A KIRCHOFF-SOBOLEV PARAMETRIX FOR THE WAVE EQUATION AND APPLICATIONS

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Abstract. We construct a first order, physical space, parametrix for solutions to covariant, tensorial, wave equations on a general Lorentzian manifold. The construction is entirely geometric; that is both the parametrix and the error terms generated by it have a purely geometric interpretation. In particular, when the background Lorentzian metric satisfies the Einstein vacuum equations, the error terms, generated at some point p of the space-time, depend, roughly, only on the flux of curvature passing through the boundary of the past causal domain of p. The virtues of or specific geometric construction becomes apparent in applications to realistic problems. Though our main application is to General Relativity , which we discuss in [Kl-Ro5], another simpler application shown here is to give a gauge invariant proof of the classical regularity result of Eardley-Moncrief [EM1]-[EM2] for the Yang -Mills equations in $\mathbb{R}^{1+3}$.

1. Introduction

We construct a first order, physical space, parametrix for covariant, tensorial, wave equations on a general Lorentzian manifold which is particularly well suited to geometric applications to the Einstein-vacuum and Yang-Mills equations. We give a purely geometric interpretation for both the parametrix and the associated error terms. This fact is particularly well suited to solutions of the Einstein vacuum equations in which case the error term can be shown to depend, roughly, only on the flux of curvature passing through the boundary of the past causal domain of a point p in space-time . Though our main application to General Relativity is not included in this paper, see [Kl-Ro5], we are able nevertheless to illustrate the effectiveness of our construction in the context of the Yang-Mills equations in the Minkowski background $\mathbb{R}^{3+1}$. We reprove the classical regularity result of Eardley-Moncrief with the help of a gauge invariant parametrix formula. This allows us to completely avoid the Crönstrom gauge, which plays an essential role in the Eardley-Moncrief proof.

It is important to emphasize that our parametrix is well suited to provide uniform curvature bounds in specific applications, such as in connection to the break-down criterion of [Kl-Ro5] or the gauge invariant proof of the Eardley-Moncrief result presented here. It is by no means well suited to other applications based on Strichartz

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or bilinear estimates, such as optimal well posedness for nonlinear wave equations for which parametrices based on Fourier integral operators, or wave packets, are clearly better suited. On the other hand, it is equally clear that Fourier space based parametrices are not appropriate in the type of applications we consider here; indeed they are not at all the right tool if one is interested in uniform bounds of solutions to geometric problems.

We give a new proof of the Eardley-Moncrief result based on our new parametrix adapted to gauge invariant wave equation. We recall that the curvature Λ of a Yang -Mills connection satisfies,

$$\square^{(\lambda)}\Lambda_{\alpha\beta} = 2[\Lambda^{\sigma}{}_{\alpha}\,,\,\Lambda_{\sigma\beta}]$$

where $\Lambda_{\alpha\beta} = \partial_\alpha\lambda_\beta - \partial_\beta\lambda_\alpha + [\lambda_\alpha, \lambda_\beta]$, λ a one form defined in Minkowski space $\mathbb{R}^{1+3}$ with values in a Lie algebra $\mathcal{G}$ and $\square^{(\lambda)}$ the corresponding gauge invariant wave operator . We recall the Eardley-Moncrief proof was based on deriving a pointwise estimate for Λ. This was done by approximating $\square^{(\lambda)}$ with the standard wave operator $\square$ and use of the classical Kirchoff formula (4). To deal with the error term $\square^{(\lambda)} - \square$ one had to rely on a particular gauge condition, called Crönstrom gauge. Our gauge invariant parametrix , which generalizes (4), allows us, instead, to estimate the value of Λ at a point p in terms of an error term expressed as an integral along the past null cone $\mathcal{N}^-(p)$ of a geometric expression which depends[1], roughly, on up to two tangential derivatives along $\mathcal{N}^-(pure)$ of the connection λ, or one derivative of its curvature Λ. This fact may seem bad enough to ruin our chances of proving the desired result[2]. The essential point here is that both derivatives are tangential to the light cone. This fact, combined with the Bianchi identities and a subtle cancellation, which depends essentially on the algebraic structure of the nonlinear equation, makes our proof go through. The proof will simply not work without a geometric, gauge invariant form of the error term.

We consider solutions to the covariant, tensorial wave equation

$$\square_{\mathbf{g}}\Psi = F, \tag{1}$$

on a given Lorentz manifold $(\mathbf{M}, \mathbf{g})$. Here Ψ and F are k tensor-fields on a $3+1$ dimensional Lorentz manifold $(\mathbf{M}, \mathbf{g})$ and $\square_{\mathbf{g}}\Psi = \mathbf{g}^{\mu\nu}\mathbf{D}_\mu\mathbf{D}_\nu\Psi$ denotes the covariant wave operator on $\mathbf{M}$, with $\mathbf{D}$ the Levi-Cevita connection defined by $\mathbf{g}$. To simplify the discussion below we consider first the scalar case

$$\square_{\mathbf{g}}\psi = f. \tag{2}$$

In Minkowski space $(\mathbb{R}^{3+1}, \mathbf{m})$ with $\mathbf{m} = \mathrm{diag}\{-1, 1, \ldots, 1\}$ the wave operator on the left hand side of (2) is the standard D'Alembertian $\square = \mathbf{m}^{\alpha\beta}\partial_\alpha\partial_\beta$. The general solution of $\square\psi = f$ can be written in the form,

$$\psi = \psi_f + \psi_0 \tag{3}$$

[1]Through the transport equation (54).

[2]In view of the energy identity we have an a-priori bound on the curvature flux through $\mathcal{N}^-(p)$ We don't have any a-priori for derivatives of Λ.

with ψ_0 a solution of the homogeneous equation $\square\psi_0 = 0$ and ψ_f given by the the Kirchoff formula,

$$\begin{aligned}\psi_f(t,x) &= (4\pi)^{-1}\int_0^t\int_{|x-y|=t-s}|x-y|^{-1}f(s,y)dsd\sigma(y)\\ &= (4\pi)^{-1}\int_{\mathbb{R}_+^{3+1}}\frac{1}{|x-y|}\delta(t-s-|x-y|)f(s,y)dsdy. \qquad (4)\end{aligned}$$

Here $d\sigma(y)$ denotes the area element of the sphere $|x-y| = t-s$ and δ represents the one dimensional Dirac measure supported at the origin. The homogeneous solution ψ_0 is fixed by initial data on the hyperplane $t = 0$.

One can also recast (4) in the form

$$\psi_f(t,x) = (4\pi)^{-1}\int_{\mathbf{R}_+^{3+1}} H(t-s)\delta\big(-(t-s)^2+|x-y|^2\big)f(s,y)dsdy \qquad (5)$$

where $H(t)$ is the Heavyside function supported on the positive real axis and the expression $|x-y|^2-(t-s)^2 = d_0(p,q)^2$ is the square of the Minkowski distance function between the vertex $p = (t,x)$ and the point $q = (s,y)$ in the causal past $\mathcal{J}^-(p)\cap\mathbf{R}_+^{3+1}$ of the point $p\in\mathbf{R}_+^{3+1}$. All attempts to extend Kirchoff's formula to a general four dimensional curved space-time are based on either (4) or (5). Thus the first term in the so called Hadamard parametrix is constructed by replacing the Minkowski distance function d_0 with the Lorentzian distance function $d(p,q)$ defined by the metric $\mathbf{g}$. Thus one can set,

$$\psi_f(p) = (4\pi)^{-1}\int_{\mathcal{J}^-(p)} r(p,q)\,\delta\big(d^2(p,q)\big)\,f(q)\,dv(q) \qquad (6)$$

with dv the volume element of the metric $\mathbf{g}$, and $r(p,q)$ a correction factor which verifies a transport equation along the null boundary of $\mathcal{J}^-(p)$ and such that $r(p,p) = 1$. The integral on the right makes sense for the portion of $\mathcal{J}^-(p)$ which belongs to a neighborhood $\mathcal{D}$ of p where the geodesic distance function $d(p,q)$ is well defined and sufficiently smooth. Typically one requires $\mathcal{D}$ to be causally geodesically convex, i.e. any two causally separated points in $\mathcal{D}$ can be joined by a unique geodesic in $\mathcal{D}$. The local parametrix in $\mathcal{D}$ is then defined

$$\psi_f(p) = (4\pi)^{-1}\int_{\mathcal{J}^-(p)\cap\mathcal{D}} r(p,q)\,\delta\big(d^2(p,q)\big)\,f(q)\,dv(q) \qquad (7)$$

The integral in (7) is supported on the portion of the boundary $\mathcal{N}^-(p)$ of $\mathcal{J}^-(p)$ included in $\mathcal{D}$.

The error term $\square_{\mathbf{g}}\psi_f - f$, however, does not vanish unless $\mathbf{g}$ is the flat metric $\mathbf{m}$. One can improve (6) by making successive corrections based on solving a series of transport equations in $\mathcal{J}^-(p)\cap\mathcal{D}$. In the process the error term can be made as smooth as we wish, for given regularity of f, at the price of requiring higher regularity of the metric $\mathbf{g}$, see [Fried]. Moreover the resulting parametrix, called Hadamard parametrix, is no longer supported just on the boundary of $\mathcal{J}^-(p)$. One obtains a solution of (2) of the form,

$$\psi(p) = \int_{\mathcal{J}^-(p)\cap\mathcal{D}} E_-(p,q)f(q)\,dv(q). \qquad (8)$$

with $E_-(p,q) = r(p,q)\delta\big(d^2(p,q)\big) + \dots$ is the retarded Green function of $\Box_{\mathbf{g}}$

The Hadamard parametrix (8), which requires both infinite smoothness of $\mathbf{g}$ and geodesic convexity for $\mathcal{D}$ is ill suited for applications to nonlinear problems. It turns out that in many situations one does not need the precise representation (8) and that in fact the first order parametrix of type (6) suffices. This fact was first made use of by S. Sobolev, see [Sob], to provide a proof of well-posedness for general second order linear wave equations with variable coefficients. A similar parametrix was later used by Y. C. Bruhat, see [Br], in her famous local existence result for the Einstein vacuum equations. Both [Sob] and [Br] construct their first order parametrices, which we refer to as Kirchoff-Sobolev, based on the flat space formula[3] (4). The generalization of (4) to a curved space-time proceeds from the observation that the function $u_p(s,y) = t - s - |x-y|$ is an optical function, i.e.

$$\mathbf{m}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \tag{9}$$

vanishing precisely on the past null cone $\mathcal{N}^-(p)$ with vertex at $p = (t,x)$ given by the equation $u_p = 0$. Letting $r = |x-y|$ one can easily check that

$$\begin{aligned} \Box\big(r^{-1}\delta(u_p)\big) &= \big(\Box r^{-1}\big)\,\delta(u_p) + (-2L(r^{-1}) + r^{-1}\Box u_p)\delta'(u) \\ &+ \big(\mathbf{m}^{a\beta}\partial_\alpha u_p \partial_\beta u_p\big)\delta''(u) = 4\pi\delta(p), \end{aligned}$$

with $\delta(p)$ the four dimensional Dirac measure supported at p. Indeed the terms involving $\delta''(u_p)$ and $\delta'(u_p)$ both vanish, the first in view of (9) and the second because,

$$-2L(r^{-1}) + r^{-1}\,\Box u_p = 0,$$

with L the null vectorfield along $\mathcal{N}^-(p)$ defined by $L = -\mathbf{m}^{\alpha\beta}\partial_\beta u_p \partial_\alpha$. On the other hand $\delta(u_p)\Box r^{-1} = \delta(u_p)\Delta r^{-1} = 4\pi\delta(p)$.

Based on this one can generalize (4) to a curved space-time by setting,

$$\psi_f(p) = \int_{\mathcal{J}^-(p)\cap\mathcal{D}} a(p,q)\,\delta(u_p(q))\,f(q)\,dv(q) \tag{10}$$

where $u_p = u_p(q)$ is the backward solution to the eikonal equation,

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u\,\partial_\beta u = 0, \tag{11}$$

vanishing on the past null cone $\mathcal{N}^-(p)$, and $a(q) = a(p,q)$ verifies the transport equation similar to that satisfied by r^{-1} in flat space. As in (6) we need to restrict ourselves to a neighborhood $\mathcal{D}$ of p in which solutions to (11) remain smooth.

To explain the restriction to the neighborhood $\mathcal{D}$ to which the integral in (10) is restricted we return for a moment to the initial value problem in flat space-time. In the Minkowski space-time model with the choice of an initial Cauchy hypersurface $\Sigma_0 = \{t = 0\}$ the Kirchoff formula

$$\psi_f(p) = (4\pi)^{-1}\int_{\mathcal{J}^-(p)\cap\mathcal{J}^+(\Sigma_0)} \frac{1}{r(p,q)}\delta\left(u_p(q)\right) f(q)\ dv(q) \tag{12}$$

[3]It is easy to show that the two constructions (6) and (13) differ in fact only by a normalization factor at the vertex p.

with $p=(t,x), q=(s,y)$, $u_p(q)=t-s-|x-y|$ and $r(p,q)=|x-y|$, coincides at point p with the solution of $\Box\psi=f$ with zero initial data at $t=0$. The representation is valid for *any* point p to the future of Σ_0 and the surface of integration

$$\mathcal{N}^-(p)\cap\mathcal{J}^+(\Sigma_0)=\{(s,y):\ t-s=|x-y|,\ s\ge 0\}$$

is smooth with exception of the vertex point p. In a flat space-time model with the Lorentzian manifold $\mathbf{M}=\mathbf{R}\times\Pi_a$, where $\Pi_a=\mathbf{R}^2\times\mathbf{R}/a\mathbf{Z}$ is a flat cylinder of "width" a, the representation (12) also coincides with the solution of the inhomogeneous wave equation with zero initial data at $t=0$, provided that we restrict ourselves to points $p=(t,x)$ such that $t\le a$. For points $p=(t,x)$ with $t>a$ formula (10) no longer[4] represents the solution of the inhomogeneous problem with zero initial data at $t=0$. The null hypersurface $\mathcal{N}^-(p)\cap\mathcal{J}^+(\Sigma_0)$ develops singularities[5] (scars) in the time interval $[0,t-a]$ due to intersecting null geodesics. This shows that the accuracy of the Kirchoff formula in this case is restricted to the neighborhood $\mathcal{D}=\{(t,x):\ 0\le t\le a\}$ of the Cauchy hypersurface Σ_0.

To describe the situation in a general space-time $(\mathbf{M},\mathbf{g})$ we assume that $\mathbf{M}$ is globally hyperbolic, i.e., there exists a Cauchy hypersurface $\Sigma\subset\mathbf{M}$ with the property that each in-extendible past (future) directed causal curve from a point p to the future (past) of Σ intersects Σ once. We denote by $\Sigma_+=\mathcal{J}^+(\Sigma)$ the future set of Σ. By finite speed of propagation the solution $\psi(p)$ of the wave equation $\Box_{\mathbf{g}}\psi=f$ at point $p\in\Sigma_+$ is completely determined by the values of f in $\mathcal{J}^-(p)\cap\Sigma_+$ and initial data for ψ on $\mathcal{J}^-(p)\cap\Sigma$.

