

Improved local well posedness for quasilinear wave equations in dimension three

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1 Introduction

The goal of this paper is to improve some recent results of Bahouri-Chemin [Ba-Ch1], [Ba-Ch2] and Tataru [Ta1], [Ta2], see also [Kl], concerning the issue of optimal well posedness for quasilinear wave equations. These results are concerned with the initial value problem for quasilinear wave equations of the form,

$$\begin{aligned} \partial_t^2 \phi - g^{ij}(\phi) \partial_i \partial_j \phi &= N(\phi, \partial\phi), \\ \phi|_{t=0} &= \phi_0, \quad \partial_t \phi|_{t=0} = \phi_1. \end{aligned} \tag{1.1}$$

The classical results on the existence and uniqueness of solutions of the quasilinear equation (1.1) state that the problem is locally well posed in the Sobolev space H^s for any $s > \frac{n}{2} + 1$. The proof is based on H^s - energy estimates and the Sobolev embedding theorem. The H^s energy estimates, obtained by standard integration by parts and commutator lemmas, take the form

$$\|\phi(t)\|_{H^s} \leq M(\|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}})$$

with M depending continuously on the integral $\int_0^t \|\partial\phi\|_{L_x^\infty}$. The Sobolev estimate $\|\partial\phi\|_{L^\infty} \leq c\|\phi\|_{H^s}$ for $s > \frac{n}{2} + 1$ allows one to conclude that for sufficiently small time t we can bound the right hand side of the above inequality purely in terms of $\|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}}$. These local apriori estimates, for $s > \frac{n}{2} + 1$, can easily be turned into a local existence and uniqueness result, see [Po-Si].

The crucial quantity $\int_0^t \|\partial\phi\|_{L_x^\infty}$ could be better controlled with the aid of Strichartz estimates.

This can be easily seen in the case of the semilinear equations $\square\phi = -\partial_t^2 \phi + \Delta\phi = N(\phi, \partial\phi)$, with N quadratic in $\partial\phi$. In this case the well-known Strichartz estimates for the linear wave equation in Minkowski space yield a far better local well posedness result¹ in the space H^s with $s > \frac{n}{2} + \frac{1}{2}$, if the space dimension $n \geq 3$, and with $s > \frac{3}{4}$ for $n = 2$. The required Strichartz estimate for the solution of the wave equation $\square\psi = -\partial_t^2 \psi + \Delta\psi = 0$ in the Minkowski space \mathbf{R}^{n+1} , $n \geq 3$, has the

¹This result turns out to be optimal, in general, according to the well-known counterexample of H. Lindblad, [Li]. His results shows that local well posedness fails for H^s data, $s \leq 2$, in dimension $n = 3$.

form²

$$\|\partial\psi\|_{L_t^2 L_x^\infty} \leq c(\|\phi_0\|_{H^{\frac{n}{2}+\frac{1}{2}+}} + \|\phi_1\|_{H^{\frac{n}{2}-\frac{1}{2}+}}).$$

In the quasilinear case the metric g^{ij} depends on the solution ϕ and can therefore have only as much regularity as ϕ has. Thus one needs to address the problem of proving the Strichartz estimates for linear wave equations $\square_g \phi = \partial_t^2 - g^{ij} \partial_i \partial_j \phi = 0$ with rough coefficients (metric).

The conditions needed for the coefficients g^{ij} of the linearized wave operator \square_g have to be consistent with the dependence on ϕ of the nonlinear coefficients $g^{ij}(\phi)$ in (1.1). In view of the expected boundedness of the Sobolev norms $\|\phi\|_{H^s}$ we may assume³ that the norms $\|g\|_{H^s}$ are bounded. Moreover, as the H^s norm of a solution of the quasilinear problem (1.1) at time t is controlled by the H^s norm of the initial data as long as $\int_0^t \|\partial\phi\|_{L_x^\infty}$ is bounded, we can inductively

assume that the metric $g^{ij} = g_{ij}(\phi)$ satisfies also the condition $\int_0^t \|\partial g\|_{L_x^\infty} \leq B_0$. In view of our experience with Strichartz estimates in flat space we do not expect have direct access to the norm of $\|\partial\phi\|_{L_t^1 L_x^\infty}$ but, locally in time this can be controlled by the $\|\partial\phi\|_{L_t^2 L_x^\infty}$. Thus, to close the argument and derive an improved local existence and uniqueness result for (1.1), we need to prove the boundedness of H^s norms as well as a local $L_t^2 L_x^\infty$ Strichartz estimate for solutions ϕ to the linearized wave equation $\square_g \phi = 0$.

The first important work concerning Strichartz type estimates for solutions to $\square_g \phi = 0$ with rough⁴ coefficients is due to H. Smith [Sm]. He showed that the standard Strichartz estimates hold for equations with $C^{1,1}$ coefficients. However the $C^{1,1}$ condition is too strong for applications to quasilinear equations. Moreover some important counterexamples of H. Smith and C. Sogge showed that the standard Strichartz estimates fail if the coefficients are rougher than $C^{1,1}$, [Sm-So].

This was the situation before the important breakthrough of H. Bahouri and J.-Y. Chemin. In [Ba-Ch1] they showed that some weaker form of the Strichartz estimates still survive for equations with coefficients rougher than $C^{1,1}$, compatible with applications to quasilinear equations. Namely, they were able to establish a Strichartz estimate with a loss

$$\|\partial\psi\|_{L_t^2 L_x^\infty} \leq c(\|\psi_0\|_{H^{\frac{n}{2}+\frac{1}{2}+\sigma}} + \|\phi_1\|_{H^{\frac{n}{2}-\frac{1}{2}+\sigma}}),$$

for solutions to the variable coefficient wave equation $\square_g \phi = 0$, with coefficients verifying assumptions of the type discussed above. As long as the loss $\sigma \leq \frac{1}{2}$ such an estimate can be applied to prove a local well posedness result for the quasilinear problem considered here, stronger than the classical one.

The results of H. Bahouri and J.-Y. Chemin, see also [Ta1] are based on Strichartz estimates with a loss of $\sigma > \frac{1}{4}$ for the linearized problem $\square_g \psi = 0$ where the metric g satisfying conditions of the type discussed above. Later, D. Tataru showed that the loss⁵ in the Strichartz estimate can be improved $\sigma > \frac{1}{6}$ [Ta2]. In fact the paper [Ta2] provides the precise relationship between

²The presence of $+$ in the H^s norms on the right hand side of the inequality means simply that the estimate holds in fact with a slightly higher norm.

³These are typical bootstrap assumptions.

⁴For C^∞ metrics the Strichartz estimates were proved by Kapitanski and by Mockenhaupt-Seeger-Sogge, [Ka], [Mo-Se-So].

⁵The immediate consequence of these results is local well posedness for the quasilinear problem (1.1) in the space H^s with $s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$, if $n \geq 3$, and $s > \frac{n}{2} + \frac{5}{6}$ for $n = 2$.

the smoothness of the metric, below $C^{1,1}$, and the corresponding loss in the Strichartz estimates. Recently H. Smith and D. Tataru[Sm-Ta] have shown that these results are also sharp and therefore there is no hope of improving Tataru's results based purely on linear theory.

In this paper we improve the well posedness results of Bahouri-Chemin and Tataru by taking into account the nonlinear character of the equation (1.1). We do not simply prove a Strichartz estimates for solutions of $\square_g \phi = 0$ with bounds on $\|g\|_{H^s}$ and $\|\partial g\|_{L_t^1 L_x^\infty}$, we make use in an essential way of the fact that the coefficients $g^{ij}(\phi)$ of the equation (1.1) verify themselves an equation of the type $\square_g g_{ij} = N$ with N depending only on ϕ and $\partial\phi$.

Our main result is included in the following theorem.

Theorem 1.1 (A) *The quasilinear initial value problem (1.1) in \mathbf{R}^{3+1} with the metric $g^{ij}(z)$ satisfying the assumptions **G** below, is locally well posed in H^{s_*} for $s_* > s_0 = 2 + \frac{2-\sqrt{3}}{2}$. Namely, for any initial data⁶ $\phi[0] \in H^{s_*}$ such that $\|\phi[0]\|_{H^{s_*}} \leq c_{s_*}^{-1} \Lambda_0$, where c_s is the Sobolev constant of the embedding $H^{s_*} \subset L^\infty$, there exists a sufficiently small interval $[0, T]$ with $T = T(\|\phi[0]\|_{H^{s_*}})$ such that problem (1.1) has a unique solution $\phi \in C([0, T], H^{s_*}) \cap C^1([0, T], H^{s_*-1})$. In addition, ϕ satisfies a Strichartz type estimate*

$$\|\partial\phi\|_{L_{[0,T]}^2 L_x^\infty} \leq c T^{s_*-s_0} \|\phi[0]\|_{H^{s_*}}. \quad (1.2)$$

The precise assumptions on the coefficients g^{ij} are given in the following:

Assumption on the coefficients g : *The family of metrics $g(z)$ is smooth and uniformly positive definite with respect to bounded values of the parameter $z \in \mathbb{R}$. More precisely there exist positive constants M_0, Λ_0 such that for a sufficiently large integer k*

$$\begin{aligned} \sup_{|z| \leq \Lambda_0} \left| \left(\frac{d}{dz} \right)^l g \right| &\leq M_0, \quad \forall 0 \leq l \leq k \\ M_0^{-1} |\xi|^2 &\leq g^{ij}(z) \xi_i \xi_j \leq M_0 |\xi|^2, \quad \forall |z| \leq \Lambda_0, \\ N(\phi, \partial\phi) &= \sum_{\alpha, \beta} N^{\alpha\beta}(\phi) \partial_\alpha \phi \partial_\beta \phi, \quad \sup_{|z| \leq \Lambda_0} \left| \left(\frac{d}{dz} \right)^l N^{\alpha\beta}(z) \right| \leq M_0, \quad \forall 0 \leq l \leq k. \end{aligned} \quad (\mathbf{G})$$

In what follows we describe the reduction of Theorem (A) to the proof of the Strichartz estimate for a linearized and microlocalized problem on a small interval of time. The first five steps are now standard, see [Ba-Ch2] and especially [Ta2]. The reduction to the last step is typical to the geometric approach of [Kl]. Most of the work of the paper goes into proving the last step, Theorem 1.8. We shall give an outline of this in Chapter 3, after we have introduced the necessary geometric concepts in Chapter 2. We also provide the reader with a detailed navigation map through the paper at the end of this Chapter. The detailed proof of the following reductions can be found in Chapter 8.

1.1 Step 1 Energy estimates

As we have already noted above the energy estimates for the quasilinear wave equation (1.1) imply that the $L_{[0,T]}^\infty H^s$ norm of a solution ϕ is controlled by the H^s norm of the initial data $\phi[0]$ provided that $\|\partial\phi\|_{L_{[0,T]}^1 L_x^\infty} \leq 1$. The next proposition is a more precise version of this statement.

⁶We denote the initial data for quasilinear equation (1.1) by $\phi[0]$ and say that $\phi[0] \in H^s$ if $(\phi_0, \phi_1) \in H^s \times H^{s-1}$.

Proposition 1.1 (Energy estimate) *Let $\phi \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$ be a solution of (1.1) on the time interval $[0, T]$ for some $s \geq 1$ obeying the condition that $\|\phi\|_{L^\infty_{[0, T]} L^\infty_x} \leq \Lambda_0$. Then ϕ verifies the following energy estimate.*

$$\|\phi\|_{L^\infty_{[0, T]} \dot{H}^s} \leq C(\|\partial\phi\|_{L^1_{[0, T]} L^\infty_x}, \Lambda_0) \|\phi[0]\|_{\dot{H}^s}. \quad (1.3)$$

Here constant $C(a, b)$ denotes the dependence on a, b and \dot{H}^s are the usual homogeneous Sobolev spaces.

1.2 Step 2 Reduction to the Strichartz type estimates

By the Cauchy-Schwartz inequality, $\|\partial\phi\|_{L^1_{[0, T]} L^\infty_x} \leq T^{\frac{1}{2}} \|\partial\phi\|_{L^2_{[0, T]} L^\infty_x}$. Thus the Strichartz inequality (1.2) and the smallness of the time interval $[0, T]$ can be used to close the energy estimates in the space H^{s^*} . This effectively yields the local well posedness of problem (1.1). Theorem (A) can be then reduced to the following bootstrap argument.

Theorem 1.2 (A1) *Let $\phi \in C([0, T], H^{s^*}) \cap C^1([0, T], H^{s^*-1})$ be a solution of (1.1) on the time interval $[0, T]$, $T \leq 1$. Assume that*

$$\|\phi\|_{L^\infty_{[0, T]} H^{s^*}} + \|\partial\phi\|_{L^2_{[0, T]} L^\infty_x} \leq B_0, \quad (1.4)$$

with the constant $B_0 \leq c_{s^*}^{-1} \Lambda_0$, where c_{s^*} is the Sobolev constant of the embedding $H^{s^*} \subset L^\infty$. Then ϕ satisfies the local in time Strichartz type estimate,

$$\|\partial\phi\|_{L^2_{[0, T]} L^\infty_x} \leq C(B_0) T^{s^*-s_0} \|\phi\|_{L^\infty_{[0, T]} H^{s^*}}. \quad (1.5)$$

1.3 Step 3 The dyadic version of the Strichartz type estimate and the paradifferential approximation

For the purpose of proving the Strichartz type estimate (1.5) we may regard the quasilinear problem (1.1) as a linear wave equation for ϕ with rough coefficients. It is advantageous to mollify the coefficients and work with the family of linear wave equations with smooth coefficients dependent on a parameter. First we introduce functions ϕ^λ obtained by restricting ϕ to the dyadic piece of frequency $\sim \lambda$ in Fourier space. More precisely, let ζ be a smooth function with support in the shell $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$. Here, ξ denotes the variable of the spatial Fourier transform. Let ζ also satisfy the condition $\sum_{k \in \mathbb{Z}} \zeta(2^k \xi) = 1, \quad \forall \xi \in \mathbb{R}^3 / \{0\}$. Let λ be a dyadic parameter $\lambda = 2^k$ with some $k \in \mathbb{Z}$ and denote by P_λ the ‘‘projector’’

$$P_\lambda f(x) = f^\lambda(x) = \int e^{-ix \cdot \xi} \zeta(\lambda^{-1} \xi) \hat{f}(\xi) d\xi.$$

Define also

$$f_{\leq \lambda} = S_\lambda f = \sum_{\mu \leq \lambda} P_\mu f.$$

Theorem (A1) then follows from the following dyadic version of the Strichartz type estimates for $\phi^\lambda = P_\lambda \phi$.

Theorem 1.3 (A2) *Let ϕ be as in Theorem (A1). Fix a large parameter Λ . Then for each $\lambda \geq \Lambda$, the function ϕ^λ satisfies the Strichartz type estimate*

$$\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} \leq C(B_0) c_\lambda T^{s_*-s_0} \|\phi\|_{L^\infty_{[0,T]}H^{s_*}}, \quad (1.6)$$

for constants c_λ such that $\sum_\lambda c_\lambda^2 \leq 1$. A similar estimate also holds for $\phi_{\leq\Lambda}$.

Remark (A2) *In the case of the low frequencies, the estimate (1.6) for $\phi_{\leq\Lambda}$ follows trivially from the Sobolev inequality.*

$$\|\partial\phi_{\leq\Lambda}\|_{L^2_{[0,T]}L^\infty_x} \leq c T^{\frac{1}{2}} \|\phi_{\leq\Lambda}\|_{L^\infty_{[0,T]}H^{\frac{5}{2}+\epsilon}} \leq c \Lambda^{\frac{5}{2}+\epsilon-s_*} T^{\frac{1}{2}} \|\phi\|_{L^\infty_{[0,T]}H^{s_*}},$$

where c is the norm of the embedding $H^{\frac{3}{2}+\epsilon}(\mathbf{R}^3) \subset L^\infty(\mathbf{R}^3)$. Since s_* is assumed to be sufficiently close to $s_0 = 2 + \frac{1}{2}(2 - \sqrt{3})$ and Λ is a fixed large parameter which could depend only upon B_0 , we have the desired bound for the low frequency part of ϕ .

We restrict the attention to the large frequencies $\lambda \geq \Lambda$. In the next proposition we show that each ϕ^λ satisfies an inhomogeneous wave equation with the smooth metric $g_{\lambda^a}^{ij} = S_{\lambda^a} g^{ij}(S_\lambda^a \phi)$ for any fixed value of the parameter $a \in [0, 1]$.

Proposition 1.2 *Let ϕ be as in Theorem (A1). Fix the value of the parameter a , $a \in [0, 1]$. Then for each $\lambda \geq \Lambda$, ϕ^λ verifies the equation*

$$\begin{aligned} \bar{\square}_{g_{\leq\lambda^a}} \phi^\lambda &= -\partial_t^2 \phi^\lambda + g_{\leq\lambda^a}^{ij} \partial_i \partial_j \phi^\lambda = R_\lambda^a, \\ \phi^\lambda|_{t=0} &= \phi_0^\lambda, \quad \partial_t \phi^\lambda|_{t=0} = \phi_1^\lambda. \end{aligned} \quad (1.7)$$

Furthermore, the Fourier support of the right-hand side R_λ^a is contained in the set $\{\xi : \lambda \leq |\xi| \leq 4\lambda\}$, and for all real $s > 1$ and an arbitrary $t \in [0, T]$

$$\|R_\lambda(t)\|_{\dot{H}^s} \leq c \lambda^s \|R_\lambda\|_{L^2_x} \lesssim C(B_0) \lambda^{1-a} c_\lambda \|\partial\phi\|_{L^\infty_x} \|\phi\|_{\dot{H}^s} \quad (1.8)$$

with the constants c_λ : $\sum_\lambda c_\lambda^2 \leq 1$.

1.4 Step 4 Strichartz estimate on the frequency dependent intervals

This step reduces the proof of Theorem (A) to the proof of the precise ⁷ Strichartz estimate for the linearized equation $\bar{\square}_{g_\lambda} \psi = 0$ on small time interval I of size $\approx T\lambda^{-(1-a)}$. The loss of regularity in the estimate (1.6) follows then as a result of summing these true Strichartz estimates over intervals I in $[0, T]$.

Theorem 1.4 (A3) *Let $\lambda \geq \Lambda$ and let ψ be a solution of the linear wave equation $\bar{\square}_{g_\lambda} \psi = 0$ with initial data $\psi[0]$ such that the supp $\hat{\psi}[0] \in \{\frac{1}{2}\lambda \leq |\xi| \leq 2\lambda\}$ and the metric g_λ is as defined in Proposition 1.2. Fix the value of the parameter a , $a = \frac{5-2s_*}{1-2s_*+2s_0} < -1 + \sqrt{3}$. Then there exists a partition $\{I\}$ of the interval $[0, T]$ into subintervals I such that the size of each I , $|I| \leq T\lambda^{-(1-a)}$, the number of the subintervals is approximately λ^{1-a} , and on each I , ψ satisfies a Strichartz estimate with a fixed, arbitrary small $\epsilon > 0$,*

$$\|P_\lambda \partial\psi\|_{L^2_I L^\infty_x} \leq C(B_0) |I|^\epsilon \|\psi[0]\|_{\dot{H}^{2+\epsilon}} \quad (1.9)$$

⁷without losses as in the flat case

Remark (A3) According to the bootstrap assumption (1.4) the solution ϕ of the quasilinear problem verifies the estimate $\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \leq B_0$. It easily follows that there exists a subpartition $\{I\}$ of the time interval $[0, T]$, with the total number of the subintervals I between λ^{1-a} and $2\lambda^{1-a}$ and the size of each I bounded by $T\lambda^{-(1-a)}$, such that on each I we have

$$\|\partial\phi\|_{L^2_I L^\infty_x} \leq \lambda^{-\frac{1-a}{2}} \|\partial\phi\|_{L^2_{[0,T]} L^\infty_x} \quad (1.10)$$

This construction defines the subpartition $\{I\}$ mentioned in Theorem (A3).

1.5 Properties of the metric g_{λ^a}

Inequality (1.9) is the Strichartz estimate for a solution of the wave equation with variable coefficients (metric) $\square_{g_{\leq\lambda^a}}\psi = 0$ on the frequency dependent intervals I .

The metric $g_{\leq\lambda^a} = S_{\lambda^a}g(S_{\lambda^a}\phi)$ depends upon the solution ϕ of the quasilinear problem. In the next proposition we state the properties of the family $g_{\leq\lambda^a}$ which follow from the bootstrap condition (1.4) on ϕ and the construction of the partition $\{I\}$ described in the Remark (A3).

Proposition 1.3 *Let $\phi \in C([0, T], H^{s_*}) \cap C^1([0, T], H^{s_*-1})$ be a solution of (1.1) on the time interval $[0, T]$, $T \leq 1$. Assume that ϕ verifies the assumption (1.4) of Theorem (A1). Consider the subpartition $\{I\}$ of the time interval $[0, T]$ as defined by Remark (A3). Then the family of metrics $g_{\leq\lambda^a} = S_{\lambda^a}g(S_{\lambda^a}\phi)$ obeys the following conditions:*

For all subintervals I and all nonnegative integers m

$$\|\partial^{1+m}g_{\leq\lambda^a}\|_{L^1_I L^\infty_x} \leq \lambda^{-(1-a)+am}\bar{B}_0, \quad (1.11)$$

$$\|\partial^{1+m}g_{\leq\lambda^a}\|_{L^2_I L^\infty_x} \leq \lambda^{-\frac{1-a}{2}+am}\bar{B}_0, \quad (1.12)$$

$$\|\partial^{1+m}g_{\leq\lambda^a}\|_{L^\infty_I L^\infty_x} \leq \lambda^{\frac{a^2}{2}+am}\bar{B}_0, \quad (1.13)$$

$$\|\partial_x^{\frac{1}{2}+m}(\partial^2g_{\leq\lambda^a})\|_{L^\infty_I L^2_x} \leq \lambda^{\frac{a^2}{2}+am}\bar{B}_0, \quad (1.14)$$

$$\|\partial^m \square_{g_{\leq\lambda^a}}g_{\leq\lambda^a}\|_{L^1_I L^\infty_x} \leq \lambda^{-(1-a)+am}\bar{B}_0. \quad (1.15)$$

The constant \bar{B}_0 depend only on the constants M_0 and B_0 .

Remark: Observe that by the construction the frequencies of the metric $g_{\leq\lambda^a}$ are truncated above λ^a only with respect to the Fourier variable dual to the spatial variable x . Therefore, each differentiation with respect to x introduces an additional factor of at most λ^a . However, using the fact that $g_{\leq\lambda^a}$ depends on the solution of the wave equation, we can make the same conclusion for the time derivatives

Theorem (A3) can be recast⁸ as a result concerning local Strichartz estimates, on a fixed small subinterval I , for solutions to a linear wave equation with the background metric $g_{\leq\lambda^a}$ satisfying the estimates of Proposition 1.3.

⁸We can therefore completely forget the origin of the metric $g_{\leq\lambda^a}$, we only need to know (1.11)-(1.15).

Theorem 1.5 (A4) *Let ψ be a solution of the linear wave equation*

$$\begin{aligned}\bar{\square}_{g_{\leq \lambda^a}} \psi &= -\partial_t^2 \psi + g_{\leq \lambda^a}^{ij} \partial_i \partial_j \psi = 0, \\ \psi|_{t=0} &= \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1\end{aligned}\tag{1.16}$$

on the time interval I of length $|I| \leq \lambda^{-(1-a)}$ with initial data $\psi[0]$ supported on the set $\{\xi : \frac{1}{2}\lambda \leq |\xi| \leq 2\lambda\}$ in Fourier space. Assume that the metric g_{λ^a} verifies (1.11)-(1.15) of Proposition 1.3 with the parameter a chosen such that $a < -1 + \sqrt{3}$. Let P_λ be the projection on the set $\{\xi : \frac{1}{2}\lambda \leq |\xi| \leq 2\lambda\}$ in Fourier space. Then for a sufficiently large parameter Λ , all dyadic $\lambda \geq \Lambda$ and a fixed $\epsilon > 0$,

$$\|P_\lambda \partial \psi\|_{L_t^2 L_x^\infty} \leq C(\bar{B}_0) |I|^\epsilon \|\psi[0]\|_{\dot{H}^{2+\epsilon}}\tag{1.17}$$

with the constant $C(\bar{B}_0)$ independent of λ .

1.6 Step 5 Rescaling

It is convenient to replace the problem (1.16) by its rescaled version, so that the support of the initial data has frequencies $|\xi| \sim 1$ and the time interval I has length $\leq \lambda^a$.

Translating the problem in time, if necessary, we can assume that the time interval I starts at $t = 0$. Introduce the family of the rescaled metrics h_λ

$$h_\lambda(t, x) = g_{\leq \lambda^a}(\lambda^{-1}t, \lambda^{-1}x)\tag{1.18}$$

Proposition 1.3 implies that h_λ obeys the following estimates⁹ on the time interval¹⁰ $I = [0, t_*]$ with $t_* \leq \lambda^a$:

$$\|\partial^{1+m} h_\lambda\|_{L_{I_\lambda}^1 L_x^\infty} \lesssim \lambda^{-(1-a)(m+1)},\tag{1.19}$$

$$\|\partial^{1+m} h_\lambda\|_{L_{I_\lambda}^2 L_x^\infty} \lesssim \lambda^{-\frac{2-a}{2}-(1-a)m},\tag{1.20}$$

$$\|\partial^{1+m} h_\lambda\|_{L_{I_\lambda}^\infty L_x^\infty} \lesssim \lambda^{-1+\frac{a^2}{2}-(1-a)m},\tag{1.21}$$

$$\|\partial_x^{\frac{1}{2}+m} (\partial^2 h_\lambda)\|_{L_{I_\lambda}^\infty L_x^2} \lesssim \lambda^{-1+\frac{a^2}{2}-(1-a)m},\tag{1.22}$$

$$\|\partial^m \bar{\square}_{h_\lambda} h_\lambda\|_{L_{I_\lambda}^1 L_x^\infty} \lesssim \lambda^{-(2-a)-(1-a)m},\tag{1.23}$$

After rescaling Theorem (A4) transforms into

Theorem 1.6 (A5) *Let ψ be a solution of the linear wave equation*

$$\begin{aligned}\bar{\square}_{h_\lambda} \psi &= -\partial_t^2 \psi + h_\lambda^{ij} \partial_i \partial_j \psi = 0, \\ \psi|_{t=0} &= \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1\end{aligned}\tag{1.24}$$

⁹According to our convention $A \lesssim B$ means $A \leq C \cdot B$ for some universal constant C . By the bootstrap assumption all the constants in our estimates may depend on the constant B_0 . Thus we can treat B_0 as a universal constant and, in what follows, replace the dependence on it by \lesssim . In addition, the choice of the large frequency Λ , as in Theorem (A3), will be determined by the condition that $A \lesssim B$ may be replaced by $A \leq \Lambda^\epsilon B$ with an arbitrary positive ϵ

¹⁰We keep the notation I for the rescaled time interval

on the time interval $[0, t_*]$ with $t_* \leq \lambda^a$. Assume that the parameter $\lambda \geq \Lambda$ for a sufficiently large constant Λ and that the metric h_λ verifies (1.19)-(1.23) with the parameter a such that $a < -1 + \sqrt{3}$. Let P be the operator of projection on the set $\{\xi : 1 \leq |\xi| \leq 2\}$ in Fourier space. Then

$$\|P \partial \psi\|_{L^2_{[0, t_*]} L^\infty_x} \lesssim |t_*|^\epsilon (\|\partial \psi_0\|_{L^2_x} + \|\psi_1\|_{L^2_x}) \quad (1.25)$$

with a constant independent of λ in the inequality \lesssim .

Remark: Note that Theorem (A5) does not contain any assumptions on the Fourier support of the initial data $\psi[0]$.

1.7 Step 6 Decay estimates

A variation of the standard TT^* type argument, see [Kl], allows us to reduce the Strichartz estimate (1.25) to a corresponding dispersive inequality, see (1.26). In the process we replace¹¹ the equation $\square_{h_\lambda} \psi = 0$ by the *geometric wave equation* $\square_{h_\lambda} \psi = -\frac{1}{\sqrt{\det h_\lambda}} \partial_t \sqrt{\det h_\lambda} \partial_t \psi + \frac{1}{\sqrt{\det h_\lambda}} \partial_i (h_\lambda^{ij} \sqrt{\det h_\lambda} \partial_j \psi) = 0$.

Theorem 1.7 (A6) *Let ψ be a solution of the linear wave equation*

$$\begin{aligned} \square_{h_\lambda} \psi &= -\frac{1}{\sqrt{\det h_\lambda}} \partial_t \sqrt{\det h_\lambda} \partial_t \psi + \frac{1}{\sqrt{\det h_\lambda}} \partial_i (h_\lambda^{ij} \sqrt{\det h_\lambda} \partial_j \psi) = 0, \\ \psi|_{t=0} &= \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1 \end{aligned} \quad (1.26)$$

on the time interval $[0, t_*]$ with $t_* \leq \lambda^a$ and with initial data $\psi[0]$ supported in the set $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ in Fourier space. We consider only large values of the parameter $\lambda \geq \Lambda$. Assume that the metric h_λ verifies (1.19)-(1.23) with the parameter a such that $a < -1 + \sqrt{3}$. Then for all $t \leq t_*$ and a fixed arbitrary small $\epsilon > 0$

$$\|P \partial \psi(t)\|_{L^\infty_x} \lesssim \frac{1}{(1 + |t|)^{1-\epsilon}} \|\psi[0]\|_{L^1_x}. \quad (1.27)$$

We make the final reduction by decomposing the initial data $\psi[0]$ in the physical space into a sum of functions with essentially disjoint supports contained in balls of radius $\frac{1}{2}$. Using the additivity of the L^1 norm and the standard Sobolev inequality we can reduce the dispersive inequality (1.27) to an $L^2 - L^\infty$ decay estimate.

Theorem 1.8 (B) *Let ψ be a solution of the linear wave equation (1.26) on the time interval $[0, t_*]$ with $t_* \leq \lambda^a$ and with initial data $\psi[0]$ supported in the ball $B_{\frac{1}{2}}(0)$ of radius $\frac{1}{2}$ centered at the origin in the physical space. We fix a big constant Λ and consider only large values of the parameter $\lambda \geq \Lambda$. Assume that the metric h_λ verifies (1.19)-(1.23) with the parameter a such that $a < -1 + \sqrt{3}$. Then for all $t \leq t_*$, an arbitrary small $\epsilon > 0$, and a sufficiently large integer $m > 0$,*

$$\|P \partial \psi(t)\|_{L^\infty_x} \lesssim \frac{1}{(1 + |t|)^{1-\epsilon}} \sum_{k=1}^m \|\partial^k \psi[0]\|_{L^2_x}. \quad (1.28)$$

¹¹The two wave operators differ only by lower order terms in so far as the Strichartz estimates are concerned.

Remark: *For convenience we shall consider the initial value problem (1.26) with initial data given at $t = 1$ rather than at $t = 0$. The reduction easily follows from the energy estimate and finite speed of propagation.*

1.8 Outline of the paper

In chapter 2 we give an introduction to our main geometric framework. Our central object is the optical function u which is defined as a solution of the Eikonal equation whose level hypersurfaces are forward null cones with vertices on the time axis. Associated to it are our null pair L, \underline{L} and the Ricci coefficients $\text{tr}\chi$, $\hat{\chi}$, and η which satisfy our main transport and Hodge type equations described in Propositions 2.2 and 2.3. The remarkable decomposition of one of the null components of the Ricci curvature discussed in Lemma 2.1 also plays a crucial role in our approach.

In chapter 3 we give an outline of the proof of Theorem B; based on generalized energy estimates with the help of the modified Morawetz vectorfield K introduced in (3.89). We thus reduce Theorem B to the Boundedness Theorem (Theorem 3.1). The proof of this latter theorem occupies most of the remaining part of the paper.

In Chapter 4 we give a sketch of the proof of the boundedness theorem; which motivates our main geometric constructions. In particular we show how the proof relies on the Asymptotics Theorem (Theorem 4.2) which describes the behavior of the Ricci coefficients $\text{tr}\chi, \hat{\chi}, \eta$.

The Asymptotics Theorem is proved in Chapter 5. The proof, which relies on an adaptation of the methods used in Chapter 9 of [Ch-Kl], takes advantage of the full geometric structure of the equations satisfied by the Ricci coefficients. As in [Ch-Kl] we use this to identify a coupled system of transport and Hodge equations which allows us to take advantage of the remarkable decomposition of one the components¹² of the Ricci tensor mentioned above. A heuristic outline of our method is given in section 5.1 We also note that the proof of this crucial theorem imposes the main constraints in the numerology of our final result.

In Chapter 6 we use the results of Chapter 5 to provide a full proof of the Boundedness Theorem. The proof relies, in addition to the Asymptotics Theorem, on a subtle integration by parts argument described in Lemma 6.4, see also [Kl].

In Chapter 7 we end the proof of Theorem B.

In Chapter 8 we provide the proof of the reductions stated in Chapter 1. Most of them are standard, see [Ba-Ch2], [Ta2], [Kl]. The main new ingredient here is Lemma 8.4 which plays an essential role in connection to the remarkable decomposition mentioned above.

Finally in Chapter 9, or the Appendix, we provide proofs of a version of the isoperimetric inequality and trace theorem which we need in Chapter 5.

¹²Note that all components of the Ricci tensor are zero for the Einstein vacuum equations. This is not at all the case for general quasilinear equations of the type we study here. It turns out that the decomposition of the \mathbf{R}_{44} component of the Ricci tensor allows us to achieve essentially the same regularizing effect for the Ricci coefficients as the complete vanishing of the Ricci curvature.

2 The geometry of the wave equation $\square_{h_\lambda}\psi = 0$

To avoid the presence of too many indices we shall denote by h any member of the family of metrics h_λ . We associate to the Riemannian metric h the space-time Lorentz metric H

$$H_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + h_{ij}dx^i dx^j. \quad (2.29)$$

Here, and throughout the paper, the greek indices α, β vary through the space-time indices $0, \dots, 3$ while the latin letters i, j are reserved for the spatial indices $1, \dots, 3$. Thus $x^0 = t$ and $dx^0 = dt$. We denote by X, Y, Z , space-time vectorfields, $X = X^\alpha \partial_\alpha$, and by $\langle X, Y \rangle = H_{\alpha\beta} X^\alpha Y^\beta$ the standard scalar product defined by H . We say that vector X is time-like if $\langle X, X \rangle < 0$, space-like if $\langle X, X \rangle > 0$, and null if $\langle X, X \rangle = 0$. Clearly the vectorfield $T = \partial_t$ is the unit timelike vectorfield to the spacelike hypersurfaces Σ_t defined by the level hypersurfaces of the time function t . We denote by \mathcal{D} the Levi-Civita space-time connection defined by H and by ∇ the induced connection on Σ_t . Recall that $\mathcal{D}H = 0$ or, given vectorfields X, Y, Z

$$\mathcal{D}_X \langle Y, Z \rangle = \langle \mathcal{D}_X Y, Z \rangle + \langle Y, \mathcal{D}_X Z \rangle.$$

Given an l -covariant tensorfield Π and vectorfields X_1, X_2, \dots, X_l we denote by $\Pi_{X_1 X_2 \dots X_l}$ the expression $\Pi(X_1, X_2, \dots, X_l)$ or, when there is no possible confusion, simply $\Pi_{12\dots l}$. We will use standard rules for raising and lowering indices with respect to the metric H (e.g. $X_\alpha = H_{\alpha\beta} X^\beta$), as well as silent summation over repeated indices. We denote by \mathcal{L} the Lie derivative and by ${}^{(X)}\pi = \mathcal{L}_X H$ the deformation tensor a vectorfield X . Recall that ${}^{(X)}\pi_{\alpha\beta} = \mathcal{D}_\alpha X_\beta + \mathcal{D}_\beta X_\alpha$.

We denote by $\Gamma_{\beta\delta}^\alpha = \frac{1}{2}H^{\alpha\gamma}(\partial_\delta H_{\gamma\beta} + \partial_\beta H_{\gamma\delta} - \partial_\gamma H_{\beta\delta})$ the Christoffel symbols of the connection \mathcal{D} . Observe that the vectorfield $T = \partial_t$ is geodesic, i.e. $\mathcal{D}_T T = 0$. We denote by k the second fundamental form of the spacelike hypersurfaces Σ_t . More precisely, for X, Y tangent to Σ_t , $k_{XY} = -\langle \mathcal{D}_X T, Y \rangle = \langle T, \mathcal{D}_X Y \rangle$. Relative to the coordinates $x^0 = t$ and $x^i, i = 1, 2, 3$, we have $k_{ij} = -\langle \mathcal{D}_i T, \partial_j \rangle = \langle T, \mathcal{D}_i \partial_j \rangle$. It is easy to check, using the commuting properties of the coordinate vectorfields, that

$$k_{ij} = -\frac{1}{2}\partial_t h_{ij} \quad (2.30)$$

We are now ready to introduce the following geometric concepts. The optical function introduced below plays a key role in our paper.

Optical function: *This is a solution u of the eikonal equation*

$$\mathcal{D}^\alpha u \mathcal{D}_\alpha u = H^{\alpha\beta} \mathcal{D}_\alpha u \mathcal{D}_\beta u = 0, \text{ or} \quad (2.31)$$

$$|\partial_t u|^2 = h^{ij} \partial_i u \partial_j u \quad (2.32)$$

*with the boundary condition $u = t$ on the time axis and whose level hypersurfaces are forward light cones with vertices on the time axis*¹³.

Null Hypersurfaces C_u : *These are the level hypersurfaces*¹⁴ C_u *of the optical function u , whose normal*¹⁵ $L' = \partial^\alpha u \cdot \partial_\alpha = -H^{\alpha\beta} \partial_\beta u \partial_\alpha$ *is null at every point of C_u .*

¹³Our choice of the optical function corresponds to $u = t - r$, $r = |x|$, in the flat case of the Minkowski spacetime.

¹⁴In Minkowski space these correspond to future light cones with vertices on the time axis. By abuse of language we shall refer to C_u as null cones with vertices on the time axis.

¹⁵Observe that the vectorfield $L' = -\partial^\alpha u \cdot \partial_\alpha$ is a null geodesic vectorfield i.e. $\langle L', L' \rangle = 0$ and $\mathcal{D}_{L'} L' = 0$, see Proposition 2.1.

Lapse function: Define the lapse function b by

$$b^{-1} = - \langle L', \partial_t \rangle = u_t. \quad (2.33)$$

Null pair: Define the null vectorfields L, \underline{L}

$$L = b L' = -b \partial^\alpha u \partial_\alpha = -b H^{\alpha\beta} \partial_\beta u \partial_\alpha, \quad (2.34)$$

$$\underline{L} = 2T - L. \quad (2.35)$$

We easily check, using (2.32), $\langle L, L \rangle = \langle \underline{L}, \underline{L} \rangle = 0$, $\langle L, \underline{L} \rangle = 2 \langle T, L \rangle = -2$.

Affine parameter s for the vectorfield L : We can parameterize any integral curve¹⁶ $x(s)$ of L , initiating on the time axis, such that $\frac{d}{ds}x(s) = L$, or $L(s) = 1$. The affine parameter s defines a function on \mathbf{R}^4 . Since $L(t) = 1$ and $L(u) = 0$, we find that $s = t - u$.

Compact surfaces $S_{t,u}$: These are the 2- surfaces¹⁷ of intersection between the null cones C_u and the spacelike hypersurfaces Σ_t . We regard $S_{t,u}$ as compact riemannian surfaces with the induced metric. We use the notation ∇ for the corresponding covariant derivative. We denote by $K = K(S_{t,u})$ their Gaussian curvature and by $A = A(S_{t,u})$ their total area. We denote by ϵ the completely antisymmetric tensor on $S_{t,u}$. Namely, for any orthonormal frame e_A , $A = 1, 2$ on $TS_{t,u}$, $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = -1$. For any tangent to $S_{t,u}$ 2-covariant tensor $\Pi_{\underline{A}}$ with components $\Pi_{A_1 A_2}$ defined relative to an orthonormal frame e_A , we define $\text{tr}\Pi = \sum_{A=1}^2 \Pi_{AA}$ to be the trace of Π .

Regarding $S_{t,u}$ as embedded in Σ_t we define the unit outward normal¹⁸ vectorfield N . Clearly,

$$N = -\frac{h^{ij} \partial_j u}{|\nabla u|_h} \partial_i = -b \partial^i u \partial_i, \quad (2.36)$$

as it can be easily checked with the help of the eikonal equation, the definition (2.33), and the fact since u is constant on the *forward* light cones, restricted to Σ_t , it decreases away from the origin. Consequently

$$L = T + N, \quad \underline{L} = T - N. \quad (2.37)$$

Thus both L and \underline{L} are orthogonal, in the sense of the spacetime metric H , to the surfaces $S_{t,u}$.

Null frame: We can complement L, \underline{L} to create a frame spanning the tangent space $T\mathbf{R}^4$. We can do this by choosing an arbitrary orthonormal frame on $S_{t,u}$ i.e. pair of unit vectorfields e_A , $A = 1, 2$ tangent to $S_{t,u}$ and orthogonal to each other.

Notation: It is often advantageous to use a different notation for the null frame e_A, L, \underline{L} . We will often denote it e_1, e_2, e_3, e_4 with $\underline{L} = e_3$ and $L = e_4$. Whenever we consider the components of various tensors with respect to the null frame e_1, e_2, e_3, e_4 we will use bold font latin indices **3**, **4** to indicate the components relative to e_3 and e_4 , and capital letters A, B for the components relative to e_1, e_2 . Hence, the components of the spacetime metric H relative to a null frame are: $H_{AB} = \delta_{AB}$, $H_{\mathbf{3}A} = H_{\mathbf{4}A} = 0$ for $A, B = 1, 2$ and $H_{\mathbf{3}\mathbf{3}} = H_{\mathbf{4}\mathbf{4}} = 0$, $H_{\mathbf{3}\mathbf{4}} = -2$.

¹⁶any such trajectory is a null geodesic

¹⁷They correspond to the spheres $|x| = t - u$ in Minkowski space.

¹⁸Normal with respect to the Riemann metric $h(t, x)$ of the slice Σ_t .

Null components of the Curvature: *The spacetime curvature tensor \mathbf{R} can be decomposed,*¹⁹ *relative to the null pair L, \underline{L} into a 2-tensor α tangent to the surfaces $S_{t,u}$, a covector β tangent to $S_{t,u}$ and a scalar γ .*

Recall that the spacetime curvature tensor $\mathbf{R} = \mathbf{R}_{\alpha\beta\gamma\delta}$ is given by

$$\mathbf{R}(X, Y)Z = \mathcal{D}_X \mathcal{D}_Y Z - \mathcal{D}_Y \mathcal{D}_X Z - \mathcal{D}_{[X, Y]} Z.$$

and has the symmetries:

$$\begin{aligned} \mathbf{R}_{\alpha\beta\gamma\delta} &= -\mathbf{R}_{\beta\alpha\gamma\delta} = -\mathbf{R}_{\alpha\beta\delta\gamma}, & \mathbf{R}_{\alpha\beta\gamma\delta} &= \mathbf{R}_{\gamma\delta\alpha\beta}, \\ \mathbf{R}_{\alpha\beta\gamma\delta} + \mathbf{R}_{\alpha\delta\beta\gamma} + \mathbf{R}_{\alpha\gamma\delta\beta} &= 0. \end{aligned}$$

Recall also the Ricci tensor, $\mathbf{R}_{\mu\nu} = H^{\alpha\beta} \mathbf{R}_{\alpha\mu\beta\nu}$. In local coordinates x^α , $\alpha = 0, \dots, 3$,

$$\mathbf{R}_{\mu\nu} = \frac{1}{2} H^{\alpha\beta} (\partial_{\mu\beta}^2 H_{\alpha\nu} + \partial_{\alpha\nu}^2 H_{\mu\beta} - \partial_{\mu\nu}^2 H_{\alpha\beta} - \partial_{\alpha\beta}^2 H_{\mu\nu}) + H^{\alpha\beta} H_{\gamma\delta} (\Gamma_{\mu\beta}^\gamma \Gamma_{\alpha\nu}^\delta - \Gamma_{\mu\nu}^\gamma \Gamma_{\alpha\beta}^\delta). \quad (2.38)$$

Note that relative to a null frame e_A, e_3, e_4

$$\mathbf{R}_{44} = H^{AB} \mathbf{R}_{4A4B} \quad (2.39)$$

since $H^{A3} = H^{A4} = 0$ and $\mathbf{R}_{A444} = 0$ by the symmetry properties of \mathbf{R} .

To define the null components α, β, γ of \mathbf{R} at a point we consider the surface $S_{t,u}$ passing through that point and an arbitrary orthonormal frame e_1, e_2 on it. Relative to the null frame e_1, e_2, e_3, e_4 we then introduce,

$$\alpha_{AB} = \mathbf{R}_{4A4B}, \quad \beta_A = \mathbf{R}_{4A34}, \quad \gamma = H^{AB} \mathbf{R}_{A43B}. \quad (2.40)$$

The traceless part²⁰ of α will be denoted by $\hat{\alpha}_{AB} = \alpha_{AB} - \frac{1}{2} \text{tr} \alpha \delta_{AB}$. Note that $\text{tr} \alpha = \mathbf{R}_{44}$.

Remarkable decomposition of \mathbf{R}_{44} : *The coordinate expression (2.38) for the components of the Ricci curvature leads to the following crucial representation for the \mathbf{R}_{44} component.*

Lemma 2.1 (Remarkable decomposition) *The $\mathbf{R}_{44} = e_4^\mu e_4^\nu \mathbf{R}_{\mu\nu}$ component of the Ricci curvature admits the following decomposition:*

$$\mathbf{R}_{44} = L(z) - \frac{1}{2} e_4^\mu e_4^\nu \square_h H_{\mu\nu} + \text{Error}, \quad (2.41)$$

with the functions z and Error obeying pointwise estimates $|z| \lesssim |\partial H|$ and $|\text{Error}| \lesssim (\partial H)^2$.

Proof: The part of the Ricci curvature involving the Cristoffel symbols has a simple pointwise bound

$$|e_4^\mu e_4^\nu H^{\alpha\beta} H_{\gamma\delta} (\Gamma_{\mu\beta}^\gamma \Gamma_{\alpha\nu}^\delta - \Gamma_{\mu\nu}^\gamma \Gamma_{\alpha\beta}^\delta)| \lesssim |\partial H|^2$$

and thus can be included in the *Error*.

¹⁹This is not a complete decomposition

²⁰with respect to the induced metric on $S_{t,u}$

Since $H^{\alpha\beta}\partial_{\alpha\beta}^2 H_{\mu\nu}$ is exactly $\square_h H_{\mu\nu}$, it remains to handle the terms

$$\frac{e_4^\mu e_4^\nu}{2} H^{\alpha\beta} (\partial_{\mu\beta}^2 H_{\alpha\nu} + \partial_{\alpha\nu}^2 H_{\mu\beta} - \partial_{\mu\nu}^2 H_{\alpha\beta}).$$

The three terms have a similar structure and can be treated in the same fashion. Note that $e_4^\mu \partial_\mu = L$. Thus $e_4^\mu e_4^\nu H^{\alpha\beta} \partial_{\mu\beta}^2 H_{\alpha\nu} = e_4^\nu H^{\alpha\beta} L(\partial_\beta H_{\alpha\nu})$. The above expression can be written as an L derivative minus the terms which can be included in the *Error*, in other words $e_4^\mu e_4^\nu H^{\alpha\beta} L(\partial_\beta H_{\alpha\nu}) = L(e_4^\nu H^{\alpha\beta} \partial_\beta H_{\alpha\nu}) - S$ with $S = L(e_4^\nu H^{\alpha\beta} \partial_\beta H_{\alpha\nu}) + e_4^\nu L(H^{\alpha\beta}) \partial_\beta H_{\alpha\nu}$. Clearly²¹, $|e_4^\nu L(H^{\alpha\beta}) \partial_\beta H_{\alpha\nu}| \lesssim |\partial H|^2$. In addition,

$$L(e_4^\nu) = L(H^{\nu\delta} \langle e_4, \partial_\delta \rangle) = L(H^{\nu\delta}) \langle e_4, \partial_\delta \rangle + H^{\nu\delta} \langle \mathcal{D}_4 e_4, \partial_\delta \rangle + H^{\nu\delta} \langle e_4, \mathcal{D}_4 \partial_\delta \rangle.$$

Observe, see Proposition 2.1 below, that $\mathcal{D}_4 e_4 = -k_{NN} e_4$ and $|k| \lesssim |\partial H|$. Also, $\mathcal{D}_4 \partial_\delta = e_4^\omega \Gamma_{\omega\delta}^\sigma \partial_\sigma$ and $|\Gamma| \lesssim |\partial H|$. Hence, the terms denoted by S are bounded pointwise by $(\partial H)^2$ and therefore, indeed, part of the terms we have called *Error*.

It remains to note that z in (2.41) is, in fact, represented by the expression

$$z = e_4^\nu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} - \frac{1}{2} H^{\alpha\beta} L(H_{\alpha\beta}).$$

and thus $|z| \lesssim |\partial H|$ as desired.

Notation: A spacetime tensor tangent to the surfaces $S_{t,u}$ will be called S -tangent in what follows. We shall use the notation R_{AB} to denote any tangent S -tangent 2-tensor obtained from the curvature \mathbf{R} . Similarly, the notation R_A is reserved for an S -tangent curvature co-vector.

S-projection, \mathcal{P} : The projection \mathcal{P} and of the connection \mathcal{D} on the surfaces $S_{t,u}$ is the map $TR^4 \times TS_{t,u} \rightarrow TS_{t,u}$ defined as follows:

$$\mathcal{P}_Y X = \langle \mathcal{D}_Y X, e_A \rangle e_A, \quad \forall X \in TS_{t,u}, \forall Y \in TR^4.$$

Note that $\mathcal{P}_Y = \nabla_Y$ for any $Y \in TS_{t,u}$. The action of the projected connection \mathcal{P} can be extended in the usual manner to any S -tangent tensor. In particular, for an S -tangent 2-tensor Π

$$\mathcal{P}_Y \Pi_{AB} = Y(\Pi_{AB}) - \Pi(\mathcal{P}_Y e_A, e_B) - \Pi(e_A, \mathcal{P}_Y e_B).$$

We shall also make use of the notation

$$\Pi_{\mathcal{P}_Y AB} = \Pi(\mathcal{P}_Y e_A, e_B).$$

Ricci coefficients: In the next proposition we introduce the Ricci, or connection, coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \xi$ of the connection \mathcal{D} relative to our null pair L, \underline{L} .

²¹Our calculations are coordinate dependent. Thus ∂_α are the coordinate vectorfields $\frac{\partial}{\partial x^\alpha}$ with respect to our “given” coordinate system x^α . Observe that the components e_4^ν of L relative to this system of coordinates are bounded.

Proposition 2.1 Define the following tensors on the surfaces $S_{t,u}$ relative to a null frame e_A, e_3, e_4 :

$$\chi_{AB} = \langle \mathcal{D}_A e_4, e_B \rangle, \quad \underline{\chi}_{AB} = \langle \mathcal{D}_A e_3, e_B \rangle, \quad (2.42)$$

$$\eta_A = \frac{1}{2} \langle \mathcal{D}_3 e_4, e_A \rangle, \quad \underline{\eta}_A = \frac{1}{2} \langle \mathcal{D}_4 e_3, e_A \rangle, \quad (2.43)$$

$$\underline{\xi}_A = \frac{1}{2} \langle \mathcal{D}_3 e_3, e_A \rangle. \quad (2.44)$$

Then the connection \mathcal{D} can be described by the following equations:

$$\mathcal{D}_A e_4 = \chi_{AB} e_B - k_{AN} e_4, \quad \mathcal{D}_A e_3 = \underline{\chi}_{AB} e_B + k_{AN} e_3, \quad (2.45)$$

$$\mathcal{D}_4 e_4 = -k_{NN} e_4, \quad \mathcal{D}_4 e_3 = 2\underline{\eta}_A e_A + k_{NN} e_3, \quad (2.46)$$

$$\mathcal{D}_3 e_4 = 2\eta_A e_A + k_{NN} e_4, \quad \mathcal{D}_3 e_3 = 2\underline{\xi}_A e_A - k_{NN} e_3, \quad (2.47)$$

$$\mathcal{D}_4 e_A = \mathcal{P}_4 e_A + \underline{\eta}_A e_4, \quad \mathcal{D}_3 e_A = \mathcal{P}_3 e_A + \eta_A e_3 + \underline{\xi}_A e_4, \quad (2.48)$$

$$\mathcal{D}_B e_A = \nabla_B e_A + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4 \quad (2.49)$$

Also,

$$\underline{\chi}_{AB} = -\chi_{AB} - 2k_{AB}, \quad (2.50)$$

$$\underline{\eta}_A = -k_{AN}, \quad (2.51)$$

$$\underline{\xi}_A = k_{AN} - \eta_A. \quad (2.52)$$

and,

$$\eta_A = b^{-1} \nabla_A b + k_{AN}. \quad (2.53)$$

Proof: The proof of Proposition 2.1 is a simple exercise. We illustrate with several examples.

Computation of $\mathcal{D}_A e_4$: The quantity $\langle \mathcal{D}_A e_4, e_B \rangle = \chi_{AB}$ by definition (2.42) of χ . Also, $\langle \mathcal{D}_A e_4, e_4 \rangle = 0$ since $\langle e_4, e_4 \rangle = 0$. The proof of the first identity in (2.45) is completed by the calculation, $\langle \mathcal{D}_A e_4, e_3 \rangle = \langle \mathcal{D}_A (T + N), T - N \rangle = \langle \mathcal{D}_A N, T \rangle - \langle \mathcal{D}_A T, N \rangle = 2k_{AN}$.

