A Commuting Vectorfields Approach to Strichartz type Inequalities and Applications to Quasilinear Wave Equations

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A large body of knowledge about wave equations can be traced down to two fundamental facts concerning the standard linear wave equations in Minkowski space-time $\mathbf{R}^{n+1}$,

$$\square\,\phi = m^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

with $m_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$ the standard Minkowski metric.

The first is the well known energy identity,

$$E[\phi](t) = E[\phi](0) \tag{0.1}$$

where,

$$E[\phi](t) = \int_{\mathbf{R}^n} \Big(|\partial_t\phi(t,x)|^2 + |\partial_1\phi(t,x)|^2 + \cdots + |\partial_n\phi(t,x)|^2\Big)dx.$$

Therefore, for $\partial\phi = (\partial_t\phi, \partial_1\phi, \dots, \partial_n\phi)$,

$$\|\partial\phi(t)\|_{L^2} \le \|\partial\phi(0)\|_{L^2} \tag{0.2}$$

The second, which I will refer to as the *basic dispersive inequality*, has the form,

$$|\phi(t)|_{L^\infty} \le ct^{-\frac{n-1}{2}}\|\nabla^{\frac{n-1}{2}}\partial\phi(0)\|_{L^1} \tag{0.3}$$

In fact 0.3 is not quite right, the correct estimate holds if we replace the L^∞ norm on the left by the BMO-norm, or, the L^1 norm on the right by the Hardy norm $\mathcal{H}^1$. The inequality 0.3 is true however, as it stands, if the Fourier transform of the data $\phi(0) = f$, $\partial_t\phi(0) = g$ have their Fourier transform supported in a dyadic shell $\frac{\lambda}{2} \le |\xi| \le 2\lambda$ for some fixed $\lambda \in 2^{\mathbf{N}}$.

Intepolating between these two basic facts one derives the so called Strichartz-Brenner result,

$$\|\phi(t)\|_{L^r} \le c|t|^{-\gamma(r)}\|\nabla^\sigma\partial\phi(0)\|_{L^{r'}}$$

with $\gamma(r) = (n-1)(\frac{1}{2} - \frac{2}{r})$, $\frac{1}{r} + \frac{1}{r'} = 1$, $r \ge 2$ and scaling condition $\frac{n}{r} = -\gamma(r) - \sigma - 1 + \frac{n}{r'}$. This leads, by a standard TT^* argument, Hardy -Littlewood-Sobolev inequalities and an application of the Littlewood-Paley theory, to the generalized Strichartz inequality,

$$\begin{aligned} \|\phi\|_{L^q_t L^r_x} &\le c\|\partial\phi(0)\|_{H^\sigma} \qquad (0.4)\\ \frac{2}{q} \le \gamma(r), q \ge 2, &\qquad (q,r,n) \ne (\infty,1,3)\\ \sigma &= n(\frac{1}{2}-\frac{2}{r}) - 1 - \frac{1}{q} \end{aligned}$$

The latter plays a crucial role in many recent advances of the theory of nonlinear wave equations. Observe that the steps involved in deriving 0.4, at fixed frequency, from the energy identity and dispersive inequality are quite soft, they can be traced back to the Duhamel's principle and uniqueness of the initial value problem[1]. Both apply to general linear wave equations with variable coefficients[2] and require very little regularity of the coefficients. Thus the main building blocks of the Strichartz type inequalities are 0.1 and 0.3.

The identity 0.1, and the corresponding L^2 estimate, can easily be derived from the Fourier representation of solutions. The beauty and power of the identity, however, is that it can be derived directly , in physical space, by a simple integration by parts argument. Thus energy type estimates are extremely versatile, they can be applied to large classes of linear and nonlinear equations. On the other hand the classic derivation of the dispersive inequality is based on the method of stationary phase applied to the specific representation of solutions as Fourier integral operators. In more complicated situations the Fourier representation of solutions, or rather approximate solutions, may be quite difficult to derive and not very natural.

The dispersive inequality provides two types of information:

1. The precise decay rate of $\|\phi(t)\|_{L^\infty}$ as $t \to \infty$.

2. Improved regularity properties of $\|\phi(t)\|_{L^\infty}$ for $t > 0$.

It is well known that as far as the asymptotic behavior is concerned 0.3 is not very useful in applications to nonlinear wave equations. A more effective procedure to derive the asymptotic properties of solutions of the wave equation is based on generalized energy estimates, obtained by the commuting vectorfields method, together with global Sobolev inequalities. We shall make a quick review of this procedure in the first part of section 1. As far as improved regularity is concerned the estimate 0.3 gains, for $t > 0$, $\frac{n-1}{2}$ derivatives when compared to the Sobolev embedding $L^\infty(\mathbf{R}^n) \subset W^{1,n}(\mathbf{R}^n)$. It thus may seem that the methods discussed in section 1, based on Sobolev estimates, are not relevant to questions concerning regularity. The main new observation of this paper, presented in the second part of section 1, is that the decay estimates based on commuting vectorfields do actually imply, after a suitable localization in phase space[3], the dispersive inequality 0.3. This simple fact allows us to achieve an unexpected connection between the modern Fourier based techniques of Strichartz and bilinear estimates on one hand and, on the

[1] See Theorem 1.2. for a straightforward derivation of 0.4 from 0.1 and 0.3

[2] The uniqueness of the I.V.P. is also a consequence of the basic energy inequality

[3] The localization method, which is the key in the proof of Theorem 1.1, was used in a different context by O. Liess [L]. The essence of his idea was that, after localization to the unit dyadic region in Fourier space, the $L^1 - L^\infty$ dispersive inequality follows from a weighted $L^2 - L^\infty$ inequality. This is done easyly by a further localization in physical space. I am grateful to T. Tao for pointing this important fact to me.

other hand, the powerful geometric methods used in the proof of the stability of the Minkowski space [C-K2]. We illustrate this point by showing how it can be used to give a different proof of the recent improved regularity results for quasilinear wave equations due to Chemin-Bahouri [B-C1], [B-C2] and D.Tataru [T1], [T2].

In section 1 of the paper we present the new approach to the derivation of 0.3, in Minkowski space, based only on energy estimates, commuting vectorfields, generalized energy estimates and an appropriate phase space localization. Though most of the material presented in the section can be found elsewhere in the literature I found it would be more convenient to the reader if the main ideas, later to be developed in a curved background, were first properly reviewed in flat space. We have divided the section in two parts. In section 1.1 we first sketch the simplest version of the vectorfield method to derive weighted L^2–L^∞ decay estimates for solutions to the homogeneous wave equation in Minkowski space. We then present in details a different method, based on the Morawetz vectorfield, of deriving similar decay estimates in dimension 3. For technical reasons this is the method we adopt later to obtain the appropriate decay estimates in curved background. The main goal of section 1.2 is to show how to derive the dispersive inequality 0.3 from the weighted L^2–L^∞ decay estimates derived in section 1.1. This is done in Theorem 1.1. In Theorem 1.2 we recall the derivation of the Strichartz estimates from the dispersive inequality 0.3 via the standard TT^* argument. We present in details an approach, based on the group properties of the wave propagation, which we shall later adapt in the proof of Theorem 2.3, see the preview below.

In section 2.1 of the paper we start by stating our main Theorems A,B,C. As explained above the statements of these Theorems are not new[4]; they are due in fact to the combined pioneering efforts of H. Smith [S1], [S2], Bahouri-Chemin [B-C1],[B-C2] and D.Tataru [T1],[T2]. The goal of the section is to first review some of the basic reductions used by the above mentioned authors. More precisely we rely on Tataru [T2] for showing how Theorem C follows from Theorem B. We also sketch his proof of how Theorem B follows from Theorem A and also the reduction of Theorem A to the case $\mu = 1$. We then sketch the main *paradifferential* type ideas needed to reduce the proof of Theorem A($\mu = 1$) to its microlocalized version in Theorem 2.1. This type of reduction plays a central role in all the above mentioned references. In the second part of the section we show how to reduce Theorem 2.1 to a dispersive type inequality stated in Theorem 2.2. This is done by the well known TT^* argument. Due to the fact that we work with the precise solutions of linear, variable coefficents, wave equations(rather then their Fourier integral parametrix representations as in [S1], [S2],[B-C1],[B-C2], [T1],[T2]) this reduction is not standard; we present it in full details in the proof of Theorem 2.3. Theorem 2.1 follows then from a simple corrolary of Theorem 2.3. Finally, in the end of the section, we show how the dispersive inequality of Theorem 2.2 follows from an $L^2 - L^\infty$ decay estimate stated in Theorem 2.4. This is the same, *essential*, phase space localization argument described in the proof of Theorem 1.1. Thus we see that the final goal of section 2 is to reduce the main Theorems A,B,C to Theorem 2.4. The rest of the paper is occupied with the proof of Theorem 2.4.

In section 2.2 we start with a simple reformulation of Theorem 2.4, which becomes Theorem 2.5. The goal of the section is to construct a curved background analogue of the Morawetz vectorfield K_0 which will be then used, in section 3, to derive generalized energy estimates analogous to those discussed in flat space. This is achieved with the help of an *optical function* u whose level

[4]Using a variation of the approach described here, Rodniansky and I(see [Kl-R]) were recently able to improve the result of Theorem C from $\sigma > \frac{1}{6}$, due to Tataru(see [T2]), to $\sigma > \frac{2-\sqrt{3}}{2}$ for $n = 3$.

hypersurfaces are outgoing null cones with vertices on the time axis. This procedure follows closely that used in [C-K2]. The main results of the section are the calculations(described in detail) of the null components of the deformation tensor of K_0, summarized in Proposition 2.4., and the asymptotic results of Theorem 2.6. which are stated without proof. The formal proof of these results is far simpler than that of the corresponding asymptotic results of [C-K2]; we provide the reader with precise references. A complete discussion of the asymptotic estimates will appear in [Kl-R]

In section 3 we make use of the vectorfield K_0 to derive generalized energy estimates for the curved background wave equation of Theorem 2.5. All the effort here goes into controlling the error terms generated by the fact that K_0 is no longer a conformal Killing vectorfield. Its failure to be conformal Killing is measured by its deformation tensor ${}^{(K_0)}\pi$. Thus the precise asymptotic properties of the null components of ${}^{(K_0)}\pi$, derived in section 2.2, play a fundamental role in controlling the error terms. In section 3.1 we give a detailed account of the boundedness of the generalized energy norm $\mathcal{E}[\phi](t)$. Its counterpart (see 1.24), in flat space is automatically bounded in view of the conseravtion part i) of Proposition 1.4 and estimate iii). In section 3.2 we sketch the derivation of the corresponding norm for the higher derivatives of ϕ. Strictly speaking our proof of Theorem A, discussed in sections 2 and 3, requires $k = 2$ and therefore $\sigma > \frac{1}{5}$ in Theorem C. This is due to the fact that in one of the error terms handled in section 3.1 we need to make an integration by parts which introduces higher derivatives on one component of the deformation tensor ${}^{(K_0)}\pi$ and therefore on some of the null components $\chi, \eta, \underline{\omega}$ of the hessian of the optical function. In section 3.3 we indicate, using once more some crucial ideas in [C-K2], how to use the nonlinear structure of the equation in Theorem C, to gain the needed regularity on those null components in order to get the result $\sigma = \frac{1}{6}$ of Tataru. These ideas can be developed further to improve the result to $\sigma > \frac{2-\sqrt{3}}{2}$ mentioned above, see [Kl-R]. The proof of Theorem A for $k = 1$(or the C^2 result of H. Smith([S1], [S2]), based on vectorfield methods, remains open. It is important to observe however that the methods presented here do not use the full force of the C^2-assumptions on the coefficients. Our method relies in fact only on bounds on uniform bounds for the components of the Riemann curvature tensor of the metric and those of the second fundamental form.

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1 The Dispersive and Strichartz Inequalities in Minkowski space

1.1 Commuting vectorfields and global Sobolev inequalities

Let ϕ be the solution of the initial value problem for the standard wave equation,

$$\begin{aligned} &\square\,\phi = 0 \\ &\phi(0) = f,\ \partial_t\phi(0) = g \end{aligned} \tag{1.5}$$

As discussed in the introduction it it is possible to show, using the explicit form of the fundamental solution as a Fourier integral operator, that for any $k \geq 0$,

$$||\nabla^k\phi(t)||_{L^\infty} \le C|t|^{-\frac{n-1}{2}} \tag{1.6}$$

as $|t|$ goes to infinity. According to 0.3 the constant C depends on the L^1 of appropriate number of derivatives of the data f, g. In what follows we review the commuting vectorfields method for deriving the decay rate 1.6. The idea is to use the energy identity 0.1 together with commuting vectorfields and a global form of the classical Sobolev inequalities.

The Minkowski space-time $\mathbf{R}^{n+1}$ is equipped with a family of Killing and conformal Killing vector fields,

$$\begin{aligned} &T_\mu = \partial_\mu \\ &O_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \\ &S = t\partial_t + x^i\partial_i \\ &K_\mu = -2x_\mu S + < x, x > \partial_\mu \end{aligned} \tag{1.7}$$

Here x^μ, denote the standard variables $x^0 = t$, $x^1, \dots, x^n$, $m^{\mu\nu} = \mathrm{diag}(-1, 1, \dots, 1)$ the Minkowski metric and $x_\mu = m_{\mu\nu}x^\nu$. The operator $\Box$ is defined by $\Box = m^{\alpha\beta}\partial_{\alpha\beta}$. The Killing vector fields T_μ and $O_{\mu\nu}$ commute with $\Box$ while S preserves the space of solutions in the sense that $\Box\,\phi = 0$ implies $\Box\, S\phi = 0$ as $[\,\Box\,, S] = 2\,\Box$. We split the operators $O_{\mu\nu}$ into the angular rotation operators ${}^{(ij)}O = x_i\partial_j - x_j\partial_i$ and the boosts ${}^{(i)}L = x_i\partial_t + t\partial_i$, for $i, j, k = 1, \dots, n$. Recall the energy norm 0.1, $E[\phi](t) = \left(\int |\partial_t\phi(t,x)|^2 + |\partial_1\phi(t,x)|^2 \cdots + |\partial_n\phi(t,x)|^2 dx\right)^{\frac{1}{2}}$. Based on the commutation properties described above we define the following "generalized energy norms"

$$E_{k+1}[\phi] = \left(\sum_{X_{i_1},..,X_{i_j}} E^2[X_{i_1}X_{i_2}...X_{i_j}\phi]\right)^{\frac{1}{2}} \tag{1.8}$$

with the sum taken over $0 \le j \le k$ and over all Killing vector fields $T, \Omega_{\mu\nu}$ as well as the scaling vector field S.

The crucial point of this method is that the quantities E_k, $k \ge 1$ are conserved by solutions to 1.5. Therefore, if for all $0 \le k \le s$ the data f, g verify,

$$\int (1+|x|)^{2k}\Big(|\nabla^{k+1}f(x)|^2 + |\nabla^k g(x)|^2\Big)dx \le C_s \tag{1.9}$$

for a constant $C_s < \infty$, then for all t,

$$E_{s+1}[\phi](t) \le C_s. \tag{1.10}$$

The desired decay estimates of solutions to 1.5 can now be derived from the following global version of the Sobolev inequalities(see [Kl1],[Kl2],[Ho]):

Proposition 1.1 *Let ϕ be an arbitrary function in R^{n+1} such that $E_s[\phi]$ is finite for some integer $s > \frac{n}{2} + 1$. Then for $t > 0$*

$$|\partial\phi(t,x)| \le (1+t+|x|)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}}E_s[u]. \tag{1.11}$$

Therefore if the data f, g in 1.5 satisfy 1.9 , for $0 \le k \le s$ with some $s > \frac{n}{2}$, then for all $t \ge 0$,

$$|\partial\phi(t,.)|_{L^\infty} \le C_s \frac{1}{(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}} \tag{1.12}$$

Clearly this estimate implies 1.6. In fact it provides more information outside the wave zone $|x| \sim t$ which fit very well with the expected propagation properties of the linear equation $\Box\phi = 0$.

Remark: *The method presented above can be refined in many directions. To start with we can derive essentially*[5] *the same information as in 1.12 by using only the scaling vectorfield S and rotation vectorfields ${}^{(ij)}O$. Moreover we can limit the number of vectorfields $S, {}^{(ij)}O$ used in the definition of the generalized energy norm E_s if we give up on the term $(1+(t-|x|))^{-\frac{1}{2}}$ on the right hand side of 1.12. Limiting the number of vectorfields S and, more importantly*[6]*, ${}^{(ij)}O$ needed to control the uniform decay of our solutions, is essential in order to derive the optimal results of Theorems A-C.*

Here is in fact a simple result(see [Kl2]) which shows how many angular momentum operators ${}^{(ij)}O$ we need to control the decay of ϕ in the *exterior region* $|x| > \frac{t}{2}$.

Proposition 1.2 *Let u be a smooth function in $\mathbf{R}^n$, vanishing sufficiently fast at infinity and s an integer larger than $\frac{n-1}{2}$. The following inequality holds for all $x \neq 0$,*

$$|u(x)| \le C_n |x|^{-\frac{n-1}{2}} \|u\|^{\frac{1}{2}}_{O,s} \cdot \|\partial_r u\|^{\frac{1}{2}}_{O,s} \tag{1.13}$$

where $\|u\|_{O,s} = \left(\sum_{0\le k\le s}\sum_{O_1,..,O_k} \|O_1 O_2 ... O_k \phi\|_{L^2(\mathbf{R}^n)}\right)^{\frac{1}{2}}$, with $O_1, \ldots, O_k$ angular momentum vectorfields.

Also[7]*, for all dimensions $n \ge 3$, and any small $\epsilon > 0$,*

$$|u(x)| \le C_{n,\epsilon} |x|^{-1+\epsilon} \|u\|^{\frac{1}{2}}_{O,1} \cdot \|\partial_r u\|^{\frac{1}{2}}_{O,1}. \tag{1.14}$$

For $n = 2$ we have,

$$|u(x)| \le C|x|^{-\frac{1}{2}} \|u\|^{\frac{1}{2}}_{O,1} \cdot \|\partial_r u\|^{\frac{1}{2}}_{O,1}. \tag{1.15}$$

Applying 1.13 to solutions ϕ of 1.5 we easily infer, as above, that

$$|\partial\phi(t,x)| \le C_n |x|^{-\frac{n-1}{2}} \|\partial\phi\|^{\frac{1}{2}}_{O,s} \cdot \|\partial^2\phi\|^{\frac{1}{2}}_{O,s}.$$

with $s > \frac{n-1}{2} + 1$ and C_n a constant which depends only on weighted L^2 norms of the data f, g.

Thus , for the exterior region $|x| \ge \frac{t}{2}$, we derive the decay estimate 1.6 with the help of $s > \frac{n-1}{2}$ angular momentum operators. Moreover, in all dimensions, we can derive a $t^{-1+\epsilon}$ decay estimate using only one angular momentum operator ${}^{(ij)}O$. The decay estimates for the *interior region*

[5] To be precise we can derive the same estimate for $\partial^2\phi$ instead of $\partial\phi$, see [Kl-Si].

[6] The construction of angular momentum vectorfields in a curved background requires more differentiability of the space-time metric than the construction of S.

[7] See Lemma 1.1 below.

$|x| \le \frac{t}{2}$ can be done by using only the scaling operator S; see discussion in Proposition 1.7 and its corollary at the end of section 1.1.

Even one angular momentum operator, however, is too much for obtaining the optimal results[8] of Theorems A, B,C, see section 2. In what follows I will present a way of deriving the $t^{-1+\epsilon}$ decay estimate, in any dimension $n \ge 3$, using, instead of scaling and angular momentum operators, the Morawetz vectorfield $K_0 = (t^2+|x|^2)\partial_t + 2tx^i\partial_i$ and its associated first order operator $K_0\phi + (n-1)t\phi$. Let

$$Q_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m_{\alpha\beta}(m^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)$$

the energy momentum tensor associated to the equation $\square\,\phi = 0$ with $m_{\mu\nu}$ the Minkowski metric of $\mathbf{R}^{n+1}$. If ϕ is a solution of the equation we have, $\partial^\beta Q_{\alpha\beta} = 0$. We recall the following classical fact, see [C-K1],

Proposition 1.3 *Let ϕ be a solution of $\square\,\phi = 0$ and $Q_{\alpha\beta}$ the corresponding energy momentum tensor. Let X be a conformal Killing vectorfield, i.e. ${}^{(X)}\pi = \mathcal{L}_X m = \Omega m$, and $tr\pi = m^{\alpha\beta}\pi_{\alpha\beta}$. It is easy to check that $\square\,\Omega = 0$; in fact, in the particular case of $X = K_0$, $\Omega = 4(n+1)t$. Let $\bar{P}_\alpha = Q_{\alpha\beta}X^\beta + \frac{n-1}{4(n-1)}tr\,{}^{(X)}\pi\phi\partial_\alpha\phi - \frac{n-1}{8(n+1)}\partial_\alpha(tr\,{}^{(X)}\pi)\phi^2$. Then, if $\square\,\phi = 0$,*

$$\partial^\alpha\bar{P}_\alpha = 0.$$

Applying the proposition to $\square\,\phi = 0$ and $X = K$ and integrating the corresponding divergence free equation on a time slab $[t_0, t] \times \mathbf{R}^n$ we infer the following[9]:

Proposition 1.4 *Let $\bar{Q}(K_0, T_0) = Q(K_0, T_0) + (n-1)t\phi\partial_t\phi - \frac{n-1}{2}\phi^2$, with $T_0 = \partial_t$ the unit normal to Σ_t and ϕ a solution to $\square\,\phi = 0$.*

i.) The following conformal conservation law holds true,

$$\int_{\Sigma_t}\bar{Q}(K_0, T_0) = \int_{\Sigma_{t_0}}\bar{Q}(K_0, T_0) \tag{1.16}$$

ii.) Moreover we have,

$$\int_{\Sigma_t}\bar{Q}(K_0, T_0) = \frac{1}{4}\Big(\int_{\Sigma_t}\underline{u}^2(L'\phi)^2 + \int_{\Sigma_t}2(t^2+r^2)|\nabla\!\!\!\!/\,\phi|^2 + \int_{\Sigma_t}u^2(\underline{L}'\phi)^2\Big) \tag{1.17}$$

where $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, $u = t - r$, $\underline{u} = t + r$ and $\underline{u}L'(\phi) = \underline{u}L(\phi) + (n-1)\phi$, $u\underline{L}'(\phi) = u\underline{L}(\phi) + (n-1)\phi$.

iii.) Also, if $n \ge 3$, there exists a constant $c > 0$ such that,

$$\int_{\Sigma_t}\bar{Q}(K_0, T_0) \ge c\Big(\int_{\Sigma_t}\underline{u}^2(L\phi)^2 + \int_{\Sigma_t}2(t^2+r^2)|\nabla\!\!\!\!/\,\phi|^2 + \int_{\Sigma_t}u^2(\underline{L}\phi)^2\Big) \tag{1.18}$$

[8] The method would be enough however to rederive the first improved regularity result for quasilinear equations, $\sigma > \frac{1}{4}$, due to Chemin-Bahouri [B-C1].

[9] Part i and ii of the proposition are due to C. Morawetz [M]. For part iii see [Kl3], pages 310–313.

To prove the second part of the proposition we first express K, T as linear combinations of the null vectorfields $L, \underline{L}$,

$$K_0 = \frac{1}{2}(\underline{u}^2 L + u^2 \underline{L}) \quad (1.19)$$

$$T_0 = \frac{1}{2}(L + \underline{L}) \quad (1.20)$$

with $u = t - r$, $\underline{u} = t + r$. Observe that u is a special solution of the Eikonal equation $m^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$. This will play an important role in the following sections.