Definition 1.1. We will say that $E_-(p,q)$ is the retarded parametrix for $\Box_{\mathbf{g}}$ at p if

$$\psi(p)=\int_{\mathcal{J}^-(p)\cap\Sigma_+}E_-(p,q)\,f(q)\,dv(q)$$

coincides with the solution of the problem $\Box_{\mathbf{g}}\psi=f$ with zero initial data on Σ. We will say that the first term in the expansion of $E_-(p,q)$ – distribution $\mathcal{K}_p^-=a(p,q)\delta\left(u_p(q)\right)$ – is the retarded Kirchoff-Sobolev parametrix.

Let $\mathcal{D}$ be a space-time neighborhood of Σ. The expression

$$\psi_f(p)=\int_{\mathcal{J}^-(p)\cap\Sigma_+}a(p,q)\delta(u_p(q))f(q)dv(q),\qquad p\in\mathcal{D}\tag{13}$$

is the Kirchoff-Sobolev approximation to the solution $\psi(p)$ of the wave equation $\Box_{\mathbf{g}}\psi=f$ with zero initial data on Σ. Clearly ψ_f fails to be a solution to (2) in the non-flat case. We write a general solution of (2) with zero initial data on Σ in the form,

$$\psi(p)=\psi_f(p)+\mathcal{E}_f(p)\tag{14}$$

with $\mathcal{E}_f$ an error term.

In this paper we will:

[4]In fact the correct representation can be obtained by lifting the problem to the covering space $\mathbf{R}\times\mathbf{R}^2\times\mathbf{R}$, applying the Kirchoff formula and taking periodization in the last variable with the period a.

[5]Note that although past null geodesics intersecting, say at $t_*=t-a$ can be extended beyond t_* they no longer belong to the boundary of the causal past of p.

(1) Provide a careful derivation of (13) and (14) for points p in a suitable neighborhood $\mathcal{D}$ of Σ and show that the error term $\mathcal{E}_f$ can be expressed in the form,

$$\mathcal{E}_f(p) = \int_{\mathcal{J}^-(p)\cap\Sigma_+} \mathcal{E}(p,q)\,\delta(u_p(q))\,\psi(q)\,dv(q) \tag{15}$$

where the smooth density $\mathcal{E}(p,q)$ depends only on geometric quantities associated to the null hypersurface $\mathcal{N}^-(p)$.

We should note that classical constructions of the Kirchoff-Sobolev parametrix establish the error term $\mathcal{E}_f$ as explicitly dependent on the metric $\mathbf{g}$ and its derivatives relative to some chosen system of coordinates. To our knowledge the fact that $\mathcal{E}_f$ is supported only on the boundary of the past set $\mathcal{J}^-(p)$ does not seem to have been fully recognized and used in applications. A similar observation was, prior to this work, communicated to us verbally by V. Moncrief. His claim, based on Friedlander's treatment of the Hadamard parametrix, was the starting point of our own investigations.

(2) Extend formulas (13) and (15) to the covariant tensorial wave equation (1).

Once again the classical treatment of the tensorial wave equation introduces additional coordinate dependent error terms. Our approach is entirely covariant.

(3) Provide a minimum set of conditions for the local geometry of $\mathbf{M}$ near p to ensure that the representation (13) and (14) holds true at p. We also make use of our recent results from [Kl-Ro4] to show that for the Einstein vacuum space-times $(\mathbf{M}, \mathbf{g})$, with vanishing Ricci curvature, formulas (13) and (14) can be extended to points p at distance t_* from Σ, with t_* dependent, essentially, only on the L^2 norm of curvature[6] of $\mathbf{g}$.

(4) Our formula can be easily adapted to gauge invariant wave equations. In section 4 of the paper we write down such a formula and show how it can be used to give a very simple proof of the Eardley-Moncrief global existence result for the Yang-Mills equation in the 3+1 dimensional Minkowski space, see [EM1],[EM2]. The remarkable fact about our approach is that it is entirely gauge independent; we don't need to specify any gauge condition[7].

The size of the neighborhood $\mathcal{D}$, mentioned above, is first and foremost constrained by the condition that the optical function u is smooth. In the case of a Riemannian manifold the distance function from a point p is smooth in a geodesically convex neighborhood of p whose size can be evaluated in terms of the C^2 norm of the metric $\mathbf{g}$, as measured in a given system of coordinates. Alternatively, by a theorem of Cheeger, the size of this neighborhood depends only on the pointwise bounds for

[6]Note that classically the construction of a Kirchoff-Sobolev parametrix could only be justified for points p such that $\mathcal{J}^-(p)\cap\Sigma$ belongs to a geodesically convex neighborhood $\mathcal{D}$ of p. As we note below this requires uniform control for at lest two derivatives of the metric.

[7]The method of [EM1],[EM2] was heavily dependent on the choice of a Crönstrom gauge.

the Riemann curvature tensor and a lower bound on the volume of a unit geodesic ball. For similar reasons the construction of a solution u_p to (11) is restricted to a geodesically convex neighborhood[8] $\mathcal{D}$ of p. Unlike the Riemannian case, however, a purely geometric characterization of the size of a geodesically convex neighborhood of a point p is not available and thus all known parametrix constructions for wave equations had to be restricted to domains $\mathcal{D}$ whose size is determined by the C^2 norm of the metric $\mathbf{g}$ in a given system of coordinates. Thus the Kirchoff-Sobolev representation would only hold for points p at maximal distance t_* from Σ with t_* dependent on the C^2 norm of the metric. As we shall explain below, such demand on the regularity of the metric would make the Kirchoff-Sobolev formula impossible to apply to realistic nonlinear situations, such as Einstein's field equations.

The importance of the classical C^2 condition becomes apparent upon examining the regularity of the null boundary $\mathcal{N}^-(p)$ of the causal past $\mathcal{J}^-(p)$. This set is ruled by past null geodesics $\gamma(s)$ originating from p and terminating at the points $\gamma(s_*)$ beyond which one can find a time-like curve connecting p and $\gamma(s)$ with $s > s_*$, see [HE]. Regularity of $\mathcal{N}^-(p)$ breaks down precisely at the terminal points $\gamma(s_*)$. There are two reasons for the existence of a terminal point $\gamma(s_*)$.

(1) $\gamma(s_*)$ is a conjugate point.
(2) $\gamma(s_*)$ is a point of intersection of two different null geodesics.

The existence of conjugate points is governed by the Jacobi equation for the Hessian $\mathbf{D}^2 u$ of the optical function u,

$$\mathbf{D_L}(\mathbf{D}^2 u) + (\mathbf{D}^2 u)^2 = \mathbf{R}(\,\cdot\,, \mathbf{L}, \,\cdot\,, \mathbf{L})$$

with $\mathbf{L} = -\mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha$ the null geodesic vectorfield along $\mathcal{N}^-(p)$ and $\mathbf{R}$ the curvature tensor of $\mathbf{g}$. This formula indicates that, at least as far as the conjugate points are concerned, the terminal value of the affine parameter s_* can be bounded below by an upper bound on sectional curvature which, in turn, can be controlled by a C^2 bound on the metric.

Uniform bounds of of the curvature tensor $\mathbf{R}$, or C^2 bounds for the metric $\mathbf{g}$, are however not very useful in applications to nonlinear wave equations. For example in the classical local existence result for the Einstein vacuum equations [Br], which is based on Kirchoff-Sobolev formula, the C^2 requirement is by itself worse[9] when compared to the result in [HKM] based on the Sobolev norm H^s, $s > 5/2$. It is for this reason alone that the Kirchoff-Sobolev parametrix has been abandoned in all rigorous work on nonlinear wave equations in favor of energy estimates and Sobolev inequalities. The main goal of our paper is to revive the Kirchoff-Sobolev parametrix by constructing it and showing that in the particular case of the Einstein vacuum equations,

$$\mathbf{R}_{\alpha\beta} = 0,$$

[8]Defined as the image of the exponential map : $T_p\mathbf{M} \to \mathbf{M}$ restricted to the largest convex subset of $T_p\mathbf{M}$ where it is a diffeomorphism.

[9]Additional losses of derivatives lead to a C^5 result in [Br].

it is well-defined under much less stringent assumptions. For this task we rely in an essential way on the results in [Kl-Ro1]–[Kl-Ro3] which show[10] that the radius of conjugacy along $\mathcal{N}^-(p)$, expressed relative to an affine parameter of $\mathbf{L}$, depends only on the size of the geodesic flux of curvature[11] $\mathcal{F}_p$ along $\mathcal{N}^-(p)$. These results are complemented by our recent work [Kl-Ro4] where we establish the remaining part of a lower bound on the radius of injectivity of $\mathcal{N}^-(p)$, i.e., control of intersecting null geodesics from p, expressed relative to a given time function. We achieve this by assuming, in addition to the above mentioned bound on the curvature flux, the existence of a coordinate system x^α in $\mathcal{D}$ relative to which the metric $\mathbf{g}$ is pointwise close to the flat Minkowski metric.

Acknowledgment. We would like to thank V. Moncrief for fruitful discussions in connection with our work. He was first to point out to us that a formula of type (14) with an error term $\mathcal{E}_f$ of the form (15), supported on the boundary of the past of p, should hold true. His derivation, based on Hadamard's parametrix construction as formulated in [Fried], differs however significantly from ours. We would also like to point out that our invariant derivation of the Eardley-Moncrief global regularity result for the $3+1$ dimensional Yang-Mills equations answers a question first raised to us by him.

2. Basic definitions and main formula

2.1. **Null cones.** Consider a spacelike hypersurface Σ, a point p to its future Σ_+ and $\mathcal{J}^-(p)$ its causal past. We start by assuming the following local hyperbolicity condition for the pair (Σ, p):

A1. *All past causal curves initiating at points in a small neighborhood of $\mathcal{J}^-(p)$ intersect Σ at precisely one point.*

Let $\mathcal{N}^-(p)$ be the null boundary of $\mathcal{J}^-(p)$. In general $\mathcal{N}^-(p)$ is an achronal, Lipschitz hypersurface. It is ruled by the null geodesics[12] from p, corresponding to all past null directions in the tangent space $T_p\mathbf{M}$. These null geodesics can be parametrized by fixing a future unit time-like vector $\mathbf{T}_p$ at p. Then, for every direction $\omega \in \mathbb{S}^2$, with $\mathbb{S}^2$ denoting the standard sphere in $\mathbb{R}^3$, consider the null vector ℓ_ω in $T_p(\mathbf{M})$,

$$\mathbf{g}(\ell_\omega, \mathbf{T}_p) = 1, \tag{16}$$

and associate to it the past null geodesic $\gamma_\omega(s)$ with initial data $\gamma_\omega(0) = p$ and $\dot\gamma_\omega(0) = \ell_\omega$. We can choose the parameter s in such a way so that $\mathbf{L} = \dot\gamma_\omega(s)$ is geodesic. Thus,

$$\mathbf{D}_\mathbf{L}\mathbf{L} = 0, \qquad \mathbf{g}(\mathbf{L},\mathbf{L}) = 0, \qquad \text{and, at point p,} \quad \mathbf{g}(\mathbf{L}, \mathbf{T}_p) = 1 \tag{17}$$

[10]Properly speaking the results in [Kl-Ro1]–[Kl-Ro3] do not consider the vertex p yet the methods used in those papers can be shown to extend to cover the case of interest here. In fact, this forms the subject of the Q. Wang's thesis, Princeton University, 2006.

[11]This is an appropriate L^2 integral of the tangential components of the curvature tensor along $\mathcal{N}^-(p)$, called curvature flux, which will be defined below.

[12]Every point in $\mathcal{N}^-(p) \setminus \{p\}$ can be reached from p by a past null geodesic in $\mathcal{N}^-(p)$.

As mentioned in the introduction the null cone $\mathcal{N}^-(p)$ is smooth as long as the exponential map $(s,\omega)\to\gamma_\omega(s)$ is a local diffeomorphism and no two geodesics, corresponding to different direction $\omega\in\mathbb{S}^2$, intersect. Thus for each $\omega\in\mathbb{S}^2$ either $\gamma_\omega(s)$ remains on the boundary of $\mathcal{J}^-(p)$ for all positive values of s or there exists a a value $s_*(\omega)$ beyond which the points $\gamma_\omega(s)$ are no longer on the boundary of $\mathcal{J}^-(p)$ but rather in its interior, see [HE]. Thus $\mathcal{N}^-(p)$ is a smooth manifold at all points except the vertex p and the terminal points of its past null geodesic generators. Indeed, at a terminal point q there exists a null geodesic through q which fails to be in $\mathcal{N}^-(p)$ past q. This implies that the tangent space $T_q(\mathcal{N}^-(p))$ contains the past tangent direction of the null geodesic but not its opposite. This means that $\mathcal{N}^-(p)$ must be singular at q. In what follows we shall denote by $\dot{\mathcal{N}}^-(p)$ the regular part of $\mathcal{N}^-(p)$, that is the part with its terminal points removed. Clearly the null geodesic vectorfield $\mathbf{L}$ is well-defined and smooth on $\dot{\mathcal{N}}^-(p)$.

The parameter s in the definition of γ_ω is an affine parameter on $\dot{\mathcal{N}}^-(p)$, i.e.

$$\mathbf{L}(s)=1,\qquad s(p)=0. \tag{18}$$

Let γ denote the degenerate metric induced by $\mathbf{g}$ on $\dot{\mathcal{N}}^-(p)$. Clearly $\gamma(\mathbf{L},X)=0$ for any $X\in T\dot{\mathcal{N}}^-(p)$. Let χ denote the null second fundamental form of $\dot{\mathcal{N}}^-(p)$,

$$\chi(X,Y)=\mathbf{g}(\mathbf{D}_X\mathbf{L},Y). \tag{19}$$

where X,Y are vector-fields tangent to $\dot{\mathcal{N}}^-(p)$ and $\mathbf{D}$ denote the covariant derivative on $(\mathbf{M},\mathbf{g})$. Clearly χ is symmetric and $\chi(\mathbf{L},X)=0$ for any $X\in T\dot{\mathcal{N}}^-(p)$. This allows us to define $\mathrm{tr}\chi$ as the trace of χ relative to γ.

Given a point $q\in\dot{\mathcal{N}}^-(p)\setminus\{p\}$, we can define a null conjugate $\underline{\mathbf{L}}$ to $\mathbf{L}$ such that,

$$\mathbf{g}(\mathbf{L},\underline{\mathbf{L}})=-2,\qquad \mathbf{g}(\mathbf{L},\mathbf{L})=\mathbf{g}(\underline{\mathbf{L}},\underline{\mathbf{L}})=0. \tag{20}$$

and further complement it by vectors (e_1,e_2) with the property that

$$\mathbf{g}(\mathbf{L},e_a)=\mathbf{g}(\underline{\mathbf{L}},e_a)=0,\qquad \mathbf{g}(e_a,e_b)=\delta_{ab},\qquad a,b=1,2. \tag{21}$$

The vectors $(\mathbf{L},\underline{\mathbf{L}},e_1,e_2)$ can be locally extended to a neighborhood of a point $q\in\dot{\mathcal{N}}^-(p)\setminus\{p\}$ to form a smooth local null frame. Relative to such a frame the only non-vanishing components of the null second fundamental form χ are $\chi_{ab}=\mathbf{g}(\mathbf{D}_{e_a}\mathbf{L},e_b)=\chi_{ba}$. We can introduce the other frame coefficients,

$$\underline{\chi}_{ab}=\mathbf{g}(\mathbf{D}_{e_a}\underline{\mathbf{L}},e_b),\qquad \zeta_a=\frac{1}{2}\mathbf{g}(\mathbf{D}_a\mathbf{L},\underline{\mathbf{L}}),\qquad \underline{\eta}_a=\frac{1}{2}\mathbf{g}(e_a,\mathbf{D}_{\mathbf{L}}\underline{\mathbf{L}}) \tag{22}$$

Note that, in general, $\underline{\chi}_{ab}$ is not symmetric.