Computation of $\mathcal{D}_4 e_4$: Recall the definition of the null geodesic vectorfield L' . $L' = -\partial u = -H^{\alpha\beta} \partial_\alpha u \partial_\beta = b^{-1} L$. The eikonal equation (2.32) can be written in the form $\langle \partial u, \partial u \rangle = \langle L', L' \rangle = 0$. The vectorfield L' is indeed geodesic ($\mathcal{D}_{L'} L' = 0$), since,

$$\begin{aligned} \langle \mathcal{D}_{L'} L', \partial_\mu \rangle &= \partial^\gamma u \langle \mathcal{D}_\gamma \partial u, \partial_\mu \rangle = \partial^\gamma u \langle \mathcal{D}_\mu \partial u, \partial_\gamma \rangle \\ &= \langle \mathcal{D}_\mu \partial u, \partial u \rangle = \frac{1}{2} \partial_\mu \langle \partial u, \partial u \rangle = 0, \quad \forall \mu. \end{aligned}$$

It follows that $\mathcal{D}_4 e_4 = b \mathcal{D}_{L'} b L' = (b^{-1} \mathcal{D}_4 b) e_4$. Thus, $\mathcal{D}_4 e_4 = (b^{-1} \mathcal{D}_4 b) e_4 = -\frac{1}{2} \langle \mathcal{D}_4 e_4, e_3 \rangle e_4$. Now ²²,

²²using that T and N are unit and orthogonal, and that T is geodesic ($\mathcal{D}_T T = 0$.)

$\langle \mathcal{D}_4 e_4, e_3 \rangle = - \langle e_4, \mathcal{D}_4 e_3 \rangle = - \langle T + N, \mathcal{D}_{T+N}(T - N) \rangle = - \langle N, \mathcal{D}_T T \rangle + \langle T, \mathcal{D}_T N \rangle - \langle T, \mathcal{D}_N T \rangle - \langle N, \mathcal{D}_N T \rangle + \langle T, \mathcal{D}_N N \rangle = 2k_{NN}$. In addition, we have also found that $\mathcal{D}_4 b = -b k_{NN}$.

Computation of $\mathcal{D}_3 e_3$: In view of the definition of $\underline{\xi}$ it suffices to calculate $\langle \mathcal{D}_3 e_3, e_4 \rangle$ which we rewrite in the form $- \langle e_3, \mathcal{D}_3 e_4 \rangle = \langle e_3, \mathcal{D}_4 e_4 \rangle - 2 \langle e_3, \mathcal{D}_T e_4 \rangle = 2k_{NN} - \langle (T - N), \mathcal{D}_T(T + N) \rangle = 2k_{NN}$.

Computation of $\underline{\eta}_A = -\mathbf{k}_{AN}$: By definition, $2\underline{\eta}_A = \langle \mathcal{D}_4 e_3, e_A \rangle$. We also know that $e_3 = -e_4 + 2T$ and $\langle \mathcal{D}_4 e_4, e_A \rangle = 0$. Thus, $2\underline{\eta}_A = 2 \langle \mathcal{D}_4 T, e_A \rangle = 2 \langle \mathcal{D}_N T, e_A \rangle = -2k_{AN}$.

Computation for (2.53): According to definition (2.43) $\eta_A = \frac{1}{2} \langle \mathcal{D}_3 e_4, e_A \rangle$. Since $e_4 = T + N$, $e_3 = T - N = e_4 - 2N$, and $\mathcal{D}_4 e_4 = -k_{NN} e_4$, we obtain $\eta_A = - \langle \mathcal{D}_N e_4, e_A \rangle = - \langle \mathcal{D}_N(T + N), e_A \rangle = k_{AN} - \langle \mathcal{D}_N N, e_A \rangle$. Since our calculations are local we can choose a system of local coordinates such that at a given point the metric h has the form $h_{ij} = \delta_{ij}$ and $\partial_k h_{ij} = 0$. Clearly, at that point $N = -b h^{ij} \partial_j u \partial_i = -b \partial_i u \partial_i$. Thus, since $e_A(u) = 0$ and $|\nabla u|_h^2 = u_i^2 = b^{-2}$,

$$\begin{aligned} \langle \mathcal{D}_N N, e_A \rangle &= e_A^k \langle \mathcal{D}_{b\partial_j u \partial_j} (b \partial_i u \partial_i), \partial_k \rangle = b e_A^k \partial_j u (\partial_j (b) \partial_k u + b \partial_{j^2}^2 u) \\ &= \frac{1}{2} b^2 e_A^k \partial_k (\partial_j u)^2 = \frac{1}{2} b^2 e_A (b^{-2}) = -b^{-1} e_A(b). \end{aligned}$$

Commutation formulas: We also record here some useful commutation formulas trivially following from the frame equations (2.45)-(2.49) and the identity $[X, Y] = \mathcal{D}_X Y - \mathcal{D}_Y X$ holding for arbitrary vectorfields X, Y .

Lemma 2.2

$$[L, \underline{L}] = 2(\underline{\eta}_A - \eta_A) e_A + k_{NN} \underline{L} - k_{NN} L, \quad (2.54)$$

$$[e_4, e_A] = \mathcal{P}_4 e_A - \chi_{AB} e_B, \quad (2.55)$$

$$[e_3, e_A] = \mathcal{P}_3 e_A - \underline{\chi}_{AB} e_B + (\eta_A - k_{AN}) \underline{L} + \underline{\xi}_A L. \quad (2.56)$$

Remark: According to our assumptions on the metric h we have a very good control on all components of the second fundamental form k . Thus, in view of the last part²³ of Proposition 2.1 we have complete control of all Ricci coefficients provided we know how to control χ and η . These have to be estimated from the Eikonal equation, task which requires a major part of this paper. In what follows we shall derive the so called null structure equations which relate various components of the Ricci coefficients. We divide these equations into transport equations, along the null generators²⁴ of C_u , for χ and η and Hodge systems on the surfaces $S_{t,u}$.

I. Transport equations. The main transport equations are given by the following:

²³See (2.50), (2.51), and (2.52).

²⁴Integration along the trajectories spanning C_u allows one to reconstruct the frame coefficients χ, η in terms of their initial data on the time axis and null components of the spacetime curvature \mathbf{R} . This was essentially the approach of [Kl]. Here, we will act more in the spirit of [Ch-Kl] and exploit other relations between the Ricci coefficients as well as special structure of the curvature.

Proposition 2.2 *Denote*

$$tr\chi = H^{AB}\chi_{AB}, \quad tr\underline{\chi} = H^{AB}\underline{\chi}_{AB}, \quad (2.57)$$

$$\hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2}tr\chi H_{AB}, \quad \hat{\underline{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2}tr\underline{\chi} H_{AB}, \quad (2.58)$$

the trace and the traceless part of tensors χ and $\underline{\chi}$ respectively.

The components $tr\chi$, $\hat{\chi}$, η and the lapse a verify the following equations²⁵:

$$L(b) = -b k_{NN}, \quad (2.59)$$

$$L(tr\chi) + \frac{1}{2}(tr\chi)^2 = -|\hat{\chi}|^2 - k_{NN}tr\chi - \mathbf{R}_{44}, \quad (2.60)$$

$$\mathcal{D}_4\hat{\chi}_{AB} + \frac{1}{2}(tr\chi)\hat{\chi}_{AB} = -k_{NN}\hat{\chi}_{AB} - \hat{\alpha}_{AB}, \quad (2.61)$$

$$\mathcal{D}_4\eta_A + \frac{1}{2}(tr\chi)\eta_A = -(k_{BN} + \eta_B)\hat{\chi}_{AB} - \frac{1}{2}(tr\chi)k_{AN} - \frac{1}{2}\beta_A, \quad (2.62)$$

Remark: The equations above can be interpreted as differential equations along null geodesics initiating on the time axis. Along each such geodesic we can use the canonical parameter $s = t - u$ defined earlier.

Proof: The equation (2.59) has already been established in the course of the calculation for \mathcal{D}_4e_4 . Among the remaining equations the most important are (2.60) and (2.61). We will show how to derive the equation for χ_{AB} and leave the derivation of (2.62) to the reader.

From (2.42), $\chi_{AB} = \langle \mathcal{D}_A e_4, e_B \rangle$. Therefore,

$$L(\chi_{AB}) = \langle \mathcal{D}_4 \mathcal{D}_A e_4, e_B \rangle + \langle \mathcal{D}_A e_4, \mathcal{D}_4 e_B \rangle. \quad (2.63)$$

Using (2.45)-(2.48), commuting \mathcal{D}_4 , \mathcal{D}_A , and invoking the definition of the curvature \mathbf{R} , we have

$$L(\chi_{AB}) = \langle \mathcal{D}_A \mathcal{D}_4 e_4, e_B \rangle + \langle \mathcal{D}_{[e_4, e_A]} e_4, e_B \rangle + \chi(e_A, \mathcal{D}_4 e_B) + \mathbf{R}_{A44B}.$$

In addition, $[e_4, e_A] = \mathcal{D}_4 e_A - \chi_{AC} e_C$. Therefore, $\langle \mathcal{D}_{[e_4, e_A]} e_4, e_B \rangle = \chi(\mathcal{D}_4 e_A, e_B) - \chi_{AC} \chi_{CB}$. Also, $\langle \mathcal{D}_A \mathcal{D}_4 e_4, e_B \rangle = -k_{NN} \chi_{AB}$. Combining, we obtain $\mathcal{D}_4 \chi_{AB} = -k_{NN} \chi_{AB} - \chi_{AC} \chi_{CB} + \mathbf{R}_{A44B}$. Taking the trace we arrive at (2.60). The traceless part of the equation is precisely (2.61).

II. Hodge systems on $S_{t,u}$. In addition to the transport equations a prominent role in our analysis will be played by certain Hodge elliptic systems on the 2-dimensional surfaces $S_{t,u}$.

Proposition 2.3 *The expressions $(div \hat{\chi})_A = \nabla^B \hat{\chi}_{AB}$, $div \eta = \nabla^B \eta_B$ and $(c\psi r l \eta)_{AB} = \nabla_A \eta_B - \nabla_B \eta_A$ verify the following equations:*

²⁵which can be interpreted as transport equations along the null geodesics generated by L .

$$(\mathcal{D}iv \hat{\chi})_A + \hat{\chi}_{AB} k_{BN} = \frac{1}{2}(\nabla_A tr \chi + k_{AN} tr \chi) - \mathbf{R}_{B4AB}, \quad \text{Codazzi equation}, \quad (2.64)$$

$$\mathcal{D}iv \eta = \frac{1}{2}(\underline{L}(tr \chi) + \frac{1}{2}tr \chi tr \underline{\chi} + tr(\hat{\chi} \cdot \underline{\hat{\chi}}) - k_{NN} tr \chi - 2|\eta|^2) - \frac{1}{2}\gamma, \quad (2.65)$$

$$curl \eta = \frac{1}{2}\epsilon^{AB} \underline{\hat{\chi}}_{AC} \hat{\chi}_{CB} - \frac{1}{2}\epsilon^{AB} \mathbf{R}_{A43B}. \quad (2.66)$$

Proof: Consider $\nabla_C \chi_{AB}$. We can expand it in the following way:

$$\begin{aligned} \nabla_C \chi_{AB} &= e_C(\chi_{AB}) - \chi(\nabla_C e_A, e_B) - \chi(e_A, \nabla_C e_B) \\ &= \langle \mathcal{D}_C \mathcal{D}_A e_4, e_B \rangle + \langle \mathcal{D}_A e_4, \mathcal{D}_C e_B \rangle - \chi(\nabla_C e_A, e_B) - \chi(e_A, \nabla_C e_B) = \langle \mathcal{D}_A \mathcal{D}_C e_4, e_B \rangle \\ &+ \langle \mathcal{D}_{[e_C, e_A]} e_4, e_B \rangle + \mathbf{R}_{B4AC} + \langle \mathcal{D}_A e_4, \mathcal{D}_C e_B \rangle - \chi(\nabla_C e_A, e_B) - \chi(e_A, \nabla_C e_B) \\ &= e_A(\chi_{CB}) - \langle \mathcal{D}_C e_4, \mathcal{D}_A e_B \rangle + \mathbf{R}_{B4AC} + \langle \mathcal{D}_A e_4, \mathcal{D}_C e_B \rangle - \chi(\nabla_A e_C, e_B) - \chi(e_A, \nabla_C e_B). \end{aligned}$$

Recall from the equations (2.45)-(2.48) that $\langle \mathcal{D}_A e_4, \mathcal{D}_C e_B \rangle = \chi(e_A, \nabla_C e_B) + k_{AN} \chi_{CB}$ and also $\langle \mathcal{D}_C e_4, \mathcal{D}_A e_B \rangle = \chi(e_C, \nabla_A e_B) + k_{CN} \chi_{AB}$. Thus, $\nabla_C \chi_{AB} = \nabla_A \chi_{CB} + k_{AN} \chi_{CB} - k_{CN} \chi_{AB} + \mathbf{R}_{B4AC}$. Taking the trace with respect to B, C we infer that, $(\mathcal{D}iv \chi)_A = \nabla_A tr \chi + k_{AN} tr \chi - \chi_{AB} k_{BN} + \mathbf{R}_{B4AB}$ as desired. Separating the $tr \chi$ we obtain

$$\begin{aligned} (\mathcal{D}iv \hat{\chi})_A - \nabla_A \hat{\chi}_{BB} &= -\frac{1}{2}(\nabla_B tr \chi \delta_{AB} - 2\nabla_A tr \chi) + k_{AN} tr \chi - \chi_{AB} k_{BN} + \mathbf{R}_{B4AB} \\ &= \frac{1}{2} \nabla_A tr \chi + k_{AN} tr \chi - \chi_{AB} k_{BN} + \mathbf{R}_{B4AB}. \end{aligned}$$

Finally, since $\hat{\chi}$ is traceless $(\mathcal{D}iv \hat{\chi})_A = \frac{1}{2} \nabla_A tr \chi + \frac{1}{2} k_{AN} tr \chi - \hat{\chi}_{AB} k_{BN} + \mathbf{R}_{B4AB}$ as desired.

To obtain the last two equations (2.65) and (2.66) consider $\mathcal{P}_3 \chi_{AB}$. We have

$$\begin{aligned} \mathcal{P}_3 \chi_{AB} &= \underline{L}(\chi_{AB}) - \chi(\mathcal{P}_3 e_A, e_B) - \chi(e_A, \mathcal{P}_3 e_B) \\ &= \langle \mathcal{D}_3 \mathcal{D}_A e_4, e_B \rangle + \langle \mathcal{D}_A e_4, \mathcal{D}_3 e_B \rangle - \chi(\mathcal{P}_3 e_A, e_B) - \chi(e_A, \mathcal{P}_3 e_B) \\ &= \langle \mathcal{D}_A \mathcal{D}_3 e_4, e_B \rangle + \langle \mathcal{D}_{[e_3, e_A]} e_4, e_B \rangle + \langle \mathcal{D}_A e_4, \mathcal{D}_3 e_B \rangle + \mathbf{R}_{B43A} - \chi(\mathcal{P}_3 e_A, e_B) \\ &- \chi(e_A, \mathcal{P}_3 e_B) = e_A(\langle \mathcal{D}_3 e_4, e_B \rangle) - \langle \mathcal{D}_3 e_4, \mathcal{D}_A e_B \rangle + \langle \mathcal{D}_{[e_3, e_A]} e_4, e_B \rangle \\ &\quad + \langle \mathcal{D}_A e_4, \mathcal{D}_3 e_B \rangle + \mathbf{R}_{B43A} - \chi(\mathcal{P}_3 e_A, e_B) - \chi(e_A, \mathcal{P}_3 e_B). \end{aligned}$$

Again, recall from (2.45)-(2.48)

$$\begin{aligned} [e_3, e_A] &= \mathcal{P}_3 e_A - \underline{\chi}_{AC} e_C + (\eta_A - k_{AN})(e_3 - e_4), & \langle \mathcal{D}_3 e_4, e_B \rangle &= 2\eta_B, \\ \langle \mathcal{D}_A e_4, \mathcal{D}_3 e_B \rangle &= \chi(e_A, \mathcal{P}_3 e_B) + 2k_{AN} \eta_B, & \langle \mathcal{D}_3 e_4, \mathcal{D}_A e_B \rangle &= 2\eta(\mathcal{P}_A e_B) - \chi_{AB} k_{NN}. \end{aligned}$$

Therefore,

$$\mathcal{P}_3 \chi_{AB} = 2\nabla_A \eta_B + \chi_{AB} k_{NN} + 2\eta_A \eta_B - \underline{\chi}_{AC} \chi_{CB} + \mathbf{R}_{B43A}. \quad (2.67)$$

Taking the trace relative to A, B we infer that

$$\underline{L}(\text{tr}\chi) = 2\text{div}\eta + \text{tr}\chi k_{NN} + 2|\eta|^2 - \text{tr}(\chi \cdot \underline{\chi}) + \gamma. \quad (2.68)$$

from which (2.65) follows.

Contracting (2.67) with ϵ we infer, $2\text{curl}\eta - \epsilon^{AB}\underline{\hat{\chi}}_{AC}\hat{\chi}_{CB} + \epsilon^{AB}\mathbf{R}_{A43B} = 0$, since $\chi, \underline{\chi}$, and $\mathcal{P}_3\chi$ are symmetric tensors.

Combining (2.67) and (2.68) we obtain a useful formula:

$$\begin{aligned} \mathcal{P}_3\hat{\chi}_{AB} &= 2\mathcal{D}_A\eta_B - \text{div}\eta\delta_{AB} + k_{NN}\hat{\chi}_{AB} + 2(\eta_A\eta_B - |\eta|^2\delta_{AB}) \\ &\quad - \frac{1}{2}\text{tr}\underline{\chi}\hat{\chi}_{AB} - \frac{1}{2}\text{tr}\chi\underline{\hat{\chi}}_{AB} + \mathbf{R}_{A43B}. \end{aligned} \quad (2.69)$$

The examination of equations (2.61)-(2.62) and (2.64)-(2.66) yields the conclusion that the system is not closed. The remaining unknown quantity is $\underline{L}(\text{tr}\chi)$. The situation is remedied by eliminating $\underline{L}(\text{tr}\chi)$ from (2.65) with the help of the application of the vectorfield L and the transport equation for $\text{tr}\chi$. The result is a coupled transport-elliptic system for η .

Proposition 2.4 *Define*

$$\mu = 2\text{div}\eta + 2\text{tr}\chi k_{NN} + 2|\eta|^2 - \text{tr}(\hat{\chi} \cdot \underline{\hat{\chi}}) + \gamma. \quad (2.70)$$

Then

$$\begin{aligned} L(\mu) + \text{tr}\chi\mu &= -2\hat{\chi}_{AB}\left(2\nabla_A\eta_B - \text{div}\eta\delta_{AB} + k_{NN}\hat{\chi}_{AB} + 2(\eta_A\eta_B - |\eta|^2\delta_{AB})\right. \\ &\quad \left. - \frac{1}{2}\text{tr}\underline{\chi}\hat{\chi}_{AB} - \frac{1}{2}\text{tr}\chi\underline{\hat{\chi}}_{AB} + \mathbf{R}_{A43B}\right) + \underline{L}(\mathbf{R}_{44}) \\ &\quad + 2(\underline{\eta}_A - \eta_A)\nabla_A(\text{tr}\chi) + \text{tr}\chi(|\hat{\chi}|^2 + (L - \underline{L})k_{NN} - \mathbf{R}_{44}). \end{aligned} \quad (2.71)$$

Proof: Note that according to equation (2.65), $\mu = \underline{L}(\text{tr}\chi) + k_{NN}\text{tr}\chi - \frac{1}{2}(\text{tr}\chi)^2$. Let $\mu_1 = \underline{L}(\text{tr}\chi) - \frac{1}{2}(\text{tr}\chi)^2$. It then follows that

$$L(\mu_1) + \text{tr}\chi\mu_1 = L\underline{L}(\text{tr}\chi) + \underline{L}(\text{tr}\chi)\text{tr}\chi - L(\text{tr}\chi)\text{tr}\chi - \frac{1}{2}(\text{tr}\chi)^3. \quad (2.72)$$

We commute L and \underline{L} and use equation (2.60) on the right hand-side of (2.72). The commutator $[L, \underline{L}] = [e_4, e_3]$ has been calculated in Lemma 2.2. We conclude that

$$\begin{aligned} L(\mu_1) + \text{tr}\chi\mu_1 &= \underline{L}L(\text{tr}\chi) + 2(\underline{\eta}_A - \eta_A)\nabla_A(\text{tr}\chi) + k_{NN}\underline{L}(\text{tr}\chi) \\ &\quad - k_{NN}L(\text{tr}\chi) + \underline{L}(\text{tr}\chi)\text{tr}\chi - L(\text{tr}\chi)\text{tr}\chi - \frac{1}{2}(\text{tr}\chi)^3. \end{aligned}$$

Substituting $L(\text{tr}\chi)$ from (2.60), we obtain

$$\begin{aligned} L(\mu_1) + \text{tr}\chi\mu_1 &= -2(\mathcal{P}_3\hat{\chi} \cdot \hat{\chi} - \underline{L}(k_{NN})\text{tr}\chi + \underline{L}(\mathbf{R}_{44}) + 2(\underline{\eta}_A - \eta_A)\nabla_A(\text{tr}\chi) \\ &\quad - L(k_{NN}\text{tr}\chi) + \text{tr}\chi k_{NN}\text{tr}\chi + \text{tr}\chi L(k_{NN}) + \text{tr}\chi(|\hat{\chi}|^2 - \mathbf{R}_{44}). \end{aligned} \quad (2.73)$$

The only unknown on the right-hand side of (2.73) is the term with $\mathcal{P}_3\hat{\chi}$. This term is available to us from (2.69). Thus, since $\mu = \mu_1 + k_{NN}\text{tr}\chi$

$$\begin{aligned} L(\mu) + \text{tr}\chi\mu &= -2\hat{\chi}_{AB}\left(2\nabla_A\eta_B - \text{div}\eta\delta_{AB} + k_{NN}\hat{\chi}_{AB} + 2(\eta_A\eta_B - |\eta|^2\delta_{AB}) - \right. \\ &\quad \left. \frac{1}{2}\text{tr}\chi\hat{\chi}_{AB} - \frac{1}{2}\text{tr}\chi\underline{\chi}_{AB} + \mathbf{R}_{A43B}\right) + \underline{L}(\mathbf{R}_{44}) \\ &\quad + 2(\underline{\eta}_A - \eta_A)\nabla_A(\text{tr}\chi) + \text{tr}\chi(|\hat{\chi}|^2 + (L - \underline{L})k_{NN} - \mathbf{R}_{44}). \end{aligned}$$

as desired

More commutation formulas: We record more important commutation formulas.

Lemma 2.3 *Let $\Pi_{\underline{A}}$ be an m -covariant tensor tangent to the surfaces $S_{t,u}$. Then*

$$\nabla_B\mathcal{P}_4\Pi_{\underline{A}} - \mathcal{P}_4\nabla_B\Pi_{\underline{A}} = \chi_{BC}\nabla_C\Pi_{\underline{A}} + \sum_i(\chi_{A_iB}k_{CN} - \chi_{BC}k_{A_iN} + \mathbf{R}_{CA_i4B})\Pi_{A_1..\check{C}..A_m}, \quad (2.74)$$

$$\begin{aligned} \mathcal{P}_3\mathcal{P}_4\Pi_{\underline{A}} - \mathcal{P}_4\mathcal{P}_3\Pi_{\underline{A}} &= 2(\eta_B - k_{BN})\nabla_B\xi_A + k_{NN}\mathcal{P}_4\xi_A - k_{NN}\mathcal{P}_3\xi_A \\ &\quad + \sum_i(2\eta_Bk_{A_iN} - 2\eta_{A_i}k_{BN} + \mathbf{R}_{BA_i43})\Pi_{A_1..\check{B}..A_m}. \end{aligned} \quad (2.75)$$

Proof: It suffices to consider the case where Π is a 1-form. Let $Proj$ denote the projection on the tangent space to $S_{t,u}$.

To prove (2.74) observe that it follows from (2.48) and (2.49) that

$$\nabla_B(\mathcal{P}_4e_A) = Proj\mathcal{D}_B(\mathcal{D}_4e_A + k_{AN}e_A) = Proj\mathcal{D}_B(\mathcal{D}_4e_A) + \chi_{BC}k_{AN}e_C$$

and $\mathcal{P}_4(\nabla_Be_A) = Proj\mathcal{D}_4(\mathcal{D}_Be_A - \frac{1}{2}\chi_{AB}e_3 - \frac{1}{2}\underline{\chi}_{AB}e_4) = Proj\mathcal{D}_4(\mathcal{D}_Be_A) + \chi_{AB}k_{CN}e_C$. As a result, we obtain for a tangent to $S_{t,u}$ co-vector ξ

$$\begin{aligned} \mathcal{P}_4\nabla_B\xi_A &= \mathcal{D}_4\mathcal{D}_B\xi_A + k_{AN}\chi_{BC}\xi_C - k_{BN}\mathcal{P}_4\xi_A, \\ \nabla_B\mathcal{P}_4\xi_A &= \mathcal{D}_B\mathcal{D}_4\xi_A + \chi_{AB}k_{CN}\xi_C + \chi_{BC}\nabla_C\xi_A - k_{BN}\mathcal{P}_4\xi_A. \end{aligned}$$

Also, $\mathcal{D}_4\mathcal{D}_B\xi_A - \mathcal{D}_B\mathcal{D}_4\xi_A = \mathbf{R}_{AC4B}\xi_C$. Combining, we have

$$\mathcal{P}_4\nabla_B\xi_A - \nabla_B\mathcal{P}_4\xi_A = -\chi_{AB}k_{CN}\xi_C + \chi_{BC}k_{AN}\xi_C + \chi_{BC}\nabla_C\xi_A + \mathbf{R}_{AC4B}\xi_C$$

For the proof of (2.75) observe that

$$\begin{aligned} \mathcal{P}_3(\mathcal{P}_4e_A) &= Proj\mathcal{D}_3(\mathcal{D}_4e_A + k_{AN}e_A) = Proj\mathcal{D}_3(\mathcal{D}_4e_A) + 2\eta_Bk_{AN}e_B, \\ \mathcal{P}_4(\mathcal{P}_3e_A) &= Proj\mathcal{D}_4(\mathcal{D}_3e_A - \eta_Ae_3 + (\eta_A - k_{AN})e_4) = Proj\mathcal{D}_4(\mathcal{D}_3e_A) + 2\eta_Ak_{BN}e_B. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{P}_4\mathcal{P}_3\xi_A &= \mathcal{D}_4\mathcal{D}_3\xi_A - 2\eta_Bk_{AN}\xi_B - 2k_{BN}\nabla_B\xi_A + k_{NN}\mathcal{P}_3\xi_A, \\ \mathcal{P}_3\mathcal{P}_4\xi_A &= \mathcal{D}_3\mathcal{D}_4\xi_A - 2\eta_Ak_{BN}\xi_B + 2\eta_B\nabla_B\xi_A + k_{NN}\mathcal{P}_4\xi_A. \end{aligned}$$

In addition, $\mathcal{D}_4\mathcal{D}_3\xi_A - \mathcal{D}_3\mathcal{D}_4\xi_A = \mathbf{R}_{AB43}\xi_B$. The identity (2.75) follows.

Corollary 2.1 *If $\Pi_{\underline{A}}$ is an m -covariant tensor tangent to the surfaces $S_{t,u}$ verifying the equation*

$$\mathcal{D}_4 \Pi_{\underline{A}} + \sigma \operatorname{tr} \chi \Pi_{\underline{A}} = F_{\underline{A}}$$

with some constant σ and a tangent k -covariant tensor $F_{\underline{A}}$, then

$$\begin{aligned} \mathcal{P}_4 \nabla_B \Pi_{\underline{A}} + \left(\sigma + \frac{1}{2}\right) \operatorname{tr} \chi \nabla_B \Pi_{\underline{A}} &= -\hat{\chi}_{BC} \nabla_C \Pi_{\underline{A}} - \sigma \nabla_B (\operatorname{tr} \chi) \Pi_{\underline{A}} + \nabla_B F_{\underline{A}} \\ &\quad - \sum_i (\chi_{A_i B} k_{CN} - \chi_{BC} k_{A_i N} + \mathbf{R}_{CA_i 4B}) \Pi_{A_1 \dots \check{C} \dots A_m}. \end{aligned} \quad (2.76)$$

Also,

$$\begin{aligned} \mathcal{P}_4 \mathcal{P}_3 \Pi_{\underline{A}} + \sigma \operatorname{tr} \chi \nabla_B \Pi_{\underline{A}} &= -k_{NN} \mathcal{P}_3 \Pi_{\underline{A}} + 2(\eta_B - k_{BN}) \nabla_B \Pi_{\underline{A}} + k_{NN} (F_{\underline{A}} - \sigma \operatorname{tr} \chi \Pi_{\underline{A}}) \\ &\quad + \sigma \mathcal{D}_3 (\operatorname{tr} \chi) \Pi_{\underline{A}} + \mathcal{P}_3 F_{\underline{A}} - \sum_i (2\eta_B k_{A_i N} - 2\eta_{A_i} k_{BN} + \mathbf{R}_{BA_i 43}) \Pi_{A_1 \dots \check{B} \dots A_m}. \end{aligned} \quad (2.77)$$

More transport equations: *Let $y = \operatorname{tr} \chi - \frac{2}{s}$ with $s = t - u$ and let the functions z and *Error* be as in Lemma 2.1. The remarkable decomposition of the \mathbf{R}_{44} component of the Ricci curvature proved in Lemma 2.1 allows us to replace the transport equation (2.60) for $\operatorname{tr} \chi$ with a more convenient equation for the function $y + z$. In addition, after differentiating the transport equation for $y + z$ with the angular derivative ∇ and applying the commutation formula (2.76), we also obtain the transport equation for $\nabla(y + z)$.*

Proposition 2.5

$$L(y+z) + \operatorname{tr} \chi (y+z) = \frac{1}{2}(y+z)^2 + \frac{2}{s}z - \frac{1}{2}z^2 - |\hat{\chi}|^2 - k_{NN} \operatorname{tr} \chi + \frac{1}{2} e_4^\nu e_4^\mu \square_h H_{\mu\nu} - \text{Error}, \quad (2.78)$$

$$\begin{aligned} \mathcal{P}_4 \nabla_A (y+z) + \frac{3}{2} \nabla_A (y+z) &= -\hat{\chi}_{AB} \nabla_B (y+z) + (y+z) \nabla_A (y+z) - (y+z) \nabla_A (\operatorname{tr} \chi) \\ &\quad + \left(\frac{2}{s} - 2z\right) \nabla_A z - 2 \nabla_A \hat{\chi} \cdot \hat{\chi} - \operatorname{tr} \chi \nabla_A k_{NN} - k_{NN} \nabla_A \operatorname{tr} \chi + \nabla_A \left(\frac{1}{2} e_4^\nu e_4^\mu \square_h H_{\mu\nu} - \text{Error}\right). \end{aligned} \quad (2.79)$$

3 Outline of the proof of Theorem (B)

As in [KL] the proof of the $L^2 - L^\infty$ decay estimates for a solution ψ of the wave equation

$$\begin{aligned} \square_h \psi &= -\frac{1}{\sqrt{\det h}} \partial_t \sqrt{\det h} \partial_t \psi + \frac{1}{\sqrt{\det h}} \partial_i (h^{ij} \sqrt{\det h} \partial_j \psi) = 0, \\ \psi|_{t=1} &= \psi_0, \quad \partial_t \psi|_{t=1} = \psi_1, \end{aligned} \quad (3.80)$$

with initial data localized in the ball $B_{\frac{1}{2}}(0)$ in physical space, is based on a curved spacetime generalization of the well known conformal energy estimates in Minkowski space. Using the optical function u and the null frame L, \underline{L} introduced in the previous Chapter we shall construct a vectorfield K which is the precise analogue of the Morawetz vectorfield in Minkowski spacetime. All the effort in our approach goes into controlling the error terms generated by the fact that our K is no longer a conformal Killing vectorfield. Its failure to be conformal is measured by its deformation tensor $^{(K_0)}\pi$. The heart of the proof of Theorem (B) is the boundedness theorem stated in section 3.4. In section 3.1 we discuss some preliminary facts concerning energy estimates for the equation (3.80).

3.1 Generalized energy estimates

The standard energy estimate for a solution of the wave equation in Minkowski space $\square \psi = -\partial_t^2 \psi + \Delta \psi = 0$ has the form

$$\int_{\Sigma_t} |\partial_t \psi|^2 + |\nabla \psi|^2 = \int_{\Sigma_0} |\partial_t \psi|^2 + |\nabla \psi|^2,$$

where Σ_t is the hyperplane $t = \text{const}$. It is a consequence of the time translation invariance of the wave equation in Minkowski space corresponding to the Killing vectorfield ∂_t . In the case of the variable coefficients it is not difficult to verify that the corresponding energy identity has the form

$$\int_{\Sigma_t} |\partial_t \psi|^2 + |\nabla \psi|^2 = \int_{\Sigma_0} |\partial_t \psi|^2 + |\nabla \psi|^2 + \int_0^t \int_{\Sigma_{t'}} k_{ij} \partial_i \psi \partial_j \psi dt', \quad (3.81)$$

where $k_{ij} = -\frac{1}{2} \partial_t h_{ij}$ is the second fundamental form of the hyperplane $t = \text{const}$. Note that we use the convention $\int_{\Sigma_t} f := \int_{\Sigma_t} f \sqrt{\det h} dx^1 dx^2 dx^3$. It follows from (3.81) that the energy norm of ψ at

time t is controlled by the corresponding norm of the initial data provided $\int_0^t \|k\|_{L_x^\infty} \leq C$ with some universal constant C .

We now address the general situation. Let X be an arbitrary time-like vectorfield with the deformation tensor $^{(X)}\pi = \mathcal{L}_X h$. Recall that $^{(X)}\pi_{\alpha\beta} = \mathcal{D}_\alpha X_\beta + \mathcal{D}_\beta X_\alpha$. Let $Q_{\alpha\beta} = \partial_\alpha \psi \partial_\beta \psi - \frac{1}{2} H_{\alpha\beta} (H^{\mu\nu} \partial_\mu \psi \partial_\nu \psi)$ be the energy momentum tensor associated to the equation $\mathcal{D}^\alpha \mathcal{D}_\alpha = \square_h \psi = F$. It is easy to check that, $\mathcal{D}^\beta Q_{\alpha\beta} = F \partial_\alpha \psi$, Therefore, setting the X -momentum 1-form $P_\alpha = Q_{\alpha\beta} X^\beta$, we have

$$\mathcal{D}^\alpha P_\alpha = \frac{1}{2} Q^{\alpha\beta} {}^{(X)}\pi_{\alpha\beta} + F X(\psi) \quad (3.82)$$

Observe that $\mathcal{D}^\alpha P_\alpha = \mathbf{Div} P$. Therefore, integrating over the time slab $[t_0, t] \times \mathbf{R}^n$, using Stokes' theorem, and noticing that ∂_t is the future unit normal to the hypersurfaces Σ_t we derive,

Proposition 3.1 *Let ψ verify $\square_h \psi = F$ and X an arbitrary vectorfield with deformation tensor ${}^{(X)}\pi$. We have,*

$$\int_{\Sigma_t} Q(X, \partial_t) = \int_{\Sigma_{t_0}} Q(X, \partial_t) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^n} Q^{\alpha\beta} {}^{(X)}\pi_{\alpha\beta} + \int_{[t_0, t] \times \mathbf{R}^n} (X\psi)F. \quad (3.83)$$

Observe that (3.83) implies the energy estimate (3.81) in the special case when $X = \partial_t$ and $F = 0$. In that case ${}^{(X)}\pi_{00} = {}^{(X)}\pi_{0i} = 0$ and ${}^{(X)}\pi_{ij} = -2k_{ij}$.

The identity (3.83) is particularly important in the case of a Killing vectorfield X , in which case the deformation tensor ${}^{(X)}\pi = 0$. In the case of a conformal Killing vectorfield²⁶ X , i.e. ${}^{(X)}\pi = \Omega H$, we need to modify the formula (3.83). This can be done as follows: For a given vectorfield X and scalar function Ω we set ${}^{(X)}\bar{\pi} = {}^{(X)}\pi - \Omega H$ and calculate,

$$\begin{aligned} \mathcal{D}^\alpha P_\alpha &= \frac{1}{2} Q^{\alpha\beta} {}^{(X)}\pi_{\alpha\beta} + FX(\psi) = \frac{1}{2} (Q^{\alpha\beta} {}^{(X)}\bar{\pi}_{\alpha\beta} + \Omega \text{tr} Q) + FX(\psi) \\ &= \frac{1}{2} (Q^{\alpha\beta} {}^{(X)}\bar{\pi}_{\alpha\beta} + \Omega \frac{1-n}{2} \partial^\mu \psi \partial_\mu \psi) + FX(\psi) \end{aligned} \quad (3.84)$$

Also,

$$\begin{aligned} \Omega \partial^\mu \psi \partial_\mu \psi &= \mathcal{D}^\mu (\Omega \psi \partial_\mu \psi) - \partial^\mu (\Omega) \psi \partial_\mu \psi - \Omega \psi \square_h \psi \\ &= \mathcal{D}^\mu (\Omega \psi \partial_\mu \psi - \frac{1}{2} \psi^2 \partial_\mu \Omega)_\mu + \frac{1}{2} \psi^2 \square_h \Omega - \Omega \psi F \end{aligned}$$

Therefore,

$$\mathcal{D}^\alpha P_\alpha + \frac{n-1}{4} (\mathcal{D}^\mu (\Omega \psi \partial_\mu \psi - \frac{1}{2} \psi^2 \partial_\mu \Omega)) = \frac{1}{2} Q^{\alpha\beta} {}^{(X)}\bar{\pi}_{\alpha\beta} - \frac{n-1}{8} \psi^2 \square_h \Omega + (X\psi + \frac{n-1}{4} \Omega \psi) F$$

or, setting

$$\bar{P}_\alpha = P_\alpha + \frac{n-1}{4} \Omega \psi \partial_\alpha \psi - \frac{n-1}{8} \psi^2 \partial_\alpha \Omega \quad (3.85)$$

we derive

$$\mathcal{D}^\alpha \bar{P}_\alpha = \frac{1}{2} Q^{\alpha\beta} {}^{(X)}\bar{\pi}_{\alpha\beta} - \frac{n-1}{8} \psi^2 \square_g \Omega + (X\psi + \frac{n-1}{4} \Omega \psi) F \quad (3.86)$$

Integrating again over the time slab we obtain,

Proposition 3.2 (Generalized energy identity) *Let ψ verify $\square_h \psi = F$ and X an arbitrary vectorfield with deformation tensor ${}^{(X)}\pi$. Let Ω be an arbitrary scalar function and ${}^{(X)}\bar{\pi} = {}^{(X)}\pi - \Omega H$. Define,*

$$\bar{Q}(X, Y) = Q(X, Y) + \frac{n-1}{4} \Omega \psi Y \psi - \frac{n-1}{8} \psi^2 Y(\Omega) \quad (3.87)$$

²⁶In the particular case of the Morawetz $K = (t^2 + r^2) \partial_t + 2tr \partial_r$ in Minkowski space we have $\Omega = 4t$

We have,

$$\begin{aligned} \int_{\Sigma_t} \bar{Q}(X, \partial_t) &= \int_{\Sigma_{t_0}} \bar{Q}(X, \partial_t) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^n} Q^{\alpha\beta(X)} \bar{\pi}_{\alpha\beta} \\ &\quad + \frac{n-1}{8} \int_{[t_0, t] \times \mathbf{R}^n} \psi^2 \square_h \Omega + \int_{[t_0, t] \times \mathbf{R}^n} (X\psi + \frac{n-1}{4} \Omega\psi) F. \end{aligned} \quad (3.88)$$

In the particular case of the Minkowski space and conformal Killing vector field $K = (t^2 + r^2)\partial_t + 2tr\partial_r$ we can choose $\Omega = 4t$ so that ${}^K\bar{\pi} = 0$. Taking also $F = 0$ we arrive at the conservation law for the conformal energy in Minkowski space.

$$\int_{\Sigma_t} \bar{Q}(K, \partial_t) = \int_{\Sigma_{t_0}} \bar{Q}(K, \partial_t)$$

We are now ready to define our spacetime analogue of the Morawetz conformal vectorfield. Let u be the optical function and $\underline{L} = e_3 = T - N$, $L = e_4 = T + N$ the corresponding null pair defined in Chapter 2.

Definition of K: Define the timelike vectorfield,

$$K = \frac{1}{2}u^2 e_3 + \frac{1}{2}\underline{u}^2 e_4, \quad \underline{u} = -u + 2t. \quad (3.89)$$

Define the modified deformation tensor ${}^K\bar{\pi} = {}^K\pi - 4tH$. The modified energy-momentum tensor $\bar{Q}(K, Y)$ then takes the form

$$\bar{Q}(K, Y) = Q(K, Y) + (n-1)t\psi Y\psi - \frac{n-1}{2}\psi^2 Y(t) \quad (3.90)$$

Remark: Note the difference between the functions u and \underline{u} in the interior of the null cone C_0 . The optical function u is positive, bounded by t on the time slice Σ_t , and uniformly bounded near C_0 . By contrast, the values of the function \underline{u} lie between t and $2t$, and $\underline{u} \sim t$ near the null cone C_0 .

Calculation of $\bar{Q}(K, \partial_t)$: The generalized energy identity with the vectorfield K involves the modified energy-momentum tensor \bar{Q} evaluated on the vectorfields K and ∂_t . The definition (3.87), (with $\Omega = 4t$) implies that $\bar{Q}(K, \partial_t) = Q(K, \partial_t) + (n-1)t\psi\partial_t\psi - \frac{n-1}{2}\psi^2$. We find the null components of the energy-momentum tensor Q :

$$Q_{44} = (\mathcal{D}_4\psi)^2, \quad Q_{34} = |\nabla\psi|^2, \quad (3.91)$$

$$Q_{33} = (\mathcal{D}_3\psi)^2, \quad Q_{3A} = \mathcal{D}_3\psi\nabla_A\psi, \quad (3.92)$$

$$Q_{4A} = \mathcal{D}_4\psi\nabla_A\psi, \quad Q_{AB} = \nabla_A\psi\nabla_B\psi + \frac{1}{2}(\mathcal{D}_3\psi\mathcal{D}_4\psi - |\nabla\psi|^2)\delta_{AB}. \quad (3.93)$$

Since K is defined in terms of the null vectorfields e_3, e_4 , and $\partial_t = \frac{1}{2}(e_3 + e_4)$,

$$\bar{Q}(K, \partial_t) = \frac{1}{4} \left(\underline{u}^2 (\mathcal{D}_4\psi)^2 + u^2 (\mathcal{D}_3\psi)^2 + (\underline{u}^2 + u^2) |\nabla\psi|^2 \right) + (n-1)t\psi\partial_t\psi - \frac{n-1}{2}\psi^2. \quad (3.94)$$

3.2 Domain of dependence

The vectorfield K is defined in terms of the optical function u which is only well defined in a neighborhood of the time axis. We will later verify that under the assumptions (1.19)- (1.23) on the metric h , the optical function u is well defined in the interior of the null cone C_0 described as the set of points verifying $u = 0$ or $s = t$, for $t \in [0, t_*]$. Consequently the conformal identity (3.88) applied to $X = K$ has to be restricted to functions ψ supported in the set $s \leq \tau$, $\tau \in [t_0, t]$ and $t_0 \leq t \leq t_*$. The following domain of dependence property shows that this is precisely what we need.

Proposition 3.3 *Let ψ be a solution of the wave equation*

$$\begin{aligned} \square_h \psi &= F, \\ \psi|_{t=1} &= \psi_0, \quad \partial_t \psi|_{t=1} = \psi_1 \end{aligned} \tag{3.95}$$

with the metric h satisfying assumptions (1.19)- (1.23) on the time interval $[0, t_]$. Assume that the initial data (ψ_0, ψ_1) at time $t = 1$ is supported in the geodesic ball $B_{\frac{1}{2}}(0)$ and that the right-hand side $F(t, \cdot)$ is supported inside the set $s \leq t - \frac{1}{2}$, $t \in [1, t_*]$. Then, for each $t \in [1, t_*]$, ψ vanishes in the exterior of the set $s \leq t$.*

Proof: This is an application of the well known finite speed of propagation result for nonlinear wave equations; it relies on standard energy estimates, like (3.81), in the exterior of the forward null cone C_0 , more precisely in the region between the initial slice Σ_1 , a future slice Σ_t , $1 \leq t \leq t_*$ and the exterior of C_0 .

3.3 The Boundedness Theorem

We are now ready to state our main technical step in the proof of the $L^2 - L^\infty$ decay estimate stated in Theorem 1.28. Let $\psi(t, \cdot)$ be a function with support contained in the interior of the set $s \leq t$ with $t \in [0, t_*]$.

Introduce²⁷

$$\mathcal{Q}_0[\psi](t) = \int_{\Sigma_t} \bar{Q}(K, \partial_t)[\psi], \quad \mathcal{Q}[\psi](t) = \mathcal{Q}_0[\psi](t) + \mathcal{Q}_0[\partial_t \psi](t) + \mathcal{Q}_0[\partial_t^2 \psi](t) \tag{3.96}$$

Theorem 3.1 *Let ψ be a solution of the wave equation*

$$\begin{aligned} \square_h \psi &= 0, \\ \psi|_{t=1} &= \psi_0, \quad \partial_t \psi|_{t=1} = \psi_1 \end{aligned} \tag{3.97}$$

with the metric h satisfying assumptions (1.19)- (1.23) on the time interval $[0, t_]$. Assume that the initial data (ψ_0, ψ_1) at time $t = 1$ has support in the geodesic ball $B_{\frac{1}{2}}(0)$. Then for any t_0, t such that $1 \leq t_0 \leq t \leq t_*$*

$$\mathcal{Q}[\psi](t) \lesssim \mathcal{Q}[\psi](t_0). \tag{3.98}$$

Proof: See Chapter 6.

²⁷Recall the definition of K in (3.89) and the modified energy momentum tensor (3.90).

3.4 Conformal norms

To prove Theorem 3.1 we need to introduce the following auxiliary energy norms.

Definition of \mathcal{E} : Let ζ be a cut-off function equal to 1 in the region $u \leq \frac{t}{2}$. Define²⁸

$$\begin{aligned}\mathcal{E}_0[\psi](t) &= \mathcal{E}_0^i[\psi](t) + \mathcal{E}_0^e[\psi](t), \\ \mathcal{E}_0^i[\psi](t) &= \int_{\Sigma_t} (t^2 |\partial\psi|^2 + \psi^2)(1 - \zeta), \\ \mathcal{E}_0^e[\psi](t) &= \int_{\Sigma_t} (\underline{u}^2 (\mathcal{D}_4\psi)^2 + u^2 (\mathcal{D}_3\psi)^2 + \underline{u}^2 |\nabla\psi|^2 + \psi^2) \zeta.\end{aligned}\tag{3.99}$$

We also define the full conformal norm of ψ

$$\mathcal{E}[\psi](t) = \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial\psi](t) + \mathcal{E}_0[\partial\psi](t).\tag{3.100}$$

Remark: Observe that due to the difference in the behavior of the functions u and \underline{u} the conformal norm $\mathcal{E}_0[\psi]$ attaches different weights to the derivatives of ψ depending on the direction. Thus the $\mathcal{D}_4\psi$ and $\nabla\psi$ are “better” than the $\mathcal{D}_3\psi$ term.

The next theorem shows that the conformal norm $\mathcal{E}[\psi](t)$ of the solution of the wave equation is controlled by the conformal energy $\mathcal{Q}[\psi](t)$.

Theorem 3.2 (Comparison Theorem) *Let ψ be a solution of the wave equation*

$$\begin{aligned}\square_h \psi &= 0, \\ \psi|_{t=1} &= \psi_0, \quad \partial_t \psi|_{t=1} = \psi_1\end{aligned}$$

with the metric h satisfying the assumptions (1.19)- (1.23) on the time interval $[0, t_*]$. Assume that the initial data $\psi[0]$ at time $t = 1$ has support in the geodesic ball $B_{\frac{1}{2}}(0)$. Then for any t such that $1 \leq t \leq t_*$

$$\mathcal{E}[\psi](t) \lesssim \mathcal{Q}[\psi](t).\tag{3.101}$$

For convenience we also state the starting estimate leading to the proof of (3.101)

$$\mathcal{E}_0[\psi](t) \lesssim \mathcal{Q}_0[\psi](t).\tag{3.102}$$

Proof: See Section 6.5.

Finally, Theorem (B) is a consequence of Theorem 3.1 Theorem 3.2, as well as the following:

Proposition 3.4 *Let ψ be a sufficiently smooth function supported in the region $s \leq t$ for $1 \leq t \leq t_*$. Then for any $\epsilon > 0$,*

$$\|\partial\psi(t)\|_{L_x^\infty} \lesssim \frac{1}{(1+t)} \mathcal{E}^{\frac{1}{2}}[\psi](t) + \frac{1}{(1+t)^{1-\frac{3}{2}\epsilon}} \mathcal{E}^{\frac{1}{2}(1-\epsilon)}[\psi](t) \left[\sum_{m=0}^3 \int_{\Sigma_1} |\partial\partial_x^m \psi|^2 \right]^\epsilon.\tag{3.103}$$

Proof: See Proposition 7.1.

²⁸Observe that $\mathcal{E}_0[\psi](t)$ is roughly equivalent to $\int_{\Sigma_t} (\underline{u}^2 (\mathcal{D}_4\psi)^2 + u^2 (\mathcal{D}_3\psi)^2 + \underline{u}^2 |\nabla\psi|^2 + \psi^2)$.

4 Sketch of the proof of the Boundedness Theorem 3.1

In this Chapter we give a rough sketch of 3.1. The sketch is supposed to help the reader understand how the main geometric constructions introduced fit in the proof of our main result. We restrict ourselves to a sketch of the estimate $\mathcal{Q}_0[\psi](t) \lesssim \mathcal{Q}_0[\psi](t_0)$.

Let ψ be a solution of the wave equation $\square_h \psi = 0$ with initial data localized in the ball $B_{\frac{1}{2}}(0)$ at time $t = 1$. Observe that the conformal energy identity (3.88) and definition (3.96) of $\mathcal{Q}_0[\psi](t)$ imply²⁹ that for all $1 \leq t_0 \leq t \leq t_*$ we have

$$\mathcal{Q}_0[\psi](t) = \mathcal{Q}_0[\psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} Q_{\alpha\beta}{}^K \bar{\pi}_{\alpha\beta} + \frac{1}{4} \int_{[t_0, t] \times \mathbf{R}^3} \psi^2 \square_h(4\tau) \quad (4.104)$$

We shall also make use of the Comparison Theorem 3.2 from which we infer that,

$$\mathcal{E}_0[\psi](t) = \int_{\Sigma_t} (\underline{u}^2 (\mathcal{D}_4 \psi)^2 + u^2 (\mathcal{D}_3 \psi)^2 + \underline{u}^2 |\nabla \psi|^2 + \psi^2) \lesssim \mathcal{Q}_0[\psi](t). \quad (4.105)$$

It is not difficult to verify that $\square_h(4\tau) = 4h^{ij}k_{ij}$, where k is the second fundamental form of the embedding $\Sigma_\tau \subset \mathbf{R}^{n+1}$. The assumption (1.19) on the metric h implies that $\int_{t_0}^t \|h^{ij}k_{ij}\|_{L_x^\infty} \lesssim \lambda^{-(1-a)}$.

Thus

$$\mathcal{Q}_0[\psi](t) \lesssim \mathcal{Q}_0[\psi](t_0) + \frac{1}{2} \left| \int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}{}^K \bar{\pi}_{\alpha\beta} \right| + \lambda^{-(1-a)} \sup_{[t_0, t]} \mathcal{E}_0[\psi](\tau).$$

4.1 The error term $\int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}{}^K \bar{\pi}_{\alpha\beta}$

To prove the boundedness theorem for $\mathcal{Q}_0[\psi](t)$, it is sufficient to establish that the error term $\int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}{}^K \bar{\pi}_{\alpha\beta}$ is controlled in the following way:

$$\int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}{}^K \bar{\pi}_{\alpha\beta} \leq \lambda^{-\epsilon} \sup_{[t_0, t]} \mathcal{E}_0[\psi](\tau) \leq \lambda^{-\epsilon} \sup_{[t_0, t]} \int_{\Sigma_\tau} (\underline{u}^2 (\mathcal{D}_4 \psi)^2 + u^2 (\mathcal{D}_3 \psi)^2 + \underline{u}^2 |\nabla \psi|^2 + \psi^2) \quad (4.106)$$

with an arbitrary $\epsilon > 0$.