We easyly check the formulas:

$$\begin{aligned} Q_{LL} = Q(L, L) &= L(\phi)^2 \\ Q_{L\underline{L}} = Q(L, \underline{L}) &= |\nabla\!\!\!/ \phi|^2 \\ Q_{\underline{L}\underline{L}} = Q(\underline{L}, \underline{L}) &= L(\phi)^2 \end{aligned}$$

where $\nabla\!\!\!/ \phi$ denotes the induced covariant derivatives on the spheres of intersection between the level surfaces of t and those of r. Thus,

$$\begin{aligned} Q(K_0, T_0) &= \frac{1}{4}\Big(\underline{u}^2 Q_{LL} + (u^2 + \underline{u}^2) Q_{L\underline{L}} + u^2 Q_{\underline{L}\underline{L}}\Big) \\ &= \frac{1}{4}\Big(\underline{u}^2 L(\phi)^2 + (u^2 + \underline{u}^2)|\nabla\!\!\!/ \phi|^2 + u^2 \underline{L}(\phi)^2\Big) \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Sigma_t} \bar{Q}(K_0, T_0) &= \int_{\Sigma_t} \frac{1}{4}\Big(\underline{u}^2 (L\phi)^2 + (u^2 + \underline{u}^2)|\nabla\!\!\!/ \phi|^2 + u^2 (\underline{L}\phi)^2\Big) \\ &+ \ (n-1)\int_{\Sigma_t} t\partial_t\phi\phi - \frac{n-1}{2}\int_{\Sigma_t} \phi^2 \end{aligned}$$

We now write, with $S = \frac{1}{2}(\underline{u}L\phi + u\underline{L}\phi)$,

$$\begin{aligned} t\partial_t\phi &= \frac{1}{2}(u + \underline{u})\partial_t\phi \\ &= \frac{1}{2}(\underline{u}L\phi + u\underline{L}\phi) - \frac{1}{2}(\underline{u} - u)\partial_r\phi \\ &= S\phi - \frac{1}{2}(\underline{u} - u)\partial_r\phi \end{aligned}$$

Hence, since $(\underline{u} - u) = 2r$

$$\begin{aligned} \int_{\Sigma_t} t\partial_t\phi\phi &= \int_{\Sigma_t} S\phi \cdot \phi - \frac{1}{4}\int_{\Sigma_t} (\underline{u} - u)\partial_r(\phi^2) \quad (1.21) \\ &= \frac{1}{2}\int_{\Sigma_t} S \cdot \phi + \frac{1}{4}\int_{\Sigma_t} \Big(\partial_r(\underline{u} - u) + \frac{n-1}{r}(\underline{u} - u)\Big)\phi^2 \\ &= \frac{1}{2}\int_{\Sigma_t} S \cdot \phi + \frac{n}{2}\int_{\Sigma_t} \phi^2 \end{aligned}$$

Thus, since $\frac{n}{2}(n-1) - \frac{n-1}{2} = \frac{(n-1)^2}{2}$,

$$\begin{aligned}\int_{\Sigma_t} \bar{Q}(K_0, T_0) &= \int_{\Sigma_t} \frac{1}{4}\Big(\underline{u}^2(L\phi)^2 + (u^2+\underline{u}^2)|\nabla\!\!\!/\phi|^2 + u^2(\underline{L}\phi)^2\Big) \\ &+ (n-1)\int_{\Sigma_t}\phi S\phi + \frac{(n-1)^2}{2}\int_{\Sigma_t}\phi^2\end{aligned}$$

Finally, writing $S\phi = \frac{1}{2}(\underline{u}L\phi + u\underline{L}\phi)$, we have

$$\begin{aligned}&\frac{1}{4}\Big(\underline{u}^2(L\phi)^2 + u^2(\underline{L}\phi)^2\Big) + (n-1)\phi S\phi + \frac{(n-1)^2}{2}\phi^2 \\ =\ & \frac{1}{4}\underline{u}^2(L\phi)^2 + \frac{n-1}{2}\underline{u}\phi L\phi + \frac{(n-1)^2}{4}\phi^2 \\ +\ & \frac{1}{4}u^2(\underline{L}\phi)^2 + \frac{n-1}{2}u\phi\underline{L}\phi + \frac{(n-1)^2}{4}\phi^2 \\ =\ & \frac{1}{4}\Big((\underline{u}L\phi + (n-1)\phi)^2 + (u\underline{L}\phi + (n-1)\phi)^2\Big)\end{aligned}$$

Therefore,

$$\int_{\Sigma_t}\bar{Q}(K_0, T_0) = \int_{\Sigma_t}\frac{1}{4}\Big(\big((\underline{u}L\phi + (n-1)\phi)^2 + (u\underline{L}\phi + (n-1)\phi)^2\big) + (u^2+\underline{u}^2)|\nabla\!\!\!/\phi|^2\Big)$$

as desired.

To prove the last part of the proposition we go back to the derivation of 1.21 and proceed somewhat differently. Denoting $\underline{S}\phi = \frac{1}{2}(\underline{u}L - u\underline{L})$ we write,

$$\begin{aligned}t\partial_t\phi &= \frac{t}{\underline{u}-u}(\underline{u}-u)\partial_t\phi \\ &= \frac{t}{\underline{u}-u}(\underline{u}L\phi - u\underline{L}\phi) - \frac{t}{\underline{u}-u}(u+\underline{u})\partial_r\phi \\ &= \frac{t}{r}\underline{S}\phi - \frac{t}{\underline{u}-u}(u+\underline{u})\partial_r\phi\end{aligned}$$

Integrating by parts we find,

$$\begin{aligned}\int_{\Sigma_t} t\partial_t\phi\phi &= \int_{\Sigma_t}\frac{t}{r}\phi\underline{S}\phi - \int_{\Sigma_t}\frac{t}{2(\underline{u}-u)}(u+\underline{u})\partial_r\phi^2 \\ &= \int_{\Sigma_t}\frac{t}{r}\phi\underline{S}\phi + \int_{\Sigma_t}\Big(\partial_r(\frac{t}{2(\underline{u}-u)}(u+\underline{u})) + \frac{n-1}{r}\frac{t}{2(\underline{u}-u)}(u+\underline{u})\Big)\phi^2 \\ &= \int_{\Sigma_t}\frac{t}{r}\phi\underline{S}\phi + \frac{n-2}{2}\int_{\Sigma_t}\frac{t^2}{r^2}\phi^2\end{aligned} \tag{1.22}$$

Consider now both identities 1.21 and 1.22,

$$\begin{aligned}\int_{\Sigma_t} t\partial_t\phi\phi &= \int_{\Sigma_t}\phi S\phi + \frac{n}{2}\int_{\Sigma_t}\phi^2 \\ \int_{\Sigma_t} t\partial_t\phi\phi &= \int_{\Sigma_t}\frac{t}{r}\phi\underline{S}\phi + \frac{n-2}{2}\int_{\Sigma_t}\frac{t^2}{r^2}\phi^2\end{aligned}$$

Observe that, $\int_{\Sigma_t}\frac{1}{4}\Big(\underline{u}^2(L\phi)^2+u^2(\underline{L}\phi)^2\Big)=\frac{1}{2}\Big((S\phi)^2+(\underline{S}\phi)^2\Big)$. Now let A, B such that $A+B=n-1$ and write,

$$\begin{aligned}
&\int_{\Sigma_t}\frac{1}{4}\Big(\underline{u}^2(L\phi)^2+u^2(\underline{L}\phi)^2\Big)+(n-1)\int_{\Sigma_t}t\partial_t\phi\phi-\frac{(n-1)}{2}\int_{\Sigma_t}\phi^2\\
=\;&\frac{1}{2}\int_{\Sigma_t}\Big((S\phi)^2+2A\phi S\phi+(An-(n-1))\phi^2\Big)\\
+\;&\frac{1}{2}\int_{\Sigma_t}\Big((\underline{S}\phi)^2+2B\frac{t}{r}\phi\underline{S}\phi+B(n-2)\frac{t^2}{r^2}\phi^2\Big)
\end{aligned}$$

If $1<A<n-1$ we can find $c_1>0$ such that

$$\Big((S\phi)^2+2A\phi S\phi+(An-(n-1))\phi^2\Big)\geq c_1\Big((S\phi)^2+\phi^2\Big).$$

Also, if $0<B<n-2$, we can find $c_2>0$ s.t.

$$\Big((\underline{S}\phi)^2+2B\frac{t}{r}\phi\underline{S}\phi+B(n-2)\frac{t^2}{r^2}\phi^2\Big)>c_2\Big((\underline{S}\phi)^2+\frac{t^2}{r^2}\phi^2\Big).$$

If $n\geq 3$, there exist A, B verifying $A+B=n-1$, $1<A<n-1$, $0<B<n-2$, and therefore, taking $c=\min(c_1,c_2)$,

$$\begin{aligned}
&\int_{\Sigma_t}\frac{1}{4}\Big(\underline{u}^2(L\phi)^2+u^2(\underline{L}\phi)^2\Big)+(n-1)\int_{\Sigma_t}t\partial_t\phi\phi-\frac{(n-1)}{2}\int_{\Sigma_t}\phi^2\\
\geq\;&c\int_{\Sigma_t}\Big(\frac{1}{2}|\phi|^2+|S(\phi)|^2+|\underline{S}\phi|^2\Big)\\
=\;&c\frac{1}{2}\int_{\Sigma_t}\Big(|\phi|^2+\underline{u}^2|L\phi|^2+u^2|\underline{L}\phi|^2\Big)
\end{aligned}$$

and therefore

$$\int_{\Sigma_t}\bar{Q}(K_0,T_0)\geq c\int_{\Sigma_t}\Big(|\phi|^2+\underline{u}^2|L\phi|^2+(t^2+r^2)|\nabla\!\!\!/\phi|^2+u^2|\underline{L}\phi|^2\Big)$$

as desired.

Lemma 1.1 *Let $u(x)$ be a smooth, compactly supported function on $\mathbf{R}^n$, $n\geq 3$. For any $p>n-1$, $\sigma\geq 1+\frac{n}{2}-\frac{n}{p}$, we have*

$$|u(x)|\leq C\frac{1}{|x|^{\frac{n-1}{p}}}\Big(\|r\nabla\!\!\!/u\|_{H^\sigma}+\|u\|_{H^\sigma}\Big)\tag{1.23}$$

where $\nabla\!\!\!/$ denotes the induced covariant derivative along the spheres $r=const.$

To prove the Lemma we write, in polar coordinates $x=r\xi$ with $\xi\in\mathbf{R}^{n-1}$,

$$u(r\xi)^p=-p\int_r^\infty\partial_r u(\lambda\xi)u^{p-1}(\lambda\xi)d\lambda$$

Hence,

$$\int_{|\xi|=1}u(r\xi)^p d\sigma(\xi)\leq c\frac{1}{r^{n-1}}\int_{\mathbf{R}^n}|\nabla u(y)||u(y)|^{p-1}dy.$$

Hence, for $\sigma \geq \frac{n}{2} - \frac{n}{p} + 1$,

$$\int_{|\xi|=1} u(r\xi)^p d\sigma(\xi) \leq c\frac{1}{r^{n-1}}(\|\nabla u\|^p_{L^p(\mathbf{R}^n)} + \|u\|^p_{L^p(\mathbf{R}^n)}) \leq c\frac{1}{r^{n-1}}\|u\|^p_{H^\sigma}.$$

Finally, using the Sobolev inequality on the unit sphere S^{n-1} we infer that, for $x = r\xi$ and $r \neq 0$,

$$|u(x)| \leq c_n\Big(\|u(r\,\cdot)\|_{L^p(S^{n-1})} + \|(r\nabla\!\!\!/ u)(r\,\cdot)\|_{L^p(S^{n-1})}\Big)$$

which combined with the inequality above proves the desired result.

Now let

$$\mathcal{E}^2[\phi](t) = \int_{\Sigma_t}\Big(|\phi|^2 + \underline{u}^2|L\phi|^2 + (t^2+r^2)|\nabla\!\!\!/\phi|^2 + u^2|\underline{L}\phi|^2\Big) \tag{1.24}$$

$$\mathcal{E}^2_{k+1}[\phi](t) = \sum_{0\leq i\leq k} \mathcal{E}^2[\nabla^i\phi] \tag{1.25}$$

In view of the Lemma 1.1 we immediately derive the result of the Proposition below in the exterior region $|x| \geq \frac{t}{2}$. For the interior region $|x| \leq \frac{t}{2}$ the result follows from the fact that,

$$\mathcal{E}^2[\phi](t) \geq ct^2\int_{|x|\leq\frac{t}{2}} |\nabla\phi|^2$$

combined with the standard $H^s(\mathbf{R}^n) \subset L^\infty, s > \frac{n}{2}$ Sobolev embedding. Thus,

Proposition 1.5 *Let $\phi(t,x)$ be a smooth function in $\mathbf{R}^{n+1}$ compactly supported in x for each fixed $t \geq 0$. The following inequality holds true for any $n \geq 3$, $p > n-1$ and $k \geq 3 + \frac{n}{2} - \frac{n}{p}$*

$$\|\partial\phi(t)\|_{L^\infty} \leq c(1+t)^{-\frac{n-1}{p}}\mathcal{E}_k(t).$$

In view of the conservation of the integrals $\int_{\Sigma_t}\bar{Q}(K_0,T_0)$ as well as[10] $\int_{\Sigma_t}\bar{Q}(T_0,T_0)$ applied to the standard derivatives $\partial_t, \partial_1, \ldots \partial_n$ of solutions to $\Box\,\phi = 0$, as well as part iii of 1.3 we obtain the following:

Proposition 1.6 *Let $\Box\,\phi = 0$ subject to the initial conditions $\phi(0) = f, \partial_t\phi(0) = g$ with f,g smooth and compactly supported in the ball $|x| \leq 2$. Then, for all $t \geq 0$, $k > 3 + \frac{n}{2} - \frac{n}{n-1}$*

$$\|\partial\phi(t)\|_{L^\infty} \leq C(1+t)^{-1+\epsilon}\Big(\|f\|_{H^k} + \|g\|_{H^{k-1}}\Big). \tag{1.26}$$

[10]This is needed to control small $t \geq 0$.

1.2 The proof of the Dispersive Inequality using the Commuting Vectorfields Method

In the previous section we have reviewed the commuting vectorfields method of deriving the asymptotic behavior of solutions to 1.5 as $t \to \infty$. The method seems quite wasteful in terms of how many derivatives are needed for the data f, g, and therefore it does not seem well suited for improved regularity results. In what follows we show that in fact the commuting vectorfields method implies the dispersive inequality 0.3. The key ingredient in the proof is a simple phase-space localization argument which I borrow from [L].

Theorem 1.1 *The commuting vectorfields method implies the dispersive inequality 0.3.*

Without loss of generality we may assume that $\partial_t \phi = g = 0$ and that the Fourier transform of $f = \phi(0)$ is supported in the shell $\frac{\lambda}{2} \le |\xi| \le 2\lambda$ for some $\lambda \in 2^{\mathbf{N}}$. By a simple scaling argument we may in fact assume $\lambda = 1$. Since $\hat{\phi}$, the Fourier transform of ϕ relative to the space variables x, is also supported in the same shell it suffices to prove the estimates for $\nabla \phi$ or $\nabla^k \phi$. Next we cover $\mathbf{R}^n$ by an union of discs D_I centered at points $I \in \mathbf{Z}^n$ with integer coordinates such that each D_I intersects at most a finite number c_n of discs D_J with c_n depending only on the dimension n. Consider a smooth partition of unity $(\chi_I)_{I \in \mathbf{Z}^n}$ with $\operatorname{supp} \chi_I \subset D_I$ and each χ_I positive. Clearly we can arrange to have, for all k,

$$\sum_{I \in \mathbf{Z}^n} |\nabla^k \chi_I(x)| \le C_{k,n} \tag{1.27}$$

uniformly in $x \in \mathbf{R}^n$. For $k = 0$ we have in fact $C_{k,n} = 1$.

Now set, $f_I = \chi_I \cdot f$, and ϕ_I the corresponding solution to 1.5 with data $\phi_I(0) = f_I, \partial_t \phi_I(0) = 0$. Clearly $f = \sum_I f_I,\ \phi = \sum_I \phi_I$. It suffices to prove that for all I,

$$\|\nabla^k \phi_I(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+k+1} \|D^j f_I\|_{L^1} \tag{1.28}$$

with a constant $C_{n,k}$ depending only on n and k. Indeed if 1.28 holds true we easily infer that,

$$\|\nabla^k \phi(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+k+1} \|\sum_I \nabla^j \chi_I\|_{L^\infty} \|f\|_{L^1}$$

and therefore, in view of 1.27

$$\|\nabla^k \phi(t)\|_{L^\infty} \le C_{n,k}(1+t)^{-\frac{n-1}{2}} \|f\|_{L^1}$$

It therefore remains to check 1.28. Without loss of generality, by performing a space translation, we may assume that $I = 0$. Applying the Prop. 1.11 to $\psi = \nabla \phi_0$ we derive, for s_* the first integer strictly larger than $\frac{n}{2} + 1$,

$$\begin{aligned} \|\psi(t)\|_{L^\infty} &\le c(1+t)^{-\frac{n-1}{2}} E_{s_*}[\phi_0](t) \\ &\le c(1+t)^{-\frac{n-1}{2}} E_{s_*}[\phi_0](0). \end{aligned}$$

Since the support of ϕ_0 is included in in the ball of radius 1 centered at the origin we have,

$$E_{s_*}[\phi_0](0) \le C_n \sum_{j=0}^{s_*+1} \|D^j f_0\|_{L^2}.$$

Finally, according to the standard Sobolev inequality in $\mathbf{R}^n$, $\|f\|_{L^2} \le c\|\nabla^{\frac{n}{2}} f\|_{L^1}$, we conclude with,

$$\|\psi(t)\|_{L^\infty} \le c(1+t)^{-\frac{n-1}{2}} \sum_{j=0}^{n+2+1} \|D^j f_0\|_{L^1}$$

as desired.

Next we show a direct proof, without using the Fourier representation of solutions[11], of the Strichartz inequality 0.4 from 0.3.

Theorem 1.2 *The dispersive inequality 0.3 implies the Strichartz inequality 0.4.*

It suffices to prove the Strichartz inequalities for initial data $\partial_t\phi(0) = i_0, \partial_t\phi(0) = i_1$ whose Fourier transform are supported in the unit dyadic shell $\frac{1}{2} \le |\xi| \le 2$. Let $\mathcal{H}$ be the real Hilbert space of vectors $I = (i_0, i_1)^t$, with the Fourier transform supported in $|\xi| \le 2$ and scalar product $< I\,;\, J >= \int_{\mathbf{R}^n} \Big(\sum_{a=1}^n \partial_a i_0(x)\partial_a j_0(x) + i_1(x)\cdot j_1(x)\Big)$. Define X to be the closed subspace of functions in $L^q_t L^r_x([0,t^*]\times\mathbf{R}^n)$ whose Fourier transform are supported also in $|\xi| \le 2$. The dual space consists of functions in $L^{q'}_t L^{r'}_x([0,t^*]\times\mathbf{R}^n)$ which have the same property. As it is well known to prove 0.4 it suffices to show that TT^* is a bounded linear operator from X^* to X.

Denote by $\phi(t,s;I(s))$, $I = (i_0,i_1)^t$, the solution at time t of $\Box\,\phi = 0$ with initial conditions, at $t = s$, given by $\phi(s) = i_0(s)$, $\partial_t\phi(s) = i_1(s)$. Let $\Phi(t,s;I(s))$ be the column vector $\Big(\phi(t,\cdot), \partial_t\phi(t,\cdot)\Big)^t$. We will make use in an essential way of the group property:

$$\Phi(t,s;\Phi(s,t_0 : I(t_0)) = \Phi(t,t_0 : I(t_0))$$

By interpolating between the $L^2 \to L^2$ and $L^\infty \to L^1$ we infer that, as long as $I(s) \in \mathcal{H}$,

$$\|\partial_t\phi(t,s;I(s))\|_{L^r_x} \le c(1+|t-s|)^{-\gamma(r)}\|I(s)\|_{L^{r'}_x}$$

where $\gamma(r) = (n-1)(\frac{1}{2}-\frac{2}{r})$, $\frac{1}{r}+\frac{1}{r'} = 1$, $r \ge 2$. Here $\|I\|_{L^{r'}_x} = \|\nabla i_0\|_{L^{r'}_x} + \|i_1\|_{L^{r'}_x}$. Let $T : \mathcal{H} \longrightarrow X = L^q_t L^r_x[0,t^*]$ be defined by $T(I)(t,x) = \partial_t\phi(t,x)$. To calculate the dual $T^* : X^* \longrightarrow \mathcal{H}$ we write, for an arbitrary $f \in X^*$,

$$< T^*(f), I >= \int_0^{t_*}\int_{\mathbf{R}^n} \partial_t\phi(t,x)f(t,x)dxdt$$

Now let ψ the solution to $\Box\,\psi = f$ with zero initial data at $t = t^*$. Therefore,

$$\begin{aligned} -< T^*(f), I > &= \int_{\mathbf{R}^n}\Big(\phi_t(0,x)\psi_t(0,x) - \phi_{tt}(0,x)\psi(0,x)\Big)dx \\ &= \int_{\mathbf{R}^n}\Big(\phi_t(0,x)\psi_t(0,x) + \sum_A \partial_A\phi(0,x)\partial_A\psi(0,x)\Big)dx \\ &= \ < \Psi(0), I > \end{aligned}$$

[11]See also [K-T]. The proof we give below however is the one we will be able to extend to time dependent variable coefficients, see Theorem 2.3 and its proof.

where $\Psi(0) = (\psi(0), \partial_t\psi(0))$. By Duhamel's principle we have, $\psi(t) = \int_{t_*}^{t} \phi(t, s; F(s))$ where $F(s) = (0, f(s))^t$. Hence $\Psi(0) = -\int_0^{t_*} \Phi(0, s; F(s))ds$ and therefore,

$$T^*(f) = \int_0^{t_*} \Phi(0, s; F(s)).$$

We infer that

$$TT^*(f) = \int_0^{t_*} \partial_t\phi\Big(t, 0; \Phi(0, s; F(s))\Big) = \int_0^{t_*} \partial_t\phi(t, s; F(s))ds.$$

Consequently,

$$\|TT^*(f)\|_{L^r_x} \leq \int_0^{t_*} (1 + |t - s|)^{-\gamma(r)} \|f(s)\|_{L^{r'}_x} ds$$

and the proof ends with the usual application of the Hausdorff-Young or Hardy -Littlewood -Sobolev inequalities.

2 Strichartz type Estimates on a Curved Background

In what follows we state the main results of this paper, Theorems A-C. As explained in the introduction they are not new they are due to H. Smith [S1], [S2], Bahouri-Chemin [B-C1],[B-C2] and D.Tataru[12]. The method of proof, however, is very different. Instead of constructing parametrices we rely on a variation of the vectorfield approach presented in the previous section. We refer to the introduction for a detailed preview of the main results and ideas discussed in this section. We recall that the final goal of the section is to show how the Theorems A, B, C can be reduced to the $L^2 - L^\infty$ decay estimate of Theorem 2.4. With the exception of the derivation of Theorem 2.3. from the dispersive inequality of Theorem 2.2, and the reduction of the latter to Theorem 2.4.(discussed in details in section 1.2.) most of the ideas presented in this section appear in the above mentioned references, especially [T2]. Therefore our presentation is sketchy; we give however full details whenever some arguments need modifications, such as the proof of Proposition 2.1 which is a minor extension of an argument in [T2].

2.1 Main Theorems and their reduction to Dispersive Inequalities

Theorem A *Consider the wave operator* $\Box'_h = -\partial_t^2 + h^{ij}\partial_i\partial_j$ *defined in a space-time slab* $\mathcal{D}_T = [0,T]\times\mathbf{R}^n$, $n\ge 3$. *Assume that the coefficients* $h = (h^{ij})_{i,j=1}^n$ *verify the following assumptions:*

A1 *For all* $(t,x)\in\mathcal{D}_T$, $\xi\in\mathbf{R}^n$,

$$C^{-1}|\xi|^2 \le h^{ij}(t,x)\xi_i\xi_j \le C|\xi|^2$$

A2 *For all* $0\le i\le k$, *and some fixed constant* $\mu\ge 1$,

$$T^i\|\partial^{1+i}h\|_{L_t^1L_x^\infty(\mathcal{D}_T)} \le c\mu^{2i}$$

Then,

$$\|\partial\phi\|_{L_t^2L_x^\infty(\mathcal{D}_T)} \le CT^\epsilon\Big(\mu^{\frac{k}{k+1}}\|\partial\phi\|_{L_t^\infty\dot H^s(\mathcal{D}_T)} + \mu^{-\frac{k}{k+1}}\|\Box'_h\phi\|_{L_t^1\dot H^s(\mathcal{D}_T)}\Big) \tag{2.29}$$

for any $s = \frac{n-1}{2}+\epsilon$.

