# Peeling properties of asymptotically flat solutions to the Einstein vacuum equations

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**Abstract**

We show that, under stronger asymptotic decay and regularity properties than those used in [Ch-Kl], [Kl-Ni], asymptotically flat initial data sets lead to solutions of the Einstein vacuum equations which have strong peeling properties consistent with the predictions of the conformal compactification approach of Penrose. More precisely we provide a systematic picture of the relationship between various asymptotic properties of the initial data sets and the peeling properties of the corresponding solutions.

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# 1 Introduction and results

One of the outstanding open questions in the wake of the global stability result of [Ch-Kl] and [Kl-Ni] is whether there exists initial data which have a "smooth" skri. Indeed the results in [Ch-Kl] and [Kl-Ni] based on general initial data lead to weaker peeling properties than those derived, see [Pe], from the assumption of asymptotic simplicity. The seminal work of Penrose, [Pe], and Penrose and Newman, [Ne-Pe2], had opened the way to many attempts to provide a constructive approach to such solutions, see [Fr1],[Fr2], [Kr]. Yet the question whether smooth skri is compatible with initial data, given on a Cauchy spacelike hypersurface, has remained wide open for a long time. An important recent result in this direction is due to P.T.Chruściel and E.Delay, [Chr-Del]. They prove, by adapting the previous result by J.Corvino, [Cor], the existence of sufficiently small initial conditions which are exactly stationary outside a sufficiently large compact set, whose Cauchy development is asymptotically simple. They do that by showing the existence of hyperboloidal hypersurfaces of the kind needed in Friedrich's stability theorem, [Fr3]. For a detailed review of this approach see [Fr4].

In this paper we find sufficient decay and regularity assumptions on the initial data sets such that the corresponding spacetimes (defined in the complement of the domain of influence of a compact set) verify peeling properties consistent with asymptotic simplicity. We do that by revisiting and adapting the global existence proof in [Kl-Ni]. More precisely we provide a systematic picture of the relationship between various asymptotic properties of the initial data sets and the peeling properties of the corresponding solutions.

We shall consider asymptotically flat initial data sets $\{\Sigma_0, g, k\}$ for which there exists a system of coordinates $x = \{x^1, x^2, x^3\}$ defined outside a sufficiently large compact set, which verify the following asymptotic assumptions:[1]

$$\begin{aligned} g_{ij} - \delta_{ij} &= \frac{2M}{r}\delta_{ij} + O_{q+1}(r^{-(\frac{3}{2}+\gamma)}) \\ k_{ij} &= O_q(r^{-(\frac{5}{2}+\gamma)}) \end{aligned} \tag{1.1}$$

In [Kl-Ni] we have constructed, under the additional smallness assumption[2] $J_K(\Sigma_0, g, k) \leq \varepsilon^2$, see [Kl-Ni] Chapter 3, definition (3.6.4), a unique devel-

[1] Here $f = O_q(r^{-a})$ means that $f$ asymptotically behaves as $O(r^{-a})$ and its partial derivatives $\partial^k f$, up to order $q$ behave as $O(r^{-a-k})$.

[2] $J_K(\Sigma_0, g, k)$ is a $L^2$ weighted norm made with the partial derivatives of the Riemann metric tensor $g_{ij}$ up to fourth order.

opment $(\mathcal{M}, \mathbf{g})$ defined outside the domain of influence of the compact set $K$, foliated by a canonical double null foliation $\{C(\lambda), \underline{C}(\nu)\}$ whose leaves are the level hypersurfaces of two optical functions $u(p)$ and $\underline{u}(p)$:

$$C(\lambda) = \{p \in \mathcal{M} | u(p) = \lambda\} \ , \ \underline{C}(\nu) = \{p \in \mathcal{M} | \underline{u}(p) = \nu\}$$

and such that the outgoing leaves $C(\lambda)$ are complete. To state our asymptotic results we associate, see [Kl-Ni] Chapter 3, to our double null foliation an adapted null frame $\{e_1, e_2, e_3, e_4\}$ where

$$e_4 = 2\Omega L \ \ ; \ \ e_3 = 2\Omega \underline{L} \tag{1.2}$$

and $L$ and $\underline{L}$ are the null geodesic vector fields generating the null hypersurfaces $C(\lambda)$ and $\underline{C}(\nu)$ respectively. $\Omega^2$ is the null shift function,

$$\Omega^2 = -(2\mathbf{g}(L, \underline{L}))^{-1} \tag{1.3}$$

and finally $\{e_1, e_2\}$ is an orthonormal moving frame tangent to the two-dimensional surfaces $S(\lambda, \nu) = C(\lambda) \cap \underline{C}(\nu)$. Using this frame we decompose the Riemann curvature tensor into its null components, in the following way:

$$\begin{aligned}
&\alpha(\mathbf{R})(X,Y) = \mathbf{R}(X, e_4, Y, e_4) \ , \ \beta(\mathbf{R})(X) = \frac{1}{2}\mathbf{R}(X, e_4, e_3, e_4) \\
&\rho(\mathbf{R}) = \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4) \quad , \quad \sigma(\mathbf{R}) = \frac{1}{4}{}^\star\mathbf{R}(e_3, e_4, e_3, e_4) \\
&\underline{\beta}(\mathbf{R})(X) = \frac{1}{2}\mathbf{R}(X, e_3, e_3, e_4) \ , \ \underline{\alpha}(\mathbf{R})(X,Y) = \mathbf{R}(X, e_3, Y, e_3)
\end{aligned}$$

where $^\star$ denotes the dual tensor.

**Theorem 1.1 (Strong peeling property)** *Under the assumptions 1.1 with $\gamma = \frac{3}{2} + \epsilon$, $\epsilon > 0$ and sufficiently large $q$ the maximal development $(\mathcal{M}, \mathbf{g})$ constructed in [Kl-Ni] verifies the following stronger asymptotic properties:*

**a)** *Along the outgoing null hypersurfaces $C(\lambda)$ the following limits hold*

$$\begin{aligned}
&\lim_{C(\lambda);\nu\to\infty} r\underline{\alpha} = \underline{A}(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^2\underline{\beta} = \underline{B}(\lambda,\omega) \\
&\lim_{C(\lambda);\nu\to\infty} r^3\rho = P(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^3\sigma = Q(\lambda,\omega) \\
&\lim_{C(\lambda);\nu\to\infty} r^4\beta = B(\lambda,\omega)
\end{aligned} \tag{1.4}$$

*with $\underline{A}(\lambda,\omega), \underline{B}(\lambda,\omega), P(\lambda,\omega), Q(\lambda,\omega), B(\lambda,\omega)$ satisfying:*

$$\begin{aligned}
&|\underline{A}(\lambda,\omega)| \leq c(1+|\lambda|)^{-(4+\epsilon)} \quad ; \quad |\underline{B}(\lambda,\omega)| \leq c(1+|\lambda|)^{-(3+\epsilon)} \\
&|(P-\overline{P})(\lambda,\omega)| \leq c(1+|\lambda|)^{-(2+\epsilon)} \ ; \ |(Q-\overline{Q})(\lambda,\omega)| \leq c(1+|\lambda|)^{-(2+\epsilon)} \\
&|B(\lambda,\omega)| \leq c(1+|\lambda|)^{-(1+\epsilon)}
\end{aligned} \tag{1.5}$$

**b)** *The Riemann components $\alpha$ and $\beta$ satisfy the following estimates, with $\epsilon' < \epsilon$:* [3]

$$\sup_{\widetilde{\mathcal{M}}} |r^5 |\lambda|^{\epsilon'} \alpha| \leq C_0 \ ; \ \sup_{\widetilde{\mathcal{M}}} |r^4 |\lambda|^{1+\epsilon'} |\beta| \leq C_0$$

**Theorem 1.2 (Weak peeling property)** *Under the assumptions 1.1 with $\gamma = \frac{3}{2} - \delta$, $\delta \in [0, \frac{3}{2})$ and sufficiently large $q$ the spacetime $(\mathcal{M}, \mathbf{g})$ constructed in [Kl-Ni] verifies the following asymptotic properties:*[4]

**a)** *Along the outgoing null hypersurfaces $C(\lambda)$ the following limits hold*

$$\begin{aligned}
&\lim_{C(\lambda);\nu\to\infty} r\underline{\alpha} = \underline{A}(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^2\underline{\beta} = \underline{B}(\lambda,\omega) \\
&\lim_{C(\lambda);\nu\to\infty} r^3\rho = P(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^3\sigma = Q(\lambda,\omega) \\
&\lim_{C(\lambda);\nu\to\infty} r^4\beta = B(\lambda,\omega) \quad \text{for } \delta \in [0,1)
\end{aligned}$$

*with $\underline{A}(\lambda,\omega)$, $\underline{B}(\lambda,\omega)$, $P(\lambda,\omega)$, $Q(\lambda,\omega)$, $B(\lambda,\omega)$ satisfying:*

$$\begin{aligned}
&|\underline{A}(\lambda,\omega)| \leq c(1+|\lambda|)^{-(4-\delta)} \quad ; \quad |\underline{B}(\lambda,\omega)| \leq c(1+|\lambda|)^{-(3-\delta)} \\
&|(P-\overline{P})(\lambda,\omega)| \leq c(1+|\lambda|)^{-(2-\delta)} \quad ; \quad |(Q-\overline{Q})(\lambda,\omega)| \leq c(1+|\lambda|)^{-(2-\delta)} \\
&|B(\lambda,\omega)| \leq c(1+|\lambda|)^{-(1-\delta)}
\end{aligned} \tag{1.6}$$

**b)** *The Riemann components $\alpha$ and $\beta$ satisfy the following estimates:*

$$\begin{aligned}
&\delta = 0 \quad ; \quad \sup_{\widetilde{\mathcal{M}}} \left| \frac{r^5 (\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}} \alpha \right| \leq C_0 \ ; \ \sup_{\widetilde{\mathcal{M}}} \left| r^4 |\lambda| \beta \right| \leq C_0 \\
&\delta \in (0,1) \ ; \quad \sup_{\widetilde{\mathcal{M}}} \left| \frac{r^{(5-\delta)} (\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}} \alpha \right| \leq C_0 \ ; \ \sup_{\widetilde{\mathcal{M}}} \left| r^4 |\lambda|^{1-\delta} \beta \right| \leq C_0 \qquad (1.7) \\
&\delta = 1 \quad ; \quad \sup_{\widetilde{\mathcal{M}}} \left| \frac{r^4 (\log|\lambda|)^\epsilon}{(\log r)^{2+\epsilon}} \alpha \right| \leq C_0 \ ; \ \sup_{\widetilde{\mathcal{M}}} \left| \frac{r^4 (\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}} \beta \right| \leq C_0 \\
&\delta \in (1, \frac{3}{2}) \ ; \quad \sup_{\widetilde{\mathcal{M}}} \left| \frac{r^{(5-\delta)} (\log|\lambda|)^\epsilon}{(\log r)^{2+\epsilon}} \alpha \right| \leq C_0 \ ; \ \sup_{\widetilde{\mathcal{M}}} \left| \frac{r^{(5-\delta)} (\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}} \beta \right| \leq C_0 \qquad (1.8)
\end{aligned}$$

**Remark:** Observe that strong peeling in the sense of Theorem 1.1, is incompatible with the presence of a nontrivial angular momentum. Indeed recall that, see [Ch-Kl], Chapter 1,

$$J_i = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} \epsilon_{iab} x^a \left( k^{bj} - g^{bj} \mathrm{tr}k \right) n_j d\sigma \ . \tag{1.9}$$

[3] Hereafter we always assume $\epsilon' < \epsilon$, wherever these two quantities appear.

[4] The precise version of this and the next theorem will be presented in Section 6.

Thus it easily follows that $\mathbf{J}=0$ if $k$ decays faster than $r^{-3}$. Therefore, in the presence of a nontrivial angular momentum we do not expect the strong peeling estimates, but rather the weaker ones consistent with $\delta=1$,

$$\sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^\epsilon}{(\log r)^{2+\epsilon}}\alpha\right|\leq C_0 \ \ ; \ \ \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}}\beta\right|\leq C_0 \tag{1.10}$$

### 1.1 The main ideas of the proof.

The proof of the stated results is based on two separate ingredients.

**Part I.)** By a modification of the construction used in the proof of the Klainerman-Nicolò global existence result, see [Kl-Ni] Chapter 3, we show that the the spacetime $\widetilde{\mathcal{M}}$ satisfies better asymptotic properties relative to $\lambda$. In particular we show that the null curvature components verify,

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{7/2}|\lambda|^\gamma|\alpha|\leq C_0\ ,\ \sup_{\mathcal{K}} r|\lambda|^{\frac{5}{2}+\gamma}|\underline{\alpha}|\leq C_0\\
&\sup_{\mathcal{K}} r^{7/2}|\lambda|^\gamma|\beta|\leq C_0\ ,\ \sup_{\mathcal{K}} r^2|\lambda|^{\frac{3}{2}+\gamma}|\underline{\beta}|\leq C_0\\
&\sup_{\mathcal{K}} r^3|\rho|\leq C_0\ ,\ \sup_{\mathcal{K}} r^3|\lambda|^{\frac{1}{2}+\gamma}|(\rho-\overline{\rho},\sigma)|\leq C_0
\end{aligned}\tag{1.11}$$

In addition we can show that the angular derivatives of order $q>0$ verify:[5]

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{7/2+q}|\lambda|^\gamma|\nabla\!\!\!\!/^q\alpha|\leq C_0\ ,\ \sup_{\mathcal{K}} r^{1+q}|\lambda|^{\frac{5}{2}+\gamma}|\nabla\!\!\!\!/^q\underline{\alpha}|\leq C_0\\
&\sup_{\mathcal{K}} r^{7/2+q}|\lambda|^\gamma|\nabla\!\!\!\!/^q\beta|\leq C_0\ ,\ \sup_{\mathcal{K}} r^{2+q}|\lambda|^{\frac{3}{2}+\gamma}|\nabla\!\!\!\!/^q\underline{\beta}|\leq C_0\\
&\sup_{\mathcal{K}} r^{3+q}|\lambda|^{\frac{1}{2}+\gamma}|\nabla\!\!\!\!/^q(\rho,\sigma)|\leq C_0
\end{aligned}\tag{1.12}$$

Remark that the only difference between these and the estimates established in [Kl-Ni] is due to the factor $|\lambda|^\gamma$. It is worthwile to point here that, exactly as in [Kl-Ni], the central part of the proof is based on the introduction of a family of energy-type norms $\tilde{\mathcal{Q}}$, based on the Bel-Robinson tensor associated to the (conformal part of the) Riemann tensor. These norms are modifications of the $\mathcal{Q}$ norms in [Kl-Ni], obtained by introducing the weight factor $|\lambda|^{2\gamma}$ in their integrand. The precise definition of these norms is given in section 2.

The fact that we can incorporate the additional weight $|\lambda|^{2\gamma}$ in the energy type norms $\tilde{\mathcal{Q}}$ is an essential ingredient of our result. One can motivate this

[5] $q$ is an integer $\leq 5$. $\nabla\!\!\!\!/$ is the covariant derivative associated to the induced metric on $S(\lambda,\nu)=C(\lambda)\cap\underline{C}(\nu)$.

fact by considering weighted energy estimates for the linear wave equations in flat space, outside a light cone. We shall do this in details at the beginning of section 3.