To achieve that we start by writing the contraction of the tensors $Q^{\alpha\beta}$ and ${}^K \bar{\pi}_{\alpha\beta}$ relative to a null frame e_A, e_3, e_4 .

$$Q^{\alpha\beta}{}^K \bar{\pi}_{\alpha\beta} = \frac{1}{4} Q_{44}{}^K \bar{\pi}_{33} + \frac{1}{4} Q_{33}{}^K \bar{\pi}_{44} + \frac{1}{2} Q_{34}{}^K \bar{\pi}_{34} - Q_{3A}{}^K \bar{\pi}_{4A} - Q_{4A}{}^K \bar{\pi}_{3A} + Q_{AB}{}^K \bar{\pi}_{AB}. \quad (4.107)$$

²⁹with $n = 3$

The components of the energy-momentum tensor Q relative to a null frame have been calculated in (3.91)-(3.93). Consider the term $Q_{33}{}^K \bar{\pi}_{44} = |\mathcal{D}_3 \psi|^2 {}^K \bar{\pi}_{44}$. We want to estimate

$$\int_{\Sigma_\tau} |\mathcal{D}_3 \psi|^2 {}^K \bar{\pi}_{44} \leq \lambda^{-\epsilon} \int_{\Sigma_\tau} \mathcal{E}_0[\psi](\tau) \approx \lambda^{-\epsilon} \tau^{-1} \int_{\Sigma_\tau} u^2 |\mathcal{D}_3 \psi|^2 + \dots$$

We see that we need the ${}^K \bar{\pi}_{44}$ component of the deformation tensor to satisfy $|{}^K \bar{\pi}_{44}| \leq u^2 \lambda^{-\epsilon} \tau^{-1}$. More generally, we seem to need the following conditions to hold:

$$|{}^K \bar{\pi}_{44}| \leq u^2 \lambda^{-\epsilon} \tau^{-1} \quad |{}^K \bar{\pi}_{34}| \leq \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}, \quad (4.108)$$

$$|{}^K \bar{\pi}_{33}| \leq \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}, \quad |{}^K \bar{\pi}_{3A}| \leq \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}, \quad (4.109)$$

$$|{}^K \bar{\pi}_{4A}| \leq u \underline{u} \lambda^{-\epsilon} \tau^{-1}, \quad |{}^K \bar{\pi}_{AB}| \leq \underline{u}^2 \lambda^{-\epsilon} \tau^{-1}. \quad (4.110)$$

The decay of the components of the deformation tensor ${}^K \bar{\pi}$ described in (4.108)-(4.110) is almost sufficient for the control of the error term $\int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta} {}^K \bar{\pi}_{\alpha\beta}$. The only remaining “dangerous” term³⁰ is the $\int_{[t_0, t] \times \mathbf{R}^3} (\text{tr} {}^K \bar{\pi} + 4u^2 \text{tr} k) \mathcal{D}_3 \psi \mathcal{D}_4 \psi$. This term requires special handling.

The components of ${}^K \bar{\pi}$ relative to a null frame are recorded in the next Lemma (to be proved in Lemma 6.1).

Lemma 4.1

$$\begin{aligned} {}^K \bar{\pi}_{44} &= -2u^2 k_{NN}, & {}^K \bar{\pi}_{34} &= 4u(1 - b^{-1}) + (u^2 + \underline{u}^2) k_{NN}, \\ {}^K \bar{\pi}_{33} &= -8\underline{u}(1 - b^{-1}) - \underline{u}^2 k_{NN}, & {}^K \bar{\pi}_{3A} &= u^2(k_{AN} - \eta_A) + \underline{u}^2(\eta_A + k_{AN}), \\ {}^K \bar{\pi}_{4A} &= -2u^2 k_{AN}, & {}^K \bar{\pi}_{AB} &= 2t(t - u)(\text{tr} \chi - \frac{2}{s})\delta_{AB} - 2u^2 \text{tr} k \delta_{AB} + u^2 \hat{\chi}_{AB} + \underline{u}^2 \hat{\chi}_{AB}. \end{aligned}$$

Observe that ${}^K \bar{\pi}$ is expressed completely³¹ in terms of frame coefficients $\chi, \underline{\chi}, \eta$, the second fundamental form k and lapse function b . In view of our assumptions on the metric h we already have good estimates for k . Most of the work³² of this paper is taken with finding the best possible asymptotic estimates³³ for χ, η and b . This will be done in Chapter 5; we outline here the most important results:

Theorem 4.2 (Asymptotics) *Let the function u be a solution of the eikonal equation $H^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ with the Lorentz metric $H = -dt^2 + h$. Assume that h satisfies conditions (1.19)-(1.23) with a sufficiently large $\lambda \geq \Lambda$ and the value of the parameter $a < -1 + \sqrt{3}$. Let e_1, e_2, e_3, e_4 be a null frame with $e_3 = \underline{L}$ and $e_4 = L$ defined in (2.35), (2.34). Then for the associated Ricci coefficients*

³⁰Here $\text{tr} {}^K \bar{\pi} = \delta^{AB} {}^K \bar{\pi}_{AB}$ is the trace of the deformation tensor ${}^K \bar{\pi}$ relative to the 2-surfaces $S_{t,u}$.

³¹Thus, the asymptotic behavior of the frame coefficients determines that of the null components of ${}^K \bar{\pi}$.

³²The refinement of the asymptotic estimates for the Eikonal equations used in [Ch-Kl], [Kl] is also the main novelty of this paper.

³³For large λ .

$\text{tr}\chi$, $\hat{\chi}$, η obey the following estimates for all values of the parameter $s = t - u$ with $0 \leq t \leq t_*$, $0 \leq u \leq t_*$, and arbitrarily small $\epsilon > 0$:

$$\sup_{S_{t,u}} \left| \text{tr}\chi - \frac{2}{s} \right| + \sup_{S_{t,u}} |\hat{\chi}| + \sup_{S_{t,u}} |\eta| \leq \lambda^{-\bar{a}+\epsilon}, \quad (4.111)$$

$$\left\| \mathcal{D} \left(\text{tr}\chi - \frac{2}{s} \right) \right\|_{L^2(S_{t,u})} + \|\mathcal{P}\hat{\chi}\|_{L^2(S_{t,u})} + \|\nabla\eta\|_{L^2(S_{t,u})} + \|\mathcal{P}_4\eta\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (4.112)$$

Here, we set $\bar{a} = 1 - \frac{1}{2}a^2$ and note that we have $\bar{a} > a$ for the values of the parameter a , $a < -1 + \sqrt{3}$.

In view of Theorem 4.2 and Lemma 4.1 it is easy to check the estimates (4.108)–(4.110). Thus,

Corollary 4.3 *Under assumptions of the Boundedness Theorem*

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta} {}^K \bar{\pi}_{\alpha\beta} - \frac{1}{2} \pi_{AA} \mathcal{D}_3 \psi \mathcal{D}_4 \psi \right| \lesssim \lambda^{-\epsilon} \sup_{[t_0, t]} \mathcal{E}[\psi](\tau).$$

4.2 The error term $\int_{[t_0, t] \times \mathbf{R}^3} (\text{tr} {}^K \bar{\pi} + 4u^2 \text{tr} k) \mathcal{D}_3 \psi \mathcal{D}_4 \psi$

We provide below a very informal discussion on how to bound the remaining term. Note that by Lemma 4.1 $\text{tr} {}^K \bar{\pi} + 4u^2 \text{tr} k = 4t(t - u)(\text{tr}\chi - \frac{2}{s})$. The Asymptotics Theorem gives the bound $|\text{tr}\chi - \frac{2}{s}| \lesssim \lambda^{-\bar{a}+\epsilon}$. We can then only conclude that³⁴

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} (\text{tr} {}^K \bar{\pi} + 4u^2 \text{tr} k) \mathcal{D}_3 \psi \mathcal{D}_4 \psi \right| \lesssim \lambda^{-\bar{a}+\epsilon} \int_{[t_0, t] \times \mathbf{R}^3} \tau^2 |\mathcal{D}_3 \psi| |\mathcal{D}_4 \psi| \lesssim \lambda^{-\epsilon} \sup_{[t_0, t]} \int_{\Sigma_\tau} (\tau^2 |\mathcal{D}_3 \psi|^2 + \tau^2 |\mathcal{D}_4 \psi|^2).$$

However, the integral $\int_{\Sigma_\tau} \tau^2 |\mathcal{D}_3 \psi|^2$ is troublesome, since it is not controlled³⁵ by the energy norm $\mathcal{E}_0[\psi](\tau)$.

This forces us to choose a different approach. As in [Kl] we integrate by parts³⁶.

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} (\text{tr} {}^K \bar{\pi} + 4u^2 \text{tr} k) \mathcal{D}_3 \psi \mathcal{D}_4 \psi &= - \int_{[t_0, t] \times \mathbf{R}^3} \mathcal{D}_3 (\text{tr} {}^K \bar{\pi} + 4u^2 \text{tr} k) \psi \mathcal{D}_4 \psi \\ &\quad - \int_{[t_0, t] \times \mathbf{R}^3} (\text{tr} {}^K \bar{\pi} + 4u^2 \text{tr} k) \psi \mathcal{D}_3 \mathcal{D}_4 \psi. \end{aligned}$$

³⁴Since $\bar{a} > a$ and $t \leq t_* \leq \lambda^a$, we can assume that $\lambda^{-\bar{a}} t \leq \lambda^{-5\epsilon}$ for some sufficiently small $\epsilon > 0$

³⁵We only have $\int_{\Sigma_\tau} u^2 |\mathcal{D}_3 \psi|^2 \lesssim \mathcal{E}_0[\psi](\tau)$

³⁶we ignore the boundary terms in what follows

Use the fact that, modulo some lower order terms, $\square_h \psi = -\mathcal{D}_3 \mathcal{D}_4 \psi + \mathbb{A} \psi = 0$. Integrating by parts once more we obtain,

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} (\operatorname{tr}^K \bar{\pi} + 4u^2 \operatorname{tr} k) \mathcal{D}_3 \psi \mathcal{D}_4 \psi &= - \int_{[t_0, t] \times \mathbf{R}^3} \mathcal{D}_3 (\operatorname{tr}^K \bar{\pi} + 4u^2 \operatorname{tr} k) \psi \mathcal{D}_4 \psi \\ &+ \int_{[t_0, t] \times \mathbf{R}^3} \nabla_A (\operatorname{tr}^K \bar{\pi} + 4u^2 \operatorname{tr} k) \psi \nabla_A \psi + \int_{[t_0, t] \times \mathbf{R}^3} (\operatorname{tr}^K \bar{\pi} + 4u^2 \operatorname{tr} k) |\nabla \psi|^2. \end{aligned}$$

The last term is easily controlled as follows:

$$\int_{[t_0, t] \times \mathbf{R}^3} (\operatorname{tr}^K \bar{\pi} + 4u^2 \operatorname{tr} k) |\nabla \psi|^2 \lesssim \lambda^{-\epsilon} \sup_{[t_0, t]} \mathcal{E}_0[\psi](\tau).$$

The first two terms can be treated in, essentially, the same way. Consider the first one and apply Leibnitz rule to $\mathcal{D}_3 (\operatorname{tr}^K \bar{\pi} + 4u^2 \operatorname{tr} k) = \mathcal{D}_3 (4t(t-u)(\operatorname{tr} \chi - \frac{2}{s}))$. The most difficult term is $4t(t-u) \mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s})$ as we do not have good pointwise estimates on the quantity $\mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s})$. However, according to (4.112), we control its L^2 norm on the surfaces $S_{\tau, u}$: $\|\mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s})\|_{L^2(S_{\tau, u})} \leq \lambda^{-\bar{a} + \epsilon}$. We now apply the adapted version³⁷ of the Sobolev inequality on $S_{t, u}$ (see Theorem 5.2): $\sup_{S_{\tau, u}} |f| \lesssim \|\nabla f\|_{L^2(S_{\tau, u})} + \frac{1}{s} \|f\|_{L^2(S_{\tau, u})}$. We have $\tau \|\mathcal{D}_4 \psi\|_{L^2(\Sigma_\tau)} \lesssim \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau)$ and also

$$\begin{aligned} \|s \mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s}) \psi\|_{L^2(\Sigma_\tau)}^2 &\lesssim \int s^2 \|\mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s})\|_{L^2(S_{\tau, u})}^2 \sup_{S_{\tau, u}} s^2 |\psi|^2 ds \\ &\lesssim \sup_u \|\mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s})\|_{L^2(S_{\tau, u})}^2 \int_{\Sigma_\tau} (s^2 |\nabla \psi|^2 + |\psi|^2) \lesssim \lambda^{-2\bar{a} + 2\epsilon} \mathcal{E}_0[\psi](\tau). \end{aligned}$$

We can now estimate

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} 2\tau s |\mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s}) \psi \mathcal{D}_4 \psi| &\lesssim \int_{t_0}^t \|s \mathcal{D}_3 (\operatorname{tr} \chi - \frac{2}{s}) \psi\|_{L^2(\Sigma_\tau)} \sup_{[t_0, t]} \tau \|\mathcal{D}_4 \psi\|_{L^2(\Sigma_\tau)} \\ &\leq \lambda^{-\bar{a} + \epsilon} (t - t_0) \sup_{[t_0, t]} \mathcal{E}_0[\psi](\tau) \leq \lambda^{-\epsilon} \sup_{[t_0, t]} \mathcal{E}_0[\psi](\tau). \end{aligned}$$

As a consequence, we conclude that $|\int_{[t_0, t] \times \mathbf{R}^3} (\operatorname{tr}^K \bar{\pi} + 4u^2 \operatorname{tr} k) \mathcal{D}_3 \psi \mathcal{D}_4 \psi| \leq \lambda^{-\epsilon} \sup_{[t_0, t]} \mathcal{E}_0[\psi](\tau)$. This completes the outline of the proof of Boundedness Theorem 3.1.

³⁷The true L^∞ estimate requires to add another factor containing an ϵ power of the L^p , $p > 2$ norm of the first derivatives

5 Proof of the Asymptotics Theorem 4.2

Recall that the optical function u is the outgoing solution of the eikonal equation $H^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ satisfying the initial condition $u = t$ on the time axis. The Lorentz metric $H = -dt^2 + h$ satisfies the assumptions following from the conditions (1.19)-(1.23) for the family of the Riemannian metrics h implicitly dependent on the parameter λ .

Notations: For convenience we denote the power $1 - \frac{a^2}{2}$ appearing in the conditions (1.21), (1.22) by \bar{a} . It is important to observe that for the values of the parameter $a < -1 + \sqrt{3}$ stated in Theorem (B), we have the inequality $\bar{a} > a$. As a result, for all sufficiently small ϵ , we have $\lambda^{-\bar{a}+\epsilon}\lambda^a \leq \lambda^{-\epsilon}$. In what follows, we shall assume that all generated small constants ϵ obey the above condition. In addition, recall that we are only interested in the large values of the parameter $\lambda \geq \Lambda$, where Λ obeys the condition that the inequality $A \lesssim B$ can be replaced by $A \leq \Lambda^\epsilon B$. Summarizing, we have for all sufficiently small $\epsilon > 0$,

$$\bar{a} = 1 - \frac{a^2}{2}, \quad \lambda^{-\bar{a}+\epsilon}\lambda^a \leq \lambda^{-\epsilon}, \quad A \lesssim B \rightarrow A \leq \Lambda^\epsilon B.$$

Thus ³⁸ on the time interval $[0, t_*]$, $t_* \leq \lambda^a$, and an arbitrary non-negative integer $m \geq 0$,

$$\|\partial^{1+m} H\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-(1-a)(m+1)}, \quad (5.113)$$

$$\|\partial^{1+m} H\|_{L^2_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\frac{2-a}{2}-(1-a)m}, \quad (5.114)$$

$$\|\partial^{1+m} H\|_{L^\infty_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\bar{a}-(1-a)m}, \quad (5.115)$$

$$\|\partial^{\frac{1}{2}+m}(\partial^2 H)\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{-\bar{a}-(1-a)m}, \quad (5.116)$$

$$\|\partial^m \square_h H\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-(2-a)-(1-a)m}. \quad (5.117)$$

Recall also the null hypersurfaces C_u , defined by the level surfaces of u , and the 2-surfaces $S_{t,u}$ of intersection between C_u and the time slices Σ_t . The outward unit normals to $S_{t,u}$, along Σ_t , are denoted by N , and $T = \partial_t$ is the unit normal to Σ_t . We have introduced our canonical null pair $L = T + N$ and $\underline{L} = T - N$. The null pair can be complemented to a null frame e_A, e_3, e_4 with $A = 1, 2$, $e_3 = \underline{L}$, and $e_4 = L$ with e_A an arbitrary orthonormal frame on $S_{t,u}$.

In any null frame the covariant derivative \mathcal{D} compatible with the Lorentz metric H can be described by the Ricci (frame) coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \xi, \underline{\xi}$, and the second fundamental form k_{ij} of the embedding $\Sigma_t \subset \mathbf{R}^4$. Recall formulas (2.45)-(2.52).

In this Chapter we prove the Asymptotics Theorem 4.2 concerning the behavior of the independent Ricci coefficients $\text{tr}\chi, \chi, \eta$, and the second fundamental form k relative to the parameter λ . We give here a complete version of the Theorem.

Theorem 5.1 (Asymptotics) *Let the function u be a solution of the eikonal equation $H^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$. Assume that H satisfies conditions (5.113)-(5.117) with a sufficiently large $\lambda \geq \Lambda$ and the parameter $a < -1 + \sqrt{3}$. Let e_1, e_2, e_3, e_4 be a null frame with $e_3 = \underline{L}$ and $e_4 = L$. Then the associated*

³⁸The operator $\bar{\square}_h$ in the last condition is replaced with the geometric wave operator \square_h . The difference between the two operators contains only terms quadratic in ∂H and, hence, controlled by (5.114).

Ricci coefficients $\text{tr}\chi$, $\hat{\chi}$, η , and the second fundamental form k^{39} obey the following estimates for an arbitrarily small $\epsilon > 0$ and all values of the parameter $s = t - u$ with $0 \leq t \leq t_*$ and $0 \leq u \leq t_*$:

$$\sup_{S_{t,u}} \left| \text{tr}\chi - \frac{2}{s} \right| + \sup_{S_{t,u}} |\hat{\chi}| + \sup_{S_{t,u}} |\eta| + \sup_{S_{t,u}} |k| \leq \lambda^{-\bar{a}+\epsilon}, \quad (5.118)$$

$$\| \mathcal{D}(\text{tr}\chi - \frac{2}{s}) \|_{L^2(S_{t,u})} + \| \mathcal{P}\hat{\chi} \|_{L^2(S_{t,u})} + \| \mathcal{D}_4\eta \|_{L^2(S_{t,u})} + \| \nabla\eta \|_{L^2(S_{t,u})} + \| \mathcal{P}k_A \|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.119)$$

$$\sup_{S_{t,u}} | \mathcal{D}_4(\text{tr}\chi - \frac{2}{s}) | + \sup_{S_{t,u}} | \mathcal{P}_4\hat{\chi} | + \sup_{S_{t,u}} | \mathcal{P}_4\eta | \lesssim \lambda^{-\bar{a}+\epsilon} s^{-1} + \lambda^{-\bar{a}-(1-a)}, \quad (5.120)$$

$$\sup_{S_{t,u}} | \mathcal{D}_3(\text{tr}\chi - \frac{2}{s}) | + \sup_{S_{t,u}} | \mathcal{P}_3\hat{\chi} | \lesssim \lambda^{-2(1-a)} s^{-1} + \lambda^{-3(1-a)}, \quad (5.121)$$

$$\sup_{S_{t,u}} | \mathcal{P}_3\eta | \lesssim \lambda^{-3(1-a)}, \quad (5.122)$$

$$\sup_{S_{t,u}} | \nabla(\text{tr}\chi - \frac{2}{s}) | + \sup_{S_{t,u}} | \nabla\hat{\chi} | + \sup_{S_{t,u}} | \nabla\eta | \lesssim \lambda^{-2(1-a)} s^{-1} + \lambda^{-3(1-a)}, \quad (5.123)$$

$$\sup_{S_{t,u}} | \mathcal{P}_4k_A | + \sup_{S_{t,u}} | \mathcal{P}_3k_A | \lesssim \lambda^{-\bar{a}-(1-a)}, \quad \sup_{S_{t,u}} | \nabla k_A | \lesssim \lambda^{-\bar{a}} s^{-1} + \lambda^{-\bar{a}-(1-a)}, \quad (5.124)$$

In addition, the derivatives of the metric H relative to the standard coordinates (t, x_1, \dots, x_3) satisfy

$$\| \partial^{2+m} H \|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}-m(1-a)+\epsilon}, \quad \forall m \geq 0. \quad (5.125)$$

The lapse function $b = u_t^{-1}$ satisfies

$$\sup_{S_{t,u}} |b - 1| \lesssim \min\{\lambda^{-(1-a)}, \lambda^{-\bar{a}} s\}, \quad \sup_{S_{t,u}} | \mathcal{D}_4 b | \lesssim \lambda^{-\bar{a}}, \quad (5.126)$$

$$\sup_{S_{t,u}} | \nabla b | \leq \lambda^{-\bar{a}+\epsilon}, \quad \sup_{S_{t,u}} | \mathcal{D}_3 b | \lesssim \lambda^{-2(1-a)}, \quad (5.127)$$

$$(5.128)$$

Remark: Note that we do not have a good estimate for $\mathcal{P}_3\eta$, see (5.119). Such an estimate, however, is not needed for the proof of the Boundedness Theorem.

Proof: The proof of Theorem 5.1 occupies Sections 5.1-5.11.

5.1 Outline of the proof of the Asymptotics Theorem

Direct attempt to estimate $\text{tr}\chi$: Proposition 2.2 provides us with the following transport equation for $\text{tr}\chi$:

$$L(\text{tr}\chi) + \frac{1}{2}(\text{tr}\chi)^2 = -|\hat{\chi}|^2 - k_{NN}\text{tr}\chi - \mathbf{R}_{44}.$$

³⁹The notation k_A will be used to denote the co-vector k_{AN} or the 2-tensor k_{AB} tangent to the surfaces $S_{t,u}$. Also recall that for an arbitrary vectorfield X and, say co-vector k_{AN} , we have $\mathcal{P}_X k_{AN} = X(k_{AN}) - k(\mathcal{P}_X e_A, N)$.

Integrating this equation along the null vectorfield L and counting only the contribution of the curvature term on the right hand-side, we obtain the simplest estimate $|\text{tr}\chi - \frac{2}{s}| \leq \int |\mathbf{R}_{44}|$. The main contribution to the curvature \mathbf{R} is given by the second derivatives of the metric $\partial^2 H$. Thus the condition (5.113) on the metric H would imply the bound $|\text{tr}\chi - \frac{2}{s}| \lesssim \lambda^{-2(1-a)}$. This estimate is far worse than the one claimed in the Asymptotics Theorem. The arguments along this line can only recover the local well posedness results of [Ta2].

Estimate for $\text{tr}\chi$ using the special structure of \mathbf{R}_{44} : The remarkable decomposition of the curvature term $\mathbf{R}_{44} = L(z) - \frac{1}{2}e_4^\nu e_4^\mu \square_h H_{\mu\nu} + \text{Error}$ proved in Lemma (2.1) and the condition (5.117) on the metric H allows us to prove a more precise estimate on $y = \text{tr}\chi - \frac{2}{s}$. The simplified version of the transport equation (2.78) has the form $L(y+z) + \text{tr}\chi(y+z) = \frac{1}{2}e_4^\nu e_4^\mu \square_h H_{\mu\nu} - \text{Error}$. Recall that $|z| \lesssim |\partial H|$ and $|\text{Error}| \lesssim (\partial H)^2$. Thus integrating the transport equation and using the conditions (5.113), (5.114), and (5.117) we could obtain $|\text{tr}\chi - \frac{2}{s}| \leq \lambda^{-\bar{a}+\epsilon}$ as desired.

Attempt to estimate $\hat{\chi}$ via the transport equation: In the first approximation, the transport equation (2.61) for $\hat{\chi}$ can be written in the form $\mathcal{D}_4 \hat{\chi} + \frac{1}{2}(\text{tr}\chi)\hat{\chi} = \hat{\alpha}$. The curvature term $\hat{\alpha}$ does not admit a decomposition similar to the one for \mathbf{R}_{44} . Thus, by integrating the transport equation we can only expect to obtain the bound $|\hat{\chi}| \leq \lambda^{-2(1-a)}$.

Codazzi equation for $\nabla \hat{\chi}$: To avoid the use of the transport equation for $\hat{\chi}$ we invoke the Codazzi equation (2.64) for the angular divergence of $\hat{\chi}$ on the surfaces $S_{t,u}$. Again, after throwing out the lower order terms we obtain the equation $(\text{div} \hat{\chi})_A = \frac{1}{2} \nabla_A \text{tr}\chi - \mathbf{R}_{B4AB}$. In view of the fact that $\hat{\chi}$ is a traceless 2-tensor we can interpret the above as an elliptic Hodge system. As a result we can estimate $\|\nabla \hat{\chi}\|_{L^2(S_{t,u})} + \frac{1}{s} \|\hat{\chi}\|_{L^2(S_{t,u})} \lesssim \|\nabla \text{tr}\chi\|_{L^2(S_{t,u})} + \|\mathbf{R}_{B4AB}\|_{L^2(S_{t,u})}$. Since we do not have good control on the $L^2(S_{t,u})$ norm of the curvature we use the trace theorem (see Theorem 5.3) to pass to the $L^2(\Sigma_t)$ norm with a loss of $\frac{1}{2}$ derivative. Again replacing the curvature term by $\partial^2 H$ we have $\|\partial^2 H\|_{L^2(S_{t,u})} \lesssim \|\partial^{\frac{1}{2}+2} H\|_{L^2(\Sigma_t)} \lesssim \lambda^{-\bar{a}}$ with the last inequality following from the assumption (5.116) on H . Thus, if we suppress the contribution of the $\nabla(\text{tr}\chi)$ term we derive the bound $\|\nabla \hat{\chi}\|_{L^2(S_{t,u})} + \frac{1}{s} \|\hat{\chi}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$.

Passage from the L^2 estimates for $\nabla \hat{\chi}$ to the L^∞ estimates for $\hat{\chi}$: This is achieved by using the Sobolev inequality $\sup_{S_{t,u}} |\hat{\chi}| \lesssim \|\nabla \hat{\chi}\|_{L^2(S_{t,u})} + \frac{1}{s} \|\hat{\chi}\|_{L^2(S_{t,u})}$ on the 2-surfaces $S_{t,u}$, see Theorem 5.2. Thus the desired estimate $|\hat{\chi}| \leq \lambda^{-\bar{a}+\epsilon}$ would follow.

Estimate for $\nabla(\text{tr}\chi)$: To include the contribution of $\nabla(\text{tr}\chi)$ in the Codazzi equation we use the special structure of the \mathbf{R}_{44} component of the curvature invoking the transport equation (2.79) for $\nabla(y+z)$.

Precise approach: In the rigorous approach, which we present below, the transport equations (2.78), (2.79) for $y+z$, $\nabla(y+z)$, and the Codazzi equation (2.64) form a coupled system involving $\text{tr}\chi$, $\hat{\chi}$, $\nabla \text{tr}\chi$ and $\nabla \hat{\chi}$. The quantities $\text{tr}\chi$ and $\hat{\chi}$ will be always estimated in the L^∞ norm while their angular derivatives will be taken in the $L^2(S_{t,u})$ norm. In addition, the Sobolev inequality of Theorem 5.2 provides the passage from the estimates for the angular derivatives to the pointwise bounds for the quantities itself.

Direct attempt to estimate η : The effort to obtain the bound on η from its transport equation (2.62) would result in the estimate $|\eta| \lesssim \lambda^{-2(1-a)}$. We can not proceed as we did for $\text{tr}\chi$ since the β component of the curvature does not admit a special decomposition.

Hodge system for η : The more efficient way to control η is via the Hodge elliptic system (2.65), (2.66) which in a simplified version has the form $\mathfrak{d}\eta = \frac{1}{2}(\mu) - \frac{1}{2}\gamma$, $\mathfrak{d}\eta = -\frac{1}{2}\epsilon^{AB}\mathbf{R}_{A43B}$. The quantity $\mu = \underline{L}(\text{tr}\chi) + k_{NN}\text{tr}\chi - \frac{1}{2}(\text{tr}\chi)^2$ depends on $\text{tr}\chi$ and thus has a ‘‘good’’ transport equation (2.71) which takes advantage of the special structure of \mathbf{R}_{44} . As a result, the Hodge system (2.65), (2.66) and the transport equation (2.71) form a coupled system for η, μ and $\nabla\eta$. We estimate η pointwise while taking the $L^2(S_{t,u})$ norm of the quantities μ and $\nabla\eta$ and use estimates for $\text{tr}\chi$ and $\hat{\chi}$ obtained on the previous step.

Remaining estimates We have just provided the outline of the proof of the estimates for L^∞ norm of $\text{tr}\chi - \frac{2}{s}$, $\hat{\chi}$, and η , and $L^2(S_{t,u})$ norm of their angular derivatives. The estimates for the remaining derivatives will then follow more easily. For example, a byproduct of the estimates for η is the bound $\|\mathcal{D}_3\text{tr}\chi\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$. The estimate for $\mathcal{D}_4(\text{tr}\chi - \frac{2}{s})$ follows immediately from the transport equation for $\text{tr}\chi$.

The estimates for the lapse function b and the second fundamental form k will be discussed in full detail in the proof of the Asymptotics Theorem.

A comparison between the estimates for $\text{tr}\chi$ and $\hat{\chi}, \eta$: There is a fundamental difference between the derivation of the estimates for $\text{tr}\chi$ and $\hat{\chi}, \eta$. The estimates for $y = \text{tr}\chi - \frac{2}{s}$ are obtained directly from the transport equation, which, as we mentioned before, takes the form $L(y+z) + \text{tr}\chi(y+z) = \frac{1}{2}\square_h H - (\partial H)^2$, if we ignore the contribution of $\hat{\chi}$. The integration of this equation produces the estimate $|y| \lesssim |z| + \lambda^{-(2-a)}$, which follows from (5.114), (5.117). If we now use the fact that $|z| \lesssim |\partial H|$ and recall the condition (5.115), we obtain the estimate $|y| \lesssim \lambda^{-\bar{a}}$ stated in the Asymptotics Theorem. Similar bounds are derived for $\hat{\chi}$ and η . However, we can also complement the pointwise bound on y with the integral estimate

$$\|y\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \|z\|_{L^1_{[0,t_*]}L^\infty_x} + t_*\lambda^{-2(1-a)} \lesssim \|\partial H\|_{L^1_{[0,t_*]}L^\infty_x} + \lambda^a\lambda^{-2(1-a)}$$

Observe that the condition (5.113), $\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{(1-a)}$. Therefore

$$\|y\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-(1-a)}.$$

It is important to observe that such an estimate would imply that the $L^1_{[0,t_*]}L^\infty_x$ norm of $\text{tr}\chi - \frac{2}{s}$ is bounded for *all values of* $a \leq 1$. If we had similar bounds for $\hat{\chi}$ and η we would be able to prove the decay estimates of Theorem (B) for all $a \leq 1$. As a result, we could obtain the local well posedness result in $H^{2+\epsilon}$.

We are not able to derive however such estimates for the Ricci coefficients $\hat{\chi}$ and η . Our approach to the derivation of the estimates for the above quantities is via the elliptic Hodge systems on 2-surfaces $S_{t,u}$. As we discussed earlier, this is already better than using the transport equations for $\hat{\chi}$ and η . The resulting estimate, say for $\hat{\chi}$, in which we only keep track of the contribution of the curvature \mathbf{R} , has the form

$$\sup_{S_{t,u}} |\hat{\chi}| \lesssim \|\nabla\hat{\chi}\|_{L^2(S_{t,u})} \lesssim \|R\|_{L^2(S_{t,u})} \approx \|\partial^2 H\|_{L^2(S_{t,u})} \lesssim \|\partial^{\frac{1}{2}+\epsilon}\partial^2 H\|_{L^2(\Sigma_t)} \leq \lambda^{-\bar{a}+\epsilon}.$$

as we have argued above. Unlike the estimate for $\text{tr}\chi - \frac{2}{s}$ we are now forced to estimate

$$\|\hat{\chi}\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \int_0^{t_*} \|\partial^{\frac{1}{2}+\epsilon}\partial^2 H\|_{L^2(\Sigma_t)} \lesssim t_* \sup_t \|\partial^{\frac{1}{2}+\epsilon}\partial^2 H\|_{L^2(\Sigma_t)} \lesssim \lambda^a \lambda^{-\bar{a}+\epsilon},$$

which is worse than $\lambda^{-(1-a)}$. We see that the above norms remain bounded only if $a \leq \bar{a}$. This dictates the range of the admissible a : $a < -1 + \sqrt{3}$. The loss occurs when we use the trace Theorem to replace $L^2(S_{t,u})$ norm by the $L^2(\Sigma_t)$.

We now state a series of results which are extensively used below. We start by introducing the notion of an admissible foliation.

5.2 Admissible foliations

Our two special functions t and u generate the following 2 important foliations. For each fixed t the 2-surfaces $S_{t,u}$ form a foliation, which we denote by $S_{t,\cdot}$, of the time slice Σ_t . For each fixed u the same 2-surfaces $S_{t,u}$ form a foliation, which we denote by $S_{\cdot,u}$, of the null cones C_u . In each case, we can describe the foliation by a map $\Phi(s, \omega)$ with values in Σ_t or C_u respectively, where $s = t - u$ and $\omega \in \mathcal{S}^2$. In addition, the foliations are formed by the level surfaces of the function $s = t - u$ restricted correspondingly to Σ_t or C_u .

In the following definition the domain Ω is assumed to be a subset of the time slice Σ_t endowed with the metric $h(t, \cdot)$.

Definition 5.1 *We say that the level surfaces of a function $v : \Omega \rightarrow \mathcal{R}$ define an admissible foliation of Ω described by a map $\Phi : [0, \max v] \times \mathcal{S}^2 \rightarrow \Omega$, if the lapse function $\mathbf{b} = |\nabla v|_h$ and the second fundamental form θ of the foliation have the following properties:*

1) *The level surface $v = 0$ consists of one point, say $x = 0$, and the norm of the tangent map $\frac{1}{C} \leq |d\Phi|_{x=0} \leq C$ for some positive constant C .*

There exist a positive constant $M \leq 2^{-5}(\max v)^{-1}$ such that for any $v \in (0, \max v]$

2) $|\mathbf{b}(v) - 1| \leq Mv$,

3) $|tr\theta - \frac{2}{v}| \leq M$.

Proposition 5.1 *Let the level surfaces of function v define an admissible foliation. Denote by S_v the level surface corresponding to the value v . Then for each $v \in [0, \max v]$ the area of the surface S_v obeys the estimate*

$$A(S_v) \approx v^2. \tag{5.129}$$

The implicit constant in the inequality above depends only on the constant C and the ellipticity constant of the metric h .

Proof: See Appendix, where we combine it with the proof of the trace Theorem 5.3.

We now describe the parameters of the foliation $S_{t,\cdot}$.

Lemma 5.1 *For a fixed $t \leq t_*$, consider the foliation $S_{t,\cdot}$ of the region in Σ_t containing the point $x = 0$, by the level surfaces of the function $s = t - u(t, \cdot)$. Assume that the foliation is described by the map $\Phi : [0, \max s] \times \mathcal{S}^2 \rightarrow \Sigma_t$. Let ϕ^A , $A = 1, 2$ be a system of local coordinates on \mathcal{S}^2 . Then the metric h in coordinates (s, ϕ_A) has the form*

$$h = \mathbf{b}^2 ds^2 + \gamma_{AB} d\phi^A d\phi^B, \tag{5.130}$$

where \mathbf{b} is the lapse function of the foliation and

$$\mathbf{b} = b = u_t^{-1} = |\nabla u|_h. \quad (5.131)$$

The second fundamental form $\theta_{AB} = \langle \nabla_A N, e_B \rangle$ of the foliation $S_{t,u}$ can be found from the identity

$$\theta_{AB} = \chi_{AB} + k_{AB}. \quad (5.132)$$

Proof: First, on a fixed time slice Σ_t , we have $ds = -du$. The representation (5.130) follows easily from the fact that ∂_u is orthogonal to ∂_{ϕ_A} . Moreover, since $du = \partial_i u dx^i$ and $|\nabla u|_h = u_t = b^{-1}$, from the eikonal equation, we have

$$\partial_u = \frac{h^{ij} \partial_j u}{|\nabla u|^2} \partial_i = b^2 \partial^i u \partial_i \quad (5.133)$$

Computing

$$h(\partial_u, \partial_u) = b^2 = \mathbf{b}^2 du^2(\partial_u, \partial_u) = \mathbf{b}^2$$

we conclude that $b = \mathbf{b}$. Clearly, $\mathbf{b} = |\nabla s|_h$, therefore \mathbf{b} is the lapse function of the foliation.

The vector N is the outward unit normal to the level surface of u on the time slice $t = \text{const}$. Hence, the second fundamental form of the foliation $S_{t,u}$ with respect to the frame $e_A, A = 1, 2$

$$\theta_{AB} = \langle \nabla_A N, e_B \rangle. \quad (5.134)$$

Thus,

$$\chi_{AB} = \langle \mathcal{D}_A e_A, e_B \rangle = \langle \nabla_A (T + N), e_B \rangle = \theta_{AB} - k_{AB} \quad (5.135)$$

5.3 Sobolev, elliptic, and trace estimates

We now formulate 3 results concerning various estimates for functions defined on 2-dimensional surfaces and admissible foliations.

Theorem 5.2 *Let S be a 2-dimensional surface embedded in \mathbb{R}^3 endowed with a Riemannian metric h . Assume that relative to the standard coordinates on \mathbf{R}^3 the metric h obeys the estimate:*

$$\sup_{\mathbb{R}^3} |\partial h| \leq \Lambda_0^{-1}, \quad (5.136)$$

for some positive constant Λ_0 . Let $A(S) = \int_S 1$ denote the area of the surface S relative to the metric h . Assume in addition that with the same constant Λ_0 as in (5.136)

$$A(S) \leq 2^{-15} \Lambda_0^2, \quad (5.137)$$

Then for any function $f : S \rightarrow \mathbb{R}$ we have the following isoperimetric inequality:

$$\int_S |f|^2 \lesssim \int_S (|\nabla f|^2 + |\text{tr}\theta||f|). \quad (5.138)$$

Here, ∇ is the covariant derivative on S compatible with the metric h and θ is the second fundamental form of the embedding $S \subset \mathbb{R}^3$. In addition, we have the following form of the Sobolev inequality on S : for any $\delta \in (0, 1)$ and $p \in (2, \infty]$

$$\sup_S |f| \lesssim \left[A(S) \right]^{\frac{\delta(p-2)}{2p+\delta(p-2)}} \left(\int_S (|\nabla f|^2 + |\text{tr}\theta|^2 |f|^2) \right)^{\frac{1}{2} - \frac{\delta p}{2p+\delta(p-2)}} \left[\int_S (|\nabla f|^p + |\text{tr}\theta|^p |f|^p) \right]^{\frac{2\delta}{2p+\delta(p-2)}}. \quad (5.139)$$

Proof: The proof of Theorem 5.2 can be found in the Appendix. Since S is a 2-dimensional surface, the $H^1 \subset L^\infty$ embedding barely fails. The estimate⁴⁰ (5.139) is a correct version of the estimate of the supremum of a function in terms of its first derivatives in L^2 and an additional small power of their L^p norm.

The second result is a trace theorem for an admissible foliation.

Theorem 5.3 *Let Ω be a subset of \mathbf{R}^3 endowed with the metric h . Assume that the level surfaces of a function $v : \Omega \rightarrow \mathbb{R}$ define an admissible foliation. Then for any level surface S_v with $v \in (0, \max v]$, arbitrary $\epsilon > 0$, and any function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $f \in H^{\frac{1}{2}+\epsilon}(\mathbb{R}^3)$, we have*

$$\|f\|_{L^2(S_v)} \lesssim \|\partial^{\frac{1}{2}-\epsilon} f\|_{L^2(\mathbb{R}^3)} + \|\partial^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R}^3)} \quad (5.140)$$

Proof: See Appendix.

The last theorem recalls the elliptic estimates for Hodge systems proved in [Ch-Kl].

Theorem 5.4 *Let S be a 2-dimensional manifold endowed with a metric h and a compatible covariant derivative ∇ . Then for any $m+1$ covariant, totally symmetric tensor ξ , a solution of the Hodge system*

$$\text{div}\xi = F, \quad (5.141)$$

$$\text{curl}\xi = G, \quad (5.142)$$

$$\text{tr}\xi = E, \quad (5.143)$$

there holds the following identity:

$$\int_S |\nabla \xi|^2 + (m+1)K|\xi|^2 = \int_S \{|F|^2 + |G|^2 + mK|E|^2\}. \quad (5.144)$$

Here operations div and curl are as defined in Proposition 2.3, and K is the Gauss curvature of S .

Remark *If ξ is a symmetric traceless tensor satisfying $\text{div}\xi = F$, then $\text{curl}\xi = {}^*F$, where *F is the Hodge dual of F : ${}^*F_A = \epsilon_{AB}F^B$. Thus, in this case the identity (5.144) becomes*

$$\int_S |\nabla \xi|^2 + 2K|\xi|^2 = 2 \int_S |F|^2. \quad (5.145)$$

⁴⁰which is a measure-theoretic consequence of the isoperimetric inequality

5.4 Main equations

We now state the equations which will be used in the proof of the Asymptotics Theorem. Recall the notations $y = \text{tr}\chi - \frac{2}{s}$, $\mu = \underline{L}(\text{tr}\chi) + k_{NN}\text{tr}\chi - \frac{1}{2}(\text{tr}\chi)^2$, and the decomposition of $\mathbf{R}_{44} = L(z) - \frac{1}{2}e_4^\nu e_4^\mu \square_h \mu H_{\mu\nu} + \text{Error}$ of Lemma 2.1. Observe that we have pointwise bounds $|e_4^\nu e_4^\mu \square_h \mu H_{\mu\nu}| \leq |\square_h H|$ and $|\text{Error}| \lesssim (\partial H)^2$. Therefore, from the point of view of the conditions (5.114) and (5.117) on the metric H these two terms have equal strength. We shall combine them into the single *Error* term and write the decomposition $\mathbf{R}_{44} = L(z) + \text{Error}$. We have

The transport equations:

$$L(b) = -bk_{NN}, \quad (5.146)$$

$$L(y+z) + \text{tr}\chi(y+z) = \frac{1}{2}(y+z)^2 + \frac{2}{s}z - \frac{1}{2}z^2 - |\hat{\chi}|^2 - k_{NN}\text{tr}\chi - \text{Error}, \quad (5.147)$$

$$\begin{aligned} \mathcal{P}_4 \nabla_A(y+z) + \frac{3}{2}\text{tr}\chi \nabla_A(y+z) &= -\hat{\chi}_{AB} \nabla_B(y+z) + (y+z) \nabla_A(y+z) - (y+z) \nabla_A(\text{tr}\chi) \\ &+ \left(\frac{2}{s} - 2z\right) \nabla_A z - 2 \nabla_A \hat{\chi} \cdot \hat{\chi} - \text{tr}\chi \nabla_A k_{NN} - k_{NN} \nabla_A \text{tr}\chi + \nabla_A \text{Error}, \end{aligned} \quad (5.148)$$

$$\begin{aligned} L(\mu) + \text{tr}\chi \mu &= -2\hat{\chi}_{AB} \left(2\nabla_A \eta_B - \not{d}^V \eta \delta_{AB} + k_{NN} \hat{\chi}_{AB} + 2(\eta_A \eta_B - |\eta|^2 \delta_{AB}) - \frac{1}{2}\text{tr}\chi \hat{\chi}_{AB} \right. \\ &\left. - \frac{1}{2}\text{tr}\chi \hat{\chi}_{AB} + \mathbf{R}_{A43B} \right) + \underline{L}(\mathbf{R}_{44}) + 2(\underline{\eta}_A - \eta_A) \nabla_A(\text{tr}\chi) + \text{tr}\chi (|\hat{\chi}|^2 + (L - \underline{L})k_{NN} - \mathbf{R}_{44}), \end{aligned} \quad (5.149)$$

$$\mathcal{P}_4 \hat{\chi}_{AB} + \frac{1}{2}(\text{tr}\chi) \hat{\chi}_{AB} = -k_{NN} \hat{\chi}_{AB} - \hat{\alpha}_{AB}, \quad (5.150)$$

$$\mathcal{P}_4 \eta_A + \frac{1}{2}(\text{tr}\chi) \eta_A = -(k_{BN} + \eta_B) \hat{\chi}_{AB} - \frac{1}{2}(\text{tr}\chi) k_{AN} - \frac{1}{2} \beta_A, \quad (5.151)$$

The equations on the surfaces $S_{t,u}$:

$$(\not{d}^V \hat{\chi})_A + \hat{\chi}_{AB} k_{BN} = \frac{1}{2}(\nabla_A \text{tr}\chi + k_{AN} \text{tr}\chi) - \mathbf{R}_{B4AB}, \quad \text{Codazzi equation}, \quad (5.152)$$

$$\not{d}^V \eta = \frac{1}{2}(\underline{L}(\mu) + \text{tr}(\hat{\chi} \cdot \hat{\chi}) - 2k_{NN} \text{tr}\chi - 2|\eta|^2) - \frac{1}{2} \gamma, \quad (5.153)$$

$$\text{curl } \eta = \frac{1}{2} \epsilon^{AB} \hat{\chi}_{AC} \hat{\chi}_{CB} - \frac{1}{2} \epsilon^{AB} \mathbf{R}_{A43B}, \quad (5.154)$$

$$K = -\frac{1}{4} \text{tr}\chi \text{tr}\underline{\chi} + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} + \frac{1}{2} \mathbf{R}_{ABAB}, \quad \text{Gauss equation.} \quad (5.155)$$

In the last equation K is the Gauss curvature of $S_{t,u}$.

5.5 Continuation argument

The proof of the Asymptotics Theorem relies on an elaborate continuation argument which we describe in this section. Recall that the optical function u solves the eikonal equation and attains the prescribed value t on the time axis. To find u , it suffices to describe its level hypersurfaces C_u . These can be obtained as the union of the null geodesics, relative to the Lorentz metric H , emanating from the vertex, on the time axis, with coordinates $(u, 0)$ and initial velocity in the direction of the vector $(1, \omega)$, with $\omega \in \mathcal{S}^2$.

Remark: It is not difficult to verify from the equation for null geodesics, written in the coordinates⁴¹ x^α , that any such geodesic $x(s)$ initiating from the point $(u, 0)$ can be extended to the value of the affine parameter $s = t_* - u$. We can also show that the equation $L(b) = -bk_{NN}$ for the lapse function b implies that the geodesic $x(s)$ intersects each time slice Σ_t with $t \leq t_*$.

Using this construction we can describe the behavior of the derivatives of the optical function in the immediate neighborhood of the time axis, and, in particular⁴², the Ricci coefficients χ and η , and the lapse function b . The following initial conditions are easy to check from our definitions and the local geometry around points on the time axis.

Initial values⁴³: There exists a positive constant⁴⁴ M such that for all $t \in [0, t_*]$ and all sufficiently small s

$$I1) \quad |\text{tr}\chi - \frac{2}{s}| + |\hat{\chi}(s)| + |\eta(s)| \leq 2M,$$

I2) The lapse function b and the area $A(S_{t,u})$ of the surfaces $S_{t,u}$ obey $|b(s) - 1| + |A(S_{t,u}) - 4\pi s^2| \rightarrow 0$ as $s \rightarrow 0$,

$$I3) \quad \|\mathcal{D}(\text{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} + \|\mathcal{D}\hat{\chi}(s)\|_{L^2(S_{t,u})} + \|\nabla\eta(s)\|_{L^2(S_{t,u})} + \|\mathcal{P}_3\eta(s)\|_{L^2(S_{t,u})} \leq 2M,$$

$$I4) \quad s^3 |\mathcal{D}(\text{tr}\chi - \frac{2}{s})| + s^2 |\mathcal{P}\hat{\chi}(s)| + s^2 |\mathcal{P}\eta(s)| + s |\mathcal{P}_3\eta(s)| \rightarrow 0 \text{ as } s \rightarrow 0.$$

The continuation argument proceeds now as follows. For each time t in the interval $[0, t_*]$ let $s(t)$ be the maximum value of the parameter $s = t - u$ such that the initial conditions with a given constant M can be extended to the neighborhood of the time axis: $t \in [0, t_*]$ and $s \in [0, s(t))$ with an additional condition that $s(t) \leq \min\{2^{-10}M^{-1}, t\}$. This means that for all $t \in [0, t_*]$ and $s \in [0, s(t))$ we have

Assumptions

$$A1) \quad |\text{tr}\chi - \frac{2}{s}| + |\hat{\chi}(s)| + |\eta(s)| \leq 2M,$$

$$A2) \quad \|\mathcal{D}(\text{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} + \|\mathcal{P}\hat{\chi}(s)\|_{L^2(S_{t,u})} + \|\nabla\eta(s)\|_{L^2(S_{t,u})} + \|\mathcal{P}_3\eta(s)\|_{L^2(S_{t,u})} \leq 2M,$$

$$A3) \quad s(t) \leq \min\{2^{-10}M^{-1}, t\}.$$

The essence of our continuity argument is to show that, under the above assumptions, the constant M defined by the initial conditions can in fact be chosen to be $M = \lambda^{-\bar{a}+\epsilon}$ for some fixed, arbitrarily small, $\epsilon > 0$. Moreover, with this choice of M , the norms described in the Assumptions remain bounded by M for all values $[0, s(t)]$. By the maximality of the interval $[0, s(t)]$, this implies that $s(t) = \min\{2^{-10}\lambda^{\bar{a}-\epsilon}, t\}$. Since $t \leq t_* \leq \lambda^a \leq 2^{-10}\lambda^{\bar{a}-\epsilon}$, we conclude that the estimates with $M = \lambda^{-\bar{a}+\epsilon}$ can be extended to the interior of the maximum cone C_0 , i.e, the region $\{(t, s) : s \in [0, t], t \in [0, t_*]\}$. This facts form the content of:

⁴¹relative to which we made our assumptions on the metric H

⁴² χ and η are simply null components of the hessian of u .

⁴³We use the affine parameter $s = t - u$ to measure the distance to the time axis.

⁴⁴which may depend on λ ! We show later that this is not the case. Here M may also depend in principle on each particular hypersurface C_u .

Theorem 5.5 *Let the frame coefficients $\text{tr}\chi$, $\hat{\chi}$, and η obey the Assumptions with a constant $2M > \lambda^{-\bar{a}+\epsilon}$ for some sufficiently small $\epsilon > 0$. Assume that the region described by $s < s(t), t \leq t_*$ is a maximal domain where the Assumptions hold true. Then the Assumptions are also satisfied with the constant $2M$ replaced by $\lambda^{-\bar{a}+\epsilon}$. As a consequence, the maximal domain is the set $\{(t, s) : s \in [0, t], t \in [0, t_*]\}$.*

Remark *Under the above Assumptions, all the Ricci coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}$ can be uniformly estimated by the quantity $\frac{2}{s}$. This is, of course, a very crude estimate, since it is only $\text{tr}\chi$ and $\text{tr}\underline{\chi}$ that have singularities at $s = 0$.*

Let Γ_M denote the domain in \mathbf{R}^4 defined in A1)-A3). For each fixed $t \in [0, t_*]$, the level surfaces of the optical function $u(t, \cdot)$ generate the foliation $S_{t, \cdot}$. The following Theorem shows that these foliations are admissible. Moreover, each of the surfaces $S_{t, u}$ admits Sobolev, elliptic, and trace estimates. Keep in mind, in what follows, that $s = t - u$, $s = 0$ on the time axis, and $s = s(t)$ describes the last surface satisfying the Assumptions.

Proposition 5.2 *For each fixed $t \in [0, t_*]$ the foliation $S_{t, \cdot}$ parametrized by the values of $s \in [0, s(t))$ is admissible. Moreover, each 2-surface $S_{t, u}$ with $s = t - u < s(t)$ satisfies the conditions of Theorems 5.2, 5.140, and 5.4*

Proof: Recall, see Lemma 5.1, the lapse $\mathbf{b} = b$ and the second fundamental form $\theta_{AB} = \chi_{AB} + k_{AB}$ of the foliation $S_{t, \cdot}$ along Σ_t . To show that the foliation is admissible it suffices⁴⁵ to verify that $|b(s) - 1| \leq M_1 s$ and $|\text{tr}\theta - \frac{2}{s}| \leq M_1$ for all $s \in [0, s(t))$ and a constant M_1 such that $M_1 s(t) \leq 2^{-5}$.

The lapse function obeys the transport equation (2.59): $L(b) = -k_{NN}b$. Integrating this equation taking into account the initial condition $b(s) \rightarrow 1$ as $s \rightarrow 0$, we obtain $b(s) = e^{-\int_0^s k_{NN}}$. The second fundamental form $k_{ij} = -\frac{1}{2}\partial_t h_{ij}$ obeys the pointwise estimate $|k| \leq |\partial H|$. Therefore, in view of the conditions (5.113) and (5.115), $|\int_0^s k_{NN}| \lesssim \min\{\lambda^{-\bar{a}+\epsilon} s, \lambda^{-(1-a)}\}$. Hence, choosing a slightly bigger value of ϵ , if necessary,

$$|b(s) - 1| \leq \min\{\lambda^{-\bar{a}+\epsilon} s, C\lambda^{-(1-a)}\} \lesssim \lambda^{-\bar{a}+\epsilon} \lambda^a \leq \lambda^{-\epsilon} \leq 2^{-5}. \quad (5.156)$$

Indeed, according to the Assumptions the largest value of s can not exceed $t \leq t_* \leq \lambda^a$. Therefore, the condition 2) of the admissible foliation is verified.