Theorem B *Assume that Theorem A holds for a fixed* $k\ge 1$. *Consider a metric* h *which verifies only the assumptions A1 and A2 for* $k=0$, $\mu=1$;

$$\|\partial h\|_{L_t^1L_x^\infty(\mathcal{D}_T)} \le c.$$

Then,

$$\|\partial\phi\|_{L_t^2L_x^\infty(\mathcal{D}_T)} \le C\Big(T^\sigma\|\,|D|^\sigma\partial\phi\|_{L^\infty\mathcal{H}^s(\mathcal{D}_T)} + T^{-\sigma}\|\,|D|^{-\sigma}\Box'_h\phi\|_{L^1\mathcal{H}^s(\mathcal{D}_T)}\Big) \tag{2.30}$$

for any $s=\frac{n-1}{2}$ *and* $\sigma > \frac{k}{2(2k+1)}$.

[12]The precise statements of Theorem A and B and the optimal result of Theorem C is due to Tataru, see [T2]. His results connected to Theorems A, B are however more general.

Here and throughout the paper whenever we write $\|\psi\|_{L^q_t B}$, or simply $\|\psi\|_{L^q B}$, we mean $(\int \|\psi(t)\|_B^q dt)^{\frac{1}{q}}$ with B a Banach norm with respect to the space variables $x = (x^1, \ldots, x^n)$. Theorem B has an immediate application to quasilinear equations of the form,

$$\Box'_{h(\phi)}\phi = N(\phi, \partial\phi) \tag{2.31}$$

subject to the initial conditions at $t = 0$,

$$\phi(0) = \varphi_0 \quad \partial_t\phi(0) = \varphi_1 \tag{2.32}$$

Here $\Box'_{h(\phi)}\phi = -\partial_t^2\phi + h^{ij}(\phi)\partial_i\partial_j\phi$. Assume that $h(\phi) = (h^{ij}(\phi))_{i,j=1}^n$ is a smooth matrix valued function of ϕ. Assume also that N is a smooth function of $\phi, \partial\phi$ and depending quadratically on $\partial\phi$.

Theorem C *Assume that Theorem A is valid for some fixed $1 \leq k$ and $\mu = 1$. Consider the initial value problem 2.32 for the quasilinear wave equation 2.31 in $\mathbf{R}^{n+1}$, $n \geq 3$ Assume that the coefficients $h_{ij}(\phi)$ verify;*

$$c^{-1}|\xi|^2 \leq h^{ij}(\phi)\xi_i\xi_j \leq c|\xi|^2 \tag{2.33}$$

uniformly for $|\phi| \leq M$ and $\xi \in \mathbf{R}^n$.

Assume also that the initial data in 2.32 verify the assumptions $(\varphi_0, \varphi_1) \in H^s \times H^{s-1}$ with $s = \frac{n}{2} + \frac{1}{2} + \sigma$ for $\sigma > \frac{k}{2(2k+1)}$. Moreover assume that $\|\varphi_0\|_{L^\infty(\mathcal{D}_T)} \leq \frac{M}{2}$. Then there exists a time $T > 0$ and a unique solution of 2.31, 2.32 verifying,

$$\begin{aligned} \phi &\in L^\infty([0,T]; H^s) \cap Lip([0,T]; H^{s-1}) \\ \partial\phi &\in L^2([0,T]; L^\infty) \end{aligned}$$

and $\|\phi\|_{L^\infty(\mathcal{D}_T)} \leq M$.

Remark 1 *A more general sharper form of Theorem A, for all wave admissible Strichartz exponents, has been proved by Tataru in [T2] for $k = 1$. In particular, the optimal known result in conection to Theorem C is $\sigma = \frac{1}{6}$. The first improved regularity result is due to Chemin-Bahouri [B-C1]. They later improved the result in [B-C2].*

I.) Sketch of the Proof of Theorem B from Theorem A:

Step 1. It clearly suffices to prove Theorem B for $T = 1$. Moreover we can reduce the proof to the following dyadic case[13],

Set $h_\lambda \approx S_{\frac{\lambda}{16}} h$ and $\phi_\lambda = \Delta_\lambda\phi$ with Δ_λ the standard frequency cut-off operator corresponding to the space-time Fourier region $\frac{1}{2}\lambda \leq |\tau| + |\xi| \leq 2\lambda$. Set $\Box'_\lambda = \Box'_{h_\lambda}$. Then, for all λ suficiently large, say $\lambda \geq 2^8$, it suffices to prove that

$$\|\partial\phi_\lambda\|_{L^2_t L^\infty_x(\mathcal{D})} \leq C\lambda^s\Big(\lambda^\sigma\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \lambda^{-\sigma}\|\Box'_\lambda\phi_\lambda\|_{L^1_t L^2_x(\mathcal{D})}\Big) \tag{2.34}$$

with $\mathcal{D} = \mathcal{D}_1$.

Step 2. Split $\Box'_\lambda\phi_\lambda = \Box'_{\lambda^a}\phi_\lambda + R_\lambda$, for some $0 \leq a \leq 1$ to be chosen later. Here $\Box'_{\lambda^a}\phi_\lambda = -\partial_t^2 + h^{ij}_{\lambda^a}\partial_i\partial_j$ with $h^a_\lambda = S_{\frac{1}{16}\lambda^a} h$. Observe that

$$\|R_\lambda\|_{L^1_t L^2_x(\mathcal{D})} \leq \lambda^{1-a}\|\partial h\|_{L^1_t L^\infty_x(\mathcal{D})}\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})}$$

[13] The reduction is standard, see [B-C1], [B-C2] and [T1], [T2].

Therefore the estimate 2.34 follows from the following,

$$\|\partial\phi_\lambda\|_{L^2_tL^\infty_x(\mathcal{D})} \le C\lambda^s\Big(\lambda^\sigma\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \lambda^{-\sigma}\|\,\Box'_{\lambda^a}\phi\|_{L^1_tL^2_x(\mathcal{D})}\Big) \tag{2.35}$$

provided that $\sigma = \frac{1-a}{2}$. On the other hand it is easy to see that the metric $h_{\lambda^a} = S_{\frac{1}{16}\lambda^a}h$ verifies the conditions, A1, A2 of Theorem A. More precisely,

$$\|\partial^{1+i}h_{\lambda^a}\|_{L^1_tL^\infty_x(\mathcal{D})} \le \lambda^{ai}\|\partial h\|_{L^1_tL^\infty_x(\mathcal{D})} \le \mu^{2i}.$$

with $\mu = \lambda^{\frac{a}{2}}$.

Finally 2.35 follows from Theorem A applied to the metric h^a_λ for $\mu = \lambda^{\frac{a}{2}}$ and a chosen such that $\frac{a}{2}\frac{k}{k+1} = \frac{1-a}{2}$, i.e. $a = \frac{k+1}{2k+1}$. Therefore $\sigma = \frac{1-a}{2} = \frac{k}{2(2k+1)}$ as desired.

II.) Reduction of Theorem A to the case $\mu = 1$.

Proposition 2.1 *For every fixed k it suffices to prove Theorem A for the special case $\mu = 1$.*

We shall follow the method of proof of Tataru, see [T2]. We assume that Theorem A has already been proved, for a fixed $k \ge 1$, with $\mu = 1$. Fix a value of $\mu \ge 1$. By a simple scaling argument it suffices to prove Theorem A for $T = \mu^2$ and,

$$\|\partial^{1+i}h\|_{L^1_tL^\infty_x(\mathcal{D}_{\mu^2})} \le 1$$

for all $0 \le i \le k$. Now divide $[0, T]$ into subintervals,

$$0 = t_0 < t_1 < \ldots < t_N = T = \mu^2$$

such that for all domains $\mathcal{D}_m = I_m \times \mathbf{R}^n$, with $I_m = [t_m, t_{m+1}]$, and all $0 \le m \le N$,

$$\|\,\Box'_h\phi\|_{L^1L^2(\mathcal{D}_m)} \le c\mu^{-\frac{2k}{k+1}}\|\,\Box'_h\phi\|_{L^1L^2(\mathcal{D}_{\mu^2})} \tag{2.36}$$

as well as,

$$|I_m|^i\cdot\|\partial^{1+i}h\|_{L^1_tL^\infty_x(\mathcal{D}_m)} \le 1 \tag{2.37}$$

for all $0 \le i \le k$. We claim the total, smallest, number of intervals needed in 2.36, and 2.37 verifies,

$$N \le C_k\mu^{\frac{2k}{k+1}}. \tag{2.38}$$

with a constant C_k depending only on k. Indeed let N_0 be the total number of intervals with equality in 2.36 and N_i the total number of intervals with equality in 2.37. Clearly $N_0 \le \mu^{\frac{2k}{k+1}}$. For an interval I_m on which equality holds for 2.37 we have, $A^i_mB_m = 1$ where $A_m = |I_m|$ and $B_m = \|\partial^{1+i}h\|_{L^1_tL^\infty_x(\mathcal{D}_m)}$. Thus $\sum_{m=0}^{N-1}A_m \le \mu^2$ and $\sum_{m=0}^{N-1}B_m = \|\partial^{1+i}h\|_{L^1_tL^\infty_x(\mathcal{D}_{\mu^2})} \le 1$.

Lemma 2.1 *Let $A, B \ge 0$ with $A^iB = 1$. Then, for all $\lambda \ge 0$, $\lambda^{-\frac{1}{i}}A + \lambda B \ge c_i$, with $0 < c_i \le 2$ depending only on i. In fact we can take $c_i = \min_{y\ge0}(y + \frac{1}{y^i})$.*

Using the Lemma, whose proof is trivial, for each A_m, B_m, $A_m^i B_m = 1$ and then summing over m we infer that $\lambda^{-\frac{1}{i}}\mu^2 + \lambda \geq c_i N_i$. Choosing $\lambda = \mu^{2\frac{i}{i+1}}$ we infer that, $c_i N_i \leq \mu^{\frac{2i}{i+1}}$. Therefore the total number of intervals $N \leq N_0 + N_1 + \cdots + N_k$ for which we can have equality in one of the estimates 2.36, 2.37 $1 \leq i \leq k$, verifies $N \leq C_k \mu^{\frac{2k}{k+1}}$. Thus the minimum number of intervals needed to satisfy 2.36, 2.37 is given by 2.38.

In view of 2.36,2.37, on each of the intervals I_m we can apply the statement of Theorem A for the particular case when $\mu = 1$. Therefore,

$$\begin{aligned} \|\partial\phi\|_{L_t^2 L_x^\infty(\mathcal{D}_m)} &\leq C|I_m|^\epsilon \Big(\|\partial\phi\|_{L^1 \dot{H}^s(\mathcal{D}_m)} + \|\,\Box'_h\phi\|_{L^1\dot{H}^s(\mathcal{D}_m)} \Big) && (2.39) \\ &\leq C|I_m|^\epsilon \Big(\|\partial\phi\|_{L^1 \dot{H}^s(\mathcal{D}_m)} + \mu^{-\frac{2k}{k+1}}\|\,\Box'_h\phi\|_{L^1\dot{H}^s(\mathcal{D}_{\mu^2})} \Big) && (2.40) \end{aligned}$$

Thus squaring and summing over all $0 \leq m \leq N-1$,

$$\|\partial\phi\|^2_{L_t^2 L_x^\infty(\mathcal{D}_{\mu^2})} \leq C C_k T^{2\epsilon} \Big(\mu^{\frac{2k}{k+1}} \|\partial\phi\|^2_{L^1\dot{H}^s(\mathcal{D}_{\mu^2})} + \mu^{-\frac{2k}{k+1}} \|\,\Box'_h\phi\|^2_{L^1\dot{H}^s(\mathcal{D}_{\mu^2})} \Big)$$

and therefore, with another constant C depending on k,

$$\|\partial\phi\|_{L_t^2 L_x^\infty(\mathcal{D}_{\mu^2})} \leq C_k T^{\epsilon} \Big(\mu^{\frac{k}{k+1}} \|\partial\phi\|_{L^1\dot{H}^s(\mathcal{D}_{\mu^2})} + \mu^{-\frac{k}{k+1}} \|\,\Box'_h\phi\|_{L^1\dot{H}^s(\mathcal{D}_{\mu^2})} \Big)$$

as desired.

III.) Sketch of the reduction of Theorem A($\mu = 1$) to a microlocalized version(see Theorem 2.1.)

Step 1. Proceeding precisely as the proof of Theorem B it suffices to prove the dyadic version of the Strichartz estimate of Theorem A($\mu = 1$) for $T = 1$ and λ sufficiently large.

$$\|\partial\phi_\lambda\|_{L_t^2 L_x^\infty(\mathcal{D})} \leq C\lambda^s \Big(\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \|\,\Box'_\lambda\phi_\lambda\|_{L_t^1 L_x^2(\mathcal{D})} \Big) \qquad (2.41)$$

with $s = \frac{n-1}{2} + \epsilon$ and $\mathcal{D} = \mathcal{D}_1$. We have, roughly , $h_\lambda = S_{\frac{\lambda}{16}} h$, $\phi_\lambda = \Delta_\lambda\phi$ and $\Box'_\lambda = \Box'_{h_\lambda}$.

Step 2. Proceeding as in step 2 of the proof of Theorem B it suffices to prove the estimate

$$\|\partial\phi_\lambda\|_{L_t^2 L_x^\infty(\mathcal{D})} \leq C\lambda^s \Big(\|\partial\phi_\lambda\|_{L^\infty L^2(\mathcal{D})} + \|\,\Box'_\lambda\phi_\lambda\|_{L_t^1 L_x^2(\mathcal{D})} \Big) \qquad (2.42)$$

for a metric $h = h^{ij}$ verifying the assumptions A1, A2 in the region $\mathcal{D} = [0,1] \times \mathbf{R}^n$ and whose space-time Fourier transform is supported in the region

$$0 \leq |\tau| + |\xi| \leq \frac{1}{16}\sqrt{\lambda}. \qquad (2.43)$$

Step 3. Let $h = h^{ij}$ verify A1, A2 as well as 2.43. Define $\tilde{h}_\lambda(t,x) = h(\frac{t}{\lambda}, \frac{x}{\lambda})$. Clearly,

$$\begin{aligned} c^{-1}|\xi|^2 \leq \tilde{h}^{ij}_\lambda \xi_i\xi_j &\leq c|\xi|^2. && (2.44) \\ \|\partial^{1+i}\tilde{h}_\lambda\|_{L_t^1 L_x^\infty(\mathcal{D}_\lambda)} &\leq C\lambda^{-i} \quad \text{for all } \ 0 \leq i \leq k && (2.45) \\ \|\partial^{1+k+j}\tilde{h}_\lambda\|_{L_t^1 L_x^\infty(\mathcal{D}_\lambda)} &\leq C\lambda^{-k-\frac{j}{2}} \quad \text{for all } \ 0 \leq j && (2.46) \end{aligned}$$

Moreover the space-time Fourier transform of $\tilde{h}_\lambda$ is supported in the region $0 \le |\tau| + |\xi| \le \frac{1}{16\sqrt{\lambda}}$. Under these conditions it suffices to prove the Strichartz inequality,

$$\|\partial\psi\|_{L_t^2 L_x^\infty(\mathcal{D}_\lambda)} \le C\lambda^\epsilon\Big(\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} + \|\,\Box'_{\tilde{h}_\lambda}\psi\|_{L^1L^2(\mathcal{D}_\lambda)}\Big) \tag{2.47}$$

for all ψ whose space-time Fourier transform is supported in the region $0 \le |\tau| + |\xi| \le 2$.

Step 4. Define the positive definite metric $g = g_\lambda$ to be the inverse of the matrix $\tilde{h}_\lambda$. Whenever there is no danger of confusion we shall also denote by $g = g_\lambda$ the Lorentzian metric with g_{ij}, $i, j = 1 \dots n$ as above and $g_{00} = -1, g_{0i} = 0$ for $i = 1 \dots n$. Denote $|g| = \det(g_{ij})$. Let $\Box_g$ be the associated wave operator,

$$\Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}\, g^{\alpha\beta}\partial_\beta) = -\frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}\partial_t) + \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\, g^{ij}\partial_j).$$

Observe that

$$\begin{aligned} \Box_{g_\lambda}\psi &= \Box'_{\tilde{h}_\lambda} + R_\lambda\psi \\ R_\lambda\psi &= -\frac{\partial_t(\sqrt{|g_\lambda|}\,)}{\sqrt{|g_\lambda|}}\partial_t\phi + \frac{\partial_i(\sqrt{|g_\lambda|}\, g_\lambda^{ij})}{\sqrt{|g_\lambda|}}\partial_j\phi \end{aligned}$$

Clearly, $\|R_\lambda\psi\|_{L^1L^\infty(\mathcal{D}_\lambda)} \le C\|\partial\tilde{h}_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)}\|\partial\psi\|_{L^\infty L^2}$. Therefore 2.47 follows easily from,

$$\|\partial\psi\|_{L_t^q L_x^\infty(\mathcal{D}\lambda)} \le C\Big(\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} + \|\,\Box_{g_\lambda}\psi\|_{L^1L^2(\mathcal{D}_\lambda)}\Big) \tag{2.48}$$

Observe also that the assumptions 2.44–2.46 for $\tilde{h}_\lambda$ remain satisfied for for the inverse metric $g_\lambda = (\tilde{h}_\lambda)^{-1}$, with different constants. This is obvious for 2.44. The others follow multiple applications of the chain rule and the following Gagliardo-Nirenberg type inequality:

Lemma 2.2 *For any integer $k \ge 1$ and compactly supported smooth functions g we have,*

$$\|\partial g\|_{L^kL^\infty} \le C\|\partial^k g\|^{\frac{1}{k}}_{L_t^1L_x^\infty}\|g\|^{1-\frac{1}{k}}_{L^\infty}. \tag{2.49}$$

The proof of Theorem A($\mu = 1$) can be thus reduced to the following:

Theorem 2.1 *Let g_λ be a familly of smooth metrics, $\lambda \ge \lambda_0 > 1$, defined in the region $\mathcal{D}_\lambda = I_\lambda \times \mathbf{R}^n$ with I_λ a time interval of length λ, in which the following assumptions are satisfied,*

$$c^{-1}|\xi|^2 \le g^\lambda_{ij}\xi^i\xi^j \le c|\xi|^2. \tag{2.50}$$

uniformly for all $(x,t) \in \mathcal{D}_\lambda$, $\xi \in \mathbf{R}^n$ and $\lambda \ge \lambda_0$.

Also,

$$\begin{aligned} \|\partial^{1+i}g_\lambda\|_{L_t^1L_x^\infty(\mathcal{D}_\lambda)} &\le C_i\lambda^{-i} \quad \textit{for all } \ 0 \le i \le k & (2.51) \\ \|\partial^{1+k+j}g_\lambda\|_{L_t^1L_x^\infty(\mathcal{D}_\lambda)} &\le C_{k,j}\lambda^{-k-\frac{j}{2}} \quad \textit{for all } \ 0 \le j & (2.52) \end{aligned}$$

Under these assumptions we have,

$$\|\partial\psi\|_{L^2_t L^\infty_x(\mathcal{D}\lambda)} \le C\Big(\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} + \|\,\Box_{g_\lambda}\psi\|_{L^1 L^2(\mathcal{D}_\lambda)}\Big) \tag{2.53}$$

for all ψ whose Fourier transform supported in the region $\frac{1}{2} \le |\tau| + |\xi| \le 2$.

IV.) Reduction of Theorem 2.1 to a microlocal dispersive inequality(see Theorem 2.2)
In what follows we show that Theorem 2.1 can be reduced to the following dispersive inequality:

Theorem 2.2 *Under the same assumptions on the metric g_λ as those of Theorem 2.1, if ϕ is a solution of the homogeneous equation $\Box_{g_\lambda}\phi = 0$, in the domain $\mathcal{D}_\lambda = I_\lambda \times \mathbf{R}^n$, $I_\lambda = [0, t_*]$, $|I_\lambda| \le \lambda$ with the Fourier transform of the data $\phi(t_0), \partial_t\phi(t_0)$ supported in $0 \le |\xi| \le 4$, then;*

$$\|\partial\phi(t)\|_{L^\infty} \le C(1 + |t - t_0|)^{-1+\epsilon}\|\partial\phi(t_0)\|_{L^1} \tag{2.54}$$

Theorem 2.2 implies the following

Theorem 2.3 *Consider the initial value problem*

$$\begin{aligned} \Box_{g_\lambda}\phi &= 0 \\ \phi(0) = \varphi_0 \quad \partial_t\phi(0) &= \varphi_1 \end{aligned}$$

in the region $\mathcal{D}_\lambda = [0, \lambda] \times \mathbf{R}^n$, in which the assumptions of Theorem 2.1 hold true. Let $q = \frac{2}{1-\epsilon}$, q' the dual exponent. Let P be the operator defined by $P\phi(t,x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi}\chi(\xi)\hat{\phi}(t,\xi)$ with $\hat{\phi}$ the space Fourier transform of ϕ and χ a compactly supported smooth function, $\chi(\xi) = 1$ for $|\xi| \le 2$, $\chi(\xi) = 0$ for $|\xi| \ge 4$.

There exists a sufficiently large M , independent on λ, such that,

$$\|\partial P\phi\|_{L^q_t L^\infty_x(\mathcal{D}\lambda)} \le M\|\partial\phi(0)\|_{L^2}. \tag{2.55}$$

Theorem 2.1 follows easily from the following Corrolary of Theorem 2.3. This is due to the fact that for the ψ of theorem 2.1 we have $P\psi = \psi$.

Corollary 2.1.1 *Under the same assumptions as above consider the inhomogeneous equation,*

$$\Box_{g_\lambda}\psi = f$$

Then,

$$\|\partial P\psi\|_{L^q_t L^\infty_x(\mathcal{D}\lambda)} \le M\Big(\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} + \|f\|_{L^1 L^2(\mathcal{D}_\lambda)}\Big) \tag{2.56}$$

The proof of the Corollary is an immediate consequence of Theorem 2.3 and the standard form of the Duhamel principle.

Proof of Theorem 2.3:

Remark 1: *Without loss of generality we may assume that the metric g_λ is flat at $t = 0$, i.e* $g_\lambda(0) = \delta$, $\partial_t g_\lambda(0) = 0$. Indeed if this is not the case we can extend it to the interval $[-1, \lambda]$ by setting $\bar{g}_\lambda(t) = \chi(t)(g_\lambda - \delta) + \delta$, with χ a smooth, compactly supported function of t, $\chi = 1$ on $[0, \lambda]$ and $\chi(t) = 0$ for $t \le -\frac{1}{2}$. Though $\bar{g}_\lambda$ does not verify the same assumptions as g_λ in the interval $[-1, 0]$ is easy to see that the conclusion of Theorem 2.2 still holds true for the equation $\Box_{\bar{g}_\lambda}\phi = 0$ in the time interval $I_\lambda = [-1, t_*]$, see the remark following Theorem 2.4. Since the solutions to $\Box_{g_\lambda}\phi = 0$ and $\Box_{\bar{g}_\lambda}\phi = 0$, with same data at $t = 0$, coincide in $[0, \lambda]$, it suffices to prove 2.55 for $\Box_{\bar{g}_\lambda}\phi = 0$. Performing a time translation this is equivalent with proving the original estimate for λ replaced by $\lambda + 1$ and $g_\lambda(0) = \delta$, $\partial_t g_\lambda(0) = 0$ as desired.

Remark 2: *It suffices to prove the Theorem under the additional smallness assumption:*
There exists a sufficiently large constant C independent of λ such that

$$C\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)} \le \frac{1}{2} \tag{2.57}$$

We can indeed divide the interval $[0, \lambda]$ into smaller intervals where the auxiliary assumption is verified. The total minimum number of such subintervals is proportional to the product of the constants C_1, in 2.51, and C, in the auxilliary assumption. The estimate 2.55 follows then easily by adding the corresponding estimates on each subinterval and the standard energy estimates, see Lemma 2.3 below. In what follows we may therefore assume that the metric g_λ verifies the additional assumption 2.57.

Remark 3: *We first prove the estimate 2.55 for $P\partial_t\phi$.*

Clearly 2.55 holds true for sufficiently small $t > 0$. Let M be a sufficiently large constant and let $J \subset [0, \lambda]$ such that for any $t \in J$ the estimate 2.55, for $\partial_t\phi$, holds true in the domain $\mathcal{D}_t$, with a fixed constant M. Let $t_* = \max_{t\in J} t$. We may assume that $t_* < \lambda$.

