

where $|\nabla u|^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t. Since the sign of $T(u)$ is irrelevant, we choose by convention:

$$T(u) = -|\nabla u| \tag{7.10}$$

so that u corresponds to $-t + x\cdot\omega$ in the flat case.

Let

$$L = bL' = T + N, \tag{7.11}$$

where L' is the space-time gradient of u (7.8), b is the *lapse of the null foliation* (or shortly null lapse)

$$b^{-1} = - < L', T >= -T(u), \tag{7.12}$$

and N is a unit vectorfield given by

$$N = \frac{\nabla u}{|\nabla u|}. \tag{7.13}$$

Note that we have the following identities.

LEMMA 7.3.

$$L(u) = 0,\ L(\partial_\omega u) = 0 \tag{7.14}$$

and

$$\mathbf{g}(N, \partial_\omega N) = 0. \tag{7.15}$$

The proof is elementary and can be found in [**46**].

7.1.3. Regularity assumptions for $u(t,x,\omega)$. We now state our assumptions for the phase $u(t,x,\omega)$. These assumptions are compatible with the regularity obtained for the function $u(t,x,\omega)$ constructed in [**44**] (see also (4.42), (4.48), (4.49)). Let $0 < \varepsilon < 1$ a small enough universal constant. b and N satisfy

$$\|b-1\|_{L^\infty} + \|\partial_\omega b\|_{L^\infty} \lesssim \varepsilon. \tag{7.16}$$

$$\|\mathbf{g}(\partial_\omega N, \partial_\omega N) - I_2\|_{L^\infty} \lesssim \varepsilon. \tag{7.17}$$

$$|N(.,\omega) - N(.,\omega')| = |\omega - \omega'|(1 + O(\varepsilon)). \tag{7.18}$$

REMARK 7.4. *In the flat case, we have $\mathcal{M} = (\mathbb{R}^{1+3}, \mathbf{m})$, where $\mathbf{m}$ is the Minkowski metric, $u(t,x,\omega) = -t + x\cdot\omega$, $b = 1$, $N = \omega$ and $L = \partial_t + \omega\cdot\partial_x$. Thus, the assumptions* (7.16) (7.17) (7.18) *are clearly satisfied with $\varepsilon = 0$.*

REMARK 7.5. *In terms of the regularity of $u(t,x,\omega)$, the assumptions* (7.16) (7.17) *correspond to*

$$\nabla u \in L^\infty \text{ and } \nabla\partial_\omega u \in L^\infty$$

which is very weak. In particular, the classical proof for obtaining Strichartz estimates for the wave equation relies on the stationary phase for an oscillatory integral involving u as a phase, and typically requires at the least one more derivative for u (see Remark 7.11).

7.1.4. A global coordinate system on Σ_t. For all $0 \le t \le 1$, and for all $\omega \in \mathbb{S}^2$, $(u(t,x,\omega), \partial_\omega u(t,x,\omega))$ is a global coordinate system on Σ_t. Furthermore, the volume element is under control in the sense that in this coordinate system, we have

$$\frac{1}{2} \le \sqrt{\det g} \le 2 \tag{7.19}$$

where g is the induced metric on Σ_t, and where $\det g$ denotes the determinant of the matrix of the coefficients of g.

REMARK 7.6. *In the flat case, we have $\Sigma_t = \{t\} \times \mathbb{R}^3$ and $u(t,x,\omega) = -t + x\cdot\omega$ so that $(u(t,x,\omega), \partial_\omega u(t,x,\omega))$ is clearly a global coordinate system on Σ_t and $\det g = 1$ in this case. These assumptions are also satisfied by the function $u(t,x,\omega)$ constructed in* [**44**].

7.1.5. Main results. We next state the main result of this chapter concerning general Strichartz inequalities in mixed space-time norms of the form $L^p_{[0,1]}L^q(\Sigma_t)$ defined as follows,

$$\|F\|_{L^p_{[0,1]}L^q(\Sigma_t)} = \left(\int_0^1 \|F(t,\cdot)\|^p_{L^p(\Sigma_t)}dt\right)^{\frac{1}{p}}.$$

THEOREM 7.7. *Let (p,q) such that $p,q \geq 2$, $q < +\infty$, and*

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}.$$

Let r defined by

$$r = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}.$$

Then, the parametrix localized at frequency j defined in (7.1) *satisfies under the assumptions* (7.6), (7.16), (7.17), (7.18) *and the assumptions in section 7.1.4 the following Strichartz inequalities*

$$\|\varphi_j\|_{L^p_{[0,1]}L^q(\Sigma_t)} \lesssim 2^{jr}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{7.20}$$

Proposition 2.32 - which corresponds to Corollary 2.8 in [**46**] - is then a simple consequence of Theorem 7.7, see [**46**] for the details.