**Part II.)** To obtain improved estimates for the null curvature components $\alpha, \beta$ we shall make use of the incoming transport equations which they satisfy. These will allow us to transfer some of the gain in powers of $|\lambda|^\gamma$ established in the first step to the desired powers of $r$. Before writing down the transport equations for $\alpha$, $\beta$ we recall briefly the definition of the connection coefficients associated to our double null foliation.

$$\begin{aligned}
&\chi_{ab} = \mathbf{g}(\mathbf{D}_{e_a}e_4, e_b) \quad , \quad \underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_{e_a}e_3, e_b)\\
&\xi_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4}e_4, e_a) \quad , \quad \underline{\xi}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3}e_3, e_a)\\
&\eta_a = -\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3}e_a, e_4) \ , \ \underline{\eta}_a = -\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4}e_a, e_3)\\
&\omega = -\frac{1}{4}\mathbf{g}(\mathbf{D}_{e_4}e_3, e_4) \quad , \quad \underline{\omega} = -\frac{1}{4}\mathbf{g}(\mathbf{D}_{e_3}e_4, e_3)\\
&\zeta_a = -\frac{1}{2}\mathbf{g}(\mathbf{D}_{e_a}e_3, e_4)
\end{aligned} \tag{1.13}$$

Moreover we observe that in the first step presented above one can show not only that the null curvature components gain decay in powers of $\lambda$ but also the connection coefficients. In particular one can prove the following estimates:

$$\begin{aligned}
&|r^2|\lambda|^{\frac{1}{2}+\gamma}\hat{\chi}| \leq C_0 \quad , \quad |r^3|\lambda|^{\frac{1}{2}+\gamma}\nabla\!\!\!\!/\,\hat{\chi}| \leq C_0\\
&|r^1|\lambda|^{\frac{3}{2}+\gamma}\underline{\hat{\chi}}| \leq C_0 \quad , \quad |r^2|\lambda|^{\frac{3}{2}+\gamma}\nabla\!\!\!\!/\,\underline{\hat{\chi}}| \leq C_0\\
&|r^2|\lambda|^{\frac{1}{2}+\gamma}\left(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}\right)| \leq C_0 \ , \ |r^2|\lambda|^{\frac{1}{2}+\gamma}\left(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}}\right)| \leq C_0\\
&|r^2|\lambda|^{\frac{1}{2}+\gamma}\eta| \leq C_0 \quad , \quad |r^2|\lambda|^{\frac{1}{2}+\gamma}\underline{\eta}| \leq C_0
\end{aligned} \tag{1.14}$$

With these preparations we are now ready to demonstrate the improved behavior of $\alpha, \beta$. [6]

**i) Improved estimate for $\beta$.** According to the null Bianchi equations, see [Kl-Ni] Chapter 3, eqs. (3.1.46), (3.1.47), (3.1.48) we have,

$$\frac{\partial\beta}{\partial\lambda} + \Omega\mathrm{tr}\underline{\chi}\beta = 2\Omega\underline{\omega}\beta + \Omega\nabla\!\!\!\!/\,\rho + \Omega{}^*\nabla\!\!\!\!/\,\sigma + \left[2\hat{\chi}\cdot\underline{\beta} + 3(\eta\rho + {}^*\eta\sigma)\right] \tag{1.15}$$

[6] A similar argument, based on the idea of using the incoming null Bianchi equations for $\alpha$ and $\beta$, has been used by D. Christodoulou in a related context, see [Ch].

where $\frac{\partial}{\partial\lambda} = \Omega \mathbf{D}\!\!\!/_3$ and $\mathbf{D}\!\!\!/_3$ denotes the projection to $S(\lambda,\nu)$ of the derivative along the null direction $e_3$. In view of the above estimates 1.11, 1.12 and 1.14 we observe that the terms in square brackets are of order $O(r^{-4}|\lambda|^{-(\frac{3}{2}+\gamma)})$ and can be neglected. Recalling,[7] see [Kl-Ni], Chapter 4, that $\frac{\partial r}{\partial\lambda} = \frac{r}{2}\overline{\Omega\mathrm{tr}\underline{\chi}}$, we obtain

$$\frac{\partial(r^2\beta_a)}{\partial\lambda} = \Big[(\overline{\Omega\mathrm{tr}\underline{\chi}} - \Omega\mathrm{tr}\underline{\chi}) + 2\Omega\underline{\omega}\Big](r^2\beta_a) + \Omega r^2 \nabla\!\!\!\!/_a\rho \tag{1.16}$$

In view of the estimates for the connection coefficients 1.14 [8] it follows immediately that the quantity $\Big[(\overline{\Omega\mathrm{tr}\underline{\chi}} - \Omega\mathrm{tr}\underline{\chi}) + 2\Omega\underline{\omega}\Big]$ is integrable in $\lambda$. An application of Gronwall's Lemma gives, with $\lambda_1(\nu) = \lambda|_{\underline{C}(\nu)\cap\Sigma_0}$,

$$|r^2\beta|(\lambda,\nu,\omega^a) \le c\bigg(|r^2\beta|_{\underline{C}(\nu)\cap\Sigma_0} + \int_{\lambda_1(\nu)}^{\lambda} |r^2\nabla\!\!\!\!/\rho|d\lambda'\bigg) \tag{1.17}$$

and, multiplying both sides by $r^2|\lambda|^{1+\epsilon}$, which is allowed in view of the fact that $r$ and $|\lambda|$ are both decreasing as we move toward the future along the incoming null hypersurface $C(\lambda)$, we obtain

$$|r^4|\lambda|^{1+\epsilon}\beta|(\lambda,\nu,\omega^a) \le c\bigg(|r^{5+\epsilon}\beta|_{\underline{C}(\nu)\cap\Sigma_0} + \int_{\lambda_1(\nu)}^{\lambda} |r^4|\lambda'|^{1+\epsilon}\nabla\!\!\!\!/\rho|d\lambda'\bigg) \tag{1.18}$$

To obtain the "strong peeling" property for $\beta$ it remains to show that the right hans side of 1.18 is bounded. The finiteness of $|r^{5+\epsilon}\beta|_{\underline{C}(\nu)\cap\Sigma_0}$ follows immediately from our initial data assumptions. To check the finiteness of the integral term we only need to make use of the asymptotic result for $\rho$ obtained in the part I. Indeed we have $\nabla\!\!\!\!/\rho = 0(r^{-4}|\lambda|^{-(\frac{1}{2}+\gamma)})$ and, therefore, for $\gamma > \frac{3}{2}+\epsilon$, the integral $\int_{\lambda_1(\nu)}^{\lambda} r^4|\lambda'|^{1+\epsilon}\nabla\!\!\!\!/\rho|d\lambda'$ is bounded.

**ii) Improved estimate for $\alpha$.** Making use of the improved estimate for $\beta$, which we have just established, we make use of the null Bianchi equation, relative to $\alpha$,

$$\frac{\partial\alpha}{\partial\lambda} + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\alpha = 4\Omega\underline{\omega}\alpha + \Omega\nabla\!\!\!\!/\widehat{\otimes}\beta + \Omega\left[-3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma) + (\zeta+4\eta)\widehat{\otimes}\beta\right] \tag{1.19}$$

Clearly the terms in in square brackets are $O(r^{-5}|\lambda|^{-(\frac{1}{2}+\gamma)})$ and can be neglected. Proceeding in the same fashion as above we write,

$$\frac{d}{d\lambda}|r\alpha| \le |4\Omega\underline{\omega} - 2^{-1}(\Omega\mathrm{tr}\underline{\chi} - \overline{\Omega\mathrm{tr}\underline{\chi}})||r\alpha| + |\Omega||r\nabla\!\!\!\!/\beta| \tag{1.20}$$

[7]Here $\overline{f}$ refers to the average of the scalar quantity $f$ over the 2-surfaces $S(\lambda,\nu)$.

[8]In fact it suffices to take the weaker estimates of [Kl-Ni] corresponding to $\gamma$ arbitrarily small. The estimate of $\underline{\omega}$ cannot be improved from the one in [Kl-Ni], but that one is sufficient.

and applying Gronwall's lemma we obtain

$$|r\alpha|(\lambda,\nu,\omega^a)\leq c\bigg(|r\alpha|_{\underline{C}(\nu)\cap\Sigma_0}+\int_{\lambda_1(\nu)}^{\lambda}|r\nabla\!\!\!\!/\,\beta|d\lambda'\bigg)\ . \tag{1.21}$$

Finally, multiplying both sides by $r^4$ we conclude that

$$|r^5\alpha|(\lambda,\nu,\omega^a)\leq c\bigg(|r^5\alpha|_{\underline{C}(\nu)\cap\Sigma_0}+\int_{\lambda_1(\nu)}^{\lambda}|r^5\nabla\!\!\!\!/\,\beta|d\lambda'\bigg) \tag{1.22}$$

To derive the strong peeling property for $\alpha$ we only have to check that the right hand side is finite. In view of our initial conditions it is immediate to check the boundedness of $|r^5\alpha|_{\underline{C}(\nu)\cap\Sigma_0}$. The boundedness of the integral term follows immediately from our improved estimate for $\beta$ discussed above and the fact that any tangential derivative produces an extra $r^{-1}$ factor.

In Section 2 we give a precise statement of the result mentioned in Part I and a sketch of its proof. We define the main norms and outline the main differences between the result presented here and that of [Kl-Ni]. Section 6 is devote to the detailed proof of our peeling result along the lines sketched above. Sections 3, 4 and 5 provide the technical details of the proof of the result presented in section 2.

## 2 The spacetime $\widetilde{\mathcal{M}}$

As discussed in the introduction the proof of our results depend on a modification of the main result proved in [Kl-Ni]. In this section we give a precise formulation of it. We recall that the main theorem in [Kl-Ni] was based on the following norms:

**1)** The initial data norms $J_K$ which we replace here by the norms $\tilde{J}_K^{(q)}$, whose precise definitions are given in the next subsection. The new norms differ from the old ones by the presence of the additional weight factor $r^\gamma$ and additional covariant derivatives up to a prescribed order.

**2)** The connection coefficient norms $\tilde{\mathcal{O}}$. These norms differ from the old ones, $\mathcal{O}$, by the presence of an additional weight factor $|\lambda|^\gamma$, except those which are non trivial in Schwarzschild spacetime,[9] and additional covariant derivatives. Their explicit expressions are given in the appendix.

**3)** The Riemann curvature norms $\tilde{\mathcal{R}}$. Again they differ from the $\mathcal{R}$ norms used in [Kl-Ni] by the presence of the additional weight factor $|\lambda|^\gamma$, except, of course, for $\overline{\rho}$ which is tied to the $ADM$ mass, and additional covariant derivatives. Their explicit expressions are given in the appendix.

**4)** The "Bel-Robinson" integral norms $\tilde{\mathcal{Q}}$. These norms differ from their analogous ones, $\mathcal{Q}$, used in [Kl-Ni] by the presence of the additional weight factor $|\lambda|^{2\gamma}$. We also need to add terms containing higher angular derivatives $\hat{\mathcal{L}}_O^i$ with $i \leq q+1$. We exhibit them below.

**5)** The definition of the canonical foliation on the last slice needs to be modified. More precisely we replace the condition $\overline{\log \Omega} = 0$, used in [Kl-Ni], with the condition

$$\mathbf{D}_3 \overline{\log \Omega} = (\overline{\mathrm{tr}\chi})^{-1} \overline{\rho}. \tag{2.1}$$

### 2.1 Initial hypersurface and final slice

In this section we discuss the initial conditions on both the initial and final slices.

#### 2.1.1 The initial data condition

We restrict ourselves to initial data sets $\{\Sigma_0, g, k\}$ with $\Sigma_0$ diffeomorphic to $R^3$; moreover we assume they are asymptotically flat in the following sense, stronger than the one used in [Ch-Kl] and in [Kl-Ni]:

[9] A more detailed discussion of this fact is in the second part of section 3.

**Definition 2.1** *An initial data set $\{\Sigma_0, g, k\}$ is "strongly asymptotically flat of order $\gamma$", see [Ch-Kl], eqs. (1.0.9a), (1.0.9b), if there exists a compact set $B$, such that its complement $\Sigma_0 \backslash B$ is diffeomorphic to the complement of the closed unit ball in $R^3$. Moreover there exists a coordinate system $(x^1, x^2, x^3)$ defined in a neighborhood of infinity such that, as $r = \sqrt{\sum_{i=1}^3 (x^i)^2} \to \infty$, we have*

$$g_{ij} = (1 + 2M/r)\delta_{ij} + O_{q+1}(r^{-(\frac{3}{2}+\gamma)}) \quad , \quad k_{ij} = O_q(r^{-(\frac{5}{2}+\gamma)}) \tag{2.2}$$

We define the global initial data smallness condition with the help of the quantity

$$\begin{aligned} \tilde{J}_0^{(q)}(\Sigma_0, g, k) &= \sup_{\Sigma_0} \left( (d_0^2 + 1)^3 |Ric|^2 \right) + \int_{\Sigma_0} \sum_{l=0}^{q} (d_0^{2(1+\gamma')} + 1)^{l+1} |\nabla^l k|^2 \\ &+ \int_{\Sigma_0} \sum_{l=0}^{q-2} (d_0^{2(1+\gamma')} + 1)^{l+3} |\nabla^l B|^2 \end{aligned} \tag{2.3}$$

with $\gamma'$ arbitrary close to $\gamma$, $\gamma' < \gamma$, where $\gamma$, see 2.2, has been introduced in Theorems 1.1 and 1.2.

**Definition 2.2** *Given an initial data set $\{\Sigma_0, g, k\}$ and a compact set $K \subset \Sigma_0$ such that $\Sigma_0 \backslash K$ is diffeomorphic to the complement of the closed unit ball in $R^3$, we define $\tilde{J}_K^{(q)}(\Sigma_0, g, k)$ as follows:*

- *We denote $\mathcal{G}$ the set of all the smooth extensions $(\tilde{g}, \tilde{k})$ of the data $(g, k)$ restricted to $\Sigma_0 \backslash K$, to the whole of $\Sigma_0$, with $\tilde{g}$ Riemannian and $\tilde{k}$ a symmetric two tensor.*

- *We denote by $\tilde{d}_0$ the geodesic distance from a fixed point $O$ in $K$ relative to the metric $\tilde{g}$.*

- *We denote* [10]

$$\tilde{J}_K^{(q)}(\Sigma_0, g, k) = \inf_{\mathcal{G}} \tilde{J}_0^{(q)}(\Sigma_0, \tilde{g}, \tilde{k}) \ . \tag{2.4}$$

**Definition 2.3** *Consider an initial data set $\{\Sigma_0, g, k\}$. Let $K$ be a compact set such that $\Sigma_0 \backslash K$ is diffeomorphic to the complement of the closed unit ball in $R^3$. We say that the initial data set satisfy the "exterior global smallness condition" if, given $\varepsilon > 0$ sufficiently small,*

$$\tilde{J}_K^{(q)}(\Sigma_0, g, k) \leq \varepsilon^2 \ .$$

[10] $\tilde{J}_0^{(q)}(\Sigma_0, \tilde{g}, \tilde{k})$ has the same expression as $\tilde{J}_0^{(q)}(\Sigma_0, g, k)$ in 2.3, with $\tilde{d}_0$ instead of $d_0$.

**Remark:** The "exterior global smallness condition" is basically the same we used in [Kl-Ni]. The difference from the previous one is the presence of different weight factors as well as the presence of additional derivatives.

### 2.1.2 The last slice canonical foliation

**Definition 2.4** *A foliation on a null incoming hypersurface $\underline{C}_*$, given by the level sets of a function $u$,*[11] *is said to be canonical if the functions $u$ and $\Omega$ satisfy the following system of equations:*

$$\begin{aligned}
&\frac{du}{dv} = (2\Omega^2)^{-1}; \quad u|_{\underline{C}_*\cap\Sigma_0} = \lambda_1 \\
&\Delta\!\!\!/ \log\Omega = \frac{1}{2}\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta} + \frac{1}{2}\left(\mathbf{K} - \overline{\mathbf{K}} + \frac{1}{4}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \overline{\mathrm{tr}\chi\mathrm{tr}\underline{\chi}})\right) \\
&\frac{d}{dv}\overline{\log\Omega} = \frac{1}{2\Omega}(\overline{\mathrm{tr}\chi})^{-1}\overline{\rho} \quad ; \quad \overline{\log 2\Omega}|_{\underline{C}_*\cap\Sigma_0} = \frac{1}{2}\log(1+r^2\overline{\rho})|_{\underline{C}_*\cap\Sigma_0}
\end{aligned} \tag{2.5}$$

The proof of the existence of the canonical foliation follows precisely the same argument as that used in Chapter 7 of [Kl-Ni] and will be omitted, see also [Ni].

## 2.2 The $\widetilde{\mathcal{Q}}$ norms

With the help of the same vector fields $S, \bar{K}$ and ${}^{(i)}O$ defined as in [Kl-Ni] and a slightly modified vector field $T_0$ we define: [12]

$$\begin{aligned}
\widetilde{\mathcal{Q}}_1(\lambda,\nu) &\equiv \int_{C(\lambda)\cap V(\lambda,\nu)} |\lambda|^{2\gamma} Q(\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} |\lambda|^{2\gamma} Q(\hat{\mathcal{L}}_{O}\mathbf{R})(\bar{K},\bar{K},T_0,e_4) \\
\widetilde{\mathcal{Q}}_2(\lambda,\nu) &\equiv \int_{C(\lambda)\cap V(\lambda,\nu)} |\lambda|^{2\gamma} Q(\hat{\mathcal{L}}_{O}\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} |\lambda|^{2\gamma} Q(\hat{\mathcal{L}}_{O}^2\mathbf{R})(\bar{K},\bar{K},T_0,e_4) \\
&\quad + \int_{C(\lambda)\cap V(\lambda,\nu)} |\lambda|^{2\gamma} Q(\hat{\mathcal{L}}_{S}\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned} \tag{2.6}$$

[11] It is the scalar function $u_*$ of [Kl-Ni].