Remark. A canonical way to define a null geodesic conjugate is to take $\underline{\mathbf{L}}$ the unique null vectorfield orthogonal to the level surfaces S_s defined by the affine parameter s. We refer to the corresponding null pair as a null geodesic pair. We can also choose e_1,e_2 to be tangent to S_s. Note that in that case $\underline{\chi}$ is symmetric. We also note that in a neighborhood of p where $\mathcal{N}^-(p)$ coincides with its regular part $\dot{\mathcal{N}}^-(p)$ the geodesic null frame defined above is smooth away from the point p.

For the purpose of constructing our Kirchoff-Sobolev parametrix we shall make, in addition to **A1** the following assumption.

A2. *We assume that $\mathcal{N}^-(q)$ coincides with $\dot{\mathcal{N}}^-(q)$ past the space-like hypersurface Σ for any point q in a neighborhood of p.*

2.2. **Optical function.** To make sense of our Kirchoff-Sobolev formula we need to define an optical function[13] u, in a neighborhood of $\dot{\mathcal{N}}^-(p)$, such that it vanishes identically on $\dot{\mathcal{N}}^-(p)$. We define u uniquely relative to the time-like vector $\mathbf{T}_p$ as follows:

Let $\epsilon > 0$ a small number and $\Gamma_\epsilon : (1-\epsilon, 1+\epsilon) \to \mathbf{M}$ denote the timelike geodesic from p such that $\Gamma_\epsilon(1) = p$ and $\Gamma'_\epsilon(1) = \mathbf{T}_p$. From every point q of Γ_ϵ let $\mathcal{N}^-(q)$ be the boundary of the past set of q. In view of assumption **A2** for all sufficiently small $\epsilon > 0$, $\mathcal{N}^-(q)$ coincides with its regular part $\dot{\mathcal{N}}^-(q)$ to the future Σ^+ of Σ.

We now set u to be the function, constant on each $\dot{\mathcal{N}}^-(q)$, such that for $q = \Gamma_\epsilon(t)$,

$$u|_{\dot{\mathcal{N}}^-(q)} = t - 1.$$

This defines a smooth function u which vanishes on $\dot{\mathcal{N}}^-(p)$ and verifies the eikonal equation (11)

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u\, \partial_\beta u = 0,$$

in a neighborhood $\mathcal{D}_\epsilon$ of $\dot{\mathcal{N}}^-(p) \cap \Sigma^+$. Observe that the null geodesic vectorfield $\mathbf{L} = \mathbf{g}^{\alpha\beta}\partial_\beta u\, \partial_\alpha$ extends the vectorfield in (17) to $\mathcal{D}$. It verifies the normalization condition,

$$\mathbf{g}(\mathbf{L}, \mathbf{T}_p) = \mathbf{T}_p(u) = 1,$$

at all points of $\mathcal{D}_\epsilon$. We can thus extend the definition (19) of the null second fundamental form χ and its trace $\mathrm{tr}\chi$ at every point in $\mathcal{D}_\epsilon$.

We can introduce local coordinates around any point in $r \in \mathcal{D}_\epsilon$ by considering the unique null geodesic $\gamma_{\omega,q}$, with $\omega \in \mathbb{S}^2$, which initiates at $q \in \Gamma_\epsilon$ and passes through r at value s of its affine parameter. Denoting by u the value corresponding to the null cone $\dot{\mathcal{N}}^-(q)$ we see that r is determined by the coordinates u, s and $\omega \in \mathbb{S}^2$.

2.3. **Dirac measure on $\dot{\mathcal{N}}^-(p)$.** Given our smooth optical function u, defined in the neighborhood $\mathcal{D}_\epsilon$ of $\dot{\mathcal{N}}^-(p) \cap \Sigma^+$, and a distribution μ on the real line $\mathbb{R}$, supported at the origin, we can define the pull-back distribution $u^*(\mu) = \mu \circ u$ on $\mathcal{D}_\epsilon \subset \mathbf{M}$ in the usual sense of distribution theory. In the particular case when μ is either the Dirac measure δ_0 or its derivatives $\delta'_0, \delta''_0, \ldots$, we denote the corresponding distributions on $\mathbf{M}$ by $\delta(u)$, $\delta'(u)$, $\delta''(u), \ldots$. We can thus make sense of calculations such as,

$$\mathbf{D}_\alpha \delta(u) = \delta'(u)\mathbf{D}_\alpha u, \qquad \mathbf{D}_\alpha \mathbf{D}_\beta\big(\delta(u)\big) = \delta''(u)\mathbf{D}_\alpha u \mathbf{D}_\beta u + \delta'(u)\mathbf{D}_\alpha \mathbf{D}_\beta u$$

[13] i.e. a function which verifies (11)

Clearly $\delta(u), \delta'(u), \ldots$ are supported on $\dot{\mathcal{N}}^-(p) \cap \mathcal{D}_\epsilon$. We can use the definition of $\delta(u)$ to define the integral along $\dot{\mathcal{N}}^-(p)$ of any continuous function f supported in $\mathcal{D}_\epsilon$ as follows.

Definition. Given a continuous function f supported in $\mathcal{D}_\epsilon$ we define its integral on $\dot{\mathcal{N}}^-(p)$ by,

$$\int_{\dot{\mathcal{N}}^-(p)} f = < \delta(u), f > \tag{23}$$

Proposition 2.4. *The definition* (23) *depends only on the restriction of f to* $\dot{\mathcal{N}}^-(p)$ *and the normalization condition* (16) *used in the definition of the null geodesic generator* $\mathbf{L}$.

Proof : We may assume without loss of generality that f is supported in the domain $\mathcal{D}_\epsilon$, which can be parametrized by the coordinates u, s and $\omega \in \mathbb{S}^2$ as described above. We can the easily calculate, according to the definition of $\delta(u)$ and coarea formula,

$$< \delta(u), f > = \int_0^\infty \int_{\mathbb{S}^2} f(0, s, \omega) ds da_s$$

where da_s denotes the area element on the 2- surfaces S_s of constant s. ■

2.5. **Kirchoff-Sobolev parametrix.** Consider $\mathbf{J}_p$ to be a fixed k-tensor at p and let $\mathbf{A}$ be the unique k -tensor-field defined along $\dot{\mathcal{N}}^-(p)$ which verifies the linear transport equation,

$$\mathbf{D_L A} + \frac{1}{2}\mathrm{tr}\chi \mathbf{A} = 0, \qquad (s\mathbf{A})(p) = \mathbf{J}_p \tag{24}$$

with s the affine parameter (18). The tensor-field $\mathbf{A}$ can be extended smoothly[14] to a small neighborhood of $\dot{\mathcal{N}}^-(p)$. We can now define the distribution, or current, in Σ^+,

$$< \mathbf{A}\delta(u), \mathbf{F} > = < \delta(u), \mathbf{g}(\mathbf{A}, \mathbf{F}) > \tag{25}$$

for an arbitrary, smooth, k-tensor-field $\mathbf{F}$ supported in Σ^+. Here $\mathbf{g}(\mathbf{A}, \mathbf{F})$ denotes the full contraction of the k -tensor-fields $\mathbf{A}$ and $\mathbf{F}$ with respect to the space-time metric $\mathbf{g}$. Observe that the current $\mathbf{A}\delta(u)$ depends only on the choice of $\mathbf{T}_p$ and $\mathbf{J}_p$ and not on the particular extensions of u and $\mathbf{A}$.

In what follows we identify the space of k-tensors at p and its dual with the help of the metric $\mathbf{g}$.

Definition. *We call* $\mathcal{K}^-_p = \mathcal{K}^-_{p,\mathbf{J}_p}$, *a k-tensor-field distribution with values in the space of k-tensors at p, defined by the formula* $\mathcal{K}^-_{p,\mathbf{J}_p} = \mathbf{A}\delta(u)$, *with* $\mathbf{A}$ *defined by* (24), *the retarded Kirchoff-Sobolev parametrix at the point p, corresponding to*

[14]We can in fact extended it canonically by solving the same transport equation along $\dot{\mathcal{N}}^-(q)$, with $q \in \Gamma_\epsilon$ and initial data $s\mathbf{A}(q) = \mathbf{J}_q$ where J_q is an arbitrary smooth tensor-field coinciding with $\mathbf{J}_p$ at $p = q$ and s the afine parameter along $\dot{\mathcal{N}}^-(q)$. Note that, so defined, the tensor-field $\mathbf{A}$ is smooth away from the axis Γ_ϵ.

$\mathbf{J}_p$. *If* Ψ *is a solution of the equation* $\square_{\mathbf{g}}\Psi = \mathbf{F}$, *with* $\mathbf{F}$ *supported in* Σ^+, *we denote by* $\Psi_{\mathbf{F},\mathbf{J}_p}(p)$ *the k-tensor at p defined by the integral,*

$$\Psi_{\mathbf{F},\mathbf{J}_p}(p) = < \mathcal{K}^-_{p,\mathbf{J}_p}, \mathbf{F} > = \int_{\dot{\mathcal{N}}^-(p)} \mathbf{g}(\mathbf{A}, \mathbf{F}). \tag{26}$$

In the case of the scalar wave equation $\square_{\mathbf{g}}\psi = f$ we can choose $\mathbf{A}$ to be the scalar solution of (24) with initial data $(s\mathbf{A})(p) = 1$. In that case we have $\mathcal{K}^-_p = \mathbf{A}\delta(u)$ and

$$\psi_f(p) = < \mathcal{K}^-_p, f > = \int_{\dot{\mathcal{N}}^-(p)} \mathbf{A}f.$$

In the particular case of Minkowski space we can easily identify $\mathbf{A} = \mathbf{A}_p(q)$ with the term $|x-y|^{-1}$ where $q = (s, y) \in \mathcal{N}^-(p)$ and $p = (t, x)$.

2.6. **Time foliation near vertex.** Returning to the construction of u in subsection (2.2) we observe that the parameter t along the geodesic Γ_ϵ can be extended to a local, equidistant[15] time foliation Σ_t, $t \in [1-\epsilon, 1+\epsilon]$ which covers a whole neighborhood of the point p, such that $p \in \Sigma_1$. Indeed, starting with a fixed spacelike hypersurface Σ_1 through p, orthogonal to the future unit timelike vectorfield $\mathbf{T}_p$, we can define this geodesic foliation using the timelike geodesics normal to Σ_1. In particular, for all $t \in [1-\epsilon, 1]$, if we denote by Ω_ϵ the set

$$\Omega_\epsilon = \big(\mathcal{J}^-(p) \cap \Sigma_+\big) \setminus \cup_{t\in[1-\epsilon,1]} \Sigma_t \tag{27}$$

then its boundary is given by

$$\partial\Omega_\epsilon = \mathcal{N}^-_\epsilon(p) \cup D_{1-\epsilon} \cup D$$

where $\mathcal{N}^-_\epsilon(p)$ is the portion of $\mathcal{N}^-(p)$ to the future of Σ and the past of $\Sigma_{1-\epsilon}$, $D_{1-\epsilon} = \mathcal{J}^-(p) \cap \Sigma_{1-\epsilon}$ and $D = \mathcal{J}^-(p) \cap \Sigma$.

Let $\mathbf{T} = \mathbf{D}t$ denote the future, unit normal to the foliation Σ_t, defined in a neighborhood of p. We define the null lapse function φ and the second fundamental form k associated to Σ_t:

$$\varphi^{-1} = \mathbf{T}(u) = \mathbf{g}(\mathbf{L}, \mathbf{T}), \quad k(X, Y) = \mathbf{g}(\mathbf{D}_X\mathbf{T}, Y), \quad \forall X, Y \in T\Sigma_t. \tag{28}$$

Clearly $\varphi(p) = 1$. Since $\mathbf{T}$ is a locally smooth vectorfield, k is a smooth symmetric 2-tensor. In particular,

$$\|k\|_{L^\infty} \le C$$

for some constant C. Similarly, since u is a smooth optical function and $\varphi(p) = 1$, the lapse φ is a smooth bounded function in a neighborhood of p. in particular,

$$|\varphi(q) - 1| \to 0, \quad q \to p. \tag{29}$$

We now recall the Raychaudhuri equation satisfied by $\mathrm{tr}\chi$ along $\dot{\mathcal{N}}^-(p)$,

$$\frac{d}{ds}(\mathrm{tr}\chi) + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2 - \mathbf{Ric}(\mathbf{L}, \mathbf{L}). \tag{30}$$

with s the afine parameter of $\mathbf{L}$ and $\hat{\chi}$ the traceless part of χ.

[15]With the lapse function of the foliation identically one.

The behavior of the function trχ at the vertex p is determined by the conditions

$$(s\mathrm{tr}\chi)(p) = 2, \qquad \hat{\chi}(p) = 0. \tag{31}$$

Integrating the Raychaudhuri equation one can easily deduce that,

$$|\mathrm{tr}\chi(q) - \frac{2}{s}| \to 0, \quad q \to p. \tag{32}$$

Consider the time function t restricted to $\mathcal{N}^-(p)$. Then

$$\frac{\partial t}{\partial s} = \mathbf{L}(t) = \mathbf{g}(\mathbf{L}, \mathbf{T}) = \varphi^{-1} \tag{33}$$

The area $|S_t(p)|$ of the 2-d surfaces $S_t(p) = \Sigma_t \cap \mathcal{N}^-(p)$ obeys the equation

$$\frac{d}{dt}|S_t(p)| = \int_{S_t(p)} \varphi \, \mathrm{tr}\chi \, da_\gamma.$$

This and the behavior of trχ and ϕ near p ($t(p) = 1$) imply that

$$|S_t(p)| = 4\pi(t-1)^2 + O(|t-1|^3) \tag{34}$$

On the other hand from (33) and (29), $t - 1 = s + o(s)$, which implies that

$$|S_t(p)| = 4\pi s^2 + o(s^2) \tag{35}$$

We shall also make use of the following simple variation of proposition 2.4.

Proposition 2.7. *Let t be a regular time function defined on $\dot{\mathcal{N}}^-(p)$ with $t(p) = 1$ and equal $t_0 < 1$ on $\Sigma_0 \cap \dot{\mathcal{N}}^-(p)$, where Σ_0 is an arbitrary spacelike hypersurface on $\dot{\mathcal{N}}^-(p) \cap \Sigma^+$. Assume that $\varphi = \frac{dt}{ds} < 0$. Then, for every test function f, compactly supported in $\mathcal{J}^+(\Sigma_0)$*

$$< \delta(u), \psi >= \int_{t_0}^1 \int_{S_t} f\varphi dt da_t \tag{36}$$

where S_t denotes the level surfaces of t and da_t the corresponding area element.