The rest of the chapter is organized as follows. In section 7.2, we use the standard TT^* argument to reduce the proof of Theorem 7.7 to an upper bound on the kernel K of a certain operator. This kernel is an oscillatory integral with a phase ϕ. In section 7.3, we prove the upper bound on the kernel K provided we have a suitable lower bound on ϕ. Finally, in section 7.4, we prove the lower bound for ϕ used in section 7.3.

7.2. Proof of the Strichartz estimates

The goal of this section is to prove Theorem 7.7. We start with the following remark.

REMARK 7.8. *Fixing a global system of coordinates $x = (x^1, x^2, x^3)$ in Σ_t, such as the one described in section 7.1.4, we note in view of* (7.19) *that* (7.20) *is equivalent with the same inequality where the norm $L^q(\Sigma_t)$ on the left-hand side is replaced by the corresponding euclidean norm in the given coordinates. More precisely we can assume from now on that*

$$\|F\|_{L^p_{[0,1]}L^q(\Sigma_t)} = \left(\int_0^1 \left(\int_{\mathbb{R}^3} |F(t,x)|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{q}}$$

which we will denote by a slight abuse of notation by

$$\|F\|_{L^p_{[0,1]}L^q(\mathbb{R}^3)}.$$

Note also that in the (t,x) coordinates $\mathcal{M} = [0,1] \times \mathbb{R}^3$.

For convenience, let us introduce the operator T_j acting on functions $f \in L^2(\mathbb{R}^3)$,

$$T_j f(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{7.21}$$

Note in particular that

$$T_j f = \varphi_j \tag{7.22}$$

where φ_j is the parametrix localized at frequency 2^j defined in (7.1). To prove Theorem 7.7, we rely on the standard TT^* argument for the Fourier integral operator (7.21). Note that the operator T_j^* takes real valued functions h on $\mathcal{M}$ to complex valued functions on $\mathbb{R}^3$

$$T_j^* h(\lambda\omega) = \psi(2^{-j}\lambda)\int_{\mathcal{M}} e^{-i\lambda u(s,y,\omega)}h(s,y)dsdy.$$

Therefore, the operator $U_j := T_j T_j^*$ is given by the formula,

$$U_j h(t,x) = \int_{\mathbb{S}^2}\int_0^\infty\int_{\mathcal{M}} e^{i\lambda u(t,x,\omega)-i\lambda u(s,y,\omega)}\psi(2^{-j}\lambda)^2 h(s,y)\lambda^2 d\lambda d\omega ds dy.$$

Note, in view of Remark 7.8 and (7.22), that (7.20) is equivalent to the following estimate

$$\|U_j h\|_{L^p_{[0,1]}L^q(\mathbb{R}^3)} \lesssim 2^{2jr}\|h\|_{L^{p'}_{[0,1]}L^{q'}(\mathbb{R}^3)}, \tag{7.23}$$

where p' (resp. q') is the conjugate exponent to p (resp. q). Observe that,

$$U_j h\left(\frac{t}{2^j},\frac{x}{2^j}\right) = 2^{-j}\int_{\mathbb{S}^2}\int_0^\infty\int_{2^j\mathcal{M}} e^{i\lambda 2^j u\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)-i\lambda 2^j u\left(\frac{s}{2^j},\frac{y}{2^j},\omega\right)}\psi(\lambda)^2 h\left(\frac{s}{2^j},\frac{y}{2^j}\right)\lambda^2 d\lambda d\omega ds dy$$

with $2^j\mathcal{M} = [0,2^j]\times\mathbb{R}^3$ relative to the rescaled variables (s,y). Thus, setting,

$$Ah(t,x) := \int_{\mathbb{S}^2}\int_0^\infty\int_{2^j\mathcal{M}} e^{i\lambda 2^j u\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)-i\lambda 2^j u\left(\frac{s}{2^j},\frac{y}{2^j},\omega\right)}\psi(\lambda)^2 h(s,y)\lambda^2 d\lambda d\omega ds dy$$

we have

$$U_j h\left(\frac{t}{2^j},\frac{x}{2^j}\right) = 2^{-j}Ah_j(t,x), \quad h_j(s,y) = h\left(\frac{s}{2^j},\frac{y}{2^j}\right).$$