We also have the formula $\text{tr}\theta = \text{tr}\chi + \text{tr}k$. Since $|k| \leq |\partial H|$, the condition (5.115) on the metric H gives $|\text{tr}k| \leq \lambda^{-\bar{a}}$. It then follows from the Assumption A1) that

$$|\text{tr}\theta - \frac{2}{s}| \leq 2M + \lambda^{-\bar{a}+\epsilon} := M_1.$$

In view of the Assumption A3) we have $M_1 s(t) \leq 2^{-9} + \lambda^{-\bar{a}} \lambda^a \leq 2^{-5}$. The foliations $S_{t, \cdot}$ are admissible with the constant M_1 .

To verify the conditions of Theorem 5.2 it remains to note that by (5.115), $\sup |\partial h| \leq \lambda^{-\bar{a}+\epsilon}$ and the Remark after the definition of the admissible foliation implies that $A(S_{t, u}) \approx s^2 \lesssim \lambda^{2a}$. Since $a + 2\epsilon < \bar{a}$, we see that Theorem 5.2 holds with the value of the parameter $\Lambda_0 = \lambda^{\bar{a}-\epsilon}$.

⁴⁵Condition 1) can be easily checked, since $S_{t, \cdot}$ is “almost” the foliation of Σ_t by geodesic spheres

5.6 Some useful Lemmas

The first Lemma provides estimates for a solution of the transport equations. Let $\Pi \cdot \Xi = \sum_{\underline{A}} \Pi_{\underline{A}} \Xi_{\underline{A}}$ denote the inner product of two S-tangent covariant tensors ⁴⁶ of the same degree ⁴⁷, and $|\Pi| = \sqrt{\Pi \cdot \Pi}$ be the corresponding norm of Π .

Lemma 5.2 *Let $\Pi_{\underline{A}}$ be an S-tangent tensorfield verifying the following transport equation with $\sigma > 0$:*

$$\mathcal{P}_4 \Pi_{\underline{A}} + \sigma \text{tr} \chi \Pi_{\underline{A}} = F_{\underline{A}}.$$

Assume that the point $(t, x) = (t, s, \omega)$ belongs to the domain Γ_M . If Π satisfies the initial condition $s^{2\sigma} \Pi_{\underline{A}}(s) \rightarrow 0$ as $s \rightarrow 0$, then

$$|\Pi(t, x)| \leq \sup_{\rho \leq s} \rho |F| \quad (5.157)$$

For the right hand-side F defined in the whole time slab $[0, t_] \times \mathbf{R}^3$, estimate (5.157) can be complemented with*

$$|\Pi(t, x)| \leq 4 \|F\|_{L^1_{[0, t_*]} L^\infty_x}. \quad (5.158)$$

In addition, if $\sigma \geq \frac{1}{2}$ and Π satisfies the initial condition $s^{2\sigma-1} \|\Pi\|_{L^2(S_{t,u})} \rightarrow 0$ as $s = t - u \rightarrow 0$, then on each surface $S_{t,u}$ with $s < s(t)$

$$\|\Pi\|_{L^2(S_{t,u})} \leq \frac{2}{s^{4\sigma-2}} \int_0^s \rho^{4\sigma-2} \|F\|_{L^2(S_{u+\rho,u})} d\rho \quad (5.159)$$

Remark: *Any null geodesic $x(s)$ tangent to the vectorfield L lies on the level surface of the optical function u . We have already observed that the affine parameter $s = t - u$. Thus, in local coordinates (t, x_1, x_2, x_3) on \mathbf{R}^4 , such trajectory is represented by $(u + s, x_1(s), x_2(s), x_3(s))$. Therefore, for a quantity F defined in the time slab $[0, t] \times \mathbf{R}^3$, we can replace its integral along a null geodesic $x(s)$ with an $L^1_t L^\infty_x$ norm of F . Namely, for $s = t - u$*

$$\int_0^s |F| d\rho = \int_0^s |F(\rho + u, x_1(\rho), x_2(\rho), x_3(\rho))| d\rho \leq \int_0^t \sup_{\Sigma_\tau} |F(\tau, \cdot)| d\tau = \|F\|_{L^1_{[0,t]} L^\infty_x}.$$

Proof: From the transport equation for Π we have the following simple identity:

$$\frac{1}{2} \frac{d}{ds} |\Pi|^2 + \frac{2\sigma}{s} |\Pi|^2 = -\sigma (\text{tr} \chi - \frac{2}{s}) |\Pi|^2 + F \cdot \Pi,$$

⁴⁶i.e., tangent to the surfaces $S_{t,u}$

⁴⁷Here for an m -covariant tensor $\Pi_{\underline{A}}$, $\underline{A} = A_1 \dots A_m$

where $\frac{d}{ds}$ is a derivative along any trajectory tangent to the vectorfield $L = e_4$ initiating on the time axis. Integrating the identity with the integrating factor $s^{4\sigma}$ and taking into account the initial condition for Π , we obtain for an arbitrary point $(t, x) \in \Gamma_M$ ⁴⁸

$$|\Pi(t, x)|^2 \leq \frac{2\sigma}{s^{4\sigma}} \int_0^s \rho^{4\sigma} |\operatorname{tr}\chi - \frac{2}{s}| |\Pi|^2 d\rho + \frac{2}{s^{4\sigma}} \int_0^s \rho^{4\sigma} F \cdot \Pi d\rho. \quad (5.160)$$

According to the Assumptions we have $|\operatorname{tr}\chi - \frac{2}{s}| \leq 2M$. Thus

$$\sup_{\rho, \rho \leq s} |\Pi|^2 \leq 2\sigma Ms \sup_{\rho, \rho \leq s} |\Pi|^2 + \frac{4}{s^{4\sigma}} \int_0^s \rho^{4\sigma} F \cdot \Pi d\rho.$$

Since $s \leq s(t)$ and $Ms(t) \leq 2^{-10}$, by the Assumption A3), we easily conclude that

$$\sup_{\rho, \rho \leq s} |\Pi|^2 \leq \frac{4}{s^{4\sigma}} \int_0^s \rho^{4\sigma} F \cdot \Pi d\rho.$$

The inequalities (5.157) and (5.158) follow. For the derivation of (5.158) we make use of the Remark.

To obtain (5.159) we integrate inequality (5.160) over a fixed surface $S_{t,u}$. Since the area $A(S_{t,u}) \approx s^2$, it suffices to have Π satisfy the initial condition $s^{4\sigma-2} \|\Pi\|_{L^2(S_{t,u})}^2 \rightarrow 0$ as $s \rightarrow 0$. Taking into account that the area $A(S_{\rho+u,u}) \approx \rho^2$, we have

$$\begin{aligned} \|\Pi\|_{L^2(S_{t,u})}^2 &\leq \frac{2\sigma}{s^{4\sigma-2}} \int_0^s \rho^{4\sigma-2} \sup |\operatorname{tr}\chi - \frac{2}{s}| \|\Pi\|_{L^2(S_{\rho+u,u})}^2 d\rho \\ &\quad + \frac{2}{s^{4\sigma-2}} \int_0^s \rho^{4\sigma-2} \|F\|_{L^2(S_{\rho+u,u})} \|\Pi\|_{L^2(S_{\rho+u,u})} d\rho. \end{aligned}$$

Estimate (5.159) follows from inequality $|\operatorname{tr}\chi - \frac{2}{s}| \leq 2M$, with $Ms(t) \leq 2^{-10}$ according to A3), and the Gronwall inequality.

Let us now record some preliminary estimates for the second fundamental form k and the components of curvature \mathbf{R} . Some of the estimates have already featured in the proof of Proposition 5.2.

Lemma 5.3 *Let Γ_M be the domain described in the Assumptions. The second fundamental form k satisfies the following estimates inside Γ_M :*

$$|k| \lesssim \lambda^{-\bar{a}}, \quad \int_0^s |k| \lesssim \min\{\lambda^{-\bar{a}} s, \lambda^{-(1-a)}\}, \quad (5.161)$$

$$\|\nabla k\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}, \quad \|\nabla(k_{NN})\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.162)$$

⁴⁸any point (t, s, ω) in the domain Γ_M can be connected by a trajectory $t - s = u = \text{const}$, $\omega = \text{const}$ to a point $(u, 0, \omega)$ on the time axis. This trajectory is parametrized by $\mathbf{s} = (s, \omega)$, is tangent to the vectorfield $L = e_4$, and is contained in Γ_M .

More generally,⁴⁹

$$\|\partial k_{NN}\|_{L^2(S_{t,u})} + \|\mathcal{D}k_{A\cdot}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.163)$$

Let R represent any component of the curvature \mathbf{R} or any quantity obeying the pointwise estimate $|r| \lesssim |\partial^2 H| + (\partial H)^2$. Then

$$\|R\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.164)$$

Remark The estimates (5.161), (5.163) establish the desired estimates (5.118), (5.119) for the second fundamental form k in the domain Γ_M .

Proof: The estimates (5.161) follow immediately from the pointwise bound $|k| \leq |\partial h|$ and the conditions (5.113) and (5.115).

To prove (5.164), note that any component of the curvature can be estimated $|R| \leq |\partial^2 H| + |\partial H|^2$. Since the foliation S_t is admissible, the Trace Theorem 5.3 implies that

$$\|\partial^2 H\|_{L^2(S_{t,u})} \lesssim \|\partial^{\frac{1}{2}-\epsilon} \partial^2 H\|_{L^2(\Sigma_t)} + \|\partial^{\frac{1}{2}+\epsilon} \partial^2 H\|_{L^2(\Sigma_t)}.$$

Both quantities on the right-hand side are controlled by the condition (5.116), $\|\partial^{\frac{1}{2}} \partial^2 H\|_{L^2(\Sigma_t)} \leq \lambda^{-\bar{a}}$ on the metric H . The presence of ϵ contributes a small extra power of λ in the estimate. Continuing to denote every small power by ϵ we obtain $\|\partial^2 H\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$. In addition, the area of the 2-surface $S_{t,u}$ obeys the estimate $A(S_{t,u}) \approx s^2$. This follows from the Remark after the definition of an admissible foliation and the fact that the family of surfaces S_t forms an admissible foliation generated by the level surfaces of the function $s = t - u$ for a fixed t . Thus $\|(\partial H)^2\|_{L^2(S_{t,u})} \lesssim s \sup(\partial H)^2 \lesssim s \lambda^{-2\bar{a}}$. The maximum possible value of s is $s(t)$ which obeys the bound $s(t) \leq t \leq t_* \leq \lambda^a$. Hence, since $\bar{a} > a$, we have $\|(\partial H)^2\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}}$. Therefore, $\|R\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$.

The first estimate in (5.162) clearly follows from (5.164), since $k_{ij} = -\frac{1}{2}\partial_t h_{ij}$. In addition, $\nabla_A(k_{NN}) = (\mathcal{D}_A k)(N, N) + 2k(\mathcal{D}_A N, N)$. Since $N = \frac{1}{2}(e_4 - e_3)$, we have $\mathcal{D}_A N = \frac{1}{2}(\chi_{AB} - \chi_{AB})e_B - k_{AN}N$. It then follows from the Remark after Assumptions and the estimates $|k| \lesssim |\partial H| \leq \lambda^{-\bar{a}+\epsilon}$, $|\mathcal{D}k| \lesssim |\partial^2 H|$ that $|\nabla_A k_{NN}| \lesssim |\partial^2 H| + \lambda^{-\bar{a}+\epsilon} s^{-1}$. Thus, $\|\nabla k_{NN}\|_{L^2(S_{t,u})} \lesssim \|\partial^2 H\|_{L^2(S_{t,u})} + A^{\frac{1}{2}}(S_{t,u}) \lambda^{-\bar{a}+\epsilon} s^{-1} \leq \lambda^{-\bar{a}+\epsilon}$. The more general estimates of (5.163) can be proved in a similar fashion. For example,

$$\begin{aligned} \mathcal{D}_3 k_{AN} &= \nabla_3 k_{AN} + k(\nabla_3 e_A - \mathcal{D}_3 e_A, N) + k(e_A, \nabla_3 N) \\ &= \nabla_3 k_{AN} + (k_{AN} - 2\eta_A)k_{NN} - (k_{BN} - 2\eta_B)k_{AB}. \end{aligned}$$

The desired estimate then follows from the Assumption A1) for η , estimate (5.161) for k , and (5.162) for ∇k .

The decomposition $\mathbf{R}_{44} = L(z) + Error$ and its use in the transport equations requires estimates on the quantities $z, Error$ and their first derivatives. The term z is a combination of the terms of the form $r = e_4^\nu H^{\alpha\beta} \partial_\beta H_{\alpha\nu} - \frac{1}{2} H^{\alpha\beta} L(H_{\alpha\beta})$ and $Error$ is represented by the following expression

$$Error = -\frac{1}{2} e_4^\mu e_4^\nu \square_h H_{\mu\nu} + e_4^\mu e_4^\nu H^{\alpha\beta} H_{\gamma\delta} (\Gamma_{\mu\beta}^\gamma \Gamma_{\alpha\nu}^\delta - \Gamma_{\mu\nu}^\gamma \Gamma_{\alpha\beta}^\delta) - L(e_4^\nu) H^{\alpha\beta} \partial_\beta H_{\alpha\nu} - e_4^\nu L(H^{\alpha\beta}) \partial_\beta H_{\alpha\nu}.$$

⁴⁹Recall that k_A denotes either the co-vector k_{AN} or the 2-tensor k_{AB}

Lemma 5.4 *Let the functions z and $Error$ be as above. Then in the domain Γ_M , i.e., for any value of the parameter $s = t - u < s(t)$ we have*

$$|z| + \|\mathcal{D}z\|_{L^2(S_{t,u})} + \lambda^{2-a-\bar{a}} \int_0^s \sup_{S_{u+\rho,\rho}} |Error| d\rho + \int_0^s \|\mathcal{D}Error\|_{L^2(S_{u+\rho,\rho})} d\rho \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.165)$$

Proof: According to Lemma 2.1 we have pointwise bounds $|z| \lesssim |\partial H|$ and $|Error| \lesssim (\partial H)^2 + |\square_h H|$. Thus the condition (5.115) on the metric H immediately implies that $|z| \lesssim \lambda^{-\bar{a}}$. In addition, in view of (5.114), (5.117), and the Remark after Lemma 5.2,

$$\int_0^s \sup_{S_{u+\rho,\rho}} |Error| d\rho \lesssim \|\partial H\|_{L^2_{[0,t_*]} L^\infty_x}^2 + \|\square_h H\|_{L^1_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-(2-a)}.$$

It remains to obtain the estimates for the derivatives of z and $Error$.

Estimates for $\nabla(z)$, $\nabla(Error)$: It is not difficult to verify that the term $\nabla_A z$ gives rise to 3 type of terms: $e_4^\nu \nabla_A(H^{\alpha\beta}) \partial_\beta H_{\alpha\nu}$, $e_4^\nu H^{\alpha\beta} \nabla_A \partial_\beta H_{\alpha\nu}$, and $\nabla_A(e_4^\nu) H^{\alpha\beta} \partial_\beta H_{\alpha\nu}$. We analyze them separately

$$1) |e_4^\nu \nabla_A(H^{\alpha\beta}) \partial_\beta H_{\alpha\nu}| \lesssim (\partial H)^2$$

Using condition (5.115) on the metric H and the fact that the area $A(S_{t,u}) \approx s^2$ and $s \leq s(t) \leq \lambda^a$, we obtain $\|(\partial H)^2\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$.

$$2) |e_4^\nu H^{\alpha\beta} \nabla_A(\partial_\beta H_{\alpha\nu})| \lesssim |\partial^2 H|$$

It follows from (5.164) of Lemma 5.3 that $\|(\partial^2 H)\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$.

$$3) |\nabla_A(e_4^\nu) H^{\alpha\beta} \partial_\beta H_{\alpha\nu}| \lesssim \frac{1}{s} |\partial H|$$

The inequality is a consequence of the following calculation

$$\nabla_A(e_4^\nu) = \nabla_A(H^{\mu\nu} \langle e_4, \partial_\mu \rangle) = \nabla_A(H^{\mu\nu}) \langle e_4, \partial_\mu \rangle + H^{\mu\nu} \langle \mathcal{D}_A e_4, \partial_\mu \rangle + H^{\mu\nu} e_A^\gamma \langle e_4, \mathcal{D}_{\partial_\gamma} \partial_\mu \rangle.$$

In view of the frame equations $\mathcal{D}_A e_4 = \chi_{AB} e_B - k_{AN} e_4$. Also, $\mathcal{D}_{\partial_\gamma} \partial_\mu = \Gamma_{\gamma\mu}^\delta \partial_\delta$. Therefore, using the estimates $|k| + |\Gamma| \leq 4|\partial H| \leq \lambda^{-\bar{a}+\epsilon}$ and $|\chi| \leq \frac{2}{s} + 2M \leq \frac{3}{s}$ with the latter following from A3), we have

$$|\nabla_A(e_4^\nu)| \lesssim \left(\frac{1}{s} + \lambda^{-\bar{a}+\epsilon}\right) \lesssim \frac{1}{s} \quad (5.166)$$

since $s \leq s(t) \leq t \leq \lambda^a \leq \lambda^{-\bar{a}+\epsilon}$. We then have $\|s^{-1} \partial H\|_{L^2(S_{t,u})} \lesssim s^{-1} A^{\frac{1}{2}}(S_{t,u}) \lambda a^{-\bar{a}+\epsilon} \leq \lambda^{-\bar{a}+\epsilon}$.

This finishes the proof of the estimate (5.165) for ∇z .

The estimate for $\nabla_A Error$ has contributions from the following terms:

⁵⁰Recall that the notation $\nabla_A(f)$ is reserved for $e_A^i \partial_i f$, while the covariant derivative of the S -tangent tensor $\Pi_{\underline{B}}$ is denoted by $\nabla_A \Pi_{\underline{B}}$. Thus $\nabla_A(H^{\alpha\beta})$ refers to the derivatives of each fixed component $H^{\alpha\beta}$ of the space-time metric.

$$1) |e_4^\mu \nabla_A(e_4^\nu) \square_h H_{\mu\nu}| \lesssim \frac{1}{s} |\square_h H|,$$

which follows from (5.166). The condition (5.117) on the metric H and the inequality $A(S_{u+\rho,\rho}) \approx \rho^2$ yield

$$\int_0^s \|\rho^{-1} \square_h H\|_{L^2(S_{u+\rho,u})} d\rho \lesssim \int_0^s \rho^{-1} A^{\frac{1}{2}}(S_{u+\rho,u}) \sup_{S_{u+\rho,u}} |\square_h H| \lesssim \|\square_h H\|_{L^1_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-(2-a)}.$$

$$2) |e_4^\mu e_4^\nu \nabla_A \square_h H_{\mu\nu}| \leq s |\partial \square_h H|$$

The condition (5.117) and the inequality $s \leq \lambda^a$ imply that

$$\int_0^s \|\partial \square_h H\|_{L^2(S_{u+\rho,u})} d\rho \lesssim s \|\partial \square_h H\|_{L^1_{[0,t_*]} L^\infty_x} \lesssim \lambda^{-3(1-a)}.$$

Note that in the interesting range of the values of a , $a < -1 + \sqrt{3}$, we have $3(1-a) > \bar{a}$. Thus, $\lambda^{-3(1-a)} \leq \lambda^{-\bar{a}-\epsilon}$.

$$3) |e_4^\mu \nabla_A(e_4^\nu) H^{\alpha\beta} H_{\gamma\delta} \Gamma_{\mu\beta}^\gamma \Gamma_{\alpha\nu}^\delta| \lesssim \frac{1}{s} (\partial H)^2$$

Hence, from the condition (5.114), $\int_0^s \|\rho^{-1} (\partial H)^2\|_{L^2(S_{u+\rho,u})} d\rho \lesssim \|\partial H\|_{L^2_{[0,t_*]} L^\infty_x}^2 \lesssim \lambda^{-(2-a)}$.

$$4) |e_4^\mu e_4^\nu \nabla_A(H^{\alpha\beta}) H_{\gamma\delta} \Gamma_{\mu\beta}^\gamma \Gamma_{\alpha\nu}^\delta| \lesssim |\partial H|^3$$

Combining (5.114) and (5.115), we obtain $\int_0^s \|(\partial H)^3\|_{L^2(S_{u+\rho,u})} d\rho \leq s \lambda^{-\bar{a}+\epsilon} \lambda^{-(2-a)} \leq \lambda^{-(2-a)}$.

$$5) |e_4^\mu e_4^\nu H^{\alpha\beta} H_{\gamma\delta} \nabla_A(\Gamma_{\mu\beta}^\gamma) \Gamma_{\alpha\nu}^\delta| \lesssim |\partial H| |\partial^2 H|$$

We have proved in Lemma 5.3 that for any surface $S_{t,u}$ with $t-u < s(t)$ there holds the estimate $\|\partial^2 H\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$. Therefore, with the help of (5.113),

$$\int_0^s \|\partial H (\partial^2 H)\|_{L^2(S_{u+\rho,u})} d\rho \leq s \lambda^{-\bar{a}+\epsilon} \|\partial H\|_{L^1_{[0,t_*]} L^\infty_x} \leq \lambda^{-\bar{a}-(1-a)+\epsilon}.$$

$$6) |\nabla_A L(e_4^\nu) H^{\alpha\beta} \partial_\beta H_{\alpha\nu}|$$

Recall the calculation of $L(e_4^\nu)$ from Lemma 2.1, $L(e_4^\nu) = L(H^{\nu\delta}) e_{4\delta} - k_{NN} e_4^\nu + H^{\nu\delta} e_4^\omega e_{4\sigma} \Gamma_{\omega\delta}^\sigma$. Taking into account that $|\nabla e_4| \lesssim \frac{1}{s}$, we obtain $|\nabla_A L(e_4^\nu)| \lesssim |\partial^2 H| + |\partial H|^2 + \frac{1}{s} |\partial H| + |\nabla k_{NN}|$. From Lemma 5.3, $\|\nabla k_{NN}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$. Hence,

$$\int_0^s \|\nabla_A L(e_4^\nu) H^{\alpha\beta} \partial_\beta H_{\alpha\nu}\|_{L^2(S_{u+\rho,u})} d\rho \lesssim \int_0^s \sup |\partial H| \left(|\partial^2 H| + |\partial H|^2 + \frac{1}{s} |\partial H| + |\nabla k_{NN}| \right) \|_{L^2(S_{u+\rho,u})} d\rho \leq \lambda^{-\bar{a}+\epsilon}.$$

$$7) |L(e_4^\nu) \nabla_A(H^{\alpha\beta}) \partial_\beta(H_{\alpha\nu})| \lesssim |\partial H|^3$$

Thus, as in 4), $\int_0^s \|(\partial H)^3\|_{L^2(S_{u+\rho,u})} d\rho \lesssim \lambda^{-(2-a)}$.

$$8) |L(e_4^\nu)H^{\alpha\beta}\nabla_A(\partial_\beta H_{\alpha\nu})| \lesssim |\partial H||\partial^2 H|$$

Thus, as in 5), $\int_0^s \|\partial H(\partial^2 H)\|_{L^2(S_{u+\rho,u})} d\rho \leq \lambda^{-\bar{a}-(1-a)+\epsilon}$.

Estimates for the remaining derivatives of z and $Error$: The treatment of the remaining L and \underline{L} derivatives is almost identical to the proof of the estimates for the angular derivatives. The only deviation is the appearance of $L(e_4^\nu)$ or $\underline{L}(e_4^\nu)$ derivatives replacing the $\nabla_A(e_4^\nu)$ terms. We have the following identities:

$$\underline{L}(e_4^\nu) = \underline{L}(H^{\mu\nu} \langle e_4, \partial_\mu \rangle) = \underline{L}(H^{\mu\nu}) \langle e_4, \partial_\mu \rangle + H^{\mu\nu} \langle \mathcal{D}_3 e_4, \partial_\mu \rangle + H^{\mu\nu} \langle e_4, \mathcal{D}_3 \partial_\mu \rangle.$$

The frame equations imply that $\mathcal{D}_3 e_4 = 2\eta_A e_A + k_{NN} e_4$. Also, $\mathcal{D}_3 \partial_\mu = e_3^\alpha \Gamma_{\alpha\mu}^\beta \partial_b$. Hence, since $|k| + |\Gamma| \leq \lambda^{-\bar{a}+\epsilon}$, and by the Assumptions $|\eta| \leq 2^{-10} s(t)^{-1}$, we can conclude that for $s < s(t)$, $|\underline{L}(e_4^\nu)| \leq 2^{-8} s^{-1}$, which is even better than the corresponding estimate for $\nabla_A(e_4^\nu)$. A similar estimate holds for $L(e_4^\nu)$. The rest of the proof remains unchanged.

Improved elliptic estimates: Armed with the estimates of Lemma 5.3 we can refine the estimates for the Hodge systems formulated in Theorem 5.4. We show that the scalar curvature K of the surface $S_{t,u}$ can be replaced⁵¹ by the quantity s^{-2} .

Lemma 5.5 *Let ξ be an $m+1$ covariant, totally symmetric tensor, a solution of the Hodge system on the surface $S_{t,u}$*

$$\begin{aligned} \text{div} \xi &= F, \\ \text{curl} \xi &= G, \\ \text{tr} \xi &= E. \end{aligned}$$

Assume that the 2-surface $S_{t,u}$ belongs to a domain Γ_M so that $s \leq s(t)$. Then ξ obeys the estimate

$$\int_{S_{t,u}} |\nabla \xi|^2 + \frac{m+1}{2s^2} |\xi|^2 \leq 2 \int_{S_{t,u}} \{|F|^2 + |G|^2 + mK|E|^2\}. \quad (5.167)$$

Proof: The Gauss curvature K of the surface $S_{t,u}$ can be expressed in terms of the Ricci coefficients of the frame e_1, e_2, e_3, e_4 and the curvature \mathbf{R} of the ambient 4-dimensional space, see (5.155) and [Ch-Kl]. Namely, $K = -\frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} + \frac{1}{2} \hat{\chi} \cdot \hat{\underline{\chi}} + \frac{1}{2} \mathbf{R}_{ABAB}$. Recall that according to (2.50), $\underline{\chi}_{AB} = -\chi_{AB} - 2k_{AB}$. Also expanding $\text{tr} \chi = \frac{2}{s} + y$ we have

$$K = \frac{1}{s^2} + \frac{1}{s} y + \frac{1}{4} y^2 + \frac{1}{s} \text{tr} k + \frac{1}{2} y \text{tr} k - \frac{1}{2} |\hat{\chi}|^2 - \hat{\chi} \cdot k + \frac{1}{2} \mathbf{R}_{ABAB}.$$

According to the Assumption A1) $|y = \text{tr} \chi - \frac{2}{s}| + |\hat{\chi}| \leq 2M$. The Assumption A3) gives $M \leq 2^{-10} s^{-1}$, and (5.161) provides the bound $|k| \leq \lambda^{-\bar{a}+\epsilon} \leq 2^{-10} s^{-1}$, where the last inequality follows from the fact that $s < s(t) \leq t \leq t_* \leq \lambda^a$ and $\bar{a} > a$. We then derive the inequality

$$K \geq \frac{1}{2s^2} - \frac{1}{2} |\mathbf{R}_{ABAB}|. \quad (5.168)$$

⁵¹We need to do this since we do not have pointwise estimates for K .

Insert inequality (5.168) into (5.144).

$$\int_{S_{t,u}} |\nabla \xi|^2 + \frac{m+1}{2s^2} |\xi|^2 \leq \int_{S_{t,u}} \{|F|^2 + |G|^2 + mK|E|^2 + |\mathbf{R}_{ABAB}||\xi|^2\}. \quad (5.169)$$

We want to absorb the last term on the right-hand side of (5.169) into its left-hand side. First, from the Hölder inequality $\int_{S_{t,u}} |\mathbf{R}_{ABAB}||\xi|^2 \leq (\int_{S_{t,u}} |\mathbf{R}_{ABAB}|^2)^{\frac{1}{2}} (\int_{S_{t,u}} |\xi|^4)^{\frac{1}{2}}$. To handle the $L^4(S_{t,u})$ norm of ξ we make use of the Sobolev inequality (5.138) applied to $f = \xi^2$.

$$\left(\int_{S_{t,u}} |\xi|^4\right)^{\frac{1}{2}} \lesssim \left(\int_{S_{t,u}} |\nabla \xi|^2\right)^{\frac{1}{2}} \left(\int_{S_{t,u}} |\xi|^2\right)^{\frac{1}{2}} + \int_{S_{t,u}} |\text{tr}\theta||\xi|^2. \quad (5.170)$$

The identity $\text{tr}\theta = \text{tr}\chi + \text{tr}k$ implies that $0 < \text{tr}\theta \leq \frac{3}{s}$. In addition, Lemma 5.3 implies that $\|\mathbf{R}_{ABAB}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$. Hence, with the help of the Cauchy-Schwartz inequality and the estimate $s \leq s(t) \leq t \leq \lambda^a$, we have

$$\begin{aligned} \int_{S_{t,u}} |\mathbf{R}_{ABAB}||\xi|^2 &\leq \lambda^{-\bar{a}+\epsilon} \int_{S_{t,u}} (s|\nabla \xi|^2 + \frac{1}{s}|\xi|^2) \leq \lambda^{-\bar{a}+\epsilon} \lambda^a \int_{S_{t,u}} (|\nabla \xi|^2 + \frac{1}{s^2}|\xi|^2) \\ &< \frac{1}{2} \int_{S_{t,u}} (|\nabla \xi|^2 + \frac{m+1}{2s^2}|\xi|^2). \end{aligned} \quad (5.171)$$

The inequality (5.167) immediately follows.

5.7 Estimates for $\text{tr}\chi$ and $\hat{\chi}$

We are now ready to prove some of the desired estimates for the Ricci coefficients $\text{tr}\chi$ and $\hat{\chi}$.

Proposition 5.6 *Let Γ_M be the neighborhood of the interval $[0, t_*]$ of the time axis described by the Assumptions. Then the frame coefficients $\text{tr}\chi$ and $\hat{\chi}$ satisfy the following estimates at any point $(t, x) = (t, s, \omega)$ and on any surface $S_{t,u} \subset \Gamma_M$ with $s = t - u < s(t)$:*

$$|\text{tr}\chi - \frac{2}{s}| + |\hat{\chi}(s)| \leq \lambda^{-\bar{a}+\epsilon}, \quad (5.172)$$

$$\|\nabla(\text{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} + \|\nabla\hat{\chi}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.173)$$

Proof: It would clearly suffice to prove the following fact:

Lemma 5.6 *Assume that the frame coefficients $\text{tr}\chi$ and $\hat{\chi}$ satisfy the following estimates at any point $(t, x) = (t, s, \omega)$ and on any surface $S_{t,u} \subset \Gamma_M$ with $s = t - u < s(t)$:*

$$\begin{aligned} |\text{tr}\chi - \frac{2}{s}| + |\hat{\chi}(s)| &\leq M_1, \\ \|\nabla(\text{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} + \|\nabla\hat{\chi}\|_{L^2(S_{t,u})} &\leq M_2. \end{aligned}$$

with some constants $M_1, M_2 \leq 2M$. Then they also obey the estimates

$$|\operatorname{tr}\chi - \frac{2}{s}| \leq \frac{M_1}{2} + \lambda^{-\bar{a}+\epsilon}, \quad |\hat{\chi}(s)| \leq \lambda^\epsilon M_2^{1-\epsilon} + \lambda^{-\bar{a}+\epsilon}, \quad (5.174)$$

$$\|\nabla(\operatorname{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} + \|\nabla\hat{\chi}\|_{L^2(S_{t,u})} \leq \frac{M_2}{2} + \lambda^{-\bar{a}+\epsilon}. \quad (5.175)$$

Proof: We divide the proof into a sequence of steps.

Step 1. Estimate for $y = \operatorname{tr}\chi - \frac{2}{s}$

Recall the transport equation (5.147) for the quantity $y + z$

$$L(y + z) + \operatorname{tr}\chi(y + z) = \frac{1}{2}(y + z)^2 + \frac{2}{s}z - \frac{1}{2}z^2 - |\hat{\chi}|^2 - k_{NN}\operatorname{tr}\chi - \text{Error},$$

where the functions z and Error appear as a result of the decomposition $\mathbf{R}_{44} = L(z) + \text{Error}$ and obey the estimates of Lemma 5.4. In particular, $|z| \lesssim \lambda^{-\bar{a}}$ and $\|\text{Error}\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-(2-a)}$.

The condition of Lemma 5.6, $|y| = |\operatorname{tr}\chi - \frac{2}{s}| \leq M_1$ and the estimate on z imply that the quantity $y + z$ verifies the initial condition $s^2(y + z) \rightarrow 0$ as $s \rightarrow 0$. Therefore, Lemma 5.2 with $\sigma = 1$ yields the estimate at a point (t, x) with $s = t - u < s(t)$

$$|(y + z)| \leq \sup_{\rho \leq s} \rho (\operatorname{tr}\chi |k| + \frac{2}{\rho}|z|) + s \sup_{\rho \leq s} (|\hat{\chi}|^2 + |y|^2 + |z|^2) + 4\|\text{Error}\|_{L^1_{[0,t_*]}L^\infty_x}.$$

As we have remarked before, in the domain Γ_M at a point corresponding to the value of the affine parameter ρ , we have $|\operatorname{tr}\chi| \leq \frac{3}{\rho}$. In addition, the conditions of Lemma 5.6 imply that $|y| + |\hat{\chi}| \leq M_1 \leq 2^{-10}s(t)$. Thus, taking into account estimates $|k| \lesssim \lambda^{-\bar{a}}$, $|z| \lesssim \lambda^{-\bar{a}}$, and $\|\text{Error}\|_{L^1_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-(2-a)}$, we obtain⁵² for $s \leq s(t)$

$$|y + z| \leq 6\lambda^{-\bar{a}+\epsilon} + 2^{-9}M_1 + \lambda^{-(2-a)+\epsilon}.$$

Using the estimate $|z| \leq \lambda^{-\bar{a}+\epsilon}$ once again, we obtain, with a slightly larger ϵ , the desired estimate $|\operatorname{tr}\chi - \frac{2}{s}| \leq 2^{-9}M_1 + \lambda^{-\bar{a}+\epsilon}$.

Step 2. Estimate for $\nabla(\operatorname{tr}\chi - \frac{2}{s})$

The $L^2(S_{t,u})$ estimates for $\nabla(\operatorname{tr}\chi - \frac{2}{s}) = \nabla y$ are derived with the help of the transport equation (5.148)

$$\begin{aligned} \mathcal{P}_4 \nabla_A(y + z) + \frac{3}{2} \operatorname{tr}\chi \nabla_A(y + z) &= -\hat{\chi}_{AB} \nabla_B(y + z) + (y + z) \nabla_A(y + z) - (y + z) \nabla_A(\operatorname{tr}\chi) \\ &\quad + \left(\frac{2}{s} - 2z\right) \nabla_A z - 2 \nabla_A \hat{\chi} \cdot \hat{\chi} - \operatorname{tr}\chi \nabla_A k_{NN} - k_{NN} \nabla_A \operatorname{tr}\chi + \nabla_A \text{Error}, \end{aligned}$$

To apply Lemma 5.2 with $\sigma = \frac{3}{2}$ to estimate $\|\nabla(y + z)\|_{L^2(S_{t,u})}$ we need to verify the initial condition $s^2 \|\nabla(y + z)\|_{L^2(S_{t,u})} \rightarrow 0$ as $s \rightarrow 0$. The conditions of Lemma 5.6 provide the bound $\|\nabla y\|_{L^2(S_{t,u})} \leq$

⁵²Recall once again that the inequality \lesssim can be replaced by \leq by adding an extra factor of λ^ϵ

M_1 , since $y = \text{tr}\chi - \frac{2}{s}$. In addition, z obeys the estimate $\|\nabla z\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$ of Lemma 5.4. Thus, the estimate (5.159) of Lemma 5.2 implies for $s = t - u < s(t)$

$$\begin{aligned} \|\nabla(y+z)\|_{L^2(S_{t,u})} &\leq \frac{2^4}{s^4} \int_0^s \rho^4 \left(\|\nabla(y+z)\|_{L^2(S_{u+\rho,u})} (|\hat{\chi}| + |y| + |z| + |k_{NN}|) + \|\nabla Error\|_{L^2(S_{u+\rho,u})} \right. \\ &\quad \left. + \|\nabla(z)\|_{L^2(S_{u+\rho,u})} (|y| + |z| + |k_{NN}| + \frac{1}{s}) + \text{tr}\chi \|\nabla k_{NN}\|_{L^2(S_{u+\rho,u})} + |\hat{\chi}| \|\nabla \hat{\chi}\|_{L^2(S_{u+\rho,u})} \right) d\rho. \end{aligned}$$

Recall the conditions of Lemma 5.6 with $y = \text{tr}\chi - \frac{2}{s}$. We have $|y| + |\hat{\chi}| \leq M_1$ and $\|\nabla y\|_{L^2(S_{t,u})} + \|\nabla \hat{\chi}\|_{L^2(S_{t,u})} \leq M_2$ on any surface $S_{t,u}$ with $s = t - u < s(t)$ with positive constants M_1, M_2 such that $(M_1 + M_2)s(t) \leq 2^{-8}$. Invoke also the estimates for ∇z , $\nabla Error$, and ∇k_{NN} from Lemmas 5.4 and 5.3. Then

$$\|\nabla \text{tr}\chi\|_{L^2(S_{t,u})} = \|\nabla(\text{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} \leq 2^{-4}(M_2 + \lambda^{-\bar{a}+\epsilon}) + 3\lambda^{-\bar{a}+\epsilon} + 2^{-4}M_2 \leq 2^{-3}M_2 + \lambda^{-\bar{a}+\epsilon}$$

We have finished the proof of the $\text{tr}\chi$ estimates in Lemma 5.6.

Remark: Using the initial condition $s^3|\nabla \text{tr}\chi| \rightarrow 0$ as $s \rightarrow 0$ we can complement the derived $L^2(S_{t,u})$ estimate for $\nabla \text{tr}\chi$ with a less precise $L^\infty(S_{t,u})$ bound. It is not difficult to obtain from (5.148) that for $s < s(t)$,

$$\sup_{S_{t,u}} |\nabla(\text{tr}\chi - \frac{2}{s})| \leq 2^{-7}s^{-2} \int_0^s \rho |\nabla \hat{\chi}| d\rho + C(s^{-1}\lambda^{-2(1-a)} + \lambda^{-3(1-a)}). \quad (5.176)$$

This will be used as weak estimate. A weak L^∞ estimate for $\nabla \text{tr}\chi$ is needed in conjunction with a similar weak estimate for $\nabla \hat{\chi}$ to establish the strong estimate for $\|\hat{\chi}\|_{L^\infty}$.

Step 3. Estimate for $\nabla \hat{\chi}$

The main tools in the estimates for $\hat{\chi}$ are the Codazzi equation and the elliptic estimates. Recall the Codazzi equation (5.152)

$$(\text{div } \hat{\chi})_A = -\hat{\chi}_{AB}k_{BN} + \frac{1}{2}(\nabla_A \text{tr}\chi + k_{AN} \text{tr}\chi) - \mathbf{R}_{B4AB}.$$

Since $\hat{\chi}$ is a traceless 2-tensor tangent to the surfaces $S_{t,u}$, the div equation constitutes a Hodge system and allows one to obtain the bounds on any angular derivative of $\hat{\chi}$. The results of Lemma 5.5 imply that on any surface $S_{t,u}$ with $s < s(t)$

$$\int_{S_{t,u}} |\nabla \hat{\chi}|^2 + \frac{3}{2s^2} |\hat{\chi}|^2 \leq 4 \int_{S_{t,u}} |\text{div } \hat{\chi}|^2.$$

Hence, using an already proved estimate $\|\nabla \text{tr}\chi\|_{L^2(S_{t,u})} \leq 2^{-3}M_2 + \lambda^{-\bar{a}+\epsilon}$, conditions of Lemma 5.6, the estimates from Lemma 5.3, and the fact that $A(S_{t,u}) \approx s^2$, we obtain

$$\begin{aligned} \int_{S_{t,u}} |\nabla \hat{\chi}|^2 + \frac{3}{2s^2} |\hat{\chi}|^2 &\leq 4 \int_{S_{t,u}} (|\hat{\chi}|^2 |k|^2 + |\nabla \text{tr}\chi|^2 + |k|^2 |\text{tr}\chi|^2 + |\mathbf{R}_{B4AB}|^2) \\ &\leq 2^{-16} \lambda^{2(-\bar{a}+\epsilon)} + 2^{-10} M_2^2 + \lambda^{2(-\bar{a}+\epsilon)} + \lambda^{2(-\bar{a}+\epsilon)} + \lambda^{2(-\bar{a}+\epsilon)} \leq 2^{-6} M_2^2 + \lambda^{2(-\bar{a}+\epsilon)}. \end{aligned}$$

The desired estimate on $\|\nabla\hat{\chi}\|_{L^2(S_{t,u})}$ immediately follows.

Step 4. Estimate for $\hat{\chi}$

In the previous step we have proved the estimate

$$\int_{S_{t,u}} |\nabla\hat{\chi}|^2 + \frac{3}{2s^2} |\hat{\chi}|^2 \leq 2^{-6} M_2^2 + \lambda^{2(-\bar{a}+\epsilon)}.$$

We now rely on the Sobolev estimate (5.139) formulated in Theorem 5.2 to prove a pointwise bound on $\hat{\chi}$. Proposition 5.2 justify the use of Theorem 5.2 on any surface $S_{t,u}$ with $s = t - u < s(t)$. We have for any $\delta > 0$ and $p \in (2, \infty]$

$$\sup_{S_{t,u}} |\hat{\chi}| \lesssim \left[A(S_{t,u}) \right]^{\frac{\delta(p-2)}{2p+\delta(p-2)}} \left(\int_{S_{t,u}} (|\nabla\hat{\chi}|^2 + |\text{tr}\theta|^2 |\hat{\chi}|^2) \right)^{\frac{1}{2} - \frac{\delta p}{2p+\delta(p-2)}} \left[\int_{S_{t,u}} (|\nabla\hat{\chi}|^p + |\text{tr}\theta|^p |\hat{\chi}|^p) \right]^{\frac{2\delta}{2p+\delta(p-2)}},$$

Since $\text{tr}\theta = \text{tr}\chi + \text{tr}k$ we infer that $\frac{1}{s} \leq |\text{tr}\theta - \frac{2}{s}| \leq \frac{3}{s}$. Thus, setting $p = \infty$ and using the estimate $A(S_{t,u}) \approx s^2 \leq \lambda^{2a}$, we obtain

$$\sup_{S_{t,u}} |\hat{\chi}| \lesssim \lambda^{\frac{2\delta a}{2+\delta}} (2^{-3} M_2 + \lambda^{-\bar{a}+\epsilon})^{1 - \frac{2\delta}{2+\delta}} \left(\sup_{S_{t,u}} |\nabla\hat{\chi}| + \left(\frac{3}{s}\right) \sup_{S_{t,u}} |\hat{\chi}| \right)^{\frac{2\delta}{2+\delta}} \quad (5.177)$$

This is a favorable estimate for large values $s \geq 1$. To take care of the region $s \leq 1$ we proceed from the transport equation (5.150) for $\hat{\chi}$

$$\mathcal{P}_4 \hat{\chi}_{AB} + \frac{1}{2} (\text{tr}\chi) \hat{\chi}_{AB} = -k_{NN} \hat{\chi}_{AB} - \hat{\alpha}_{AB},$$

According to Lemma 5.2 and the initial condition $|\hat{\chi}| \leq M$ as $s \rightarrow 0$ we derive for $s < s(t)$

$$|\hat{\chi}(s)| \leq s(\sup |k_{NN}| |\hat{\chi}| + \sup |\hat{\alpha}|).$$

In view of the assumptions of Lemma 5.6, $|\hat{\chi}| \leq M_1 \leq 2^{-10} s(t)$. We also have $|k| \leq \lambda^{-\bar{a}+\epsilon}$ and since $\hat{\alpha}$ is a component of curvature \mathbf{R} , $|\hat{\alpha}| \leq \lambda^{-\bar{a}-(1-a)}$ by (5.115). Hence,

$$|\hat{\chi}(s)| \lesssim \lambda^{-\bar{a}+\epsilon} + s \lambda^{-\bar{a}-(1-a)}.$$

which completes the proof of Lemma 5.6 in the region $s \leq 1$. For the values of s in the interval $1 \leq s \leq s(t)$ we can use the estimate (5.177) which yields⁵³

$$\sup_{S_{t,u}} |\hat{\chi}| \lesssim \lambda^{\frac{2\delta a}{2+\delta}} (2^{-3} M_2 + \lambda^{-\bar{a}+\epsilon})^{1 - \frac{2\delta}{2+\delta}} \left(\sup_{S_{t,u}} |\nabla\hat{\chi}| + \lambda^{-\bar{a}+\epsilon} \right)^{\frac{2\delta}{2+\delta}}.$$

Since δ can be chosen to be an arbitrary positive number, the desired estimate on $\hat{\chi}$ stated in Lemma 5.6 would follow if we prove a weak estimate of the type $|\nabla\hat{\chi}| \leq \lambda^\alpha$ for some even positive power α .

Step 5. Weak estimate for $\nabla\hat{\chi}$

⁵³with the help of the inequality $\frac{1}{s} |\hat{\chi}| \lesssim \frac{1}{s} (\lambda^{-\bar{a}+\epsilon} + s \lambda^{-\bar{a}-(1-a)}) \leq \lambda^{-\bar{a}+\epsilon}$, since $s \geq 1$.

To obtain such an estimate we shall make use of the transport equation for $\nabla\hat{\chi}$ which is obtained by differentiating the transport equation for $\hat{\chi}$ and using the commutation formula of Corollary 2.1. Thus,

$$\begin{aligned} \mathcal{P}_4\nabla_C\hat{\chi}_{AB} + \text{tr}\chi\nabla_C\hat{\chi}_{AB} &= -\hat{\chi}_{CD}\nabla_D\hat{\chi}_{AB} - k_{NN}\nabla_C\hat{\chi}_{AB} - \frac{1}{2}\nabla_C(\text{tr}\chi)\hat{\chi}_{AB} - \hat{\chi}_{AB}\nabla_Ck_{NN} - \nabla_C\hat{\alpha}_{AB} \\ &\quad -(\chi_{AC}k_{DN} - \chi_{CD}k_{AN} + \mathbf{R}_{DA4C})\hat{\chi}_{DB} + (\chi_{BC}k_{DN} - \chi_{CD}k_{BN} + \mathbf{R}_{DB4C})\hat{\chi}_{DA}. \end{aligned}$$

Lemma 5.2 with $\sigma = 1$ and the initial condition $s^2|\nabla\hat{\chi}| \rightarrow 0$ as $s \rightarrow 0$ provides an estimate for the $\sup_{S_{t,u}}|\nabla\hat{\chi}|$ on any surface $S_{t,u}$ with $s = t - u < s(t)$. As in the Remark at the end of the Step 2, it is not difficult to show that

$$\sup_{S_{t,u}}|\nabla\hat{\chi}(s)| \leq 2^{-8}s^{-2} \int_0^s \rho |\nabla\text{tr}\chi| d\rho + C(s^{-1}\lambda^{-2(1-a)} + \lambda^{-3(1-a)}). \quad (5.178)$$

The only non-trivial part of the estimate (5.178) is the analysis of the term involving $\nabla_C\hat{\alpha}_{AB}$. We provide it below.

Estimate for $\nabla_C\hat{\alpha}_{AB}$: Recall that $\hat{\alpha}$ is a traceless 2-tensor tangent to the surfaces $S_{t,u}$ generated by the curvature \mathbf{R} according to the formula $\hat{\alpha}_{AB} = \mathbf{R}_{4A4B} - \frac{1}{2}\mathbf{R}_{4C4C}\delta_{AB}$. It clearly suffices to consider the term $\nabla_C\mathbf{R}_{4A4B}$. We have a simple identity relating the intrinsic angular derivative to its counterpart in the ambient space.

$$\begin{aligned} \nabla_C\mathbf{R}_{4A4B} &= \mathcal{D}_C\mathbf{R}_{4A4B} + \langle \mathbf{R}(e_A, e_B)e_A, \mathcal{D}_Ce_A \rangle + \langle \mathbf{R}(\mathcal{D}_Ce_A, e_B)e_A, e_A \rangle \\ &\quad + \langle \mathbf{R}(e_A, e_B)(\mathcal{D}_Ce_A - \nabla_Ce_A), e_A \rangle + \langle \mathbf{R}(e_A, (\mathcal{D}_Ce_B - \nabla_Ce_B)e_A, e_A \rangle. \end{aligned}$$

Hence, with the help of the frame equations (2.45)-(2.49) and the estimates $|\chi| + |k| \leq 4s^{-1}$ following from the assumptions of Lemma 5.6, we obtain

$$|\nabla_C\mathbf{R}_{4A4B}| \leq |\mathcal{D}\mathbf{R}| + 2^4s^{-1}|\mathbf{R}|.$$

The estimate (5.178) requires control of the integral $s^{-2} \int_0^s \rho^2 |\nabla_C\hat{\alpha}_{AB}|$. Since the curvature \mathbf{R} can be estimated pointwise as $|\mathbf{R}| \lesssim |\partial^2 H| + |\partial H|^2$, and $|\mathcal{D}\mathbf{R}| \lesssim |\partial^3 H| + |\partial^2 H||\partial H|$, we easily conclude from the conditions (5.113), (5.114) on the metric H that $s^{-2} \int_0^s \rho^2 |\nabla_C\hat{\alpha}_{AB}| \lesssim s^{-1}\lambda^{-2(1-a)} + \lambda^{-3(1-a)}$.

Estimates (5.176) and (5.178) combined together yield the desired weak bound

$$\sup_{S_{t,u}}|\nabla(\text{tr}\chi - \frac{2}{s})| + \sup_{S_{t,u}}|\nabla\hat{\chi}| \lesssim s^{-1}\lambda^{-2(1-a)} + \lambda^{-3(1-a)}.$$

Hence, for the values of the parameter $s \in [1, s(t)]$

$$\sup_{S_{t,u}}|\hat{\chi}| \lesssim \lambda^{\frac{2\delta a}{2+\delta}}(2^{-3}M_2 + \lambda^{-\bar{a}+\epsilon})^{1-\frac{2\delta}{2+\delta}}(\lambda^{-2(1-a)} + \lambda^{-\bar{a}+\epsilon})^{\frac{2\delta}{2+\delta}}.$$

The appropriate choice of sufficiently small constants δ and ϵ concludes the proof of the estimate for $\hat{\chi}$ and of Lemma 5.6.

5.8 Estimates for η

Proposition 5.7 *Let Γ_M be the neighborhood of the interval $[0, t_*]$ of the time axis described in the Assumptions. Then the frame coefficient η satisfy the following estimates at any point $(t, x) = (t, s, \omega)$ and any surface $S_{t,u} \subset \Gamma_M$ with $s = t - u \leq s(t)$:*

$$|\eta(s)| + \|\nabla\eta\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.179)$$

Remark: *During the proof of Proposition 5.7 we can supplement the Assumptions with the estimates for $\text{tr}\chi$ and $\hat{\chi}$ which we have already proved in Proposition 5.6.*

Proof: The proof follows the same pattern as that of Proposition 5.6. The role of $\text{tr}\chi$ is played here by $\mu = \underline{L}(\text{tr}\chi) + k_{NN}\text{tr}\chi - \frac{1}{2}(\text{tr}\chi)^2$ while η plays the role of $\hat{\chi}$. Indeed recall that μ verifies the transport equation⁵⁴ (5.149),

$$\begin{aligned} L(\mu) + \text{tr}\chi\mu &= -2\hat{\chi}_{AB} \left(2\nabla_A\eta_B - \text{div}\eta\delta_{AB} + k_{NN}\hat{\chi}_{AB} + 2(\eta_A\eta_B - |\eta|^2\delta_{AB}) \right. \\ &\quad \left. - \frac{1}{2}\text{tr}\chi\hat{\chi}_{AB} - \frac{1}{2}\text{tr}\chi\hat{\chi}_{AB} + \mathbf{R}_{A43B} \right) + \underline{L}(\mathbf{R}_{44}) \\ &\quad + 2(\underline{\eta}_A - \eta_A)\nabla_A(\text{tr}\chi) + \text{tr}\chi(|\hat{\chi}|^2 + (L - \underline{L})k_{NN} - \mathbf{R}_{44}). \end{aligned}$$

while η verifies the divergence-curl system⁵⁵ (5.153), (5.154),

$$\begin{aligned} \text{div}\eta &= \frac{1}{2}(\mu + \text{tr}(\hat{\chi} \cdot \underline{\hat{\chi}}) - 2k_{NN}\text{tr}\chi - 2|\eta|^2) - \frac{1}{2}\gamma, \\ \text{curl}\eta &= \frac{1}{2}\epsilon^{AB}\hat{\chi}_{AC}\hat{\chi}_{CB} - \frac{1}{2}\epsilon^{AB}\mathbf{R}_{A43B}. \end{aligned}$$

We shall slightly change however the scheme of the proof; first, we establish the bound for $\|\nabla\eta\|_{L^2(S_{t,u})}$ in terms of $\|\mu\|_{L^2(S_{t,u})}$ then the desired bound for the latter, and last, with the help of a weak estimate for the L^∞ norm of η and the Sobolev inequality, we prove the desired result for η .