Let $i_0, i_1 \in L^2(\mathbf{R}^n)$, $I = (i_0, i_1)$ and $\Phi(t, s; I)$ be the vector $(\phi, \partial_t\phi)$ where $\phi(t, s; I)$ denotes the solution at time t of the homogeneous equation $\Box_{g_\lambda}\phi = 0$ subject to the initial data at time s, $\phi(s, s; I) = i_0, \partial_t\phi(s, s; I) = i_1$. By a standard uniqueness argument, which depends only on the assumptions 2.50 and $\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)} \le C_1$, we can easily prove the following:

$$\Phi\Big(t, s; \Phi(s, t_0; I_0)\Big) = \Phi(t, t_0; I_0) \tag{2.58}$$

Denote by $\mathcal{H}$ be the set of vector functions $I = (i_0, i_1)$ with $i_0, i_1 \in L^2(\mathbf{R}^n)$. The scalar product in $\mathcal{H}$ is defined by

$$< I, J >= \int_{\Sigma_0} \Big(i_1 \cdot j_1 + \delta^{ab}\partial_a i_0 \cdot \partial_b j_0\Big) dx$$

with Σ_0 the initial hypersurface $t = 0$. Let $X = L^qL^\infty(\mathcal{D}_{t_*})$ and its dual $X' = L^{q'}L^1(\mathcal{D}_{t_*})$. Let $\mathcal{T}$ be the operator from $\mathcal{H}$ to X defined by:

$$\mathcal{T}(I) = -P\partial_t\phi(t, 0; I) \tag{2.59}$$

The adjoint $\mathcal{T}^*$ is defined from X' to $\mathcal{H}$. To prove Proposition 2.3 it suffices to prove that $\mathcal{T}\cdot\mathcal{T}^*$ is a bounded operator from X' to X. In view of the definition of t_* we have $\|\mathcal{T}\|_{\mathcal{H}\to X} = M$ where $\|\mathcal{T}\|_{\mathcal{H}\to X}$ denotes the operator norm of $\mathcal{T}$. Thus,

$$\|\mathcal{T}\cdot\mathcal{T}^*\|_{X'\to X} = M^2.$$

To calculate $\mathcal{T}^*$ we write,

$$\begin{aligned} < T^* f, I > &:= < f, T(I) >= -\int\int_{\mathcal{D}_{t_*}} \partial_t\phi P f dt dx \\ &= -\int\int_{\mathcal{D}_{t_*}} \partial_t\phi\, \bar{\Box}_{g_\lambda}\psi \end{aligned}$$

where ψ is the unique solution to the equation

$$\begin{aligned} \bar{\Box}_{g_\lambda}\psi &= Pf \\ \psi(t_*) &= \partial_t\psi(t_*) = 0 \end{aligned} \tag{2.60}$$

with

$$\bar{\Box}_{g_\lambda} = -\partial_t^2 + \partial_i(g_\lambda^{ij}\partial_j)$$

Observe that,

$$\Box_{g_\lambda} = \bar{\Box}_{g_\lambda} - \frac{1}{\sqrt{|g_\lambda|}}\Big(\partial_t(\sqrt{|g_\lambda|})\partial_t - \partial_i(\sqrt{|g_\lambda|})\partial_i\Big).$$

Consequently, integrating by parts,

$$\begin{aligned} < \mathcal{T}^* f, I > &= -\int\int_{\mathcal{D}_{t_*}} \bar{\Box}_{g_\lambda}\partial_t\phi\cdot\psi dt dx \\ &+ \int_{\Sigma_0}\Big(\partial_t\phi(0)\partial_t\psi(0) - \partial_t^2\phi(0)\psi(0)\Big)dx \\ &= -\int\int_{\mathcal{D}_{t_*}} \bar{\Box}_{g_\lambda}\partial_t\phi\cdot\psi dt dx \\ &+ \int_{\Sigma_0}\Big(\partial_t\phi(0)\partial_t\psi(0) + \delta^{ij}\partial_i\phi(0)\partial_j\psi(0)\Big)dx \\ &= < I, \Psi(0) + R(f) >_{\mathcal{H}} \end{aligned}$$

with $\Psi(0) = (\psi(0), \partial_t\psi(0))$, $\phi = \phi(t, 0; I)$ and $R(f)$ the linear operator defined from X' to $\mathcal{H}$ by the formula,

$$< I, R(f) >= -\int\int_{\mathcal{D}_{t^*}} \bar{\Box}_{g_\lambda}\partial_t\phi\cdot\psi dt dx.$$

Therefore,

$$\mathcal{T}\mathcal{T}^* f = T\Psi(0) + \mathcal{T}R(f) \tag{2.61}$$

Observe that $\Box_{g_\lambda}\psi = Pf + e$ with $e = \frac{1}{\sqrt{|g_\lambda|}}\Big(\partial_t(\sqrt{|g_\lambda|})\partial_t\psi - \partial_i(\sqrt{|g_\lambda|})\partial_i\psi\Big)$. Thus we can write $\psi = \psi_1 + \psi_2$ with,

$$\begin{aligned} \Box_{g_\lambda}\psi_1 &= Pf \\ \Box_{g_\lambda}\psi_2 &= e \end{aligned}$$

with both ψ_1, ψ_2 verifying the zero initial conditions in 2.60. Now $T\Psi(0) = T\Psi_1(0) + T\Psi_2(0)$ and $T\Psi_1(0) = -P\partial_t\phi(t, 0; \Psi_1(0))$. According to the Duhamel Principle we have, $\Psi_1(t) = \int_{t_*}^t \Phi(t, s; F(s))ds$ with $F(s) = (0, Pf(s))$ and therefore,

$$\Psi_1(0) = -\int_0^{t_*} \Phi(0, s; F(s))ds$$

and,

$$\begin{aligned}\mathcal{T}\Psi_1(0) &= P\partial_t\phi\Big(t, 0; \int_0^{t_*} \Phi(0, s; F(s))ds\Big) \\ &= P\int_0^{t_*} \partial_t\phi(t, s; F(s))ds.\end{aligned}$$

We are now in a position to apply the dispersive inequality of Theorem 2.2. Indeed, since the space Fourier transform of $F(s) = (0, Pf(s))$ is supported in the region $|\xi| \le 2$,

$$\|\partial_t\phi(t, s; F(s))\|_{L^\infty} \le C(1 + |t - s|)^{-1+\epsilon}\|Pf(s)\|_{L^1}.$$

Therefore, by the Hardy-Littlewood-Sobolev inequality,

$$\|\mathcal{T}\Psi_1(0)\|_{L_t^q L_x^\infty(\mathcal{D}_{t_*})} \le C\|f\|_{L_t^{q'} L_x^1(\mathcal{D}_{t_*})} \tag{2.62}$$

with C a constant, independent of t_* and λ. It depends in fact only on the H-L-S constant, $\epsilon > 0$ and the constant C in the dispersive inequality of Theorem 2.2.

To estimate $\mathcal{T}\Psi_2(0)$ we apply the Strichartz inequality with bound M,

$$\|\mathcal{T}\Psi_2(0)\|_{L^q L^\infty(\mathcal{D}_{t_*})} \le M\|\Psi_2(0)\|_{\mathcal{H}}$$

where,

$$\|\Psi_2(0)\|_{\mathcal{H}} = \sup_{\|I\|_{\mathcal{H}} \le 1} | < I, \Psi_2(0) >_{\mathcal{H}} | \le \|\partial\psi_2(0)\|_{L^2}.$$

To estimate this we shall make use of the following standard energy estimate to which we have alluded before:

Lemma 2.3 *Any solution of the inhomogeneous problem $\Box_{g_\lambda}\psi = H$, subject to the initial conditions $\psi(t_0) = i_0$, $\partial_t\psi(t_0) = i_1$ verifies the following estimate, for all $t, t_0 \in [0, \lambda]$,*

$$\|\partial\psi(t)\|_{L^2} \le \Big(\|I\|_{\mathcal{H}_{t_0,\lambda}} + \|H\|_{L^1L^2(\mathcal{D}_\lambda)}\Big) \exp C\|\partial_t g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)} \tag{2.63}$$

with C a constant independent of λ. Here $\mathcal{H}_{t_0,\lambda}$ is defined by the scalar product $< I, J >_{\mathcal{H}_{t_0,\lambda}} = \int_{\Sigma_{t_0}} \Big(i_1(x)j_1(x) + g_\lambda(t_0)^{ab}\partial_a i_0(x)\partial_b j_0(x)\Big)dx$.

Applying the Lemma to $\Box_{g_\lambda}\psi_2 = e$, $\psi_2(t_*) = \partial_t\psi_2(t_*) = 0$ and taking into account the auxiliary assumption $C\|\partial_t g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)} \le \frac{1}{2}$ we deduce,

$$\begin{aligned}\|\partial\psi_2(0)\|_{L^2} &\le C\|e\|_{L^1L^2(\mathcal{D}_\lambda)} \\ &\le C\|\partial g\|_{L^1L^\infty(\mathcal{D}_\lambda)}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}\end{aligned}$$

Therefore,

$$\|\mathcal{T}\Psi_2(0)\|_{L^qL^\infty(\mathcal{D}_{t_*})} \le CM\|\partial g\|_{L^1L^\infty(\mathcal{D}_\lambda)}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})} \tag{2.64}$$

We shall now estimate the other error term $\mathcal{T}Rf$. Since the operator norm of $\mathcal{T}$ is bounded by M,

$$\|\mathcal{T}R(f)\|_{L^qL^\infty(\mathcal{D}_{t_*})} \le M\|R(f)\|_{\mathcal{H}}.$$

On the other hand,

$$\begin{aligned}\|R(f)\|_{\mathcal{H}} &= \sup_{\|I\|_{\mathcal{H}}\le 1} < I, R(f) >_{\mathcal{H}} \\ &= \sup_{\|I\|_{\mathcal{H}}\le 1} \int\int_{\mathcal{D}_{t_*}} \bar{\Box}_{g_\lambda}\partial_t\phi\cdot\psi.\end{aligned}$$

Clearly,

$$\begin{aligned}\bar{\Box}_{g_\lambda}\partial_t\phi &= \partial_t\bar{\Box}_{g_\lambda}\phi - \partial_i(\partial_t g^{ij}\partial_j)\phi \\ &= \partial_t\Big[\Box_{g_\lambda}\phi - \frac{1}{\sqrt{|g_\lambda|}}\Big(\partial_t(\sqrt{|g_\lambda|})\partial_t\phi - \partial_i(\sqrt{|g_\lambda|})\partial_i\phi\Big)\Big] - \partial_i(\partial_t g^{ij}\partial_j)\phi \\ &= \partial_t\Big[\frac{1}{\sqrt{|g_\lambda|}}\Big(\partial_t(\sqrt{|g_\lambda|})\partial_t\phi - \partial_i(\sqrt{|g_\lambda|})\partial_i\phi\Big)\Big] - \partial_i(\partial_t g^{ij}\partial_j)\phi\end{aligned}$$

Therefore, integrating by parts, recalling that g is flat at $t=0$, and estimating in a straightforward manner we derive,

$$\|R(f)\|_{\mathcal{H}} \le C\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_{t_*})}\|\partial\phi\|_{L^\infty L^2(\mathcal{D}_{t_*})}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}.$$

To estimate $\|\partial\phi\|_{L^\infty L^2(\mathcal{D}_{t_*})}$ we rely on Lemma 2.3.

Applying it and using $\|I\|_{\mathcal{H}}\le 1$, as well as the auxiliary assumption $C\|\partial_t g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)} \le \frac{1}{2}$ we infer that,

$$\|\partial\phi\|_{L^\infty L^2(\mathcal{D}_{t_*})} \le 2.$$

Therefore,

$$\|\mathcal{T}R(f)\|_{L^qL^\infty(\mathcal{D}_{t_*})} \le MC\|\partial g_\lambda\|_{L^\infty(\mathcal{D}_{t_*})}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}. \tag{2.65}$$

To estimate $\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}$ we rely on the following:

Lemma 2.4 *Under the assumption $C\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)} \le \frac{1}{2}$, for sufficiently large C independent of λ, the solution ψ of the equation $\bar{\Box}_{g_\lambda}\psi = Pf$, $\psi(t_*) = \partial_t\psi(t_*) = 0$ verifies the estimate,*

$$\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t^*})} \le 2M\|f\|_{L^{q'}L^1(\mathcal{D}_{t^*})} \tag{2.66}$$

Posponing the proof for a moment we gather together 2.62,2.64,2.65 and 2.66 to infer that,

$$\begin{aligned}\|\mathcal{T}\mathcal{T}^*f\|_X &= \|\mathcal{T}(\Psi_1(0)+\Psi_2(0)+R(f))\|_{L^qL^\infty(\mathcal{D}_{t_*})} \\ &\le C(1+M^2\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_{t^*})})\|f\|_{L^{q'}L^1(\mathcal{D}_{t_*})}\end{aligned}$$

Therefore, in view of 2.61,

$$M^2 = \|\mathcal{T}\mathcal{T}^*\|_{X'\to X} \le C(1 + M^2\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_{t^*})}).$$

Thus, since $C\|\partial_t g_\lambda\|_{L^1L^\infty(\mathcal{D}_{t^*})} \le \frac{1}{2}$, we infer that M is bounded independently of both λ and t_*, as desired. This proves the desired Strichartz inequality for $P\partial_t\phi$ in the entire region $\mathcal{D}_\lambda$.

It only remains to prove the Lemmas 2.3 and 2.4. The first Lemma is the standard energy inequality. To prove the second we proceed as follows,

Let t be fixed in the interval $[0, t_*]$. We rewrite the equation $\Box_{g_\lambda}\phi = 0$ in the form,

$$\bar{\Box}_{g_\lambda}\phi = h = \frac{1}{\sqrt{|g_\lambda|}}\Big(\partial_t(\sqrt{|g_\lambda|})\partial_t\phi - \partial_i(\sqrt{|g_\lambda|})\partial_i\phi\Big) \tag{2.67}$$

with initial data $\phi(t) = i_0, \partial_t\phi(t) = i_1$, and $(i_0, i_1) = I \in \mathcal{H}_{t,\lambda}$, $\|I\|_{\mathcal{H}_{t,\lambda}} \le 1$. We also recall that, see 2.60,

$$\bar{\Box}_{g_\lambda}\psi = Pf \tag{2.68}$$

with initial data $\psi_1(t_*) = \partial_t\psi_1(t_*) = 0$. Multiplying 2.67 by $\partial_t\psi$ and 2.68 by $\partial_t\phi$ and integrating in the region $[t, t_*] \times \mathbf{R}^n$ we derive the identity,

$$\begin{aligned}\int_{\Sigma_t}\Big(\partial_t\phi\partial_t\psi + g_\lambda^{ij}\partial_i\phi\partial_j\psi\Big)dx &= \int_t^{t_*}\int_{\Sigma_s}\partial_t\phi\cdot Pf\,dsdx &(2.69)\\ &+ \int_t^{t_*}\int_{\Sigma_s}\partial_t\psi\cdot h\,dsdx &(2.70)\\ &+ \int_t^{t_*}\int_{\Sigma_s}\partial_t(g_\lambda^{ij})\partial_i\phi\partial_j\psi\,dsdx\end{aligned}$$

Therefore,

$$\begin{aligned}\|\partial\psi(t)\|_{L^2} &\le \|P\partial_t\phi\|_{L^qL^\infty(\mathcal{D}_{t_*})}\|f\|_{L^{q'}L^1(\mathcal{D}_{t_*})}\\ &+ C\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_{t_*})}\|\partial\phi\|_{L^\infty L^2(\mathcal{D}_{t_*})}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}\end{aligned}$$

We recall that according to our assumption $\|P\partial_t\phi\|_{L^qL^\infty(\mathcal{D}_{t_*})} \le M\|I\|_{\mathcal{H}_{t,\lambda}} \le M$. Also according to Lemma 2.3, $\|\partial_t\phi\|_{L^\infty L^2(\mathcal{D}_{t_*})} \le 2\|I\|_{\mathcal{H}_{t,\lambda}} \le 2$. Therefore,

$$\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})} \le M\|f\|_{L^{q'}L^1(\mathcal{D}_{t_*})} + C\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}$$

and therefore, since $C\|\partial g_\lambda\|_{L^1L^\infty(\mathcal{D}_\lambda)} \le \frac{1}{2}$, we conclude that,

$$\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_\lambda)} \le 2M\|f\|_{L^{q'}L^1(\mathcal{D}_{t_*})}$$

as desired.

To prove the Strichartz estimate for the spatial derivatives we rely on the proof, given above, for $P\partial_t\phi$. We thus assume that the estimate 2.4 holds true for $P\partial_t\phi$ with an M independent of λ.

To estimate $\|P\partial_a\phi\|_{L^qL^\infty(\mathcal{D}_{t_*})}$ it suffices to estimate the integral, $\mathcal{I} = \int\int_{\mathcal{D}_{t_*}} P\partial_a\phi\cdot f dtdx$ for functions f with $\|f\|_{L^{q'}L^\infty(\mathcal{D}_{t_*})}\le 1$. Let ψ verify the equation $\bar\Box_{g_\lambda}\psi = Pf$ with $\psi(t_*) = \partial_t\psi(t_*) = 0$. Therefore,

$$\begin{aligned}\mathcal{I} &= \int\int_{\mathcal{D}_{t_*}} \partial_a\phi\,\bar\Box_{g_\lambda}\psi dtdx\\ &= \int\int_{\mathcal{D}_{t_*}} \bar\Box_{g_\lambda}\partial_a\phi\cdot\psi dtdx + \int_{\mathbf{R}^n}\Big(\partial_a\phi(0)\partial_t\psi(0)+\partial_t\phi(0)\partial_a\psi(0)\Big)\end{aligned}$$

Proceeding as before we show that,

$$|\int\int_{\mathcal{D}_{t_*}} \bar\Box_{g_\lambda}\partial_a\phi\cdot\psi dtdx| \le C\|\partial g\|_{L^1L^\infty(\mathcal{D}_{t_*})}\|\partial\phi\|_{L^\infty L^2(\mathcal{D}_{t_*})}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}$$

Also,

$$\int_{\mathbf{R}^n}\Big(\partial_a\phi(0)\partial_t\psi(0)+\partial_t\phi(0)\partial_a\psi(0)\Big)\le\|\partial\phi(0)\|_{L^2}\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}.$$

According to the Lemma 2.3 $\|\partial\phi\|_{L^\infty L^2(\mathcal{D}_{t_*})}\le C\|\partial\phi(0)\|_{L^2}$. According to the Lemma 2.4 we have,

$$\|\partial\psi\|_{L^\infty L^2(\mathcal{D}_{t_*})}\le 2M\|f\|_{L^qL^\infty(\mathcal{D}_{t_*})}.$$

Observe that the M in Lemma 2.4 depends only on the Strichartz estimate 2.4 for $P\partial_t\phi$ which we have assumed to be independent of λ. Therefore,

$$|\mathcal{I}|\le CM\|\partial\phi(0)\|_{L^2}(1+\|\partial g\|_{L^1L^\infty(\mathcal{D}_{t_*})})\|f\|_{L^qL^\infty(\mathcal{D}_{t_*})}\le CM\|\partial\phi(0)\|_{L^2}$$

which implies,

$$\|P\partial_a\phi\|_{L^qL^\infty(\mathcal{D}_{t_*})}\le CM\|\partial\phi(0)\|_{L^2}$$

as desired.

V.) Final Reduction to Theorem 2.4.

In the previous step we have reduced the proof of Theorem 2.1 to that of Theorem 2.2. We can perform one more reduction based on the phase space localization described in the proof of Theorem 1.1. Therefore Theorem 2.2, as well as the Theorems A,B,C, is a consequence of the following:

Theorem 2.4 *Assume that the metric g_λ verifies the same assumptions as those of Theorem 2.1. Consider solutions of the homogeneous wave equation $\Box_{g_\lambda}\phi = 0$ in the domain $\mathcal{D}_\lambda$, with initial data $\phi(t_0), \partial_t\phi(t_0)$, $t_0\in I_\lambda$, supported in a ball of radius 2. Then, for a sufficiently large positive integer N,*

$$\|\partial\phi(t)\|_{L^\infty}\le C(1+|t-t_0|)^{-1+\epsilon}\|\partial\phi(t_0)\|_{H^N(\mathbf{R}^n)} \tag{2.71}$$

The proof of Theorem 2.4 will occupy the remaning part of the whole paper.

Remark: *The result of Theorem 2.4 remains valid if the assumptions 2.45, 2.52 of Theorem 2.1 are not verified in the finite time interval $[0,1]$. In that interval it suffices to have the weaker assumption,*

$$\|\partial^{1+i}g_\lambda\|_{L^1_tL^\infty_x([0,1]\times\mathbf{R}^n)}\le C_i \quad \textit{for all}\ \ 0\le i\le N.$$

Indeed, since there is no need to prove a decay estimate in $[0,1]$, it suffices to rely on energy estimates, see Lemma 2.3 applied to sufficiently many derivatives of ϕ, and the standard $H^s - L^\infty$ Sobolev inequalities.

2.2 Decay Estimates on a Curved Background

In this section we provide the main ingredients in the proof of Theorem 2.4. For the convenience of the reader we restate below the result in a somewhat different form, see Theorem 2.5 below.

We assume given a family of Lorentz metrics[14] g^Λ, $\Lambda \geq 1$, of the form $-dt^2 + g^\Lambda_{ij} dx^i dx^j$, defined in a slab region $\mathcal{D}_I = I \times \mathbf{R}^n \subset \mathbf{R}^{n+1}$, $I = I_\Lambda = [0, t_*]$, of length $|I| \leq \Lambda^{2-\epsilon}$ for some small fixed constant $\epsilon > 0$, and verifying the following properties:

$$
\begin{aligned}
C_0^{-1}|\xi|^2 \leq g^\Lambda_{ij}\xi^i\xi^j &\leq C_0|\xi|^2 && (2.72)\\
\|\partial g^\Lambda\|_{L^1 L^\infty} &\leq C_1 && (2.73)\\
\|\partial^{1+i} g^\Lambda\|_{L^1 L^\infty} &\leq C_{1+i}\Lambda^{-2i} \quad \text{for all } 0 \leq i \leq k. && (2.74)\\
\|\partial^{1+k+j} g^\Lambda\|_{L^1 L^\infty} &\leq C_{1+k+j}\Lambda^{-2k-j} \quad \text{for all } 0 \leq j. && (2.75)
\end{aligned}
$$

We may also assume without loss of generality[15] that $g(t) - \delta$ vanishes identically for $t \leq 0$.

Throughout this section, whenever there is no danger of confusion, we denote by g a fixed metric g^Λ. We denote by D the covariant derivative defined by g. The wave operator $\Box_g = g^{\alpha\beta} D_\alpha D_\beta$ takes the usual form relative to the coordinates $x^0 = t, x = (x^1, \ldots x^n)$, $\Box_g = -\frac{1}{|g|^{\frac{1}{2}}}\partial_t(|g|^{\frac{1}{2}}\partial_t\phi) + \triangle_g$ with $\triangle_g = \frac{1}{|g|^{\frac{1}{2}}}\partial_i(g^{ij}|g|^{\frac{1}{2}}\partial_j)$. Given a vectorfield $X = X^\alpha\partial_\alpha$ we define its deformation tensor ${}^{(X)}\pi$ to be the Lie derivative $\mathcal{L}_X g$. Recall that ${}^{(X)}\pi_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha$, $\alpha\beta = 0, 1 \ldots n$.