We easily infer that (7.23) is equivalent to the estimate,

$$\|Ah\|_{L^p_{[0,2^j]}L^q(\mathbb{R}^3)} \lesssim \|h\|_{L^{p'}_{[0,2^j]}L^{q'}(\mathbb{R}^3)}. \tag{7.24}$$

We introduce the kernel K of A

$$K(t,x,s,y) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda 2^j u\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)-i\lambda 2^j u\left(\frac{s}{2^j},\frac{y}{2^j},\omega\right)}\psi(\lambda)^2\lambda^2 d\lambda d\omega. \tag{7.25}$$

REMARK 7.9. *In the flat case, we have $u(t,x,\omega) = -t + x\cdot\omega$ so that*

$$2^j u\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right) = u(t,x,\omega).$$

In particular, K is independent of j

$$K(t,x,s,y)=\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)-i\lambda u(s,y,\omega)}\psi(\lambda)^2\lambda^2 d\lambda d\omega.$$

We have the following proposition.

PROPOSITION 7.10. *The kernel K of the operator A satisfies the dispersive estimates,*

$$|K(t,x,s,y)|\lesssim\frac{1}{|t-s|},\ \ \forall(t,x)\in 2^j\mathcal{M},\ \ \forall(s,y)\in 2^j\mathcal{M}. \tag{7.26}$$

The proof of Proposition 7.10 is postponed to section 7.3. We now conclude the proof of Theorem 7.7. (7.24) follows from (7.26) using interpolation and the Hardy-Littlewood inequality according to the standard procedure, see for example [**38**] and [**39**]. Finally, in view of the discussion above, (7.24) yields (7.23) which in turn implies (7.20) in view of (7.22). This concludes the proof of Theorem 7.7.

7.3. Upper bound on the kernel K

The goal of this section is to prove Proposition 7.10. Let ϕ the scalar function on $\mathcal{M}\times\mathcal{M}\times\mathbb{S}^2$ defined as

$$\phi(t,x,s,y,\omega)=u(t,x,\omega)-u(s,y,\omega). \tag{7.27}$$

In view of (7.25), we may rewrite K as

$$K(t,x,s,y)=\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda 2^j\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)}\lambda^2 d\lambda d\omega.$$

After integrating by parts twice in λ, and using the size of the support of ψ, this yields

$$|K(t,x,s,y)|\lesssim\int_{\mathbb{S}^2}\frac{1}{1+2^{2j}\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)^2}d\omega. \tag{7.28}$$

The next section is dedicated to the obtention of a lower bound on $|\phi|$ which will allow us to deduce (7.26) from (7.28).

REMARK 7.11. *It is at this stage that we depart from the standard strategy for proving Strichartz estimates. Indeed, the usual method consists in using the stationary phase method to derive* (7.26)*. To this end, one considers the neighborhood in $\mathbb{S}^2$ of stationary points ω_0, i.e. such that $\partial_\omega\phi_{|\omega=\omega_0}=0$. One then needs an identity of the type*

$$\phi=(s-t)A(\omega-\omega_0)\cdot(\omega-\omega_0)+o\left((s-t)(\omega-\omega_0)^2\right) \tag{7.29}$$

for ω in the neighborhood of ω_0 and for some 3×3 invertible matrix A. (7.29) *then allows to perform a change of variables in ω which ultimately leads to* (7.26)*. In particular, the standard method requires at the least*[2] *$\partial_{t,x}\partial_\omega^2 u\in L^\infty$ just to derive* (7.29)*.*

[2]One also needs to take care of the contribution to K of the angles $\omega\in\mathbb{S}^2$ corresponding to the exterior of the neighborhood of stationary points which may increase the needed regularity.

Our assumptions correspond only to $\partial_{t,x}\partial_\omega u \in L^\infty$*. Thus, in order to obtain* (7.26)*, we instead integrate by parts in* λ *to obtain* (7.28)*, and then look for a suitable lower bound on* $|\phi|$*. In particular, we obtain lower bounds of the following type (see details in Lemma 7.19)*

$$|\phi| \gtrsim |s-t||\omega-\omega_0|^2 \tag{7.30}$$

for ω *in the neighborhood of some* $\omega_0 \in \mathbb{S}^2$*. The fundamental observation is that, as it turns out, the inequality* (7.30) *requires less regularity than the equality* (7.29)*.*