Step 1. Estimate for $\nabla\eta$

We apply the elliptic estimates proved in Lemma 5.5 for the $\text{div} - \text{curl}$ system verified by η to derive that, on any surface $S_{t,u}$ with $s < s(t)$,

$$\int_{S_{t,u}} |\nabla\eta|^2 + \frac{1}{s^2}|\eta|^2 \leq \int_{S_{t,u}} \{|\mu|^2 + 2|\hat{\chi}|^2|\underline{\hat{\chi}}|^2 + 2|k|^2|\text{tr}\chi|^2 + |\eta|^4 + |\hat{\chi}|^2 + 2|\mathbf{R}|^2\}.$$

We have $\|\mathbf{R}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$ from Lemma 5.3. In addition, $|\hat{\chi}| + |\underline{\hat{\chi}}| + |k| \leq \lambda^{-\bar{a}+\epsilon}$, $|\text{tr}\chi| \leq 3s^{-1}$, and $|\eta| \leq M \leq 2^{-10}s^{-1}$. Thus,

$$\int_{S_{t,u}} |\nabla\eta|^2 + \frac{1}{2s^2}|\eta|^2 \leq \int_{S_{t,u}} |\mu|^2 + \lambda^{2(-\bar{a}+\epsilon)}. \quad (5.180)$$

Hence the estimate on $\|\nabla\eta\|_{L^2(S_{t,u})}$ requires control of the $L^2(S_{t,u})$ norm of μ . In the next step we shall prove the following:

⁵⁴One might be tempted to compare it with the transport equation for $y = \text{tr}\chi - \frac{2}{s}$; in fact it should be compared with the transport equation verified by the angular derivatives ∇y .

⁵⁵to be compared with Codazzi equation verified by $\hat{\chi}$.

Lemma 5.7 *Let $\mu = \underline{L}(\text{tr}\chi) + k_{NN}\text{tr}\chi - \frac{1}{2}(\text{tr}\chi)^2$. Then on any surface $S_{t,u}$ with $s < s(t)$*

$$\int_{S_{t,u}} |\mu|^2 \leq \lambda^{2(-\bar{a}+\epsilon)}$$

Lemma 5.7 immediately implies the estimate

$$\int_{S_{t,u}} |\nabla\eta|^2 + \frac{1}{2s^2}|\eta|^2 \leq \lambda^{2(-\bar{a}+\epsilon)}, \quad (5.181)$$

as desired.

Step 2. Proof of Lemma 5.7

To obtain the bound on μ in $L^2(S_{t,u})$ for any surface $S_{t,u}$ with $s < s(t)$ we use the transport equation for μ and Lemma 5.2 with $\sigma = 1$. First, however, we have to modify the equation for μ by taking advantage of the special structure of the term $\underline{L}(\mathbf{R}_{44})$. Recall that $\mathbf{R}_{44} = L(z) + \text{Error}$ with z and Error satisfying estimates of Lemma 5.4. In particular, we have

$$\|\mathcal{D}z\|_{L^2(S_{t,u})} + \int_0^s \|\mathcal{D}\text{Error}\|_{L^2(S_{u+\rho,\rho})} \leq \lambda^{-\bar{a}+\epsilon}. \quad (5.182)$$

Using the commutator formula

$$[\underline{L}, L] = \mathcal{D}_3 e_4 - \mathcal{D}_4 e_3 = 2(\eta - \underline{\eta})e_A + k_{NN}L - k_{NN}\underline{L}$$

see Lemma 2.2, we obtain

$$\underline{L}(\mathbf{R}_{44}) = L(\underline{L}(z)) + 2(\eta - \underline{\eta})\nabla_A(z) + k_{NN}(L - \underline{L})r + \underline{L}(\text{Error})$$

This leads to the following transport equation for $\mu - \underline{L}(z)$

$$L(\mu - \underline{L}(z)) + \text{tr}\chi(\mu - \underline{L}(z)) = F, \quad (5.183)$$

where F is given by

$$\begin{aligned} F = & -2\hat{\chi}_{AB} \left(2\nabla_A \eta_B - \text{d}\!i\!v \eta \delta_{AB} + k_{NN}\hat{\chi}_{AB} + 2(\eta_A \eta_B - |\eta|^2 \delta_{AB}) - \frac{1}{2}\text{tr}\chi \hat{\chi}_{AB} \right. \\ & \left. - \frac{1}{2}\text{tr}\chi \hat{\chi}_{AB} + \mathbf{R}_{A43B} \right) + 2(\eta - \underline{\eta})\nabla_A(r) + k_{NN}(L - \underline{L})(r) + \underline{L}(\text{Error}) \\ & - \text{tr}\chi \underline{L}(z) + 2(\underline{\eta}_A - \eta_A)\nabla_A(\text{tr}\chi) + \text{tr}\chi(|\hat{\chi}|^2 + (L - \underline{L})k_{NN} - \mathbf{R}_{44}). \end{aligned}$$

To apply Lemma 5.2 to equation (5.183) we need to verify the initial condition $s\|\mu - \underline{L}(z)\|_{L^2(S_{t,u})} \rightarrow 0$ as $s \rightarrow 0$. To do this it suffices to observe that $\mu = 2\text{d}\!i\!v \eta + 2k_{NN}\text{tr}\chi + 2|\eta|^2 - \text{tr}(\hat{\chi} \cdot \hat{\chi}) + R$. Hence, using the Assumptions for η , $\text{tr}\chi$, and $\hat{\chi}$, and the estimate $\|R\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$ following from Lemma 5.3, we obtain

$$s\|\mu(s)\|_{L^2(S_{t,u})} \leq s^2(4M + \lambda^{-\bar{a}+\epsilon} + 2^5 M^2 s).$$

In addition, (5.182) provides the estimate $\|\underline{L}(z)\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$, from which also $s\|\underline{L}(z)\|_{L^2(S_{t,u})} \rightarrow 0$ as $s \rightarrow 0$ as desired.

Hence, on any surface $S_{t,u}$ with $s < s(t)$, there holds the estimate

$$\|\mu - \underline{L}(r)\|_{L^2(S_{t,u})} \leq 2s^{-2} \int_0^s \rho^2 \|F\|_{L^2(S_{t,u})}.$$

The remaining part of the proof is a tedious estimate of the contributions of the each term involved in the definition of F . The following estimates are trivial consequences of estimates (5.172), (5.173) on $\text{tr}\chi$ and $\hat{\chi}$ proved in Proposition 5.6, estimate (5.182) on $\mathcal{D}(z)$, $\underline{L}(\text{Error})$, Lemma 5.3, and the estimates $|\eta(s)| + \|\nabla\eta\|_{L^2(S_{t,u})} \leq 2^{-10}s^{-1}$, which follow from the Assumptions.

- 1) $s^{-2} \int_0^s \rho^2 \|\hat{\chi} \nabla\eta\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$,
- 2) $s^{-2} \int_0^s \rho^2 \|\hat{\chi} |\eta|^2\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$
- 3) $s^{-2} \int_0^s \rho^2 \|\hat{\chi}^2 \text{tr}\chi\|_{L^2(S_{t,u})} \leq \lambda^{2(-\bar{a}+\epsilon)} s$
- 4) $s^{-2} \int_0^s \rho^2 \|\text{tr}\chi \mathbf{R}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$
- 5) $s^{-2} \int_0^s \rho^2 \|\eta \mathcal{D}(r)\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$
- 6) $s^{-2} \int_0^s \rho^2 \|\underline{L}(\text{Error})\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$
- 7) $s^{-2} \int_0^s \rho^2 \|\text{tr}\chi(L - \underline{L})k_{NN}\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$

These, allow us to conclude that $\|\mu - \underline{L}(z)\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$. Finally, using estimate (5.182) once again, we have the bound $\|\underline{L}(z)\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$, from which the conclusion of Lemma 5.7 now follows.

Step 3. Estimate for η

As in the case of the pointwise estimate for $\hat{\chi}$ we shall make use of the Sobolev estimate (5.139) of Theorem 5.2 to prove that $|\eta| \leq \lambda^{-\bar{a}+\epsilon}$. For any $\delta > 0$ and $p \in (2, \infty]$ we have ⁵⁶

$$\sup_{S_{t,u}} |\eta| \lesssim \left[A(S_{t,u}) \right]^{\frac{\delta(p-2)}{2p+\delta(p-2)}} \left(\int_{S_{t,u}} (|\nabla\eta|^2 + \frac{1}{s^2} |\eta|^2) \right)^{\frac{1}{2} \left(1 - \frac{2\delta}{2p+\delta(p-2)} \right)} \left[\int_{S_{t,u}} (|\nabla\eta|^p + \frac{1}{s^p} |\eta|^p) \right]^{\frac{2\delta}{2p+\delta(p-2)}},$$

Setting $p = \infty$ and using estimates (5.181) and $A(S_{t,u}) \lesssim \lambda^{2a}$ we infer that

$$\sup_{S_{t,u}} |\eta| \lesssim \lambda^{\frac{2\delta a}{2+\delta}} \lambda^{(-\bar{a}+\epsilon) \left(1 - \frac{2\delta}{2p+\delta(p-2)} \right)} \left(\sup_{S_{t,u}} |\nabla\eta| + \frac{1}{s} \sup_{S_{t,u}} |\eta| \right)^{\frac{2\delta}{2+\delta}} \quad (5.184)$$

Therefore, it only remains to establish some weak estimates for η and $\nabla\eta$.

Step 4. Weak estimates for η and $\nabla\eta$

Recall the transport equation (5.151) for η .

$$\mathcal{P}_4 \eta_A + \frac{1}{2} (\text{tr}\chi) \eta_A = -(k_{BN} + \eta_B) \hat{\chi}_{AB} - \frac{1}{2} (\text{tr}\chi) k_{AN} - \frac{1}{2} \beta_A$$

⁵⁶Take into account the estimate $\theta \leq \frac{3}{s}$

and apply ⁵⁷ to it Lemma 5.2 with $\sigma = \frac{1}{2}$. We have for any $s < s(t)$,

$$\sup_{S_{t,u}} |\eta(s)| \lesssim \lambda^{-\bar{a}+\epsilon} + s\lambda^{-\bar{a}-(1-a)}. \quad (5.185)$$

In the region $s \leq 1$ this already gives the desired estimate $|\eta| \leq \lambda^{-\bar{a}+\epsilon}$. To estimate $\nabla\eta$ we need to take the tangential derivatives of the above transport equation for η and apply Corollary 2.1. We have

$$\begin{aligned} \mathcal{P}_4 \nabla_B \eta_A + \text{tr}\chi \nabla_B \eta_A &= -\hat{\chi}_{BC} \nabla_C \eta_A - \hat{\chi}_{AC} \nabla_B \eta_C - \eta_C \nabla_B \hat{\chi}_{AC} - \frac{1}{2} \nabla_B (\text{tr}\chi) \eta_A \\ &\quad + \nabla_B F_A + (\chi_{AB} k_{CN} - \chi_{BC} k_{AN} + \mathbf{R}_{CA4B}) \eta_C, \end{aligned}$$

with $F_A = -k_{CN} \hat{\chi}_{AC} - \frac{1}{2} (\text{tr}\chi) k_{AN} - \frac{1}{2} \beta_A$. We now proceed as in the case of the weak estimate for $\nabla\hat{\chi}$. It is not difficult to show, by using Lemma 5.2 with $\sigma = 1$ and checking the initial condition $s^2 |\nabla\eta(s)| \rightarrow 0$ as $s \rightarrow 0$, the following estimate:

$$\sup_{S_{t,u}} |\nabla\eta| \lesssim s^{-1} \lambda^{-2(1-a)} + \lambda^{-3(1-a)}. \quad (5.186)$$

Hence, in the region $1 \leq s \leq s(t) \leq \lambda^a$ we have the estimates $\sup_{S_{t,u}} |\nabla\eta| \lesssim \lambda^{-2(1-a)}$ and $s^{-1} |\eta(s)| \lesssim s^{-1} (\lambda^{-\bar{a}+\epsilon} + s\lambda^{-\bar{a}-(1-a)}) \leq \lambda^{-\bar{a}+\epsilon}$. We then infer from (5.184) that for $1 \leq s \leq s(t)$

$$\sup_{S_{t,u}} |\eta| \lesssim \lambda^{\frac{2\delta a}{2+\delta}} \lambda^{(-\bar{a}+\epsilon)(1-\frac{2\delta}{2+\delta})} (\lambda^{-2(1-a)} + \lambda^{-\bar{a}+\epsilon})^{\frac{2\delta}{2+\delta}}.$$

Since δ can be chosen to be arbitrarily close to 0, we conclude that for all $s < s(t)$

$$\sup_{S_{t,u}} |\eta| \leq \lambda^{(-\bar{a}+\epsilon)}.$$

5.9 Remaining estimates for $\text{tr}\chi$, $\hat{\chi}$, and η

To estimate the remaining derivatives of $\text{tr}\chi$, $\hat{\chi}$, and η we invoke the equations

$$\begin{aligned} \mathcal{D}_4 (\text{tr}\chi - \frac{2}{s}) &= -\frac{1}{2} ((\text{tr}\chi)^2 - \frac{4}{s^2}) - |\hat{\chi}|^2 - k_{NN} \text{tr}\chi + \mathbf{R}_{44}, \\ \mathcal{P}_4 \hat{\chi}_{AB} &= -\frac{1}{2} (\text{tr}\chi) \hat{\chi}_{AB} - k_{NN} \hat{\chi}_{AB} + \hat{\mathbf{R}}_{A44B}, \\ \mathcal{P}_4 \eta_A &= -\frac{1}{2} (\text{tr}\chi) \eta_A - (k_{BN} + \eta_B) \hat{\chi}_{AB} - \frac{1}{2} (\text{tr}\chi) k_{AN} + \frac{1}{2} \mathbf{R}_{A443}, \\ \mathcal{D}_3 (\text{tr}\chi - \frac{2}{s}) &= 2\mathfrak{d}\eta + \text{tr}\chi k_{NN} + 2|\eta|^2 - \frac{1}{2} (\text{tr}\chi \text{tr}\underline{\chi} + \frac{4}{s^2}) + \frac{4}{s^2} (1 - b^{-1}) - \text{tr}(\hat{\chi} \cdot \hat{\chi}) + \mathbf{R}_{A43A}, \\ \mathcal{P}_3 \hat{\chi}_{AB} &= 2\mathcal{P}_A \eta_B - \mathfrak{d}\eta \delta_{AB} + k_{NN} \hat{\chi}_{AB} + 2(\eta_A \eta_B - |\eta|^2 \delta_{AB}) - \frac{1}{2} \text{tr}\underline{\chi} \hat{\chi}_{AB} - \frac{1}{2} \text{tr}\chi \hat{\chi}_{AB} + \mathbf{R}_{A43B}. \end{aligned}$$

The pointwise estimates and the $L^2(S_{t,u})$ estimates for the angular derivatives of $\hat{\chi}$, $\text{tr}\chi$, and η obtained in the previous 2 sections allow us to show that for any $s < s(t)$

⁵⁷Observe that the initial condition $s|\eta(s)| \rightarrow 0$ is satisfied in view of the Assumptions. We also make use of the estimates $|k| + |\hat{\chi}| \leq \lambda^{-\bar{a}+\epsilon}$, $|\eta(\rho)| \leq 2^{-10} \rho^{-1}$, and $|\beta| \lesssim \lambda^{-\bar{a}-(1-a)}$.

$$\begin{aligned} & \|\mathcal{D}_4(\text{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} + \|\mathcal{D}_3(\text{tr}\chi - \frac{2}{s})\|_{L^2(S_{t,u})} + \|\mathcal{P}_4\hat{\chi}\|_{L^2(S_{t,u})} \\ & + \|\mathcal{P}_3\hat{\chi}\|_{L^2(S_{t,u})} + \|\mathcal{P}_4\eta\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}. \end{aligned} \quad (5.187)$$

In addition, using the weak pointwise estimate $|\nabla\eta| \leq \lambda^{-2(1-a)}s^{-1} + \lambda^{-3(1-a)}$ for the angular derivative $\nabla\eta$, we can conclude from the equations above that

$$\begin{aligned} & |\mathcal{D}_4(\text{tr}\chi - \frac{2}{s})| + |\mathcal{P}_4\hat{\chi}| + |\mathcal{P}_4\eta| \lesssim \lambda^{-\bar{a}+\epsilon}s^{-1} + \lambda^{-\bar{a}-(1-a)}, \\ & |\mathcal{D}_3(\text{tr}\chi - \frac{2}{s})| + |\mathcal{P}_3\hat{\chi}| \lesssim \lambda^{-2(1-a)}s^{-1} + \lambda^{-3(1-a)}. \end{aligned}$$

5.10 Proof of Theorem 5.5

The Propositions 5.6 and 5.7, and the estimates (5.187) immediately imply Theorem 5.5.

5.11 The end of the proof of the Asymptotics Theorem

The Proposition 5.5 provides the most important estimates of the Asymptotics Theorem. Observe also that we have already established the weaker pointwise estimates on all the derivatives of $\text{tr}\chi$, $\hat{\chi}$, and the angular and the \mathcal{P}_4 derivatives of η of the form

$$\begin{aligned} & \sup_{S_{t,u}} |\nabla(\text{tr}\chi - \frac{2}{s})| + \sup_{S_{t,u}} |\nabla\hat{\chi}| + \sup_{S_{t,u}} |\nabla\eta| \lesssim \lambda^{-2(1-a)}s^{-1} + \lambda^{-3(1-a)}, \\ & \sup_{S_{t,u}} |\mathcal{D}_4(\text{tr}\chi - \frac{2}{s})| + \sup_{S_{t,u}} |\mathcal{P}_4\hat{\chi}| + \sup_{S_{t,u}} |\mathcal{P}_4\eta| \lesssim \lambda^{-\bar{a}+\epsilon}s^{-1} + \lambda^{-\bar{a}-(1-a)}, \\ & \sup_{S_{t,u}} |\mathcal{D}_3(\text{tr}\chi - \frac{2}{s})| + \sup_{S_{t,u}} |\mathcal{P}_3\hat{\chi}| \lesssim \lambda^{-2(1-a)}s^{-1} + \lambda^{-3(1-a)}. \end{aligned}$$

To obtain the weak estimate on the remaining \mathcal{P}_3 derivative of η we use the transport equation for $\eta_A + k_{AN}$, which can be easily deduced from (5.151),

$$\mathcal{P}_4(\eta_A + k_{AN}) + \frac{1}{2}(\text{tr}\chi)(\eta_A + k_{AN}) = -(k_{BN} + \eta_B)\hat{\chi}_{AB} + \mathcal{P}_4k_{AN} - \frac{1}{2}\beta_A =: F_A.$$

Integrating this equation we infer a weak estimate $|\eta + k_{AN}| \lesssim s\lambda^{-\bar{a}-(1-a)}$. We then derive the transport equation for $\mathcal{P}_3(\eta_A + k_{AN})$ via formula (2.77) of Corollary 2.1.

$$\begin{aligned} \mathcal{P}_4\mathcal{P}_3(\eta_A + k_{AN}) + \frac{1}{2}(\text{tr}\chi)\mathcal{P}_3(\eta_A + k_{AN}) &= -k_{NN}\mathcal{P}_3(\eta_A + k_{AN}) + 2(\eta_B - k_{BN})\nabla_B(\eta_A + k_{AN}) + \\ & k_{NN}(F_A - \frac{1}{2}\text{tr}\chi(\eta_A + k_{AN})) - \frac{1}{2}\mathcal{D}_3(\text{tr}\chi)(\eta_A + k_{AN}) + \mathcal{P}_3F_A \\ & - (2\eta_Bk_{AN} - 2\eta_Ak_{BN} + \mathbf{R}_{BA43})(\eta_B + k_{BN}) \end{aligned}$$

The most ‘‘dangerous’’ term here is $\mathcal{D}_3(\text{tr}\chi)(\eta_A + k_{AN})$ since $\mathcal{D}_3(\text{tr}\chi) \approx 2s^{-2}$ as $s \rightarrow 0$. The weak estimate for $\eta_A + k_{AN}$ however allows us to reduce the singularity at $s = 0$ to that of s^{-1} . Now

integrating the transport equation we can obtain the weak pointwise estimate $|\mathcal{D}_3(\eta + k_{AN})| \lesssim \lambda^{-3(1-a)}$. The estimate $|\mathcal{D}_3\eta| \lesssim \lambda^{-3(1-a)}$ easily follows from the estimate $|\mathcal{D}_3k_{AN}| \lesssim \lambda^{-\bar{a}-(1-a)}$, see below.

The remaining estimates for the derivatives of the lapse function can be obtained from its transport equation $\mathcal{D}_4b = -k_{NN}b$. This yields $|\mathcal{D}_4b| \leq \lambda^{-\bar{a}+\epsilon}$. The angular derivatives of b can be obtained from the equation $\eta_A = b^{-1}\nabla_A b + k_{AN}$. Finally, the \mathcal{D}_3 derivative can be estimated from its transport equations obtained with the help of identity (2.77) of Corollary 2.1.

As we already noted in the Remark after Lemma 5.3, the estimates (5.161) and (5.163) yield the most important bounds on the second fundamental form k

$$\sup_{S_{t,u}} |k| + \|\partial k_{NN}\|_{L^2(S_{t,u})} + \|\mathcal{D}k_A\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}.$$

However, following the proof of Lemma 5.3 and using already established estimates on $\text{tr}\chi$, $\hat{\chi}$, and η , it is not difficult to obtain the additional weaker pointwise estimates:

$$\begin{aligned} \sup_{S_{t,u}} |\mathcal{D}_4k_A| + \sup_{S_{t,u}} |\mathcal{D}_3k_A| &\lesssim \lambda^{-\bar{a}-(1-a)}, \\ \sup_{S_{t,u}} |\nabla k_A| &\lesssim \lambda^{-2(1-a)}s^{-1} + \lambda^{-3(1-a)}. \end{aligned}$$

6 Proof of the Boundedness Theorem

In this Chapter we give a full proof of the crucial Boundedness Theorem 3.1 stated in section 3.3. We first recall some of the basic terminology, introduced before, which shall be used throughout the Chapter.

Energy-momentum tensor: $Q_{\alpha\beta} = \partial_\alpha\psi\partial_\beta\psi - \frac{1}{2}H_{\alpha\beta}(H^{\mu\nu}\partial_\mu\psi\partial_\nu\psi)$

Null components⁵⁸ of Q :

$$Q_{44} = (\mathcal{D}_4\psi)^2, \quad Q_{34} = |\nabla\psi|^2, \quad (6.188)$$

$$Q_{33} = (\mathcal{D}_3\psi)^2, \quad Q_{3A} = \mathcal{D}_3\psi\nabla_A\psi, \quad (6.189)$$

$$Q_{4A} = \mathcal{D}_4\psi\nabla_A\psi, \quad Q_{AB} = \nabla_A\psi\nabla_B\psi + \frac{1}{2}(\mathcal{D}_3\psi\mathcal{D}_4\psi - |\nabla\psi|^2)\delta_{AB}. \quad (6.190)$$

Modified energy-momentum tensor: $\bar{Q}_{\alpha\beta} = Q_{\alpha\beta} + \frac{n-1}{4}\Omega\psi\partial_\beta\psi - \frac{n-1}{8}\psi^2\partial_\beta(\Omega)$

Modified Morawetz' vectorfield: $K = \frac{1}{2}(\underline{u}^2L + u^2\underline{L})$

Modified Deformation tensor: $\bar{\pi} = {}^K\pi - 4tH$

Conformal energy: $\mathcal{Q}_0[\psi](t) = \int_{\Sigma_t} \bar{Q}(K, \partial_t)[\psi]$, with

$$\bar{Q}(K, \partial_t)[\psi] = \frac{1}{4}(\underline{u}^2(\mathcal{D}_4\psi)^2 + u^2(\mathcal{D}_3\psi)^2 + (\underline{u}^2 + u^2)|\nabla\psi|^2) + (n-1)t\psi\partial_t\psi - \frac{n-1}{2}\psi^2,$$

Conformal identity for a solution $\square_h\psi = F$:

$$\mathcal{Q}_0[\psi](t) = \mathcal{Q}_0[\psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}\bar{\pi}_{\alpha\beta} + \frac{n-1}{2} \int_{[t_0, t] \times \mathbf{R}^3} \psi^2\square_h(t) + \int_{[t_0, t] \times \mathbf{R}^3} (K(\psi) + (n-1)t\psi)F. \quad (6.191)$$

Full conformal energy: $\mathcal{Q}[\psi](t) = \mathcal{Q}_0[\psi](t) + \mathcal{Q}_0[\partial_t\psi](t) + \mathcal{Q}_0[\partial_t^2\psi](t)$

We rewrite the Boundedness Theorem, stated in Chapter 3, in the following form:

Theorem 6.1 (Boundedness Theorem) *Let ψ be a solution of the wave equation*

$$\begin{aligned} \square_H\psi &= 0, \\ \psi|_{t=1} &= \psi_0, \quad \partial_t\psi|_{t=1} = \psi_1 \end{aligned} \quad (6.192)$$

with the metric H satisfying assumptions (5.113)- (5.117) on the time interval $[0, t_]$. Assume that the initial data (ψ_0, ψ_1) at time $t = 1$ has support in the geodesic ball $B_{\frac{1}{2}}(0)$. Then for any t_0, t , such that $1 \leq t_0 \leq t \leq t_*$,*

$$\mathcal{Q}_0[\psi](t) \lesssim \mathcal{Q}_0[\psi](t_0), \quad (6.193)$$

$$\mathcal{Q}_0[\partial_t\psi](t) \lesssim \mathcal{Q}_0[\psi](t_0) + \mathcal{Q}_0[\partial_t\psi](t_0), \quad (6.194)$$

$$\mathcal{Q}_0[\partial_t^2\psi](t) \lesssim \mathcal{Q}_0[\psi](t_0) + \mathcal{Q}_0[\partial_t\psi](t_0) + \mathcal{Q}_0[\partial_t^2\psi](t_0), \quad (6.195)$$

⁵⁸These formulae can be easily checked from the definition of Q .

Proof: The proof of the Boundedness Theorem occupies sections 6.1-6.5.

In the proof of the Theorem 6.1 we need to use the following auxiliary norms.

Basic Conformal norm: $\mathcal{E}_0[\psi](t) = \mathcal{E}_0^i[\psi](t) + \mathcal{E}_0^e[\psi](t)$, where,

$$\mathcal{E}_0^i[\psi](t) = \int_{\Sigma_t} (t^2 |\partial\psi|^2 + \psi^2)(1 - \zeta), \quad \mathcal{E}_0^e[\psi](t) = \int_{\Sigma_t} (\underline{u}^2 (\mathcal{D}_4\psi)^2 + u^2 (\mathcal{D}_3\psi)^2 + \underline{u}^2 |\nabla\psi|^2 + \psi^2) \zeta,$$

with ζ a smooth cut-off function equal to 1 in the region $u \leq \frac{t}{2}$.

The full conformal norm: $\mathcal{E}[\psi](t) = \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial\psi](t) + \mathcal{E}_0[\partial^2\psi](t)$.

We also make use of the partial conformal norms $\mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t\psi](t) + \mathcal{E}_0[\partial_t^2\psi](t)$. The following propositions are important to establish the relationship between the conformal energy $\mathcal{Q}[\psi](t)$ and conformal norm $\mathcal{E}[\psi](t)$:

Proposition 6.1 *Let ψ be a function with support in the interior of the cone C_0 and the metric H satisfy the assumptions (5.113)- (5.117) on the time interval $[0, t_*]$. Then for any $t \leq t_*$*

$$\mathcal{E}_0[\psi](t) \lesssim \mathcal{Q}_0[\psi](t). \quad (6.196)$$

As a consequence, we also have

$$\mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t\psi](t) + \mathcal{E}_0[\partial_t^2\psi](t) \lesssim \mathcal{Q}[\psi](t). \quad (6.197)$$

Proof: See section 6.1.

Proposition 6.2 *Let ψ be a solution of the wave equation $\square_h\psi = 0$ under the assumptions of the Boundedness Theorem. Then for any $t \leq t_*$ the full conformal norm of ψ , $\mathcal{E}[\psi](t) = \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial\psi](t) + \mathcal{E}_0[\partial^2\psi](t)$ obeys the estimate*

$$\mathcal{E}[\psi](t) \lesssim \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t\psi](t) + \mathcal{E}_0[\partial_t^2\psi](t) \quad (6.198)$$

As a corollary of the last two propositions we have,

$$\mathcal{E}[\psi](t) \lesssim \mathcal{Q}[\psi](t) \quad (6.199)$$

Proof: See section 6.5.

6.1 Preliminaries

We start with a Lemma in which we calculate the components of the modified deformation tensor $\bar{\pi}$ relative to an arbitrary null frame.

Lemma 6.1 *Relative to an arbitrary null frame e_A, e_3, e_4 , $A = 1, 2$, complementing the null pair $L = e_4$, $\underline{L} = e_3$, the modified deformation tensor $\bar{\pi} = {}^K\pi - 4tH$ has the following components:*

$$\bar{\pi}_{44} = -2u^2 k_{NN}, \quad \bar{\pi}_{34} = 4u(1 - b^{-1}) + (u^2 + \underline{u}^2)k_{NN}, \quad (6.200)$$

$$\bar{\pi}_{33} = -8\underline{u}(1 - b^{-1}) - \underline{u}^2 k_{NN}, \quad \bar{\pi}_{3A} = u^2(k_{AN} - \eta_A) + \underline{u}^2(\eta_A + k_{AN}), \quad (6.201)$$

$$\bar{\pi}_{4A} = -2u^2 k_{AN}, \quad \bar{\pi}_{AB} = 2t(t - u)\left(\text{tr}\chi - \frac{2}{s}\right)\delta_{AB} - 2u^2 \text{tr}k\delta_{AB} + u^2 \underline{\chi}_{AB} + \underline{u}^2 \hat{\chi}_{AB}. \quad (6.202)$$

Proof: The deformation tensor ${}^K\pi$ evaluated on the vectors X, Y can be found from the formula

$$\begin{aligned} {}^K\pi(X, Y) &= \langle \mathcal{D}_X K, Y \rangle + \langle \mathcal{D}_Y K, X \rangle = \\ &X \langle K, Y \rangle + Y \langle K, X \rangle - \langle K, \mathcal{D}_X Y \rangle - \langle K, \mathcal{D}_Y X \rangle. \end{aligned}$$

Therefore, with the help of the structure equations (2.45)-(2.52)

$$\begin{aligned} {}^K\pi_{44} &= 2 \langle \mathcal{D}_4 K, e_4 \rangle = 2\mathcal{D}_4 \langle K, e_4 \rangle - 2k_{NN} \langle K, e_4 \rangle = \\ &- 2\mathcal{D}_4(u^2) + 2u^2 k_{NN} = 2u^2 k_{NN}, \end{aligned}$$

with the last equality holding since $e_4(u) = 0$.

$$\begin{aligned} {}^K\pi_{34} &= \langle \mathcal{D}_4 K, e_3 \rangle + \langle \mathcal{D}_3 K, e_4 \rangle = -\mathcal{D}_4(\underline{u}^2) - \mathcal{D}_3(u^2) + \underline{u}^2 k_{NN} + u^2 k_{NN}, \\ {}^K\pi_{33} &= 2 \langle \mathcal{D}_3 K, e_3 \rangle = -2\mathcal{D}_3(\underline{u}^2) - 2\underline{u}^2 k_{NN}, \\ {}^K\pi_{3A} &= \langle \mathcal{D}_3 K, e_A \rangle + \langle \mathcal{D}_A K, e_3 \rangle = -\mathcal{D}_A(\underline{u}^2) + \underline{u}^2 \eta_A + u^2 \underline{\xi}_A + \underline{u}^2 k_{AN} = \\ &\underline{u}^2(\eta_A + k_{AN}) + u^2 \underline{\xi}_A, \end{aligned}$$

since $\underline{u} = -u + 2t$ and $e_A(u) = e_A(t) = 0$.

$$\begin{aligned} {}^K\pi_{4A} &= \langle \mathcal{D}_4 K, e_A \rangle + \langle \mathcal{D}_A K, e_4 \rangle = -\mathcal{D}_A(u^2) + u^2 \underline{\eta}_A - u^2 k_{AN} = u^2(\underline{\eta}_A - k_{AN}), \\ {}^K\pi_{AB} &= 2 \langle \mathcal{D}_A K, e_B \rangle = \underline{u}^2 \chi_{AB} + u^2 \underline{\chi}_{AB}. \end{aligned}$$

We now compute $\mathcal{D}_3(u^2)$, $\mathcal{D}_3(\underline{u}^2)$, and $\mathcal{D}_4(\underline{u}^2)$.

$$\begin{aligned} \mathcal{D}_3(u^2) &= -\mathcal{D}_4(u)^2 + 2T(u^2) = 4b^{-1}u, \\ \mathcal{D}_3(\underline{u}^2) &= 2\underline{u}\mathcal{D}_3(-u + 2t) = -4b^{-1}\underline{u} + 4u(T - N)(t) = 4u(1 - b^{-1}), \\ \mathcal{D}_4(\underline{u}^2) &= 4\underline{u}. \end{aligned} \quad (6.203)$$

Let us rewrite ${}^K\pi$ taking into account (2.50)-(2.52), (6.203), and the identity $\underline{u} = -u + 2t$.

$$\begin{aligned} {}^K\pi_{\mathbf{44}} &= -2u^2k_{NN}, & {}^K\pi_{\mathbf{34}} &= -8t + 4u(1 - b^{-1}) + (u^2 + \underline{u}^2)k_{NN}, \\ {}^K\pi_{\mathbf{33}} &= -8\underline{u}(1 - b^{-1}) - \underline{u}^2k_{NN}, & {}^K\pi_{\mathbf{3A}} &= u^2(k_{AN} - \eta_A) + \underline{u}^2(\eta_A + k_{AN}), \\ {}^K\pi_{\mathbf{4A}} &= -2u^2k_{AN}, & {}^K\pi_{\mathbf{AB}} &= 2t(t - u)\text{tr}\chi\delta_{AB} - 2u^2\text{tr}k\delta_{AB} + u^2\hat{\chi}_{AB} + \underline{u}^2\hat{\chi}_{AB}. \end{aligned}$$

The components $H_{\mathbf{34}} = -2$ and $H_{AA} = 1$ are the only non zero components of the metric H in the null frame. Subtracting $4tH$ from the deformation tensor ${}^K\pi$ we obtain the desired expression for $\bar{\pi}$.

We next record the following integration by parts formulae.

Lemma 6.2 *Let X be a vectorfield tangent to the hyperplane Σ_t . Denote by ∇ the covariant derivative obtained by restricting \mathcal{D} to Σ_t . Then for any smooth functions $F, G : \Sigma_t \rightarrow \mathbf{R}$*

$$\int_{\Sigma_t} FX(G) = - \int_{\Sigma_t} (X(F) + \text{div}XF)G. \quad (6.204)$$

In particular,

$$\int_{\Sigma_t} FN(G) = - \int_{\Sigma_t} (N(F) + \text{tr}\theta)G. \quad (6.205)$$

In addition, if V is a vectorfield in $T\Sigma_t$ tangent to the surfaces $S_{t,u}$ then

$$\int_{\Sigma_t} F \mathcal{D}\text{iv}V = - \int_{\Sigma_t} (\nabla F + b^{-1}\nabla bF) \cdot V. \quad (6.206)$$

Proof: The Identity (6.204) follows easily once we choose geodesic coordinates in the neighborhood of a given point x_0 so that $h_{ij}(x_0) = \delta_{ij}$, $\partial h(x_0) = 0$, and $\text{div}X = \partial_i X^i$. To compute $\text{div}N$ in (6.205) we use the orthonormal frame e_A, N , $A = 1, 2$. Then $\text{div}N = \langle \nabla_A N, e_A \rangle = \text{tr}\theta$. Finally, recall that the metric h has the form $h = b^2 du^2 + \gamma_{AB} d\phi^A d\phi^B$, where ϕ^A are local coordinates on the surfaces $S_{t,u}$, see Lemma 5.1. As a consequence,

$$\int_{\Sigma_t} F = \int_{S_{t,u}} b \left(\int F \right) du, \quad \text{coarea formula}, \quad (6.207)$$

from which the identity (6.206) is now an obvious consequence.

We shall next make use of the above integration by parts formulae to prove Proposition 6.1.

Proof of Proposition 6.1: Define two additional vectorfields S and \underline{S} .

$$S = \frac{1}{2}(\underline{u}e_4 + ue_3), \quad \underline{S} = \frac{1}{2}(\underline{u}e_4 - ue_3). \quad (6.208)$$

Then since $\underline{u} = -u + 2t$, $e_3 = \partial_t - N$, $e_4 = \partial_t + N$

$$\begin{aligned} t\partial_t &= \frac{1}{2}(u + \underline{u})\partial_t = S - \frac{1}{4}(u - \underline{u})(e_4 - e_3) = S - (t - u)N, \\ t\partial_t &= \frac{1}{2}t(e_3 + e_4) = \frac{t}{t - u}\underline{S} - \frac{t^2}{t - u}N. \end{aligned}$$

Therefore, with the help of the identities (6.205), and $N(t) = 0$, $N(u) = -b^{-1}$

$$\begin{aligned} (n-1) \int_{\Sigma_t} \psi t \partial_t \psi &= (n-1) \int_{\Sigma_t} (\psi(S\psi) - \frac{1}{2}(t-u)N(\psi^2)) \\ &= (n-1) \int_{\Sigma_t} \psi(S\psi) + \frac{n-1}{2} \int_{\Sigma_t} (\text{tr}\theta(t-u) + b^{-1})\psi^2, \\ (n-1) \int_{\Sigma_t} \psi t \partial_t \psi &= (n-1) \int_{\Sigma_t} (\psi \frac{t}{t-u}(\underline{S}\psi) - \frac{1}{2} \frac{t^2}{t-u}N(\psi^2)) \\ &= (n-1) \int_{\Sigma_t} \psi \frac{t}{t-u}(\underline{S}\psi) + \frac{n-1}{2} \int_{\Sigma_t} \frac{t^2}{(t-u)^2} (\text{tr}\theta(t-u) - b^{-1})\psi^2. \end{aligned}$$

The second fundamental form $\theta_{AB} = \chi_{AB} + k_{AB}$. According to the Asymptotics Theorem 5.1, with $n = 3$, we have $|\text{tr}\chi - \frac{n-1}{t-u}| \leq \lambda^{-\bar{a}+\epsilon}$, $|k| \leq \lambda^{-\bar{a}+\epsilon}$, and $|b-1| \lesssim \lambda^{-(1-a)}$. Therefore,

$$\begin{aligned} (n-1) \int_{\Sigma_t} \psi t \partial_t \psi &= n-1 \int_{\Sigma_t} (\psi(S\psi) + \frac{n(n-1)}{2} \int_{\Sigma_t} (1 + O(\lambda^{-\bar{a}+a+\epsilon}))\psi^2, \\ (n-1) \int_{\Sigma_t} \psi t \partial_t \psi &= n-1 \int_{\Sigma_t} (\psi \frac{t}{t-u}(\underline{S}\psi) + \frac{(n-1)(n-2)}{2} \int_{\Sigma_t} \frac{t^2}{(t-u)^2} (1 + O(\lambda^{-\bar{a}+a+\epsilon}))\psi^2. \end{aligned}$$

Since

$$\bar{Q}(K, \partial_t)[\psi] = \frac{1}{4}(\underline{u}^2(\mathcal{D}_4\psi)^2 + u^2(\mathcal{D}_3\psi)^2 + (\underline{u}^2 + u^2)|\nabla\psi|^2) + (n-1)t\psi\partial_t\psi - \frac{n-1}{2}\psi^2,$$

and

$$\frac{1}{4}(\underline{u}^2(\mathcal{D}_4\psi)^2 + u^2(\mathcal{D}_3\psi)^2) = \frac{1}{2}((S\psi)^2 + (\underline{S}\psi)^2)$$

we can introduce positive constants $A, B : A + B = n - 1$ such that

$$\begin{aligned} \mathcal{Q}_0[\psi](t) &= \frac{1}{2} \int_{\Sigma_t} \left\{ \left((S\psi)^2 + 2A\psi(S\psi) + (An - 1 + O(\lambda^{-\bar{a}+a+\epsilon}))\psi^2 \right) \right. \\ &\quad \left. + \left((\underline{S}\psi)^2 + 2B\psi \frac{t}{t-u}(\underline{S}\psi) + (B(n-2) + O(\lambda^{-\bar{a}+a+\epsilon})) \frac{t^2}{(t-u)^2} \psi^2 \right) + (\underline{u}^2 + u^2)|\nabla\psi|^2 \right\}. \end{aligned}$$

For any values of A, B such that $1 < A < n - 1$ and $0 < B < n - 2$ it is possible to find positive constants c_1, c_2 such that

$$\begin{aligned} (S\psi)^2 + 2A\psi(S\psi) + (An - 1)\psi^2 &\geq c_1((S\psi)^2 + \psi^2), \\ (\underline{S}\psi)^2 + 2B\psi \frac{t}{t-u}(\underline{S}\psi) + B(n-2) \frac{t^2}{(t-u)^2} \psi^2 &\geq c_2((\underline{S}\psi)^2 + \frac{t^2}{(t-u)^2} \psi^2). \end{aligned}$$

The above constraints on A and B can be satisfied for any $n \geq 3$. Therefore, since $\lambda^{-\bar{a}+a+\epsilon}$ is arbitrarily small for all sufficiently large values of λ , we conclude that

$$\mathcal{Q}_0[\psi](t) \gtrsim \int_{\Sigma_t} (\underline{u}^2 (\mathcal{D}_4 \psi)^2 + u^2 (\mathcal{D}_3 \psi)^2 + (u^2 + \underline{u}^2) |\nabla \psi|^2 + \psi^2).$$

It remains to note that $\underline{u} = -u + 2t = t + s \geq t$.

The following Lemma will be very useful throughout this section.

Lemma 6.3 *Let $V, w : \mathbb{R}^3 \rightarrow \mathbb{R}$ be smooth functions defined in the region $Ext_\tau = \{x : \frac{\tau}{4} \leq s \leq \tau\}$ with $\tau : 1 \leq \tau \leq \lambda^a$. For any $p : 1 \leq p < \infty$,*

$$\begin{aligned} \int_{Ext_\tau} V^2 w^2 &\lesssim \tau^{\frac{2}{p}} \sup_{Ext_\tau} |V|^{\frac{2}{p}} \sup_{0 \leq u \leq \frac{3\tau}{4}} \|V\|_{L^2(S_{\tau,u})}^{\frac{2}{p}} (\|\nabla w\|_{L^2(\Sigma_\tau)}^2 + \frac{1}{\tau^2} \|\psi\|_{L^2(\Sigma_\tau)}^2) \\ &\lesssim \tau^{\frac{2}{p}-2} \sup_{Ext_\tau} |V|^{\frac{2}{p}} \sup_{0 \leq u \leq \frac{3\tau}{4}} \|V\|_{L^2(S_{\tau,u})}^{\frac{2}{p}} \mathcal{E}_0[w](\tau). \end{aligned} \quad (6.209)$$

Remark: *The best rate of decay in τ in the above lemma is attained for large values of p . We shall apply Lemma 6.3 in situations when all the corresponding norms of V are bounded by powers of λ . Thus choosing p to be sufficiently large we can write (6.209) in the simplified form.*

$$\int_{Ext_\tau} V^2 w^2 \leq \tau^{-2} \lambda^\epsilon \sup_{0 \leq u \leq \frac{3\tau}{4}} \|V\|_{L^2(S_{\tau,u})}^2 \mathcal{E}_0[w](\tau). \quad (6.210)$$

Proof: Start with the coarea formula (6.207), $\int_{Ext_\tau} V^2 w^2 = \int_{\frac{\tau}{4}}^\tau b \left(\int_{S_{\tau,u}} V^2 w^2 \right) du$. and apply to it the

Hölder inequality for some $p \geq 1$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\int_{S_{\tau,u}} V^2 w^2 \lesssim \left(\int_{S_{\tau,u}} |V|^{2p'} \right)^{\frac{1}{p'}} \left(\int_{S_{\tau,u}} |w|^{2p} \right)^{\frac{1}{p}}. \quad (6.211)$$

Recall⁵⁹ the isoperimetric inequality (5.138), $\left(\int_{S_{\tau,u}} |f|^2 \right)^{\frac{1}{2}} \lesssim \int_{S_{\tau,u}} (|\nabla f| + |\text{tr}\theta||f|)$. Substituting $f = f^p$ and applying Hölder inequality once more we obtain for any $1 \leq p < \infty$

$$\left(\int_{S_{\tau,u}} |f|^{2p} \right)^{\frac{1}{2p}} \lesssim \left(\int_{S_{\tau,u}} |\nabla f|^{\frac{2p}{p+1}} + |\text{tr}\theta|^{\frac{2p}{p+1}} |f|^{\frac{2p}{p+1}} \right)^{\frac{p+1}{2p}}. \quad (6.212)$$

⁵⁹stated in Theorem 5.2

Apply (6.212) to the function w in (6.211).

$$\begin{aligned} \int_{S_{\tau,u}} V^2 w^2 &\lesssim \left(\int_{S_{\tau,u}} |V|^{2p'} \right)^{\frac{1}{p'}} \left(\int_{S_{t',u}} |\nabla w|^{\frac{2p}{p+1}} + |\operatorname{tr}\theta|^{\frac{2p}{p+1}} |w|^{\frac{2p}{p+1}} \right)^{\frac{p+1}{p}} \\ &\lesssim \sup_{S_{\tau,u}} |V|^{2\frac{p'-1}{p'}} \left(\int_{S_{\tau,u}} |V|^2 \right)^{\frac{1}{p'}} \left(\int_{S_{\tau,u}} 1 \right)^{\frac{1}{p}} \left(\int_{S_{\tau,u}} |\nabla w|^2 + |\operatorname{tr}\theta|^2 |w|^2 \right). \end{aligned}$$

Note that $\frac{p'-1}{p'} = p^{-1}$, $\left| \int_{S_{\tau,u}} 1 \right| \lesssim s^2 \leq \tau^2$, and $\operatorname{tr}\theta \leq 3s^{-1}$. Therefore,

$$\int_{Ext_{\tau}} V^2 w^2 \lesssim \tau^{\frac{2}{p}} \sup |V|^{\frac{2}{p}} \sup_u \left(\int_{S_{\tau,u}} |V|^2 \right)^{\frac{1}{p'}} \int_{Ext_{\tau}} (|\nabla w|^2 + \frac{1}{\tau^2} |w|^2)$$

as desired.

We now formulate a crucial lemma, based on integration by parts.

Lemma 6.4 *Let ψ be a solution of the equation $\square_h \psi = F$ and G a function verifying the following properties:*

$$\|G\|_{L^\infty(S_{\tau,u})} + \|\mathcal{D}_3 G\|_{L^2(S_{\tau,u})} + \|\nabla G\|_{L^2(S_{\tau,u})} \leq \tau^2 \lambda^{-\bar{a}+\epsilon}. \quad (6.213)$$

Then, for $1 \leq t_0 \leq t \leq \lambda^a$ and a cut-off function ζ equal to 1 in the region $s = \tau - u \geq \frac{\tau}{2}$, we have

$$\int_{[t_0,t] \times \mathbf{R}^3} G \mathcal{D}_4 \psi \mathcal{D}_3 \psi \zeta = \int_{[t_0,t] \times \mathbf{R}^3} G F \psi \zeta + \lambda^{-\epsilon} \sup_{\tau \in [t_0,t]} \mathcal{E}[\psi](\tau). \quad (6.214)$$

In addition,

$$\left| \int_{[t_0,t] \times \mathbf{R}^3} \tau^{-1} G \mathcal{D}_3 \psi \psi \zeta \right| \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0,t]} \mathcal{E}[\psi](\tau). \quad (6.215)$$

Proof: We shall use the convention $\int_{[t_0,t] \times \mathbf{R}^3} = \int_{[t_0,t] \times \mathbf{R}^3} \sqrt{\det h} dx d\tau$. Recall that $\partial_t \sqrt{\det h} =$

$\operatorname{Tr} k \sqrt{\det h}$, where $\operatorname{Tr} k = h^{ij} k_{ij}$. We also have $e_3 = T - N = \partial_t$. Integrating by parts first with respect to ∂_t and also with respect to N , with the help of Lemma 6.2, we obtain

$$\begin{aligned} \int_{[t_0,t] \times \mathbf{R}^3} G \mathcal{D}_3 \psi \mathcal{D}_4 \psi \zeta &= - \int_{[t_0,t] \times \mathbf{R}^3} G (\mathcal{D}_3 \mathcal{D}_4 \psi) \psi \zeta \\ &+ \int_{[t_0,t] \times \mathbf{R}^3} \left(-\mathcal{D}_3(G \zeta) + (\operatorname{tr}\theta - \operatorname{Tr} k) G \zeta \right) \mathcal{D}_4 \psi \psi + \int_{\Sigma_t} G \psi \mathcal{D}_4 \psi \zeta - \int_{\Sigma_{t_0}} G \psi \mathcal{D}_4 \psi \zeta. \end{aligned}$$

Using the estimates $|G(\tau)| \leq \tau^2 \lambda^{-\bar{a}+\epsilon}$, $|\mathcal{D}\zeta(\tau)| \leq 5\tau^{-1}$, $|\text{tr}\theta| \leq \frac{3}{s}$, and $|k| \leq \lambda^{-\bar{a}+\epsilon}$ we can easily take care of the boundary terms, the term involving $\mathcal{D}_3(\zeta)$, as well as the term involving $(\text{tr}\theta - \text{Tr } k)$. We derive the formula:

$$\int_{[t_0, t] \times \mathbf{R}^3} G \mathcal{D}_3 \psi \mathcal{D}_4 \psi \zeta = - \int_{[t_0, t] \times \mathbf{R}^3} \mathcal{D}_3 G \mathcal{D}_4 \psi \psi \zeta - \int_{[t_0, t] \times \mathbf{R}^3} G \mathcal{D}_3 \mathcal{D}_4 \psi \psi \zeta + \text{l.o.t.}, \quad (6.216)$$

where the lower order terms l.o.t. can be estimated by $|\text{l.o.t.}| \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau)$. We shall show in fact that the remaining terms on the right hand side are also of the form of lower order terms.

1) *Estimate for* $\int_{[t_0, t] \times \mathbf{R}^3} \mathcal{D}_3 G \mathcal{D}_4 \psi \psi \zeta$: First apply Cauchy-Schwartz to write,

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} \mathcal{D}_3 G \mathcal{D}_4 \psi \psi \zeta \right| \lesssim \int_{[t_0, t]} \tau^{-1} \|\mathcal{D}_3 G \psi \zeta\|_{L^2(\Sigma_\tau)} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau).$$

We apply⁶⁰ Lemma 6.3 for $V = \zeta \mathcal{D}_3 G$ and $w = \psi$ to obtain,

$$\|\mathcal{D}_3 G \psi \zeta\|_{L^2(\Sigma_\tau)} \leq \tau^{-2} \lambda^\epsilon \sup_{0 \leq u \leq \frac{3\tau}{4}} \|\mathcal{D}_3 G\|_{L^2(S_{\tau, u})} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau) \leq \lambda^{-\bar{a}+\epsilon} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau)$$

where we have used the hypothesis $\|\mathcal{D}_3 G\|_{L^2(S_{\tau, u})} \leq \tau^2 \lambda^{-\bar{a}+\epsilon}$ of Lemma 6.4. Consequently,

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} \mathcal{D}_3 G \mathcal{D}_4 \psi \psi \zeta \right| \lesssim \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau),$$

as desired.

2) *Estimate for* $\int_{[t_0, t] \times \mathbf{R}^3} G (\mathcal{D}_3 \mathcal{D}_4 \psi) \psi \zeta$: We shall use now the fact that ψ is a solution of the

wave equation. This allows us to express the $\mathcal{D}_3 \mathcal{D}_4$ derivative of ψ in terms of angular derivatives, the right hand-side F , and lower order terms. Expressed relative to a null frame the wave operator $\square_h \psi$ takes the form

$$\square_h \psi = H^{\alpha\beta} \psi_{;\alpha\beta} = -\psi_{;\mathbf{43}} + \psi_{;\mathbf{AA}},$$

where $\psi_{;e_i e_j} = e_j(e_i(\psi)) - \mathcal{D}_{e_i} e_j(\psi)$. We use the Ricci formulas: $\mathcal{D}_3 e_4 = 2\eta_A e_A + k_{NN} e_4$, and $\mathcal{D}_B e_A = \nabla_B e_A + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4$ to derive

$$\square_h \psi = -\mathcal{D}_3 \mathcal{D}_4 \psi + \mathcal{A} \psi + 2\eta_A \nabla_A \psi + \frac{1}{2} \text{tr} \chi \mathcal{D}_3 \psi + \left(\frac{1}{2} \text{tr} \underline{\chi} + k_{NN}\right) \mathcal{D}_4 \psi. \quad (6.217)$$

⁶⁰We apply in fact the Remark following Lemma 6.3.