Proof : The result follows easily by first extending t and u to a neighborhood $\mathcal{D}$ of $\dot{\mathcal{N}}^-(p) \cap \Sigma^+$ and the applying the coarea formula as above. ∎

2.8. **Statement of the result.** We consider a space-like hypersurface $\Sigma \subset \mathbf{M}$ and a point $p \in \Sigma^+ = \mathcal{J}^+(\Sigma)$ such that the assumptions **A1-A2** are satisfied.

Theorem 2.9. *Let Ψ be a solution of the equation $\square_{\mathbf{g}}\Psi = \mathbf{F}$ with $\mathbf{F}$ a k-tensor-field supported in Σ^+. Then for any k-tensor $\mathbf{J}_p$ at p,*

$$\Psi(p) = \Psi_{\mathbf{F},\mathbf{J}_p}(p) + \int_{\dot{\mathcal{N}}^-(p)} \mathbf{g}(\mathcal{E}, \Psi), \tag{37}$$

where

$$\Psi_{\mathbf{F},\mathbf{J}_p}(p) =< \mathcal{K}^-_{p,\mathbf{J}_p}, \mathbf{F} >= \int_{\dot{\mathcal{N}}^-(p)} \mathbf{g}(\mathbf{A}, \mathbf{F})$$

and $\mathbf{A}$ verifies (24). The smooth error term $\mathcal{E}$ depends only on $\mathbf{J}_p$, the geometry of the truncated null cone $\dot{\mathcal{N}}^-(p) \cap \Sigma^+ \subset \mathbf{M}$, and the ambient spacetime curvature $\mathbf{R}$ restricted to $\dot{\mathcal{N}}^-(p)$.

In the particular case of a scalar wave equation $\square_{\mathbf{g}}\psi = f$, $\mathbf{A}$ and $\mathcal{E}$ are scalar functions on $\dot{\mathcal{N}}^-(p)$ and,

$$\psi(p) = \psi_f(p) + \int_{\mathcal{N}^-(p)} \mathcal{E}\psi,$$

The precise expression of the error term $\mathcal{E}$ will be given in Theorem 3.11

3. Derivation of Kirchoff-Sobolev formula

3.1. **Covariant derivatives of space-time tensors.** As is well known there is no canonical way to define a restriction of the space-time covariant derivative $\mathbf{D}$ to a null hypersurface. This is due to the absence of a canonical projection of a tangent space $T_q\mathbf{M}$, $q \in \dot{\mathcal{N}}^-(p)$, onto the tangent space $T_q(\dot{\mathcal{N}}^-(p))$. This projection can be fixed, however, by a choice of a null conjugate $\underline{\mathbf{L}}$, i.e. a null vector such that $\mathbf{g}(\mathbf{L}, \underline{\mathbf{L}}) = -2$. With this choice we define an induced covariant derivative ${}^{(\underline{\mathbf{L}})}D$ on $\dot{\mathcal{N}}^-(p)$:

$${}^{(\underline{\mathbf{L}})}D_X Y = \mathbf{D}_X Y + \frac{1}{2}\chi(X,Y)\underline{\mathbf{L}}, \qquad \forall X, Y \in T\dot{\mathcal{N}}^-(p)$$

For example, if we choose X, Y to be the elements $(e_a)_{a=1,2}$ of a null frame $(\mathbf{L}, \underline{\mathbf{L}}, e_1, e_2)$,

$$\mathbf{D}_a e_b = {}^{(\underline{\mathbf{L}})}D_a e_b + \frac{1}{2}\chi_{ab}\,\underline{\mathbf{L}}.$$

We now make sense of covariant derivatives of space-time tensors along $\dot{\mathcal{N}}^-(p)$. We start by defining a covariant derivative $\hat{\mathbf{D}}$ of a space-time 1-form A_μ defined on $\dot{\mathcal{N}}^-(p)$. Thus we view A as a section of the vector bundle[16] $T^*\mathbf{M}$ over $\dot{\mathcal{N}}^-(p)$, endowed with the induced covariant derivative ${}^{(\underline{\mathbf{L}})}D$. We interpret the covariant derivative $\hat{\mathbf{D}}A$ of A along $\mathcal{N}^-(p)$ as a 1-form on $\dot{\mathcal{N}}^-(p)$ with values in $T^*\mathbf{M}$. Thus, for every vectorfield $X \in T\mathcal{N}^-(p)$ and any vectorfield Z in $T\mathbf{M}$,

$$\hat{\mathbf{D}}A(X;Z) = \hat{\mathbf{D}}_X A(Z) := X\big(A(Z)\big) - A(\mathbf{D}_X Z)$$

We also write,

$$(\hat{\mathbf{D}}_X A)_\mu = X^a \mathbf{D}_a A_\mu, \qquad \forall X \in T\dot{\mathcal{N}}^-(p).$$

We define $\hat{\mathbf{D}}^2 A$, an $\dot{\mathcal{N}}^-(p)$ 2-tensor of second covariant derivatives of A along $\dot{\mathcal{N}}^-(p)$ with values in $T^*\mathbf{M}$, by the formula,

$$\hat{\mathbf{D}}^2 A(X,Y;Z) = (\hat{\mathbf{D}}_X \hat{\mathbf{D}}A)(Y;Z) = X(\hat{\mathbf{D}}A(Y;Z)) - \hat{\mathbf{D}}A({}^{(\underline{\mathbf{L}})}D_X Y;Z) - \hat{\mathbf{D}}A(Y;\mathbf{D}_X Z)$$

or simply,

$$\hat{\mathbf{D}}^2 A_\mu(X,Y) = (\hat{\mathbf{D}}_X(\hat{\mathbf{D}}_Y A))_\mu - (\hat{\mathbf{D}}_{{}^{(\underline{\mathbf{L}})}D_X Y} A)_\mu$$

These definitions can be easily extended to higher covariant derivatives along $\dot{\mathcal{N}}^-(p)$ and to higher order tensors A.

[16]with the covariant derivative denoted by $\mathbf{D}$

3.2. **Kirchoff-Sobolev current.** Consider the current $\mathcal{K}^-_{p,\mathbf{J}_p} = \mathbf{A}\delta(u)$ defined in (25). Recall that $\mathbf{J}_p$ is an arbitrary k-tensor at p and $\mathbf{A}$ is a k-tensor-field verifying the transport equation, along $\dot{\mathcal{N}}^-(p)$,

$$\mathbf{D_L A} + \frac{1}{2}\mathbf{A}\mathrm{tr}\chi = 0, \qquad s\mathbf{A}(s)|_{s=0} = \mathbf{J}_p. \tag{38}$$

Also $\mathbf{L} = \mathbf{g}^{\mu\nu}\partial_\nu u\,\partial_\mu$, $\mathbf{g}^{\mu\nu}\partial_\nu u\,\partial_\mu u = 0$ and $\mathbf{L}(s) = 1$. Let $\underline{\mathbf{L}}$ be an arbitrary local null conjugate to $\mathbf{L}$, i.e. $\underline{\mathbf{L}}(u) = \mathbf{g}(\mathbf{L},\underline{\mathbf{L}}) = -2$. Calculating relative to an arbitrary null frame we easily check that $\square_{\mathbf{g}} u = \mathrm{tr}\chi$. Formally we thus have,

$$\begin{aligned}
\square_{\mathbf{g}}\big(\mathbf{A}\delta(u)\big) &= \square_{\mathbf{g}}\mathbf{A}\delta(u) + (\mathbf{g}^{\mu\nu}\mathbf{D}_\mu\mathbf{A}\mathbf{D}_\nu u + \mathbf{A}\square_{\mathbf{g}} u)\delta'(u) + \mathbf{A}(\mathbf{g}^{\mu\nu}\partial_\nu u\,\partial_\mu u)\delta''(u)\\
&= \square_{\mathbf{g}}\mathbf{A}\delta(u) + (-\underline{\mathbf{L}}(u)\mathbf{D_L A} + \mathbf{A}\square_{\mathbf{g}} u)\delta'(u)\\
&= \square_{\mathbf{g}}\mathbf{A}\delta(u) + 2(\mathbf{D_L A} + \frac{1}{2}\mathbf{A}\mathrm{tr}\chi)\delta'(u)
\end{aligned}$$

Hence,

$$\square_{\mathbf{g}}\big(\mathbf{A}\delta(u)\big) = \square_{\mathbf{g}}\mathbf{A}\delta(u) + 2(\mathbf{D_L A} + \frac{1}{2}\mathbf{A}\mathrm{tr}\chi)\delta'(u). \tag{39}$$

Observe that the above calculation does not depend on the choice of $\underline{\mathbf{L}}$.

Since

$$\mathbf{D_L}(s\mathbf{A}) = -\frac{1}{2}(\mathrm{tr}\chi - \frac{2}{s})s\mathbf{A} \tag{40}$$

we have, in view of (32), that along $\dot{\mathcal{N}}^-(p)$,

$$|s\mathbf{A}(q) - \mathbf{J}_p| \to 0, \qquad s \to 0. \tag{41}$$

We shall next apply $\mathcal{K}^-_{p,\mathbf{J}_p} = \mathbf{A}\,\delta(u)$ to the equation $\square\psi = \mathbf{F}$ in the sense of distributions,

$$\int_{\Sigma^+}\mathbf{g}(\mathbf{A}\,\delta(u),\square_{\mathbf{g}}\Psi) = \int_{\Sigma^+}\mathbf{g}(\square\big(\mathbf{A}\,\delta(u)\big),\mathbf{F}) \tag{42}$$

where $\Sigma^+ = \mathcal{J}^+(\Sigma)$ is the future of the initial hypersurface Σ. We assume that Ψ has zero data on Σ and that $\mathbf{F}$ is supported in Σ^+. Our next goal is to integrate by parts on the left hand side of (42). We first decompose $\square_{\mathbf{g}}\mathbf{A}$, for an arbitrary tensor-field $\mathbf{A}$, relative to our null frame (20) - (21). For simplicity we assume that $\mathbf{A}_\mu$ is a one tensor, the general case can be treated in the same manner. We recall the definition of the Ricci coefficients (22),

$$\underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_{e_a}\underline{\mathbf{L}}, e_b), \qquad \zeta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_a\mathbf{L},\underline{\mathbf{L}}), \qquad \underline{\eta}_a = \frac{1}{2}\mathbf{g}(e_a,\mathbf{D_L}\underline{\mathbf{L}})$$

and also introduce,

$$\omega = -\frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{\mathbf{L}}}\underline{\mathbf{L}},\mathbf{L}) \tag{43}$$

which is well defined in a neighborhood $\mathcal{D}_\epsilon$ of $\dot{\mathcal{N}}^-(p)$, see subsection 2.2. Using also the notation in subsection (3.1) we derive:

$$\square_{\mathbf{g}} A_\mu \;=\; \mathbf{g}^{\alpha\beta}\mathbf{D}^2_{\alpha\beta}A_\mu = -\frac{1}{2}\mathbf{D}^2_{\mathbf{L}\underline{\mathbf{L}}}A_\mu - \frac{1}{2}\mathbf{D}^2_{\underline{\mathbf{L}}\mathbf{L}}A_\mu + \delta^{ab}\mathbf{D}^2_{ab}A_\mu$$

Now, $\mathbf{D}_b A_\mu = \hat{\mathbf{D}}_b A_\mu$ and, since $\mathbf{D}_a e_b = {}^{(\mathbf{L})}D_a e_b + \frac{1}{2}\chi_{ab}\underline{\mathbf{L}}$,

$$\begin{aligned}\mathbf{D}^2_{ab} A_\mu &= e_a(\hat{\mathbf{D}}_b A_\mu) - \mathbf{D}_{\mathbf{D}_a e_b} A_\mu\\ &= e_a(\hat{\mathbf{D}}_b A_\mu) - \mathbf{D}_{{}^{(\mathbf{L})}D_a e_b} A_\mu - \frac{1}{2}\chi_{ab}\mathbf{D}_{\underline{\mathbf{L}}} A_\mu\\ &= \hat{\mathbf{D}}^2_{ab} A_\mu - \frac{1}{2}\chi_{ab}\mathbf{D}_{\underline{\mathbf{L}}} A_\mu\end{aligned}$$

Hence, denoting $\hat{\Delta} A_\mu = \delta^{ab}\,\hat{\mathbf{D}}^2_{ab} A_\mu$,

$$\delta^{ab}\mathbf{D}^2_{ab} A_\mu = \hat{\Delta} A_\mu - \frac{1}{2}\mathrm{tr}\chi\mathbf{D}_{\underline{\mathbf{L}}} A_\mu$$

On the other hand,

$$\begin{aligned}\mathbf{D}^2_{\mathbf{L}\underline{\mathbf{L}}} A_\mu &= \mathbf{D}^2_{\underline{\mathbf{L}}\mathbf{L}} A_\mu + \mathbf{R}_\mu{}^\lambda{}_{\mathbf{L}\underline{\mathbf{L}}} A_\lambda\\ &= \mathbf{D}_{\underline{\mathbf{L}}}\mathbf{D}_{\mathbf{L}} A_\mu - 2\zeta_a\,\hat{\mathbf{D}}_a A_\mu + 2\omega\,\hat{\mathbf{D}}_{\mathbf{L}} A_\mu + \mathbf{R}_\mu{}^\lambda{}_{\mathbf{L}\underline{\mathbf{L}}} A_\lambda\end{aligned}$$

Henceforth,

$$\begin{aligned}\square_{\mathbf{g}} A_\mu &= -\mathbf{D}_{\underline{\mathbf{L}}}\mathbf{D}_{\mathbf{L}} A_\mu + \hat{\Delta} A_\mu + \zeta_a \cdot \hat{\mathbf{D}}_a A_\mu\\ &- \omega\,\hat{\mathbf{D}}_{\mathbf{L}} A_\mu - \frac{1}{2}\mathrm{tr}\chi\mathbf{D}_{\underline{\mathbf{L}}} A_\mu - \frac{1}{2}\mathbf{R}_\mu{}^\lambda{}_{\mathbf{L}\underline{\mathbf{L}}} A_\lambda\end{aligned} \tag{44}$$

Remark 3.3. In the case when $\mathbf{A}$ is a scalar formula (44) becomes, simply,

$$\square_{\mathbf{g}} A_\mu = -\mathbf{D}_{\underline{\mathbf{L}}}\mathbf{D}_{\mathbf{L}} A + \hat{\Delta} A + \zeta_a \cdot \hat{\mathbf{D}}_a A - \omega\,\hat{\mathbf{D}}_{\mathbf{L}} A - \frac{1}{2}\mathrm{tr}\chi\mathbf{D}_{\underline{\mathbf{L}}} A \tag{45}$$

3.4. **Integration by parts.** In view of (39) we have,

$$\square_{\mathbf{g}}\big(\mathbf{A}\delta(u)\big) = \square_{\mathbf{g}}\mathbf{A}\,\delta(u) + (2\,\hat{\mathbf{D}}_{\mathbf{L}} A + \mathrm{tr}\chi\mathbf{A})\delta'(u) \tag{46}$$

where for $u = 0$,

$$\hat{\mathbf{D}}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}\mathbf{A}\mathrm{tr}\chi = 0.$$

According to section 2.6 we have defined a time foliation Σ_t, $t \in [1-\epsilon, 1+\epsilon]$, in a neighborhood of the vertex p with $p \in \Sigma_1$ such that the boundary of the set $\Omega_\epsilon = \mathcal{J}^-(p) \setminus \cup_{t\in[1-\epsilon,1]}\Sigma_t$ is given by,

$$\partial\Omega_\epsilon = \dot{\mathcal{N}}^-_\epsilon(p) \cup D_{1-\epsilon} \cup D.$$

Here $\dot{\mathcal{N}}^-_\epsilon(p)$ is the portion of $\dot{\mathcal{N}}^-(p)$ to the future of $\Sigma_{1-\epsilon}$, $D_{1-\epsilon} = \mathcal{J}^-(p) \cap \Sigma_{1-\epsilon}$ and $D = \mathcal{J}^-(p) \cap \Sigma$. As before we denote by $\mathbf{T}$ the future unit normal to the surfaces $\Sigma_0 = \Sigma$ and $\Sigma_{1-\epsilon}$. Note that $\Sigma_{1-\epsilon}$ needs only be defined locally, for small $\epsilon > 0$. Note also that, for $\epsilon = 0$, $\mathbf{T}$ coincides with $\mathbf{T}_p$ as defined in section 2.1. Clearly,

$$\int_\Omega \mathbf{g}(\mathbf{A}\,\delta(u), \square_{\mathbf{g}}\Psi) = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\mathbf{g}(\mathbf{A}\,\delta(u), \square_{\mathbf{g}}\Psi)$$

where $\Omega = \mathcal{J}^-(p) \cap \Sigma_+$. Due to the presence of $\delta(u)$ and the fact that ψ is supported in Σ_+ we may assume in what follows that all functions we deal with in the calculation below are supported in the set $\Omega = \mathcal{J}^-(p) \cap J^+(\Sigma)$. Thus the boundary of the intersection of their supports with Ω_ϵ is included in $\Sigma_{1-\epsilon}$.