7.3.1. The key lemma. Let (t,x) and (s,y) in $\mathcal{M}$, and let $\omega \in \mathbb{S}^2$. In this section, we obtain a lower bound on $\phi(t,x,s,y,\omega)$. We may assume

$$0 \leq t < s \leq 1.$$

Definition 7.12. *For any* $\omega \in \mathbb{S}^2$ *and* $\sigma \in \mathbb{R}$*, let* $\gamma_\omega(\sigma)$ *denote the null geodesic parametrized by the time function and with initial data*

$$\gamma_\omega(0) = (t,x),\ \gamma'_\omega(0) = b^{-1}(t,x,\omega)L(t,x,\omega).$$

Definition 7.13. *Let us define the subset* S *of* Σ_s *as*

$$S = \bigcup_{\omega\in\mathbb{S}^2}\{\gamma_\omega(s-t)\}. \tag{7.31}$$

We also define for all $(s,z) \in \Sigma_s$

$$m(s,z) = \max_{\omega\in\mathbb{S}^2}(u(s,z,\omega) - u(t,x,\omega)). \tag{7.32}$$

We have the following lemma characterizing the zeros of m (see [**46**] for a proof).

Lemma 7.14. *We have*

$$S = \{p \in \Sigma_s,\ /\, m(p) = 0\}.$$

Next, we define the following two subsets of Σ_s

$$A_{int} = \{p \in \Sigma_s \,/\, m(p) < 0\},\ A_{ext} = \{p \in \Sigma_s \,/\, m(p) > 0\}. \tag{7.33}$$

Note in view of Lemma 7.14 that

$$\Sigma_s = S \sqcup A_{int} \sqcup A_{ext}. \tag{7.34}$$

Remark 7.15. *In the flat case, the picture is the following:*

(1) *The null geodesics*[3] γ_ω *span the light cone from* (t,x)*. In particular, the null geodesics* γ_ω *do not intersect except at* (t,x)*.*
(2) S *is the intersection*[4] *of the forward light cone from* (t,x) *with* $\{s\}\times\mathbb{R}^3$*.*
(3) A_{int} *and* A_{ext} *correspond respectively to the interior and the exterior of* S*.*

[3]which are straight lines in this case
[4]S is a sphere in this case

Note that we do not need to prove these statements in our case. This is fortunate since these statements - while probably true in our general setting - would be delicate to establish (see for instance [**25**] *for a proof of (1) on a space-time* $(\mathcal{M}, \mathbf{g})$ *with limited regularity).*

Next, we introduce some further notations. First, we denote by m_0 the value of m at (s, y), i.e.

$$m_0 = \max_{\omega\in\mathbb{S}^2}(u(s, y, \omega) - u(t, x, \omega)). \tag{7.35}$$

We also denote by ω_0 an angle in $\mathbb{S}^2$ where the maximum in (7.35) is achieved, i.e.

$$m_0 = u(s, y, \omega_0) - u(t, x, \omega_0). \tag{7.36}$$

REMARK 7.16. *In the flat case,* ω_0 *is unique and corresponds to the angle of the projection of* (s, y) *on* S*. Again, while this may be also true in our general setting, we do not need to prove this statement in our case.*

Note that if $(s, y) \in A_{ext}$, the function $u(s, y, \omega) - u(t, x, \omega)$ may change sign as ω varies on $\mathbb{S}^2$. We define

$$D = \{\omega \in \mathbb{S}^2 \,/\, u(t, x, \omega) = u(s, y, \omega)\}. \tag{7.37}$$

The following lemma gives a precise description of D (see [**46**] for a proof).

LEMMA 7.17. *Let* $(s, y) \in A_{ext}$*. Let* D *defined as in* (7.37)*. Let* (θ, φ) *denote the spherical coordinates with axis* ω_0*. Then, there exists a* C^1 2π*-periodic function*

$$\theta_1 : [0, 2\pi) \to (0, \pi)$$

such that in the coordinate system (θ, φ)*,* D *is parametrized by*

$$D = \{\theta = \theta_1(\varphi),\ 0 \leq \varphi < 2\pi\}.$$

REMARK 7.18. *In the flat case, recall that* $u(t, x, \omega) = -t + x \cdot \omega$*. In this case, one easily checks that* D *is a circle of axis* ω_0 *on the sphere* S *which is generated by the tangents to* S *through* y *(see figure 1).*

Let $\omega \in \mathbb{S}^2$. According to Lemma 7.17, the great half circle on $\mathbb{S}^2$ originating at ω_0 and containing ω intersects D at a fixed point ω_1. Let θ and θ_1 respectively denote the positive angles between ω_0 and ω (resp. ω_0 and ω_1).

In order to obtain a lower bound for $|\phi|$, we will argue differently according to whether (s, y) belongs to the region S, A_{int} or A_{ext}.