Theorem 2.5 *Assume that the metric $g = g^\Lambda$ verifies the assumptions 2.72–2.75, with $k = 2$. Let ϕ be a solution of the wave equation*

$$
\begin{aligned}
\Box_g \phi &= 0 && (2.76)\\
\phi(t_0) = \varphi_0 \quad &, \quad \partial_t\phi(t_0) = \varphi_1, && (2.77)
\end{aligned}
$$

in the domain $\mathcal{D}_I = I \times \mathbf{R}^n$, $I = [t_0, t_*]$ with $t_0 = 2$. Assume that the initial conditions (φ_0, φ_1) at $t = t_0 = 2$ are supported in a ball $|x| \leq 2$. Then, for a fixed positive integer $s_0 \geq 3 + \frac{n}{2} - \frac{n}{n-1}$,

$$
\|\partial\phi(t)\|_{L^\infty} \leq C(1 + |t|)^{-1+\epsilon}\Big(\|\varphi_0\|_{H^{s_0+1}(\mathbf{R}^n)} + \|\varphi_1\|_{H^{s_0}(\mathbf{R}^n)}\Big) \tag{2.78}
$$

Remark 1 *Throughout most of the paper we need $k = 1$ in the above set of assumptions. Only in a few places(see 3.130, 3.131, 3.133,3.134 and 3.150) in our construction we need to rely on $k = 2$ for the assumptions 2.72–2.75. It is important to remark however that our proof does not use the full power of the above assumptions. In fact the only assumptions we need for the derivatives of the metric can be reformulated in terms of the second fundamental form, $k_{ij} = -\frac{1}{2}\partial_t g_{ij}$, and the Riemann curvature tensor of the space-time metric g_Λ.*

[14] Here Λ corresponds to λ^2 in 2.1.

[15] See Remark 1 following Theorem 2.3 and the Remark following Theorem 2.4.

Moreover we shall show, see the discussion in the last section of the paper, that under additional assumptions on the Ricci curvature of the metric we may fall back on the $k = 1$ case. This additional assumptions are automatically satisfied in the case of quasilinear equations of the form 2.31. Therefore our method recovers the optimal result, $\sigma = \frac{1}{6}$, obtained by D. Tataru [T2] in connection to Theorem C.

Remark 2 Without loss of generality we may assume that the constants C_1, C_2 are sufficiently small. The general case follows by dividing I into a finite number of subintervals for which the corresponding constants C_1, C_2 are sufficiently small.

Remark 3 *For technical reason we need to make a stronger assumption concerning $\|\partial g\|_{L^1L^\infty}$,*

$$\|\partial g^\Lambda\|_{L^1L^\infty} \leq \Lambda^{-\epsilon} \tag{2.79}$$

The effect of this stronger assumption is to lose an additional $\epsilon > 0$ derivatives in the statement of Theorem C.

We plan to prove the Theorem 2.5 following the same strategy as in the proof of Proposition 1.6. The main ingredient in the proof is the construction of a vectorfield K_0 which is the analogue of the Morawetz vectorfield used in the derivation of the generalized energy estimates described in Proposition 1.4. We start with a sequence of Lemmas and Propositions which we shall need in the proof of Theorem 2.5.

The first Lemma is simply a reformulation of Lemma 2.3.

Lemma 2.5 *Consider the wave equation $\square_g \phi = F$ and let $k_{ij} = -\frac{1}{2}\partial_t g_{ij}$ the second fundamental form[16] of the time slices Σ_t. We have ,*

$$\|\partial\phi(t)\|_{L^2} \leq \Big(\|\partial\phi(t_0)\|_{L^2} + \int_{t_0}^t \|F(\tau)\|_{L^2} d\tau\Big) \exp C \int_2^t \|k(t')\|_{L^\infty}.$$

with the constant C depending only on n and the constant C_0 in 2.72.

The proof of the Lemma is standard and will be omitted. The Lemma can in fact be viewed as a special case of the following more general energy type estimates associated to arbitrary time-like vectorfields X.

Let X be an arbitrary timelike vectorfield with deformation tensor ${}^{(X)}\pi = \mathcal{L}_X g$, We write $\pi = {}^{(X)}\pi$ in the form $\tilde{\pi} = \pi - \Omega g$ with Ω a given scalar function. Let $Q_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}(g^{\mu\nu}\partial_\mu\phi\partial_\nu)$ be the energy momentum tensor associate to $\square_g\phi = F$. If ϕ is a solution to the equation we have

$$D^\beta Q_{\alpha\beta} = F\partial_\alpha\phi.$$

Therefore, setting the $X-$ momentum 1-form $P_\alpha = Q_{\alpha\beta}X^\beta$, we have

$$\begin{aligned} D^\alpha P_\alpha &= Q^{\alpha\beta}\pi_{\alpha\beta} + FX(\phi) \\ &= \frac{1}{2}(Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} + \Omega \mathrm{tr} Q) + FX(\phi) \\ &= \frac{1}{2}(Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} + \Omega\frac{1-n}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) + FX(\phi) \end{aligned}$$

[16]The definition is $k_{ij} = - < D_{e_i}T, e_j >$ relative to an orthonormal frame e_i on Σ_t.

Now,

$$
\begin{aligned}
\Omega g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi &= D^\mu(\Omega\phi\partial_\mu\phi) - \partial^\mu(\Omega)\phi\partial_\mu\phi - \Omega\phi\,\Box_g\phi \\
&= D^\mu(\Omega\phi\partial_\mu\phi - \frac{1}{2}\phi^2\partial_\mu\Omega) + \frac{1}{2}\phi^2\,\Box_g\Omega - \Omega\phi F
\end{aligned}
$$

Therefore,

$$
D^\alpha P_\alpha = \frac{1}{2}Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} - \frac{n-1}{4}\Big(D^\mu(\Omega\phi\partial_\mu\phi - \frac{1}{2}\phi^2\partial_\mu\Omega) + \frac{1}{2}\phi^2\,\Box_g\Omega\Big) + (X\phi + \frac{n-1}{4}\Omega\phi)F
$$

or, setting

$$
\bar{P}_\alpha = P_\alpha + \frac{n-1}{4}\Omega\phi\partial_\alpha\phi - \frac{n-1}{8}\phi^2\partial_\alpha\Omega \tag{2.80}
$$

we derive

$$
D^\alpha\bar{P}_\alpha = \frac{1}{2}Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta} - \frac{n-1}{8}\phi^2\,\Box_g\Omega + (X\phi + \frac{n-1}{4}\Omega\phi)F \tag{2.81}
$$

Now, integrating on the time slab $[t_0, t]\times\mathbf{R}^n$, and observing that ∂_t is the future unit normal to the hypersurfaces Σ_t we derive,

Proposition 2.2 *Let ϕ verify $\Box\phi = F$ and X an arbitrary vectorfield with deformation tensor ${}^{(X)}\pi = \pi$. Let Ω be an arbitrary scalar function and $\tilde{\pi} = \pi - \Omega g$. Define, for another vectorfield Y,*

$$
\bar{Q}(X,Y) = Q(X,Y) + \frac{n-1}{4}\Omega\phi Y\phi - \frac{n-1}{8}\phi^2 Y(\Omega)
$$

We have,

$$
\begin{aligned}
\int_{\Sigma_t}\bar{Q}(X,\partial_t)dv_g &= \int_{\Sigma_{t_0}}\bar{Q}(X,\partial_t)dv_g + \frac{1}{2}\int_{[t_0,t]\times\mathbf{R}^n} Q^{\alpha\beta}\tilde{\pi}_{\alpha\beta}dtdv_g \\
&- \frac{n-1}{8}\int_{[t_0,t]\times\mathbf{R}^n}\phi^2\,\Box_g\Omega dtdv_g + \int_{[t_0,t]\times\mathbf{R}^n}(X\phi + \frac{n-1}{4}\Omega\phi)Fdtdv_g
\end{aligned} \tag{2.82}
$$

Observe that 2.82 implies the energy identity 1.16 in the particular case of the Minkowski space, $X = K_0$ and $F = 0$. In that case $\Omega = \frac{1}{n+1}\mathrm{tr}\pi = 4t$, $\mathrm{tr}\pi = g^{\alpha\beta}\pi_{\alpha\beta}$ and $\tilde{\pi} = \hat{\pi} = 0$. Observe also that the Lemma 2.5 is a consequence of the above proposition in the special case when $X = \partial_t$. In that case ${}^{(X)}\pi_{00} = {}^{(X)}\pi_{0i} = 0$ and ${}^{(X)}\pi_{ij} = -2k_{ij}$.

In the next Proposition we record a commutation formula between $\Box_g$ and vectorfields X expressed relative to the deformation tensor ${}^{(X)}\pi$.

Proposition 2.3 *Consider an arbitrary vectorfield X with deformation tensor π, trace $tr\pi = g^{\alpha\beta}\pi_{\alpha\beta}$ and traceless part $\hat{\pi}_{\alpha\beta} = \pi_{\alpha\beta} - \frac{1}{n+1}tr\pi g_{\alpha\beta}$. Denote by D the covariant derivative corresponding to the metric g. We have the following commutation formula with $\Box_g$, the wave operator in 2.77.*

$$
\begin{aligned}
\Box_g(X\phi) - X\,\Box_g\phi &= \hat{\pi}^{\alpha\beta}D_\alpha\mathcal{D}_\beta\phi + \frac{1}{n+1}tr\pi\,\Box_g\phi \\
&+ D^\alpha\hat{\pi}_{\alpha\lambda}D^\lambda\phi + (\frac{1}{n+1} - \frac{1}{2})D^\lambda tr\pi D_\lambda\phi
\end{aligned} \tag{2.83}
$$

The proposition is an immediate consequence of the following more general formula, see Lemma 7.1.3 in [C-K2].

Lemma 2.6 *Let $X = X^\alpha \partial_\alpha$ be a vectorfield with deformation tensor π, $V = V_\alpha dx^\alpha$ an arbitrary 1-form on a spacetime manifold with metric g and corresponding connection D. We have,*

$$D_\sigma(\mathcal{L}_X V_\alpha) = \mathcal{L}_X(D_\sigma V_\alpha) + {}^{(X)}\Gamma_{\alpha\sigma\lambda} V^\lambda$$

where,

$${}^{(X)}\Gamma_{\alpha\sigma\lambda} = \frac{1}{2}\Big(D_\alpha \pi_{\sigma\lambda} + D_\sigma \pi_{\alpha\lambda} - D_\lambda \pi_{\alpha\sigma}\Big)$$

We plan to construct vectorfields X which are the analogue of the Killing vectorfield $T_0 = \partial_t$, and conformal Killing vectorfields $K_0 = (t^2 + r^2)\partial_t + \sum_i 2tx^i \partial_i$ in Minkowski space.

We shall do this with the help of a special solution u of the Eikonal equation

$$(\partial_t u)^2 - g^{ij}(t,x)\partial_i u \partial_j u = 0 \tag{2.84}$$

whose level hypersurfaces are forward light cones, C_u, with vertices on the time axis G given by the points of coordinates $(t, 0)$. The optical function u can be viewed as the analogue of the function $t - |x|$ in Minkowski space. It corresponds to the interior optical function introduced in section 9.2 of [C-K2].

Denote by $S_{t,u}$ the (n-1)-surface of intersection between the Σ_t hypersurface and the null cone C_u. Let ∂_t be the unit normal to Σ_t and N the exterior unit normal to $S_{t,u}$ tangent to Σ_t. Let L to be the null generator vectorfield of C_u

$$L = -g^{\mu\nu}\partial_\nu u \partial_\mu = \partial_t u \partial_t - (g^{ij}\partial_i u)\,\partial_j \tag{2.85}$$

Clearly, $L(u) = 0$, $< L, L >_g = 0$ and $D_L L = 0$. Also, $L = a^{-1}(\partial_t + N)$ with $a^{-1} = - < L, \partial_t >_g$. L is tangent to the null geodesics generating C_u. Denoting by s the affine parameter of L, i.e. $L(s) = 1$, $s = 0$ on G we can write

$$D_L = \frac{d}{ds}.$$

Define also the incoming null vector $\underline{L} = a(\partial_t - N)$. We have $< L, \underline{L} >_g = -2$. We record all these formulas below,

$$\begin{aligned} L &= a^{-1}(\partial_t + N) \\ \underline{L} &= a(\partial_t - N) \\ < L, \underline{L} >_g &= -2 \\ -a^{-1} &= < L, \partial_t >_g = -\partial_t(u) \end{aligned} \tag{2.86}$$

At every point of the space-time there passes a unique codimension 2 surface $S_{t,u}$. Define r such that the total area of the surface $S_{t,u}$ be equal to $w_{n-1} r^{n-1}$ with w_{n-1} the total area of the standard S^{n-1} sphere. We associate to any point a null frame formed by the null pair $(L, \underline{L})$ and a frame e_A, $A = 1, 2, \ldots, n-1$ which is orthonormal on $S_{t,u}$. We shall also use the notation $\underline{L} = e_n$ and $L = e_{n+1}$ and e_μ, $\mu = 1, \ldots, n-1, n, n+1$ for the full null frame. The components of the space-time metric relative to the null frame are $g_{nn} = g_{(n+1)(n+1)} = g_{nA} = g_{(n+1)A} = 0$ and $g_{n(n+1)} = -2$, $g_{AB} = \delta_{AB}$. For the inverse metric g^{-1} we have $g^{nn} = g^{(n+1)(n+1)} = g^{nA} = g^{(n+1)A} = 0$, $g^{n(n+1)} = -\frac{1}{2}$.

The tensor of projection to the surfaces $S_{t,u}$ is given by the formula

$$\Pi^\mu_\nu = \delta^\mu_\nu + \frac{1}{2}(L^\mu \underline{L}_\nu + \underline{L}^\mu L_\nu).$$

We denote by $\nabla\!\!\!/$ the induced covariant differentiation on the surfaces $S_{t,u}$. We also denote by D the space-time covariant derivative associated to the metric g and by ∇ that induced on the hypersurfaces Σ_t.

Let e_A, $A = 1, \ldots, n-1$ be an orthonormal frame to the unit sphere in the tangent space of a point on the centerline (G). We extend this frame according to the equation,

$$\frac{d}{ds} e_A = \underline{\eta}_A L \tag{2.87}$$

$$< \partial_t, e_A > = 0 \tag{2.88}$$

with s the affine parameter of L defined above. Now $0 = \frac{d}{ds} < \partial_t, e_A >=< D_L \partial_t, e_A >= -\underline{\eta}_A a^{-1}$. Therefore, $\underline{\eta}_A = a < D_L \partial_t, e_A >$ and since $\partial_t = \frac{1}{2}(aL + a^{-1}\underline{L})$ and $D_{\partial_t}\partial_t = 0$,

$$\underline{\eta}_A = \frac{1}{2} < D_L \underline{L}, e_A >= k_{AN}.$$

In view of the fact that L is geodesic we infer that e_A remain orthogonal to L. In particular the e_A's remain tangent to the surfaces $S_{t,u}$. Moreover

$$\frac{d}{ds} < e_A, e_B >= \underline{\eta}_A < L, e_B > + \underline{\eta}_B < L, e_A >= 0,$$

and therefore $< e_A, e_B >= \delta_{AB}$ everywhere on C_u.

We introduce the "frame coefficients"

$$\begin{array}{llll} < D_A L, e_B > & = \chi_{AB} & < D_A \underline{L}, e_B > & = \underline{\chi}_{AB} \\ < D_L L, e_A > & = 0 & < D_{\underline{L}} \underline{L}, e_A > & = 2\underline{\xi}_A \\ < D_{\underline{L}} L, e_A > & = 2\eta_A & < D_L \underline{L}, e_A > & = 2\underline{\eta}_A \\ < D_L L, \underline{L} > & = 0 & < D_{\underline{L}} \underline{L}, L > & = 4\underline{\omega} \end{array} \tag{2.89}$$

Observe also that $< D_A L, \underline{L} >=< D_{\underline{L}} L, e_A >= 2\eta_A$ and $< D_{\underline{L}} L, \underline{L} >= - < D_{\underline{L}} \underline{L}, L >= -4\underline{\omega}$. It turns out that we can express the bar quantities $\underline{\chi}, \underline{\eta}$ and $\underline{\xi}$ in terms of the second fundamental form k and the basic quantities χ, η, $\underline{\omega}$. Indeed, since $L = a^{-1}(\partial_t + N)$ and $\underline{L} = a(\partial_t - N)$, we have $\underline{L} = -a^2 L + 2a\partial_t$. Therefore setting $k_{AB} = - < D_A \partial_t, e_B >$, $k_{NA} = - < D_N \partial_t, e_A >$, $k_{NN} = - < D_N \partial_t, N >$ we derive

$$\begin{array}{rcl} \underline{\chi}_{AB} & = & -a^2 \chi_{AB} - 2a k_{AB} \\ \underline{\xi}_A & = & -a^2(k_{AN} + \eta_A) \\ \underline{\eta}_A & = & k_{AN} \\ L(a) & = & -k_{NN} \end{array} \tag{2.90}$$

In view of the above definitions we can express the covariant derivatives of the null frame $L, \underline{L}, e_A$ defined above as follows

$$\begin{array}{llll} \mathbf{D}_A \underline{L} & = \underline{\chi}_{AB} e_B + \eta_A \underline{L} & \mathbf{D}_A L & = \chi_{AB} e_B - \eta_A L \\ \mathbf{D}_{\underline{L}} \underline{L} & = 2\underline{\xi}_A e_A - 2\underline{\omega} \underline{L} & \mathbf{D}_{\underline{L}} L & = 2\eta_A e_A + 2\underline{\omega} L \\ \mathbf{D}_L \underline{L} & = 2\underline{\eta}_A e_A & \mathbf{D}_L L & = 0 \end{array} \tag{2.91}$$

Also,

$$\begin{aligned}
\mathbf{D}_B e_A &= \nabla\!\!\!/_B e_A + \frac{1}{2}\chi_{AB}\underline{L} + \frac{1}{2}\underline{\chi}_{AB}L \\
\mathbf{D}_{\underline{L}} e_A &= \nabla\!\!\!/_{\underline{L}} e_A + \eta_A\underline{L} + \underline{\xi}_A L \qquad (2.92)\\
\mathbf{D}_L e_A &= \nabla\!\!\!/_L e_A + \underline{\eta}_A L
\end{aligned}$$

As we have mentioned above the optical function u is a solution of the Eikonal equation 2.84. The boundary conditions for u are specified as follows:

Condition 1: On the timelike geodesic (G) we have $u = t$. Here t is the arclength along (G) as measured from the initial slice $t = 0$.

Condition 2: The level sets of C_u are future null geodesic cones with vertices on (G). We choose an afine parameter s, $L(s) = 1$ on C_u such that $s|_{(G)} = 0$. We also set, on (G), $a|_{(G)} = 1$. Finally assume that $\mathrm{tr}\chi - \frac{n-1}{s}, \hat{\chi}, \zeta$ stay bounded as $s \to 0$, where $\mathrm{tr}\chi$ is the trace and $\hat{\chi}$ is the traceless part of χ relative to the $n-1$ spheres $S_{t,u}$. Using this assumption one can in fact prove the better result, $\mathrm{tr}\chi - \frac{n-1}{s} = 0(s)$, $|\hat{\chi}|, |\zeta| = 0(s)$ as $s \to 0$, see section 9.2 in [C-K2]. Now $\underline{\omega} = a^2 < D_{\partial_t}L, \partial_t >= a^2\partial_t < L, \partial_t >$, in view of the fact that $D_{\partial_t}\partial_t = 0$. Hence $\underline{\omega} = a^2\partial_t(-a^{-1}) = \partial_t a = 0$ on (G).

Using the above Ricci formulas we calculate the deformation tensors of L and $\underline{L}$.

$$\begin{aligned}
{}^{(L)}\pi_{LL} &= 2 < D_L L\,,\, L >= 0\\
{}^{(L)}\pi_{\underline{L}L} &= < D_L L\,,\, \underline{L} > + < D_{\underline{L}} L\,,\, L >= 0\\
{}^{(L)}\pi_{\underline{L}\,\underline{L}} &= 2 < D_{\underline{L}} L\,,\, \underline{L} >= -8\underline{\omega}\\
{}^{(L)}\pi_{LA} &= < D_L L\,,\, e_A > + < D_A L\,,\, L >= 0\\
{}^{(L)}\pi_{\underline{L}A} &= < D_{\underline{L}} L\,,\, e_A > + < D_A L\,,\, \underline{L} >= 4\eta_A\\
{}^{(L)}\pi_{AB} &= 2 < D_{e_A} L\,,\, e_B >= 2\chi_{AB}
\end{aligned}$$

Also,

$$\begin{aligned}
{}^{(\underline{L})}\pi_{LL} &= 2 < D_L\underline{L}\,,\, L >= 0\\
{}^{(\underline{L})}\pi_{\underline{L}L} &= < D_L\underline{L}\,,\, \underline{L} > + < D_{\underline{L}}\underline{L}\,,\, L >= 4\underline{\omega}\\
{}^{(\underline{L})}\pi_{\underline{L}\,\underline{L}} &= 2 < D_{\underline{L}}\underline{L}\,,\, \underline{L} >= 0\\
{}^{(\underline{L})}\pi_{LA} &= < D_L\underline{L}\,,\, e_A > + < D_A\underline{L}\,,\, L >= 2(\underline{\eta}_A - \eta_A)\\
{}^{(\underline{L})}\pi_{\underline{L}A} &= < D_{\underline{L}}\underline{L}\,,\, e_A > + < D_A\underline{L}\,,\, \underline{L} >= 2\underline{\xi}_A\\
{}^{(\underline{L})}\pi_{AB} &= 2 < D_{e_A}\underline{L}\,,\, e_B >= 2\underline{\chi}_{AB}
\end{aligned}$$

With the help of the functions t, u and the null pairs $L, \underline{L}$ defined above, we can now construct the analogue of the vectorfields T_0, K_0 we have used in the flat case.

$$T_0 = \frac{1}{2}\Big(\underline{L} + L\Big) \qquad (2.93)$$

$$K_0 = \frac{1}{2}\Big(u^2\underline{L} + \underline{u}^2 L\Big) \qquad (2.94)$$

where $\underline{u} = 2t - u$.