As a result of this calculation

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} G \mathcal{D}_3 \mathcal{D}_4 \psi \psi \zeta &= \int_{[t_0, t] \times \mathbf{R}^3} G \psi \zeta \left(\mathbb{A} \psi + 2\eta_A \nabla_A \psi + \frac{1}{2} \text{tr} \chi \mathcal{D}_3 \psi \right. \\ &\quad \left. + \left(\frac{1}{2} \text{tr} \underline{\chi} + k_{NN} \right) \mathcal{D}_4 \psi - F \right). \end{aligned} \quad (6.218)$$

Take now into account the estimates $|\eta| + |k| + \tau^{-2}|G| \leq \lambda^{-\bar{a}+\epsilon}$ and $|\text{tr} \chi| + |\text{tr} \underline{\chi}(s)| \leq \frac{6}{s}$. Note also that, for $0 \leq u \leq \frac{3\tau}{4}$, the values of $s = \tau - u$ lie in the interval $s \in [\frac{\tau}{4}, \tau]$. Using these we easily derive,

$$\begin{aligned} \left| \int_{[t_0, t] \times \mathbf{R}^3} G \left(2\eta_A \nabla_A \psi + \left(\frac{1}{2} \text{tr} \underline{\chi} + k_{NN} \right) \mathcal{D}_4 \psi \right) \psi \zeta \right| &\leq \lambda^{-2(\bar{a}+\epsilon)} t \int_{[t_0, t] \times \mathbf{R}^3} (\tau^2 |\mathcal{D}_4 \psi|^2 + \tau^2 |\nabla \psi|^2 + |\psi|^2) \\ &\leq \lambda^{-\bar{a}+\epsilon} \int_{[t_0, t]} \mathcal{E}_0[\psi](\tau), \end{aligned}$$

as desired. The terms with $\mathcal{D}_3 \psi$ and $\mathbb{A} \psi$ have to be once more integrated by parts.

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} G \text{tr} \chi \mathcal{D}_3 \psi \psi \zeta &= \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} \left(-\mathcal{D}_3(G \text{tr} \chi \zeta) + (\text{tr} \theta - \text{Tr} k) G \zeta \right) \text{tr} \chi (\psi)^2 \\ &\quad + \frac{1}{2} \int_{\Sigma_t} G \text{tr} \chi (\psi)^2 \zeta - \frac{1}{2} \int_{\Sigma_{t_0}} G \text{tr} \chi (\psi)^2 \zeta \end{aligned}$$

Clearly, $|G \text{tr} \chi \zeta(\tau)| \leq \lambda^{-\bar{a}+\epsilon} \tau$ and $|\text{tr} \chi (\text{tr} \theta - \text{Tr} k) G \zeta| \leq \lambda^{-\bar{a}+\epsilon}$. Thus,

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} G \text{tr} \chi \mathcal{D}_3 \psi \psi \zeta \right| \leq \frac{1}{2} \int_{[t_0, t]} \|\mathcal{D}_3(G \text{tr} \chi \zeta) \psi\|_{L^2(\Sigma_\tau)} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau) + \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau).$$

We shall now apply Lemma 6.3 to the remaining integral on the right-hand side with $V = \mathcal{D}_3(G \text{tr} \chi \zeta)$ and $w = \psi$. In view of the Asymptotics Theorem 5.1 we have the estimate $\|\mathcal{D}_3(\text{tr} \chi - \frac{2}{s})\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}$. Using our assumptions on G and the estimates $|\mathcal{D}_3 s| = |1 - 2a^{-1}| \leq 2$, $|\mathcal{D} \zeta| \leq 5\tau^{-1}$, we easily conclude on any surface $S_{\tau,u}$ with $0 \leq u \leq \frac{3\tau}{4}$ and $1 \leq \tau \leq t_*$, $\|\mathcal{D}_3(G \text{tr} \chi \zeta)\|_{L^2(S_{\tau,u})} \leq \lambda^{-\bar{a}+\epsilon} \tau$. Using Lemma 6.3 we infer that $\|\mathcal{D}_3(G \text{tr} \chi \zeta) \psi\|_{L^2(\Sigma_\tau)} \lesssim \lambda^{\epsilon+\frac{\epsilon}{2}} \lambda^{a-\bar{a}} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau) \leq \lambda^{-\epsilon} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau)$, and therefore

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} G \text{tr} \chi \mathcal{D}_3 \psi \psi \zeta \right| \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau).$$

as desired.

Finally we estimate the last term in (6.1) by integration by parts as follows:

$$\int_{[t_0, t] \times \mathbf{R}^3} G \mathbb{A} \psi \psi \zeta = - \int_{[t_0, t] \times \mathbf{R}^3} G |\nabla \psi|^2 - \int_{[t_0, t] \times \mathbf{R}^3} b^{-1} \nabla_A (b G \zeta) \nabla_A \psi \psi.$$

The first integral can be easily estimated by $\lambda^{-\bar{a}+\epsilon}\lambda^a \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau) \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi]$, as desired. To estimate the second we use Lemma 6.3 once more.

$$\begin{aligned} \left| \int_{[t_0, t] \times \mathbf{R}^3} b^{-1} \nabla_A (b G \zeta) \nabla_A \psi \psi \right| &\leq \int_{[t_0, t]} \tau^{-1} \|b^{-1} \nabla_A (b G \zeta)\|_{L^2(\Sigma_\tau)} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau) \\ &\leq \int_{[t_0, t]} \tau^{-2} \sup_{u \in [0, \frac{3\tau}{4}]} \|b^{-1} \nabla_A (b G \zeta)\|_{L^2(S_{\tau, u})} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau). \end{aligned}$$

It thus suffices to show that $\tau^{-2} \sup_{u \in [0, \frac{3\tau}{4}]} \|b^{-1} \nabla_A (b G \zeta)\|_{L^2(S_{\tau, u})} \leq \lambda^{-\bar{a}+\epsilon}$. This follows trivially from the assumptions⁶¹ of Lemma 6.4 together with the estimates $|b| \leq 2$, $|\nabla b| \leq \lambda^{-\bar{a}+\epsilon}$ of the Asymptotics Theorem 5.1, and $|\nabla \zeta| \leq 5\tau^{-1}$. Thus, back to (6.1), we infer that,

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} G \mathcal{D}_3 \mathcal{D}_4 \psi \psi \zeta \right| \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau)$$

as desired.

The proof of the second part of Lemma 6.4, estimate (6.215), is much easier. It suffices to observe that $\mathcal{D}_3 \psi \psi = \frac{1}{2} \mathcal{D}_3(\psi^2)$ and use the integration by parts argument along the lines of the proof of (6.214).

This ends the proof of Lemma 6.4.

6.2 Proof of Theorem 6.1; Estimates for ψ :

The goal of this section is to prove the estimate (6.193) of Theorem 6.1. Let ψ be a solution of the wave equation $\square_h \psi = 0$ with initial data supported in the geodesic ball $B_{\frac{1}{2}}(0)$ at time $t_0 \geq 1$. Then ψ verifies the conformal identity (6.191)

$$\mathcal{Q}_0[\psi](t) = \mathcal{Q}_0[\psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} \left(Q^{\alpha\beta} \bar{\pi}_{\alpha\beta} - 2\psi^2 \square_h(\tau) \right) d\tau = \mathcal{Q}_0[\psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} I_1.$$

In view of Proposition⁶² 6.1 it suffices to prove the following fact:

Proposition 6.3 *Under the assumptions of the Boundedness Theorem, for all $t \in [t_0, t_*]$, we have:*

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} I_1 \right| \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau). \quad (6.219)$$

⁶¹we have $|G| + \|\nabla G\|_{L^2(S_{\tau, u})} \leq \lambda^{-\bar{a}+\epsilon}\tau^2$.

⁶²according to which $\mathcal{E}_0[\psi](t) \leq \mathcal{Q}_0[\psi](t)$

More generally, the following identity holds for a solution of the wave equation $\square_h \psi = F$:

$$\int_{[t_0, t] \times \mathbf{R}^3} \left(Q^{\alpha\beta} \bar{\pi}_{\alpha\beta} - 2\psi^2 \square_h(\tau) \right) = 2 \int_{[t_0, t] \times \mathbf{R}^3} \tau(\tau - u) \left(\text{tr}\chi - \frac{2}{s} \right) F \psi \zeta + \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau). \quad (6.220)$$

Here, ζ is a cut-off function equal to 1 in the region described by the condition $s \geq \frac{\tau}{2}$ and vanishing for $s \leq \frac{\tau}{4}$.

Proof: We start by observing that $\square_h(t) = -\frac{1}{\sqrt{\det h}} \partial_t(\sqrt{\det h}) = -\frac{1}{2} h^{ij} \partial_t h_{ij} = -\frac{1}{2} h^{ij} k_{ij}$. Since $|k| \lesssim \lambda^{-\bar{a}}$ we have

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} \psi^2 \square_h(\tau) \right| \lesssim \lambda^{-\bar{a}+a} \sup_{\tau \in [t_0, t]} \mathcal{E}[\psi](\tau)$$

It thus remains to estimate the term $\int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}[\psi] \bar{\pi}_{\alpha\beta}$. Decomposing relative to a null frame, making use of (6.188)-(6.190), we find⁶³,

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}[\psi] \bar{\pi}_{\alpha\beta} &= \int_{[t_0, t] \times \mathbf{R}^3} \left(\frac{1}{4} \bar{\pi}_{\mathbf{33}} (\mathcal{D}_4 \psi)^2 + \frac{1}{4} \bar{\pi}_{\mathbf{44}} (\mathcal{D}_3 \psi)^2 + \frac{1}{2} \bar{\pi}_{\mathbf{34}} |\nabla \psi|^2 \right. \\ &\quad \left. - \bar{\pi}_{\mathbf{4A}} \mathcal{D}_3 \psi \nabla_A \psi - \bar{\pi}_{\mathbf{3A}} \mathcal{D}_4 \psi \nabla_A \psi + \bar{\pi}_{AB} \nabla_A \psi \nabla_B \psi + \text{tr} \bar{\pi} \left(\frac{1}{2} \mathcal{D}_3 \psi \mathcal{D}_4 \psi - |\nabla \psi|^2 \right) \right). \end{aligned}$$

Recall the expressions (6.200)-(6.202) for the components of the deformation tensor $\bar{\pi}$. With the help of the estimates for the frame coefficients stated in the Asymptotics Theorem, and taking into account that $u \leq \underline{u}$ and $\tau \leq \underline{u} \leq 2\tau$ we have

$$|\bar{\pi}_{\mathbf{44}}| \lesssim \lambda^{-\bar{a}} \underline{u}^2, \quad |\bar{\pi}_{\mathbf{34}}| \lesssim (\lambda^{-(1-a)} \tau^{-1} + \lambda^{-\bar{a}}) \underline{u}^2, \quad (6.221)$$

$$|\bar{\pi}_{\mathbf{33}}| \lesssim (\lambda^{-(1-a)} \tau^{-1} + \lambda^{-\bar{a}}) \underline{u}^2, \quad |\bar{\pi}_{\mathbf{3A}}| \lesssim \lambda^{-\bar{a}+\epsilon} \underline{u}^2, \quad (6.222)$$

$$|\bar{\pi}_{\mathbf{4A}}| \lesssim \lambda^{-\bar{a}} \underline{u}^2, \quad |\bar{\pi}_{AB}| \lesssim \lambda^{-\bar{a}+\epsilon} \underline{u}^2. \quad (6.223)$$

Since $\mathcal{E}[\psi](\tau) = \int_{\Sigma_\tau} (\underline{u}^2 (\mathcal{D}_4 \psi)^2 + u^2 (\mathcal{D}_3 \psi)^2 + \underline{u}^2 |\nabla \psi|^2 + \psi^2)$ it easily follows that

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}[\psi] \bar{\pi}_{\alpha\beta} \right| \lesssim \int_{[t_0, t] \times \mathbf{R}^3} (\lambda^{-(1-a)} \tau^{-1} + \lambda^{-\bar{a}}) \mathcal{E}[\psi](\tau) + 2 \left| \int_{[t_0, t] \times \mathbf{R}^3} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi \right|$$

where, $\Xi = \tau(\tau - u) \left(\text{tr}\chi - \frac{2}{s} \right)$.

The last term presents difficulties; indeed since we cannot do any better than $|\Xi| \leq \lambda^{-\bar{a}+\epsilon} \underline{u}^2$, the straightforward estimate, $\left| \int_{\Sigma_\tau} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi \right| \leq \lambda^{-\bar{a}+\epsilon} \int_{\Sigma_\tau} (\underline{u}^2 (\mathcal{D}_3 \psi)^2 + \underline{u}^2 (\mathcal{D}_4 \psi)^2)$, is not convenient

⁶³Recall that $\text{tr} \bar{\pi} = \sum_{A=1}^2 \bar{\pi}_{AA}$ is the trace relative to the surfaces $S_{t,u}$

as $\int_{\Sigma_\tau} \underline{u}^2 (\mathcal{D}_3 \psi)^2 \not\leq \mathcal{E}[\psi](\tau)$. Therefore, this term requires special treatment; Lemma 6.4 was proved for this reason. Using it we derive the estimate

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi \zeta \right| \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}[\psi](\tau), \quad (6.224)$$

and thus end the proof of Proposition 6.3.

We break the integral on the left into the interior and exterior parts parts as follows:

$$\int_{[t_0, t] \times \mathbf{R}^3} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi = \int_{[t_0, t] \times \mathbf{R}^3} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi (1 - \zeta) + \int_{[t_0, t] \times \mathbf{R}^3} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi \zeta,$$

where as before ζ is a cut-off function equal to 1 in the region $s \geq \frac{r}{2}$. The first integral can be estimated by the interior part of the conformal norm $\mathcal{E}_0^i[\psi]$.

$$\int_{[t_0, t] \times \mathbf{R}^3} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi (1 - \zeta) \leq \lambda^{-\bar{a} + \epsilon} \int_{[t_0, t] \times \mathbf{R}^3} \tau^2 |\partial \psi|^2 (1 - \zeta) \leq \lambda^{-\bar{a} + a + \epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0^i[\psi](\tau).$$

We concentrate on the remaining exterior part. Note⁶⁴ that the function $\Xi = \tau(\tau - u)(\text{tr}\chi - \frac{2}{s})$ verifies the estimate $\|\Xi\|_{L^\infty(S_{t,u})} + \|\mathcal{D}\Xi\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a} + \epsilon} \tau^2$.

We can therefore apply Lemma 6.4 to end the proof of the estimate (6.224). Indeed, since Ξ satisfies the conditions of Lemma 6.4 and $\square_h \psi = 0$, we conclude that

$$\int_{[t_0, t] \times \mathbf{R}^3} \Xi \mathcal{D}_3 \psi \mathcal{D}_4 \psi \zeta \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}[\psi](\tau).$$

as desired.

6.3 Proof of Theorem 6.1; Estimates for $\partial_t \psi$

We shall now try to establish the estimate (6.194) of the Boundedness Theorem 6.1.

Notation: Let Π be an arbitrary tensor (say of rank 2) on \mathbf{R}^4 . We define the absolute value $|\Pi|$ by setting $|\Pi| = \sum_{\alpha, \beta} |\Pi_{\alpha\beta}|$, where $\Pi_{\alpha\beta}$ are the components of Π relative to the standard coordinates⁶⁵. Observe that $|\Pi| \approx \sum_{\alpha, \beta} |\Pi^{\alpha\beta}|$, since the metric H is bounded.

We also use notation $i_X \Pi = \Pi_{\alpha\beta} X^\beta$ to denote the contraction of the tensor Π with a vectorfield X . In particular, for the tensor $\mathcal{D}^2 \psi$ of the second covariant derivatives of a function ψ we have $i_X \mathcal{D}^2 \psi$ is a co-vector $(i_X \mathcal{D}^2 \psi)_\alpha = \mathcal{D}^2_{\alpha\beta} \psi X^\beta$. Throughout the section we shall use the following:

⁶⁴This easily follows from the Asymptotics Theorem 5.1 and the trivial bound $|\mathcal{D}\tau| + |\mathcal{D}u| \leq 10$.

⁶⁵Relative to which we have our assumptions on the metric H

Estimates for H : We recall the estimates for the metric H following from the conditions (5.113)-(5.117) and the Asymptotics Theorem.

$$\|\partial H\|_{L^\infty_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\bar{a}}, \quad \|(\partial^2 H)\|_{L^\infty_{[0,t_*]}L^\infty_x} \lesssim \lambda^{-\bar{a}-(1-a)}, \quad (6.225)$$

$$\left\| \left(|\partial^2 H| + (\partial H)^2 \right) \right\|_{L^2(S_{t,u})} \leq \lambda^{-\bar{a}+\epsilon}, \quad (6.226)$$

In addition, as a consequence of (6.226) and the Remark following Lemma 6.3 we also have for any $\tau \in [0, t_*]$

$$\tau \| (|\partial^2 H| + (\partial H)^2) \partial \psi \zeta \|_{L^2(\Sigma_\tau)} \leq \lambda^{-\bar{a}+\epsilon} \mathcal{E}_0^{\frac{1}{2}}[\partial \psi](\tau). \quad (6.227)$$

We now state a useful lemma concerning the conformal norm $\mathcal{E}_0[\partial \psi]$.

Lemma 6.5 *The following bound holds true for any $t \in [0, t_*]$:*

$$\int_{\Sigma_t} t^2 (|i_4 \mathcal{D}^2 \psi|^2 + \sum_{A=1}^2 |i_A \mathcal{D}^2 \psi|^2) + u^2 |\mathcal{D}_{\mathbf{33}}^2 \psi|^2 + |\partial \psi|^2 \leq 4\mathcal{E}_0[\partial \psi](t) \quad (6.228)$$

Remark: Lemma 6.5 implies that $\mathcal{D}_{\mathbf{33}}^2$ is the only “bad” second covariant derivative of ψ . Observe also that the expression (6.228) does not depend on the particular choice of the frame e_A on $S_{t,u}$.

Proof: The proof follows almost immediately from the following observation:

$$\begin{aligned} \mathcal{E}_0[\partial \psi](t) &\geq \int_{\Sigma_t} t^2 (|\mathcal{D}_4(\partial \psi)|^2 + |\nabla(\partial \psi)|^2) + u^2 |\mathcal{D}_3(\partial \psi)|^2 + |\partial \psi|^2 \geq \\ &\frac{1}{2} \int_{\Sigma_t} t^2 (|i_4 \mathcal{D}^2 \psi|^2 + \sum_A |i_A \mathcal{D}^2 \psi|^2) + u^2 |\mathcal{D}_{\mathbf{33}}^2 \psi|^2 + |\partial \psi|^2 - \int_{\Sigma_t} t^2 |\Gamma|^2 |\partial \psi|^2. \end{aligned}$$

It only remains to note that the Cristoffel symbols Γ obey the estimate $|\Gamma| \leq 4|\partial H| \leq \lambda^{-\bar{a}+\epsilon}$. Thus, $\int_{\Sigma_t} t^2 |\Gamma|^2 |\partial \psi|^2 \leq \int_{\Sigma_t} |\partial \psi|^2 \leq \mathcal{E}_0[\partial \psi](t)$.

Our next task is to derive an equation for $\partial_t \psi$. Since $\square_h \psi = 0$ it follows that $\square_h \partial_t \psi = [\square_h, \partial_t] \psi$. Since ∂_t is a special coordinate vectorfield we can easily calculate in coordinates,

$$[\square_h, \partial_t] = [-\partial_t^2 + h^{ij} \partial_i \partial_j + \text{Tr } k \partial_t + \Gamma_{ii}^j \partial_j, \partial_t] = -\partial_t h^{ij} \partial_i \partial_j - \partial_t (\text{Tr } k) \partial_t - \partial_t (\Gamma_{ii}^j) \partial_j. \quad (6.229)$$

Therefore,

Proposition 6.4 *The function $\partial_t \psi$ verifies the inhomogeneous wave equation,*

$$\square_h \partial_t \psi = F(t) = -\partial_t h^{ij} \partial_i \partial_j \psi - \partial_t (\text{Tr } k) \partial_t \psi - \partial_t (\Gamma_{ii}^j) \partial_j \psi = \mathcal{W}^{\alpha\beta} \psi_{;\alpha\beta} + \mathcal{V}^\alpha \partial_\alpha \psi. \quad (6.230)$$

Here \mathcal{W} denotes a spacetime 2-tensor verifying ⁶⁶ $|\mathcal{W}| \lesssim |\partial H|$. Similarly, \mathcal{V} denotes a spacetime vector whose components, in coordinates, verify $|\mathcal{V}| \lesssim |\partial^2 H| + |\partial H|^2$.

⁶⁶Here, in coordinates, $|\mathcal{W}| = \sum_{\alpha\beta} |\mathcal{W}_{\alpha\beta}|$ and $|\partial H| = \sum_{\alpha\beta\gamma} |\partial_\gamma H_{\alpha\beta}|$. In fact, in this case, $\mathcal{W}^{\alpha\beta} = -\partial_t H^{\alpha\beta}$

As a solution of the wave equation (6.230) function $\partial_t \psi$ verifies the conformal identity⁶⁷ (6.191)

$$\begin{aligned} \mathcal{Q}_0[\partial_t \psi](t) &= \mathcal{Q}_0[\partial_t \psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} (Q^{\alpha\beta}[\partial_t \psi] \bar{\pi}_{\alpha\beta} - 2(\partial_t \psi)^2 \square_h(\tau)) d\tau \\ &+ \int_{[t_0, t] \times \mathbf{R}^3} (K(\partial_t \psi) + 2\tau \partial_t \psi) F_{(t)} d\tau = \mathcal{Q}_0[\partial_t \psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} I_2. \end{aligned}$$

In view of Propositions 6.1 and 6.2 in order to establish the estimate (6.194) of Theorem 6.1 it suffices to prove the following

Proposition 6.5 *Under assumptions of the Boundedness Theorem the following estimates hold for all $t \in [t_0, t_*]$ and $\partial_t \psi$ solution of the equation (6.230):*

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} I_2 \right| \leq \lambda^{-\epsilon} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau) \right) \quad (6.231)$$

Proof: We decompose $I_2 = Q^{\alpha\beta}[\partial_t \psi] \bar{\pi}_{\alpha\beta} - 2(\partial_t \psi)^2 \square_h(\tau) - 2I_{21}$ with the term $I_{21} = \left(K(\partial_t \psi) + 2\tau \partial_t \psi \right) F_{(t)}$. In view of the second part of Proposition 6.3 with the function $\partial_t \psi$ replacing ψ , we have

$$\int_{[t_0, t] \times \mathbf{R}^3} Q^{\alpha\beta}[\partial_t \psi] \bar{\pi}_{\alpha\beta} - 2(\partial_t \psi)^2 \square_h(\tau) = 2 \int_{[t_0, t] \times \mathbf{R}^3} \Xi F_{(t)} \partial_t \psi \zeta + \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial_t \psi](\tau)$$

with $\Xi = \tau(\tau - u)(\text{tr}\chi - \frac{2}{s})$. Thus to establish Proposition 6.5 it only remains to show that for $J = (\Xi \partial_t \psi \zeta - K(\partial_t \psi) - 2\tau \partial_t \psi) F_{(t)}$ we have

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} J \right| \leq \lambda^{-\epsilon} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau) \right).$$

We now break the term $F_{(t)}$ into two parts $F_{(t)} = \mathcal{V}^\alpha \partial_\alpha \psi + \mathcal{W}^{\alpha\beta} \mathcal{D}_{\alpha\beta}^2 \psi = F_{(t)}^1 + F_{(t)}^2$ according to (6.230). We then also have a corresponding decomposition $J = J^1 + J^2$.

Estimate for J^1 : The term $J^1 = (\Xi \partial_t \psi \zeta - K(\partial_t \psi) - 2\tau \partial_t \psi) F_{(t)}^1$ contains the part of $F_{(t)}$ involving only the first derivatives of ψ .

Observe that it follows from a pointwise inequality $|F_{(t)}^1| \lesssim (|\partial^2 H| + |\partial H|^2) |\partial \psi|$ and (6.225) that $|F_{(t)}^1| \lesssim \lambda^{-\bar{a} - (1-a)} |\partial \psi|$. Recall also the estimate $|\Xi| \leq \lambda^{-\bar{a} + \epsilon} \tau^2 \leq \tau$. Write

$$J^1 = (\Xi - 2\tau) F_{(t)}^1 \partial_t \psi \zeta - (K(\partial_t \psi) + 2\tau \partial_t \psi (1 - \zeta)) F_{(t)}^1 = J_1^1 - J_2^1$$

⁶⁷Here, $Q^{\alpha\beta}[\partial_t \psi]$ is the energy-momentum tensor Q associated to $\partial_t \psi$.

We have

$$\int_{[t_0, t] \times \mathbf{R}^3} |J_1^1| \lesssim \int_{[t_0, t]} \tau \left((|\partial^2 H| + |\partial H|^2) |\partial \psi| \right)^2 \zeta \lesssim \int_{[t_0, t]} \tau \| (|\partial^2 H| + |\partial H|^2) \partial \psi \zeta \|_{L^2(\Sigma_\tau)} \mathcal{E}_0^{\frac{1}{2}}[\partial \psi](\tau).$$

It then immediately follows from (6.227) that

$$\int_{[t_0, t] \times \mathbf{R}^3} |J_1^1| \leq \lambda^{-\bar{a} + \epsilon t} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau) \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau).$$

To treat the remaining term J_2^1 we break it into its interior and exterior parts

$$J_2^1 = (\underline{u}^2 \mathcal{D}_4 \partial_t \psi + u^2 \mathcal{D}_3 \partial_t \psi) F_{(t)}^1 + 2\tau \partial_t \psi (1 - \zeta) + (\underline{u}^2 \mathcal{D}_4 \partial_t \psi + u^2 \mathcal{D}_3 \partial_t \psi) F_{(t)}^1 \zeta = J_2^{1i} + J_2^{1e}$$

We first estimate with the help of (6.225)

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} |J_2^{1i}| &\lesssim \int_{[t_0, t] \times \mathbf{R}^3} (\tau^2 |\partial^2 \psi| + \tau |\partial \psi|) (|\partial^2 H| + |\partial H|^2) |\partial \psi| (1 - \zeta) \lesssim \\ &\lambda^{-\bar{a} - (1-a)} \int_{[t_0, t] \times \mathbf{R}^3} (\tau^2 |\partial^2 \psi|^2 + \tau^2 |\partial \psi|^2) (1 - \zeta) \leq \lambda^{-(1-a)} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau) \right). \end{aligned}$$

The exterior integral of J_2^{1e} can be treated again with the help of the estimate (6.227).

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} |J_2^{1e}| &\lesssim \int_{[t_0, t]} \left(\int_{\Sigma_\tau} \underline{u}^2 (\mathcal{D}_4 \partial_t \psi)^2 + u^2 (\mathcal{D}_3 \partial_t \psi)^2 \right)^{\frac{1}{2}} \times \\ &\tau \| (|\partial^2 H|^2 + |\partial H|^2) \partial \psi \zeta \|_{L^2(\Sigma_\tau)} \leq \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau). \end{aligned}$$

We shall now estimate the remaining error term containing J^2 .

Estimate for J^2 : We consider the remaining part of the error term:

$$\int_{[t_0, t] \times \mathbf{R}^3} J^2 = \int_{[t_0, t] \times \mathbf{R}^3} (\Xi \partial_t \psi \zeta - K(\partial_t \psi) - 2\tau \partial_t \psi) F_{(t)}^2.$$

According to (6.234) the term $F_{(t)}^2 = \mathcal{W}^{\alpha\beta} \mathcal{D}_{\alpha\beta}^2 \psi$ and $|\mathcal{W}| \lesssim |\partial H| \lesssim \lambda^{-\bar{a}}$. We have already noted in Lemma 6.5 that all second covariant derivatives $i_4 \mathcal{D}^2 \psi$ and $i_A \mathcal{D}^2 \psi$ containing at least one derivative in the directions e_4 or e_A , $A = 1, 2$ exhibit the best decay, i.e.,

$$\int_{\Sigma_\tau} \tau^2 (|i_4 \mathcal{D}^2 \psi| + \sum_A |i_A \mathcal{D}^2 \psi|^2) \lesssim \mathcal{E}_0[\partial \psi](\tau).$$

It is then a simple exercise to check that the part of $F_{(t)}^2$ corresponding to such covariant derivatives contributes the term with the bound $\lambda^{-\bar{a} + a} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau)$ to the estimate for the error term.

Therefore, it only suffices to consider the term $\mathcal{W}^{33}\mathcal{D}_{33}^2\psi$. Furthermore, since $e_3 = -e_4 + 2T$, and $\mathcal{D}_3T = k_{AN}e_A + k_{NN}N$ we can assume that $F_{(t)}^2 = \mathcal{W}^{33}\mathcal{D}_3\partial_t\psi$. We write

$$\begin{aligned} \int_{[t_0,t]\times\mathbf{R}^3} J_1^2 &= \int_{[t_0,t]\times\mathbf{R}^3} \left((\Xi - 2\tau)\mathcal{W}^{33}\mathcal{D}_3\partial_t\psi \partial_t\psi \zeta - \underline{u}^2\mathcal{D}_4\partial_t\psi\mathcal{W}^{33}\mathcal{D}_3\partial_t\psi \zeta \right) \\ &- \int_{[t_0,t]\times\mathbf{R}^3} \underline{u}^2\mathcal{D}_4\partial_t\psi\mathcal{W}^{33}\mathcal{D}_3\partial_t\psi (1 - \zeta) - \int_{[t_0,t]\times\mathbf{R}^3} u^2\mathcal{D}_3\partial_t\psi\mathcal{W}^{33}\mathcal{D}_3\partial_t\psi = \int_{[t_0,t]\times\mathbf{R}^3} (J_1^{2e} - J_1^{2i} - J_2^2) \end{aligned}$$

The estimate for J_2^2 is trivial, since it follows from the estimate $|\mathcal{W}| \lesssim \lambda^{-\bar{a}}$ that

$$\int_{[t_0,t]\times\mathbf{R}^3} |J_2^2| \lesssim \lambda^{-\bar{a}}t \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\partial\psi](\tau) \leq \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\partial\psi](\tau).$$

The interior term J_1^{2i} produces the estimate

$$\int_{[t_0,t]\times\mathbf{R}^3} |J_1^{2i}| \lesssim \lambda^{-\bar{a}} \int_{[t_0,t]\times\mathbf{R}^3} \tau^2 |\partial^2\psi|^2 (1 - \zeta) \leq \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\partial\psi](\tau).$$

It remains to address the exterior term J_1^{2e} . We split

$$J_1^{2e} = (\Xi - 2\tau)\mathcal{W}^{33}\mathcal{D}_3\partial_t\psi \partial_t\psi \zeta - \underline{u}^2\mathcal{D}_4\partial_t\psi\mathcal{W}^{33}\mathcal{D}_3\partial_t\psi \zeta = j_1 - j_2$$

The integral $\int_{[t_0,t]\times\mathbf{R}^3} j_1$ will be estimated with the help of (6.215) of Lemma 6.4. We take $G = \tau(\Xi - 2\tau)\mathcal{W}^{33}$ and replace ψ by $\partial_t\psi$ in Lemma 6.4. Then, provided we can verify the conditions of Lemma 6.4 for thus chosen G , we obtain from (6.215)

$$\left| \int_{[t_0,t]\times\mathbf{R}^3} j_1 \right| \leq \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\partial\psi](\tau).$$

The second term j_2 is reduced to yet another term with the help of identity (6.214) of Lemma 6.4. We take $G = \underline{u}^2\mathcal{W}^{33}$ and replace ψ by $\partial_t\psi$ observing that $\partial_t\psi$ verifies the equation $\square_h\partial_t\psi = F_{(t)}$. Thus, if G obeys the conditions of Lemma 6.4, we have

$$\int_{[t_0,t]\times\mathbf{R}^3} j_2 = \int_{[t_0,t]\times\mathbf{R}^3} \underline{u}^2\mathcal{W}^{33}F_{(t)}\partial_t\psi \zeta + \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\psi](\tau) = \int_{[t_0,t]\times\mathbf{R}^3} j_3 + \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\psi](\tau)$$

with $j_3 = \underline{u}^2\mathcal{W}^{33}F_{(t)}\partial_t\psi \zeta$. Before tackling the last error term j_3 let us verify that the functions $\tau(\Xi - 2\tau)\mathcal{W}^{33}$ and $\underline{u}^2\mathcal{W}^{33}$ obey the conditions of Lemma 6.4 in the region $u \in [0, \frac{3\tau}{4}]$. We need the following properties:

$$1) \quad |\tau(\Xi - 2\tau)\mathcal{W}^{33}\zeta| + |\underline{u}^2\mathcal{W}^{33}\zeta| \leq \lambda^{-\bar{a}+\epsilon}\tau^2$$

$$2) \quad \|\mathcal{D}_3(\tau(\Xi - 2\tau)\mathcal{W}^{33}\zeta)\|_{L^2(S_{\tau,u})} + \|\nabla(\tau(\Xi - 2\tau)\mathcal{W}^{33}\zeta)\|_{L^2(S_{\tau,u})} \leq \lambda^{-\bar{a}+\epsilon}\tau^2$$

$$3) \quad \|\mathcal{D}_3(\underline{u}^2\mathcal{W}^{33}\zeta)\|_{L^2(S_{\tau,u})} + \|\nabla(\underline{u}^2\mathcal{W}^{33}\zeta)\|_{L^2(S_{\tau,u})} \leq \lambda^{-\bar{a}+\epsilon}\tau^2$$

The first property easily follows from the fact that $|\mathcal{W}| \leq |\partial H| \leq \lambda^{-\bar{a}}$, $|\Xi| \leq \lambda^{-\bar{a}+\epsilon}\tau^2$, and $\underline{u} \leq 2\tau$. The other two are the consequences of the pointwise bounds mentioned above, the already established estimates for Ξ : $\|\mathcal{D}_3\Xi\|_{L^2(S_{\tau,u})} + \|\nabla\Xi\|_{L^2(S_{\tau,u})} \leq \lambda^{-\bar{a}+\epsilon}\tau^2$, the trivial bounds $|\mathcal{D}\tau| + |\mathcal{D}\underline{u}| \leq 10$, $|\mathcal{D}\zeta| \leq 5\tau^{-1}$, and the following estimates for the derivatives of \mathcal{W}^{33} . First, $\mathcal{W}^{33} = \frac{1}{4}\mathcal{W}_{44}$. Furthermore, we have from the frame equations

$$\begin{aligned} \mathcal{D}_3(\mathcal{W}_{44}) &= \mathcal{D}_3\mathcal{W}_{44} + 2\eta_A\mathcal{W}_{A4} + k_{NN}\mathcal{W}_{44} \\ \nabla_A(\mathcal{W}_{44}) &= \mathcal{D}_A\mathcal{W}_{44} + \chi_{AB}\mathcal{W}_{B4} - k_{AN}\mathcal{W}_{44} \end{aligned}$$

Thus, since \mathcal{W} is essentially ⁶⁸ the tensor of the first derivatives of the metric H , both of the \mathcal{D}_3 and ∇ derivatives are bounded pointwise by $|\partial^2 H| + s^{-1}|\partial H|$. Hence, the Asymptotics Theorem implies that $\|\mathcal{D}_3(\mathcal{W}_{44})\|_{L^2(S_{\tau,u})} + \|\nabla(\mathcal{W}_{44})\|_{L^2(S_{\tau,u})} \leq \lambda^{-\bar{a}+\epsilon}$. Properties 1) - 3) allow us to apply Lemma 6.4 to the integrals of j_1 and j_2 .

$$\text{The remaining term is } \int_{[t_0,t]\times\mathbf{R}^3} j_3 = \int_{[t_0,t]\times\mathbf{R}^3} \underline{u}^2\mathcal{W}^{33}F_{(t)}\partial_t\psi\zeta.$$

Remark: Note that we have already obtained the estimate for the integral of j_1

$$\left| \int_{[t_0,t]\times\mathbf{R}^3} j_1 \right| = \left| \int_{[t_0,t]\times\mathbf{R}^3} (\Xi - 2\tau)\mathcal{W}^{33}\mathcal{D}_3\partial_t\psi\partial_t\psi\zeta \right| \leq \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\partial\psi](\tau)$$

Combining this with the estimates for the integral of J^1 we can conclude that we have, in fact, already established the estimate

$$\left| \int_{[t_0,t]\times\mathbf{R}^3} (\Xi - 2\tau)F_{(t)}\partial_t\psi\zeta \right| \leq \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\partial\psi](\tau) \quad (6.232)$$

From the point of view of the asymptotic behavior the function $\underline{u}^2\mathcal{W}^{33}$ is slightly "better" than $\Xi - 2\tau$ since

$$\begin{aligned} |\Xi - 2\tau| &\leq 3\tau, & |\underline{u}^2\mathcal{W}^{33}| &\lesssim \lambda^{-\bar{a}}\tau^2, \\ \|\mathcal{D}(\Xi - 2\tau)\|_{L^2(S_{\tau,u})} &\lesssim \tau, & \|\mathcal{D}(\underline{u}^2\mathcal{W}^{33})\|_{L^2(S_{\tau,u})} &\lesssim \lambda^{-\bar{a}}\tau^2. \end{aligned}$$

The Remark above implies that we can repeat the arguments that led to the estimate (6.232) with $\underline{u}^2\mathcal{W}^{33}$ in place of $\Xi - 2\tau$ to show that $\left| \int_{[t_0,t]\times\mathbf{R}^3} j_3 \right| \leq \lambda^{-\epsilon} \sup_{\tau\in[t_0,t]} \mathcal{E}_0[\partial\psi](\tau)$, thus completing the proof of Proposition 6.5.

Remark The results of the computation above can be summarized in a slightly more abstract form which will prove to be useful in the next section.

⁶⁸Recall that in this case $\mathcal{W}^{\alpha\beta} = -\partial_t H^{\alpha\beta}$

Assume that $\partial_t \psi$ verifies the equation $\square_h \partial_t \psi = F$ and let $F_{(t)}$ be as in (6.230). Then the following identity holds true:

$$\int_{[t_0, t] \times \mathbf{R}^3} \Xi F_{(t)} \partial_t \psi \zeta - \int_{[t_0, t] \times \mathbf{R}^3} (K(\partial_t \psi) + 2\tau \partial_t \psi) F_{(t)} = - \int_{[t_0, t] \times \mathbf{R}^3} \underline{u}^2 \mathcal{W}^{33} (F - F_{(t)}) \partial_t \psi \zeta + \lambda^{-\epsilon} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}[\psi](\tau) \right). \quad (6.233)$$

Here $\Xi = \tau(\tau - u)(\text{tr} \chi - \frac{2}{s})$ and the tensor $\mathcal{W}^{\alpha\beta} = -\partial_t H^{\alpha\beta}$.

6.4 Proof of Theorem 6.1; Estimates for $\partial_{tt}^2 \psi$

Differentiating (6.230) and commuting \square_h with ∂_t we derive the equation for $\partial_t^2 \psi$.

$$\begin{aligned} \square_h \partial_t^2 \psi &= F_{(tt)} = F_{(tt)}^1 + F_{(tt)}^2, \\ F_{(tt)}^1 &= -2(2k^{ij} \partial_i \partial_j \partial_t \psi + \partial_t(\text{tr} k) \partial_t^2 \psi + \partial_t(\Gamma_{ii}^j) \partial_j \partial_t \psi), \\ F_{(tt)}^2 &= -(2\partial_t(k^{ij}) \partial_i \partial_j \psi + \partial_t^2(\text{tr} k) \partial_t \psi + \partial_t^2(\Gamma_{ii}^j) \partial_j \psi). \end{aligned} \quad (6.234)$$

The conformal identity for $\partial_t^2 \psi$ has the form $\mathcal{Q}[\partial_t^2 \psi](t) = \mathcal{Q}[\partial_t^2 \psi](t_0) + \int_{[t_0, t] \times \mathbf{R}^3} I_3$ with

$$I_3 = -\frac{1}{2} Q^{\alpha\beta} [\partial_t^2 \psi] \bar{\pi}_{\alpha\beta} + (\partial_t^2 \psi)^2 \square_h(\tau) + (K(\partial_t^2 \psi) + 2\tau \partial_t^2 \psi) F_{(tt)}.$$

Proposition 6.6 *Under the assumptions of the Boundedness Theorem we have the following estimate for all $t \in [t_0, t_*]$:*

$$\left| \int_{[t_0, t] \times \mathbf{R}^3} I_3 \right| \leq \lambda^{-\epsilon} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial^2 \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\psi](\tau) \right). \quad (6.235)$$

The Propositions 6.1 and 6.2 imply that Proposition 6.6 completes the proof of Theorem 6.1 and, as a result, the proof of the Boundedness Theorem.

Proof: According to the second part of Proposition 6.3 with $\partial_t^2 \psi$ instead of ψ , we have

$$\int_{[t_0, t] \times \mathbf{R}^3} -\frac{1}{2} Q^{\alpha\beta} [\partial_t^2 \psi] \bar{\pi}_{\alpha\beta} + (\partial_t^2 \psi)^2 \square_h(\tau) = - \int_{[t_0, t] \times \mathbf{R}^3} \Xi F_{(tt)} \partial_t^2 \psi \zeta + \lambda^{-\epsilon} \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial_t^2 \psi](\tau) \quad (6.236)$$

Observe that in the decomposition $F_{(tt)} = F_{(tt)}^1 + F_{(tt)}^2$ the term $F_{(tt)}^1 = 2F_{(t)}$ from the previous section with $\partial_t \psi$ replacing ψ . Therefore, we are in a situation described by the last Remark of the previous section with $\partial_t \psi$ instead of ψ . We have $\square_h \partial_t^2 \psi = F_{(tt)}$ and

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} \Xi F_{(tt)}^1 \partial_t^2 \psi \zeta - \int_{[t_0, t] \times \mathbf{R}^3} (K(\partial_t^2 \psi) + 2\tau \partial_t^2 \psi) F_{(tt)}^1 &= - \int_{[t_0, t] \times \mathbf{R}^3} \underline{u}^2 \mathcal{W}^{33} F_{(tt)}^2 \partial_t^2 \psi \zeta \\ &+ \lambda^{-\epsilon} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial_t \partial \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0[\partial_t \psi](\tau) \right) \end{aligned}$$

We are left with estimating the term

$$\int_{[t_0, t] \times \mathbf{R}^3} \left(\Xi \zeta + \underline{u}^2 \mathcal{W}^{33} \zeta - 2\tau \right) F_{(tt)}^2 \partial_t^2 \psi - \int_{[t_0, t] \times \mathbf{R}^3} \left(\underline{u}^2 \mathcal{D}_4 \partial_t^2 \psi + u^2 \mathcal{D}_3 \partial_t^2 \psi \right) F_{(tt)}^2.$$

We now act by brute force. First, since $|\Xi| + |\mathcal{W}| \leq \lambda^{-\bar{a}+\epsilon} \tau^2 \leq \tau$, we can bound the expression above by

$$\int_{[t_0, t]} \tau \|F_{(tt)}^2\|_{L^2(\Sigma_\tau)} \sup_{\tau \in [t_0, t]} \mathcal{E}_0^{\frac{1}{2}}[\partial_t^2 \psi](\tau) \quad (6.237)$$

The function $F_{(tt)}^2$ can be schematically written in the form $F_{(tt)}^2 = \partial^2 H \partial^2 \psi + \partial^3 H \partial \psi$. In the interior all derivatives multiplied by the weight τ are bounded by the corresponding conformal norm. Therefore, using the bounds $|\partial^2 H| \lesssim \lambda^{-\bar{a}-(1-a)}$ and $|\partial^3 H| \lesssim \lambda^{-\bar{a}-2(1-a)}$ following from the assumptions on the metric H , we obtain

$$\begin{aligned} \int_{[t_0, t]} \tau \|F_{(tt)}^2(1-\zeta)\|_{L^2(\Sigma_\tau)} &\lesssim \int_{[t_0, t]} \left(\lambda^{-\bar{a}-(1-a)} \|\tau \partial^2 \psi(1-\zeta)\|_{L^2(\Sigma_\tau)} + \lambda^{-\bar{a}-2(1-a)} \|\tau \partial \psi(1-\zeta)\|_{L^2(\Sigma_\tau)} \right) \\ &\lesssim \lambda^{-\bar{a}+a-(1-a)} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0^{\frac{1}{2}}[\partial^2 \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0^{\frac{1}{2}}[\psi](\tau) \right) \end{aligned}$$

The exterior part is treated as follows. We have

$$\int_{\Sigma_\tau} \tau^2 |F_{(tt)}^2|^2 \zeta \lesssim \int_{\Sigma_\tau} \tau^2 (|\partial^2 H|^2 |\partial^2 \psi|^2 + |\partial^3 H|^2 |\partial \psi|^2) \zeta.$$

Lemma 6.3 yields

$$\int_{\Sigma_\tau} \tau^2 |\partial^2 H|^2 |\partial^2 \psi|^2 \leq \lambda^{-2\bar{a}+2\epsilon} \mathcal{E}_0[\partial^2 \psi](\tau)$$

since $\int_{S_{\tau, u}} |\partial^2 H|^2 \leq \lambda^{-2\bar{a}+2\epsilon}$ and $|\partial^2 H| \lesssim \lambda^{-\bar{a}-(1-a)}$.

In addition, from (5.125) and (5.115) we also have $\int_{S_{\tau, u}} |\partial^3 H|^2 \leq \lambda^{-2\bar{a}-2(1-a)+2\epsilon}$ and $|\partial^3 H| \lesssim \lambda^{-\bar{a}-2(1-a)}$. Thus by Lemma 6.3

$$\int_{\Sigma_\tau} \tau^2 |\partial^3 H|^2 |\partial \psi|^2 \leq \lambda^{-2\bar{a}-2(1-a)+\epsilon} \mathcal{E}_0[\partial \psi](\tau)$$

Taking the square root and integrating in time the last 2 inequalities we obtain

$$\int_{[t_0, t]} \tau \|F_{(tt)}^2 \zeta\|_{L^2(\Sigma_\tau)} \leq \lambda^{-\bar{a}+a+\epsilon} \left(\sup_{\tau \in [t_0, t]} \mathcal{E}_0^{\frac{1}{2}}[\partial^2 \psi](\tau) + \sup_{\tau \in [t_0, t]} \mathcal{E}_0^{\frac{1}{2}}[\partial \psi](\tau) \right)$$

6.5 Proof of Proposition 6.2

We now address the issue of the control of the full conformal norm $\mathcal{E}[\psi]$ of a solution of the wave equation $\square_h \psi = 0$ by the partial conformal norm $\mathcal{E}_0[\psi] + \mathcal{E}_0[\partial_t \psi] + \mathcal{E}_0[\partial_t^2 \psi]$. Intuitively, this statement is almost obvious since the spatial derivatives of ψ can be controlled from the elliptic equation $\Delta_h \psi = \partial_t^2 \psi +$ lower order terms. Let us recall the precise statement of Proposition 6.2.

Proposition 6.7 *Let*

$$\mathcal{E}_0[\psi](t) = \int_{\Sigma_t} t^2 (|\mathcal{D}_4 \psi|^2 + |\nabla \psi|^2) + u^2 |\mathcal{D}_3 \psi|^2 + |\psi|^2$$

Recall that $\mathcal{E}[\psi](t) = \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t \psi](t) + \mathcal{E}_0[\partial_t^2 \psi](t)$ is the full conformal energy of ψ . If ψ is a solution of the wave equation $\square_h \psi = 0$ and the assumptions of the Boundedness Theorem are verified, then for any $t \leq t_$*

$$\mathcal{E}[\psi](t) \lesssim \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t \psi](t) + \mathcal{E}_0[\partial_t^2 \psi](t) \quad (6.238)$$

Proof: In what follows we shall say that a term is a l.o.t.₁ (l.o.t.₂)⁶⁹ if it can be estimated by $c \mathcal{E}_0[\psi] + \epsilon \mathcal{E}[\psi]$ (respectively $c(\mathcal{E}_0[\psi] + \mathcal{E}_0[\partial_t \psi]) + \epsilon \mathcal{E}[\psi]$) with some positive constant c and a sufficiently small positive ϵ . To prove Proposition 6.7 it clearly suffices to establish that

$$\begin{aligned} \mathcal{E}_0[\partial_t \psi] &\leq \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t \psi](t) + \mathcal{E}_0[\partial_t^2 \psi](t) + l.o.t_1, \\ \mathcal{E}_0[\partial_t^2 \psi] &\leq \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t \psi](t) + \mathcal{E}_0[\partial_t^2 \psi](t) + l.o.t_2. \end{aligned}$$

Recall that the covariant derivative ∇ is the restriction of the covariant derivative \mathcal{D} to Σ_t . We formulate a useful commutation Lemma which can be proved by a tedious application of the Codazzi equation and is very similar to the proof of Lemma 2.3 of Chapter 2. We also include a simple integration by parts formula. Recall that for a tensor U tangent to Σ_t the covariant derivatives $\nabla_{\mathbf{3},4} U$ denote the projections of the space-time derivatives $\mathcal{D}_{\mathbf{3},4} U$ to Σ_t .