LEMMA 7.19 (Key lemma). $|\phi|$ *satisfies the following lower bounds*

(1) *If* $(s, y) \in S$*, we have*

$$|\phi(t, x, s, y, \omega)| \geq \frac{1}{4}|t - s||\omega - \omega_0|^2. \tag{7.38}$$

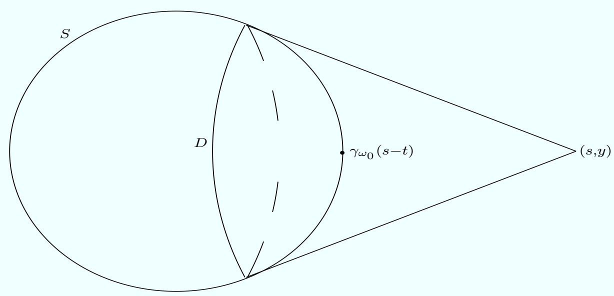

FIGURE 1. Representation of D in the flat case

(2) *If* $(s,y) \in A_{int}$, *we have*

$$|\phi(t,x,s,y,\omega)| \geq \frac{1}{8}|t-s||\omega-\omega_0|^2. \tag{7.39}$$

(3) *If* $(s,y) \in A_{ext}$ *and* $\theta_1 \leq \theta \leq \pi$, *we have*

$$|\phi(t,x,s,y,\omega)| \geq \frac{1}{4}|t-s||\omega-\omega_1|^2. \tag{7.40}$$

(4) *If* $(s,y) \in A_{ext}$ *and* $0 \leq \theta \leq \theta_1$, *we have*

$$|\phi(t,x,s,y,\omega)| \gtrsim \sqrt{\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta_1)}}m_0 \tag{7.41}$$

The proof of Lemma 7.19 is postponed to section 7.4.

REMARK 7.20. *The proof of Lemma 7.19 is inspired by the overlap estimates for wave packets derived in* [**35**] *and* [**36**] *in the context of Strichartz estimates respectively for* $C^{1,1}$ *and* $H^{2+\varepsilon}$ *metrics. Note however that the estimates in these papers rely heavily on a direct comparison of various quantities with the corresponding ones in the flat case. Such direct comparisons do not hold in our framework. Here, the closeness to the flat case manifests itself in the small constant* ε *in the right-hand side of* (7.16), (7.17) *and* (7.18), *and in the existence of the global coordinates systems of section 7.1.4.*

7.3.2. Proof of Proposition 7.10. Recall that we need to show that the kernel K defined in (7.25) satisfies the upper bound (7.26). To this end, we will use the estimate (7.28) for K together with the estimates provided by Lemma 7.19. We argue differently according according to whether (s,y) belongs to S, A_{int} or A_{ext}.

If (s,y) belongs to S, we have the lower bound (7.38) for $|\phi|$

$$|\phi(t,x,s,y,\omega)| \geq \frac{1}{4}|t-s||\omega-\omega_0|^2,$$

where $\omega_0 \in \mathbb{S}^2$ is an angle satisfying (7.36). Then, we deduce

$$2^j\left|\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)\right| \geq \frac{1}{4}|t-s||\omega-\omega_0|^2.$$

Together with (7.28), this yields

$$|K(t,x,s,y)| \lesssim \int_{\mathbb{S}^2} \frac{d\omega}{1+|t-s|^2|\omega-\omega_0|^4}.$$

Using the spherical coordinates (θ,φ) with axis ω_0, we obtain

$$|K(t,x,s,y)| \lesssim \int_0^\pi \frac{\sin(\theta)d\theta}{1+|t-s|^2(1-\cos(\theta))^2}.$$

Performing the change of variables

$$z = |t-s|(1-\cos(\theta))$$

we obtain

$$|K(t,x,s,y)| \lesssim \frac{1}{|t-s|}\int_0^{+\infty} \frac{dz}{1+z^2}.$$

This implies

$$|K(t,x,s,y)| \lesssim \frac{1}{|t-s|}, \ \ \forall (t,x)\in 2^j\mathcal{M}, \ \ \forall\left(\frac{s}{2^j},\frac{y}{2^j}\right)\in S \tag{7.42}$$

which is the desired estimate.

The estimates corresponding to the cases where (s,y) belongs to A_{int} or A_{ext} are similar (see [**46**] for the details). This concludes the proof of Proposition 7.10.

7.4. Lower bound for $|\phi|$

The goal of this section is to prove Lemma 7.19. The main ingredients of the proof are already present in the flat case. Thus, to simplify the exposition, we will prove Lemma 7.19 for the phase function $u=-t+x\cdot\omega$ of the flat case. We will then explain what are the modifications in the general case (see Remark 7.22). We refer to [**46**] for the proof in the general case.