Clearly the deformation tensor of T_0 can be easily calculated from those of L and $\underline{L}$, ${}^{(T_0)}\pi = \frac{1}{2}({}^{(L)}\pi + {}^{(\underline{L})}\pi)$. Therefore, relative to the null frame, $e_{n+1} = L, e_n = \underline{L}$ and e_A, $A = 1, \dots n-1$.

$$\begin{aligned}
{}^{(T_0)}\pi_{LL} &= 0 \\
{}^{(T_0)}\pi_{L\underline{L}} &= 2\underline{\omega} \\
{}^{(T_0)}\pi_{\underline{L}\underline{L}} &= -4\underline{\omega} \\
{}^{(T_0)}\pi_{LA} &= \underline{\eta}_A - \eta_A \\
{}^{(T_0)}\pi_{\underline{L}A} &= 2\eta_A + \underline{\xi}_A \\
{}^{(T_0)}\pi_{AB} &= \chi_{AB} + \underline{\chi}_{AB}
\end{aligned} \tag{2.95}$$

The deformation tensor ${}^{(K_0)}\pi$ of K can be expressed in the form

$$\begin{aligned}
2\,{}^{(K_0)}\pi_{\alpha\beta} &= \underline{u}^2\,{}^{(L)}\pi_{\alpha\beta} + u^2\,{}^{(\underline{L})}\pi_{a\beta} \\
&+ e_\alpha(u^2) < \underline{L}, e_\beta > + e_\beta(u^2) < \underline{L}, e_\alpha > \\
&+ e_\alpha(\underline{u}^2) < L, e_\beta > + e_\beta(\underline{u}^2) < L, e_\alpha >
\end{aligned} \tag{2.96}$$

Expressing ${}^{(K_0)}\pi$ relative to the null frame $e_{n+1} = L, e_n = \underline{L}$ and $e_1, \dots e_{n-1}$ and taking into account the fact that $L(u) = 0$, $e_A(u) = e_A(\underline{u}) = 0$, $< L, L >=< \underline{L}, \underline{L} >=< L, e_A >=< \underline{L}, e_A >= 0$, and $< L, \underline{L} >= -2$ we derive,

$$\begin{aligned}
2\,{}^{(K_0)}\pi_{LL} &= \underline{u}^2\,{}^{(L)}\pi_{LL} + u^2\,{}^{(\underline{L})}\pi_{LL} \\
2\,{}^{(K_0)}\pi_{\underline{L}L} &= \underline{u}^2\,{}^{(L)}\pi_{\underline{L}L} + u^2\,{}^{(\underline{L})}\pi_{\underline{L}L} - 2(\underline{L}(u) + L(\underline{u})) \\
2\,{}^{(K_0)}\pi_{\underline{L}\underline{L}} &= \underline{u}^2\,{}^{(L)}\pi_{\underline{L}\underline{L}} + u^2\,{}^{(\underline{L})}\pi_{\underline{L}\underline{L}} - 4\underline{L}(\underline{u}) \\
2\,{}^{(K_0)}\pi_{LA} &= \underline{u}^2\,{}^{(L)}\pi_{LA} + u^2\,{}^{(\underline{L})}\pi_{LA} \\
2\,{}^{(K_0)}\pi_{\underline{L}A} &= \underline{u}^2\,{}^{(L)}\pi_{\underline{L}A} + u^2\,{}^{(\underline{L})}\pi_{\underline{L}A} - 2e_A(\underline{u}^2) \\
2\,{}^{(K_0)}\pi_{AB} &= \underline{u}^2\,{}^{(L)}\pi_{AB} + u^2\,{}^{(\underline{L})}\pi_{AB}
\end{aligned}$$

Hence, since $\nabla\!\!\!/(\underline{u}) = 0$,

$$\begin{aligned}
{}^{(K_0)}\pi_{LL} &= 0 \\
{}^{(K_0)}\pi_{\underline{L}L} &= 2u^2\underline{\omega} - (\underline{L}(u^2) + L(\underline{u}^2)) \\
{}^{(K_0)}\pi_{\underline{L}\underline{L}} &= -4\underline{u}^2\underline{\omega} - 2\underline{L}(\underline{u}^2) \\
{}^{(K_0)}\pi_{LA} &= u^2(\underline{\eta}_A - \eta_A) \\
{}^{(K_0)}\pi_{\underline{L}A} &= 2\underline{u}^2\eta_A + u^2\underline{\xi}_A \\
{}^{(K_0)}\pi_{AB} &= u^2\underline{\chi}_{AB} + \underline{u}^2\chi_{AB}
\end{aligned}$$

Also,

$$\mathrm{tr}\,{}^{(K_0)}\pi = -\,{}^{(K_0)}\pi_{L\underline{L}} + \delta^{AB}\,{}^{(K_0)}\pi_{AB} = -2u^2\underline{\omega} + 2((\underline{L}(u^2) + L(\underline{u}^2)) + u^2\mathrm{tr}\underline{\chi} + \underline{u}^2\mathrm{tr}\chi. \tag{2.97}$$

Observe that $\underline{L}(u) = - < L\,, \underline{L} >= 2$ and $L(\underline{u}) = L(2t-u) = 2L(t) = -2 < L\,, \partial_t >= 2a^{-1}$. Therefore

$$\underline{L}(u) + L(\underline{u}) = 2 + 2a^{-1}.$$

We now introduce an approximate traceless tensor ${}^{(K_0)}\tilde{\pi}$ defined by the formula,

$$\begin{aligned} {}^{(K_0)}\tilde{\pi}_{\alpha\beta} &= {}^{(K_0)}\pi_{\alpha\beta} - \Omega g_{\alpha\beta} &(2.98)\\ \Omega &= 4t &(2.99) \end{aligned}$$

Recall that $4t$ is the exact value of $\frac{1}{n+1}\mathrm{tr}\,{}^{(K_0)}\pi$ in the particular case of the Minkowski space. We have to calculate ${}^{(K_0)}\tilde{\pi}_{L\underline{L}}$ and ${}^{(K_0)}\tilde{\pi}_{AB}$.

$$\begin{aligned} {}^{(K_0)}\tilde{\pi}_{L\underline{L}} &= {}^{(K_0)}\pi_{L\underline{L}} + 2\Omega \\ &= 2u^2\underline{\omega} - (\underline{L}(u^2) + L(\underline{u}^2)) + 2\Omega \\ &= 2u^2\underline{\omega} - 4(u + a^{-1}\underline{u}) + 8t \\ &= 2u^2\underline{\omega} - 4a^{-1}(u + \underline{u} - (1-a)u) + 8t \\ &= 2u^2\underline{\omega} + 8t(1-a^{-1}) - 4(1-a^{-1})u \\ &= 2u^2\underline{\omega} + 4(1-a^{-1})(2t-u) \end{aligned}$$

Thus,

$${}^{(K_0)}\tilde{\pi}_{L\underline{L}} = 2u^2\underline{\omega} + 4(1-a^{-1})(2t-u) \tag{2.100}$$

Also,

$${}^{(K_0)}\tilde{\pi}_{AB} = u^2\underline{\chi}_{AB} + \underline{u}^2\chi_{AB} - 4t\delta_{AB}$$

Taking the trace with respect to the $S_{t,u}$ surfaces we find,

$$\delta^{AB}\,{}^{(K_0)}\tilde{\pi}_{AB} = u^2\mathrm{tr}\underline{\chi} + \underline{u}^2\mathrm{tr}\chi - 4(n-1)t$$

Recall that, $\underline{\chi}_{AB} = -a^2\chi_{AB} - 2ak_{AB}$. Setting

$$\mu = \delta^{AB}k_{AB} \tag{2.101}$$

we find,

$$\begin{aligned} \mathrm{tr}\underline{\chi} &= -a^2\mathrm{tr}\chi - 2a\mu \\ &= -\mathrm{tr}\chi + (1-a^2)\mathrm{tr}\chi - 2a\mu \end{aligned}$$

Hence,

$$\begin{aligned} \delta^{AB}\,{}^{(K_0)}\tilde{\pi}_{AB} &= (\underline{u}^2 - u^2)\mathrm{tr}\chi - 4(n-1)t + \Big((1-a^2)\mathrm{tr}\chi - 2a\mu\Big)u^2 \\ &= 2t\Big((\underline{u} - u)\mathrm{tr}\chi - 2(n-1)\Big) + \Big((1-a^2)\mathrm{tr}\chi - 2a\mu\Big)u^2 \end{aligned}$$

We write,

$$\begin{aligned}(\underline{u}-u)\mathrm{tr}\chi-2(n-1) &= (\underline{u}-u)(\mathrm{tr}\chi-\frac{n-1}{r})+\frac{n-1}{r}(\underline{u}-u)-2(n-1)\\ &= \frac{n-1}{r}(\underline{u}-u-2r)+(\underline{u}-u)(\mathrm{tr}\chi-\frac{n-1}{r})\\ &= \frac{2(n-1)}{r}(-u+t-r)+(\underline{u}-u)(\mathrm{tr}\chi-\frac{n-1}{r})\end{aligned}$$

Consequently,

$$\begin{aligned}\delta^{AB\,(K_0)}\tilde{\pi}_{AB} &= \frac{4(n-1)t}{r}(-u+t-r)+2t(\underline{u}-u)(\mathrm{tr}\chi-\frac{n-1}{r})\\ &+ \Big((1-a^2)\mathrm{tr}\chi-2a\mu\Big)u^2\end{aligned}$$

or setting $\delta^{AB\,(K_0)}\tilde{\pi}_{AB}=\Xi$ we have

$$\Xi = \Xi_1+\Xi_2 \tag{2.102}$$

$$\Xi_1 = \Big((1-a^2)\mathrm{tr}\chi-2a\mu\Big)u^2-4tu(\mathrm{tr}\chi-\frac{n-1}{r}) \tag{2.103}$$

$$\Xi_2 = 4t^2(\mathrm{tr}\chi-\frac{n-1}{r})+\frac{4(n-1)t}{r}(-u+t-r) \tag{2.104}$$

We also write, using the fact that $\chi_{AB}=\hat{\chi}_{AB}+\frac{1}{n-1}\mathrm{tr}\chi\delta_{AB}$,

$$\begin{aligned}{}^{(K_0)}\tilde{\pi}_{AB} &= u^2\underline{\chi}_{AB}+\underline{u}^2\chi_{AB}-4t\delta_{AB}\\ &= u^2(-a^2\chi_{AB}-2ak_{AB})+\underline{u}^2\chi_{AB}-4t\delta_{AB}\\ &= (\underline{u}^2-a^2u^2)\chi_{AB}-4t\delta_{AB}-2ak_{AB}u^2\\ &= (\underline{u}^2-a^2u^2)\hat{\chi}_{AB}+\Big(\frac{1}{n-1}\mathrm{tr}\chi(\underline{u}^2-a^2u^2)-4t\Big)\delta_{AB}-2ak_{AB}u^2\end{aligned}$$

Now, since $u+\underline{u}=2t$,

$$\begin{aligned}\frac{1}{n-1}\mathrm{tr}\chi(\underline{u}^2-a^2u^2)-4t &= \frac{1}{n-1}\mathrm{tr}\chi(\underline{u}^2-u^2)-4t+(1-a^2)\frac{1}{n-1}\mathrm{tr}\chi u^2\\ &= 4t\Big(\frac{1}{n-1}\mathrm{tr}\chi(t-u)-1\Big)+(1-a^2)\frac{1}{n-1}\mathrm{tr}\chi u^2\\ &= 4t\Big((t-u)\frac{t-u}{n-1}(\mathrm{tr}\chi-\frac{n-1}{r})+\frac{t-r-u}{r}\Big)+(1-a^2)\frac{1}{n-1}\mathrm{tr}\chi u^2\end{aligned}$$

Therefore,

$$\begin{aligned}{}^{(K_0)}\tilde{\pi}_{AB} &= (\underline{u}^2-a^2u^2)\hat{\chi}_{AB}-2ak_{AB}u^2\\ &+ \Big(4\frac{t(t-u)}{n-1}(\mathrm{tr}\chi-\frac{n-1}{r})+\frac{4t(t-r-u)}{r}+(1-a^2)\frac{1}{n-1}\mathrm{tr}\chi u^2\Big)\delta_{AB}\end{aligned} \tag{2.105}$$

We recall our calculations for the deformation tensors of T_0, K_0 in the the following proposition:

Proposition 2.4 *Consider the vectorfields*

$$T_0 = \frac{1}{2}\Big(\underline{L} + L\Big) \tag{2.106}$$

$$K_0 = \frac{1}{2}\Big(u^2\underline{L} + \underline{u}^2 L\Big) \tag{2.107}$$

i.) The null components of the deformation tensor of T_0 are given by the formulas:

$$\begin{aligned}
{}^{(T_0)}\pi_{LL} &= 0 \\
{}^{(T_0)}\pi_{L\underline{L}} &= 2\underline{\omega} \\
{}^{(T_0)}\pi_{\underline{L}\underline{L}} &= -4\underline{\omega} \\
{}^{(T_0)}\pi_{LA} &= \underline{\eta}_A - \eta_A \\
{}^{(T_0)}\pi_{\underline{L}A} &= 2\eta_A + \underline{\xi}_A \\
{}^{(T_0)}\pi_{AB} &= \chi_{AB} + \underline{\chi}_{AB} = (1-a^2)\chi_{AB} - 2ak_{AB}
\end{aligned} \tag{2.108}$$

ii.) The null components of ${}^{(K_0)}\tilde{\pi}$ verify the following properties:

$$\begin{aligned}
{}^{(K_0)}\tilde{\pi}_{LL} &= 0 \\
{}^{(K_0)}\tilde{\pi}_{L\underline{L}} &= 2u^2\underline{\omega} + 4(1-a^{-1})(2t-u) \\
{}^{(K_0)}\tilde{\pi}_{\underline{L}\underline{L}} &= -4\underline{u}^2\underline{\omega} - 2\underline{L}(\underline{u}^2) \\
{}^{(K_0)}\tilde{\pi}_{LA} &= u^2(\underline{\eta}_A - \eta_A) \\
{}^{(K_0)}\tilde{\pi}_{\underline{L}A} &= 2\underline{u}^2\eta_A + u^2\underline{\xi}_A \\
{}^{(K_0)}\tilde{\pi}_{AB} &= (\underline{u}^2 - a^2u^2)\hat{\chi}_{AB} - 2ak_{AB}u^2 \\
&+ \Big(4\frac{t(t-u)}{n-1}(tr\chi - \frac{n-1}{r}) + \frac{4t(t-r-u)}{r} + (1-a^2)\frac{1}{n-1}tr\chi u^2\Big)\delta_{AB}
\end{aligned}$$

Moreover the trace $\delta^{AB}\,{}^{(K_0)}\tilde{\pi}_{AB}$ relative to the $S_{t,u}$ surfaces, which we denote Σ, verifies the formula

$$\begin{aligned}
\Xi &= \Xi_1 + \Xi_2 \\
\Xi_1 &= \Big((1-a^2)tr\chi - 2a\mu\Big)u^2 - 4tu(tr\chi - \frac{n-1}{r}) \\
\Xi_2 &= 4t^2(tr\chi - \frac{n-1}{r}) + \frac{4(n-1)t}{r}(-u+t-r)
\end{aligned}$$

Observe that all the null components of ${}^{(T_0)}\pi$ and ${}^{(K_0)}\tilde{\pi}$ are determined by $a, \chi, \underline{\chi}, \eta, \underline{\eta}, \underline{\xi}, \underline{\omega}$. In view of the formulas 2.90 it suffices to estimate a, $\chi, \eta, \underline{\omega}$, and the components of k. In view of the assumption 2.79 we have,

$$\|k\|_{L^1L^\infty(\mathcal{D}_I)} \le C\Lambda^{-\epsilon}$$

Since we can write $k(t) = -2\int_0^t \partial_t^2 g(s)ds$ we also have,

$$\|k\|_{L^\infty(\mathcal{D}_I)} \le C\Lambda^{-2}$$

for all $t \in I$.

Similarly, in view of the fact that $L(a) = k_{NN}$ we infer, integrating with respect to the affine parameter s of L, ($L(s) = 1$,with $s = 0$ on the intersection of the vertex of C_u , or $t = 0$,) and taking $a(0) = 1$

$$\|a - 1\|_{L^\infty} \leq C\Lambda^{-\epsilon}.$$

To estimate $\chi, \eta, \underline{\omega}$ we shall use the eikonal equation 2.84. Observe that χ, η, $\underline{\omega}$ are the nonvanishing components of the hessian D^2u which verifies the Jacobi equation,

$$D_L(D^2u)_{\alpha\beta} + (D^2u)_\alpha^{\ \mu}(D^2u)_{\mu\beta} = R_{\alpha L\beta L} \tag{2.109}$$

Decomposing 2.109 relative to our null pair $L, \underline{L}$, and proceeding as in chapter 9 of [C-K2] we derive, relative to the orthonormal frames e_A, defined in 2.88,

$$\begin{aligned} \frac{d}{ds}\chi_{AB} + \chi_{AC}\chi_{AC} &= -\alpha_{AB} && (2.110)\\ \frac{d}{ds}\eta_A - \chi_{AB}(\eta + \underline{\eta}) &= -\beta_A && (2.111)\\ \frac{d}{ds}\underline{\omega} - \eta\cdot(\eta + 2\underline{\eta}) &= \rho && (2.112) \end{aligned}$$

where $\alpha_{AB} = R_{ALBL}$, $\beta_A = \frac{1}{2}R_{AL\underline{L}L}$ $\rho = \frac{1}{4}R_{L\underline{L}L\underline{L}}$. Splitting χ into its trace(relative to the (n-1)-surfaces $S_{t,u}$), $\mathrm{tr}\chi = \delta^{AB}\chi_{AB}$ and trace less part $\hat{\chi}$,

$$\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{n-1}\mathrm{tr}\chi\delta_{AB}$$

we, rewrite 2.110 in the form,

$$\begin{aligned} \frac{d}{ds}\mathrm{tr}\chi + \frac{1}{n-1}(tr\chi)^2 &= -|\hat{\chi}|^2 - \mathrm{tr}\alpha && (2.113)\\ \frac{d}{ds}\hat{\chi}_{AB} + \frac{2}{n-1}\mathrm{tr}\chi\hat{\chi}_{AB} &= -(\hat{\chi}_{AC}\hat{\chi}_{CB} - \frac{1}{n-1}|\hat{\chi}|^2)\delta_{AB} - \hat{\alpha}_{AB} && (2.114) \end{aligned}$$

where $\alpha_{AB} = \hat{\alpha}_{AB} + \frac{1}{n-1}\mathrm{tr}\alpha\delta_{AB}$.

Theorem 2.6 i. *Under the assumptions 2.72–2.75, with $k = 1$, and the additional assumption 2.79 we have the following estimates:*

$$\begin{aligned} \|tr\chi - \frac{n-1}{r}\|_{L^\infty(\mathcal{D}_I)} &\leq C\Lambda^{-2}\\ \|\hat{\chi}\|_{L^\infty(\mathcal{D}_I)} &\leq C\Lambda^{-2}\\ \|\eta\|_{L^\infty(\mathcal{D}_I)} &\leq C\Lambda^{-2}\\ \|\underline{\omega}\|_{L^\infty(\mathcal{D}_I)} &\leq C\Lambda^{-2}. \end{aligned}$$

Moreover each time we take a covariant derivative, relative to the null frame $L, \underline{L}, \nabla\!\!\!/$, of the quantities $tr\chi - \frac{n-1}{r}$, $\hat{\chi}$ η and $\underline{\omega}$ we improve their asymptotic behavior by Λ^{-1}. Under the same assumptions we also have,

$$
\begin{aligned}
\frac{|u-t+r|}{r} &\le C\Lambda^{-\epsilon}\\
|a-1| &\le C\Lambda^{-\epsilon}\\
\frac{|a-1|}{r} &\le C(\Lambda^{-\epsilon}t^{-1}+\Lambda^{-2})\\
|\underline{L}\,a| &\le C\Lambda^{-2}\\
|L\,a| &\le C\Lambda^{-2}\\
|\nabla\!\!\!/\,a| &\le C\Lambda^{-2}
\end{aligned}
$$

All higher covariant derivatives with respect to the null frame improve by Λ^{-1}.

ii. *Assume now 2.72–2.75 for $k=2$ and 2.79. Then all first covariant derivatives, with respect to our null frame, of the quantities $tr\chi - \frac{n-1}{r}$, $\hat{\chi}$, η and $\underline{\omega}$ we improve their asymptotic behavior by Λ^{-2}. All further derivatives with respect to the frame improve by Λ^{-1}.*

iii. *Assume again 2.72–2.75 for $k=1$ and 2.79. Assume in addition that $g=g^{\Lambda}$ are Einstein metrics, i.e. the Ricci tensor $Ric(g^{\Lambda})$ vanishes identically. Then,*

$$
\begin{aligned}
\|\nabla\!\!\!/\, tr\chi\|_{L^\infty(\mathcal{D}_I)} &\le C\Lambda^{-4}\\
\|\underline{L}(tr\chi - \frac{n-1}{r})\|_{L^\infty(\mathcal{D}_I)} &\le C\Lambda^{-4}\\
\|d\!\!\!/iv\eta\|_{L^\infty(\mathcal{D}_I)} &\le C\Lambda^{-4}
\end{aligned}
$$

where $d\!\!\!/iv\eta = \nabla\!\!\!/^A\eta_A$.

The proof of the first two parts of the Theorem is a much simpler version of that used in chapters 9 and 13 of [C-K2] to derive the asymptotic properties of various components of the Hessian of the optical function and their higher derivatives. To avoid making the paper too lengthy I will assume Theorem 2.6 without proof. Together with I. Rodniansky we plan to present a formal proof of these asymptotic results in a following paper [Kl-R].

Part iii. of the proof relies on the methods presented in section 13.1 of [C-K2]. A short sketch of the method and its adaptation to quasilinear equations is discussed in the last section of this paper.

Combining proposition 2.4 with theorem 2.6 we easily derive the following:

Theorem 2.7 *Under the assumptions 2.72–2.75, with $k=1$, and the additional assumption 2.79 we have the following estimates for the deformation tensors ${}^{(T_0)}\pi$ and ${}^{(K_0)}\tilde{\pi}$:*

$$
\begin{aligned}
\|{}^{(T_0)}\pi_{L\,A}(t)\|_{L^\infty} &\le C\Lambda^{-2}\\
\|{}^{(T_0)}\pi_{L\,\underline{L}}(t)\|_{L^\infty} &\le C\Lambda^{-2}\\
\|{}^{(T_0)}\pi_{\underline{L}\,\underline{L}}(t)\|_{L^\infty} &\le C\Lambda^{-2}\\
\|{}^{(T_0)}\pi_{\underline{L}\,A}(t)\|_{L^\infty} &\le C\Lambda^{-2}\\
\|{}^{(T_0)}\pi_{A\,B}(t)\|_{L^\infty} &\le C(\Lambda^{-2}+\Lambda^{-\epsilon}t^{-1})
\end{aligned}
$$

$$
\begin{aligned}
|\,{}^{(K_0)}\tilde{\pi}_{A\,L}(t,x)| &\leq C\Lambda^{-2}|u|^2 \\
\|\,{}^{(K_0)}\tilde{\pi}_{L\,\underline{L}}(t)\|_{L^\infty} &\leq C(\Lambda^{-2}|u|^2 + \Lambda^{-\epsilon}t) \\
\|\,{}^{(K_0)}\tilde{\pi}_{\underline{L}\,\underline{L}}(t)\|_{L^\infty} &\leq C(\Lambda^{-2}|u|^2 + \Lambda^{-\epsilon}t) \\
\|\,{}^{(K_0)}\tilde{\pi}_{\underline{L}\,A}(t)\|_{L^\infty} &\leq C\Lambda^{-2}t^2 \\
\|\,{}^{(K_0)}\tilde{\pi}_{A\,B}(t)\|_{L^\infty} &\leq C(t^2\Lambda^{-2} + \Lambda^{-\epsilon}t) \\
|\Xi_1(t,x)| &\leq C(\Lambda^{-2}tu + \Lambda^{-\epsilon}u) \\
\|\Xi_2(t)\|_{L^\infty} &\leq C(t^2\Lambda^{-2} + \Lambda^{-\epsilon}t)
\end{aligned}
$$

All higher derivatives improve by Λ^{-1}.

3 Conformal Energy Estimates

3.1 First Derivative Estimates

Our goal in this section is to sketch the main steps in the proof of the Theorem 2.5 using the conformal energy method and the vectorfields K_0, T_0 defined in the previous section. We first recall Proposition 2.2 applied to the equation $\Box_g\phi = F$ and the vectorfield $X = K_0 = \frac{1}{2}(\underline{u}^2 L + u^2\underline{L})$ which we have defined in the previous section. Setting also $\Omega = 4t$, ${}^{(K_0)}\pi = {}^{(K_0)}\tilde{\pi} + \Omega g$ and observing that

$$\Box_g t = -\frac{1}{\sqrt{|g|}}\partial_t\sqrt{|g|} = -\frac{1}{2}g^{ij}\partial_t k_{ij} = \mathrm{tr}k$$

we derive the formula,

$$\begin{aligned}\int_{\Sigma_t}\bar{Q}(K_0,\partial_t) &= \int_{\Sigma_{t_0}}\bar{Q}(K_0,\partial_t) + \frac{1}{2}\int_{t_0}^t\int_{\Sigma_\tau}Q^{\alpha\beta\ (K_0)}\tilde{\pi}_{\alpha\beta}d\tau \\ &- \frac{n-1}{2}\int_{t_0}^t\int_{\Sigma_\tau}\mathrm{tr}k\phi^2 d\tau + \int_{t_0}^t\int_{\Sigma_\tau}(K_0\phi + (n-1)t\phi)F d\tau\end{aligned} \tag{3.115}$$

where,

$$\begin{aligned}\bar{Q}(K_0,\partial_t) &= Q(K_0,\partial_t) + (n-1)t\phi\partial_t\phi - \frac{n-1}{2}\phi^2 \\ Q(K_0,\partial_t) &= \frac{1}{4}Q(\underline{u}^2 L + u^2 L\,,\ aL + a^{-1}\underline{L}) \\ &= \frac{1}{4}\Big(a\underline{u}^2(L\phi)^2 + (a^{-1}u^2 + a\underline{u}^2)|\nabla\!\!\!/\phi|^2 + a^{-1}u^2(\underline{L}\phi)^2\Big)\end{aligned}$$

Throughout this section we denote by $\int_{\Sigma_t}F$ the integral $\int F\sqrt{|g|}(t,x)dx$ and by $\int_{\mathcal{D}_t}F$ the space-time integral $\int_{t_0}^t\int_{\Sigma_\tau}F d\tau = \int_{t_0}^t\int F\sqrt{|g|}(t,x)dtdx$. We can integrate by parts on Σ_t for given tangent vectorfield X according to the formula,

$$\int_{\Sigma_t}F\cdot X(G) = -\int_{\Sigma_t}(X(F) + F\mathrm{div}X)\cdot G \tag{3.116}$$

with $\mathrm{div}X = g^{ij}D_iX_j$. In the particular case of the vectorfield N, the unit normal, in Σ_t, to $S_{t,u}$ we have, $\mathrm{div}N = \mathrm{tr}\theta$ where $\theta_{AB} = <\nabla\!\!\!/_A N, e_B>$. with the trace defined relative to $S_{t,u}$. Since $N = aL - T$ we have $\mathrm{tr}\theta = a\mathrm{tr}\chi - \mu$, $\mu = \delta^{AB}k_{AB}$. Therefore we have

$$\int_{\Sigma_t}F\cdot N(G) = -\int_{\Sigma_t}(N(F) + (a\mathrm{tr}\chi + 0(\Lambda^{-2})F)\cdot G \tag{3.117}$$

We also record the following formulae, see Remark 2 on page 86 of [C-K2],

Lemma 3.1 *Let F be a scalar function V a vectorfield on Σ_t tangent to the surfaces $S_{t,u}$. Then*

$$\begin{aligned}\int_{\Sigma_t}F\cdot \mathrm{d\!\!\!/iv}V &= -\int_{\Sigma_t}(\nabla\!\!\!/F + Fa^{-1}\nabla\!\!\!/a)\cdot V \\ \int_{\Sigma_t}\nabla\!\!\!/F\cdot V &= -\int_{\Sigma_t}F\mathrm{d\!\!\!/iv}V - \int_{\Sigma_t}Fa^{-1}\nabla\!\!\!/a\cdot V\end{aligned}$$

We record below a proposition which is analogous to the part iii) of proposition 1.4.