Lemma 6.6 *Let $U_{\underline{I}}$ be an m -covariant tensor tangent to Σ_t . Then in the frame N, e_A , $A = 1, 2$ on Σ_t*

$$\begin{aligned} \nabla_4 \nabla_A U_{\underline{I}} - \nabla_A \nabla_4 U_{\underline{I}} &= -\chi_{AB} \nabla_B U_{\underline{I}} + k_{AN} \nabla_N U_{\underline{I}} + \sum_p (k_{I_p N} k_{AJ} - k_{I_p A} k_{JN} + \mathbf{R}_{I_p J 4 A}) U_{I_1 \dots \check{J} \dots I_m} \\ \nabla_4 \nabla_N U_{\underline{I}} - \nabla_N \nabla_4 U_{\underline{I}} &= \eta_A \nabla_A U_{\underline{I}} + k_{NN} \nabla_N U_{\underline{I}} + \sum_p \mathbf{R}_{I_p J 4 N} U_{I_1 \dots \check{J} \dots I_m} \\ \nabla_3 \nabla_A U_{\underline{I}} - \nabla_A \nabla_3 U_{\underline{I}} &= -\underline{\chi}_{AB} \nabla_B U_{\underline{I}} + k_{AN} \nabla_N U_{\underline{I}} + \sum_p (k_{I_p A} k_{JN} - k_{I_p N} k_{AJ} + \mathbf{R}_{I_p J 3 A}) U_{I_1 \dots \check{J} \dots I_m}, \\ \nabla_3 \nabla_N U_{\underline{I}} - \nabla_N \nabla_3 U_{\underline{I}} &= (2k_{AN} - \eta_A) \nabla_A U_{\underline{I}} + k_{NN} \nabla_N U_{\underline{I}} + \sum_p \mathbf{R}_{I_p J 3 N} U_{I_1 \dots \check{J} \dots I_m} \end{aligned} \quad (6.239)$$

⁶⁹lower order term

We also have that for any vectorfield X tangent to Σ_t

$$\nabla_X \nabla_A U_{\underline{I}} - \nabla_A \nabla_X U_{\underline{I}} = \sum_p \bar{\mathbf{R}}_{I_p J X A} U_{I_1 \dots j \dots I_m}. \quad (6.240)$$

Here, $\bar{\mathbf{R}}$ is the curvature of the covariant derivative ∇ on Σ_t . In addition, let e_i be a frame on Σ_t and let $h_{ij} = h(e_i, e_j)$. Then for any scalar function U and any co-vectorfield V_j tangent to Σ_t the following formula of integration by parts holds true:

$$\int_{\Sigma_t} \sum_{i,j} h^{ij} \nabla_i U V_j = - \int_{\Sigma_t} \sum_{i,j} h^{ij} U \nabla_i V_j. \quad (6.241)$$

Estimates for $\mathcal{E}_0[\partial\psi]$: Clearly we only need to consider the spatial derivatives ∇ . In this section we shall show that

$$\mathcal{E}_0[\nabla\psi](t) \leq \mathcal{E}_0[\psi](t) + \mathcal{E}_0[\partial_t\psi](t) + \mathcal{E}_0[\partial_t^2\psi](t) + \text{l.o.t.}_1$$

Note that it suffices to consider only the exterior part of the conformal norm

$$\mathcal{E}_0^e[\nabla\psi](t) = \int_{\Sigma_t} \left(t^2 (|\mathcal{D}_4(\nabla\psi)|^2 + |\nabla(\nabla\psi)|^2) + u^2 |\mathcal{D}_3(\nabla\psi)|^2 + |\nabla\psi|^2 \right) \zeta$$

with a cut-off function ζ equal to 1 in the region $s \geq \frac{t}{2}$. This can be explained as follows. The interior part of the norm $\mathcal{E}_0^i[\nabla\psi](t)$ is simply equivalent to the weighted H^1 norm of $\nabla\psi$.

$$\mathcal{E}_0^i[\nabla\psi](t) \approx \int_{\Sigma_t} (t^2 |\partial(\nabla\psi)|^2 + |\nabla\psi|^2) (1 - \zeta).$$

Integrate by parts in the first term relative to the standard spatial derivatives ∂_i and note that the terms where the derivatives falls on the cut-off function $(1 - \zeta)$ are of the type

$$\int_{\Sigma_t} t^2 \partial\psi \partial(\partial_i\psi) \partial_i\zeta \leq \epsilon^{-1} \int_{\Sigma_t} |\partial\psi|^2 (1 - \tilde{\zeta}) + \epsilon \int_{\Sigma_t} t^2 |\partial(\partial_i\psi)|^2 (1 - \tilde{\zeta}) \leq \epsilon^{-1} \mathcal{E}_0[\psi](t) + \text{l.o.t.}_1,$$

where $\tilde{\zeta}$ is a cut-off function with a slightly larger support. Thus

$$\mathcal{E}_0^i[\nabla\psi](t) \leq \int_{\Sigma_t} t^2 \partial\psi \partial(\Delta\psi) (1 - \zeta) + 2\epsilon^{-1} \mathcal{E}_0[\psi](t) + \text{l.o.t.}_1.$$

Since ψ verifies the wave equation, we have $\Delta\psi = \partial_t^2\psi + (\partial H)\partial\psi$, where $(\partial H)\partial\psi$ schematically denotes the term containing the first derivatives of the metric H and function ψ . Then

$$\mathcal{E}_0^i[\nabla\psi](t) \leq \int_{\Sigma_t} t^2 \partial\psi \partial(\partial H \partial\psi) (1 - \zeta) + \mathcal{E}_0[\partial_t^2\psi] + 2\epsilon^{-1} \mathcal{E}_0[\psi](t) + \text{l.o.t.}_1.$$

Since by the assumptions on the metric $|\partial H| \lesssim \lambda^{-\bar{a}}$ and $|\partial^2 H| \lesssim \lambda^{-\bar{a}-(1-a)}$ we obtain

$$\mathcal{E}_0^i[\nabla\psi](t) \leq \epsilon^{-1}\mathcal{E}_0[\partial_t^2\psi] + 2\epsilon^{-1}\mathcal{E}_0[\psi](t) + \text{l.o.t.}_1.$$

In order to simplify the exposition we shall only treat the $\int_{\Sigma_t} t^2 |\mathcal{D}_4(\nabla\psi)|^2 \zeta$ part of the exterior conformal norm $\mathcal{E}_0^e[\nabla\psi](t)$. Using the fact that $\nabla_4 \partial_j = e_4^k \bar{\Gamma}_{kj}^l \partial_l$, where $\bar{\Gamma}$ are the Cristoffel symbols of the metric h on the slice Σ_t , and the bound $|\bar{\Gamma}| \lesssim \lambda^{-\bar{a}}$, we have

$$\int_{\Sigma_t} t^2 |\mathcal{D}_4(\nabla\psi)|^2 = \int_{\Sigma_t} t^2 h^{ij} \nabla_4 \nabla_i \psi \nabla_4 \nabla_j \psi \zeta + \text{l.o.t.}_1.$$

Here we work in the standard coordinates x_1, \dots, x_3 on Σ_t . Commuting the covariant derivatives

$$\int_{\Sigma_t} t^2 h^{ij} \nabla_i \nabla_4 \psi \nabla_4 \nabla_j \psi \zeta + \text{l.o.t.}_1.$$

Integration by parts yields

$$- \int_{\Sigma_t} t^2 h^{ij} \nabla_4 \psi \nabla_i \nabla_4 \nabla_j \psi \zeta + \epsilon^{-1}\mathcal{E}_0[\psi](t) + \text{l.o.t.}_1.$$

The expression above does not depend on the choice of the frame on Σ_t . Hence, we take an orthonormal frame N, e_A , $A = 1, 2$ and commute ∇_i with ∇_4 . These commutators are completely described in Lemma 6.6. Its analysis shows that the “worst” possible term produced is of the type $s^{-1}\nabla$. Since we work in the exterior region we have $s \geq \frac{t}{2}$, and thus obtain

$$- \int_{\Sigma_t} t^2 \nabla_4 \psi \nabla_4 (\Delta_h \psi) \zeta + 2\epsilon^{-1}\mathcal{E}_0[\psi](t) + \text{l.o.t.}_1.$$

Again, since ψ verifies $\square_h \psi = 0$, we have $\Delta_h \psi = \partial_t^2 \psi + (\partial H)(\partial\psi)$. The same argument as in the interior case then produces the bound

$$\mathcal{E}_0[\partial_t^2\psi](t) + \epsilon^{-1}\mathcal{E}_0[\psi](t) + \text{l.o.t.}_1$$

Estimates for $\mathcal{E}_0[\partial^2\psi]$: Again the more difficult part is the exterior norm

$$\mathcal{E}_0^e[\partial^2\psi](t) = \int_{\Sigma_t} \left(t^2 (|\mathcal{D}_4(\partial^2\psi)|^2 + |\nabla(\partial^2\psi)|^2 + u^2 |\mathcal{D}_3(\partial^2\psi)|^2) \zeta \right).$$

There are 2 possible cases. If the both derivatives in ∂^2 are spatial, then choosing again the term $\mathcal{D}_4(\partial^2\psi)$ we have

$$\int_{\Sigma_t} t^2 |\mathcal{D}_4(\nabla^2\psi)|^2 \zeta = \int_{\Sigma_t} t^2 |\nabla_4 \nabla^2 \psi|^2 \zeta + \text{l.o.t.}_2 = \int_{\Sigma_t} t^2 \nabla_i \nabla_4 \nabla_k \psi \nabla_4 \nabla_k \nabla_i \psi \zeta + \text{l.o.t.}_2,$$

where the last identity follows from the commutation formula (6.239) with an orthonormal frame $e_i = N, e_A, A = 1, 2$. Integration by parts according to Lemma 6.6 and another commutation with ∇_4 yields

$$- \int_{\Sigma_t} t^2 \nabla_4 \nabla_k \psi \nabla_4 \nabla_i \nabla_k \nabla_i \psi \zeta + \text{l.o.t.}_2.$$

The commutation between ∇_k and ∇_i produces the curvature term $\mathcal{D}_4 \overline{\mathbf{R}}_{liki} \nabla_l \psi$. The curvature obeys a pointwise estimate $|\overline{\mathbf{R}}| \leq \lambda^{-\bar{a}-(1-a)}$ and its covariant derivative $|\nabla_4 \overline{\mathbf{R}}| \leq \lambda^{-\bar{a}-2(1-a)}$. Hence, after an additional commutation of ∇_4 and ∇_k , we obtain

$$- \int_{\Sigma_t} t^2 \nabla_4 \nabla_k \psi \nabla_k \nabla_4 \nabla_i \nabla_i \psi \zeta + \text{l.o.t.}_2.$$

Another integration by parts and commutation gives

$$\int_{\Sigma_t} t^2 |\nabla_4(\Delta_h \psi)|^2 \zeta + \text{l.o.t.}_2.$$

Using the fact that $\Delta_h \psi = \partial_t^2 \psi + (\partial H)(\partial \psi)$ the estimate can be finished as in the end of the previous section.

The second case is when we have a mixed derivative $\partial_t \nabla \psi$. Again the most difficult term is $\int_{\Sigma_t} t^2 |\mathcal{D}_4(\partial_t \nabla \psi)|^2 \zeta$. Since $\partial_t = e_3 + N$ and the term with N falls into a category that has just been treated, it suffices to consider $\int_{\Sigma_t} t^2 |\mathcal{D}_4 \mathcal{D}_3(\nabla \psi)|^2 \zeta$. Note that the expression $\square_h(\nabla \psi)$ can be written relative to a null frame in the form⁷⁰

$$\square_h \nabla \psi = -\mathcal{D}_3 \mathcal{D}_4(\nabla \psi) + \mathbb{A}(\nabla \psi) + 2\eta_A \mathbb{V}_A(\nabla \psi) + \frac{1}{2} \text{tr} \chi \mathcal{D}_3(\nabla \psi) + \left(\frac{1}{2} \text{tr} \underline{\chi} + k_{NN}\right) \mathcal{D}_4(\nabla \psi).$$

and that $\square_h(\nabla \psi) = (\partial^2 H)(\partial \psi) + (\partial H)(\partial^2 H)$ if ψ verifies the wave equation. Hence, using the estimates for the Ricci coefficients $|\eta| + |\text{tr} \chi - \frac{2}{s}| + |\text{tr} \underline{\chi} - \frac{2}{s}| \leq \lambda^{-\bar{a}+\epsilon}$ and $|k| + |\partial H| \lesssim \lambda^{-\bar{a}}$, $|\partial^2 H| \lesssim \lambda^{-\bar{a}-(1-a)}$, we obtain

$$\int_{\Sigma_t} t^2 |\mathbb{A}(\nabla \psi)|^2 \zeta + \text{l.o.t.}_2$$

The expression under the integral sign above is clearly dominated by $|\mathbb{V}(\nabla^2 \psi)|^2$. Thus the case of a mixed derivative is reduced to the situation with 2 spatial derivatives considered above.

⁷⁰see (6.217)

7 Proof of Theorem (B)

As we have already observed, Theorem (B) is a consequence of the Boundedness Theorem 3.1, the Comparison Theorem 3.2, and the following:

Proposition 7.1 *Let ψ be a sufficiently smooth function supported in the region $s \leq t$ for $1 \leq t \leq t_*$. Then for any $\epsilon > 0$,*

$$\|\partial\psi(t)\|_{L_x^\infty} \lesssim \frac{1}{(1+t)} \mathcal{E}^{\frac{1}{2}}[\psi](t) + \frac{1}{(1+t)^{1-\frac{3}{2}\epsilon}} \mathcal{E}^{\frac{1}{2}(1-\epsilon)}[\psi](t) \left[\sum_{m=0}^3 \int_{\Sigma_1} |\partial\partial_x^m \psi|^2 \right]^{\frac{\epsilon}{2}} \quad (7.242)$$

Therefore, the proof of Theorem (B) will be complete with the proof of Proposition 7.1.

Proof: Recall that $\mathcal{E}[\psi] = \mathcal{E}_0[\psi] + \mathcal{E}_0[\partial\psi] + \mathcal{E}_0[\partial^2\psi]$. Also the conformal norm $\mathcal{E}_0[\psi] = \mathcal{E}_0^i[\psi] + \mathcal{E}_0^e[\psi]$,

$$\mathcal{E}_0^i[\psi](t) = \int_{\Sigma_t} \left(t^2 |\partial\psi|^2 + |\psi|^2 \right) (1 - \zeta), \quad \mathcal{E}_0^e[\psi](t) = \int_{\Sigma_t} \left(\underline{u}^2 |\mathcal{D}_4\psi|^2 + \underline{u}^2 |\nabla\psi|^2 \right) + u^2 |\mathcal{D}_3\psi|^2 + |\psi|^2 \zeta$$

with a cut-off function ζ equal to 1 in the region $s \geq \frac{t}{2}$. As a result, the interior part of the $\mathcal{E}[\psi]$,

$$\mathcal{E}^i[\psi](t) = \int_{\Sigma_t} \left(t^2 \sum_{m=1}^3 |\partial^m \psi|^2 + |\psi|^2 \right) (1 - \zeta).$$

Estimate for $\psi(1 - \zeta)$: We start with the interior estimate. Since in dimension $n = 3$ we have the embedding $H^2(\Sigma_t) \rightarrow L^\infty$, we derive

$$\|\partial(\psi(t)(1 - \zeta))\|_{L_x^\infty} \lesssim \|\partial^3(\psi(1 - \zeta))\|_{L^2(\Sigma_t)} + \|\partial(\psi(1 - \zeta))\|_{L^2(\Sigma_t)}.$$

It remains to observe that the derivatives of the cut-off function ζ are supported in the region $\frac{t}{4} \leq s \leq \frac{t}{2}$ and obey the estimates $|\partial^m \zeta| \lesssim t^{-m}$. Therefore ⁷¹, $\|\partial(\psi(t)(1 - \zeta))\|_{L_x^\infty}^2 \lesssim \frac{1}{t^2} \mathcal{E}^i[\psi](t)$ as desired.

Estimate for $\psi \zeta$: We have the identity $\partial(\psi \zeta) = \partial\psi \zeta + \psi \partial\zeta$. The term $\psi \partial\zeta$ obeys the estimate $\|\psi(t) \partial\zeta\|_{L_x^\infty} \lesssim \frac{1}{t} \|\psi(t)(1 - \zeta)\|_{L_x^\infty}$ with a slightly different cut-off function $\tilde{\zeta}$ and, thus can be easily treated with the help of the interior part of $\mathcal{E}[\psi](t)$.

Thus, the problem is reduced to establishing the estimate for $|\partial\psi(t, x)|$ at any point (t, x) with $\frac{t}{4} \leq s \leq t$. We shall show that for any such s

$$\sup_{S_{t,u}} |\partial\psi(t, x)|^2 \lesssim t^{-(2-3\epsilon)} \mathcal{E}^{1-\epsilon}[\psi](t) \left[\sum_{m=0}^3 \int_{\Sigma_1} |\partial\partial_x^m \psi|^2 \right]^\epsilon.$$

⁷¹with perhaps a slightly different ζ in the definition of $\mathcal{E}^i[\psi]$

According to the Sobolev inequality (5.139) with $p = 4$ of Theorem 5.2 we have ⁷² for any positive $\delta < 1$,

$$\sup_{S_{t,u}} |\partial\psi|^2 \lesssim t^{\frac{4\delta}{4+\delta}} \left(\int_{S_{t,u}} (|\nabla\partial\psi|^2 + \frac{1}{t^2} |\partial\psi|^2) \right)^{1-\frac{4\delta}{4+\delta}} \left[\int_{S_{t,u}} (|\nabla\partial\psi|^4 + \frac{1}{t^4} |\partial\psi|^4) \right]^{\frac{2\delta}{4+\delta}}.$$

We can also estimate

$$\begin{aligned} \left(\int_{S_{t,u}} |\partial\psi|^4 \right)^{\frac{1}{2}} &\lesssim \left(\int_{S_{t,u}} |\nabla\partial\psi|^2 \right)^{\frac{1}{2}} \left(\int_{S_{t,u}} |\partial\psi|^2 \right)^{\frac{1}{2}} + \frac{1}{t} \int_{S_{t,u}} |\partial\psi|^2, \\ \left(\int_{S_{t,u}} |\nabla\partial\psi|^4 \right)^{\frac{1}{2}} &\lesssim \left(\int_{S_{t,u}} |\nabla^2\partial\psi|^2 \right)^{\frac{1}{2}} \left(\int_{S_{t,u}} |\nabla\partial\psi|^2 \right)^{\frac{1}{2}} + \frac{1}{t} \int_{S_{t,u}} |\nabla\partial\psi|^2. \end{aligned}$$

In addition, observe that the trace theorem 5.3 for the foliation $S_{t,\cdot}$ parametrized by $s = t - u$ provides us with the estimate ⁷³

$$\int_{S_{t,u}} |f|^2 \lesssim \left(\int_{\Omega_{\frac{1}{4}s,s}} |N(f)|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_{\frac{1}{4}s,s}} |f|^2 \right)^{\frac{1}{2}} + \frac{1}{s} \int_{\Omega_{\frac{1}{4}s,s}} |f|^2.$$

Here, $\Omega_{\frac{1}{4}s,s} = \cup_{\rho \in [\frac{1}{4}s,s]} S_{t,t-\rho}$ and N is the vectorfield of the unit normals to $S_{t,t-\rho}$.

Thus, setting $\epsilon = \frac{4\delta}{4+\delta}$, using the fact that $\frac{1}{4} \leq \frac{t}{4} \leq s \leq t$, and applying the Hölder inequality, we obtain

$$\begin{aligned} \sup_{S_{t,u}} |\partial\psi|^2 &\lesssim t^\epsilon \left(\int_{\Omega_{\frac{1}{4}s,s}} |\nabla_N \nabla \partial\psi|^2 + |\nabla \partial\psi|^2 + \frac{1}{t^2} (|N(\partial\psi)|^2 + |\partial\psi|^2) \right)^{1-\epsilon} \\ &\times \left[\int_{\Omega_{\frac{1}{4}s,s}} |\nabla_N \nabla^2 \partial\psi|^2 + |\nabla^2 \partial\psi|^2 + |\nabla_N \nabla \partial\psi|^2 + |\nabla \partial\psi|^2 + \frac{1}{t^4} (|N(\partial\psi)|^2 + |\partial\psi|^2) \right]^\epsilon. \end{aligned} \quad (7.243)$$

We make the following two observations:

1) The second factor can be crudely bounded by $\sum_{m=0}^3 \int_{\Sigma_t} |\partial\partial_x^m \psi|^2$. Using the standard ⁷⁴ energy estimate, it is not difficult to show that the sum above is bounded $\sum_{m=0}^3 \int_{\Sigma_1} |\partial\partial_x^m \psi|^2$.

2) Using the fact that $N = \frac{1}{2}(e_4 - e_3)$ and the frame equations which imply that $\mathcal{D}_N e_A - \nabla_N e_A = \eta_A N - k_{AN} e_4$, we can conclude that $\nabla_N \nabla(\partial\psi) = \nabla N(\partial\psi) + \eta_A N(\partial\psi) - k_{AN} e_4(\psi)$. According to the Asymptotics Theorem, we have the bound $|\eta| + |k| \leq \lambda^{-\bar{a}+\epsilon}$. Since $t \leq t_* \leq \lambda^a \leq \lambda^{\bar{a}-\epsilon}$, we can estimate the first term in (7.243) by

$$\int_{\Sigma_t} \left(|\nabla(\partial^2\psi)|^2 + |\nabla(\partial\psi)|^2 + \frac{1}{t^2} |\partial^2\psi| + |\partial\psi|^2 \right) \leq t^{-2} \mathcal{E}[\psi](t).$$

⁷²taking into account that $\text{tr}\theta \leq \frac{3}{s}$, $A(S_{t,u}) \lesssim s^2$, and $\frac{t}{4} \leq s \leq \frac{t}{2}$

⁷³The tensor version of the estimate requires the covariant ∇_N derivative

⁷⁴based on the commutation with the time vectorfield ∂_t only

Combining all the estimates we obtain for the values of $s = t - u$, $\frac{t}{4} \leq s \leq t$

$$\sup_{S_{t,u}} |\partial\psi|^2 \lesssim t^{-2+3\epsilon} \mathcal{E}[\psi]^{1-\epsilon} \left[\sum_{m=0}^3 \int_{\Sigma_1} |\partial\partial_x^m \psi|^2 \right]^\epsilon,$$

as desired

8 Proof of the Reductions

In this Chapter we fill in the proofs of the reductions of the local well posedness for the quasilinear equation 1.1 stated in Theorem (A) to the proof of the decay estimate formulated in Theorem (B).

8.1 Step 1 Proof of the energy estimate

We start with the lemma providing the equation for ϕ^λ , a dyadic piece of ϕ . Recall that the notation $f_{\leq \lambda} = S_\lambda f$ refers to the truncation of function f in Fourier space *above* frequencies of size λ .

$$f_{\leq \lambda} = \sum_{\mu \leq \lambda} f^\mu.$$

Lemma 8.1 *Let ϕ satisfy the conditions of Proposition 1.1. Then for each dyadic λ , ϕ^λ verifies the equation*

$$\partial_t^2 \phi^\lambda - g_{\leq \lambda}^{ij}(\phi) \partial_i \partial_j \phi^\lambda = R_\lambda, \quad (8.244)$$

where for any $s > 1$ and $t \in [0, T]$ the right hand-side R_λ obeys the estimate

$$\left(\sum_\lambda \|R_\lambda(t)\|_{\dot{H}^{s-1}}^2 \right)^{\frac{1}{2}} \leq C \|\phi(t)\|_{\dot{H}^s} \|\partial \phi(t)\|_{L_x^\infty}. \quad (8.245)$$

with C a constant depending only on $\|\phi\|_{L_{[0,T]}^\infty L_x^\infty} \leq \Lambda_0$ and M_0 .

Proof of Lemma 8.1

Remark: Throughout this chapter we shall often ignore universal constants independent of M_0 and Λ_0 .

The proof of this lemma is based on the technique of the paradifferential calculus. Recall that P_λ denotes the projection on the frequencies of size λ , so that $\phi^\lambda = P_\lambda \phi$. Then

$$\partial_t^2 \phi^\lambda - P_\lambda(g^{ij}(\phi) \partial_i \partial_j \phi) = 0.$$

We introduce the notation

$$G \cdot \partial^2 \phi = g^{ij}(\phi) \partial_i \partial_j \phi.$$

Then

$$\begin{aligned} P_\lambda(G \cdot \partial^2 \phi) &= P_\lambda \sum_{\mu, \nu} G^\mu \cdot \partial^2 \phi^\nu = P_\lambda \sum_{\mu < \frac{1}{2}\nu, \nu} G^\mu \cdot \partial^2 \phi^\nu + \\ &P_\lambda \sum_{\nu < \frac{1}{2}\mu, \mu} G^\mu \cdot \partial^2 \phi^\nu + P_\lambda \sum_{\frac{1}{2}\nu \leq \mu \leq 2\nu, \nu} G^\mu \cdot \partial^2 \phi^\nu = E_1 + E_2 + E_3. \end{aligned}$$

It is clear that in the case when of one frequencies μ or ν dominate, the projection P_λ on the frequencies of size λ forces the dominant frequency to be of the same size. We say that $\mu \sim \lambda$ if $\frac{1}{4}\lambda \leq \mu \leq 4\lambda$.

Treatment of E_1

$$E_1 = \sum_{\mu < \frac{1}{2}\lambda} G^\mu \cdot \partial^2 \phi^\lambda + \sum_{\nu \sim \lambda} [P_\lambda, G_{\frac{1}{2}\nu} \cdot] \partial^2 \phi^\nu.$$

The first term is precisely the term to keep⁷⁵. To estimate the second we need a simple commutator lemma.

Lemma 8.2 *Let $p(\xi)$ defines a multiplier P and $m(x)$ defines a multiplication operator M . Then for any q such that $1 \leq q \leq \infty$ and any function $f \in L_x^q$*

$$\|[P, M]f\|_{L_x^q} \leq \|P'\| \|\partial m\|_{L_x^\infty} \|f\|_{L_x^q}. \quad (8.246)$$

Here P' is the multiplier corresponding to $\partial_\xi p$ and $\|P'\|$ is its norm.

Proof of Lemma 8.2

To prove this lemma it suffices to note that

$$\begin{aligned} f(x) &= \int P(x-y)(m(y) - m(x))f(y) dy \\ &= - \int P(x-y)(x-y)^i \int_0^1 \partial_i m(\tau y + (1-\tau)x) d\tau f(y) dy, \end{aligned}$$

and that the convolution with the $P(x)x^i$ corresponds to the multiplier with the symbol $\partial_\xi p(\xi)$.

Observe that the symbol of P_λ is the function $\zeta(\lambda^{-1}\xi)$. Therefore⁷⁶,

$$\begin{aligned} \left\| \sum_{\nu \sim \lambda} [P_\lambda, G_{\frac{1}{2}\nu} \cdot] \partial^2 \phi^\nu \right\|_{\dot{H}^{s-1}} &\approx \lambda^{s-1} \left\| \sum_{\nu \sim \lambda} [P_\lambda, G_{\frac{1}{2}\nu} \cdot] \partial^2 \phi^\nu \right\|_{L_x^2} \\ &\leq \lambda^s \sum_{\nu \sim \lambda} M_0 \|\partial \phi\|_{L^\infty} \|\phi^\nu\|_{L_x^2} \leq \sum_{\nu \sim \lambda} M_0 \|\partial \phi\|_{L^\infty} \|\phi^\nu\|_{\dot{H}^s}. \end{aligned}$$

Squaring and summing over λ we obtain the bound

$$8M_0 \|\partial \phi\|_{L^\infty} \|\phi\|_{\dot{H}^s}.$$

Treatment of E_2

$$E_2 = P_\lambda \sum_{\mu \sim \lambda} G^\mu \cdot \partial^2 \phi_{\leq \frac{1}{2}\mu}$$

Hence,

$$\|E_2\|_{\dot{H}^{s-1}} \leq \lambda^{s-1} \sum_{\mu \sim \lambda} \|G^\mu\|_{L_x^2} \|\partial^2 \phi_{\leq \frac{1}{2}\mu}\|_{L_x^\infty} \leq \sum_{\mu \sim \lambda} \lambda^{s-1} \mu \|G^\mu\|_{L_x^2} \|\partial \phi_{\leq \frac{1}{2}\mu}\|_{L_x^\infty} \leq \sum_{\mu \sim \lambda} M_0 \|G^\mu\|_{\dot{H}^s} \|\partial \phi\|_{L_x^\infty}.$$

Thus, squaring and summing over λ we obtain

$$\|E_2\|_{\dot{H}^{s-1}} \leq 8M_0 \|G\|_{\dot{H}^s} \|\partial \phi\|_{L_x^\infty}.$$

We need an additional standard result.

⁷⁵Observe that $\sum_{\mu < \frac{1}{2}\lambda} G^\mu \cdot \partial^2 \phi^\lambda = g_{\leq \frac{1}{2}\lambda}^{ij} \partial_i \partial_j \phi^\lambda - \sum_{\mu = \frac{1}{2}\lambda}^\lambda G^\mu \cdot \partial^2 \phi^\lambda$ and the second term is of the type E_3

⁷⁶Recall that the metric g obeys the condition $\sup_{|z| \leq \Lambda_0} \left| \left(\frac{d}{dz} \right)^l g \right| \leq M_0$ for all integer l : $0 \leq l \leq k$ and a sufficiently large k

Lemma 8.3 *Let $g(z)$ be a smooth function with the property that $|g|_{C^k} \leq M_0$ for a sufficiently large $k > 0$ and all $|z| \leq \Lambda_0$. Assume also that $\phi \in \dot{H}^s$ and $\|\phi\|_{L_x^\infty} \leq \Lambda_0$. Then*

$$\|g(\phi)\|_{\dot{H}^s} \leq C(M_0, \Lambda_0)\|\phi\|_{\dot{H}^s} \quad (8.247)$$

We can then conclude that

$$\|E_2\|_{\dot{H}^{s-1}} \leq C(M_0, \Lambda_0)\|\phi\|_{\dot{H}^s}\|\partial\phi\|_{L_x^\infty}.$$

Treatment of E_3

$$E_3 = P_\lambda \sum_{\nu \sim \mu, \nu > \frac{1}{4}\lambda} G^\mu \cdot \partial^2 \phi^\nu.$$

Hence,

$$\begin{aligned} \|E_3\|_{\dot{H}^{s-1}} &\leq \lambda^{s-1} \sum_{\nu \sim \mu, \nu > \frac{1}{4}\lambda} \|G^\mu\|_{L_x^\infty} \|\partial^2 \phi^\nu\|_{L_x^2} \leq \sum_{\nu \sim \mu, \nu > \frac{1}{4}\lambda} \left(\frac{\lambda}{\nu}\right)^{s-1} \|\partial G^\mu\|_{L_x^\infty} \|\phi^\nu\|_{\dot{H}^s} \\ &\leq M_0 \sum_{\nu, \nu > \frac{1}{4}\lambda} \left(\frac{\lambda}{\nu}\right)^{s-1} \|\phi^\nu\|_{\dot{H}^s} \|\partial\phi\|_{L_x^\infty}. \end{aligned}$$

Clearly, if $s > 1$ then the convolution with $\nu^{(1-s)}$ maps $l^2 \rightarrow l^2$. Thus,

$$\|E_3\|_{\dot{H}^{s-1}} \leq 8M_0\|\phi\|_{\dot{H}^s}\|\partial\phi\|_{L_x^\infty}.$$

We are ready to finish the proof of Proposition 1.1. Choose a large parameter Λ in such a way that for any $\lambda \geq \Lambda$ the metric $g_{\leq \lambda}^{ij}(\phi)$ is elliptic with a constant of ellipticity $2M_0$. This is always possible since S_λ is an approximation of the identity and the original metric $g^{ij}(\phi)$ is elliptic with a constant M_0 .

For the values of the dyadic parameter $\lambda < \Lambda$ rewrite the equation for ϕ^λ in the form

$$\partial_t^2 \phi^\lambda - g_{\leq \Lambda}^{ij} \partial_i \partial_j \phi^\lambda = R_\lambda$$

noting that the change of the metric introduces the error term of the type E_2 .

For $\lambda \geq \Lambda$ we keep the form of the equation as in Lemma 8.1

$$\partial_t^2 \phi^\lambda - g_{\leq \lambda}^{ij} \partial_i \partial_j \phi^\lambda = R_\lambda$$

In either case, the standard H^1 energy estimate for the wave equation yields

$$\|\phi^\lambda\|_{L_{[0,T]}^\infty \dot{H}^1} \leq 2M_0(\|\phi^\lambda[0]\|_{\dot{H}^1} + \|R_\lambda\|_{L_{[0,T]}^1 L_x^2}).$$

Using Lemma 8.1 and the Gronwall inequality we immediately obtain for $s > 1$

$$\|\phi\|_{L_{[0,T]}^\infty \dot{H}^s} \leq C(\Lambda_0, M_0) \exp(\|\partial\phi\|_{L_{[0,T]}^1 L_x^\infty}) \|\phi[0]\|_{\dot{H}^s}.$$

The estimate for $s = 1$ is standard and does not require the paradifferential decomposition.

8.2 Step 3 Proof of Proposition 1.2

The paradifferential approximation needed in Proposition 1.2 can be obtained from the paradifferential approximation of Lemma 8.1 by a further truncation (mollification) of the frequencies of the metric $g^{ij}(\phi)$. In particular, it is easy to see that

$$\bar{\square}_{g_{\leq \lambda^a}} \phi^\lambda = -\partial_t^2 \phi^\lambda + g_{\leq \lambda^a}^{ij} \partial_i \partial_j \phi^\lambda = -R_\lambda + (g_{\leq \lambda}^{ij}(\phi) - g_{\leq \lambda^a}^{ij}) \partial_i \partial_j \phi^\lambda.$$

The term R_λ satisfies in fact a better estimate than stated in Proposition 1.2. It remains to handle $(g_{\leq \lambda}^{ij}(\phi) - g_{\leq \lambda^a}^{ij}) \partial_i \partial_j \phi^\lambda$.

First, we have

$$S_\lambda g^{ij}(\phi) - S_{\lambda^a} g^{ij}(S_{\lambda^a} \phi) = S_\lambda g^{ij}(\phi) - S_{\lambda^a} g^{ij}(\phi) + S_{\lambda^a} g^{ij}(\phi) - S_{\lambda^a} g^{ij}(S_{\lambda^a} \phi).$$

The Fourier support of the term $(S_\lambda - S_{\lambda^a})g^{ij}(\phi)$ belongs to the region $\lambda^a \leq |\xi| \leq \lambda$. Therefore,

$$\begin{aligned} \|(S_\lambda - S_{\lambda^a})g^{ij}(\phi) \partial_i \partial_j \phi^\lambda\|_{\dot{H}^{s-1}} &\leq \lambda^{s-1} \|(S_\lambda - S_{\lambda^a})g^{ij}(\phi)\|_{L_x^\infty} \|\partial_i \partial_j \phi^\lambda\|_{L_x^2} \\ &\leq \lambda^{1-a} M_0 \|\partial \phi\|_{L_x^\infty} \|\phi^\lambda\|_{\dot{H}^s} \leq M_0 \lambda^{1-a} c_\lambda \|\partial \phi\|_{L_x^\infty} \|\phi\|_{\dot{H}^s}. \end{aligned}$$

We also have

$$S_{\lambda^a} g^{ij}(\phi) - S_{\lambda^a} g^{ij}(S_{\lambda^a} \phi) = S_\lambda \int_0^1 g^{ij'}(\tau \phi + (1-\tau)S_{\lambda^a} \phi) d\tau (Id - S_{\lambda^a}) \phi.$$

and the function $(Id - S_{\lambda^a})\phi$ has its Fourier support only for $|\xi| \geq \lambda^a$. Therefore,

$$\begin{aligned} \|S_\lambda (g^{ij}(\phi) - g^{ij}(S_{\lambda^a} \phi)) \partial_i \partial_j \phi^\lambda\|_{\dot{H}^{s-1}} &\leq \lambda^{s-1} M_0 \|(Id - S_{\lambda^a})\phi\|_{L_x^\infty} \|\partial_i \partial_j \phi^\lambda\|_{L_x^2} \\ &\leq \lambda^{1-a} M_0 \|\partial \phi\|_{L_x^\infty} \|\phi^\lambda\|_{\dot{H}^s} \leq M_0 \lambda^{1-a} c_\lambda \|\partial \phi\|_{L_x^\infty} \|\phi\|_{\dot{H}^s}. \end{aligned}$$

8.3 Proof of the implication: Theorem (A3) \rightarrow Theorem (A2)

We assume that Theorem (A3) has been already proved with the partition I as described in the Remark (A3). The Duhamel formula immediately implies the inhomogeneous version of the Strichartz estimate. Namely, for any function ψ such that the supp $\widehat{\bar{\square}_{g_{\leq \lambda^a}} \psi} \subset \{\frac{1}{4}\lambda \leq |\xi| \leq 4\lambda\}$ we have on each time interval I

$$\|P_\lambda \partial \psi\|_{L_t^2 L_x^\infty} \leq C(B_0) |I|^\epsilon (\|\partial \psi[0]\|_{\dot{H}^{1+\epsilon}} + \|\bar{\square}_{g_{\lambda^a}} \psi\|_{L_t^1 \dot{H}^{1+\epsilon}}).$$

Proposition 1.2 implies that the estimate above can be applied to ϕ^λ , the λ -dyadic piece of the solution of the quasilinear equation. Note that $\bar{\square}_{g_{\lambda^a}} \phi^\lambda = R_\lambda^a$ and for any $s > 1$ we have the estimate $\|R_\lambda^a(t)\|_{\dot{H}^s} \leq c_\lambda \lambda^{1-a} \|\partial \phi(t)\|_{L_x^\infty} \|\phi(t)\|_{\dot{H}^s}$. From the Remark (A3) and the Hölder inequality it follows that on each time interval I we have $\|\partial \phi(t)\|_{L_t^1 L_x^\infty} \leq \lambda^{-(1-a)} B_0$. Therefore,

$$\|R_\lambda^a\|_{L_t^1 \dot{H}^{1+\epsilon}} \leq B_0 \lambda^{2+\epsilon-s^*} c_\lambda \|\phi\|_{L_{[0,T]}^\infty \dot{H}^{s^*}}$$

Hence, from the Strichartz estimate

$$\|\partial \phi^\lambda\|_{L_t^2 L_x^\infty} \leq C(B_0) |I|^\epsilon (\|\partial \phi^\lambda\|_{L_t^\infty \dot{H}^{1+\epsilon}} + B_0 \lambda^{2+\epsilon-s^*} c_\lambda \|\phi\|_{L_{[0,T]}^\infty \dot{H}^{s^*}}).$$

Squaring and summing over I we obtain after taking into account that the number of intervals $\#I \sim \lambda^{1-a}$ and that $|I| \leq T\lambda^{-(1-a)}$

$$\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} \leq C(B_0) T^\epsilon \lambda^{-\epsilon(1-a)} \lambda^{\frac{1-a}{2}} (\|\partial\phi^\lambda\|_{L^\infty_{[0,T]}\dot{H}^{1+\epsilon}} + \lambda^{2+\epsilon-s_*} c_\lambda \|\phi\|_{L^\infty_{[0,T]}\dot{H}^{s_*}}).$$

Therefore, we obtain the desired estimate

$$\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} \leq C(B_0) T^\epsilon c_\lambda \|\phi\|_{L^\infty_{[0,T]}\dot{H}^{s_*}},$$

provided that $s_* + \epsilon(1-a) - 2 - \epsilon - \frac{1}{2}(1-a) \geq 0$. This requires that $a \geq \frac{5-2s_*}{1-2\epsilon}$. Since in the Strichartz estimate of Theorem (A2) we have $T^{s_*-s_0}$, parameter ϵ should be chosen $\epsilon = s_* - s_0$ and $a = \frac{5-2s_*}{1-2s_*+2s_0}$. It is easy to check that since $s_* > s_0 = 2 + \frac{1}{2}(2 - \sqrt{3})$, the value of the parameter $a < -1 + \sqrt{3}$.

8.4 Proof of the properties of the metric $g_{\leq\lambda^a}$

In this section we provide the proof of Proposition 1.3. First we note that it is sufficient to verify (1.11)-(1.15) with $m = 0$. This is due to the fact that $\partial^{1+m}g_{\leq\lambda^a} = (\partial^m S_{\lambda^a})(\partial g(S_{\lambda^a}\phi))$ and each differentiation of the projector S_{λ^a} produces roughly a factor λ^a . The only exception is (1.15) since it is nonlinear in $g_{\leq\lambda^a}$. In this case we can argue that $\bar{\square}_{g_{\leq\lambda^a}}g_{\leq\lambda^a}$ is essentially a product of two terms such that the Fourier transforms of each of them are supported in the region $|\xi| \leq \lambda^a$. Therefore, the result also has support in roughly the same region of Fourier space. Hence each differentiation again introduces the factor of λ^a .

Remark: *To be precise, the argument above works only for the spatial derivatives ∂_x^m , since S_{λ^a} truncates the frequencies $\geq \lambda^a$ only with respect to the space variable x . However, using the fact that g depends on ϕ , a solution of the wave equation, one can recover the corresponding estimates for the time derivatives. Let us illustrate this by proving the estimate (1.11) with $m = 1$. This is one of the few estimates with $m \neq 0$ that we actually use. We assume that we have already proved (1.11)-(1.15) for $m = 0$. Then, clearly the derivatives $\partial_x^2 g_{\leq\lambda^a}$ and $\partial_x \partial_t g_{\leq\lambda^a}$ can be estimated with an additional factor of λ^a . It remains to address the derivative $\partial_t^2 g_{\leq\lambda^a}$. Observe that $\partial_t^2 = \bar{\square}_{g_{\leq\lambda^a}} + g_{\leq\lambda^a}^{ij} \partial_i \partial_j$. Therefore, using the condition (1.15) with $m = 0$ and the fact that ∂_i, ∂_j are spatial derivatives, we obtain*

$$\|\partial_t^2 g_{\leq\lambda^a}\|_{L^1_t L^\infty_x} \leq \lambda^{-(1-a)} \bar{B}_0 + \lambda^{-(1-a)+a} \bar{B}_0 \leq \lambda^{-(1-a)+a} \bar{B}_0.$$

Proof of (1.11)-(1.15) for $m = 0$: The inequality (1.12) with $m = 0$ follows immediately from inequality (1.10) defining the partition I , since

$$\|\partial g_{\leq\lambda^a}\|_{L^2_t L^\infty_x} \leq M_0 \|\partial\phi\|_{L^2_t L^\infty_x} \leq \lambda^{-\frac{1-a}{2}} M_0 B_0$$

The Hölder inequality yields (1.11) from (1.11).

To prove (1.13) we have the following chain of estimates. Recall that the value of the parameter a is chosen such that $a > 5 - 2s_*$ and that $s_* > 2 + \frac{1}{2}(2 - \sqrt{3})$. Let c be the constant of the embedding $H^{\frac{3}{2}+\epsilon}(\mathbf{R}^3) \subset L^\infty(\mathbf{R}^3)$. Then

$$\begin{aligned} \|\partial(S_{\lambda^a} g(S_{\lambda^a} \phi))\|_{L^1_t L^\infty_x} &\leq M_0 \|\partial(S_{\lambda^a} \phi)\|_{L^1_t L^\infty_x} \leq M_0 c \|\partial(S_{\lambda^a} \phi)\|_{L^1_t H^{\frac{3}{2}+\epsilon}} \\ &\leq M_0 c \lambda^{a(\frac{5}{2}+\epsilon-s_*)} \|\partial\phi\|_{L^1_t H^{s_*}} \lesssim \lambda^{\frac{a}{2}} M_0 c B_0. \end{aligned} \tag{8.248}$$

The estimate (1.14) is scale equivalent to (1.13). We use Lemma 8.3 to obtain

$$\|\partial^{\frac{3}{2}+1} g_{\leq \lambda^a}\|_{L_x^2} \leq C(\Lambda_0, M_0) \|\partial^{\frac{3}{2}+1} S_{\lambda^a} \phi\|_{L_x^2}.$$

The proof now proceeds in the same way as the proof of (1.13).

The most interesting part of the Lemma is estimate (1.15). It is a true reflection of the fact that metric $g_{\leq \lambda^a}$ depends on the solution of the quasilinear wave equation. To emphasize its importance we formulate it as a separate lemma.

In the following, for brevity, we will suppress the argument $S_{\lambda^a} \phi$ of the metric $g_{\leq \lambda^a}$.

Lemma 8.4 *Each component of the metric $g_{\leq \lambda^a}^{ij} = S_{\lambda^a} g^{ij}(S_{\lambda^a}(\phi))$ obeys the estimate*

$$\|\bar{\square}_{g_{\leq \lambda^a}} g_{\leq \lambda^a}\|_{L_t^1 L_x^\infty} \leq \lambda^{-(1-a)} \bar{B}_0.$$

Here, ϕ is a solution of the quasilinear problem (1.1) and the time interval I described in the Remark (A3) after Theorem 1.4.

Proof: We compute

$$\begin{aligned} \bar{\square}_{g_{\leq \lambda^a}} g_{\leq \lambda^a} &= -\partial_t^2 g_{\leq \lambda^a} + g_{\leq \lambda^a}^{ij} \partial_i \partial_j g_{\leq \lambda^a} \\ &= -S_{\lambda^a} g''(S_{\lambda^a}(\partial_t \phi))^2 + g_{\leq \lambda^a}^{ij} S_{\lambda^a}(g'' \partial_i(S_{\lambda^a} \phi) \partial_j(S_{\lambda^a} \phi)) \\ &\quad + S_{\lambda^a}(g'(S_{\lambda^a}(-\partial_t^2 \phi))) + g_{\leq \lambda^a}^{ij} S_{\lambda^a}(g'(S_{\lambda^a}(\partial_i \partial_j \phi))). \end{aligned} \quad (8.249)$$

The first two terms on the right-hand side of (8.249) are quadratic in $\partial \phi$. Thus inequality (1.10) imply that its $L_t^1 L_x^\infty$ -norm are bounded by $\lambda^{-(1-a)} M_0 B_0^2$. In the remaining two terms we need to do some commutation and use the fact that ϕ is a solution of the equation $-\partial_t^2 \phi + g^{ij} \partial_i \partial_j \phi = 0$.

As in the proof of Proposition 1.2 we have

$$\begin{aligned} g_{\leq \lambda^a}^{ij} - g^{ij} &= S_{\lambda^a} g^{ij}(S_{\lambda^a} \phi) - g^{ij}(\phi) = (S_{\lambda^a} - Id) g^{ij}(S_{\lambda^a} \phi) + g^{ij}(S_{\lambda^a} \phi) - g^{ij}(\phi) \\ &= (S_{\lambda^a} - Id) g^{ij}(S_{\lambda^a} \phi) + \int_0^1 g^{ij, \tau}(\tau S_{\lambda^a} \phi + (1-\tau)\phi) d\tau (S_{\lambda^a} - Id) \phi. \end{aligned} \quad (8.250)$$

Note again that $Id - S_{\lambda^a}$ is a projector removing frequencies $|\xi| \leq \lambda^a$. Therefore,

$$\|g_{\leq \lambda^a}^{ij} - g^{ij}\|_{L_t^2 L_x^\infty} \leq \lambda^{-a} M_0 \|\partial \phi\|_{L_t^2 L_x^\infty} \leq \lambda^{-a} \lambda^{-\frac{1-a}{2}} M_0 B_0. \quad (8.251)$$

Then

$$\begin{aligned} \|(g_{\leq \lambda^a}^{ij} - g^{ij}) S_{\lambda^a}(g'(\bar{S}_{\lambda^a}(\partial_i \partial_j \phi)))\|_{L_t^1 L_x^\infty} &\leq \|g_{\leq \lambda^a}^{ij} - g^{ij}\|_{L_t^2 L_x^\infty} \|S_{\lambda^a}(g'(S_{\lambda^a}(\partial_i \partial_j \phi)))\|_{L_t^2 L_x^\infty} \leq \\ &\leq \lambda^{-a} \lambda^{-\frac{1-a}{2}} \lambda^a M_0^2 B_0 \|\partial \phi\|_{L_t^2 L_x^\infty} \leq \lambda^{-(1-a)} M_0^2 B_0^2. \end{aligned} \quad (8.252)$$

Therefore, we consider the term $g^{ij} S_{\lambda^a}(g'(S_{\lambda^a}(\partial_i \partial_j \phi)))$. We have

$$g^{ij} S_{\lambda^a}(g'(S_{\lambda^a}(\partial_i \partial_j \phi))) = [g^{ij}, S_{\lambda^a}](g'(S_{\lambda^a}(\partial_i \partial_j \phi))) + S_{\lambda^a}(g^{ij} g'(S_{\lambda^a}(\partial_i \partial_j \phi))).$$

Using Lemma 8.2 we conclude that

$$\|[g^{ij}, S_{\lambda^a}](g'(S_{\lambda^a}(\partial_{ij}\phi)))\|_{L^1_t L^\infty_x} \leq \lambda^{-a} M_0^2 \|\partial\phi\|_{L^2_t L^\infty_x} \|S_{\lambda^a}(\partial_i \partial_j \phi)\|_{L^2_t L^\infty_x} \leq \lambda^{-(1-a)} M_0^2 B_0^2.$$

Clearly, g^{ij} and g' commute. Consider the identity

$$g^{ij} S_{\lambda^a}(\partial_i \partial_j \phi) = [g^{ij}, S_{\lambda^a} \partial_i] \partial_j \phi + S_{\lambda^a}(\partial_i g^{ij} \partial_j \phi).$$

The second term combined together with the third term of (8.249) produces $\bar{\square}_g \phi$ which is zero. To estimate the first term we use Lemma 8.2 again. Note that $S_{\lambda^a} \partial_i$ is a multiplier determined by the function $\zeta(\lambda^{-a} \xi) \xi_i$, where ζ is a cut-off function equal to one on the unit disk. Therefore,

$$\|[g^{ij}, S_{\lambda^a} \partial_i] \partial_j \phi\|_{L^1_t L^\infty_x} \lesssim M_0 \|\partial\phi\|_{L^2_t L^\infty_x}^2 \leq \lambda^{-(1-a)} M_0 B_0^2.$$

Gathering all estimates together

$$\|\bar{\square}_{g \leq \lambda^a} g \leq \lambda^a\|_{L^1_t L^\infty_x} \leq \lambda^{-(1-a)} M_0^2 B_0^2.$$

8.5 Step 6 Proof of the implication: Theorem (B) \rightarrow Theorem (A6)

Assume that Theorem (B) has been already proved. Consider function ψ verifying the geometric wave equation $\square_{h_\lambda} \psi = 0$ with initial data $\psi[0]$ supported in the set $|\xi| \in [\frac{1}{2}, 2]$ in Fourier space.

Let functions $\chi_J(x)$ form a partition of unity subordinate to a covering of \mathbf{R}^3 by balls $B_{\frac{1}{2}}(x_J)$ of radius $\frac{1}{2}$ such that at most 5 balls intersect at one point. In addition, assume that

$$|\chi_J|_{C^{m+2}} \leq C \tag{8.253}$$

for some sufficiently large integer m uniformly in J .

Let $\psi_J(t)$ denote the solution of the geometric wave equation $\square_{h_\lambda} \psi_J = 0$ with initial data $\psi_J[0] = \chi_J \psi[0]$. By Theorem (B)

$$\|P \partial \psi_J(t)\|_{L^\infty_x} \lesssim \frac{1}{(1+|t|)^{1-\epsilon}} \sum_{k=0}^m \|\partial^k \psi_J[0]\|_{L^2_x},$$

for some sufficiently large integer m . Since $\psi_J[0]$ has compact support of size one in physical space, by Sobolev inequality

$$\|\partial^k \psi_J[0]\|_{L^2_x} \lesssim \|\partial^{k+2} \psi_J[0]\|_{L^1_x} \lesssim \sum_{r+s=k+2} \|\partial^r \chi\|_{L^\infty} \|\partial^s \psi[0]\|_{L^1(B_{\frac{1}{2}}(x_J))}$$

Using the property of the covering and (8.253) we obtain after summing over J

$$\|P \partial \psi(t)\|_{L^\infty_x} \lesssim \frac{1}{(1+|t|)^{1-\epsilon}} \sum_{k=0}^{m+2} \|\partial^k \psi[0]\|_{L^1_x},$$

The fact that the Fourier support of $\psi[0]$ lies in the dyadic shell of size 1 allows one to complete the proof of the implication.

8.6 Proof of the implication Theorem (A6) \rightarrow Theorem (A5); Decay \rightarrow Strichartz

On this step of the reduction of Theorem (A) to Theorem (B) we show that Theorem (A6) implies Theorem (A5). Thus, we shall assume that the family of metrics h_λ satisfies conditions (1.19)-(1.23) and that any solution of the geometric wave equation $\square_{h_\lambda} \psi = 0$, with Fourier support of the initial data in the set $\{|\xi| \in [\frac{1}{2}, 2]\}$, obeys the decay estimate $\|P \partial \psi(t)\|_{L_x^\infty} \lesssim \frac{1}{(1+|t|)^{1-\epsilon}} \|\psi[0]\|_{L_x^1}$. We need to show that under these assumptions any solution⁷⁷ of the wave equation $\square_{h_\lambda} \phi = 0$ satisfies the Strichartz estimate $\|P \partial \phi\|_{L_{[0,t_*]}^2 L_x^\infty} \lesssim |t_*|^\epsilon \|\psi[0]\|_{L_x^2}$.

First, observe that it suffices to prove the following estimate:

$$\|P \partial \phi\|_{L_{[0,t_*]}^q L_x^\infty} \leq M \|\phi[0]\|_{L_x^2} \quad (8.254)$$

with $\epsilon = 1 - \frac{2}{q}$ and a constant M independent of ϕ and λ .

Observe also that the solutions of either the geometric wave equation $\square_{h_\lambda} \psi = F$ or the equation $\bar{\square}_{h_\lambda} \psi = F$ obey the following energy inequality for any $t, t_0 \in [t_0, t_*]$:

$$\begin{aligned} \|\partial \psi(t)\|_{L_x^2} &\leq \exp(C \|\partial h_\lambda\|_{L_{[0,t_*]}^1 L_x^\infty}) \left(\|\partial \psi(t_0)\|_{L_x^2} + \|F\|_{L_{[0,t_*]}^1 L_x^2} \right) \\ &\leq 2 \left(\|\partial \psi(t_0)\|_{L_x^2} + \|F\|_{L_{[0,t_*]}^1 L_x^2} \right), \end{aligned} \quad (8.255)$$

where the last inequality follows⁷⁸ from the condition (1.19) on the metric h_λ .

Furthermore, since⁷⁹

$$\square_{h_\lambda} = \bar{\square}_{h_\lambda} + \frac{1}{\sqrt{|h_\lambda|}} \partial_t (\sqrt{|h_\lambda|}) \partial_t + \partial_i (\sqrt{|h_\lambda|} h^{ij}) \partial_j,$$

it is also not difficult to show⁸⁰ that it suffices to establish (8.254) for a solution the geometric wave equation. We shall now prove a stronger result.

Proposition 8.1 *Let ϕ verifies the wave equation $\square_h \phi = 0$. Assume that the metric h is elliptic⁸¹ and satisfies the condition*

$$C \|\partial h\|_{L_{[0,t_*]}^1 L_x^\infty} \leq \frac{1}{2} \quad (8.256)$$

for some sufficiently large positive constant C . Then for any $q > 2$ and for some universal constant M ,

$$\|P \partial \phi\|_{L_{[0,t_*]}^q L_x^\infty} \leq M \|\phi[0]\|_{L_x^2}, \quad (8.257)$$

provided that the conclusions of Theorem (A6) hold true.

⁷⁷remark that we don't require the Fourier support assumption on the initial data. This is due to the presence of the projection P in the estimate.

⁷⁸Recall that we consider $\lambda \geq \Lambda$ for a sufficiently large constant Λ

⁷⁹ $|h_\lambda| = \det h_\lambda$

⁸⁰By the Duhamel Principle we would obtain

$$\|P \partial \phi\|_{L_{[0,t_*]}^q L_x^\infty} \leq M (\|\phi[0]\|_{L_x^2} + \|\partial h\|_{L_{[0,t_*]}^1 L_x^\infty} \|\partial \phi\|_{L_{[0,t_*]}^\infty L_x^2})$$

and the condition (1.19) together with the energy inequality for ϕ would imply (8.254).

⁸¹for simplicity we can assume that the ellipticity constant is 2

Remark: We first prove the estimate (8.257) for $P \partial_t \phi$.

Proof: Clearly, we can assume that (8.257) holds true for a sufficiently large constant M dependent on λ . We shall establish a uniform bound on M .