7.4.1. A lower bound for $|\phi|$ when $(s,y)\in S$ (proof of (7.38)). In view of the definition of m_0 in (7.35) and ω_0 in (7.36), we have in the flat case

$$u(t,x,\omega) = -t+x\cdot\omega, \ \omega_0 = \frac{y-x}{|y-x|}, \ m_0 = -(s-t)+|y-x|. \tag{7.43}$$

If $(s,y)\in S$, we have $|y-x| = s-t$. Together with (7.43), this yields

$$\begin{aligned} u(s,y,\omega)-u(t,x,\omega) &= -s+t+(y-x)\cdot\omega \\ &= -(s-t)+(s-t)\omega_0\cdot\omega \\ &= -\frac{1}{2}(s-t)|\omega-\omega_0|^2 \end{aligned} \tag{7.44}$$

which is the desired estimate (7.38).

7.4.2. A lower bound for $|\phi|$ when $(s,y)\in A_{int}$ (proof of (7.39)). (7.43) yields

$$\begin{aligned} u(s,y,\omega)-u(t,x,\omega) &= -(s-t)+(y-x)\cdot\omega \\ &= -(s-t)+|y-x|\omega\cdot\omega_0 \\ &= -(s-t)+|y-x|-\frac{1}{2}|x-y||\omega-\omega_0|^2. \end{aligned} \tag{7.45}$$

Now, if $|x-y|\le\frac{1}{4}(s-t)$ we have,

$$u(s,y,\omega)-u(t,x,\omega) \le -\frac{3}{4}(s-t)+\frac{1}{4}(s-t)\frac{1}{2}|\omega-\omega_0|^2\le -\frac{1}{2}(s-t).$$

On the other hand, if $|x-y|\ge\frac{1}{4}(s-t)$

$$u(s,y,\omega)-u(t,x,\omega)\le -\frac{1}{2}|x-y||\omega-\omega_0|^2\le -\frac{1}{4}(s-t)|\omega-\omega_0|^2.$$

Thus, in both cases,

$$u(s,y,\omega)-u(t,x,\omega)\le -\frac{1}{2}|x-y||\omega-\omega_0|^2\le -\frac{1}{4}(s-t)|\omega-\omega_0|^2$$

which is the desired estimate (7.39).

7.4.3. A lower bound for $|\phi|$ when $(s,y)\in A_{ext}$ (proof of (7.40) (7.41)). (7.43) yields

$$\begin{aligned} u(s,y,\omega)-u(t,x,\omega) &= -(s-t)+(y-x)\cdot\omega \\ &= -(s-t)+|x-y|\omega\cdot\omega_0 \\ &= -\frac{1}{2}(s-t)|\omega-\omega_0|^2+m_0\omega\cdot\omega_0. \end{aligned} \tag{7.46}$$

Recall that we have defined the set D by

$$D=\{\omega\in\mathbb{S}^2\,/\,u(t,x,\omega)=u(s,y,\omega)\}.$$

Also, for fixed ω_0,ω we defined $\omega_1\in D$ to lie on the same plane great circle of $\mathbb{S}^2$ as ω_0,ω. Clearly, since $\omega_1\in D$, and in view of (7.43), (7.46) and the definition of D, we have

$$\omega_1\cdot\omega_0=\frac{(s-t)}{|x-y|} \tag{7.47}$$

Fix now $\omega_1\in D$ and let $z=\gamma_{\omega_1}(s-t)$, i.e.

$$z=x+(s-t)\omega_1\in S. \tag{7.48}$$

Note that in view of the definition of D,

$$(y-z)\cdot\omega_1=-(s-t)+(y-x)\cdot\omega_1=u(s,y,\omega_1)-u(t,x,\omega_1)=0.$$

Hence, with the notation

$$v_0=y-z,$$

we obtain

$$v_0 \cdot \omega_1 = 0. \tag{7.49}$$

Now, we have

$$\begin{aligned} u(s,y,\omega) - u(t,x,\omega) &= u(s,y,\omega) - u(s,z,\omega) + u(s,z,\omega) - u(t,x,\omega) \\ &= v_0 \cdot \omega + u(s,z,\omega) - u(t,x,\omega) \\ &= v_0 \cdot (\omega - \omega_1) + u(s,z,\omega) - u(t,x,\omega). \end{aligned} \tag{7.50}$$