[12] If $q-2<2$ the $\widetilde{\mathcal{Q}}_{(q)}(\lambda,\nu)$ and $\underline{\widetilde{\mathcal{Q}}}_{(q)}(\lambda,\nu)$ terms are absent.

$$
\begin{aligned}
\widetilde{\mathcal{Q}}_{(q)}(\lambda,\nu) &\equiv \sum_{i=2}^{q-2}\bigg\{\int_{C(\lambda)\cap V(\lambda,\nu)}|\lambda|^{2\gamma}Q(\hat{\mathcal{L}}_{T_0}\hat{\mathcal{L}}_O^i\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4)\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)}|\lambda|^{2\gamma}Q(\hat{\mathcal{L}}_O^{i+1}\mathbf{R})(\bar{K},\bar{K},T_0,e_4)\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)}|\lambda|^{2\gamma}Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}\hat{\mathcal{L}}_O^{i-1}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_4)\bigg\}
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
\underline{\widetilde{\mathcal{Q}}}_1(\lambda,\nu) &\equiv \sup_{V(\lambda,\nu)\cap\Sigma_0}|r^3\overline{\rho}|^2+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\\
&+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},T_0,e_3)\\
\underline{\widetilde{\mathcal{Q}}}_2(\lambda,\nu) &\equiv \int_{\underline{C}(\nu)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\\
&+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_O^2\mathbf{R})(\bar{K},\bar{K},T_0,e_3)\\
&+\int_{\underline{C}(\nu)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
\underline{\widetilde{\mathcal{Q}}}_{(q)}(\lambda,\nu) &\equiv \sum_{i=2}^{q-2}\bigg\{\int_{C(\lambda)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_{T_0}\hat{\mathcal{L}}_O^i\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_O^{i+1}\mathbf{R})(\bar{K},\bar{K},T_0,e_3)\\
&+\int_{C(\lambda)\cap V(\lambda,\nu)}|u|^{2\gamma}Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}\hat{\mathcal{L}}_O^{i-1}\mathbf{R})(\bar{K},\bar{K},\bar{K},e_3)\bigg\}\ .
\end{aligned} \tag{2.9}
$$

We define also the global norm

$$
\widetilde{\mathcal{Q}}=\sup_{V(\lambda,\nu)\subset\tilde{\mathcal{M}}}\widetilde{\mathcal{Q}}(\lambda,\nu) \tag{2.10}
$$

where[13]

$$
\begin{aligned}
\widetilde{\mathcal{Q}}(\lambda,\nu) &= \widetilde{\mathcal{Q}}_1(\lambda,\nu)+\widetilde{\mathcal{Q}}_2(\lambda,\nu)+\underline{\widetilde{\mathcal{Q}}}_1(\lambda,\nu)+\underline{\widetilde{\mathcal{Q}}}_2(\lambda,\nu)\\
&\left[\widetilde{\mathcal{Q}}_{(q)}(\lambda,\nu)+\underline{\widetilde{\mathcal{Q}}}_{(q)}(\lambda,\nu)\right]\ .
\end{aligned} \tag{2.11}
$$

**Remarks:**

[13] $V(\lambda,\nu)$ is defined as $V(\lambda,\nu)=J^-(S(\lambda,\nu))$.

1) Observe that the factor $|u|^{2\gamma}$ is constant along the $C(\lambda)$ null hypersurfaces, while it varies along the $\underline{C}(\nu)$ hypersurfaces.
2) The vector field $T$ appearing in $\hat{\mathcal{L}}_T\mathbf{R}$, $\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\mathbf{R}$, $\hat{\mathcal{L}}_S\hat{\mathcal{L}}_T\mathbf{R}$ in the $\mathcal{Q}$ norms introduced in [Ch-Kl] and in [Kl-Ni], is substituted by the vector field

$$T_0 = \Omega(e_3 + e_4) = 2\Omega T \tag{2.12}$$

where $\Omega$ is defined in equation 1.3. The reason of the choice 3.8 will be made clear in the discussion on the error estimates in section 3.[14]
We are now ready to state the improved version of the main result of [Kl-Ni].

**Theorem 2.1** *Assume the initial data set* $\{\Sigma_0, g, k\}$ *"strongly asymptotically flat of order* $\gamma$*" and* $\tilde{J}_0^{(q)}(\Sigma_0, g, k) < \infty$*.*
*Choosing* $\epsilon_0$ *sufficiently small, there exists a compact region* $K$ *containing the origin, such that* $\tilde{J}_K^{(q)}(\Sigma_0, g, k) < \varepsilon^2$ *with* $\varepsilon < \epsilon_0$ *and the initial data set has a unique development* $(\widetilde{\mathcal{M}}, \mathbf{g})$*, defined outside the domain of influence of* $K$*, with the following properties:*

*i)* $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}^+ \cup \widetilde{\mathcal{M}}^-$ *where* $\widetilde{\mathcal{M}}^+$ *consists of the part of* $\widetilde{\mathcal{M}}$ *which is in the future of* $\Sigma_0 \backslash K$*,* $\widetilde{\mathcal{M}}^-$ *the one to the past.*

*ii)* $(\widetilde{\mathcal{M}}^+, g)$ *can be foliated by a canonical double null foliation* $\{C(\lambda), \underline{C}(\nu)\}$ *whose outgoing leaves* $C(\lambda)$ *are complete* [15] *for all* $|\lambda| \geq |\lambda_0|$*. The boundary of* $K$ *can be chosen to be the intersection of* $C(\lambda_0) \cap \Sigma_0$*.*

*iii) The norms* $\mathcal{O}$*,* $\mathcal{D}$ *and* $\mathcal{R}$ *are bounded by a constant* $\leq c\varepsilon$*.* [16]

*iv) The null Riemann components have the following asymptotic behaviour:*

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{\frac{7}{2}}|\lambda|^{\gamma}|\alpha| \leq c\varepsilon \ , \ \sup_{\mathcal{K}} r|\lambda|^{\frac{5}{2}+\gamma}|\underline{\alpha}| \leq c\varepsilon \\
&\sup_{\mathcal{K}} r^{\frac{7}{2}}|\lambda|^{\gamma}|\beta| \leq c\varepsilon \ , \ \sup_{\mathcal{K}} r^2|\lambda|^{\frac{3}{2}+\gamma}|\underline{\beta}| \leq c\varepsilon \\
&\sup_{\mathcal{K}} r^3|\rho| \leq c\varepsilon \ , \ \sup_{\mathcal{K}} r^3|\lambda|^{\frac{1}{2}+\gamma}|(\rho - \overline{\rho}, \sigma)| \leq c\varepsilon
\end{aligned} \tag{2.13}$$

[14] Observe that, multiplying them by $2\Omega$, we could have modified also the vector fields $\bar{K}$ and $T$ in the entries $(\bar{K}, \bar{K}, T, e_4)$ of 2.6 and 1.2, but this is not needed as it will be clear during the proof of Theorem 2.2.

[15] By this we mean that the null geodesics generating $C(\lambda)$ can be indefinitely extended toward the future.

[16] $\mathcal{D}$ are norms for the various components of the deformation tensor of the rotation vector fields. They are defined in [Kl-Ni] Chapter 3.

The definition of double null foliation and of canonical double null foliation are given in [Kl-Ni], Chapter 3. The possibility of endowing the spacetime $(\widetilde{\mathcal{M}}, \mathbf{g})$ with a canonical double null foliation is crucial to obtain the result. The proof of Theorem 2.1 follows precisely the same steps as the proof of the "Main Theorem" (Theorem 3.7.1) in [Kl-Ni]. The result proved in [Kl-Ni] corresponds to the $\gamma = 0$ case.[17] As in that case the main body of the work can be separated into three different steps:

**Theorem 2.2** *Assume that in $(\widetilde{\mathcal{M}}, \mathbf{g})$ the following inequalities hold:*

$$\tilde{\mathcal{R}} \leq \epsilon_0 \ , \ \tilde{\mathcal{O}} \leq \epsilon_0 \ . \tag{2.14}$$

*Then choosing $\epsilon_0$ sufficiently small there exists a constant $c > 0$ such that, for any arbitrary region $\mathcal{K} = V(\lambda, \nu) \subset \widetilde{\mathcal{M}}$, the following inequality holds*

$$\widetilde{\mathcal{Q}}_{\mathcal{K}} \leq c\widetilde{\mathcal{Q}}_{\Sigma_0 \cap \mathcal{K}}. \tag{2.15}$$

*Here $\widetilde{\mathcal{Q}}_{\Sigma_0 \cap \mathcal{K}}$ denotes an integral norm analogous to the one defined in 2.11, but relative to $\Sigma_0 \cap \mathcal{K}$.*

The proof of this theorem is in Section 3.

**Theorem 2.3** *Assume that $\tilde{\mathcal{O}} \leq \epsilon_0$, then the following inequality holds*

$$\tilde{\mathcal{R}} \leq c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \tag{2.16}$$

*with c a positive constant.*

The proof of this theorem is obtained repeating the proof of the corresponding theorem in Chapter 5 of [Kl-Ni] using the $\tilde{\mathcal{R}}$, $\tilde{\mathcal{O}}$ and $\widetilde{\mathcal{Q}}_{\mathcal{K}}$ norms introduced here. A sketch of its proof is in Section 4.

**Theorem 2.4** *Let the "strongly asymptotically flat of order $\gamma$" initial data be such that $\tilde{J}_K^{(q)}(\Sigma_0, g, k) < \varepsilon^2$, assume that*

$$\widetilde{\mathcal{Q}}_{\mathcal{K}} \leq c\widetilde{\mathcal{Q}}_{\Sigma_0 \cap \mathcal{K}} \quad , \quad \tilde{\mathcal{R}} \leq c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \ ,$$

*then the following inequality holds*

$$\tilde{\mathcal{O}} \leq c\varepsilon \quad . \tag{2.17}$$

A sketch of its proof is in Section 5.
The previous three theorems combined with a bootstrap argument, as described in full detail in Chapter 3 of [Kl-Ni] allows to prove Theorem 2.1.

[17] In this case also $\gamma' = 0$, see equation 2.3.

## 3 The $\tilde{\mathcal{Q}}$ integral norms, proof of Theorem 2.2.

Theorem 2.4 below provides the crucial step in the proof of Theorem 2.1.

**Theorem 2.4:**
*Assume that in $(\widetilde{\mathcal{M}}, \mathbf{g})$ the following inequalities hold:*

$$\tilde{\mathcal{R}} \leq \epsilon_0 \ , \ \tilde{\mathcal{O}} \leq \epsilon_0 \ .$$

*Then choosing $\epsilon_0$ sufficiently small there exists a constant $c > 0$ such that, for any arbitrary region $\mathcal{K} = V(\lambda, \nu) \subset \widetilde{\mathcal{M}}$, the following inequality holds*

$$\widetilde{\mathcal{Q}}_{\mathcal{K}} \leq c \widetilde{\mathcal{Q}}_{\Sigma_0 \cap \mathcal{K}}. \tag{3.1}$$

*Here $\widetilde{\mathcal{Q}}_{\Sigma_0 \cap \mathcal{K}}$ denotes an integral norm analogous to the one defined in 2.11, but relative to $\Sigma_0 \cap \mathcal{K}$.*

**Proof:** The proof of Theorem 2.4 requires the main new technical ingredient of the paper. The idea is to introduce a factor $|\lambda|^{2\gamma}$ in the "energy density" of our main quantity $\tilde{\mathcal{Q}}$. We shall illustrate below how this can be done in the simple case of the wave equation in Minkowski spacetime, $M^4$. The crucial fact is that the weight factor $|\lambda|^{2\gamma}$ leads, in the exterior of the "light cone" $C(\lambda_0) = \{p \in M^4 | u(p) \leq \lambda_0 < 0\}$, to an energy inequality with a favorable sign. We then apply the same idea to the integral norms $\tilde{\mathcal{Q}}$.

The new problem which confronts us is to control the error terms which are generated by this procedure. They differ from the ones treated in [Kl-Ni], Chapter 6, (mainly) by the presence of the weight factor $|\lambda|^{2\gamma}$. Most of these error terms are easy to treat, but we have to pay special attention to those involving $\bar{\rho}$ and those connection coefficients, such as $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}$ which are non-trivial in the particular case of the Schwarszchild metric and, consequently, cannot decay any better relative to powers of $|\lambda|$. We prove that, nevertheless, these terms come up only in combination with other curvature and connection coefficients terms for which we have improved decay. Compared to [Kl-Ni] this argument also requires a modification of the vector field $T$ and of the canonical foliation on the last slice.

### 3.1 Main energy identities

#### 3.1.1 Wave Equation in flat space

Let us consider first the simpler case of the linear scalar wave equation $\Box \Phi = 0$. Its $L^2$ energy norm $\left( \int_{R^n} [|\partial_0 \Phi|^2(\cdot, t) + |\nabla \Phi|^2(\cdot, t)] \right)^{\frac{1}{2}}$ is conserved as can be immediately seen with the help of the energy-momentum tensor

$$Q_{\mu\nu}(\Phi) = 2\partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\left(g^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi\right)$$

and the local conservation laws $\partial^\beta Q_{\alpha\beta}(\Phi) = 2\partial_\alpha\Phi\,\Box\Phi = 0$ . Thus, in $M^4$,

$$0 = \partial_0 Q_{00} - \partial_i Q_{0i} \tag{3.2}$$

where $$Q_{00}(\Phi) = (\partial_0\Phi)^2 + (\nabla\Phi)^2 \quad ; \quad Q_{0i}(\Phi) = 2\partial_0\Phi\partial_i\Phi \ . \tag{3.3}$$

Consider the optical functions $u = t - r$ , $\underline{u} = t + r$, using the formulae [18]

$$\partial_0|u|^{2\gamma} = -2\gamma|u|^{2\gamma-1} \ , \ \partial_i|u|^{2\gamma} = 2\gamma|u|^{2\gamma-1}\frac{x_i}{r} \ ,$$

it follows

$$\begin{aligned}
&\partial_0(|u|^{2\gamma}Q_{00}) - \partial_i(|u|^{2\gamma}Q_{0i}) = -2\gamma|u|^{2\gamma-1}\left(Q_{00} + \frac{x_i}{r}Q_{0i}\right)\\
&= -2\gamma|u|^{2\gamma-1}\left((\partial_0\Phi)^2 + (\nabla\Phi)^2 + 2\frac{x_i}{r}\partial_0\Phi\partial_i\Phi\right)\\
&= -2\gamma|u|^{2\gamma-1}\left((\partial_0\Phi)^2 + (\partial_r\Phi)^2 + 2\partial_0\Phi\partial_r\Phi + (\not\nabla\Phi)^2\right)\\
&= -2\gamma|u|^{2\gamma-1}\left((\partial_0\Phi + \partial_r\Phi)^2 + (\not\nabla\Phi)^2\right)
\end{aligned} \tag{3.4}$$

where $\partial_r$ is the radial derivative and $\not\nabla$ denotes the derivatives tangential to $S(r) = \{p \in \Sigma_t | r(p) = r\}$. Therefore, in the region where $u(p) < 0$, we have

$$\partial_t(|u|^{2\gamma}Q_{00}) - \partial_i(|u|^{2\gamma}Q_{0i}) \leq 0 \ . \tag{3.5}$$

Let us consider the standard null pair $\{e_3, e_4\}$, where

$$e_3 = \frac{\partial}{\partial t} - \frac{\partial}{\partial r} \quad , \quad e_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \quad , \tag{3.6}$$

and the unit time like vector field

$$T_0 = \frac{1}{2}(e_3 + e_4) = \frac{\partial}{\partial t} \ . \tag{3.7}$$

Denoting $X$ an arbitrary timelike Killing vector field and defining the quantity $Q_\beta(X) = Q_{\alpha\beta}X^\alpha$ it follows:

$$\partial^\beta Q_\beta(X) = \partial^\beta Q_{\alpha\beta}X^a = (\partial^\beta Q_{\alpha\beta})X^a + Q_{\alpha\beta}\partial^\beta X^a = 0 \ . \tag{3.8}$$

[18] Valid in the exterior of $C(0)$.