Let $(w_0, w_1) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$, $w = (w_0, w_1)$ and $\Phi(t, s; w)$ be the vector $(\phi, \partial_t \phi)$, where $\phi(t, s; w)$ denotes the solution at time t of the homogeneous equation $\square_h \phi = 0$ subject to the initial data at time s , $\phi(s, s; w) = w_0, \partial_t \phi(s, s; w) = w_1$. By a standard uniqueness argument ⁸² we can easily prove the following:

$$\Phi\left(t, s; \Phi(s, t_0; w)\right) = \Phi(t, t_0; w) \quad (8.258)$$

Denote by \mathcal{H} be the set of vector functions $w = (w_0, w_1)$ with $(w_0, w_1) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$. The scalar product in \mathcal{H} is defined by

$$\langle w, v \rangle = \int_{\Sigma_0} \left(w_1 \cdot v_1 + h^{ij} \partial_j w_0 \cdot \partial_j v_0 \right)$$

Let $X = L^q_{[0, t_*]} L^\infty_x$ and its dual $X' = L^{q'}_{[0, t_*]} L^1_x$. Let \mathcal{T} be the operator from \mathcal{H} to X defined by:

$$\mathcal{T}(w) = -P \partial_t \phi(t, 0; w) \quad (8.259)$$

The adjoint \mathcal{T}^* is defined from X to \mathcal{H} . To prove estimate (8.257) it suffices to prove that $\mathcal{T} \cdot \mathcal{T}^*$ is a bounded operator from X' to X . By our bootstrap assumption we have $\|\mathcal{T}\|_{\mathcal{H} \rightarrow X} = M$ where $\|\mathcal{T}\|_{\mathcal{H} \rightarrow X}$ denotes the operator norm of \mathcal{T} . Thus,

$$\|\mathcal{T} \cdot \mathcal{T}^*\|_{X' \rightarrow X} = M^2.$$

To calculate \mathcal{T}^* we write,

$$\langle \mathcal{T}^* f, w \rangle := \langle f, \mathcal{T}(w) \rangle = - \int_{[0, t_*] \times \mathbf{R}^3} \partial_t \phi P f dt dx = - \int_{[0, t_*] \times \mathbf{R}^3} \partial_t \phi \tilde{\square}_h \psi,$$

where ψ is the unique solution to the equation

$$\begin{aligned} \tilde{\square}_h \psi &= \partial_t^2 \psi - \partial_i (h^{ij} \partial_j \psi) = P f, \\ \phi(t_*) &= \partial_t \phi(t_*) = 0. \end{aligned} \quad (8.260)$$

Consequently, integrating by parts, and observing that

$$\partial_t^2 \phi = \frac{1}{\sqrt{|h|}} \left(\partial_i (\sqrt{|h|} h^{ij} \partial_j \phi) - \partial_t (\sqrt{|h|}) \partial_t \phi \right)$$

⁸²which follows from the energy estimate (8.255), which still holds under assumption (8.256) on the metric h

we obtain

$$\begin{aligned}
\langle \mathcal{T}^* f, w \rangle &= \int_{\Sigma_0} \left(\partial_t \phi(0) \partial_t \psi(0) - \partial_t^2 \phi(0) \psi(0) \right) - \int_{[0, t_*] \times \mathbf{R}^3} \tilde{\square}_h \partial_t \phi \psi \\
&= \int_{\Sigma_0} \left(\partial_t \phi(0) \partial_t \psi(0) + h^{ij} \partial_i \phi(0) \partial_j \psi(0) \right) - \int_{\Sigma_0} \frac{1}{\sqrt{|h|}} \left(h^{ij} \partial_i (\sqrt{|h|}) \partial_j \phi(0) \right. \\
&\quad \left. - \partial_t (\sqrt{|h|}) \partial_t \phi(0) \right) \psi(0) - \int_{[0, t_*] \times \mathbf{R}^3} \tilde{\square}_h \partial_t \phi \psi \\
&= \langle w, \phi[0] + R(f) \rangle_{\mathcal{H}}
\end{aligned}$$

with $\phi[0] = (\phi(0), \partial_t \phi(0))$ and $R(f)$ the linear operator defined from X' to \mathcal{H} by the formula,

$$\langle I, R(f) \rangle = - \int_{\Sigma_0} \frac{1}{\sqrt{|h|}} \left(h^{ij} \partial_i (\sqrt{|h|}) \partial_j \phi(0) - \partial_t (\sqrt{|h|}) \partial_t \phi(0) \right) \psi(0) - \int_{[0, t_*] \times \mathbf{R}^3} \tilde{\square}_h \partial_t \phi \psi$$

Therefore,

$$\mathcal{T} \mathcal{T}^* f = \mathcal{T} \phi[0] + \mathcal{T} R(f) \tag{8.261}$$

Observe that $\square_{h_\lambda} \psi = Pf + e$ with $e = \frac{1}{\sqrt{|h|}} \left(\partial_t (\sqrt{|h|}) \partial_t \psi - \partial_i (\sqrt{|h|}) h^{ij} \partial_j \psi \right)$. Thus we can write $\psi = \psi_1 + \psi_2$ with,

$$\begin{aligned}
\square_h \psi_1 &= Pf, \\
\square_h \psi_2 &= e
\end{aligned}$$

with both ψ_1, ψ_2 verifying the zero initial conditions in (8.260). Now $\mathcal{T} \psi[0] = \mathcal{T} \psi_1[0] + \mathcal{T} \psi_2[0]$ and $\mathcal{T} \psi_1[0] = -P \partial_t \Phi(t, 0; \psi_1[0])$. According to the Duhamel Principle we have, $\psi_1(t) = \int_{t_*}^0 \Phi(t, s; F(s)) ds$ with $F(s) = (0, Pf(s))$ and therefore,

$$\psi_1[0] = - \int_0^{t_*} \Phi(0, s; F(s)) ds$$

and,

$$\mathcal{T} \psi_1[0] = P \partial_t \Phi \left(t, 0; \int_0^{t_*} \Phi(0, s; F(s)) ds \right) = P \int_0^{t_*} \partial_t \Phi(t, s; F(s)) ds.$$

We are now in a position to apply the dispersive inequality of Theorem **(A6)**. Indeed, since the space Fourier transform of $F(s) = (0, Pf(s))$ is supported in the region $|\xi| \in [\frac{1}{2}, 2]$,

$$\|P \partial_t \Phi(t, s; F(s))\|_{L^\infty} \leq C(1 + |t - s|)^{-1+\epsilon} \|Pf(s)\|_{L^1}.$$

Therefore, by the Hardy-Littlewood-Sobolev inequality,

$$\|\mathcal{T}\psi_1[0]\|_{L^q_{[0,t_*]L^\infty_x}} \leq C\|f\|_{L^{q'}_{[0,t_*]L^1_x}} \quad (8.262)$$

with C a constant, independent of λ . It depends in fact only on the H-L-S constant, $\epsilon > 0$, and the constant C in the dispersive inequality of Theorem (A6).

To estimate $\mathcal{T}\psi_2[0]$ we apply the Strichartz inequality with bound M ,

$$\|\mathcal{T}\psi_2[0]\|_{L^q_{[0,t_*]L^\infty_x}} \leq M\|\psi_2[0]\|_{\mathcal{H}}$$

where,

$$\|\psi_2[0]\|_{\mathcal{H}} = \sup_{\|w\|_{\mathcal{H}} \leq 1} \langle w, \psi_2[0] \rangle_{\mathcal{H}} \leq \|\partial\psi_2(0)\|_{L^2}.$$

To estimate we shall make use of the energy estimate (8.255) for ψ_2 verifying the equation $\square_h\psi_2 = e$, subject to the initial conditions $\psi_2(t_*) = \partial_t\psi_2(t_*) = 0$,

$$\|\partial\psi_2(0)\|_{L^2_x} \leq C\|e\|_{L^1_{[0,t_*]L^2_x}} \leq C\|\partial h\|_{L^1_{[0,t_*]L^\infty_x}} \|\partial\psi\|_{L^\infty_{[0,t_*]L^2_x}}$$

Therefore, with the help of the condition (8.256), we have

$$\|\mathcal{T}\psi_2[0]\|_{L^q_{[0,t_*]L^\infty_x}} \leq \frac{1}{4}M\|\partial\psi\|_{L^\infty_{[0,t_*]L^2_x}} \quad (8.263)$$

We shall now estimate the other error term $\mathcal{T}Rf$. Since the operator norm of \mathcal{T} is bounded by M ,

$$\|\mathcal{T}R(f)\|_{L^q_{[0,t_*]L^\infty_x}} \leq M\|R(f)\|_{\mathcal{H}}.$$

On the other hand,

$$\begin{aligned} \|R(f)\|_{\mathcal{H}} &= \sup_{\|w\|_{\mathcal{H}} \leq 1} \langle w, R(f) \rangle_{\mathcal{H}} \\ &= - \sup_{\|w\|_{\mathcal{H}} \leq 1} \left[\int_{\Sigma_0} \frac{1}{\sqrt{|h|}} \left(h^{ij} \partial_i(\sqrt{|h|}) \partial_j \phi(0) - \partial_t(\sqrt{|h|}) \partial_t \phi(0) \right) \psi(0) + \int_{[0,t_*] \times \mathbf{R}^3} \tilde{\square}_h \partial_t \phi \psi \right] \end{aligned}$$

Observe that since ϕ verifies the equation $\square_h\phi = 0$, we have

$$\tilde{\square}_h \partial_t \phi = \partial_t \tilde{\square}_h \phi + \partial_i \left(\partial_t(h^{ij}) \partial_j \phi \right) = \partial_t \left[\frac{1}{\sqrt{|h|}} \left(h^{ij} \partial_i(\sqrt{|h|}) \partial_j \phi - \partial_t(\sqrt{|h|}) \partial_t \phi \right) \right] + \partial_i \left(\partial_t(h^{ij}) \partial_j \phi \right).$$

Therefore, integrating by parts, we obtain

$$\|R(f)\|_{\mathcal{H}} = \sup_{\|w\|_{\mathcal{H}} \leq 1} \int_{[0,t_*] \times \mathbf{R}^3} \frac{1}{\sqrt{|h|}} \left(h^{ij} \partial_i(\sqrt{|h|}) \partial_j \phi - \partial_t(\sqrt{|h|}) \partial_t \phi \right) \partial_t \psi + \partial_t(h^{ij}) \partial_j \phi \partial_i \psi.$$

Estimating in a straightforward manner we derive,

$$\|R(f)\|_{\mathcal{H}} \leq C\|\partial h\|_{L^1_{[0,t_*]L^\infty_x}} \|\partial\phi\|_{L^\infty_{[0,t_*]L^2_x}} \|\partial\psi\|_{L^\infty_{[0,t_*]L^2_x}}.$$

We use the energy inequality (8.255) to estimate $\|\partial\phi\|_{L_{[0,t_*]}^\infty L_x^2}$.

Applying it and using $\|w\|_{\mathcal{H}} \leq 1$ we infer that, $\|\partial\phi\|_{L_{[0,t_*]}^\infty L_x^2} \leq C$. Therefore, with the help of (8.256), we have

$$\|\mathcal{T}R(f)\|_{L_{[0,t_*]}^q L_x^\infty} \leq \frac{1}{4}M\|\partial\psi\|_{L_{[0,t_*]}^\infty L_x^2}. \quad (8.264)$$

To estimate $\|\partial\psi\|_{L_{[0,t_*]}^\infty L_x^2}$ we rely on the following:

Lemma 8.5 *The solution ψ of the equation $\tilde{\square}_h\psi = Pf$, $\psi(t_*) = \partial_t\psi(t_*) = 0$ verifies the estimate,*

$$\|\partial\psi\|_{L_{[0,t_*]}^\infty L_x^2} \leq 2M\|f\|_{L_{[0,t_*]}^{q'} L_x^1} \quad (8.265)$$

Gathering together (8.262),(8.263),(8.264) and (8.265) we infer that,

$$\|\mathcal{T}\mathcal{T}^*f\|_X = \|\mathcal{T}(\psi_1[0] + \psi_2[0] + R(f))\|_{L_{[0,t_*]}^q L_x^\infty} \leq (C + \frac{1}{2}M^2)\|f\|_{L_{[0,t_*]}^{q'} L_x^1}$$

Therefore, in view of (8.261),

$$M^2 = \|\mathcal{T}\mathcal{T}^*\|_{X' \rightarrow X} \leq (C + \frac{1}{2}M^2).$$

Thus we infer that M is a universal constant, as desired.

It only remains to prove the Lemma 8.5. We proceed as follows. Let t be fixed in the interval $[0, t_*]$. We rewrite the equation $\square_h\phi = 0$ in the form,

$$\tilde{\square}_h\phi = F = -\frac{1}{\sqrt{|h|}}\left(\partial_t(\sqrt{|h|})\partial_t\phi - \partial_i(\sqrt{|h|})\partial_i\phi\right) \quad (8.266)$$

with initial data $\phi(t) = w_0$, $\partial_t\phi(t) = w_1$, and $(w_0, w_1) = w \in \mathcal{H}_t$, $\|w\|_{\mathcal{H}_t} \leq 1$. Here, the space \mathcal{H}_t is defined by the scalar product $\langle w, v \rangle_{\mathcal{H}_t} = \int_{\Sigma_t} w_1 v_1 + h^{ij}\partial_i w_0 \partial_j v_0$. We also recall that, see (8.260),

$$\tilde{\square}_h\psi = Pf \quad (8.267)$$

with initial data $\psi_1(t_*) = \partial_t\psi_1(t_*) = 0$. Multiplying (8.266) by $\partial_t\psi$ and (8.267) by $\partial_t\phi$ and integrating in the region $[t, t_*] \times \mathbf{R}^n$ we derive the identity,

$$\int_{\Sigma_t} \left(\partial_t\phi\partial_t\psi + h^{ij}\partial_i\phi\partial_j\psi\right) = -\int_t^{t_*} \int_{\Sigma_\tau} \partial_t\phi Pf + \partial_t\psi F + \partial_t(h^{ij})\partial_i\phi\partial_j\psi.$$

Therefore,

$$\|\partial\psi(t)\|_{L^2} \leq \|P\partial_t\phi\|_{L_{[0,t_*]}^q L_x^\infty} \|f\|_{L_{[0,t_*]}^{q'} L_x^1} + C\|\partial h\|_{L_{[0,t_*]}^1 L_x^\infty} \|\partial\phi\|_{L_{[0,t_*]}^\infty L_x^2} \|\partial\psi\|_{L_{[0,t_*]}^\infty L_x^2}$$

We recall that according to our assumption $\|P \partial_t \phi\|_{L^q_{[0,t_*]} L^\infty_x} \leq M \|w\|_{\mathcal{H}_\perp} \leq M$. Also according to the energy estimate, $\|\partial_t \phi\|_{L^\infty_{[0,t_*]} L^2_x} \leq 2 \|w\|_{\mathcal{H}_t} \leq 2$. Therefore,

$$\|\partial \psi\|_{L^\infty_{[0,t_*]} L^2_x} \leq M \|f\|_{L^{q'}_{[0,t_*]} L^1_x} + C \|\partial h\|_{L^1_{[0,t_*]} L^\infty_x} \|\partial \psi\|_{L^\infty_{[0,t_*]} L^2_x}$$

and therefore, since $C \|\partial g_\lambda\|_{L^1_{[0,t_*]} L^\infty_x} \leq \frac{1}{2}$, we conclude that,

$$\|\partial \psi\|_{L^\infty_{[0,t_*]} L^2_x} \leq 2M \|f\|_{L^{q'}_{[0,t_*]} L^1_x}$$

as desired.

To prove the Strichartz estimate for the spatial derivatives we rely on the proof, given above, for $P \partial_t \phi$. We thus assume that the estimate (8.5) holds true for $P \partial_t \phi$ with a universal constant M .

To estimate $\|P \partial_a \phi\|_{L^q_{[0,t_*]} L^\infty_x}$ it suffices to estimate the integral, $\mathcal{I} = \int_{[0,t_*] \times \mathbf{R}^3} P \partial_a \phi f dt dx$ for functions f with $\|f\|_{L^{q'}_{[0,t_*]} L^\infty_x} \leq 1$. Let ψ verify the equation $\tilde{\square}_h \psi = P f$ with $\psi(t_*) = \partial_t \psi(t_*) = 0$. Therefore,

$$\mathcal{I} = \int_{[0,t_*] \times \mathbf{R}^3} \partial_a \phi \tilde{\square}_h \psi = \int_{[0,t_*] \times \mathbf{R}^3} \tilde{\square}_h \partial_a \phi \psi + \int_{\Sigma_0} \left(\partial_a \phi(0) \partial_t \psi(0) + \partial_t \phi(0) \partial_a \psi(0) \right)$$

Proceeding as before we show that,

$$\left| \int_{[0,t_*] \times \mathbf{R}^3} \tilde{\square}_h \partial_a \phi \psi \right| \leq C \|\partial g\|_{L^1_{[0,t_*]} L^\infty_x} \|\partial \phi\|_{L^\infty_{[0,t_*]} L^2_x} \|\partial \psi\|_{L^\infty_{[0,t_*]} L^2_x}$$

Also,

$$\int_{\Sigma_0} \left(\partial_a \phi(0) \partial_t \psi(0) + \partial_t \phi(0) \partial_a \psi(0) \right) \leq \|\partial \phi(0)\|_{L^2} \|\partial \psi\|_{L^\infty_{[0,t_*]} L^2_x}.$$

The energy estimate (8.255) gives $\|\partial \phi\|_{L^\infty_{[0,t_*]} L^2_x} \leq 2 \|\partial \phi(0)\|_{L^2}$. According to the Lemma 8.5 we have,

$$\|\partial \psi\|_{L^\infty_{[0,t_*]} L^2_x} \leq 2M \|f\|_{L^{q'}_{[0,t_*]} L^1_x}.$$

Observe that the M in Lemma 8.5 depends only on the Strichartz estimate (8.257) for $P \partial_t \phi$ which we have already proved. Therefore,

$$|\mathcal{I}| \leq CM \|\partial \phi(0)\|_{L^2} (1 + \|\partial h\|_{L^1_{[0,t_*]} L^\infty_x}) \|f\|_{L^{q'}_{[0,t_*]} L^1_x} \leq CM \|\partial \phi(0)\|_{L^2}$$

which implies, $\|P \partial_a \phi\|_{L^q_{[0,t_*]} L^\infty_x} \leq CM \|\partial \phi(0)\|_{L^2}$ as desired.

9 Appendix

In this Chapter we provide the proof of the two theorems formulated and employed throughout Chapter 5. The first theorem, see Theorem 5.2, is the isoperimetric inequality on a 2-surface embedded in \mathbb{R}^3 endowed with metric h with minimal control of its derivatives.

Theorem 9.1 *Let S be a 2-dimensional surface without boundary smoothly embedded in \mathbb{R}^3 endowed with a smooth Riemannian metric h . Assume that relative to the standard coordinates on \mathbf{R}^3 , the metric h satisfies the following condition:*

$$\sup_{\mathbb{R}^3} |\partial h| \leq \Lambda_0^{-1} \quad (9.268)$$

for some positive constant Λ_0 . Assume also that the area $A(S)$ of the surface S induced by h verifies

$$A(S) = \int_S 1 \leq 2^{-15} \Lambda_0^2. \quad (9.269)$$

Then for any smooth function $f : S \rightarrow \mathbb{R}$ we have the following isoperimetric inequality⁸³:

$$\left(\int_S |f|^2 \right)^{\frac{1}{2}} \lesssim \int_S (|\nabla f| + |\text{tr}\theta||f|). \quad (9.270)$$

Here, ∇ is a covariant derivative on S compatible with the restriction of metric h on S and θ is a second fundamental form of the embedding $S \subset \mathbb{R}^3$. In addition, the following Sobolev inequality holds on S : for any $\delta \in (0, 1)$ and p from the interval $p \in (2, \infty]$

$$\sup_S |f| \lesssim \left[A(S) \right]^{\frac{\epsilon(p-2)}{2p+\delta(p-2)}} \left(\int_S (|\nabla f|^2 + |\text{tr}\theta|^2 |f|^2) \right)^{\frac{1}{2} - \frac{\delta p}{2p+\delta(p-2)}} \left[\int_S (|\nabla f|^p + |\text{tr}\theta|^p |f|^p) \right]^{\frac{2\delta}{2p+\delta(p-2)}}, \quad (9.271)$$

where $A(S)$ is the area of S .

Remark: The isoperimetric inequality in the form (5.138) on a submanifold M of a Riemannian manifold N has been known to hold in the following cases:

- 1) $N = \mathbb{R}^n$, h is the Euclidean metric, [Si].
- 2) M , and N are arbitrary, and there is a pointwise bound on the curvature of N , $|\mathbf{R}_N| \leq C$, [Ho-Sp].

Theorem 5.2 is an intermediate result between 1) and 2). Note that in the context (5.113)-(5.117), where we will apply this result, the uniform pointwise bounds on the curvature are not even available.

Proof: The proof is an adaption of the method used in [Si] and [Ho-Sp].

Let ∇ be a covariant derivative on \mathbf{R}^3 compatible with the metric h and let ∇ denote the corresponding covariant derivative on S . Choose an orthonormal frame e_A , $A = 1, 2$ on S . For any

⁸³independent of Λ_0

vectorfield $X : S \rightarrow T\mathbb{R}^3$ we can define $\operatorname{div}X = \sum_A \langle \nabla_A X, e_A \rangle$. The inner product \langle, \rangle is taken with respect to the metric h : $\langle X, Y \rangle = h_{ij} X^i Y^j$. Any vectorfield X can be decomposed, $X = X^T + X^n N$, where X^T is tangent to S and N is a unit normal vectorfield to S . Then

$$\operatorname{div}X = \sum_A \langle \nabla_A X^T, e_A \rangle + X^n \sum_A \langle \nabla_A N, e_A \rangle = \mathfrak{d}\operatorname{iv}X^T + X^n \operatorname{tr}\theta,$$

where $\mathfrak{d}\operatorname{iv}X^T$ denotes the intrinsic divergence of the tangent to S vectorfield X^T and θ is a second fundamental form of the embedding $S \subset \mathbb{R}^3$. Integrating the last identity over the surface S and noting that by the Stokes' theorem $\int_S \mathfrak{d}\operatorname{iv}X^T = 0$, we have

$$\int_S \operatorname{div}X = \int_S X^n \operatorname{tr}\theta. \quad (9.272)$$

Fix a point x_0 on S . Define the distance function $r(x, x_0)$ on \mathbb{R}^3 according to the formula

$$r^2(x, x_0) = h_{ij}(x)(x - x_0)^i (x - x_0)^j \quad (9.273)$$

The distance function r is equivalent to the standard euclidean distance on \mathbb{R}^3 . It is also equivalent⁸⁴ to the geodesic distance d_h generated by the metric h . Let f be a positive function on S . Fix two positive parameters s and ρ and define a radial vectorfield X

$$X = \zeta_1(f(x) - s) \zeta_2(\rho - r(x, x_0)) Z, \quad Z = (x - x_0)^i \partial_i. \quad (9.274)$$

Here ζ_1, ζ_2 are 2 identical cut-off functions such that $\zeta_{1,2}(\tau) = 0$ for $\tau < 0$ and $\zeta_{1,2}(\tau) = 1$ for $\tau \geq \epsilon$ for some sufficiently small $\epsilon > 0$. Also $\zeta_{1,2}$ are chosen in such a way that the first derivative $\zeta'_{1,2} \geq 0$.

The smallness of ϵ is determined by the condition that the set $\{x \in S : r(x, x_0) < \epsilon\}$ is contained in the geodesic disk⁸⁵ $D_{r_0}(x_0)$ of radius r_0 on S . We require that r_0 be less than the injectivity radius of S and that

$$\int_{D_r(x_0)} 1 \geq \frac{\pi r^2}{2}, \quad \forall r \leq r_0 \quad (9.275)$$

Clearly, the smallness of ϵ could depend on the value of the parameter Λ_0 and various constants defining the embedding $S \subset \mathbb{R}^3$. However, it can be made independent of the choice of the point x_0 . The dependence on Λ_0 is irrelevant since we will take the limit $\epsilon \rightarrow 0$.

We already observed that the distance function r is equivalent to the geodesic distance on \mathbf{R}^3 . Since the geodesic distance can only increase after passing to a submanifold, we can conclude that the set $\{x \in S : r(x, x_0) < \epsilon\}$ contains the geodesic disk $D_{\frac{1}{2}\epsilon}(x_0)$. Thus, the inequality (9.275) implies that

$$\int_S \zeta_2(2\epsilon - r(x, x_0)) \geq \frac{\pi \epsilon^2}{8}. \quad (9.276)$$

⁸⁴The constant relating the two distance functions clearly depends on the ellipticity constant of the metric h . Without loss of generality we can assume that $\frac{1}{2}d_h(x, x_0) \leq r(x, x_0) \leq 2d_h(x, x_0)$ for all $x_0 \in \mathbf{R}^3$ and all points x such that $d_h(x, x_0) < \epsilon$ for a sufficiently small ϵ

⁸⁵with respect to the geodesic distance on S

Since f is a smooth function there exists a positive constant C , which in fact is determined by the L^∞ norm of ∇f , such that wherever $f(x_0) > s + C\epsilon$ we have $f(x) > s + \epsilon$ for all x such that $r(x, x_0) < \epsilon$. We will assume that the point x_0 has the property that $f(x_0) > s + C\epsilon$.

We calculate $\operatorname{div} X$. Using the silent summation rule over the repeated indices we obtain

$$\begin{aligned} \operatorname{div} X &= \zeta'_1 \zeta_2 \nabla_A f \langle Z, e_A \rangle - \zeta_1 \zeta'_2 \nabla_A r \langle Z, e_A \rangle \\ &+ \zeta_1 \zeta_2 \nabla_A (x - x_0)^i \langle \partial_i, e_A \rangle + \zeta_1 \zeta_2 (x - x_0)^i \langle \nabla_A \partial_i, e_A \rangle. \end{aligned}$$

Clearly, $\nabla_A (x - x_0)^i = e_A^k \partial_k (x - x_0)^i = e_A^i$. Therefore,

$$\nabla_A (x - x_0)^i \langle \partial_i, e_A \rangle = \langle e_A, e_A \rangle = 2.$$

Furthermore, $\langle \nabla_A \partial_i, e_A \rangle = e_A^k e_A^j \Gamma_{ikj}$, where Γ denotes the Cristoffel symbols for the metric h . We also have that $|(x - x_0)^i| \leq Cr$ for some positive constant C which depends on the ellipticity constant of the metric h . Therefore, it follows from condition (9.268) that

$$|\zeta_1 \zeta_2 (x - x_0)^i \langle \nabla_A \partial_i, e_A \rangle| \leq \zeta_1 \zeta_2 \Lambda_0^{-1} r(x, x_0) \leq \zeta_1 \zeta_2 \Lambda_0^{-1} \rho$$

since $r(x, x_0) \leq \rho$ on the support of ζ_2 . Continuing the calculation we have

$$\nabla_A r(x, x_0) = e_A^k \partial_k r = \frac{h_{mk}(x - x_0)^m e_A^k}{r} + e_A^k \frac{\partial_k (h_{mj})(x - x_0)^m (x - x_0)^j}{2r}.$$

Thus,

$$\nabla_A (r) \langle Z, e_A \rangle = \sum_A \frac{\langle Z, e_A \rangle^2}{r} + \sum_A \nabla_A (h_{mj}) \frac{(x - x_0)^m (x - x_0)^j}{2r} \langle Z, e_A \rangle.$$

Note that $r(x, x_0)$ is precisely the length of the vectorfield Z with respect to the metric h . It follows that $\sum_A \langle Z, e_A \rangle^2 \leq r^2(x, x_0)$. In addition, $|(x - x_0)^i (x - x_0)^j| \lesssim Cr^2(x, x_0)$. Hence, using condition (9.268)

$$|\zeta_1 \zeta_2 \nabla_A (r) \langle Z, e_A \rangle| \leq \zeta_1 \zeta'_2 (r + \Lambda_0^{-1} r^2) \leq \zeta_1 \zeta'_2 (\rho + \Lambda_0^{-1} \rho^2).$$

We conclude that

$$\zeta_1 \zeta_2 (2 - \Lambda_0^{-1} \rho) - \zeta_1 \zeta'_2 (\rho + \Lambda_0^{-1} \rho^2) \leq \operatorname{div} X + \zeta'_1 \zeta_2 |\nabla_A f| \rho. \quad (9.277)$$

Integrating (9.277) over S using the identity (9.272) together with the fact that since $|X|_h = \zeta_1 \zeta_2 r$ the normal component $|X^n| \leq \zeta_1 \zeta_2 \rho$, we obtain

$$(2 - \Lambda_0^{-1} \rho) \int_S \zeta_1 \zeta_2 - (\rho + \Lambda_0^{-1} \rho^2) \frac{d}{d\rho} \int_S \zeta_1 \zeta_2 \leq \rho \int_S \left(\zeta'_1 \zeta_2 |\nabla_A f| + \zeta_1 \zeta_2 |\operatorname{tr} \theta| \right). \quad (9.278)$$

It is not difficult to verify that (9.278) is equivalent to

$$-\frac{d}{d\rho} \left[\rho^{-2} (1 + \Lambda_0^{-1} \rho)^3 \int_S \zeta_1 \zeta_2 \right] \leq \rho^{-2} (1 + \Lambda_0^{-1} \rho)^2 \int_S \zeta'_1 \zeta_2 |\nabla_A f| + \zeta_1 \zeta_2 |\operatorname{tr} \theta|. \quad (9.279)$$

Integration with respect to ρ over the interval $[\sigma, \rho]$ with $0 < \sigma < \rho$ yields

$$\begin{aligned} \sigma^{-2}(1 + \Lambda_0^{-1}\sigma)^3 \int_S \zeta_1 \zeta_2(\sigma - r(x, x_0)) &\leq \rho^{-2}(1 + \Lambda_0^{-1}\rho)^3 \int_S \zeta_1 \zeta_2(\rho - r(x, x_0)) \\ &+ \int_\sigma^\rho \tau^{-2}(1 + \Lambda_0^{-1}\tau)^2 \int_S \zeta_1' \zeta_2(\tau - r(x, x_0)) |\nabla_A f| + \zeta_1 \zeta_2(\tau - r(x)) |\text{tr}\theta|. \end{aligned} \quad (9.280)$$

Let

$$\begin{aligned} \alpha(\tau) &:= (1 + \Lambda_0^{-1}\tau)^3 \int_S \zeta_1(f(x) - s) \zeta_2(\tau - r(x, x_0)), \\ \beta(\tau) &:= (1 + \Lambda_0^{-1}\tau)^2 \int_S \zeta_1'(f(x) - s) \zeta_2(\tau - r(x, x_0)) |\nabla_A f| + \zeta_1(f(x) - s) \zeta_2(\tau - r(x, x_0)) |\text{tr}\theta|. \end{aligned}$$

Note that functions $\alpha(\tau)$ and $\beta(\tau)$ are nondecreasing. The identity (9.280) takes the form

$$\sigma^{-2}\alpha(\sigma) \leq \rho^{-2}\alpha(\rho) + \int_\sigma^\rho \tau^{-2}\beta(\tau). \quad (9.281)$$

Recall that at the point x_0 , $f(x_0) > s + C\epsilon$. The choice of the constant C ensures that $f(x) > s + \epsilon$ for all x such that $r(x, x_0) < \epsilon$. It follows that $\zeta_1(f(x) - s) \zeta_2(2\epsilon - r(x, x_0)) = 1$ on the set $r(x, x_0) < \epsilon$. According to (9.276), the area of this set is bounded from below by $\pi\epsilon^2/8$. Thus,

$$(2\epsilon)^{-2}\alpha(2\epsilon) \geq (1 + \Lambda_0^{-1}2\epsilon)^3 2^{-5}\pi \geq 2^{-5}\pi. \quad (9.282)$$

Observe also that

$$\int_S \zeta_1(f(x) - s) \zeta_2(\rho - r(x, x_0)) \leq \int_S \zeta_1(f(x) - s) \leq A(S) \leq 2^{-15}\Lambda_0^2.$$

for arbitrary ρ and s , which follows from (9.269). Define

$$\rho_0 = \left(2^{14}\pi^{-1} \int_S \zeta_1(f(x) - s) \right)^{\frac{1}{2}} \quad (9.283)$$

Clearly, $\rho_0 \leq \frac{1}{5}\Lambda_0$. We then have the following inequality for $\alpha(\tau)$:

$$(\rho_0)^{-2}\alpha(\tau) \leq 2^{-9}\pi, \quad \forall 0 \leq \tau \leq 5\rho_0. \quad (9.284)$$

We formulate an auxiliary lemma which is a variant of the well-known result, see [Si] (Lemma 18.7).

Lemma 9.1 *Let $\alpha(\tau)$, $\beta(\tau)$ be two positive nondecreasing functions satisfying (9.281), (9.282), and (9.284) with ρ_0 defined in (9.283). Then there exists a positive ρ in the interval $2\epsilon \leq \rho \leq \rho_0$ such that*

$$\alpha(5\rho) \leq 20\rho_0\beta(\rho). \quad (9.285)$$

Proof: We argue by contradiction. Assume that for all ρ in the interval $2\epsilon \leq \rho \leq \rho_0$ we have $\alpha(5\rho) \geq 20\rho_0\beta(\rho)$. Inequalities (9.281), (9.282), and (9.284) then yield

$$\begin{aligned}
2^{-5}\pi &\leq \sup_{2\epsilon \leq \sigma \leq \rho_0} (\sigma)^{-2}\alpha(\sigma) \leq \rho_0^{-2}\alpha(\rho_0) + \frac{1}{20\rho_0} \int_{2\epsilon}^{\rho_0} \tau^{-2}\alpha(5\tau) \quad \text{using (9.281), (9.282)} \\
&\leq \rho_0^{-2}\alpha(\rho_0) + \frac{1}{4\rho_0} \int_{10\epsilon}^{\rho_0} \tau^{-2}\alpha(\tau) + \frac{1}{4\rho_0} \int_{\rho_0}^{5\rho_0} \tau^{-2}\alpha(\tau) \\
&\leq \rho_0^{-2}\alpha(\rho_0) + \frac{1}{4} \sup_{2\epsilon \leq \sigma \leq \rho_0} (\sigma)^{-2}\alpha(\sigma) + \frac{1}{5}(\rho_0)^{-2}\alpha(5\rho_0) \\
&< \frac{1}{4} \sup_{2\epsilon \leq \sigma \leq \rho_0} (\sigma)^{-2}\alpha(\sigma) + 2^{-8}\pi \quad \text{using (9.284)}.
\end{aligned}$$

Contradiction.

Since $\rho \leq \rho_0 \leq \frac{1}{5}\Lambda_0$, the inequality (9.285) can be interpreted as follows:

$$\begin{aligned}
\int_S \zeta_1(f(x) - s) \zeta_2(5\rho - r(x, x_0)) &\leq 20 \cdot 2^2 \left(2^{14}\pi^{-1} \int_S \zeta_1(f(x) - s) \right)^{\frac{1}{2}} \\
&\times \int_S \zeta_1'(f(x) - s) \zeta_2(\rho - r(x, x_0)) |\nabla_A f| + \zeta_1(f(x) - s) \zeta_2(\rho - r(x, x_0)) |\text{tr}\theta|.
\end{aligned} \tag{9.286}$$

The same estimate holds for any choice of the point x_0 , with perhaps different $\rho(x_0)$, as long as x_0 satisfies the condition that $f(x_0) > s + C\epsilon$.

For any positive τ define the set

$$M_\tau := \{x \in S : f(x) > \tau\}.$$

In this notation, inequality (9.286) holds for any point $x_0 \in M_{s+C\epsilon}$. The union of 3-d euclidean balls $B_{\rho(x_0)}(x_0)$, $x_0 \in M_{s+C\epsilon}$ is a covering of $M_{s+C\epsilon}$. By the Vitali covering lemma for \mathbf{R}^3 we can find a disjoint subcollection $B_{\rho(x_0)}(x_0)$, $x_0 \in M'_{s+C\epsilon}$, where $M'_{s+C\epsilon} \subset M_{s+C\epsilon}$ such that $M_{s+C\epsilon} \subset \cup_{x_0 \in M'_{s+C\epsilon}} B_{4\rho(x_0)}(x_0)$.

As we already mentioned above, the ellipticity of the metric h implies that on \mathbf{R}^3 the euclidean ball $B_{\rho(x_0)}(x_0)$ is essentially the same as the set $r(x, x_0) \leq \rho(x_0)$. Without loss of generality, we can assume that they are the same.

Now observe that the function $\zeta_2(5\rho(x_0) - r(x, x_0)) = 1$ on the set where $r(x, x_0) \leq 5\rho(x_0) - \epsilon$. Since $\rho(x_0) \geq 2\epsilon$, the above set is contained in $r(x, x_0) \leq 4\rho(x_0) \approx B_{4\rho(x_0)}(x_0)$. Thus, summing (9.286) over $x_0 \in M'_{s+C\epsilon}$ we obtain ⁸⁶

$$\int_{M_{s+C\epsilon}} \zeta_1(f(x) - s) \lesssim \left(\int_S \zeta_1(f(x) - s) \right)^{\frac{1}{2}} \int_S \zeta_1'(f(x) - s) |\nabla f| + \zeta_1(f(x) - s) |\text{tr}\theta| \tag{9.287}$$

⁸⁶We replace the inequality \leq by \lesssim with a universal constant independent of ϵ, Λ_0 . In this case, the constant is $20 \cdot 2^9 \pi^{-1}$.

We can rewrite (9.287) as follows:

$$A(M_{s+C\epsilon}) \lesssim A^{\frac{1}{2}}(M_s) \left(-\frac{d}{ds} \left[\int_S \zeta_1(f(x) - s) |\nabla f| \right] + \int_{M_s} |\text{tr}\theta| \right). \quad (9.288)$$

Multiply (9.288) by $s + C\epsilon$, note that $(s + C\epsilon)^2 A(M_s) \leq 2 \int_S f^2 + 2(C\epsilon)^2 A(S)$, and integrate with respect to s over the interval $(0, \infty)$. We easily obtain the following estimate

$$\int_{f(x) > C\epsilon} f^2 \lesssim \left(\int_S f^2 \right)^{\frac{1}{2}} \int_S (|\nabla f| + f|\text{tr}\theta|) + C\epsilon (A(S))^{\frac{1}{2}} \int_S (|\nabla_A f| + f|\text{tr}\theta|).$$

Taking the limit as $\epsilon \rightarrow 0$ and simplifying yields

$$\left(\int_S f^2 \right)^{\frac{1}{2}} \lesssim \int_S (|\nabla f| + f|\text{tr}\theta|). \quad (9.289)$$

To obtain estimate (9.271) we integrate (9.288) over the interval (t, ∞) using the fact that $A(M_s) \leq A(M_t)$ for all $s \geq t$. Taking the limit as $\epsilon \rightarrow 0$ we obtain

$$\int_t^\infty A(M_s) \lesssim A^{\frac{1}{2}}(M_t) \int_{M_t} (|\nabla f| + (f(x) - t)|\text{tr}\theta|). \quad (9.290)$$

Define the function $m(t) = \int_t^\infty A(M_s)$. Applying the Cauchy-Schwartz inequality in (9.290) we obtain

$$m(t) \lesssim -m'(t) \left(\int_{M_t} |\nabla f|^2 + (f(x) - t)^2 |\text{tr}\theta|^2 \right)^{\frac{1}{2}}. \quad (9.291)$$

The application of another Cauchy-Schwartz inequality yields for an arbitrary $\delta \in (0, 1)$ and $p \in (2, \infty]$

$$\begin{aligned} \left(\int_{M_t} |\nabla f|^2 + |\text{tr}\theta|^2 (f(x) - t)^2 \right)^{\frac{1}{2}} &\lesssim [A(M_t)]^{\frac{\delta(p-2)}{2p}} \left(\int_{M_t} |\nabla^2 f|^2 + |\text{tr}\theta|^2 |f - t|^2 \right)^{\frac{1}{2}(1-\delta)} \\ &\quad \times \left(\int_{M_t} |\nabla f|^p + |\text{tr}\theta|^p |f - t|^p \right)^{\frac{\delta}{p}}. \end{aligned} \quad (9.292)$$

Recall that $A(M_t) = -m'(t)$ and strengthen the inequality by replacing the first integral over the set M_t on the right hand-side with the integral over the whole surface S . Combining (9.291) and (9.292) we obtain

$$m(t) \lesssim [-m'(t)]^{1 + \frac{\delta(p-2)}{2p}} C \quad (9.293)$$

with the constant C defined by

$$C := \left(\int_S |\nabla^2 f|^2 + |\operatorname{tr}\theta|^2 |f|^2 \right)^{\frac{1}{2}(1-\delta)} \left(\int_{M_t} |\nabla f|^p + |\operatorname{tr}\theta|^p |f|^p \right)^{\frac{\delta}{p}}. \quad (9.294)$$

Rewrite inequality (9.293) in the form

$$-\frac{m'(t)}{[m(t)]^{\frac{2p}{2p+\delta(p-2)}}} \gtrsim C^{-\frac{2p}{2p+\delta(p-2)}}$$

and integrate with respect to t over the interval $(0, t_{\max})$, where $t_{\max} = \sup_S f$. Clearly, t_{\max} is the smallest value of t where $m(t) = 0$. In addition, $m(0) = \int_0^\infty A(M_s) = \int_S f$. Therefore,

$$\sup_S f \lesssim C^{\frac{2p}{2p+\delta(p-2)}} \left[\int_S f \right]^{\frac{\delta(p-2)}{2p+\delta(p-2)}}$$

Observe that the Hölder inequality, estimate (9.289), and definition of the constant C imply that

$$\int_S f \lesssim A^{\frac{1}{2}}(S) \left(\int_S f^2 \right)^{\frac{1}{2}} \lesssim A(S) \left(\int_S |\nabla f|^2 + |\operatorname{tr}\theta|^2 f^2 \right)^{\frac{1}{2}}$$

Using the definition of the constant C we finally conclude that

$$\sup_S f \lesssim \left[A(S) \right]^{\frac{\delta(p-2)}{2p+\delta(p-2)}} \left(\int_S |\nabla f|^2 + |\operatorname{tr}\theta|^2 |f|^2 \right)^{\frac{1}{2} - \frac{\delta p}{2p+\delta(p-2)}} \left(\int_S |\nabla f|^p + |\operatorname{tr}\theta|^p |f|^p \right)^{\frac{2\delta}{2p+\delta(p-2)}}$$

The second result proved in this Appendix is the trace theorem for foliations, see Theorem 5.3.

Theorem 9.2 *Let $\Omega \subset \mathbf{R}^3$ be a subset of \mathbf{R}^3 endowed with a Riemannian metric h . For simplicity assume that the ellipticity constant of h is equal to 2, i.e,*

$$\frac{1}{2}|X|^2 \leq h_{ij}X^iX^j \leq 2|X|^2$$

for any vector $X \in \mathbf{R}^3$. Assume that the level surfaces of a smooth function v define an admissible foliation $\Phi : [0, \max v] \times \mathbb{S}^2 \rightarrow \Omega$ which means that the lapse function $\mathbf{b} = |\nabla v|$ and the second fundamental form θ of the foliation have the following properties:

- 1) The level surface $v = 0$ consists of one point, say $x = 0$, and the norm of the tangent map $\frac{1}{C} \leq |d\Phi|_{x=0}| \leq C$ for some positive constant C .
- 2) $|\mathbf{b}(v) - 1| \leq Mv$,
- 3) $|\operatorname{tr}\theta - \frac{2}{v}| \leq M$

for some constant M such that $M \max v \leq 2^{-5}$.

Then for any level surface S_v with $v \in (0, \max v]$, arbitrary $\epsilon > 0$, and any function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $f \in H^{\frac{1}{2}+\epsilon}(\mathbb{R}^3)$, we have

$$\|f\|_{L^2(S_v)} \lesssim \|\partial^{\frac{1}{2}-\epsilon} f\|_{L^2(\mathbb{R}^3)} + \|\partial^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbb{R}^3)}. \quad (9.295)$$

In addition, let $\Omega_{\frac{1}{4}v,v} = \cup_{w \in [\frac{1}{4}v,v]} S_w$ and let $N = \mathbf{b}^{-1} \partial_w$ be the vectorfield of the unit normals to the foliation. Then

$$\|f\|_{L^2(S_v)}^2 \lesssim \|N(f)\|_{L^2(\Omega_{\frac{1}{4}v,v})} \|f\|_{L^2(\Omega_{\frac{1}{4}v,v})} + \frac{1}{v} \|f\|_{L^2(\Omega_{\frac{1}{4}v,v})}^2. \quad (9.296)$$

The constants in (9.295), (9.296) depend on the constant C .

Proof: On Ω the metric h in coordinates v, ϕ^1, ϕ^2 associated with the foliation Φ has the form

$$h = \mathbf{b}^2 dv^2 + \gamma_{AB} d\phi^A d\phi^B,$$

where γ is the family dependent on v of metrics on \mathbb{S}^2 . The second fundamental form of the foliation, i.e., the second fundamental form of the embedding $S_v \subset \mathbb{R}^3$, is determined from the formula

$$\theta_{AB} = \frac{1}{2\mathbf{b}} \partial_v \gamma_{AB}.$$

Therefore

$$\text{tr}\theta = \frac{1}{2\mathbf{b}} \partial_v \ln(\det \gamma). \quad (9.297)$$

Note that condition 1) implies that the metric h in coordinates v, ϕ^1, ϕ^2 has the property that for the values of $v \rightarrow 0$, we have $\frac{\mathbf{b}(v)}{2C} \leq v^{-4} \det h(v) \leq 2\mathbf{b}(v)C$. Using condition 2) we conclude that

$$\frac{1}{4C} \leq v^{-4} \det h(v) \leq 4C, \quad v \rightarrow 0. \quad (9.298)$$

Thus integrating (9.297) with respect to v and using the conditions 2) and 3) we obtain

$$\ln \frac{\det \gamma(v)}{\det \gamma(v_0)} = \int_{v_0}^v 2\mathbf{b} \text{tr}\theta dw = \int_{v_0}^v \frac{4}{w} \pm \int_{v_0}^v (6M + 2M^2 w) dw$$

Hence

$$\ln \frac{v^4}{v_0^4} - 2^{-2} \leq \ln \frac{\det \gamma(v)}{\det \gamma(v_0)} \leq \ln \frac{v^4}{v_0^4} + 2^{-2}.$$

Exponentiating, taking the limit as $v_0 \rightarrow 0$, and using (9.298) we conclude that

$$\frac{1}{Ce} v^4 \leq \det \gamma(v) \leq C e v^4.$$

Since the area of the level surface S_v can be found as $A(S_v) = \int_{\mathbb{S}^2} \sqrt{\det \gamma}$, it follows that for any $v \in [0, \max v]$

$$v^2 \lesssim A(S_v) \lesssim v^2$$

with implicit constants dependent only on the constant C and the ellipticity constant of the metric h . This proves Proposition 5.1.

Consider function $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ from the Sobolev space $H^{\frac{1}{2}+\epsilon}(\mathbf{R}^3)$. Decompose f into its Fourier dyadic pieces $f = \sum_{\lambda} f^{\lambda}$, where $\lambda = 2^k$, $k \in \mathbb{Z}$ is a dyadic parameter, and the Fourier support of the function f^{λ} belongs to the set $\{|\xi| \in [\frac{1}{2}\lambda, 2\lambda]\}$. Fix the value of $v \in (0, \max v]$ and consider the restriction of f^{λ} to the domain $\Omega_{\frac{1}{4}v, v} = \cup_{w \in [\frac{1}{4}v, v]} S_w$. Let $\zeta_v(w)$ be a cut-off function such that $\zeta_v(w) = 1$ for $w \geq \frac{1}{2}v$, $\zeta_v(w) = 0$ for $w \leq \frac{1}{4}v$, and $|\zeta'_v| \leq 5v^{-1}$.

Consider the integral of the function $|f^{\lambda} \zeta_v|^2$ over the level surface S_v . In the coordinates v, ϕ^a the integral has the form

$$\int_{\mathbb{S}^2} |f^{\lambda}(v, \cdot) \zeta_v|^2 \sqrt{\det \gamma(v)}.$$

Integrating by parts relative to the radial variable w and taking into account that $\zeta_v(\frac{1}{4}v) = 0$, we obtain

$$\begin{aligned} \int_{\mathbb{S}^2} |f^{\lambda}(v, \cdot) \zeta_v|^2 \sqrt{\det \gamma(v)} &= \int_{\frac{1}{4}v}^v \int_{\mathbb{S}^2} (2(\partial_w f^{\lambda}(w, \cdot)) f^{\lambda}(w, \cdot) \zeta_v^2 + 2\partial_w(\zeta_v) |f^{\lambda}(w, \cdot)|^2 \zeta_v \\ &\quad + \frac{1}{2}(\partial_w \ln \sqrt{\det \gamma(w)}) |f^{\lambda}(v, \cdot) \zeta_v|^2) \sqrt{\det \gamma(w)}. \end{aligned}$$

Note that $\zeta_v(v) = 1$, $|\zeta'_v| \leq 5v^{-1}$, and $\partial_w \ln \sqrt{\det \gamma(w)} = 2\mathbf{b}(w)\text{tr}\theta(w)$. In addition, the volume form on Ω , relative to the metric h in coordinates w, ϕ^A , has the density $\mathbf{b}(w)\sqrt{\det \gamma(w)}$. Thus, using the conditions 2) and 3) we have

$$\int_{S_v} |f^{\lambda}|^2 \leq \int_{\Omega_{\frac{1}{4}v, v}} (2|\partial_w f^{\lambda}| |f^{\lambda}| + 10v^{-1}|f^{\lambda}|^2).$$

Note that the inequality above obviously holds also for the function f . Therefore, estimate (9.296) follows from a simple application of the Hölder inequality and condition 2).

Using the Hölder inequality and the fact that the volume of $\Omega_{\frac{1}{4}v, v}$ is $\approx v^3$, which follows from the condition 2) and (9.298), we also obtain

$$\int_{S_v} |f^{\lambda}|^2 \lesssim \int_{\Omega_{\frac{1}{4}v, v}} |\partial_w f^{\lambda}| |f^{\lambda}| + \left(\int |f^{\lambda}|^3 \right)^{\frac{2}{3}}.$$

The directional derivative ∂_w can be expressed as a combination of the standard ⁸⁷ derivatives. Since the length of the vectorfield ∂_w relative to the metric h is $|\partial_w|_h = b$ and h is the Riemannian metric on \mathbf{R}^3 with the uniformly bounded ellipticity constant ⁸⁸, it follows that $|\partial_w f| \leq 2|\partial f|$. The

⁸⁷relative to the standard coordinates on \mathbf{R}^3

⁸⁸Recall that we assumed for simplicity that the ellipticity constant = 2

same reasoning also allows one to replace the integral on the right hand-side with the integral over the whole space \mathbf{R}^3 relative to the standard euclidean metric. Thus

$$\int_{S_v} |f^\lambda|^2 \lesssim \|\partial f^\lambda\|_{L^2} \|f^\lambda\|_{L^2} + \|f^\lambda\|_{L^3}^2 \lesssim \|\partial^{\frac{1}{2}} f^\lambda\|_{L^2}^2, \quad (9.299)$$

where we used the fact that there is an embedding $\dot{H}^{\frac{1}{2}}(\mathbf{R}^3) \subset L^3(\mathbf{R}^3)$ and that, since f^λ has only frequencies of size λ , $\|\partial f^\lambda\|_{L^2} \|f^\lambda\|_{L^2} \leq \lambda^{\frac{1}{2}} \|\partial^{\frac{1}{2}} f^\lambda\|_{L^2} \|f^\lambda\|_{L^2} \leq \|\partial^{\frac{1}{2}} f^\lambda\|_{L^2}^2$. We can also replace the right hand-side of (9.299) by $\lambda^{2\epsilon} \|\partial^{\frac{1}{2}-\epsilon} f^\lambda\|_{L^2}$ or $\lambda^{-2\epsilon} \|\partial^{\frac{1}{2}+\epsilon} f^\lambda\|_{L^2}$. The first replacement provides some decay for the low frequencies λ and the second is good for the high frequencies. Taking the square root of (9.299) and summing over all dyadic frequencies λ , we obtain

$$\|f\|_{L^2(S_v)} \lesssim \|\partial^{\frac{1}{2}-\epsilon} f\|_{L^2(\mathbf{R}^3)} + \|\partial^{\frac{1}{2}+\epsilon} f\|_{L^2(\mathbf{R}^3)}.$$

References

- [Ba-Ch2] H. Bahouri and J. Y. Chemin. *Équations d'ondes quasilineaires et effet dispersif*. IMRN, vol. 21; (1999), pp. 1141–1178
- [Ba-Ch1] H. Bahouri and J. Y. Chemin. *Équations d'ondes quasilineaires et estimation de Strichartz*. Amer. J. Math., vol. 121; (1999), pp. 1337–1777
- [Ch-Kl] D. Christodoulou and S. Klainerman. *The Global Nonlinear Stability of the Minkowski Space*. Princeton Mathematical Series, 41. Princeton University Press, 1993
- [Ho-Sp] D. Hoffman and J. Spruck. *Sobolev and isoperimetric inequalities for Riemannian submanifolds*. CPAM, vol. 27; (1974), pp. 715–727
- [Ka] L. Kapitanski. *Estimates for norms in Besov and Lizorkin-Triebel spaces for solutions of second-order linear hyperbolic equations*. J. Soviet Math., vol. 49; (1990), pp. 1166–1186
- [Kl] S. Klainerman. *A commuting vectorfield approach to Strichartz type inequalities and applications to quasilinear wave equations*. IMRN. To appear
- [Li] H. Linblad. *Counterexamples to local existence for semilinear wave equations*. AJM, vol. 118; (1996), pp. 1–16
- [Po-Si] G. Ponce and T. Sideris. *Local regularity of non linear wave equations in three space dimensions*. CPDE, vol. 18; (1993), pp. 169–177
- [Mo-Se-So] G. Mockenhaupt, A. Seeger, and C. Sogge. *Local smoothing of Fourier integral operators and Carleson-Sjölin estimates*. J. Amer. Math. Soc., vol. 6; (1993), pp. 65–130
- [Si] L. Simon. *Lectures on geometric measure theory*. Proceedings of the Centre for Mathematical Analysis, vol 3. Australian National University, 1983
- [Sm] H. Smith. *A parametrix construction for wave equations with $C^{1,1}$ coefficients*. Annales de L'Institut Fourier, vol. 48; (1998), pp. 797–835
- [Sm-So] H. Smith and C. Sogge. *On Strichartz and eigenfunction estimates for low regularity metrics*. Math. Res. Lett., vol. 1; (1994), pp. 729–737
- [Sm-Ta] H. Smith and D. Tataru. *Sharp counterexamples for Strichartz estimates for low regularity metrics*. Preprint
- [Ta2] D. Tataru. *Strichartz estimates for second order hyperbolic operators with non smooth coefficients*. Preprint
- [Ta1] D. Tataru. *Strichartz estimates for operators with non smooth coefficients and the nonlinear wave equation*. Amer. J. Math., vol. 122; (2000), pp. 349–376