Proposition 3.1 *If $n \geq 3$, there exists a constant $c > 0$, independent of Λ such that for all $t \in I$*

$$\int_{\Sigma_t} \bar{Q}(K_0, \partial_t) \geq \mathcal{E}^2[\phi](t) \tag{3.118}$$

where

$$\mathcal{E}^2[\phi](t) = \int_{\Sigma_t} \Big(|\phi|^2 + t^2 (L\phi)^2 + t^2 |\nabla\!\!\!/ \phi|^2 + u^2 (\underline{L}\phi)^2 \Big) \tag{3.119}$$

Sketch of Proof Follows precisely that given in Proposition 1.4. First observe, following the same steps as before 1.21, that we can write the term $t\partial_t\phi$ in the form,

$$\begin{aligned} t\partial_t\phi &= \frac{1}{2}(u + \underline{u})\partial_t\phi \\ &= S\phi - \frac{1}{2}(u - \underline{u})N \\ S &= \frac{1}{2}(a\underline{u}L + a^{-1}u\underline{L}) \end{aligned} \tag{3.120}$$

Also,

$$\begin{aligned} t\partial_t\phi &= \frac{t}{\underline{u} - u}(\underline{u} - u)\partial_t\phi \\ &= \frac{t}{\underline{u} - u}\Big(2\underline{S}\phi - (u + \underline{u})N\phi\Big) \\ &= \frac{2t}{\underline{u} - u}\underline{S}\phi - \frac{2t^2}{\underline{u} - u}N\phi \\ \underline{S} &= \frac{1}{2}(a\underline{u}L\phi - a^{-1}u\underline{L}\phi) \end{aligned}$$

This is due to the fact that $L = a^{-1}(\partial_t + N)$, $\underline{L} = a(\partial_t - N)$ with N the unit normal to $S_{t,u}$ tangent to Σ_t.

Using 3.120, just as in 1.21, we integrate by parts with the help of the formula 3.116,3.117

$$\begin{aligned} \int_{\Sigma_t} t\partial_t\phi\phi &= \int_{\Sigma_t} S\phi \cdot \phi - \frac{1}{4}\int_{\Sigma_t} (\underline{u} - u)N(\phi^2) \\ &= \frac{1}{2}\int_{\Sigma_t} S\phi \cdot \phi + \frac{1}{4}\int_{\Sigma_t} \Big(N(\underline{u} - u) + \nu_N(\underline{u} - u)\Big)\phi^2 \end{aligned}$$

where $\nu_N = a\mathrm{tr}\chi + \mu$. Expressing N in the form $N = \frac{1}{2}(aL - a^{-1}\underline{L})$, $\underline{u} = 2t - u$ and using theorem 2.6 we deduce,

$$N(\underline{u} - u) + \nu_N(\underline{u} - u) = 2n + (\Lambda^{-\epsilon})$$

and we infer that,

$$\int_{\Sigma_t} \bar{Q}(K_0, T_0) = \int_{\Sigma_t} \frac{1}{4}\Big(a\underline{u}^2(L\phi)^2 + (a^{-1}u^2 + a\underline{u}^2)|\nabla\!\!\!/\phi|^2 + a^{-1}u^2(\underline{L}\phi)^2\Big) + (n-1)\int_{\Sigma_t}\phi S\phi + (\frac{(n-1)^2}{2} + 0(\Lambda^{-\epsilon}))\int_{\Sigma_t}\phi^2 \tag{3.121}$$

Similarly using 3.121 we proceed as in 1.22,

$$\begin{aligned}\int_{\Sigma_t} t\partial_t\phi\phi &= \int_{\Sigma_t}\frac{2t}{\underline{u}-u}\phi\underline{S}\phi - \int_{\Sigma_t}\frac{t^2}{\underline{u}-u}N(\phi^2) \\ &= \int_{\Sigma_t}\frac{2t}{\underline{u}-u}\phi\underline{S}\phi + \int_{\Sigma_t}\Big(N(\frac{t^2}{(\underline{u}-u)}) + \nu_N\frac{t^2}{\underline{u}-u}\Big)\phi^2\end{aligned} \tag{3.122}$$

We claim that

$$N(\frac{t^2}{(\underline{u}-u)}) + \nu_N\frac{t^2}{\underline{u}-u} = 2(n-2)\frac{t^2}{(\underline{u}-u)^2} + 0(\Lambda^{-\epsilon})$$

The remaining part of the argument is now precisely as in proposition 1.4.

We now proceed to prove the boundedness of the generalized energy norm $\mathcal{E}[\phi](t)$, $t \in I = I_\Lambda$, with ϕ solution to our basic equation $\Box_g\phi = 0$, $g = g^\Lambda$. To do this we have to estimate the error terms $\int_{\mathcal{D}} \mathrm{tr}k\phi^2$ and $\int_{\mathcal{D}} Q^{\alpha\beta\,(K_0)}\tilde{\pi}_{\alpha\beta}$, see 3.115 with respect to $\mathcal{E}[\phi]$ and thus derive the desired bound for the latter by the Gronwall inequality. The first error term is very easy to estimate in view of the fact that $\|k\|_{L^\infty(\mathcal{D}_t)} \le C\Lambda^{-2}$. To estimate the second we proceed as follows:

The null components of the energy momentum tensor $Q_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)$ are given by,

$$\begin{aligned} Q_{LL} &= |L\phi|^2, & Q_{L\underline{L}} &= |\nabla\!\!\!/\phi|^2 \\ Q_{\underline{L}\underline{L}} &= |\underline{L}\phi|^2, & Q_{LA} &= L(\phi)\nabla\!\!\!/_A\phi \\ Q_{\underline{L}A} &= \underline{L}(\phi)\nabla\!\!\!/_A\phi, & Q_{AB} &= \nabla_A\phi\nabla_B\phi - \frac{1}{2}\Big(-L(\phi)\underline{L}(\phi) + |\nabla\!\!\!/\phi|^2\Big)\delta_{AB}\end{aligned} \tag{3.123}$$

Recalling(see Proposition 2.4) that $\delta^{AB\,(K_0)}\tilde{\pi}_{AB} = \Xi$ we write,

$$\begin{aligned} Q^{\alpha\beta\,(K_0)}\tilde{\pi}_{\alpha\beta} &= \frac{1}{2}{}^{(K_0)}\tilde{\pi}_{L\underline{L}}|\nabla\!\!\!/\phi|^2 + \frac{1}{4}\pi_{\underline{L}\underline{L}}|L\phi|^2 \\ &- \sum_{A=1}^{n-1}{}^{(K_0)}\tilde{\pi}_{LA}\underline{L}(\phi)\nabla\!\!\!/_A\phi - \sum_{A=1}^{n-1}{}^{(K_0)}\tilde{\pi}_{\underline{L}A}L(\phi)\nabla\!\!\!/_A\phi \\ &+ \sum_{A,B=1}^{n-1}{}^{(K_0)}\tilde{\pi}_{AB}\nabla\!\!\!/_A\phi\nabla\!\!\!/_B\phi - \frac{1}{2}\Xi|\nabla\!\!\!/\phi|^2 \\ &+ \frac{1}{2}\Xi L(\phi)\underline{L}(\phi)\end{aligned} \tag{3.124}$$

We now recall the results of theorem 2.7. In view of

$$\|{}^{(K_0)}\tilde{\pi}_{L\underline{L}}(t)\|_{L^\infty} \le C(t^2\Lambda^{-2} + t\Lambda^{-\epsilon})$$

we infer that

$$\int_{\mathcal{D}_t} {}^{(K_0)}\tilde{\pi}_{L\underline{L}}|\nabla\!\!\!/\phi|^2 \le \int_{t_0}^t \frac{1}{(1+\tau)^2}\|\,{}^{(K_0)}\tilde{\pi}(t)\|_{L^\infty}\mathcal{E}(\tau)^2 dt$$
$$\le C\Big(\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau\Big)$$

with $\mathcal{D}_t = [t_0, t]\times \mathbf{R}^n$.

Since $\|\,{}^{(K_0)}\tilde{\pi}_{\underline{L}\,\underline{L}}(t)\|_{L^\infty} \le Ct^2\Lambda^{-2}$ we also easily deduce that

$$\int_{\mathcal{D}_t} {}^{(K_0)}\tilde{\pi}_{\underline{L}\,\underline{L}}|L\phi|^2 \le C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau$$

In the same vein,

$$\int_{\mathcal{D}_t} {}^{(K_0)}\tilde{\pi}_{\underline{L}A}L(\phi)\nabla\!\!\!/_A\phi \le C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau$$

To estimate the term $\int_{\mathcal{D}_t} {}^{(K_0)}\tilde{\pi}_{LA}\underline{L}(\phi)\nabla\!\!\!/_A\phi$ we make use of $|\,{}^{(K_0)}\tilde{\pi}_{LA}| \le u^2\Lambda^{-2}$,

$$\int_{\mathcal{D}_t} {}^{(K_0)}\tilde{\pi}_{LA}\underline{L}(\phi)\nabla\!\!\!/_A\phi \le C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau$$

Also,

$$\int_{\mathcal{D}_t} {}^{(K_0)}\tilde{\pi}_{AB}\nabla\!\!\!/_A\phi\nabla\!\!\!/_B\phi \le C\Big(\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau\Big).$$

Recalling the definition of $\Xi = \Xi_1 + \Xi_2$, see Proposition 2.4 we easily check that,

$$\int_{\mathcal{D}_t} \Xi_1 \text{Ł}(\phi)\underline{L}(\phi) \le C\Big(\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau\Big).$$

The only difficult term to estimate is $R = \int_{\mathcal{D}_t} \Xi_2 \text{Ł}(\phi)\underline{L}(\phi)$. This is due to the fact that if we use the obvious bound $|\Xi_2(t,x)| \le t^2\Lambda^{-2}$ we run into a serious obstacle as $\underline{L}\phi$ may not absorb t as a weight. For all other terms we find,

$$|\int_{\mathcal{D}_t} Q^{\alpha\beta}\,{}^{(K_0)}\tilde{\pi}_{\alpha\beta} - R| \le C\Big(\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau\Big). \tag{3.125}$$

We can therefore concentrate our attention to R. Introduce a truncation function $\theta_0 = \theta_0(t,x)$ with the property that $\theta_0 = 1$ in the region where $u \le \frac{t}{2}$ and $\theta_0 = 0$ in the region $u \ge \frac{t}{4}$. Moreover $|\partial\theta_0| \le Ct^{-1}$ for $t \ge 2$. Clearly there is no difficulty to estimate the term $\int_{\mathcal{D}_t} \Xi_2(1-\theta_0)\text{Ł}(\phi)\underline{L}(\phi)$ where $u \ge \frac{t}{2}$. Thus,

$$\int_{\mathcal{D}_t} \Xi_2(1-\theta_0)\text{Ł}(\phi)\underline{L}(\phi) \le C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau. \tag{3.126}$$

To estimate the remaining term we first write $g^{\mu\nu}\phi_\mu\phi_\nu = -L(\phi)\underline{L}(\phi) + |\nabla\!\!\!/\phi|^2$, or, since $\square_g\phi = 0$,

$$L(\phi)\underline{L}(\phi) = |\nabla\!\!\!/\phi|^2 - \frac{1}{2}\square_g(\phi^2).$$

There is no problem to estimate $\int_{\mathcal{D}_t} \Xi_2\theta_0|\nabla\!\!\!/\phi|^2$. Therefore the only nontrivial term remaining to be estimated is

$$R_1 = -\frac{1}{2}\int_{\mathcal{D}_t} \Xi_2\theta_0 \,\Box_g(\phi^2)dv_g \tag{3.127}$$

Now, $\Box_g\psi = -\psi_{;L\underline{L}} + \delta^{AB}\psi_{;AB}$. Using the formulas 2.91 and 2.92 we calculate,

$$\begin{aligned}
\psi_{;L\underline{L}} &= \underline{L}L(\psi) - 2\eta\cdot\nabla\!\!\!/\psi - 2\underline{\omega}L(\psi)\\
\psi_{;AB} &= \nabla\!\!\!/_B\nabla\!\!\!/_A\psi - \frac{1}{2}\underline{\chi}_{AB}L\psi - \frac{1}{2}\chi_{AB}\underline{L}\psi
\end{aligned}$$

Therefore,

$$\Box\,\psi = -\underline{L}L\psi + \Delta\!\!\!/\psi - \frac{1}{2}\mathrm{tr}\underline{\chi}L\psi - \frac{1}{2}\mathrm{tr}\chi\underline{L}\psi + 2\eta\cdot\nabla\!\!\!/\psi + 2\underline{\omega}L\psi.$$

In view of the fact that

$$\|\Xi_2(t)\|_{L^\infty} \le C(t^2\Lambda^{-2} + \Lambda^{-\epsilon}t),$$

it is easy to see that the only nontrivial term remaining to be estimated has the form,

$$B = -\frac{1}{2}\int_{\mathcal{D}} \Xi_2\theta_0\Big(- \underline{L}L(\phi^2) + \Delta\!\!\!/(\phi^2) - \frac{1}{2}\mathrm{tr}\chi\underline{L}(\phi^2)\Big) = B_1 + B_2 + B_3 \tag{3.128}$$

To estimate B_1 we first integrate by parts using the formula 3.116. Recalling that $\nu_{\underline{L}} = \frac{1}{2}g^{ij}\underline{L}(g_{ij})$ we have:

$$B_1 = \int_{\mathcal{D}_t} \Xi_2\theta_0\underline{L}L(\phi^2) = -\int_{\mathcal{D}_t}\Big(\underline{L}(\Xi_2\theta_0) + \nu_{\underline{L}}\Xi_2\theta_0\Big)L(\phi^2) + \mathcal{B}_1$$

where $\mathcal{B}_1$ is the boundary term,

$$\mathcal{B}_1 = \int_{\Sigma_t} \Xi_2\theta_0 aL(\phi^2) - \int_{\Sigma_{t_0}} \Xi_2\theta_0 aL(\phi^2).$$

Clearly, $\int_{\Sigma_t} \Xi_2\theta_0 aL(\phi^2) \le C|t|\Lambda^{-2}\mathcal{E}^2(t)$, or, in view of the fact that $t \le \Lambda^{2-\epsilon}$,

$$\int_{\Sigma_t} \Xi_2\theta_0 aL(\phi^2) \le C\Lambda^{-\epsilon}\mathcal{E}^2(t).$$

Thus,

$$|\mathcal{B}_1| \le C(\Lambda^{-\epsilon}\mathcal{E}^2(t) + \mathcal{E}(2)). \tag{3.129}$$

Clearly, since $u \le \frac{t}{2}$ on our domain of integration and $u = t - r + tO(\Lambda^{-\epsilon})$ we have $r \ge \frac{t}{2} + tO(\Lambda^{-\epsilon})$. Therefore, for Λ sufficiently large, $|\nu_{\underline{L}}| \le C(\frac{1}{t} + \Lambda^{-2})$ and thus,

$$\int_{\mathcal{D}_t} \nu_{\underline{L}}\Xi_2\theta_0 L(\phi^2) \le C\Big(\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau\Big).$$

We also have,

$$\int_{\mathcal{D}_t} \Xi_2\underline{L}(\theta_0)L(\phi^2) \le C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau$$

The only term of B_1 which remains to estimate is

$$\int_{\mathcal{D}_t} \underline{L}(\Xi_2)\theta_0 L(\phi^2) \tag{3.130}$$

To estimate B_2 we write with the help of the Lemma 3.1,

$$B_2 = \int_{\mathcal{D}} \Xi_2\theta_0 \not{\Delta}(\phi^2) = -\int_{\mathcal{D}} \Big(\not{\nabla}_A(\Xi_2\theta_0) + a^{-1}\not{\nabla}_A a \Xi_2\theta_0\Big)\not{\nabla}_A(\phi^2).$$

Recall that, $|a^{-1}\not{\nabla}_A a| \le \Lambda^{-2}$. Thus the only term which is not straightforward to estimate is

$$\int_{\mathcal{D}_t} \not{\nabla}_A(\Xi_2)\theta_0\not{\nabla}_A(\phi^2). \tag{3.131}$$

Leaving 3.130, 3.131 aside for a moment it remains to estimate $B_3 = \int_{\mathcal{D}_t} \Xi_2\theta_0 \mathrm{tr}\chi \underline{L}(\phi^2)$. Integrating once more by parts we have

$$B_3 = -\int_{\mathcal{D}_t} \Big(\underline{L}(\Xi_2\theta_0\mathrm{tr}\chi)\phi^2 + \nu_{\underline{L}}\Xi_2\theta_0\mathrm{tr}\chi\Big)\phi^2 + \mathcal{B}_3$$

with $\mathcal{B}_3$ the boundary term,

$$\mathcal{B}_3 = \int_{\Sigma_t} \Xi_2\theta_0\mathrm{tr}\chi a\phi^2 - \int_{\Sigma_{t_0}} \Xi_2\theta_0\mathrm{tr}\chi a\phi^2.$$

As above,

$$|\mathcal{B}_3| \le C(\Lambda^{-\epsilon}\mathcal{E}^2(t) + \mathcal{E}(2)). \tag{3.132}$$

Using the straightforward estimate $\|\Xi_2(t)\|_{L^\infty} \le C(t^2\Lambda^{-2} + \Lambda^{-\epsilon}t)$ we easily deduce that,

$$\int_{\mathcal{D}_t} \nu_{\underline{L}}\Xi_2\theta_0\mathrm{tr}\chi\phi^2 \le C\Big(\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau\Big).$$

The term corresponding to the integrand $\Xi_2\underline{L}(\theta_0)\mathrm{tr}\chi\phi^2$ can be estimate in precisely the same way. Therefore the only terms connected to B_3 which remain to be estimated are

$$\int_{\mathcal{D}_t} \underline{L}(\Xi_2)\theta_0\mathrm{tr}\chi\phi^2 \tag{3.133}$$

and

$$\int_{\mathcal{D}_t} \Xi_2\theta_0\underline{L}(\mathrm{tr}\chi)\phi^2 \tag{3.134}$$

To conclude that,

$$B \le C\Lambda^{-\epsilon}\mathcal{E}^2(t) + C\mathcal{E}^2(t_0) + C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau \tag{3.135}$$

it only remains to estimate the terms 3.130, 3.131, 3.133, 3.134. Recalling that $\Xi_2 = -4t^2(\mathrm{tr}\chi - \frac{n-1}{r}) + \frac{4(n-1)t}{r}(u - t + r)$ it is easy to see that the only difficulty with these terms appear when the derivatives $\not{\nabla}$, $\underline{L}$ fall on $\mathrm{tr}\chi$. In these cases we need the pointwise estimates, see part ii) of Theorem 2.6,

$$\|\not{\nabla}(\mathrm{tr}\chi - \frac{n-1}{r})\|_{L^\infty(\mathcal{D}_t)} \le c\Lambda^{-4} \tag{3.136}$$

$$\|\underline{L}(\mathrm{tr}\chi - \frac{n-1}{r})\|_{L^\infty(\mathcal{D}_t)} \le c\Lambda^{-4}. \tag{3.137}$$

Remark: *The estimates 3.136, 3.137 which are needed in connection to the terms 3.130, 3.131, 3.133, 3.134, see also 3.150 below, are the only ones which require $k = 2$ in the assumptions 2.72–2.75 of our proof of Theorem 2.5.* Using 3.136, 3.137 we derive,

$$\|\underline{L}(\Xi_2)\|_{L^\infty(\mathcal{D}_t)} \leq Ct^2\Lambda^{-4} \tag{3.138}$$
$$\|\nabla\!\!\!/(\Xi_2)\|_{L^\infty(\mathcal{D}_t)} \leq Ct^2\Lambda^{-4} \tag{3.139}$$

Therefore,

$$\int_{\mathcal{D}_t} \underline{L}(\Xi_2)\theta_0 L(\phi^2) \leq C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau$$
$$\int_{\mathcal{D}_t} \nabla\!\!\!/_A(\Xi_2)\theta_0\nabla\!\!\!/_A(\phi^2) \leq C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau$$

Also,

$$\int_{\mathcal{D}_t} \Xi_2\theta_0\underline{L}(\mathrm{tr}\chi)\phi^2 \leq C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau$$

and therefore we have established 3.135 and the same estimate for R. Combining this with 3.125 we infer that,

Proposition 3.2 *Under the assumptions 2.72–2.75 for $k = 2$ as well as 2.79 we have for all $t \in I$,*

$$\int_{\mathcal{D}_t} Q^{\alpha\beta\,(K_0)}\tilde{\pi}_{\alpha\beta} \leq C\Lambda^{-\epsilon}\mathcal{E}^2(t) + C\mathcal{E}^2(t_0) + C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2(\tau)d\tau \tag{3.140}$$

Therefore, in view of the identity 3.115, proposition 3.1 and Gronwall inequality we conclude with the following:

Proposition 3.3 *Consider the equation $\Box_g\phi = 0$ in the domain $\mathcal{D}_I$ with $g = g^\Lambda$ verifying the assumptions 2.72–2.75 for $k = 2$ and 2.79. Then, with the $\mathcal{E}(t) = \mathcal{E}[\phi](t)$ defined as in 3.119, and C a constant independent of Λ,*

$$\mathcal{E}(t) \leq C\mathcal{E}(t_0)$$

for all $t \in I$.

3.2 Estimates for Higher derivatives

In the previous section we have shown how to bound the integral $\mathcal{E}[\phi](t)$ on the domain $\mathcal{D}$ in terms of only the initial data. To implement the strategy, which we have discussed in details in the flat situation of section 1.2.(see in particular proposition 1.5, for deriving decay estimates for $\partial\phi$, we need also to estimate the norm $\mathcal{E}$ for higher derivatives of ϕ. For this reason we define the quantity

$$\mathcal{E}^2_{s+1}[\phi](t) = \|\phi\|^2_{L^2(\Sigma_t)} + \sum_{a=1}^{s+1}\|u\underline{L}^a\phi\|^2_{L^2(\Sigma_t)} + \sum_{1\leq a+b+c\leq s+1;\, b\cdot c\neq 0} t^2\|\nabla\!\!\!/^c L^b\underline{L}^a\phi\|^2_{L^2(\Sigma_t)} \tag{3.141}$$

with $\nabla\!\!\!/^a L^b\underline{L}^c\phi$ representing the a-th covariant derivative of the scalar $L^b\underline{L}^c\phi$ along the $S_{t,u}$ surfaces.

The goal is to prove that $\mathcal{E}_{s+1}(t)$ is bounded for $t \in I_\Lambda$. Our strategy is to estimate first the time derivatives of ϕ, using the norms $\mathcal{E}[\partial_t^j \phi]$ by commuting with the equation $\Box_g \phi = 0$ and then derive the estimates for all other derivatives of ϕ from the equation $\partial_t^2 \phi = \triangle_g \phi + \partial_t(\log \sqrt{|g|})\partial_t \phi$.