We define also the spacetime region $V(\lambda,\nu)\subset\{p\in M^4|u(p)\leq 0\}$,

$$V(\lambda,\nu)=\{p\in M^4|\underline{u}(p)\in[\nu_0,\nu],u(p)\in[\lambda_1,\lambda]\} \tag{3.9}$$

where $\lambda_1=u|_{\underline{C}(\nu)\cap\Sigma_0}=-\nu$ , $\nu_0=\underline{u}|_{C(\lambda)\cap\Sigma_0}$ and $\nu_0<\nu$. The boundary of $V(\lambda,\nu)$ consists of a portion of the incoming null cone $\underline{C}(\nu)$, a portion of the outgoing null cone $C(\lambda)$ and a portion of the initial hypersurface:

$$\partial V(\lambda,\nu)=\underline{C}(\nu;[\lambda_1,\lambda])\cup C(\lambda;[\nu_0,\nu])\cup\Sigma_0([\nu_0,\nu_1])\ , \qquad \text{where}$$

$$\begin{aligned}\underline{C}(\nu;[\lambda_1,\lambda])&=\{p\in\underline{C}(\nu)|u(p)\in[\lambda_1,\lambda])\}\\ C(\lambda;[\nu_0,\nu])&=\{p\in C(\lambda)|\underline{u}(p)\in[\nu_0,\nu])\}\\ \Sigma_0([\nu_0,\nu_1])&=\{p\in M^4|r(p)\in[\nu_0,\nu]\}\ .\end{aligned} \tag{3.10}$$

Integrating over $V(\lambda,\nu)$ the analogous of 3.8, with $Q_\beta(X)$ substituted by $|u|^{2\gamma}Q_\beta(X)$, and using 3.4 we obtain immediately

$$\int_{\underline{C}(\nu;[\lambda_1,\lambda])}|u|^{2\gamma}Q(K,e_3)+\int_{C(\lambda;[\nu_0,\nu])}|u|^{2\gamma}Q(K,e_4)\leq\int_{\Sigma_0}|u|^{2\gamma}Q(K,T_0) \tag{3.11}$$

which, in view of the positivity of $Q(X,Y)$ for $X,Y$ timelike or null vector fields, implies the boundedness of the "flux quantities" $\int_{\underline{C}(\nu;[\lambda_1,\lambda])}|u|^{2\gamma}Q(K,e_3)$ and $\int_{\underline{C}(\lambda;[\nu_0,\nu])}|u|^{2\gamma}Q(K,e_4)$.

### 3.1.2 Energy inequality for $\widetilde{\mathcal{Q}}$

The same argument can be adapted to the vacuum Einstein curved spacetime with the help of the Bel-Robinson tensor. Denoting

$$P_\alpha=(|u|^\sigma Q)_{\alpha\beta\gamma\delta}X^\beta Y^\gamma Z^\delta \tag{3.12}$$

with $\sigma>0$, it follows [19]

$$\begin{aligned}DivP&=Div(|u|^\sigma Q)_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta\\ &+\frac{1}{2}|u|^\sigma Q^{\alpha\beta\gamma\delta}\left({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta+{}^{(Y)}\pi_{\alpha\beta}Z_\gamma X_\delta+{}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta\right)\ .\end{aligned} \tag{3.13}$$

Moreover

$$\begin{aligned}Div(|u|^\sigma Q)_{\beta\gamma\delta}&=g^{\epsilon\alpha}D_\epsilon|u|^\sigma Q_{\alpha\beta\gamma\delta}=|u|^\sigma(DivQ)_{\beta\gamma\delta}+(g^{\epsilon\alpha}D_\epsilon|u|^\sigma)Q_{\alpha\beta\gamma\delta}\\ &=|u|^\sigma(DivQ)_{\beta\gamma\delta}-\sigma|u|^{\sigma-1}(g^{\epsilon\alpha}D_\epsilon u)Q_{\alpha\beta\gamma\delta}\\ &=|u|^\sigma(DivQ)_{\beta\gamma\delta}+\sigma|u|^{\sigma-1}L^\alpha Q_{\alpha\beta\gamma\delta}\\ &=|u|^\sigma(DivQ)_{\beta\gamma\delta}+\sigma|u|^{\sigma-1}(2\Omega)^{-1}e_4^\alpha Q_{\alpha\beta\gamma\delta}\end{aligned} \tag{3.14}$$

[19]The function $u(p)$, whose level hypersurfaces $\{p\in\mathcal{M}|u(p)=\lambda\}$ are the null hypersurfaces $C(\lambda)$, is the analogous of the quantity $t-r$ in Minkowski spacetime.

where we have used the relation $L=(2\Omega)^{-1}e_4$, see 1.2. Applying Stokes theorem in the $V(u,\underline{u})$ region, we obtain:

$$
\begin{aligned}
&\left\{\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,e_3)+\int_{C(u)\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,e_4)\right.\\
&\left.-\int_{\Sigma_0\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,T)\right\} \qquad (3.15)\\
=\ &-\int_{V_{(u,\underline{u})}}|u|^\sigma\left[DivQ(W)_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta+\frac{1}{2}Q^{\alpha\beta\gamma\delta}(W)\left({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta\right.\right.\\
&\left.\left.+\ {}^{(Y)}\pi_{\alpha\beta}Z_\gamma X_\delta+{}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta\right)\right]-\sigma\int_{V_{(u,\underline{u})}}(2\Omega)^{-1}|u|^{\sigma-1}Q(W)(X,Y,Z,e_4)\ .
\end{aligned}
$$

Due to the fact that the last integral in the right hand side is positive we conclude

$$
\begin{aligned}
&\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,e_3)+\int_{C(u)\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,e_4)\\
&\leq\int_{\Sigma_0\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,T)-\int_{V_{(u,\underline{u})}}|u|^\sigma\Big[DivQ(W)_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta \qquad (3.16)\\
&+\frac{1}{2}Q^{\alpha\beta\gamma\delta}(W)\left({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta+{}^{(Y)}\pi_{\alpha\beta}Z_\gamma X_\delta+{}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta\right)\Big]\ .
\end{aligned}
$$

The flux integrals

$$
\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,e_3)\ \text{and}\ \int_{C(u)\cap V(u,\underline{u})}|u|^\sigma Q(W)(X,Y,Z,e_4)
$$

are bounded provided that we can control the following quantity we call "Error term",

$$
\begin{aligned}
\tilde{\mathcal{E}}(W)(X,Y,Z)\ =\ &-\int_{V_{(u,\underline{u})}}|u|^\sigma\Big[DivQ(W)_{\beta\gamma\delta}X^\beta Y^\gamma Z^\delta \qquad (3.17)\\
&+\ \frac{1}{2}Q^{\alpha\beta\gamma\delta}(W)\left({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta+{}^{(Y)}\pi_{\alpha\beta}Z_\gamma X_\delta+{}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta\right)\Big]
\end{aligned}
$$

where ${}^{(X)}\pi_{\alpha\beta}$ is the deformation tensor relative to the vector field $X$, see [Kl-Ni] subsection (3.4.2).
Applying exactly the same argument to $\widetilde{\mathcal{Q}}(\lambda,\nu)$ defined in equations 2.11, one shows that $\widetilde{\mathcal{Q}}_1(\lambda,\nu)$ , $\widetilde{\mathcal{Q}}_2(\lambda,\nu)$ , $\underline{\widetilde{\mathcal{Q}}}_1(\lambda,\nu)$ , $\underline{\widetilde{\mathcal{Q}}}_2(\lambda,\nu)$, $[\widetilde{\mathcal{Q}}_{(q)}(\lambda,\nu)+\underline{\widetilde{\mathcal{Q}}}_{(q)}(\lambda,\nu)]$ are bounded if we can control the following error term $\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\tilde{\mathcal{E}}_{(q)}$ where

$$
\tilde{\mathcal{E}}_1\ =\ \tilde{\mathcal{E}}(\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K})+\tilde{\mathcal{E}}(\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},T)
$$

$$\begin{aligned}
\tilde{\mathcal{E}}_2 &= \tilde{\mathcal{E}}(\hat{\mathcal{L}}_O^2\mathbf{R})(\bar{K},\bar{K},T)+\tilde{\mathcal{E}}(\hat{\mathcal{L}}_{T_0}\hat{\mathcal{L}}_O\mathbf{R})(\bar{K},\bar{K},\bar{K}) \\
&\quad +\tilde{\mathcal{E}}(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}\mathbf{R})(\bar{K},\bar{K},\bar{K}) \\
\tilde{\mathcal{E}}_{(q)} &= \sum_{i=2}^{q-2}\Big\{\tilde{\mathcal{E}}(\hat{\mathcal{L}}_O^{i+1}\mathbf{R})(\bar{K},\bar{K},T)+\tilde{\mathcal{E}}(\hat{\mathcal{L}}_{T_0}\hat{\mathcal{L}}_O^i\mathbf{R})(\bar{K},\bar{K},\bar{K}) \\
&\quad +\tilde{\mathcal{E}}(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}\hat{\mathcal{L}}_O^{i-1}\mathbf{R})(\bar{K},\bar{K},\bar{K})\Big\}
\end{aligned} \tag{3.18}$$

### 3.2 Preliminary steps concerning the estimate of the Error terms

The strategy to estimate the "Error terms" is essentially the one used in [Kl-Ni]. Nevertheless there are some differences which we shall describe below. For simplicity we consider only $\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2$. The remaining error terms do not present additional difficulties.
We start examining the general structure of $\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2$. Apart from the new weights, they are exactly the same as in [Kl-Ni], Chapter 6. Symbolically we can write:

$$\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2=\sum\int_{V(\lambda,\nu)}|\lambda|^{2\gamma}\tau_-^a\tau_+^b(D^\alpha\mathbf{R})\cdot(D^\beta\mathbf{R})\cdot(D^\delta\Pi) \tag{3.19}$$

where $\alpha+\beta+\delta=2$ and the factor $\tau_-^a\tau_+^b$ represents, with appropriate $a$ and $b$, the same weights which were already present in the analogous terms in [Kl-Ni]. The factor $|\lambda|^{2\gamma}$ is the extra weight factor which we have to cope with.
At first glance we might expect that each of the terms $D^\alpha\mathbf{R}, D^\beta\mathbf{R}, D^\delta\Pi$ appearing on the right hand side of 3.19, [20] can absorb a factor $|\lambda|^\gamma$. This will more than compensate for the presence of the weight $|\lambda|^{2\gamma}$ appearing in our formula. Unfortunately this is not quite true, indeed the symbolic expression $D^\alpha\mathbf{R}\cdot D^\beta\mathbf{R}\cdot D^\delta\Pi$ hides the presence of terms such as $\rho(\mathbf{R})$ or $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$, $\omega$, $\underline{\omega}$ which cannot absorb any additional weight factors. Nevertheless, making a small modification of the vector field $T$ and of the canonical foliation,[21] we can arrange that these terms only appear linearly, that is multiplied by terms which have better behaviour.

[20] $D^\delta\Pi$ denotes the covariant derivative of an arbitrary deformation tensor.

[21] Together with some inspired modifications of the quantities which are estimated by integration along null geodesics.

### 3.2.1 The approximate Killing vector fields and their deformation tensors.

One of the important features of the result in [Kl-Ni], see Chapter 8, is that all connection coefficients which are identically zero in the Schwarzschild spacetime allow an additional factor $\tau_-^{1/2} = (1+u^2)^{1/4}$ in the corresponding $\mathcal{O}$ norms. This is due to the fact that the evolution equations used to estimate them do not depend on the $\rho$ component[22] of the curvature tensor. An analogous argument can be used now to infer, using the assumption $\tilde{\mathcal{R}} \leq \epsilon_0$, that the corresponding connection coefficients gain an extra decay factor $|\lambda|^\gamma$.[23] As a consequence we deduce that the components of the deformation tensors, corresponding to those vector fields which are precisely Killing or conformal Killing in Schwarzschild gain also the decay factor $|\lambda|^\gamma$.

As explained earlier we need to modify the vector field $T = \frac{1}{2}(e_3+e_4)$ which was used in [Kl-Ni]. Indeed that choice turns out to be unacceptable, in view of the remarks above, since it does not coincide with the precise timelike Killing vector field of the Schwarzschild metric. To see this observe that in the Schwarzschild spacetime the null pair $\{e_3, e_4\}$, corresponding to our definition (see [Ni-In]) is given by

$$e_4 = \Phi^{-1}(\frac{\partial}{\partial t} + \frac{\partial}{\partial r_*}) = \Phi^{-1}\frac{\partial}{\partial t} + \Phi\frac{\partial}{\partial r}$$
$$e_3 = \Phi^{-1}(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}) = \Phi^{-1}\frac{\partial}{\partial t} - \Phi\frac{\partial}{\partial r},$$

where[24] $\Phi = \sqrt{(1-\frac{2M}{r})}$. Thus the choice $T = \frac{1}{2}(e_3+e_4) = \Phi^{-1}\frac{\partial}{\partial_t}$ does not correspond to the correct Killing vector field of Schwarzschild spacetime which is $\frac{\partial}{\partial_t}$. The correct choice[25] which we shall use below is,

$$T_0 = \Omega(e_3+e_4) \tag{3.20}$$

With this choice of $T_0$ we calculate the null components of the traceless part of its deformation tensor ${}^{(T_0)}\pi$, see [Kl-Ni] Chapter 3, subsection 3.4.2,

$${}^{(T_0)}\mathbf{i}_{ab} = 2\Omega\left(\hat{\chi}_{ab} + \underline{\hat{\chi}}_{ab} + \delta_{ab}(\omega + \underline{\omega})\right)$$

[22]Recall that $\rho$, the only Riemann component different from zero in Schwarzschild spacetime, carries the information about the $ADM$ mass and therefore, does not allow any improved decay in $|\lambda|$.

[23]To derive this improvement one needs also to take advantage of new definition of the canonical foliation.

[24]Observe that $\Phi$ corresponds precisely to $2\Omega$, see definition 1.3.

[25]There is no point in trying to change the definitions of $S$ and $K_0$; there are in fact no analogous conformal Killing vectorfields in Schwarzschild spacetime.

$$
\begin{aligned}
&{}^{(T_0)}\mathbf{j} = \Omega(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + 4\Omega(\omega + \underline{\omega}) \\
&{}^{(T_0)}\mathbf{m}_a = -4\Omega\zeta_a \ , \ {}^{(T_0)}\underline{\mathbf{m}}_a = 4\Omega\zeta_a \\
&{}^{(T_0)}\mathbf{n} = 0 \ , \ {}^{(T_0)}\underline{\mathbf{n}} = 0 \ .
\end{aligned} \tag{3.21}
$$

Observe that, although each of the components of the pair $(\omega, \underline{\omega})$, respectively $(\mathrm{tr}\chi, \mathrm{tr}\underline{\chi})$, are nontrivial in Schwarzschild [26] and, therefore, cannot have better decay in factors in $|\lambda|$, the combination $\omega + \underline{\omega}$, respectively $\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}$, do in fact cancel out their nontrivial Schwarzschild parts and behave better. These facts are included in the following proposition:

**Proposition 3.1** *Assume*[27] *that* $\tilde{J}^{(q)}_K(\Sigma_0, g, k) < \varepsilon^2$ *and*

$$
\tilde{\mathcal{R}} \le \epsilon_0 \ , \ \tilde{\mathcal{O}} \le \epsilon_0 \ .
$$

*Then the following inequalities hold*

$$
\begin{aligned}
&|r^2|\lambda|^\gamma({}^{(T_0)}\mathbf{i}, {}^{(T_0)}\mathbf{j})|_\infty \le c\epsilon_0 \\
&|r^2|\lambda|^\gamma({}^{(T_0)}\mathbf{m}, {}^{(T_0)}\underline{\mathbf{m}})|_\infty \le c\epsilon_0
\end{aligned} \tag{3.22}
$$

**Proof:** We prove the first line of 3.22. The second line is an immediate consequence of the assumption $\tilde{\mathcal{O}} \le \epsilon_0$ and the definition of the $\tilde{\mathcal{O}}$ norms. The estimates are proved in a finite region $V(\lambda_0, \nu_*)$ as in the proof of the Main Theorem in [Kl-Ni].[28]

1) **Estimate of** $\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}$:

From the structure equations, see [Kl-Ni], equations (3.1.46),..., (3.1.48), we have

$$
\begin{aligned}
&\mathbf{D}_3\mathrm{tr}\chi = -\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi - (\mathbf{D}_3\log\Omega)\mathrm{tr}\chi + 2\rho + \left[-\underline{\hat{\chi}}\cdot\hat{\chi} + 2|\eta|^2 + 2(\not{\Delta}\log\Omega + \not{d}\mathrm{iv}\,\zeta)\right] \\
&\mathbf{D}_3\mathrm{tr}\underline{\chi} = -\frac{1}{2}(\mathrm{tr}\underline{\chi})^2 + (\mathbf{D}_3\log\Omega)\mathrm{tr}\underline{\chi} + \left[|\underline{\hat{\chi}}|^2\right]
\end{aligned} \tag{3.23}
$$

and from them

$$
\begin{aligned}
\mathbf{D}_3(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) \ = \ & -\frac{1}{2}\mathrm{tr}\underline{\chi}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + (\mathbf{D}_3\log\Omega)(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi}) + 2\left(\rho - (\mathbf{D}_3\log\Omega)\mathrm{tr}\chi\right) \\
& + \left[-\underline{\hat{\chi}}\cdot\hat{\chi} + 2|\eta|^2 + 2(\not{\Delta}\log\Omega + \not{d}\mathrm{iv}\,\zeta) + |\underline{\hat{\chi}}|^2\right]
\end{aligned} \tag{3.24}
$$

[26] Obviously $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ are different from zero also in Minkowski spacetime, but in Schwarzschild they depend on the $ADM$ mass.