Note that since $z \in S$ we can apply the estimate obtained in section 7.4.1. Since the maximum in $m(s,z)$ is attained at $\omega = \omega_1$, we have

$$u(s,z,\omega) - u(t,x,\omega) = -\frac{1}{2}(s-t)|\omega - \omega_1|^2 \tag{7.51}$$

and we infer that,

$$u(s,y,\omega) - u(t,x,\omega) = -\frac{1}{2}(s-t)|\omega - \omega_1|^2 + v_0 \cdot (\omega - \omega_1). \tag{7.52}$$

Recall that we have also denoted by θ and θ_1 the positive angles between ω_0, ω and respectively ω_0, ω_1. If $\theta_1 \le \theta \le \pi$ - which corresponds to $v_0 \cdot (\omega - \omega_1) \le 0$ - we have in view of (7.52)

$$u(s,y,\omega) - u(t,x,\omega) \le -\frac{1}{2}(s-t)|\omega - \omega_1|^2 \tag{7.53}$$

which is the desired estimate (7.40).

The delicate case is when $0 \le \theta < \theta_1$ which corresponds to

$$v_0 \cdot (\omega - \omega_1) > 0. \tag{7.54}$$

In the rest of the proof, we assume (7.54), and we focus on the remaining estimate (7.41). In view of the definition of ω_0 and (7.52), we have

$$\begin{aligned} -(s-t) + |x-y| &= u(s,y,\omega_0) - u(t,x,\omega_0) \\ &= -\frac{1}{2}(s-t)|\omega_0 - \omega_1|^2 + v_0 \cdot (\omega_0 - \omega_1). \end{aligned}$$

Thus,

$$m_0 = -(s-t) + |x-y| = -\frac{1}{2}(s-t)|\omega_0 - \omega_1|^2 + v_0 \cdot (\omega_0 - \omega_1). \tag{7.55}$$

Since $m_0 > 0$ we deduce,

$$v_0 \cdot (\omega_0 - \omega_1) > 0. \tag{7.56}$$

Let α, α_1 be the positive angles between v_0 and $(\omega - \omega_1)$ and, respectively v_0 and $(\omega_0 - \omega_1)$. In view of (7.54) and (7.56) we infer that

$$0 < \alpha, \alpha_1 < \pi/2. \tag{7.57}$$

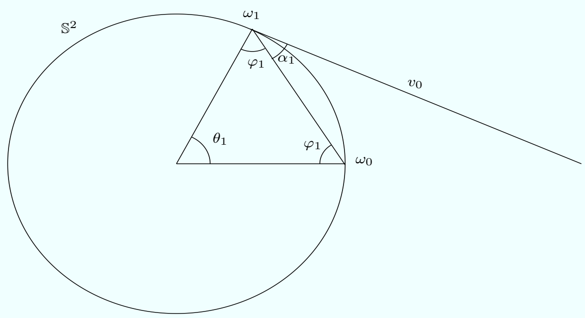

FIGURE 2. Definition of the angles θ_1 and α_1

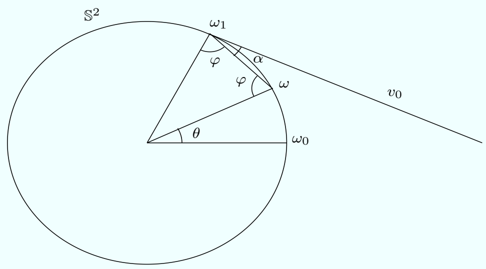

FIGURE 3. Definition of the angles θ and α

Also, in view of (7.47) we have,

$$0 < \theta_1 < \pi/2.$$

Simple considerations on angles imply[5] (see figure 2),

$$\theta_1 = 2\alpha_1. \tag{7.58}$$

Therefore,

$$m_0 \quad = \quad -\frac{1}{2}(s-t)|\omega_0 - \omega_1|^2 + |z-y||\omega_0 - \omega_1| \cos\left(\frac{\theta_1}{2}\right)$$

and

$$|v_0| = \frac{m_0 + \frac{1}{2}(s-t)|\omega_0 - \omega_1|^2}{|\omega_0 - \omega_1| \cos\left(\frac{\theta_1}{2}\right)}. \tag{7.59}$$

Using the same type of argument[6] as in (7.58) we also deduce (see figure 3)

$$\alpha = \frac{\theta_1 - \theta}{2}. \tag{7.60}$$

Therefore, according to (7.52), (7.59) and (7.60), we obtain

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[5]Let φ_1 the angle defined on figure 2. Then $2\varphi_1 + \theta_1 = \pi$, and $\varphi_1 + \alpha_1 = \frac{\pi}{2}$. Hence $\theta_1 = 2\alpha_1$