Lemma 3.5. *Let F, G be two tensor-fields of the same rank and F is a distribution supported in Ω. Then*

$$\int_{\Omega_\epsilon} \mathbf{g}(F, \square_{\mathbf{g}} G) = \int_{\Omega_\epsilon} \mathbf{g}(\square_{\mathbf{g}} F, G) - \int_{D_t} \big(\mathbf{g}(F, \mathbf{D_T} G) - \mathbf{g}(G, \mathbf{D_T} F)\big)\Big|_{t=0}^{t=1-\epsilon},$$

where $D_0 = D$.

Proof : Indeed,

$$\mathbf{g}(F, \square_{\mathbf{g}} G) - \mathbf{g}(\square_{\mathbf{g}} F, G) = \mathbf{D}^\alpha \mathbf{g}(F, \mathbf{D}_\alpha G) - \mathbf{D}^\alpha \mathbf{g}(\mathbf{D}_\alpha F, G)$$

Thus,

$$\begin{aligned}
\int_{\Omega_\epsilon} \big(\mathbf{g}(F, \square_{\mathbf{g}} G) - \mathbf{g}(\square_{\mathbf{g}} F, G)\big) &= \int_{\Omega_\epsilon} \mathbf{D}^\alpha \big(\mathbf{g}(F, \mathbf{D}_\alpha G) - \mathbf{g}(\mathbf{D}_\alpha F, G)\big) \\
&= -\int_{D_t} \mathbf{T}^\alpha \big(\mathbf{g}(F, \mathbf{D}_\alpha G) - \mathbf{g}(\mathbf{D}_\alpha F, G)\big)\Big|_0^{1-\epsilon}
\end{aligned}$$

■

We now write,

$$\begin{aligned}
\int_{\Omega_\epsilon} \mathbf{g}\left(\mathbf{A}\, \delta(u), \square_{\mathbf{g}} \Psi\right) &= \int_{\Omega_\epsilon} \mathbf{g}\left(\square_{\mathbf{g}} \big(\mathbf{A}\, \delta(u)\big), \Psi\right) \\
&- \int_{D_t} \mathbf{g}(\mathbf{A}\delta(u), \mathbf{D_T} \Psi)\Big|_0^{1-\epsilon} + \int_{D_t} \mathbf{g}\left(\mathbf{D_T}\big(\mathbf{A}\delta(u)\big), \Psi\right)\Big|_0^{1-\epsilon} \\
&= \int_{\Omega_\epsilon} \mathbf{g}\left(\square_{\mathbf{g}} \big(\mathbf{A}\, \delta(u)\big), \Psi\right) + I_\epsilon + J_\epsilon,
\end{aligned}$$

where I_ϵ and J_ϵ denote the boundary terms on $D_{1-\epsilon}$. The term corresponding to D_0 vanishes due to the zero data assumption for Ψ.

Proposition 3.6. *We have*

$$I_\epsilon \to 0, \qquad J_\epsilon \to -4\pi \mathbf{g}(\Psi(p), \mathbf{J}_p) \quad as \quad \epsilon \to 0.$$

Thus,

$$\int_{\Omega} \mathbf{g}\left(\mathbf{A}\, \delta(u), \square_{\mathbf{g}} \Psi\right) = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \mathbf{g}\left(\square_{\mathbf{g}} \big(\mathbf{A}\, \delta(u)\big), \Psi\right) - 4\pi \mathbf{g}\left(\Psi(p), \mathbf{J}_p\right) \tag{47}$$

Proof : We analyze the boundary terms I_ϵ, J_ϵ. Clearly,

$$I_\epsilon = -\int_{D_{1-\epsilon}} \mathbf{g}\left(\mathbf{A}\delta(u), \mathbf{D_T} \Psi\right) = -\int_{\dot{\mathcal{N}}^-(p) \cap D_{1-\epsilon}} \mathbf{g}(\mathbf{A}, \mathbf{D_T} \Psi)\, \varphi\, da_\gamma$$

where, see (28), $\varphi = |\mathbf{D_T} u|^{-1}$ is the null lapse and da_γ is the area element of the 2-surface $S_{1-\epsilon}(p) = \dot{\mathcal{N}}^-(p) \cap D_{1-\epsilon}$. Recall that according to (34) the area $|S_{1-\epsilon}(p)| \lesssim \epsilon^2$.

Now,

$$\begin{aligned}|I_\epsilon| &\lesssim \|\varphi\|_{L^\infty}\Big(\int_{S_{1-\epsilon}}|\mathbf{A}|^2 da_\gamma\Big)^{1/2}\Big(\int_{S_{1-\epsilon}}|\mathbf{D_T}\Psi|^2 da_\gamma\Big)^{1/2}\\ &\lesssim \|\varphi\|_{L^\infty}\|\mathbf{D_T}\Psi\|_{L^\infty}\|\mathbf{A}\|_{L^2(S_{1-\epsilon}(p))}|S_{1-\epsilon}(p)|^{1/2}\\ &\lesssim \epsilon\,\|\varphi\|_{L^\infty}\|\mathbf{D_T}\Psi\|_{L^\infty}\|\mathbf{A}\|_{L^2(S_{1-\epsilon}(p))}\end{aligned}$$

Recalling (41) and (34) we easily see that $\|\mathbf{A}\|_{L^2(S_{1-\epsilon})}$ is bounded as $\epsilon \to 0$. Thus, for a smooth tensor-field Ψ, we clearly have,

$$I_\epsilon \to 0, \qquad \text{as} \quad \epsilon \to 0.$$

We now consider the second boundary term,

$$\begin{aligned}J_\epsilon &= \int_{D_{1-\epsilon}} \mathbf{g}\left(\mathbf{D_T}\big(\mathbf{A}\delta(u)\big), \Psi\right)\\ &= \int_{D_{1-\epsilon}} \delta(u)\,\mathbf{g}\left(\mathbf{D_T}\mathbf{A}, \Psi\right) + \int_{D_{1-\epsilon}} \delta'(u)\mathbf{D_T}u\ \mathbf{g}(\mathbf{A}, \Psi)\\ &= \int_{D_{1-\epsilon}} \delta(u)\,\mathbf{g}\left(\mathbf{D_T}\mathbf{A}, \Psi\right) + \int_{D_{1-\epsilon}} \delta'(u)\,\varphi^{-1}\,\mathbf{g}(\mathbf{A}, \Psi)\\ &= J^1_\epsilon + J^2_\epsilon\end{aligned}$$

If $N = \varphi\mathbf{L} + \mathbf{T}$ denotes the unit normal[17] to $S_{1-\epsilon} = \dot{\mathcal{N}}^-(p) \cap \Sigma_{1-\epsilon}$ in $\Sigma_{1-\epsilon}$, then $\mathbf{D}_N\delta(u) = \delta'(u)\mathbf{D}_N u$ and $\mathbf{D}_N u = \mathbf{D_T}u = \varphi^{-1}$. Hence,

$$J^2_\epsilon = \int_{\Sigma_{1-\epsilon}} \delta'(u)\varphi^{-1}\,\mathbf{g}(\mathbf{A}, \Psi) \quad = \quad \int_{\Sigma_{1-\epsilon}} \mathbf{D}_N\delta(u)\,\mathbf{g}(\mathbf{A}, \Psi)$$

We next record the following integration by parts formulae.

Lemma 3.7. *Let X be a vectorfield tangent to the hyperplane Σ_t and let f, g be two scalar functions on Σ_t. Denote by ∇ the covariant derivative restricted to Σ_t. Then,*

$$\int_{\Sigma_t} fX(g) = -\int_{\Sigma_t} (X(f) + div\ Xf)g. \tag{48}$$

In particular,

$$\int_{\Sigma_t} fN(g) = -\int_{\Sigma_t} (N(f) + tr\theta\, f)g, \tag{49}$$

where $tr\theta$ is the mean curvature of the 2-d surfaces $S_{t,u} = \{u = const\} \cap \Sigma_t$.

Proof : Formula (48) is standard. To prove (49) observe div $N = \mathbf{g}(\nabla_N N, N) + \sum_a \mathbf{g}(\nabla_a N, e_a) = \mathrm{tr}\theta$ where θ is the second fundamental form of the surfaces $S_{t,u} \subset \Sigma_t$. ■

---

[17]A priori, the vectorfield N is defined only on $\dot{\mathcal{N}}^-(p) \cap \Sigma_{1-\epsilon}$, it can however be extended locally as a unit normal to the foliation of 2-d surfaces $\{u = const\} \cap \Sigma_{1-\epsilon}$.

Using the lemma we infer that,

$$
\begin{aligned}
J_\epsilon^2 &= -\int_{\Sigma_{1-\epsilon}} \delta(u)\big(N\mathbf{g}(\mathbf{A},\Psi)+\mathrm{tr}\theta\,\mathbf{g}(\mathbf{A},\Psi)\big)\\
&= -\int_{\Sigma_{1-\epsilon}} \delta(u)\big(\mathbf{g}(\mathbf{D}_N\mathbf{A},\Psi)+\mathrm{tr}\theta\,\mathbf{g}(\mathbf{A},\Psi)\big)-\int_{\Sigma_{1-\epsilon}}\delta(u)\,\mathbf{g}(\mathbf{A},\mathbf{D}_N\Psi)
\end{aligned}
$$

Now, proceeding as for I_ϵ, it is easy to check that $\int_{\Sigma_{1-\epsilon}}\delta(u)\,\mathbf{g}(\mathbf{A},\mathbf{D}_N\Psi)\to 0$ as $\epsilon\to 0$. Hence,

$$
\lim_{\epsilon\to 0} J_\epsilon^2 = -\lim_{\epsilon\to 0}\int_{\Sigma_{1-\epsilon}} \delta(u)\big(\mathbf{g}(\mathbf{D}_N\mathbf{A},\Psi)+\mathrm{tr}\theta\,\mathbf{g}(\mathbf{A},\Psi)\big)
$$

Or,

$$
\lim_{\epsilon\to 0} J_\epsilon = -\lim_{\epsilon\to 0}\int_{\Sigma_{1-\epsilon}} \delta(u)\big(\mathbf{g}(\mathbf{D}_{N-\mathbf{T}}\mathbf{A},\Psi)+\mathrm{tr}\theta\,\mathbf{g}(\mathbf{A},\Psi)\big)
$$

Now observe that $\mathbf{L}=\varphi^{-1}(N-\mathbf{T})$. Hence, $\mathbf{D}_\mathbf{L}\mathbf{A}=\varphi^{-1}\mathbf{D}_{N-\mathbf{T}}\mathbf{A}$. Since $\mathbf{D}_\mathbf{L}\mathbf{A}+\frac{1}{2}\mathbf{A}\mathrm{tr}\chi=0$ we infer,

$$
\mathbf{D}_{N-\mathbf{T}}\mathbf{A}=\varphi\,\mathbf{D}_\mathbf{L}\mathbf{A}=-\frac{1}{2}\varphi\mathbf{A}\mathrm{tr}\chi.
$$

Hence,

$$
\lim_{\epsilon\to 0} J_\epsilon = -\lim_{\epsilon\to 0}\int_{\Sigma_{1-\epsilon}} \delta(u)\big(-\frac{1}{2}\varphi\mathrm{tr}\chi+\mathrm{tr}\theta\big)\mathbf{g}(\mathbf{A},\Psi)
$$

On the other hand $\theta_{ab}=g(\nabla_a N,e_b)=\mathbf{g}(\mathbf{D}_a(\varphi\mathbf{L}+\mathbf{T}),e_b)=\varphi\chi_{ab}+k_{ab}$. Therefore, $\mathrm{tr}\theta=\varphi\mathrm{tr}\chi+\delta^{ab}k_{ab}$ and we deduce,

$$
\lim_{\epsilon\to 0} J_\epsilon = -\frac{1}{2}\lim_{\epsilon\to 0}\int_{\Sigma_{1-\epsilon}}\varphi\mathrm{tr}\chi\delta(u)\,\mathbf{g}(\mathbf{A},\Psi)-\lim_{\epsilon\to 0}\int_{\Sigma_{1-\epsilon}}(\delta^{ab}k_{ab})\delta(u)\,\mathbf{g}(\mathbf{A},\Psi)
$$

It is easy to see that the second term of the right hand side converges to zero for $\epsilon\to 0$. Indeed,

$$
\begin{aligned}
\Big|\int_{\Sigma_{1-\epsilon}}(\delta^{ab}k_{ab})\delta(u)\,\mathbf{g}(\mathbf{A},\Psi)\Big| &= \Big|\int_{S_{1-\epsilon}(p)}(\delta^{ab}k_{ab})\,\mathbf{g}(\mathbf{A},\Psi)\,\varphi\, da_\gamma\Big|\\
&\lesssim |S_{1-\epsilon}(p)|^{\frac{1}{2}}\|k\|_{L^\infty}\|\Psi\|_{L^\infty}\|\varphi|_{L^\infty}\|\mathbf{A}\|_{L^2(S_{1-\epsilon}(p))}
\end{aligned}
$$

Therefore,

$$
\lim_{\epsilon\to 0} J_\epsilon = -\frac{1}{2}\lim_{\epsilon\to 0}\int_{S_{1-\epsilon}(p)}\varphi^2\,\mathrm{tr}\chi\,\mathbf{g}(\mathbf{A},\Psi)\,da_\gamma
$$

It is easy to check that,

$$
\frac{1}{2}\lim_{\epsilon\to 0}\int_{S_{1-\epsilon}(p)}\varphi^2\,\mathrm{tr}\chi\,\mathbf{g}\left(\mathbf{A},(\Psi-\Psi(p)\right)da_\gamma=0
$$