[27] We shall also make use of the canonical nature of our double null foliation.

[28] Then, proceeding in the construction of $\tilde{\mathcal{M}}$ as in [Kl-Ni], one proves that this region coincides with the whole spacetime.

From assumptions 2.14 of Theorem 2.2 it follows that the term in square brackets behaves as $O(r^{-3}|\lambda|^{-(\frac{1}{2}+\gamma)})$, therefore it is a good term from the point of view of the presence of the new decay factor $|\lambda|^\gamma$. Hereafter we denote with $[Good]$ all the terms which have, at least, the following asymptotic behaviour: [29]

$$[Good]=O\bigg(\frac{1}{r^3|\lambda|^\gamma}\bigg)+O\bigg(\frac{1}{r^2|\lambda|^{1+\gamma}}\bigg)\ . \tag{3.25}$$

In view of this notation equation 3.24 can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial\lambda}(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})+\frac{\overline{\Omega\mathrm{tr}\underline{\chi}}}{2}(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})&=\Big[2^{-1}\Big(\overline{\Omega\mathrm{tr}\underline{\chi}}-\Omega\mathrm{tr}\underline{\chi}\Big)+(\mathbf{D}_3\log\Omega)\Big](\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})\\&\quad+2\left(\rho-(\mathbf{D}_3\log\Omega)\mathrm{tr}\chi\right)+[Good]\end{aligned} \tag{3.26}$$

The only term we have to take care of is $2(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)$. Its estimate is provided by the following lemma.

**Lemma 3.1** *Under the assumptions $\tilde{\mathcal{R}}\leq\epsilon_0$ , $\tilde{\mathcal{O}}\leq\epsilon_0$, assuming that the last slice $\underline{C}(\nu_*)$ is endowed by a canonical foliation such that on it*

$$\mathbf{D}_3\overline{\log\Omega}=(\overline{tr\chi})^{-1}\overline{\rho}\ , \tag{3.27}$$

*then in the whole region $V(\lambda_0,\nu_*)$ the quantity $(\rho-tr\chi\mathbf{D}_3\log\Omega)$ satisfy the estimate:* [30]

$$\sup_{V(\lambda_0,\nu_*)}|r^3|\lambda|^\gamma(\rho-tr\chi\mathbf{D}_3\log\Omega)|\leq c\epsilon_0\ . \tag{3.28}$$

Using the result of Lemma 3.1 we obtain immediately with a simple application of the Gronwall's lemma, the following estimate for $(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})$:

$$|r^{2-\frac{2}{p}}|\lambda|^\gamma(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})|_{p,S}(\lambda,\nu)\leq|r^{2-\frac{2}{p}}|\lambda|^\gamma(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})|_{p,\underline{C}(\nu)\cap\Sigma_0}+c\epsilon_0\ . \tag{3.29}$$

and using the stronger assumptions on the initial data on $\Sigma_0$ we obtain the result, for $p\in[2,4]$,

$$|r^{2-\frac{2}{p}}|\lambda|^\gamma(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})|_{p,S}(\lambda,\nu)\leq c\epsilon_0\ . \tag{3.30}$$

[29] In fact the terms which appear in $[Good]$ behave better due to an extra $|\lambda|^{\frac{1}{2}}$ factor. Definition 3.24 is nevertheless, sufficient.

[30] This estimate can be improved by a factor $|\lambda|^{\frac{1}{2}}$.

Exactly the same argument can be redone for the first tangential derivatives obtaining, for $p\in[2,4]$,

$$|r^{3-\frac{2}{p}}|\lambda|^{\gamma}\nabla\!\!\!/(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})|_{p,S}(\lambda,\nu)\leq c\epsilon_0\ . \tag{3.31}$$

Using inequalities 3.30 and 3.31 together with the Sobolev inequality for the sphere, see Lemma 4.1.3 of [Kl-Ni], we obtain

$$\sup_{V(\lambda,\nu)}|r^2|\lambda|^{\gamma}(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})|\leq c\epsilon_0\ . \tag{3.32}$$

**Proof of Lemma 3.1:** It is in order to estimate $(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)$ that we are obliged to introduce a new canonical foliation on the last slice satisfying

$$\mathbf{D}_3\overline{\log\Omega}=(\overline{\mathrm{tr}\chi})^{-1}\overline{\rho}\ , \tag{3.33}$$

see equation 2.1.[31] We estimate $(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)$ by deriving its evolution equation along $C(\lambda)$ using the structure equations, see [Kl-Ni] equations (3.1.46),...,(3.1.48) of [Kl-Ni] and the null Bianchi equations. We obtain

$$\frac{\partial}{\partial\nu}(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)=\Omega\mathbf{D}_4\rho-[\Omega(\mathbf{D}_3\log\Omega)\mathbf{D}_4\mathrm{tr}\chi+\Omega\mathrm{tr}\chi\mathbf{D}_4\mathbf{D}_3\log\Omega]\ . \tag{3.34}$$

where $\frac{\partial}{\partial\nu}=\Omega\mathbf{D}_4$. Recalling the explicit expression of the Bianchi equations, see [Kl-Ni], equations (3.2.8) we have

$$\begin{aligned}\Omega\mathbf{D}_4\rho &= -\frac{3}{2}\Omega\mathrm{tr}\chi\rho+[\mathrm{d}\!\!/\mathrm{iv}\,\beta-(-\frac{1}{2}\underline{\hat{\chi}}\cdot\alpha-\zeta\cdot\underline{\beta}+2\eta\cdot\underline{\beta})]\\ &= -\frac{3}{2}\Omega\mathrm{tr}\chi\rho+[\mathit{Good}]\end{aligned} \tag{3.35}$$

The second term in the right hand side of 3.34 can be written, using the transport equation for $\mathrm{tr}\chi$, as:

$$\begin{aligned}&-[\Omega(\mathbf{D}_3\log\Omega)\mathbf{D}_4\mathrm{tr}\chi+\Omega\mathrm{tr}\chi\mathbf{D}_4\mathbf{D}_3\log\Omega]\\ &=-\Omega\mathrm{tr}\chi\left(-\rho-(\mathbf{D}_3\log\Omega)(\mathbf{D}_4\log\Omega)+\left[\underline{\eta}\cdot\eta-2\zeta^2-2\zeta\cdot\nabla\!\!\!/\log\Omega\right]\right)\\ &\ \ -\Omega\mathbf{D}_3\log\Omega\left(-\frac{1}{2}\mathrm{tr}\chi^2+\mathrm{tr}\chi(\mathbf{D}_4\log\Omega)+|\hat{\chi}|^2\right)\\ &=-\frac{1}{2}\Omega\mathrm{tr}\chi(-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)+\Omega\mathrm{tr}\chi\rho+\left[\underline{\eta}\cdot\eta-2\zeta^2-2\zeta\cdot\nabla\!\!\!/\log\Omega+|\hat{\chi}|^2\right]\\ &=-\frac{1}{2}\Omega\mathrm{tr}\chi(-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)+\Omega\mathrm{tr}\chi\rho+[\mathit{Good}]\end{aligned} \tag{3.36}$$

[31] In [Kl-Ni] we were able to obtain 3.33 only asymptotically as $\nu\to\infty$. In order to control the quantity $(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)$ in the whole spacetime we are forced to make the choice 3.33 on the last slice.

Collecting 3.35 and 3.36, evolution equation 3.34 can be written as

$$\frac{\partial}{\partial\nu}(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega) = -\frac{1}{2}\Omega\mathrm{tr}\chi(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)+[Good] \tag{3.37}$$

where, in this case it is easy to see from the explicit expressions 3.35, 3.36, that $[Good]$ decays at least as $O(r^{-4}|\lambda|^{-(\gamma+\frac{1}{2})})$. Applying Gronwall's lemma we obtain

$$|r^{3-\frac{2}{p}}|\lambda|^{\gamma}(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)|_{p,S}(\lambda,\nu)\leq c|r^{3-\frac{2}{p}}|\lambda|^{\gamma}(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)|_{p,S}(\lambda,\nu_*)+c\epsilon_0\ . \tag{3.38}$$

In view of the canonicity of our double null foliation and observing that

$$\begin{aligned}(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega) &= -\mathrm{tr}\chi(\mathbf{D}_3\log\Omega-\overline{\mathbf{D}_3\log\Omega})-(\mathrm{tr}\chi-\overline{\mathrm{tr}\chi})\overline{\mathbf{D}_3\log\Omega}\\ &\quad+(\rho-\overline{\mathrm{tr}\chi}\ \overline{\mathbf{D}_3\log\Omega})\\ &= 2(\rho-\overline{\mathrm{tr}\chi}\ \overline{\mathbf{D}_3\log\Omega})+[Good]\end{aligned} \tag{3.39}$$

we conclude that, for $p\in[2,4]$,

$$|r^{3-\frac{2}{p}}|\lambda|^{\gamma}(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)|_{p,S}(\lambda,\nu)\leq c\epsilon_0 \tag{3.40}$$

Repeating exactly the same argument for the first tangential derivative $\nabla\!\!\!/(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)$ one obtains the analogous estimate

$$|r^{4-\frac{2}{p}}|\lambda|^{\gamma}\nabla\!\!\!/(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)|_{p,S}(\lambda,\nu)\leq c\epsilon_0 \tag{3.41}$$

and using both 3.40 and 3.41 together with Lemma 4.1.3 of [Kl-Ni] one obtains the estimate for the *sup* norm,

$$\sup_{V(\lambda,\nu)}|r^3|\lambda|^{\gamma}(\rho-\mathrm{tr}\chi\mathbf{D}_3\log\Omega)|\leq c\epsilon_0 \tag{3.42}$$

proving the lemma.

2) **Estimate of** $\omega+\underline{\omega}$:

The result of Lemma 3.1 implies immediately the following relation:

$$\underline{\omega}=-(2\mathrm{tr}\chi)^{-1}\rho+O\left(\frac{1}{r^2|\lambda|^{\gamma}}\right) \tag{3.43}$$

Using this relation we can estimate $\omega+\underline{\omega}$ from the estimate of $(\omega-(2\mathrm{tr}\chi)^{-1}\rho)$. In fact

$$\omega+\underline{\omega}=\left(\omega-(2\mathrm{tr}\chi)^{-1}\rho\right)+O\left(\frac{1}{r^2|\lambda|^{\gamma}}\right) \tag{3.44}$$

To estimate of $(\omega-(2\mathrm{tr}\chi)^{-1}\rho)$ we write the evolution equation for this quantity along $\underline{C}(\nu)$. We observe that, from equation (4.3.58) in [Kl-Ni],

$$\begin{aligned}
\mathbf{D}_3\omega &= -\frac{1}{2}\mathbf{D}_3\mathbf{D}_4\log\Omega=-\frac{1}{2\Omega}\mathbf{D}_3\Omega\mathbf{D}_4\log\Omega+\frac{1}{2}(\mathbf{D}_3\log\Omega)\mathbf{D}_4\log\Omega\\
&= -(\mathbf{D}_3\log\Omega)\omega+\frac{1}{2}\rho-\frac{1}{2\Omega}\hat{F}=-(\mathbf{D}_3\log\Omega)\omega-\mathrm{tr}\chi\underline{\omega}+[Good]\\
&= -(\mathbf{D}_3\log\Omega)\omega+\mathrm{tr}\underline{\chi}\underline{\omega}-(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})\underline{\omega}+[Good]\\
&= -(\mathbf{D}_3\log\Omega)\omega+\mathrm{tr}\underline{\chi}\underline{\omega}+[Good] \qquad (3.45)
\end{aligned}$$

where $\hat{F}\equiv 2\Omega\zeta\cdot\nabla\!\!\!/\log\Omega+\Omega(\underline{\eta}\cdot\eta-2\zeta^2)$.
From the structure equations and Bianchi equations, see [Kl-Ni], Chapter 3,

$$\begin{aligned}
&\mathbf{D}_3\Big(-(2\mathrm{tr}\chi)^{-1}\rho\Big)=\frac{1}{2}\frac{\rho}{\mathrm{tr}\chi^2}(\mathbf{D}_3\mathrm{tr}\chi)-\frac{1}{2\mathrm{tr}\chi}\mathbf{D}_3\rho\\
&=\Big(-\underline{\omega}+O(\frac{1}{r^2|\lambda|^\gamma})\Big)\frac{1}{\mathrm{tr}\chi}\mathbf{D}_3\mathrm{tr}\chi-\frac{1}{2\mathrm{tr}\chi}\Big(-\frac{3}{2}\mathrm{tr}\underline{\chi}\rho+[Good]\Big)\\
&=-\underline{\omega}\frac{1}{\mathrm{tr}\chi}\mathbf{D}_3\mathrm{tr}\chi+\frac{1}{2\mathrm{tr}\chi}\frac{3}{2}\mathrm{tr}\underline{\chi}\rho+\Big[O(\frac{1}{r^2|\lambda|^\gamma})\frac{1}{\mathrm{tr}\chi}\mathbf{D}_3\mathrm{tr}\chi+\frac{1}{2\mathrm{tr}\chi}[Good]\Big]\\
&=-\underline{\omega}\frac{1}{\mathrm{tr}\chi}\Big(-\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi-(\mathbf{D}_3\log\Omega)\mathrm{tr}\chi+2\rho-\Big[\hat{\underline{\chi}}\cdot\hat{\chi}-2\mathrm{d}\!\!\!/\mathrm{iv}\,\zeta-2\triangle\!\!\!\!/\log\Omega-2|\eta|^2\Big]\Big)\\
&+\frac{1}{2\mathrm{tr}\chi}\frac{3}{2}\mathrm{tr}\underline{\chi}\rho+\Big[O\Big(\frac{1}{r^2|\lambda|^\gamma}\Big)\frac{1}{\mathrm{tr}\chi}\mathbf{D}_3\mathrm{tr}\chi+\frac{1}{2\mathrm{tr}\chi}[Good]\Big]\\
&=\frac{1}{2}\mathrm{tr}\underline{\chi}\underline{\omega}+(\mathbf{D}_3\log\Omega)\underline{\omega}-\underline{\omega}\frac{1}{\mathrm{tr}\chi}2\rho+\frac{1}{2\mathrm{tr}\chi}\frac{3}{2}\mathrm{tr}\underline{\chi}\rho+[Good]\\
&=\frac{1}{2}\mathrm{tr}\underline{\chi}\underline{\omega}+(\mathbf{D}_3\log\Omega)\underline{\omega}-\frac{1}{2}(\mathbf{D}_3\log\Omega)4\underline{\omega}-\frac{3}{2}\mathrm{tr}\underline{\chi}\underline{\omega}+[Good] \qquad (3.46)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{D}_3\Big(-(2\mathrm{tr}\chi)^{-1}\rho\Big) &= \frac{1}{2}\mathrm{tr}\underline{\chi}\underline{\omega}-(\mathbf{D}_3\log\Omega)\underline{\omega}-\frac{3}{2}\mathrm{tr}\underline{\chi}\underline{\omega}+[Good] \qquad (3.47)\\
&= -\mathrm{tr}\underline{\chi}\underline{\omega}-(\mathbf{D}_3\log\Omega)\Big(-(2\mathrm{tr}\chi)^{-1}\rho\Big)+[Good]
\end{aligned}$$

which finally implies, together with 3.45,

$$\mathbf{D}_3\Big(\omega-(2\mathrm{tr}\chi)^{-1}\rho\Big)=-(\mathbf{D}_3\log\Omega)\Big(\omega-(2\mathrm{tr}\chi)^{-1}\rho\Big)+[Good] \qquad (3.48)$$

Repeating the same calculation for the first tangential derivatives, applying again Gronwall's lemma and using the initial data assumptions we conclude that

$$\sup_{V(\lambda,\nu)}\Big|r^2|\lambda|^\gamma\Big(\omega-(2\mathrm{tr}\chi)^{-1}\rho\Big)\Big|\le c\epsilon_0 \qquad (3.49)$$

so that, together with 3.44, we obtain

$$\sup_{V(\lambda,\nu)} \left| r^2 |\lambda|^\gamma \left(\omega + \underline{\omega}\right) \right| \leq c\epsilon_0 \ . \tag{3.50}$$

Estimates 3.32 and 3.50 prove Proposition 3.1.