The simple strategy of applying the estimates derived in the previous section to time derivatives of the equation $\Box_g \phi = 0$ will not work however because of the error terms generated whenever we commute with the wave operator. Indeed, in view of the Lemma 2.3, we have

$$\Box_g \partial_t \phi = F = \pi^{\alpha\beta} D_\alpha \mathcal{D}_\beta \phi + D^\alpha \pi_{\alpha\lambda} D^\lambda \phi$$

where $\pi = -2k$ is the deformation tensor of ∂_t, k the second fundamental form of the Σ_t surfaces. To get a bound for $\mathcal{E}[\partial_t \phi]$ we need to estimate $\int_{\mathcal{D}_t}(K_0\phi + (n-1)t\phi)F$. This requires that π_{LL} behaves like $0(\Lambda^{-4})$ which is not true in general.

To get around this difficulty we replace ∂_t by the vectorfield $T_0 = (L + \underline{L})$. Recall that ${}^{(T_0)}\pi_{LL} = 0$. In what follows we will sketch the proof for the boundedness of $\mathcal{E}_2[\phi]$.

According to Lemma 2.3 we have,

$$\Box_g (T_0 \phi) = F = {}^{(T_0)}\pi^{\alpha\beta} D_\alpha \mathcal{D}_\beta \phi + D^{\alpha\, (T_0)}\pi_{\alpha\lambda} D^\lambda \phi \tag{3.142}$$

with ${}^{(T_0)}\pi$ the deformation tensor of T_0. We now apply 3.115 to the equation $\Box_g (T_0\phi) = F$ and derive

$$\begin{aligned}\int_{\Sigma_t} \bar{Q}[T_0\phi](K_0, \partial_t) &= \int_{\Sigma_{t_0}} \bar{Q}[T_0\phi](K_0, \partial_t) + \frac{1}{2}\int_{\mathcal{D}_t} Q^{\alpha\beta}[T_0\phi]\, {}^{(K_0)}\tilde{\pi}_{\alpha\beta} \\ &- \int_{\mathcal{D}_t} (K_0 T_0 \phi + (n-1)t T_0\phi)F - \frac{n-1}{2}\int_{\mathcal{D}_t} \mathrm{tr}k |T_0\phi|^2\end{aligned} \tag{3.143}$$

In view of the proposition 3.1 we have

$$\int_{\Sigma_t} \bar{Q}[T_0\phi](K_0, T) \geq \mathcal{E}^2[T_0\phi](t) \tag{3.144}$$

Proceeding exactly as in the previous section we can show that,

$$\begin{aligned}\int_{\mathcal{D}_t} Q^{\alpha\beta}[T_0\phi]\, {}^{(K_0)}\tilde{\pi}_{\alpha\beta} &\leq C\Lambda^{-\epsilon}\mathcal{E}^2[T_0\phi](t) + C\mathcal{E}^2[T_0\phi](t_0) \\ &+ C\Lambda^{-2}\int_{t_0}^t \mathcal{E}^2[T_0\phi](\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}^2[T_0\phi](\tau)d\tau\end{aligned} \tag{3.145}$$

The only new type of terms which we have to estimate are of the form

$$J = \int_{\mathcal{D}_t} (K_0(T_0\phi) + (n-1)t(T_0\phi))F.$$

with,

$$\begin{aligned} F &= F_1 + F_2 \\ F_1 &= {}^{(T_0)}\pi^{\alpha\beta} D_\alpha \mathcal{D}_\beta \phi \\ F_2 &= D^{\alpha\, (T_0)}\pi_{\alpha\lambda} D^\lambda \phi \end{aligned}$$

Let,

$$J = J_i + J_e$$

where,

$$J_e = \int_{\mathcal{D}_t} \theta_0 (K_0(T_0\phi) + (n-1)t(T_0\phi))F.$$

and θ_0 a cut-off function defined as 3.126. In what follows we will concentrate only on J_e, the interior integral J_i is far easier to handle.

For F_1 we write,

$$\begin{aligned} F_1 &= \frac{1}{4}\,{}^{(T_0)}\pi_{L\,L}\phi_{;\underline{L}\,\underline{L}} + \frac{1}{4}\,{}^{(T_0)}\pi_{\underline{L}\,\underline{L}}\phi_{;LL} - \frac{1}{2}\,{}^{(T_0)}\pi_{L\,\underline{L}}\phi_{;L\underline{L}} \\ &- {}^{(T_0)}\pi_{\underline{L}\,A}\phi_{;LA} - {}^{(T_0)}\pi_{L\,A}\phi_{;\underline{L}A} + {}^{(T_0)}\pi_{AB}\phi_{;AB} \end{aligned}$$

Using the formulas 2.91 and 2.91 we calculate,

$$\begin{aligned} \phi_{;LL} &= L^2\phi \\ \phi_{;\underline{L}L} &= L\underline{L}(\phi) - 2\underline{\eta}\cdot\nabla\!\!\!/(\phi) \\ \phi_{;\underline{L}\underline{L}} &= \underline{L}^2\phi + 2\underline{\omega}\underline{L}\phi - 2\underline{\xi}_A e_A(\phi) \\ \phi_{;LA} &= e_A(L\phi) - \chi_{AB}\phi_B + \eta_A L\phi \\ \phi_{;\underline{L}A} &= e_A(\underline{L}\phi) - \underline{\chi}_{AB}\phi_B + \eta_A\underline{L}\phi \\ \phi_{;AB} &= \nabla\!\!\!/_A\nabla\!\!\!/_B\phi - \frac{1}{2}\underline{\chi}_{AB}L\phi - \frac{1}{2}\chi_{AB}\underline{L}\phi \end{aligned}$$

Therefore, since ${}^{(T_0)}\pi_{L\,L} = 0$,

$$\begin{aligned} F_1 &= \frac{1}{4}\,{}^{(T_0)}\pi_{\underline{L}\,\underline{L}}L^2(\phi) - \frac{1}{2}\,{}^{(T_0)}\pi_{L\,\underline{L}}\Big(\Delta\!\!\!/\phi - \frac{1}{2}\mathrm{tr}\chi\underline{L}(\phi) - \frac{1}{2}\mathrm{tr}\underline{\chi}L(\phi)\Big) \\ &- {}^{(T_0)}\pi_{\underline{L}\,A}\Big(\nabla\!\!\!/_A(L(\phi)) - \chi_{AB}\nabla\!\!\!/_B\phi + \eta_A L(\phi)\Big) \\ &- {}^{(T_0)}\pi_{L\,A}\Big(\nabla\!\!\!/_A(\underline{L}(\phi)) - \underline{\chi}_{AB}\nabla\!\!\!/_B\phi - \eta_A\underline{L}(\phi)\Big) \\ &+ {}^{(T_0)}\pi_{AB}\Big(\nabla\!\!\!/_A\nabla\!\!\!/_B\phi - \frac{1}{2}\chi_{AB}\underline{L}(\phi) - \frac{1}{2}\underline{\chi}_{AB}L(\phi)\Big) \end{aligned}$$

The most dangerous terms in the integral $\int_{\mathcal{D}_t}\theta_0\Big(K_0(T_0\phi) + (n-1)t(T_0\phi)\Big)F_1$ are of the form

$$\int_{\mathcal{D}_t}\theta_0\underline{u}^2 LT_0(\phi)F_1.$$

Consider for example the integral,

$$\int_{\mathcal{D}_t}\theta_0\underline{u}^2 LT_0(\phi)\cdot{}^{(T_0)}\pi_{L\,A}\cdot\Big(\nabla\!\!\!/_A(\underline{L}(\phi)) - \underline{\chi}_{AB}\nabla\!\!\!/_B\phi - \eta_A\underline{L}(\phi)\Big)$$

Since $\|{}^{(T_0)}\pi_{L\,A}(t)\|_{L^\infty} \le C\Lambda^{-2}$. we infer that,

$$\int_{\mathcal{D}_t}\theta_0\underline{u}^2 L\phi\,{}^{(T_0)}\pi_{L\,A}\cdot\Big(\nabla\!\!\!/_A(\underline{L}(\phi)) - \underline{\chi}_{AB}\nabla\!\!\!/_B\phi - \eta_A\underline{L}(\phi)\Big) \le C\Lambda^{-2}\int_{t_0}^t \mathcal{E}[T_0\phi](\tau))\mathcal{E}_2(\tau)d\tau \qquad (3.146)$$

All other terms can be estimated in the same manner. Thus,

$$\int_{\mathcal{D}_t}(K_0(T_0\phi)+(n-1)tT_0(\phi))F_1 \leq C\Lambda^{-2}\int_{t_0}^t \mathcal{E}[T_0\phi](\tau))\mathcal{E}_2(\tau)d\tau + \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}[T_0\phi](\tau))\mathcal{E}_2(\tau)d\tau. \tag{3.147}$$

On the other hand we have

$$\begin{aligned} F_2 &= D^{\alpha\,(T_0)}\pi_{\alpha\lambda}D^\lambda\phi \\ &= -\frac{1}{2}D^{\alpha\,(T_0)}\pi_{\alpha\underline{L}}L\phi - \frac{1}{2}D^{\alpha\,(T_0)}\pi_{\alpha L}\underline{L}\phi + D^{\alpha\,(T_0)}\pi_{\alpha A}\nabla\!\!\!/_A\phi \\ &= -\frac{1}{2}(-\frac{1}{2}D_{\underline{L}}{}^{(T_0)}\pi_{L\underline{L}} - \frac{1}{2}D_L{}^{(T_0)}\pi_{\underline{L}\underline{L}} + D^{A\,(T_0)}\pi_{A\underline{L}})L\phi \\ &- \frac{1}{2}(-\frac{1}{2}D_{\underline{L}}{}^{(T_0)}\pi_{LL} - \frac{1}{2}D_L{}^{(T_0)}\pi_{\underline{L}L} + D^{A\,(T_0)}\pi_{AL})\underline{L}\phi \\ &+ (-\frac{1}{2}D_{\underline{L}}{}^{(T_0)}\pi_{LA} - \frac{1}{2}D_L{}^{(T_0)}\pi_{\underline{L}A} + D^{B\,(T_0)}\pi_{AB})\nabla\!\!\!/_A\phi \end{aligned}$$

The most dangerous terms in the integral

$$\int_{\mathcal{D}_t}\theta_0\Big(K_0(T_0\phi)+(n-1)tT_0(\phi)\Big)F_2$$

are of the form

$$\int_{\mathcal{D}_t}\theta_0\underline{u}^2LT_0\phi\cdot(-\frac{1}{2}D_{\underline{L}}{}^{(T_0)}\pi_{LL} - \frac{1}{2}D_L{}^{(T_0)}\pi_{\underline{L}L} + D^{A\,(T_0)}\pi_{AL})\cdot\underline{L}\phi \tag{3.148}$$

Using 2.91 and 2.92 we calculate,

$$\begin{aligned} D_{\underline{L}}{}^{(T_0)}\pi_{LL} &= \underline{L}({}^{(T_0)}\pi_{LL}) - 4\eta^{A\,(T_0)}\pi_{AL} - 4\underline{\omega}^{(T_0)}\pi_{LL} = -4\eta^{A\,(T_0)}\pi_{AL} \\ D_L{}^{(T_0)}\pi_{\underline{L}L} &= L({}^{(T_0)}\pi_{\underline{L}L}) - 2\underline{\eta}^{A\,(T_0)}\pi_{A\underline{L}} \\ D^{A\,(T_0)}\pi_{AL} &= \nabla\!\!\!/^A({}^{(T_0)}\pi_{AL}) - \chi_{AB}{}^{(T_0)}\pi_{L\underline{L}} - {}^{(T_0)}\pi_{AC}\chi_{BC} + {}^{(T_0)}\pi_{AL}\eta^A \end{aligned}$$

We have, $L({}^{(T_0)}\pi_{\underline{L}L}) = 2L(\underline{\omega})$ and $\nabla\!\!\!/^A({}^{(T_0)}\pi_{AL}) = \nabla\!\!\!/^A(\underline{\eta}_A - \eta_A)$. Thus the only term which depends of the third derivatives of the metric is $\mathrm{d\!\!/iv}\,\eta$. Returning to 3.148 we have to estimate,

$$\int_{t_0}^t\int_{\Sigma_t}\theta_0\underline{u}^2\mathrm{d\!\!/iv}\,\eta\cdot LT_0\phi\cdot\underline{L}\phi.$$

To do this we need the estimate, see part ii) of Theorem 2.6,

$$|\mathrm{d\!\!/iv}\,\eta| \leq C\Lambda^{-4}. \tag{3.149}$$

Then,

$$\int_{t_0}^t\int_{\Sigma_t}\theta_0\underline{u}^2\mathrm{d\!\!/iv}\,\eta\cdot LT_0\phi\cdot\underline{L}\phi \leq C\Lambda^{-2}\int_{t_0}^t\mathcal{E}[T_0\phi](s)\mathcal{E}[\phi](s)ds \tag{3.150}$$

All other terms are easier to estimate. We find,

$$\int_{t_0}^t \int_{\Sigma_t} (K_0(T_0\phi) + (n-1)tT_0(\phi))F_2 \le C\Lambda^{-2} \int_{t_0}^t \mathcal{E}[T_0\phi](s))(\mathcal{E}[\phi](s) + \mathcal{E}[\phi](s))ds. \tag{3.151}$$

Thus, combining the estimates, 3.147 with 3.151 and also 3.142, 3.118 we derive,

$$\begin{aligned}\mathcal{E}^2[T_0\phi](t) &\le \mathcal{E}[T_0\phi](0) + \Lambda^{-2}\int_{t_0}^t \mathcal{E}[T_0\phi](\tau))\mathcal{E}_2(\tau)d\tau \\ &+ \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}[T_0\phi](\tau))\mathcal{E}_2(\tau)d\tau.\end{aligned} \tag{3.152}$$

Similarly, proceeding in the same way,

$$\begin{aligned}\mathcal{E}^2[T_0^2\phi](t) &\le \mathcal{E}[T_0^2\phi](0) + \mathcal{E}^2[T_0^2\phi](0) + \Lambda^{-2}\int_{t_0}^t \mathcal{E}[T_0^2\phi](\tau))\mathcal{E}_3(\tau)d\tau \\ &+ \Lambda^{-\epsilon}\int_{t_0}^t \frac{1}{1+\tau}\mathcal{E}[T_0^2\phi](\tau))\mathcal{E}_3(\tau)d\tau.\end{aligned} \tag{3.153}$$

On the other hand, expressing

$$\partial_t = \frac{1}{a+a^{-1}}(T_0 + (a-a^{-1})(L - \underline{L}))$$

in the equation,

$$\partial_t^2\phi + \partial_t \log\sqrt{|g|}\partial_t\phi = \triangle_g\phi$$

and using L^2 elliptic theory, based on simple integration by parts arguments, we deduce

$$\mathcal{E}_3(t) \le C\Big(\mathcal{E}[T_0^2\phi](t) + \mathcal{E}[T_0\phi](t) + \mathcal{E}[\phi](t)\Big).$$

Therefore,

Proposition 3.4 *Consider the equation $\Box_g\phi = 0$ in the domain $\mathcal{D}_I$ with $g = g^\Lambda$ verifying the assumptions 2.72–2.75 for $k = 2$ and 2.79. Then with C a constant independent of Λ,*

$$\mathcal{E}_3(t) \le C\mathcal{E}_3(t_0)$$

for all $t \in I$.

The same estimates can be proved for the norms $\mathcal{E}_s$, s positive integer. We can then proceed exactly as in the proof of propositions 1.5 and 1.6 to deduce the required decay estimates of theorem 2.5.

3.3 Further Improvements

The arguments presented so far in this paper only prove Theorem C (see section 2.1) with a loss $\sigma > \frac{1}{5}$. To obtain Tataru's better result $\sigma > \frac{1}{6}$ we need to deal with the integrals 3.130,3.131,3.133,3.134 and 3.150 with the limited regularity assumptions 2.72–2.75 for $k = 1$. In what follows I will sketch an argument which shows that by using the special structure of the nonlinear equation we can derive Tataru's result. The key observation is that for an Einstein metric, i.e. a metric with flat Ricci curvature, the quantities[17] $\underline{L}(\Xi_2), \nabla\!\!\!/(\Xi_2), \text{d}\!\!\!/\text{iv}\,\eta$ have the same regularity properties as the curvature tensor of the metric, rather than the covariant derivative of the curvature(as one expects in general). This fact played an important role in [C-K2] and is explained in details in the section 13.1 of the book. Though our metric $g^\Lambda = g^\Lambda(\phi)$ is not an Einstein metric we can show that $Ric(L, L)$ is better behaved. This fact suffices in order to use the improved regularity estimates of section 13.1 in [C-K2].

Recall that

$$R_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\Big(\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\nu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\mu\nu}\Big) + \text{Qr}.$$

with Qr denoting terms quadratic in g, ∂g

Thus,

$$Ric(L, L) = L\Big(g^{\alpha\beta}(\partial_\alpha g_{\beta\mu} L^\mu - \frac{1}{2} L(g_{\alpha\beta})\Big) - \frac{1}{2} L^\mu L^\nu (g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}) + \text{Qr}.$$

i.e. schematically,

$$Ric(L, L) = L(m) + n \tag{3.154}$$

with m an expression depending only on the first derivatives of the metric and n a term which depends[18] on the second derivatives of the metric through the wave operator $g^{\alpha\beta}\partial_\alpha\partial_\beta$. In view of the fact that $g = g^\Lambda(\phi)$ and ϕ verifies the original nonlinear equation $\square'_h \phi = N(\phi, \partial\phi)$, see 2.31, it is easy to see that n is a lower order term. Now consider the equation 2.113

$$\frac{d}{s}\text{tr}\chi + \frac{1}{n-1}(tr\chi)^2 = -|\hat{\chi}|^2 - \text{tr}\alpha \tag{3.155}$$

with $\alpha_{AB} = R_{ALBL}$ and $\text{tr}\alpha = \delta^{AB}\alpha_{AB}$. Thus $\text{tr}\alpha = g^{\mu\nu}R_{\mu L\nu L} = Ric(L, L)$ and we can rewrite the equation 3.155 in the form,

$$\frac{d}{s}(\text{tr}\chi - m) + \frac{1}{n-1}(tr\chi)^2 = -|\hat{\chi}|^2 - n. \tag{3.156}$$

To estimate[19] the angular derivatives of χ one differentiates 3.156 as in the formula 13.1.6a of [C-K2] to obtain,

$$\begin{aligned}
\frac{d}{s}(\nabla\!\!\!/_A \text{tr}\chi - \nabla\!\!\!/_A m) + \frac{3}{2})\nabla\!\!\!/_A tr\chi \quad &= \quad \nabla\!\!\!/_A n - \hat{\chi}_{AB}\nabla\!\!\!/_B \text{tr}\chi - 2\hat{\chi}_{BC}\nabla\!\!\!/_A\hat{\chi}_{BC} && (3.157)\\
&- \quad (\eta_A + \underline{\eta}_A)(|\hat{\chi}|^2 + \frac{1}{2}(\text{tr}\chi)^2). && (3.158)
\end{aligned}$$

We consider this equation coupled with the Codazzi equation for $\hat{\chi}$, see formula 13.1.2l in [C-K2],

$$(\text{d}\!\!\!/\text{iv}\,\hat{\chi})_A + \hat{\chi}_{AB}\eta_B = \frac{1}{2}(\nabla\!\!\!/_A \text{tr}\chi + \eta_A \text{tr}\chi) - \beta_A \tag{3.159}$$

[17]These quantities depend on $\underline{L}(\text{tr}\chi), \nabla\!\!\!/\text{tr}\chi$ and $\text{d}\!\!\!/\text{iv}\,\eta$ which depend, in general, on the third derivatives of the metric, see part iii of the theorem 2.6.

[18]It also depends quadratically on ∂g.

[19]For simplicity we assume in what follows $n = 3$ which is the case considered in [C-K2].

which can be viewed as an elliptic system on the 2-surfaces $S_{t,u}$, see proposition 2.2.2 in [C-K2]. One can thus proceed as in proposition 13.1.1 of [C-K2] and estimate the angular derivatives of both $\mathrm{tr}\chi$ and $\hat{\chi}$ in terms of β, n, $\nabla\!\!\!/ m$ and the undifferentiated quantities $\chi, \eta, \underline{\eta}$.

To estimate $\underline{L}\mathrm{tr}\chi$ and $\mathrm{d}\!\!/\mathrm{iv}\,\eta$ one has to proceed as in the proof of proposition 13.1.2 in [C-K2]. The clue is the following equation for $\underline{L}\mathrm{tr}\chi$, see formula 13.1.2h in [C-K2],

$$\underline{L}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\underline{\omega}\mathrm{tr}\chi = \mathrm{d}\!\!/\mathrm{iv}\,\eta - \hat{\chi}\cdot\underline{\hat{\chi}} + 2|\eta|^2 + 2\tilde{\rho} \tag{3.160}$$

Also, see formula 13.1.2i,

$$\mathrm{c}\!\!/\mathrm{url}\,\eta = \tilde{\sigma} - \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}}.$$

Here $\tilde{\rho}, \tilde{\sigma}$ are components of the curvature tensor R. Now introduce, as in 13.1.10c of [C-K2], $\mu = -\mathrm{d}\!\!/\mathrm{iv}\,\eta + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \tilde{\rho} + |\zeta|^2$. From 3.160 we have

$$\mu = -\frac{1}{2}(\underline{L}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\underline{\omega}\mathrm{tr}\chi).$$

We estimate $\nabla\!\!\!/\eta$ and μ from the elliptic Hodge system on $S_{t,u}$,

$$\mathrm{d}\!\!/\mathrm{iv}\,\eta = -\mu + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \tilde{\rho} \tag{3.161}$$

$$\mathrm{c}\!\!/\mathrm{url}\,\eta = \tilde{\sigma} - \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}} \tag{3.162}$$

and a propagation equation for μ, similar to that in formula 13.1.11 of [C-K2]. The crucial point of this procedure is that in the propagation equation for μ there are no terms which depend on derivatives of the curvature R.

References

[B-C1] H. Bahouri and J.Y.Chemin *"Equations d'ondes quasilineaires et effect dispersif"* American Journal of Mathematics 121(1999), 1337–1377.

[B-C2] H. Bahouri and J.Y.Chemin *"Equations d'ondes quasilineaires et estimations de Strichartz"* International Mathematics Research Notices 21(1999) 1141–1177.

[C-K1] D.Christodoulou, S.Klainerman, *"Asymptotic properties of linear field equations in Minkowski space"*. Comm.Pure Appl.Math. XLIII,(1990), 137-199.

[C-K2] D.Christodoulou, S.Klainerman, *"The global non linear stability of the Minkowski space"*. Princeton Mathematical series, 41 (1993).

[Ho] L.Hormander, *"Lectures on Nonlinear Hyperbolic Equations* Mathematics and Applications 26, Springer-Verlag (1987).

[K-T] Keel-Tao, "Endpoint Strichartz estimates", Amer. J. Math. 120 (1998) 955-980.

[Kl1] S.Klainerman *"Uniform decay estimates and the Lorentz invariance of the classical wave equation"*. Comm.Pure.Appl.Math. 38, (1985), 321-332.

[Kl2] S.Klainerman, *"Remarks on the global Sobolev inequalities in Minkowski Space"*. Comm.Pure.Appl.Math. 40, (1987), 111-117.

[Kl3] S.Klainerman, *"The null condition and global existence to nonlinear wave equations"*. Lect. Appl. Math. 23, (1986), 293-326.

[Kl-R] S. Klainerman, I. Rodniansky *"Improved regularity results for quasilinear wave equations"* in preparation.

[Kl-Si] S.Klainerman, T. Sideris *"On Almost Global Existence for Nonrelativistic Wave Equations in 3D"* Comm.Pure.Appl.Math. 49 1996, 307-321

[L] O. Liess *"Decay Estimates in Crystal Optics"* Assymptotic Analysis 4(1991), 61-95.

[M] C. Morawetz "The Limiting Amplitude Principle" Comm.Pure.Appl.Math. 15, 1962, 349-362.

[S1] H. Smith *"A parametrix construction for wave equations with $C^{1,1}$ coefficients"* Ann Inst Fourier de Grenoble 48(3) 1994, 797-835.

[S2] H. Smith *" Strichartz and Nullform Estimates for Metrics of Bounded Curvature"* preprint.

[T1] D. Tataru *"Strichartz Estimates for operators with nonsmooth coefficients and the nonlinear wave equation"* To appear in AJM

[T2] D. Tataru *'Strichartz Estimates for second order hyperbolic operators with nonsmooth coefficients III"* preprint