[6]Let φ the angle defined on figure 3. Then $2\varphi + |\theta_1 - \theta| = \pi$, and $\varphi + \alpha = \frac{\pi}{2}$. Hence $|\theta_1 - \theta| = 2\alpha$

$$\begin{aligned} u(s,y,\omega)-u(t,x,\omega) &= -\frac{1}{2}(s-t)|\omega-\omega_1|^2+|v_0|\,|\omega-\omega_1|\cos\left(\frac{\theta_1-\theta}{2}\right)\\ &= \frac{|\omega-\omega_1|}{|\omega_0-\omega_1|}\frac{\cos\left(\frac{\theta_1-\theta}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)}m_0+\frac{1}{2}(s-t)|\omega-\omega_1|A(\omega), \end{aligned} \tag{7.61}$$

where $A(\omega)$ is given by

$$A(\omega)=-|\omega-\omega_1|+\frac{\cos\left(\frac{\theta_1-\theta}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)}|\omega_0-\omega_1|. \tag{7.62}$$

We have the following lemma (see [**46**] for a proof).

LEMMA 7.21. *For all* $0\le\theta\le\theta_1$, *we have*

$$A(\omega)\ge 0. \tag{7.63}$$

Back to (7.61), we thus derive,

$$u(s,y,\omega)-u(t,x,\omega) \ \ge\ \frac{|\omega-\omega_1|}{|\omega_0-\omega_1|}\frac{\cos\left(\frac{\theta_1-\theta}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)}m_0.$$

Using our angle restriction

$$0\le\theta\le\theta_1<\frac{\pi}{2},$$

we deduce

$$u(s,y,\omega)-u(t,x,\omega)\ge\frac{\sqrt{2}}{2}\frac{|\omega-\omega_1|}{|\omega_0-\omega_1|}m_0. \tag{7.64}$$

Since θ is the angle between ω and ω_0, and θ_1 is the angle between ω_1 and ω_0, we have

$$|\omega-\omega_1|=\sqrt{2}\sqrt{1-\cos(\theta_1-\theta)},\ |\omega_1-\omega_0|=\sqrt{2}\sqrt{1-\cos(\theta_1)}. \tag{7.65}$$

In view of (7.65), we can rewrite (7.64) in the form,

$$\phi(t,x,s,y,\omega)\gtrsim m_0\sqrt{\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta_1)}} \tag{7.66}$$

which is the desired estimate (7.41). This concludes the proof of Lemma 7.19 in the flat case.

REMARK 7.22. *Let us indicate how to prove Lemma 7.19 in the general case. The whole point is to realize that the only estimates for which the precise regularity of u matters are the ones corresponding to* (7.44), (7.45), (7.46), (7.50) *and* (7.51)*. Indeed, once this has been achieved, the rest of the argument is then essentially the one of the flat case.*

Now, to prove the estimates corresponding to (7.44), (7.45), (7.46), (7.50) *and* (7.51) *in the general case, one needs the following two additional ingredients (see* [**46**] *for the details):*

(1) *These estimates are obtained by using the following standard identity*

$$u(\eta(1),\omega) = u(\eta(0),\omega) + \int_0^1 \mathbf{g}(L_{\eta(\sigma)}, \eta'(\sigma))d\sigma, \tag{7.67}$$

where η is a curve in $\mathcal{M}$, and where L denotes the space-time gradient of u. It turns out that one may chose suitable curves[7] *η allowing us to deduce from* (7.67) *the estimates corresponding to* (7.44)*,* (7.45)*,* (7.46)*,* (7.50) *and* (7.51) *under our assumptions* (7.16)*,* (7.17) *and* (7.18)*. This changes the constants in the inequalities due to the presence of additional $O(\varepsilon)$ terms, but does not change the nature of the estimates for $\varepsilon > 0$ small enough.*

(2) *The above mentioned curves η start on S, and a crucial point is to check that such curves end up exactly at (s, y). To this end, one uses the global coordinate system $(u(t, x, \omega_0), \partial_\omega u(t, x, \omega_0))$ of section 7.1.4 on Σ_s for a well-chosen angle $\omega_0 \in \mathbb{S}^2$, which allows us to identify (s, y) as the unique point p on Σ_s such that*

$$u(p, \omega_0) = u(s, y, \omega_0) \text{ and } \partial_\omega u(p, \omega_0) = \partial_\omega u(s, y, \omega_0).$$

[7] In the flat case, the corresponding curves η are straight lines.

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