### 3.3 Error estimates

As we discussed in the previous section the terms which we have to take care of in the error terms are those depending on $\rho(\mathbf{R})$. Therefore we look at the various terms appearing in Chapter 6 of [Kl-Ni], but focusing our attention to the parts which depend on $\rho(\mathbf{R})$.

#### 3.3.1 Estimate of $\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} DivQ(\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\beta\gamma\delta}(\bar{K}^\beta, \bar{K}^\gamma, \bar{K}^\delta)$

This requires to estimate the following four integrals:[32]

$$B_1 \equiv \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 D(T_0,\mathbf{R})_{444} \quad , \ B_2 \equiv \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_-^2\tau_+^4 D(T_0,\mathbf{R})_{344}$$
$$B_3 \equiv \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_-^4\tau_+^2 D(T_0,\mathbf{R})_{334} \ , \ B_4 \equiv \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_-^6 D(T_0,\mathbf{R})_{333}$$

As discussed in [Kl-Ni] the more delicate term is the first one, $B_1$. To estimate it we have to control the integrals:

$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \alpha(\hat{\mathcal{L}}_{T_0}\mathbf{R}) \cdot \Theta(T_0,\mathbf{R}) \quad , \ \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \beta(\hat{\mathcal{L}}_{T_0}\mathbf{R}) \cdot \Xi(T_0,\mathbf{R}) \tag{3.51}$$

**Estimate of the $B_1$ integrals**

From the decomposition $J(T_0;\mathbf{R}) = J^1(T_0;\mathbf{R}) + J^2(T_0;\mathbf{R}) + J^3(T_0;\mathbf{R})$, see [Kl-Ni] Chapter 6, equation (6.1.6), it follows

$$\Theta(T_0,\mathbf{R}) = \Theta^{(1)}(T_0,\mathbf{R}) + \Theta^{(2)}(T_0,\mathbf{R}) + \Theta^{(3)}(T_0,\mathbf{R})$$
$$\Xi(T_0,\mathbf{R}) = \Xi^{(1)}(T_0,\mathbf{R}) + \Xi^{(2)}(T_0,\mathbf{R}) + \Xi^{(3)}(T_0,\mathbf{R})$$

We write the two integrals in 3.51 as sums of three terms:

$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \alpha(\hat{\mathcal{L}}_{T_0}\mathbf{R}) \cdot \Theta(T_0,\mathbf{R}) = \sum_{i=1}^{3} \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \alpha(\hat{\mathcal{L}}_{T_0}\mathbf{R}) \cdot \Theta^{(i)}(T_0,\mathbf{R})$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \beta(\hat{\mathcal{L}}_{T_0}\mathbf{R}) \cdot \Xi(T_0,\mathbf{R}) = \sum_{i=1}^{3} \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \beta(\hat{\mathcal{L}}_{T_0}\mathbf{R}) \cdot \Xi^{(i)}(T_0,\mathbf{R})$$

[32] We use here as extra weight factor $\tau_-^{2\gamma}$ which is equivalent to $|\lambda|^{2\gamma}$.

Let us consider the first integral in the first line. Using the coarea formulas

$$\int_{V(u,\underline{u})} F = \int_{u_0}^{u} du' \int_{C(u')\cap V(u,\underline{u})} F$$
$$\int_{V(u,\underline{u})} F = \int_{\underline{u}_0}^{\underline{u}} d\underline{u}' \int_{\underline{C}(\underline{u}')\cap V(u,\underline{u})} F \ , \tag{3.52}$$

Cauchy-Schwartz inequality and Theorem 2.3 we have

$$\begin{aligned}
&\left| \int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \alpha(\hat{\mathcal{L}}_{T_0}\mathbf{R})\cdot\Theta(T_0,\mathbf{R})^{(1)} \right| \\
&\le c\int_{u_0}^{u} du' \left( \int_{C(u';[\underline{u}_0,\underline{u}])} |u|'^{2\gamma}\underline{u}'^6 |\alpha(\hat{\mathcal{L}}_{T_0}\mathbf{R})|^2 \right)^{\frac{1}{2}} \\
&\quad \left( \int_{C(u';[\underline{u}_0,\underline{u}])} |u'|^{2\gamma}\underline{u}'^6 |\Theta^{(1)}(T_0,\mathbf{R})|^2 \right)^{\frac{1}{2}} \\
&\le c\tilde{\mathcal{Q}}^{\frac{1}{2}} \int_{u_0}^{u} du' \left( \int_{C(u';[\underline{u}_0,\underline{u}])} |u'|^{2\gamma}\underline{u}'^6 |\Theta^{(1)}(T_0,\mathbf{R})|^2 \right)^{\frac{1}{2}}
\end{aligned} \tag{3.53}$$

We are left to prove that the remaining integral is bounded. The difference from the corresponding situation in [Kl-Ni] is the presence of the factor $|u'|^{2\gamma}$. Recall that $\Theta^{(1)}(T_0,\mathbf{R})$ is quadratic, bilinear in the null Riemann components and in structure coefficients which can be expressed in terms of connection coefficients:

$$\begin{aligned}
\Theta^{(1)}(T_0,\mathbf{R}) &= \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}}\,;\,\nabla\!\!\!/\alpha\right] + \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{n}}\,;\,\alpha_4\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{j}\,;\,\alpha_3\right] \\
&+ \mathrm{Qr}\left[{}^{(T_0)}\mathbf{i}\,;\,\nabla\!\!\!/\beta\right] + \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}}\,;\,\beta_4\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{m}\,;\,\beta_3\right] \\
&+ \mathrm{Qr}\left[{}^{(T_0)}\mathbf{m}\,;\,\nabla\!\!\!/(\rho,\sigma)\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{j}\,;\,(\rho_4,\sigma_4)\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{n}\,;\,(\rho_3,\sigma_3)\right] \\
&+ \mathrm{tr}\chi\left(\mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{n}}\,;\,\alpha\right] + \mathrm{Qr}\left[{}^{(T_0)}\underline{\mathbf{m}}\,;\,\beta\right] + \mathrm{Qr}\left[({}^{(T_0)}\mathbf{i},{}^{(T_0)}\mathbf{j})\,;\,(\rho,\sigma)\right]\right. \\
&+ \left.\mathrm{Qr}\left[{}^{(T_0)}\mathbf{m}\,;\,\underline{\beta}\right]\right) + \mathrm{tr}\underline{\chi}\left(\ \mathrm{Qr}\left[{}^{(T_0)}\mathbf{j}\,;\,\alpha\right] + \mathrm{Qr}\left[{}^{(T_0)}\mathbf{m}\,;\,\beta\right]\right. \\
&+ \left. +\mathrm{Qr}\left[{}^{(T_0)}\mathbf{n}\,;\,(\rho,\sigma)\right]\right) + \text{l.o.t.}\ .
\end{aligned}$$

From the definition of $\tilde{\mathcal{R}}$ and the assumption $\tilde{\mathcal{R}} \le \epsilon_0$ it follows that all the bounded norms of the null Riemann coefficients have an extra $|u'|^{\gamma}$ with the exception of the norm relative to $\rho$. Therefore, for the terms in $\Theta^{(1)}(T,W)$ which do not contain $\rho$, analogous estimates to those of Chapter 6 of [Kl-Ni] hold, without even considering that the most of the connection coefficients satisfy better decay estimates. Looking at the integral

$$\int_{u_0}^{u} du' \left( \int_{C(u';[\underline{u}_0,\underline{u}])} u'^{2\gamma} \underline{u}'^6 |\Theta^{(1)}(T_0,\mathbf{R})|^2 \right)^{\frac{1}{2}}$$

and the expression of $\Theta^{(1)}(T,\mathbf{R})$ we observe that $\rho(\mathbf{R})$ appears only in

$$\int_{u_0}^{u} du' \left( \int_{C(u';[\underline{u}_0,\underline{u}])} |u'|^{2\gamma} \underline{u}'^6 (\mathrm{tr}\chi)^2 |({}^{(T_0)}\mathbf{i}, {}^{(T_0)}\mathbf{j})|^2 |\rho(\mathbf{R})|^2 \right)^{\frac{1}{2}} \tag{3.54}$$

Both terms ${}^{(T_0)}\mathbf{i}$ and ${}^{(T_0)}\mathbf{j}$ satisfy, as proved in Proposition 3.1, the estimate 3.22,

$$|r^2|\lambda|^\gamma({}^{(T_0)}\mathbf{i}, {}^{(T_0)}\mathbf{j})|_\infty \leq c\epsilon_0$$

from which it follows

$$\begin{aligned} &\int_{C(u';[\underline{u}_0,\underline{u}])} |u'|^{2\gamma} \underline{u}'^6 (\mathrm{tr}\chi)^2 |({}^{(T_0)}\mathbf{i}, {}^{(T_0)}\mathbf{j})|^2 |\rho(\mathbf{R})|^2 \\ &\leq c \int d\underline{u} |u'|^{2\gamma} \underline{u}'^8 \frac{1}{r^8} \frac{1}{r^4 |u'|^{2\gamma}} \leq c \int d\underline{u} \frac{1}{r^4} \leq c \frac{1}{|u'|^3} \end{aligned} \tag{3.55}$$

and, therefore,

$$\left( \int_{C(u';[\underline{u}_0,\underline{u}])} |u'|^{2\gamma} \underline{u}'^6 (\mathrm{tr}\chi)^2 |({}^{(T_0)}\mathbf{i}, {}^{(T_0)}\mathbf{j})|^2 |\rho(\mathbf{R})|^2 \right)^{\frac{1}{2}} \leq c \frac{1}{|u'|^{\frac{3}{2}}} \tag{3.56}$$

which is integrable in $u'$.

**Remark:** *It is important to recognize that the term* $Qr\left[{}^{(T_0)}\mathbf{n}\,;\,(\rho,\sigma)\right]$ *of* $\Theta^{(1)}(T_0,W)$ *is absent as* ${}^{(T_0)}\mathbf{n}=0$*. Indeed had we used the previous vector field* $T$*, used in [Kl-Ni], we would have to deal with a contribution of the form*

$$\int_{u_0}^{u} du' \left( \int_{C(u';[\underline{u}_0,\underline{u}])} |u'|^{2\gamma} \underline{u}'^6 (\mathrm{tr}\chi)^2 |{}^{(T)}\mathbf{n}|^2 |\rho(\mathbf{R})|^2 \right)^{\frac{1}{2}}$$

*which would have been impossible to control.*[33]

[33] In fact in this case, see again Chapter 3 of [Kl-Ni], ${}^{(T)}\mathbf{n} = -4\omega$, and $\omega$ cannot acquire a factor $|\lambda|^\gamma$ as its evolution equation depends on $\rho$ as proved in Chapter 4 of [Kl-Ni].

Let us now examine the integrals

$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6 \alpha(\hat{\mathcal{L}}_{T_0}\mathbf{R}) \cdot \Theta^{(i)}(T_0, W)$$

for $i \in \{2,3\}$. To control the integral with $i=2$ we recall that

$$\Theta^{(2)}(T_0,W) = \mathrm{Qr}\left[{}^{(T_0)}p_3\,;\,\alpha\right] + \mathrm{Qr}\left[{}^{(T_0)}\not{p}\,;\,\beta\right] + \mathrm{Qr}\left[{}^{(T_0)}p_4\,;\,(\rho,\sigma)\right]\ .$$

As the only terms to control are those with $\rho(\mathbf{R})$, we are led to estimate the integral

$$\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_-^{2\gamma}\tau_+^6 |{}^{(T_0)}p_4|^2|\rho(\mathbf{R})|^2 \le C_0 \int_{C(u';[\underline{u}_0,\underline{u}])} \tau_-^{2\gamma}|{}^{(T_0)}p_4|^2$$

where ${}^{(T_0)}p_4$ has the following expression, see [Kl-Ni] Chapter 6, equations (6.1.24),...,(6.1.26):

$${}^{(T_0)}p_4 = \not{d}\mathrm{iv}\,{}^{(T_0)}\mathbf{m} - \frac{1}{2}\not{\mathbf{D}}_4{}^{(T_0)}\mathbf{j} + (2\eta+\underline{\eta}+\zeta)\cdot{}^{(T_0)}\mathbf{m} - \hat{\underline{\chi}}\cdot{}^{(T_0)}\mathbf{i} - \frac{1}{2}\mathrm{tr}\underline{\chi}(\mathrm{tr}\,{}^{(T_0)}\mathbf{i} + {}^{(T_0)}\mathbf{j})$$

As already discussed, all the traceless deformation tensor components of $T_0$ do not depend on $\rho(\mathbf{R})$ and therefore when estimated in terms of the connection coefficients have the extra decay factor $|\lambda|^{-\gamma}$ which compensates the factor $\tau_-^{2\gamma}$ in the integral. Therefore the final estimate is exactly the same as in [Kl-Ni]. The same holds for the part containing $\rho(\mathbf{R})$ of $\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma}\tau_+^6\alpha(\hat{\mathcal{L}}_{T_0}\mathbf{R})\cdot\Theta^{(3)}(T_0,W)$:

$$\int_{C(u';[\underline{u}_0,\underline{u}])} \tau_-^{2\gamma}\tau_+^6|\Theta({}^{(T_0)}q)|^2|\rho(\mathbf{R})|^2 \tag{3.57}$$

as follows immediately looking at the explicit expression of $\Theta({}^{(T_0)}q)$. We have, therefore, proved that the error term

$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} DivQ(\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\beta\gamma\delta}(\bar{K}^\beta,\bar{K}^\gamma,\bar{K}^\delta)$$

is under control.

### 3.3.2 Estimate of $\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Q(\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$

The argument is, in this case, very simple. Recall, as mentioned above, that the null components of ${}^{(\bar{K})}\pi^{\alpha\beta}$ must contain terms which depend on $\rho(\mathbf{R})$.