Therefore,

$$
\lim_{\epsilon\to 0} J_\epsilon = -\frac{1}{2}\lim_{\epsilon\to 0}\int_{S_{1-\epsilon}(p)}\varphi^2\mathrm{tr}\chi\,\mathbf{g}\left(\mathbf{A},\Psi(p)\right)\,d\alpha_\gamma
$$

Or, since $\sup_{S_{1-\epsilon}(p)}|\varphi-1|\to 0$, and $\sup_{S_{1-\epsilon}(p)}|\mathrm{tr}\chi-\frac{2}{s}|\to 0$ as $\epsilon\to 0$, we infer that

$$
\lim_{\epsilon\to 0} J_\epsilon = -\epsilon^{-2}\lim_{\epsilon\to 0}\int_{S_{1-\epsilon}(p)}\mathbf{g}\left(\Psi(p),(s\mathbf{A})\right)=-4\pi\lim_{r\to 0}\mathbf{g}\left(\Psi(p),(r\mathbf{A})(r)\right)
$$

Thus, using the initial condition $\lim_{s\to 0} s\mathbf{A}(s) = \mathbf{J}_p$, we obtain

$$\lim_{\epsilon\to 0} J_\epsilon = -4\pi \mathbf{g}(\Psi(p), \mathbf{J}_p)$$

∎

We now analyze the term $\int_{\Omega_\epsilon} \mathbf{g}\left(\square_{\mathbf{g}}\big(\mathbf{A}\,\delta(u)\big), \Psi\right)$ on the right hand side of (47). In view of (46) we have,

$$\int_{\Omega_\epsilon} \mathbf{g}\left(\square_{\mathbf{g}}\big(\mathbf{A}\,\delta(u)\big), \Psi\right) = \int_{\Omega_\epsilon} \delta(u)\,\mathbf{g}\left((\square_{\mathbf{g}}\mathbf{A}), \Psi\right) + \int_{\Omega_\epsilon} \delta'(u)\,\mathbf{g}\left((2\mathbf{D}_{\mathbf{L}}A + \mathrm{tr}\chi\mathbf{A}), \Psi\right).$$

Given the normalization $\underline{\mathbf{L}}(u) = -2$ we have $\delta'(u) = -\frac{1}{2}\mathbf{D}_{\underline{\mathbf{L}}}\delta(u)$. Integrating by parts we obtain

$$\begin{aligned}\int_{\Omega_\epsilon} \delta'(u)\,\mathbf{g}\left((2\mathbf{D}_{\mathbf{L}}A + \mathrm{tr}\chi\mathbf{A}), \Psi\right) &= \frac{1}{2}\int_{\Omega_\epsilon} \delta(u)\,\mathbf{g}\left(\mathbf{D}_{\underline{\mathbf{L}}}(2\mathbf{D}_{\mathbf{L}}A + \mathrm{tr}\chi\mathbf{A}), \Psi\right)\\ &+ \frac{1}{2}\int_{\Omega_\epsilon} \delta(u)\,\mathbf{g}\left((2\mathbf{D}_{\mathbf{L}}A + \mathrm{tr}\chi\mathbf{A}), \big(\mathbf{D}_{\underline{\mathbf{L}}}\Psi + \mathbf{D}^\alpha\underline{\mathbf{L}}_\alpha\Psi\big)\right)\\ &+ \int_{D_t} \delta(u)\,\mathbf{g}(((2\mathbf{D}_{\mathbf{L}}A + \mathrm{tr}\chi\mathbf{A}), \Psi)\ \mathbf{g}(\underline{\mathbf{L}}, \mathbf{T})\,|_{t=0}^{t=1-\epsilon}\end{aligned}$$

Recall that $2\mathbf{D}_{\mathbf{L}}A + \mathrm{tr}\chi\mathbf{A} = 0$ on the surface $u = 0$. Therefore the last two terms vanish and we derive,

$$\int_{\Omega_\epsilon} \delta'(u)\,\mathbf{g}\left((2\mathbf{D}_{\mathbf{L}}A + \mathrm{tr}\chi\mathbf{A}), \Psi\right) = \int_{\Omega_\epsilon} \delta(u)\,\mathbf{g}\left(\mathbf{D}_{\underline{\mathbf{L}}}(\mathbf{D}_{\mathbf{L}}A + \frac{1}{2}\mathrm{tr}\chi\mathbf{A}), \Psi\right)$$

Therefore,

$$\int_{\Omega_\epsilon} \mathbf{g}\left(\square_{\mathbf{g}}\big(\mathbf{A}\,\delta(u)\big), \Psi\right) = \int_{\Omega_\epsilon} \delta(u)\,\mathbf{g}\left(\big(\square_{\mathbf{g}}\mathbf{A} + \mathbf{D}_{\underline{\mathbf{L}}}(\mathbf{D}_{\mathbf{L}}A + \frac{1}{2}\mathrm{tr}\chi\mathbf{A})\big), \Psi\right)$$

We now recall (44),

$$\begin{aligned}\square_{\mathbf{g}}\mathbf{A} &= -\mathbf{D}_{\underline{\mathbf{L}}}\mathbf{D}_{\mathbf{L}}\mathbf{A} + \hat{\Delta}\mathbf{A} + \zeta_a\cdot\hat{\mathbf{D}}_a\mathbf{A}\\ &- \omega\,\hat{\mathbf{D}}_{\mathbf{L}}\mathbf{A} - \frac{1}{2}\mathrm{tr}\chi\mathbf{D}_{\underline{\mathbf{L}}}\mathbf{A} - \frac{1}{2}\mathbf{R}(\cdot, \mathbf{A}, \mathbf{L}, \underline{\mathbf{L}})\end{aligned}$$

Therefore,

$$\begin{aligned}\square_{\mathbf{g}}\mathbf{A} + \mathbf{D}_{\underline{\mathbf{L}}}(\mathbf{D}_{\mathbf{L}}A + \frac{1}{2}\mathrm{tr}\chi\mathbf{A}) &= \hat{\Delta}\mathbf{A} + \zeta_a\,\hat{\mathbf{D}}_a\mathbf{A} - \frac{1}{2}\mathrm{tr}\chi\mathbf{D}_{\underline{\mathbf{L}}}\mathbf{A}\\ &- \omega\,\hat{\mathbf{D}}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}(\mathbf{D}_{\underline{\mathbf{L}}}\mathrm{tr}\chi)\mathbf{A} - \frac{1}{2}\mathbf{R}(\cdot, \mathbf{A}, \mathbf{L}, \underline{\mathbf{L}})\end{aligned}$$

while since $\mathbf{D}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}\mathrm{tr}\chi\mathbf{A} = 0$ on $\dot{\mathcal{N}}^-(p)$,

$$-\frac{1}{2}\mathrm{tr}\underline{\chi}\mathbf{D}_{\mathbf{L}}\mathbf{A} - \omega\mathbf{D}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}(\mathbf{D}_{\underline{\mathbf{L}}}\mathrm{tr}\chi)\mathbf{A} = \frac{1}{2}(\mathbf{D}_{\underline{\mathbf{L}}}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\omega\mathrm{tr}\chi)\mathbf{A}$$

Hence, we have proved the following,

Proposition 3.8. *In the case of a one form* $\mathbf{A}$ *verifying* (38),

$$\begin{aligned}\square_{\mathbf{g}}\mathbf{A} + \mathbf{D}_{\underline{\mathbf{L}}}(\mathbf{D}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}tr\chi\mathbf{A}) &= \Delta\mathbf{A} + \zeta_a\mathbf{D}_a\mathbf{A}\\ &+ \frac{1}{2}(\mathbf{D}_{\underline{\mathbf{L}}}tr\chi + \frac{1}{2}tr\chi tr\underline{\chi} + 2\omega tr\chi)\mathbf{A} - \frac{1}{2}\mathbf{R}(\cdot, \mathbf{A}, \mathbf{L}, \underline{\mathbf{L}}),\end{aligned}$$

where

$$\Delta\mathbf{A} = \hat{\Delta}\mathbf{A} + \frac{1}{2}tr\chi\,\hat{\mathbf{D}}_{\mathbf{L}}\mathbf{A} = e_a(\mathbf{D}_a A) - \mathbf{g}(\mathbf{D}_a e_a, e_b)\mathbf{D}_b A$$

is a Laplace-Beltrami type operator which coincides with the standard surface Laplace-Beltrami operator in the case when the frame $\{e_a\}_{a=1,2}$ *spans a tangent space of a 2-dimensional surface*[18].

In the scalar case we have instead, see remark 3.3,

$$\begin{aligned}\Box_{\mathbf{g}}\mathbf{A} + \mathbf{D}_{\underline{\mathbf{L}}}(\mathbf{D}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}tr\chi\mathbf{A}) &= \Delta\mathbf{A} + \zeta_a\mathbf{D}_a\mathbf{A} \\ &+ \frac{1}{2}(\mathbf{D}_{\underline{\mathbf{L}}}tr\chi + \frac{1}{2}tr\chi tr\underline{\chi} + 2\omega tr\chi)\mathbf{A},\end{aligned}$$

Using the above proposition we infer that,

$$\begin{aligned}\int_{\Omega_\epsilon}\mathbf{g}\left(\Box_{\mathbf{g}}\big(\mathbf{A}\,\delta(u)\big),\Psi\right) &= \int_{\Omega_\epsilon}\delta(u)\mathbf{g}\left(\big(\Delta\mathbf{A}+\zeta_a\mathbf{D}_a\mathbf{A}\big),\Psi\right) \\ &+ \frac{1}{2}\int_{\Omega_\epsilon}\delta(u)\left(\mathbf{D}_{\underline{\mathbf{L}}}\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\omega\mathrm{tr}\chi\right)\mathbf{g}(\mathbf{A},\Psi) \\ &+ \frac{1}{2}\int_{\Omega_\epsilon}\delta(u)\mathbf{R}(\Psi,\mathbf{A},\underline{\mathbf{L}},\mathbf{L})\end{aligned}$$

We now make use of the following,

Proposition 3.9. *Introduce the mass aspect function as in (13.1.10b) of* [C-K],

$$\mu = \mathbf{D}_{\underline{\mathbf{L}}}tr\chi + \frac{1}{2}tr\chi tr\underline{\chi} + 2\omega tr\chi \tag{50}$$

The following formula holds true relative to the standard geodesic foliation on $\dot{\mathcal{N}}^-(p)$,

$$\mu = 2div\,\zeta - \hat{\chi}\cdot\underline{\hat{\chi}} + 2|\zeta|^2 + \mathbf{R}_{\underline{\mathbf{L}}\mathbf{L}} + \frac{1}{2}\mathbf{R}(\underline{\mathbf{L}},\mathbf{L},\underline{\mathbf{L}},\mathbf{L}) \tag{51}$$

Proof : See [C-K]. ■

Remark 3.10. Note that according to (51) the mass aspect function μ depends only on the null hypersurface $\dot{\mathcal{N}}^-(p)$ and the ambient curvature $\mathbf{R}$.

We have therefore proved the following precise version of theorem 2.9,

Theorem 3.11. *Let* $\mathbf{A}$ *be a vectorfield verifying,*

$$\mathbf{D}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}\mathbf{A}tr\chi = 0, \quad s\mathbf{A}(p) = \mathbf{J}_p \qquad on \quad u = 0$$

where $\mathbf{J}_p$ *is a fixed vector at* p. *Then solution* Ψ *of an inhomogeneous vector equation* $\Box_{\mathbf{g}}\Psi = \mathbf{F}$, *in a globally hyperbolic spacetime* $(\mathbf{M},\mathbf{g})$ *satisfying* $\mathbf{A1},\mathbf{A2}$, *with*

[18]As in the case of the geodesic foliation.

zero initial data on a Cauchy hypersurface Σ *can be represented by the following formula at point* p *with* $\Omega = \mathcal{J}^-(p) \cap \mathcal{J}^+(\Sigma)$,

$$\begin{aligned} 4\pi \mathbf{g}(\Psi(p), \mathbf{J}_p) &= -\int_\Omega \delta(u)\, \mathbf{g}(\mathbf{A}, \mathbf{F}) \\ &- \frac{1}{2}\int_\Omega \delta(u)\, \mathbf{R}(\Psi, \mathbf{A}, \underline{\mathbf{L}}, \mathbf{L}) + \int_\Omega \delta(u)\, \mathbf{g}\left((\Delta \mathbf{A} + \zeta_a \mathbf{D}_a \mathbf{A}), \Psi\right) \\ &+ \frac{1}{2}\int_\Omega \delta(u)\, \mu\, \mathbf{g}(\mathbf{A}, \Psi) \end{aligned} \tag{52}$$

where,

$$\mathbf{R}(\Psi, \mathbf{A}, \underline{\mathbf{L}}, \mathbf{L}) = \mathbf{R}_{\alpha\beta\gamma\delta} \underline{\mathbf{L}}^\gamma \mathbf{L}^\delta \Psi^\alpha \mathbf{A}^\beta,$$

with $\mathbf{R}_{\alpha\beta\gamma\delta}$ *the components of the curvature tensor* $\mathbf{R}$ *relative to an arbitrary frame.*

Remark 3.12. Theorem 3.11 implies that the error term $\mathcal{E}$ in (37) has the following representation

$$\mathcal{E}_a = -\frac{1}{2} \mathbf{R}_{a\lambda\gamma\delta} \mathbf{A}^\lambda \underline{\mathbf{L}}^\gamma \mathbf{L}^\delta + (\Delta \mathbf{A} + \zeta_a \mathbf{D}_a \mathbf{A})_\alpha + \frac{\mu}{2} \mathbf{A}_\alpha$$

in the case of a vectorial wave equation. For the scalar wave equation

$$\mathcal{E} = (\Delta \mathbf{A} + \zeta_a \mathbf{D}_a \mathbf{A}) + \frac{\mu}{2} \mathbf{A}$$

Remark 3.13. Formula (52) can easily be generalized to higher order tensor wave equations. Indeed if both Ψ and $\mathbf{F}$ are tensor-fields of order k then $\mathbf{J}_p$, $\mathbf{A}$ and $\mathcal{E}$ are also of order k and,

$$\begin{aligned} \mathbf{g}(\mathcal{E}, \Psi) &= \mathbf{g}\big((\Delta \mathbf{A} + \zeta_a \mathbf{D}_a \mathbf{A}), \Psi\big) + \frac{\mu}{2} \mathbf{g}(\mathbf{A}, \Psi) \\ &+ \mathbf{R}(\cdot, \cdot, \underline{\mathbf{L}}, \mathbf{L}) \# \Psi \# \mathbf{A} \end{aligned} \tag{53}$$

where the last term denotes a scalar contraction of $\mathbf{R}(\cdot, \cdot, \underline{\mathbf{L}}, \mathbf{L})$ with Ψ and $\mathbf{A}$.

4. Wave equation for sections of vector bundles and applications to the Yang-Mills equations

Now let $\mathbf{V}$ be a vector bundle over $(\mathbf{M}, \mathbf{g})$ with a positive definite scalar product $<,>$ and a compatible connection λ. We may assume that $\mathbf{V}$ is a vector bundle associated to a principal bundle P so that $\mathbf{V} = P \times_G E$ with G a compact Lie group and a vector space E. Let $\mathcal{G}$ denote the Lie algebra of G. The connection λ is a $\mathcal{G}$ valued 1-form on V, which, locally can be viewed as a $\mathcal{G}$ valued 1-form on $\mathbf{M}$.

We define the gauge wave operator $\Box_{\mathbf{g}}^{(\lambda)}$ for sections $\Psi : \mathbf{M} \to \mathbf{V}$

$$\Box_{\mathbf{g}}^{(\lambda)} \Psi = \mathbf{g}^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu \Psi,$$

where $\mathcal{D}_\mu = \mathbf{D}_\mu + [\lambda_\mu, \cdot\,]$ denotes the gauge covariant derivative. We denote by Λ the curvature of the connection, i.e. the $\mathcal{G}$ valued 2-form on $\mathbf{M}$,

$$\Lambda_{\alpha\beta} = \partial_\alpha \lambda_\beta - \partial_\beta \lambda_\alpha + [\lambda_\alpha, \lambda_\beta].$$

As before we construct a Kirchoff-Sobolev parametrix $\mathcal{K}_p^-$ for $\square_{\mathbf{g}}^{(\lambda)}$ by defining $\mathcal{K}_p^- = \mathcal{K}_{p,\mathbf{J}_p}^- = \mathbf{A}\,\delta(u)$, where $\mathbf{A}$ is a section of $\mathbf{V}$ which verifies the covariant transport equation[19]

$$\mathcal{D}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}\mathrm{tr}\chi\mathbf{A} = 0 \tag{54}$$

with initial data $(s\mathbf{A})|_{s=0} = \mathbf{J}_p$ and $\mathbf{J}_p$ is a fixed element of the fiber $\mathbf{V}_p$. As before we assume that $(\mathbf{M}, \mathbf{g})$ is globally hyperbolic and satisfies $\mathbf{A1}, \mathbf{A2}$. We also assume that u is a solution of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\,\partial_\beta u = 0$ with u, vanishing on the boundary $\mathcal{N}^-(p)$ of the past of p in $\mathbf{M}$. Repeating our calculations of section 3, we obtain the following analog[20] of Theorem 3.11.

Theorem 4.1. *Let* $\mathbf{A}$ *be a section of a vector bundle* $\mathbf{V}$ *over* $(\mathbf{M}, \mathbf{g})$ *verifying,*

$$\mathcal{D}_{\mathbf{L}}\mathbf{A} + \frac{1}{2}\mathbf{A}\,tr\chi = 0, \quad s\mathbf{A}(p) = \mathbf{J}_p \qquad on \quad \mathcal{N}^-(p)$$

where $\mathbf{J}_p$ *is a fixed element of* $\mathbf{V}_p$. *The solution* Ψ *of the inhomogeneous gauge equation* $\square_{\mathbf{g}}^{(\lambda)}\Psi = \mathbf{F}$, *with zero initial data on a Cauchy hypersurface* Σ *can be represented by the following formula at point* p *with* $\Omega = \mathcal{J}^-(p) \cap \mathcal{J}^+(\Sigma)$,

$$\begin{aligned} 4\pi < \Psi(p), \mathbf{J}_p > \;&=\; -\int_\Omega \delta(u) \, < \mathbf{A}, \mathbf{F} > \\ &-\; \frac{1}{2}\int_\Omega \delta(u) \, < [\Lambda_{\underline{\mathbf{L}}\mathbf{L}}, \mathbf{A}], \Psi > + \int_\Omega \delta(u) \, \Big\langle \big(\Delta^{(\lambda)}\mathbf{A} + \zeta_a \mathcal{D}_a^{(\lambda)}\mathbf{A}\big), \Psi \Big\rangle \\ &+\; \frac{1}{2}\int_\Omega \delta(u)\, \mu \, < \mathbf{A}, \Psi > \end{aligned} \tag{55}$$

In particular, in Minkowski space, for general initial data, we have the following representation:

$$\begin{aligned} 4\pi < \Psi(p), \mathbf{J}_p > \;&=\; -\int_\Omega \delta(u) \, < \mathbf{A}, F > - \frac{1}{2}\int_\Omega \delta(u) \, < [\Lambda_{\underline{\mathbf{L}}\mathbf{L}}, \mathbf{A}], \Psi > \\ &+\; \int_\Omega \delta(u) \, < \Delta^{(\lambda)}\mathbf{A}, \Psi > + \int_\Sigma (< \mathbf{A}\delta(u), \mathbf{D_T}\Psi > - < \mathbf{D_T}(\mathbf{A}\delta(u)), \Psi >) \end{aligned} \tag{56}$$

The last term represents contribution of the initial data on Σ.

Remark 4.2. The formula (55) can be naturally extended to consider sections of the bundle $TM \otimes \ldots \otimes TM \otimes V$. In this case terms of the form $\int_\Omega \delta(u) < \mathbf{R}(\cdot, \cdot, \mathbf{L}, \underline{\mathbf{L}})\mathbf{A}, \Psi >$, where $\mathbf{R}$ is the Riemann curvature tensor of $\mathbf{g}$, need to be added. The corresponding extension of (56) does not therefore introduce any additional terms.

4.3. **Yang-Mills equations.** We now assume that λ is a Yang-Mills connection on a 4-dimensional Lorentzian manifold $(\mathbf{M}, \mathbf{g})$, i.e., it verifies the equations

$$\mathcal{D}^\alpha \Lambda_{\alpha\beta} = 0. \tag{57}$$

[19]Note that here transport of $\mathbf{A}$ along integral curves of $\mathbf{L}$ is modulated by the action of the gauge potential λ.

[20]The only new term in the formula (55) below is due to the commutator $(\mathbf{D}_L^{(\lambda)}\mathbf{D}_{\underline{L}}^{(\lambda)} - \mathbf{D}_{\underline{L}}^{(\lambda)}\mathbf{D}_{\mathbf{L}}^{(\lambda)})\mathbf{A} = [\Lambda_{L\underline{L}}, \mathbf{A}]$ in deriving formula (44)

The Yang-Mills equations are hyperbolic in nature and admit a Cauchy formulation, in which the connection λ is prescribed on a Cauchy hypersurface[21] Σ and then extended as a solution of the problem (57). The uniqueness and global existence for the Yang-Mills equations with smooth initial data in 4-dimensional Minkowski space-time was established by Eardley-Moncrief, [EM1], [EM2]. This result was later extended to the Yang-Mills equations on a smooth 4-dimensional globally hyperbolic Lorentzian space-time by Chruściel-Shatah, [CS]. A different proof in Minkowski space, allowing for initial data with only finite energy , was given by Klainerman-Machedon, [KM].

All of the above approaches were manifestly non-covariant; as they required a choice of a gauge condition for the connection λ. The approach of Eardley-Moncrief was based on the fundamental solution for a scalar wave equation in Minkowski space (Kirchoff formula) and made use of Cronström gauges: For any point p the connection λ can be chosen to satisfy the condition

$$(p-q)^\alpha\, \lambda_\alpha(q) = 0$$

The work of Chruśicel-Shatah relied on the Friedlander's representation of the fundamental solution of a scalar wave equation in a curved space-time and a local analog of the Cronström gauge. Finally, Klainerman-Machedon's proof was based on a Fourier representation of the fundamental solution of a scalar wave equation in Minkowski space, bilinear estimates and the use of the Coulomb gauge:

$$\partial^i \lambda_i = 0$$

Below we present a new simple *gauge independent* proof of the global existence and uniqueness result for the $3+1$-dimensional Yang-Mills equations. The main new ingredient is the use of a gauge covariant first order Kirchoff-Sobolev parametrix described in Theorem 4.1.

Differentiating the Bianchi identities $\mathcal{D}_{[\alpha}\Lambda_{\beta\sigma]} = 0$ and using the equations we infer that the curvature Λ is a solution of a covariant gauge wave equation

$$\square_{\mathbf{g}}^{(\lambda)}\Lambda_{\alpha\beta} = 2[\Lambda^\sigma{}_\alpha, \Lambda_{\sigma\beta}] + 2\mathbf{R}_{\sigma\alpha\gamma\beta}\Lambda^{\sigma\gamma} + \mathbf{R}_{\alpha\sigma}\Lambda_\beta{}^\sigma + \mathbf{R}_{\beta\sigma}\Lambda^\sigma{}_\beta$$

For simplicity we consider the problem in Minkowski space, although our results can easily be extended to the general case of a globally hyperbolic smooth Lorentzian manifold, as in [CS]. The equations then simplify,

$$\square^{(\lambda)}\Lambda_{\alpha\beta} = 2[\Lambda^\sigma{}_\alpha\,,\, \Lambda_{\sigma\beta}] \tag{58}$$

Recall that the curvature Λ can be decomposed into its electric and magnetic parts $E_i = \Lambda_{0i}$ and $H_i = {}^\star\Lambda_{0i}$. We also recall that the total energy

$$\mathcal{E}_0 = \int_{\Sigma_t} \left(|E|^2 + |H|^2\right).$$

is conserved. Moreover, adapting the energy identity to to the past $\mathcal{J}^-(p)$ of a point $p \in \Sigma^+$ we also get a bound on the flux of energy along $\mathcal{N}^-(p)$. More precisely we derive,

$$\mathcal{F}_p^- \le \mathcal{E}_0.$$

[21]This requires space-time $\mathbf{M}$ to be globally hyperbolic.

where the backward null energy flux $\mathcal{F}_p^-$ is defined with the help of a null frame $(\mathbf{L}, \underline{\mathbf{L}}, e_a)$ centered at p. Without loss of generality we may assume that $p = (t, 0)$ and denote $r = |y|$. Then $\mathbf{L} = \partial_r - \partial_t$, $\underline{\mathbf{L}} = -\partial_t - \partial_r$ and e_a is a frame on a standard sphere $\mathbb{S}_r$. With these notations

$$\mathcal{F}_p^- = \int_{\mathcal{N}^-(p)} \left(|\Lambda_{\mathbf{L}\underline{\mathbf{L}}}|^2 + \sum_{a=1}^{2} |\Lambda_{\mathbf{L}a}|^2 \right)$$

As in the original approach of Eardley-Moncrief the key element of the proof of global existence is a pointwise bound on curvature Λ. Once this bound is established the remaining steps concerning existence, propagation of regularity and uniqueness are very standard and will be omitted. The precise statement concerning an L^∞ bound on Λ is as follows:

Lemma 4.4. *There exists $\tau_* > 0$ dependent only on $\mathcal{E}_0$ such that for any point $p = (t, 0)$ we have*

$$|\Lambda(p)| \le C_{t-\tau_*},$$

where the constant $C_{t-\tau_*}$ depends only on the solution Λ on a hypersurface $\Sigma_{t-\tau_*}$.

Remark 4.5. Iterations of Lemma 4.4 leads to a pointwise bound on the curvature Λ in terms of the initial data.

Proof : We fix $\tau_* > 0$, whose is to be determined later, and apply the representation formula (56) in the domain $\Omega = \mathcal{J}^-(p) \cap \mathcal{J}^+(\Sigma_{t-\tau_*})$,

$$\begin{aligned} 4\pi < \Lambda(p), \mathbf{J} > \;&=\; -2 \int_\Omega \delta(u) < \mathbf{A}, [\Lambda, \Lambda] > - \frac{1}{2} \int_\Omega \delta(u) < [\Lambda_{\underline{\mathbf{L}}\mathbf{L}}, \mathbf{A}], \Lambda > \\ &+ \int_\Omega \delta(u) < \Delta^{(\lambda)} \mathbf{A}, \Lambda > \\ &+ \int_{\Sigma_{t-\tau_*}} (< \mathbf{A}\delta(u), \mathbf{D_T}\Lambda > - < \mathbf{D_T}(\mathbf{A}\delta(u)), \Lambda >) \end{aligned} \tag{59}$$

Here $\mathbf{J}$ is an arbitrary $\mathcal{G}$ valued anti-symmetric 2-tensor on $\mathbb{R}^{3+1}$, $\mathbf{A}$ is a $\mathcal{G}$ valued 2-form on $\mathbb{R}^{3+1}$ verifying the equation[22]

$$\mathcal{D}_\mathbf{L} \mathbf{A} + r^{-1} \mathbf{A} = 0, \qquad (r\mathbf{A})|_{r=0} = \mathbf{J}$$

and $<,>$ denotes a positive definite scalar product on $\Lambda^2(\mathbb{R}^{3+1}) \otimes \mathcal{G}$. The last term in (59) depends only on the solution Λ on $\Sigma_{t-\tau_*}$ and therefore is consistent with the claim of Lemma 4.4. We now observe that for $a, b \in \Lambda^2(\mathbb{R}^{3+1}) \otimes \mathcal{G}$ we have

$$| < a, b > | \le |a|\, |b|,$$

where $|a|$ denotes the absolute value of an element in $\Lambda^2(\mathbb{R}^4) \otimes \mathcal{G}$ relative to the positive definite scalar product[23]. In what follows all the norms will be understood to involve the absolute value $|\cdot|$ on $\Lambda^2(\mathbb{R}^4) \otimes \mathcal{G}$. We denote by $\mathcal{N}^-_{\tau_*}(p)$ the null

[22] Recall that $\mathbf{L} = \partial_r - \partial_t$, $\mathcal{D}_\mathbf{L} = \partial_r - \partial_t + [\lambda_\mathbf{L}, \cdot]$ and $r = |y|$.

[23] $\mathbb{R}^4$ here stands for the Euclidean 4-dimensional space.

boundary of Ω to the future of Σ_{τ_*}. Then

$$
\begin{aligned}
|\int_\Omega \delta(u) \, < \mathbf{A}, [\Lambda, \Lambda] > | &\le \|r\mathbf{A}\|_{L^\infty(\mathcal{N}^-_{\tau_*}(p))} \|r^{-1}[\Lambda,\Lambda]\|_{L^1(\mathcal{N}^-_{\tau_*}(p))},\\
|\int_\Omega \delta(u) \, < [\Lambda_{\mathbf{L}\underline{\mathbf{L}}}, \mathbf{A}], \Lambda > | &\le \|r\mathbf{A}\|_{L^\infty(\mathcal{N}^-_{\tau_*}(p))} \|r^{-1}\Lambda_{\mathbf{L}\underline{\mathbf{L}}}\|_{L^1(\mathcal{N}^-_{\tau_*}(p))} \|\Lambda\|_{L^\infty(\mathcal{N}^-_{\tau_*}(p))},\\
|\int_\Omega \delta(u) \, < \Delta^{(\lambda)}\mathbf{A}, \Lambda > | &\le \|\Lambda\|_{L^\infty(\mathcal{N}^-_{\tau_*}(p))} \|\Delta^{(\lambda)}\mathbf{A}\|_{L^1(\mathcal{N}^-_{\tau_*}(p))}
\end{aligned}
$$

It is easy to see, see e.g. [EM2] that

$$
|[\Lambda, \Lambda]| \le |\Lambda| \left(|\Lambda_{\mathbf{L}\underline{\mathbf{L}}}| + \sum_{a=1}^{2} |\Lambda_{\mathbf{L}a}| \right)
$$

and therefore

$$
\|r^{-1}[\Lambda,\Lambda]\|_{L^1(\mathcal{N}^-_{\tau_*}(p))} \le \tau_*^{\frac12} \|\Lambda\|_{L^\infty(\mathcal{N}^-_{\tau_*}(p))} \left(\mathcal{F}^-_p\right)^{\frac12}
$$

Similarly,

$$
\|r^{-1}\Lambda_{\mathbf{L}\underline{\mathbf{L}}}\|_{L^1(\mathcal{N}^-_{\tau_*}(p))} \le \tau_*^{\frac12} \left(\mathcal{F}^-_p\right)^{\frac12}
$$

To prove Lemma 4.4 it would be sufficient to show that

$$
\|r\mathbf{A}\|_{L^\infty(\mathcal{N}^-_{\tau_*}(p))} \lesssim |\mathbf{J}|, \qquad \|\Delta^\lambda \mathbf{A}\|_{L^1(\mathcal{N}^-_{\tau_*}(p))} \lesssim |\mathbf{J}| \left(\tau_*^{\frac32} \left(\mathcal{F}^-_p\right)^{\frac12} + \tau_* \mathcal{F}^-_p\right) \tag{60}
$$

and then choose $\tau_* << (1+\mathcal{E}_0)^{-1}$ as $\mathcal{F}^-_p \le \mathcal{E}_0$.