This implies that, in our case not all the components of ${}^{(\bar{K}_0)}\hat{\pi}$ have the decay factor $|\lambda|^{-\gamma}$.
This is not a problem here as $Q(\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\alpha\beta\gamma\delta}$ is quadratic in the components of the Riemann tensor $\hat{\mathcal{L}}_{T_0}\mathbf{R}$. Indeed the Lie derivative $\hat{\mathcal{L}}_{T_0}$ cancels the dependance on the $ADM$ mass $M$.[34] This implies that $Q(\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\alpha\beta\gamma\delta}$ comes with a factor $|\lambda|^{-2\gamma}$ which compensates the $\tau_-^{2\gamma}$ factor in the integral.
To see it in a more formal way we look at the relation between $\rho(\hat{\mathcal{L}}_{T_0}\mathbf{R})$ and $\mathbf{D}_{T_0}\rho(\mathbf{R})$. From Chapter 5 of [Kl-Ni] we have

$$\begin{aligned}\rho(\hat{\mathcal{L}}_{T_0}\mathbf{R}) &= \mathbf{D}_{T_0}\rho(\mathbf{R}) - \frac{1}{8}\mathrm{tr}^{(T_0)}\pi\rho(\mathbf{R}) - \frac{1}{2}\left({}^{(T_0)}\underline{P}_a + {}^{(T_0)}\underline{Q}_a\right)\beta(W)_a \\ &+ \frac{1}{2}\left({}^{(T_0)}P_a + {}^{(T_0)}Q_a\right)\underline{\beta}(W)_a \end{aligned} \tag{3.58}$$

The term $\mathrm{tr}^{(T_0)}\pi\rho(\mathbf{R})$ has a factor $|\lambda|^{-\gamma}$ due to $\mathrm{tr}^{(T_0)}\pi$. The term $\mathbf{D}_{T_0}\rho(\mathbf{R})$ has also the right behaviour as, in view of the null Bianchi equations,

$$\mathbf{D}_{T_0}\rho(\mathbf{R}) = \Omega(\mathbf{D}_3\rho(\mathbf{R}) + \mathbf{D}_4\rho(\mathbf{R})) = -\frac{3}{2}(\mathrm{tr}\chi + \mathrm{tr}\underline{\chi})\rho(\mathbf{R}) + ...... \tag{3.59}$$

and we have already proved in Proposition 3.1 that $(\mathrm{tr}\chi+\mathrm{tr}\underline{\chi})$ has the decay factor $|\lambda|^{-\gamma}$. The remaining terms have the same decay factor which allows us to conclude that the error term $\int_{V_{(u,\underline{u})}} Q(\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$ can be bounded by $c\epsilon_0\tilde{\mathcal{Q}}$ as required. The next error terms to examine are

$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Div Q(\hat{\mathcal{L}}_O W)_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Q(\hat{\mathcal{L}}_O W)_{\alpha\beta\gamma\delta}({}^{(T_0)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$$

The argument to prove that these error terms behave correctly is similar to the previous one. For the first integral one has to convince oneself that the components ${}^{(O)}\pi_{\alpha\beta}$ have the right behaviour. This follows easily recalling that in the Schwarzschild case the rotation vector fields are Killing. For the second and the third integral it suffices to observe that $\rho(\hat{\mathcal{L}}_O\mathbf{R})$ does not depend on $M$ (the part depending on the $ADM$ mass is spherically

[34] Indeed $M$ is time independent.

symmetric in the Schwarzschild case). Finally the remaining error terms

$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} DivQ(\hat{\mathcal{L}}_O^2\mathbf{R})_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma T_0^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} DivQ(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} DivQ(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\beta\gamma\delta}(\bar{K}^\beta\bar{K}^\gamma\bar{K}^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Q(\hat{\mathcal{L}}_O^2\mathbf{R})_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma T_0^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Q(\hat{\mathcal{L}}_O^2\mathbf{R})_{\alpha\beta\gamma\delta}({}^{(T_0)}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$$
$$\int_{V_{(u,\underline{u})}} \tau_-^{2\gamma} Q(\hat{\mathcal{L}}_S\hat{\mathcal{L}}_{T_0}\mathbf{R})_{\alpha\beta\gamma\delta}({}^{(\bar{K})}\pi^{\alpha\beta}\bar{K}^\gamma\bar{K}^\delta)$$

can be estimated in the same manner as there is always or a term $\hat{\mathcal{L}}_{T_0}\mathbf{R}$ or $\hat{\mathcal{L}}_0\mathbf{R}$. The final conclusion is that the error term is bounded by $c\epsilon_0\widetilde{\mathcal{Q}}$ and, therefore, the $\widetilde{\mathcal{Q}}$ integral norms are bounded and Theorem 2.2 is proved.

## 4 Proof of Theorem 2.3

We sketch in this section the proof of Theorem 2.3 which we restate here.

**Theorem 2.3:** *Assume that $\tilde{\mathcal{O}} \le \epsilon_0$, then the following inequality holds*

$$\tilde{\mathcal{R}} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \tag{4.1}$$

*with c a positive constant.*

We write explicitely some inequalities implicit in 4.1, which are used in Section 6.

$$\begin{aligned}
&\sup_{\mathcal{K}} r^{7/2}|\lambda|^{\gamma}|\alpha| \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \ , \ \sup_{\mathcal{K}} r|\lambda|^{\frac{5}{2}+\gamma}|\underline{\alpha}| \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \\
&\sup_{\mathcal{K}} r^{7/2}|\lambda|^{\gamma}|\beta| \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \ , \ \sup_{\mathcal{K}} r^2|\lambda|^{\frac{3}{2}+\gamma}|\underline{\beta}| \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \\
&\sup_{\mathcal{K}} r^3|\rho| \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \ , \ \sup_{\mathcal{K}} r^3|\lambda|^{\frac{1}{2}+\gamma}|(\rho-\overline{\rho},\sigma)| \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&|r^{3-\frac{2}{p}}|\lambda|^{2\gamma}\nabla\!\!\!/\,\underline{\beta}|_{p,S} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \ , \ |r^{4-\frac{2}{p}}|\lambda|^{\frac{1}{2}+\gamma}\nabla\!\!\!/\,(\rho,\sigma)|_{p,S} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \\
&\|r^3|\lambda|^{1+\gamma}\nabla\!\!\!/^2\underline{\beta}\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \\
&\|r^4|\lambda|^{\gamma}\nabla\!\!\!/^2(\rho,\sigma)\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \ .
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\|r^2|\lambda|^{\gamma}\alpha\|_{2,C(\lambda)\cap V(\lambda,\nu)} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \ , \ \|r^2|\lambda|^{\gamma}\beta\|_{2,C(\lambda)\cap V(\lambda,\nu)} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}} \\
&\|r^2|\lambda|^{\gamma}\beta\|_{2,\underline{C}(\nu)\cap V(\lambda,\nu)} \le c\widetilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}}
\end{aligned}$$

**Proof:** The proof follows exactly the argument used in the proof of the corresponding result in [Kl-Ni], see Chapter 5. The only terms where one has to pay special attention are, again, those involving $\rho(\mathbf{R})$. They appear for example through the formula, see [Kl-Ni] Chapter 5 page 211,

$$\mathcal{L}_O\rho(W) = \rho(\hat{\mathcal{L}}_O W) + \frac{1}{8}\mathrm{tr}{}^{(O)}\pi\rho(W) + \frac{1}{2}{}^{(O)}\underline{P}\cdot\beta(W) \ . \tag{4.4}$$

The term $\mathrm{tr}{}^{(O)}\pi\rho(W)$ decays, however, in the proper fashion relative to powers of $|\lambda|$. Indeed, in view of the $\tilde{\mathcal{O}}$ estimates, the ${}^{(O)}\pi$ factor can absorb the $|\lambda|^{\gamma}$ factor as

$$|r|\lambda|^{\gamma}\mathrm{tr}{}^{(O)}\pi| \le c\epsilon_0 \quad , \quad |r^2|\lambda|^{\gamma}\nabla\!\!\!/\,\mathrm{tr}{}^{(O)}\pi| \le c\epsilon_0 \ . \tag{4.5}$$

A similar argument can be done for the term ${}^{(O)}\underline{P}\rho(W)$ which appear in the formula, see [Kl-Ni] Proposition 5.1.1,

$$\not{\mathcal{L}}_O\underline{\beta}(W)_a = \underline{\beta}(\hat{\mathcal{L}}_O W)_a + {}^{(O)}C_{ab}\underline{\beta}(W)_b - \frac{3}{4}{}^{(O)}\underline{P}_a\rho(W) + \frac{3}{4}\in_{ab}{}^{(O)}\underline{P}_b\sigma(W) \quad (4.6)$$

Again, in view of the $\tilde{\mathcal{O}}$ estimates, the ${}^{(O)}\underline{P}_a$ factor can absorb the $|\lambda|^\gamma$ factor as the following estimates hold

$$|r|\lambda|^{\gamma(O)}\underline{P}| \leq c\epsilon_0 \ \ , \ \ |r^2|\lambda|^\gamma \nabla\!\!\!/ {}^{(O)}\underline{P}| \leq c\epsilon_0 \ . \quad (4.7)$$

# 5 Proof of Theorem 2.4

We discuss in this section Theorem 2.4 which we recall.

**Theorem 2.4:** *Let the initial data be such that* $\tilde{J}^{(q)}_K(\Sigma_0,g,k)<\varepsilon^2$*, assume that*

$$\widetilde{\mathcal{Q}}_{\mathcal{K}}\leq c\widetilde{\mathcal{Q}}_{\Sigma_0\cap\mathcal{K}}\ \ ;\ \ \tilde{\mathcal{R}}\leq c\widetilde{\mathcal{Q}}^{\frac{1}{2}}_{\mathcal{K}}$$

*Then the following inequality holds*

$$\tilde{\mathcal{O}}\leq c\varepsilon \tag{5.1}$$

We write explicitely some of the inequalities implicit in 5.1, relative to those connection coefficients norms which will be used in the next section.

$$\begin{aligned}
&||\lambda|^{\frac{1}{2}+\gamma}r^{2-2/p}(\mathrm{tr}\chi-\overline{\mathrm{tr}\chi})|_{p,S}(\lambda,\nu)\leq c\varepsilon\\
&||\lambda|^{\frac{1}{2}+\gamma}r^{2-2/p}\hat{\chi}|_{p,S}(\lambda,\nu)\leq c\varepsilon\ ,\ ||\lambda|^{\frac{1}{2}+\gamma}r^{3-2/p}\nabla\!\!\!\!/\,\hat{\chi}|_{p,S}(\lambda,\nu)\leq c\varepsilon\\
&|r^{2-2/p}|\lambda|^{\frac{1}{2}+\gamma}\eta|_{p,S}(\lambda,\nu)\leq c\varepsilon\ ,\ |r^{2-2/p}|\lambda|^{\frac{1}{2}+\gamma}\underline{\eta}|_{p,S}(\lambda,\nu)\leq c\varepsilon\\
&|r^{3-2/p}|\lambda|^{\frac{1}{2}+\gamma}\nabla\!\!\!\!/\,\eta|_{p,S}(\lambda,\nu)\leq c\varepsilon\ ,\ |r^{3-2/p}|\lambda|^{\frac{1}{2}+\gamma}\nabla\!\!\!\!/\,\underline{\eta}|_{p,S}(\lambda,\nu)\leq c\varepsilon
\end{aligned} \tag{5.2}$$

with $p\in[2,\infty]$ for the non derived coefficients, and $p\in[2,4]$ for the remaining ones. Moreover the following quantities, which can be expressed in terms of the null connection coefficients, see [Kl-Ni] Chapter 3 and Chapter 6, satisfy the inequalities

$$|r^2|\lambda|^{\gamma}\nabla\!\!\!\!/\,\mathrm{tr}^{(O)}\pi|\leq c\varepsilon\ ,\ |r^2|\lambda|^{\gamma}\nabla\!\!\!\!/^{(O)}\underline{P}|\leq c\varepsilon\ . \tag{5.3}$$

**Proof:** The proof of this theorem follows precisely the same steps as the corresponding one proved in Chapter 4 of [Kl-Ni], without any additional complications.

To illustrate how the connection coefficients (except those different from zero in Schwarzschild spacetime) get the additional decay factor $|\lambda|^{-\gamma}$ we write schematically the main strategy of deriving the estimates for the first derivatives of the connection coefficients used in [Kl-Ni] Chapter 4.

1.) We denote by $M$, $\underline{M}$ quantities, such as

$$\begin{aligned}
\psi\!\!\!/ &= \Omega^{-1}(\nabla\mathrm{tr}\chi+\mathrm{tr}\chi\zeta),\qquad \underline{\psi\!\!\!/}=\Omega^{-1}\nabla\mathrm{tr}\chi+\mathrm{tr}\underline{\chi}\zeta\\
\widetilde{\mu} &= -\mathrm{d}\!\!\!/\mathrm{iv}\,\eta+\frac{1}{2}(\chi\cdot\underline{\chi}-\overline{\chi\cdot\underline{\chi}})-(\rho-\bar{\rho})\\
\underline{\widetilde{\mu}} &= -\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\eta}+\frac{1}{2}(\chi\cdot\underline{\chi}-\overline{\chi\cdot\underline{\chi}})-(\rho-\bar{\rho})
\end{aligned}$$

which satisfy transport equations of the type,

$$\frac{\partial M}{\partial \nu} + \text{tr}\chi M = H \cdot M + R + [error] \tag{5.4}$$
$$\frac{\partial \underline{M}}{\partial \lambda} + \text{tr}\underline{\chi}\underline{M} = \underline{H} \cdot \underline{M} + R + [error]$$

2.) We denote by $H$ and $\underline{H}$ those connection coefficients, such as $\hat{\chi}, \eta$, $\underline{\hat{\chi}}, \underline{\eta}$ which are estimated through elliptic Hodge systems of the type

$$\begin{aligned} \mathcal{D}H &= R + M + [error] \\ \mathcal{D}\underline{H} &= R + \underline{M} + [error] \end{aligned} \tag{5.5}$$

and which are shown to be integrable, in the uniform norm.
3.) We denote by $R$ any null curvature component.
4.) We denote by $[error]$ all other terms, quadratic or highr order, which appear in the precise equations.

To see that both the quantities $M, \underline{M}$ and connection coefficients acquire the decay factors $|\lambda|^{-\gamma}$ we may assume, based on the previously established estimates, that both the source terms $R$ and the initial conditions on $\Sigma_0$ have these decay factors. Then, solving the coupled system 5.4, 5.5, it is very easy to check, by a standard bootstrap argument, that $M$ and $H$ have the desired behaviour in the whole spacetime, provided that this holds true for the last slice.

The only part which remains to check is, therefore, that the data on the last slice $\underline{C}_*$, for $M$ and $H$ have the right decay factor. This argument goes exactly as the corresponding one in [Kl-Ni] where it was proved that the "last slice data" have a factor $\tau_-^{1/2}$ if the foliation of $\underline{C}_*$ is the canonical one. The rigorous proof of it is in Chapter 7 of [Kl-Ni], see also [Ni]. The same happens here with the new last slice canonical foliation.

We illustrate the above discussion by showing how to derive the estimates for $\nabla\!\!\!/ \text{tr}\chi$ and $\nabla\!\!\!/ \hat{\chi}$.