To prove (60) we introduce a new $\mathcal{G}$ valued 2-form $\mathbf{B} = r\mathbf{A}$ so that

$$
\mathcal{D}_{\mathbf{L}}\mathbf{B} = 0, \qquad \mathbf{B}|_{r=0} = \mathbf{J}.
$$

Considering the components of the 2-form $\mathbf{B}$ it suffices to assume that $\mathbf{B}$ is a $\mathcal{G}$ valued function on $\mathbb{R}^{3+1}$, in fact on $\mathcal{N}^-_{\tau_*}(p)$.

Commuting[24] the transport equation $\mathcal{D}_{\mathbf{L}}\mathbf{B} = 0$ with $r^2\Delta^{(\lambda)}$ we obtain

$$
\mathcal{D}_{\mathbf{L}}(r^2\Delta^{(\lambda)}\mathbf{B}) = r^2\,[\Lambda_{\mathbf{L}}^{\ a}, \nabla^{(\lambda)}_a \mathbf{B}] + r^2\,\nabla^{(\lambda)}_a [\Lambda_{\mathbf{L}}^{\ a}, \mathbf{B}]
$$

We also have the equation

$$
\mathcal{D}_{\mathbf{L}}(r\nabla^{(\lambda)}_a \mathbf{B}) = r\,[\Lambda_{\mathbf{L}a}, \mathbf{B}]
$$

We combine these equations into the system:

$$
\begin{aligned}
&\mathcal{D}_{\mathbf{L}}\mathbf{B} = 0, \qquad\qquad (61)\\
&\mathcal{D}_{\mathbf{L}}(r\nabla^{(\lambda)}_a \mathbf{B}) = r\,[\Lambda_{\mathbf{L}a}, \mathbf{B}],\\
&\mathcal{D}_{\mathbf{L}}(r^2\Delta^{(\lambda)}\mathbf{B}) = 2r^2\,[\Lambda_{\mathbf{L}}^{\ a}, \nabla^{(\lambda)}_a \mathbf{B}] + r^2\,[\nabla^{(\lambda)}_a \Lambda_{\mathbf{L}}^{\ a}, \mathbf{B}].
\end{aligned}
$$

The first equation immediately implies that

$$
\sup_{\mathcal{N}^-_{\tau_*}(p)} |\mathbf{B}| \le |\mathbf{J}|,
$$

[24]Recall that $\nabla^{(\lambda)}$ is a gauge covariant derivative acting on sections $S_r \to P \times_{Ad} \mathcal{G}$ and $\Delta^{(\lambda)}$ is the corresponding gauge Laplace-Beltrami operator on a standard sphere S_r of radius r.

as the covariant derivative $\mathbf{D}_\alpha = \partial_\alpha + [\lambda.\cdot]$ is compatible with a scalar product on $\mathcal{G}$. We infer from the second equation that

$$\|\nabla^{(\lambda)}\mathbf{B}\|_{L^2(S_r)} \le |\mathbf{J}| \sum_{a=1,2} \int_0^r \left(\int_{S_\rho} |\Lambda_{\mathbf{L}a}|^2 d\sigma_s \right)^{\frac12} d\rho$$

where $d\sigma_s$ is the are element of a 2-dimensional sphere S_ρ of radius ρ.

To treat the last equation in (61) we need to worry about the term $[\nabla_a^{(\lambda)}\Lambda_{\mathbf{L}}^{\ a}, \mathbf{B}]$ which contains derivatives of Λ. Recall that the flux only allows us to estimate the tangential components of Λ and none if its derivatives. We get around this difficulty by expressing the Yang -Mills equations $\mathbf{D}^\alpha \Lambda_{\alpha\beta} = 0$ relative to the null frame $\mathbf{L}, \underline{\mathbf{L}}, e_1, e_2$. In particular, $\mathcal{D}^a \Lambda_{\mathbf{L}a} + \mathcal{D}^{\underline{\mathbf{L}}} \Lambda_{\mathbf{L}\underline{\mathbf{L}}} = 0$. This in turn implies that

$$\nabla_a^{(\lambda)}\Lambda_{\mathbf{L}}^{\ a} = \frac12 \mathcal{D}_{\mathbf{L}} \Lambda_{\mathbf{L}\underline{\mathbf{L}}} + \frac1r \Lambda_{\mathbf{L}\underline{\mathbf{L}}}.$$

Thus

$$\mathcal{D}_{\mathbf{L}}(r^2 \Delta^{(\lambda)}\mathbf{B} - \frac12 r^2 [\Lambda_{\mathbf{L}\underline{\mathbf{L}}}, \mathbf{B}]) = 2r^2 \, [\Lambda_{\mathbf{L}}^{\ a}, \nabla_a^{(\lambda)}\mathbf{B}]$$

Therefore,

$$\|\Delta^{(\lambda)}\mathbf{B}\|_{L^1(S_r)} \le |\mathbf{J}| \left(\frac12 \|\Lambda_{\mathbf{L}\underline{\mathbf{L}}}\|_{L^1(S_r)} + \left(\sum_{a=1,2} \int_0^r \left(\int_{S_\rho} |\Lambda_{\mathbf{L}a}|^2 d\sigma_s \right)^{\frac12} d\rho \right)^2 \right)$$

Integrating with respect to r we obtain

$$\|\Delta^{(\lambda)}\mathbf{B}\|_{L^1(\mathcal{N}^-_{\tau_*}(p))} \lesssim |\mathbf{J}| \left(\tau_*^{\frac32} \|\Lambda_{\mathbf{L}\underline{\mathbf{L}}}\|_{L^2(\mathcal{N}^-_{\tau_*}(p))} + \tau_* \sum_{a=1,2} \|\Lambda_{\mathbf{L}a}\|^2_{L^2(\mathcal{N}^-_{\tau_*}(p))} \right).$$

and the result follows. ∎

5. Applications to General Relativity

In this section we specialize our results to Einstein vacuum space-times $(\mathbf{M}, \mathbf{g})$:

$$\mathbf{R}_{\alpha\beta} - \frac12 \mathbf{R}\, \mathbf{g}_{\alpha\beta} = 0 \tag{62}$$

Equations (62) combined with the Bianchi identities imply that the Riemann curvature tensor $\mathbf{R}_{\alpha\beta\mu\nu}$ of an Einstein vacuum metric $\mathbf{g}$ satisfies a covariant wave equation

$$\square_{\mathbf{g}} \mathbf{R}_{\alpha\beta\mu\nu} = (\mathbf{R} \star \mathbf{R})_{\alpha\beta\mu\nu} \,,$$

where the quadratic term $\mathbf{R} \star \mathbf{R}$ is obtained by a contraction [25] of the curvature tensor $\mathbf{R}_{\alpha\beta\mu\nu}$ with itself.

[25]These contractions result in a special structure of the quadratic term, crucial to the analysis in [Kl-Ro5] where we investigate a breakdown criterion in General Relativity. The structure of this term is somewhat analogous to the corresponding term in the Yang-Mills theory, see previous section.

Theorem 5.1. *Let p be a point to the future of a space-like hypersurface Σ in an Einstein vacuum space-time* $(\mathbf{M}, \mathbf{g})$. *We assume that assumptions* $\mathbf{A1}, \mathbf{A2}$ *are verified at p. Let* $\mathbf{A}$ *be a 4-tensor verifying,*

$$\mathbf{D_L A} + \frac{1}{2}\mathbf{A}\, tr\chi = 0, \quad s\mathbf{A}(p) = \mathbf{J}_p \qquad on \quad u = 0$$

where $\mathbf{J}_p$ *is a fixed 4-tensor at p. Then the curvature tensor* $\mathbf{R}_{\alpha\beta\mu\nu}$ *of* $\mathbf{g}$ *can be represented by the following formula at point p with* $\Omega = \mathcal{J}^-(p) \cap \mathcal{J}^+(\Sigma)$,

$$\begin{aligned} 4\pi \mathbf{g}(\mathbf{R}(p), \mathbf{J}_p) &= -\int_\Omega \delta(u)\, \mathbf{g}(\mathbf{A}, \mathbf{R} \star \mathbf{R}) \\ &- \frac{1}{2}\int_\Omega \delta(u)\, \mathbf{R}(\cdot, \cdot, \underline{\mathbf{L}}, \mathbf{L}) \# \mathbf{R} \# \mathbf{A} + \int_\Omega \delta(u)\, \mathbf{g}\left((\Delta \mathbf{A} + \zeta_a \mathbf{D}_a \mathbf{A}), \mathbf{R}\right) \\ &+ \frac{1}{2}\int_\Omega \delta(u)\, \mu\, \mathbf{g}(\mathbf{A}, \mathbf{R}) + \int_\Sigma \left(\mathbf{g}(\mathbf{A}\delta(u), \mathbf{D_T R}) - \mathbf{g}(\mathbf{D_T}(\mathbf{A}\delta(u)), \mathbf{R})\right) \end{aligned} \tag{63}$$

where # denotes a contraction operation between tensors. The last term represents the contribution of the initial data on Σ.

Representation (63) opens the possibility of proving a pointwise bound on the curvature tensor in terms of initial data on Σ and, as in the Yang-Mills case, the flux of curvature along the null boundary $\mathcal{N}^-(p)$ of the set Ω. However, as opposed to the Yang-Mills equations on Minkowski background, where the curvature flux is bounded by the L^2-norm of the curvature of initial data, no such a priori bounds are available for the Einstein vacuum equations. This suggests the use of the L^2 based curvature norms to deduce a breakdown criteria in General Relativity, i.e. to show that the space-time can be continued as long as such norms remain finite. For the Yang-Mills problem in Minkowski space the underlying reason for having an a- priori bounds on the flux of curvature is due to the presence of the Killing vectorfield $\partial_t = \frac{\partial}{\partial t}$. In the case of the Yang-Mills equations on a smooth curved background, such as in [CS], the result remains true even though ∂_t is no longer Killing; it suffices that its deformation tensor is bounded. We call such a vectorfield *approximately Killing*.

These considerations suggest the following question. Assume that the space-time $(\mathbf{M}, \mathbf{g})$ possesses an *approximately Killing*, unit, vectorfield $\mathbf{T}$, orthogonal to a space-like Cauchy hypersurface Σ, with deformation tensor $\pi(X, Y) = \mathbf{g}(\mathbf{D}_X \mathbf{T}, Y)$. We also assume that the slices Σ_t obtained by following integral curves of $\mathbf{T}$ from Σ_0 have constant mean curvature. Can the space-time be extended as long as π remains finite in the uniform norm?

The finiteness of the deformation tensor π allows one to control, via energy estimates based on the Bel-Robinson tensor, both the L^2 norms of the curvature $\mathbf{R}$ along Σ_t and the flux of curvature along the null boundaries $\mathcal{N}^-(p)$. The key step in the remaining analysis is to derive a pointwise curvature based on the representation formula (63). In [Kl-Ro5] we give an affirmative answer to the question raised above by showing that the size of the region of validity of the formula (63) depends only on the assumed L^∞- bounds on π and reasonable assumptions on the initial data on Σ. Such estimates follow from our work in [Kl-Ro4]. More precisely we show that for a space-time metric $\mathbf{g}$ in the form

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j,$$

where n is the lapse function of the t foliation and the vectorfield $\mathbf{T}$ is orthogonal to Σ_t, the following result holds true.

Theorem 5.2. *Assume that $(\mathbf{M}, \mathbf{g})$ is a globally hyperbolic Einstein vacuum space-time with Σ_0 a Cauchy hypersurface. Let the lapse function n and the deformation tensor π of $\mathbf{T}$ satisfy*

$$N_0^{-1} \le n \le N_0, \qquad \|\pi\|_{L^\infty} \le \mathcal{K}_0$$

Assume also that $\mathbf{M}$ contains a future, compact set $\mathcal{D} \subset \mathbf{M}$ such that for any point $q \in \mathcal{D}^c$ the radius of injectivity of $\mathcal{N}^-(q)$ is at least $\delta_0 > 0$.

Let Σ be one of the slices of the t foliation. Then assumptions **A1**, **A2** *of this paper are satisfied for all points p at distance $\le \delta_*$ from Σ, where δ_* depends only on the Cauchy data on Σ_0, N_0, $\mathcal{K}_0$ and δ_0. In particular, the representation formula* (63) *holds for all such points.*

6. Open questions

All the results of this paper have been derived under assumption **A2** which requires that, for any point p, the boundary $\mathcal{N}^-(p)$ of the causal past $\mathcal{J}^-(p)$ remains smooth at least until it reaches the space-like hypersurface Σ. It is only under this assumption that we can guarantee that the Kirchoff-Sobolev parametrix of Theorem 3.11 gives a faithful representation of a solution of the wave equation. We have already discussed the two obstructions to smoothness of $\mathcal{N}^-(p)$: conjugate points of the congruence of past directed null geodesics from p and intersection of two distinct past directed null geodesics from p. The second obstruction can be easily demonstrated on a space-time $\mathbf{M} = \mathbb{R} \times \mathbb{T}^3$ or $\mathbf{M} = \mathbb{R} \times \Pi_a$ based on a flat torus $\mathbb{T}^3$ or a flat cylinder Π_a of width a. In those cases, however, an (exact) parametrix can be easily constructed by lifting the problem to the covering space $\mathbb{R} \times \mathbb{R}^3$. On the other hand, the examples above are very special. Conjugate points can not be *removed* by a simple[26] lifting and the quantity $\mathrm{tr}\chi$, which features prominently in our representation formulas, diverges to $-\infty$ at a conjugate point. However, the same focusing phenomenon shrinks the volume of a *conjugate point region* thus leaving open a possibility that, perhaps, with some additional assumptions on the structure and strength of conjugate points, the integral quantities appearing in our representation formulas remain finite. It may thus be that the Kirchoff-Sobolev parametrix remains valid even beyond the region of formation of conjugate points. A good place to start investigating this issue would be product manifolds $\mathbf{M} = \mathbb{R} \times M$ with metrics of the form $\mathbf{g} = -dt^2 + g_{ij} dx^i dx^j$. A particularly interesting class to consider are product manifolds with M collapsing in the sense of Cheeger-Gromov.

[26]Conjugate points can be *desingularized* however by lifting to the cotangent space

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