### 5.1 Estimate of $\nabla\!\!\!/ \text{tr}\chi$ and $\nabla\!\!\!/ \hat{\chi}$

First we define the quantity

$$\psi\!\!\!/ = \Omega^{-1}\nabla\!\!\!/ \text{tr}\chi + \Omega^{-1}\text{tr}\chi\zeta = U + \Omega^{-1}\text{tr}\chi\zeta \tag{5.6}$$

which, as discussed in [Kl-Ni] Chapter 4, satisfies the evolution equation

$$\frac{d}{d\underline{u}}\psi\!\!\!/_a + \frac{3}{2}\Omega\text{tr}\chi\psi\!\!\!/_a = \Omega\hat{\chi}\cdot\psi\!\!\!/ - \nabla\!\!\!/|\hat{\chi}|^2 - \text{tr}\chi\beta + \tilde{F\!\!\!/}_a \tag{5.7}$$

where

$$\tilde{F} = -\eta|\hat{\chi}|^2 + \text{tr}\chi\ \hat{\chi} \cdot \underline{\eta} \tag{5.8}$$

Integrating along $C(\nu)$, more precisely applying to 5.7 the Evolution Lemma (Lemma 4.1.5) of [Kl-Ni], we obtain

$$\begin{aligned}
&|r^{3-\frac{2}{p}}\Psi|_{p,S}(u,\underline{u}) \leq c\Big(|r^{3-\frac{2}{p}}\Psi|_{p,S}(u,\underline{u}_*) \qquad (5.9)\\
&+\int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}\hat{\chi}\cdot\Psi|_{p,S} + \int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}\nabla|\hat{\chi}|^2|_{p,S} + \int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}\tilde{F}|_{p,S}\Big)
\end{aligned}$$

We then apply elliptic $L^p$ estimates to the the Codazzi equation (3.1.47) of [Kl-Ni], expressed relative to the tensor $\Psi$,

$$\text{div}\,\hat{\chi} + \zeta\cdot\hat{\chi} = \frac{1}{2}\Omega\Psi - \beta \tag{5.10}$$

and derive, using the bootstrap assumptions,

$$|r^{3-2/p}\nabla\hat{\chi}|_{p,S} \leq c\left(|r^{3-\frac{2}{p}}\Psi|_{p,S} + |\lambda|^{-(\frac{1}{2}+\gamma)}\varepsilon\right) \tag{5.11}$$

Therefore equation 5.9 can be rewritten, using 5.11 as

$$\begin{aligned}
&|r^{3-\frac{2}{p}}\Psi|_{p,S}(u,\underline{u}) \leq c\Big(|r^{3-\frac{2}{p}}\Psi|_{p,S}(u,\underline{u}_*)\\
&+\frac{1}{|\lambda|^{\frac{1}{2}+\gamma}}\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}\left(|r^{3-\frac{2}{p}}\Psi|_{p,S} + c\epsilon_0\right) + \frac{1}{|\lambda|^{\frac{1}{2}+\gamma}}\int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}\tilde{F}|_{p,S}\Big)\\
&\leq c\Big(|r^{3-\frac{2}{p}}\Psi|_{p,S}(u,\underline{u}_*) + \frac{1}{|\lambda|^{\frac{1}{2}+\gamma}}\int_{\underline{u}}^{\underline{u}_*}\frac{1}{r^2}\left(|r^{3-\frac{2}{p}}\Psi|_{p,S} + c\varepsilon\right)\Big) \qquad (5.12)
\end{aligned}$$

neglecting the integral $\int_{\underline{u}}^{\underline{u}_*}|r^{3-\frac{2}{p}}\tilde{F}|_{p,S}$ which contain the non linear term $\tilde{F}$. Finally we apply the Gronwall inequality to 5.12 and obtain

$$|r^{3-2/p}\Psi|_{p,S}(u,\underline{u}) \leq |r^{3-\frac{2}{p}}\Psi|_{p,S}(u,\underline{u}_*) + c\varepsilon\frac{1}{|\lambda|^{\frac{1}{2}+\gamma}}$$

From the assumption that the inequalities $\tilde{\mathcal{O}} \leq c\varepsilon$ are satisfied on the last slice $\underline{C}(\nu_*)$ the inequality

$$||\lambda|^{\frac{1}{2}+\gamma}r^{3-\frac{2}{p}}\Psi|_{p,S}(u,\underline{u}_*) \leq c\varepsilon \tag{5.13}$$

follows. This estimate, together with the previous elliptic $L^p$ estimate applied to 5.10, implies, for $p \in [2,4]$,

$$||\lambda|^{\frac{1}{2}+\gamma} r^{3-2/p} \nabla\!\!\!/ \hat{\chi}|_{p,S}(u,\underline{u}) \leq c\varepsilon$$
$$||\lambda|^{\frac{1}{2}+\gamma} r^{2-2/p} \hat{\chi}|_{p,S}(u,\underline{u}) \leq c\varepsilon$$

completing the proof for $\hat{\chi}$. Using again the definition of $\psi\!\!\!/$ one obtains also the estimate

$$||\lambda|^{\frac{1}{2}+\gamma} r^{3-2/p} \nabla\!\!\!/ \hat{\chi}|_{p,S}(u,\underline{u}) \leq c\varepsilon \tag{5.14}$$

We are left to proving estimate 5.13. To estimate on $\underline{C}(\nu_*)$

$$\psi\!\!\!/ = \Omega^{-1} \nabla\!\!\!/ \mathrm{tr}\chi + \Omega^{-1} \mathrm{tr}\chi\zeta$$

we have to estimate first the asymptotic behaviour of $\zeta$. The equation $\zeta$ satisfies on $\underline{C}_*$ is

$$\mathbf{D}\!\!\!\!/_3 \zeta + 2\underline{\chi} \cdot \zeta - \mathbf{D}_3 \nabla\!\!\!/ \log\Omega = -\underline{\beta}$$

In [Kl-Ni] we had that, on $\underline{C}_*$, $\zeta$ satisfies the inequality

$$|r^{2-2/p} \tau_-^{\frac{1}{2}} \zeta|_{p,S_*} \leq c\varepsilon$$

We have now better estimates for $\underline{\beta}$ following from better $\tilde{\mathcal{R}}$ norms, in fact $\underline{\beta} = O(r^{-2}\tau_-^{\frac{3}{2}+\gamma})$. $\mathbf{D}_3 \nabla\!\!\!/ \log\Omega$ also has a better behaviour in view of the fact that on $\underline{C}_*$ $\mathbf{D}_3 \log\Omega$ satisfies the elliptic equation

$$\triangle\!\!\!/(\Omega \mathbf{D}_3 \log\Omega) = \mathrm{d}\!\!\!/\mathrm{iv}\, F_1 + G_1 - \overline{G_1} \tag{5.15}$$

$$\text{where } \; F_1 = \Omega\underline{\beta} + \tilde{F}_1 \quad , \quad \tilde{F}_1 = \left(\frac{3}{2}\Omega\eta \cdot \underline{\hat{\chi}} + \frac{1}{4}\Omega\eta \mathrm{tr}\underline{\chi}\right)$$
$$G_1 = H + \frac{1}{4}\Omega \mathbf{D}_3(\hat{\chi}\cdot\underline{\hat{\chi}}) - \frac{1}{2}(\Omega tr\underline{\chi})(\rho - \overline{\rho}) + \frac{1}{4}(\Omega \mathrm{tr}\underline{\chi})(\hat{\chi}\cdot\underline{\hat{\chi}} - \overline{\hat{\chi}\cdot\underline{\hat{\chi}}}) \ .$$

and it follows immediately that

$$|r^{2-\frac{2}{p}} |\lambda|^{\frac{3}{2}+\gamma} \nabla\!\!\!/ \mathbf{D}\!\!\!\!/_3 \log\Omega|_{p,S} \leq c\varepsilon \tag{5.16}$$

(once that we have proved the behaviour for $\hat{\chi}$ and $\underline{\hat{\chi}}$). Then once we have proved 5.16 we have also proved the estimate for $\zeta$

$$|r^{2-2/p} \tau_-^{\frac{1}{2}+\gamma} \zeta|_{p,S_*} \leq c\varepsilon \ . \tag{5.17}$$

An analogous estimate for $\nabla\!\!\!/ \mathrm{tr}\chi$ can be easily obtained from its evolution equation along $\underline{C}_*$, see [Kl-Ni] chapter 7, equation 7.4.19. The estimate for $\zeta$, together with the analogous estimate for $\nabla\!\!\!/ \mathrm{tr}\chi$, allows to prove the following inequality for $\Psi\!\!\!/$,

$$||\lambda|^{\frac{3}{2}+\gamma}r^{3-\frac{2}{p}}\Psi\!\!\!/|_{p,S}(\lambda,\nu_*) \leq c\varepsilon \ . \tag{5.18}$$

which satisifies our request.

# 6 Proof of the peeling properties

Using the results of Theorems 2.4 and 2.3 we give a detailed proof of the peeling results discussed in the introduction. This result is a consequence of the initial data assumptions needed to prove Theorems 2.4 and 2.3. We formulate it in a slightly more general version:

**Theorem 6.1 (Strong peeling properties)**
*Assume initial data sets $\{\Sigma_0, g, k\}$ satisfying the following conditions*

$$\begin{aligned} g_{ij} - \delta_{ij} &= \frac{2M}{r}\delta_{ij} + O_{q+1}(r^{-(\frac{3}{2}+\gamma)}) \\ k_{ij} &= O_q(r^{-(\frac{5}{2}+\gamma)}) \end{aligned} \tag{6.1}$$

*with* $\gamma = \frac{3}{2} + \epsilon$, $\epsilon > 0$ *and*

$$\tilde{J}_K^{(q)}(\Sigma_0, g, k) < \varepsilon^2 \ . \tag{6.2}$$

*Under these assumptions we prove that, depending on the choice of $q$ in $\tilde{J}_K^{(q)}(\Sigma_0, g, k)$, the following holds:*

**a)** *Along the outgoing null hypersurfaces $C(\lambda)$ the following limits hold*

$$\begin{aligned} &q \geq 3 \qquad \lim_{C(\lambda);\nu\to\infty} r\underline{\alpha} = \underline{A}(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^2\underline{\beta} = \underline{B}(\lambda,\omega) \\ &q \geq 3 \qquad \lim_{C(\lambda);\nu\to\infty} r^3\rho = P(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^3\sigma = Q(\lambda,\omega) \\ &q \geq 5 \qquad \lim_{C(\lambda);\nu\to\infty} r^4\beta = B(\lambda,\omega) \end{aligned} \tag{6.3}$$

*with* $\underline{A}(\lambda,\omega), \underline{B}(\lambda,\omega), P(\lambda,\omega), Q(\lambda,\omega), B(\lambda,\omega)$ *satisfying:*

$$\begin{aligned} &|\underline{A}(\lambda,\omega)| \leq c(1+|\lambda|)^{-(\frac{5}{2}+\gamma)} \quad ; \quad |\underline{B}(\lambda,\omega)| \leq c(1+|\lambda|)^{-(\frac{3}{2}+\gamma)} \\ &|(P-\overline{P})(\lambda,\omega)| \leq c(1+|\lambda|)^{-(\frac{1}{2}+\gamma)} \quad ; \quad |(Q-\overline{Q})(\lambda,\omega)| \leq c(1+|\lambda|)^{-(\frac{1}{2}+\gamma)} \\ &|B(\lambda,\omega)| \leq c(1+|\lambda|)^{-(-\frac{1}{2}+\gamma)} \end{aligned} \tag{6.4}$$

**b)** *The Riemann components $\alpha$ and $\beta$ satisfy the following estimates, with $\epsilon' < \epsilon$ and $C_0$ a positive constant depending on the initial data:* [35]

$$q = 3 \qquad \sup_{\widetilde{\mathcal{M}}} |r^{5-\frac{2}{p}}|\lambda|^{\epsilon'}\alpha|_{p,S} \leq C_0, \quad p=2 \quad ; \quad \sup_{\widetilde{\mathcal{M}}} |r^{4-\frac{2}{p}}|\lambda|^{1+\epsilon'}|\beta| \leq C_0, \quad p \in [2,4]$$

[35] Hereafter we always assume $\epsilon' < \epsilon$, wherever these two quantities appear.

$$q=4 \qquad \sup_{\widetilde{\mathcal{M}}}|r^{5-\frac{2}{p}}|\lambda|^{\epsilon'}\alpha|_{p,S}\le C_0, \quad p\in[2,4] \ \ ; \ \ \sup_{\widetilde{\mathcal{M}}}|r^4|\lambda|^{1+\epsilon'}|\beta|\le C_0 \tag{6.5}$$

$$q=5 \qquad \sup_{\widetilde{\mathcal{M}}}|r^5|\lambda|^{\epsilon'}\alpha|\le C_0 \ ; \ \sup_{\widetilde{\mathcal{M}}}|r^4|\lambda|^{1+\epsilon'}|\beta|\le C_0$$

**Theorem 6.2 (Weak peeling properties)** *Assume initial data sets $\{\Sigma_0, g, k\}$ satisfying the following conditions*

$$\begin{aligned} g_{ij}-\delta_{ij} &= \frac{2M}{r}\delta_{ij}+O_{q+1}(r^{-(\frac{3}{2}+\gamma)}) \\ k_{ij} &= O_q(r^{-(\frac{5}{2}+\gamma)}) \end{aligned} \tag{6.6}$$

*with $\gamma=\frac{3}{2}-\delta$, $\delta\in[0,\frac{3}{2})$ and*

$$\tilde{J}_K^{(q)}(\Sigma_0,g,k)<\varepsilon^2 \ . \tag{6.7}$$

*Under these assumptions we prove that, depending on the choice of $q$ in $\tilde{J}_K^{(q)}(\Sigma_0,g,k)$, the following holds:*

**a)** *Along the outgoing null hypersurfaces $C(\lambda)$ the following limits hold*

$$\begin{aligned} &\delta\in[0,\tfrac{3}{2}) \ ; \ q\ge 3 \qquad \lim_{C(\lambda);\nu\to\infty} r\underline{\alpha}=\underline{A}(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^2\underline{\beta}=\underline{B}(\lambda,\omega) \\ &\delta\in[0,\tfrac{3}{2}) \ ; \ q\ge 3 \qquad \lim_{C(\lambda);\nu\to\infty} r^3\rho=P(\lambda,\omega) \quad , \quad \lim_{C(\lambda);\nu\to\infty} r^3\sigma=Q(\lambda,\omega) \\ &\delta\in[0,1) \ ; \ q\ge 5 \qquad \lim_{C(\lambda);\nu\to\infty} r^4\beta=B(\lambda,\omega) \end{aligned} \tag{6.8}$$

*with $\underline{A}(\lambda,\omega)$, $\underline{B}(\lambda,\omega)$, $P(\lambda,\omega)$, $Q(\lambda,\omega)$, $B(\lambda,\omega)$ satisfying:*

$$\begin{aligned} &|\underline{A}(\lambda,\omega)|\le c(1+|\lambda|)^{-(4-\delta)} \quad ; \quad |\underline{B}(\lambda,\omega)|\le c(1+|\lambda|)^{-(3-\delta)} \\ &|(P-\overline{P})(\lambda,\omega)|\le c(1+|\lambda|)^{-(2-\delta)} \ ; \ |(Q-\overline{Q})(\lambda,\omega)|\le c(1+|\lambda|)^{-(2-\delta)} \\ &|B(\lambda,\omega)|\le c(1+|\lambda|)^{-(1-\delta)} \end{aligned} \tag{6.9}$$

**b)** *The Riemann components $\alpha$ and $\beta$ satisfy the following estimates, with $\epsilon>0$:*

**Case 1:** $(q=3)$

$$\delta=0 \quad ; \quad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^5(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right|_{p=2,S}\le C_0 \ ; \ \sup_{\widetilde{\mathcal{M}}}\left|r^4|\lambda|\beta\right|_{p,S}\le C_0 \ , \ p\in[2,4]$$

$$\delta \in (0,1)\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right|_{p=2,S} \leq C_0\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|r^4|\lambda|^{1-\delta}\beta\right|_{p,S} \leq C_0\ ,\ p\in[2,4] \qquad (6.10)$$

$$\delta = 1\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right|_{p=2,S} \leq C_0\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right|_{p,S} \leq C_0\ ,\ p\in[2,4]$$

$$\delta \in (1,\frac{3}{2})\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{2+\epsilon}}\alpha\right|_{p=2,S} \leq C_0\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right|_{p,S} \leq C_0\ ,\ \ p\in[2,4]$$

**Case 2:** $(q=4)$

$$\delta = 0\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^5(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right|_{p,S} \leq C_0\ \ p\in[2,4]\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|r^4|\lambda|\beta\right| \leq C_0$$

$$\delta \in (0,1)\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right|_{p,S} \leq C_0\ p\in[2,4]\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|r^4|\lambda|^{1-\delta}\beta\right| \leq C_0 \qquad (6.11)$$

$$\delta = 1\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right|_{p,S} \leq C_0\ \ p\in[2,4]\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right| \leq C_0$$

$$\delta \in (1,\frac{3}{2})\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{2+\epsilon}}\alpha\right|_{p,S} \leq C_0\ \ p\in[2,4]\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right| \leq C_0$$

**Case 3:** $(q=5)$

$$\delta = 0\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^5(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right| \leq C_0\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|r^4|\lambda|\beta\right| \leq C_0$$

$$\delta \in (0,1)\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right| \leq C_0\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|r^4|\lambda|^{1-\delta}\beta\right| \leq C_0 \qquad (6.12)$$

$$\delta = 1\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha\right| \leq C_0\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^4(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right| \leq C_0$$

$$\delta \in (1,\frac{3}{2})\ ; \qquad \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{(2+\epsilon)}}\alpha\right| \leq C_0\ ;\ \sup_{\widetilde{\mathcal{M}}}\left|\frac{r^{(5-\delta)}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right| \leq C_0$$

**Proof of Theorem 6.1 (Strong peeling properties):** We prove the theorem assuming the lowest regularity for the metric we are able to handle, that is $q=3$. This is, in fact the more complicated case, the cases with higher values of $q$ follow immediately.

From the Bianchi equations, see (3.2.8) of [Kl-Ni], it follows that $\beta$ satisfies, along the incoming null hypersurface $\underline{C}(\nu)$, the evolution equation

$$\mathbf{D}\!\!\!/_3\beta + \mathrm{tr}\underline{\chi}\beta = \nabla\!\!\!\!/\rho + \left[2\underline{\omega}\beta + {}^{\star}\nabla\!\!\!\!/\sigma + 2\hat{\chi}\cdot\underline{\beta} + 3(\eta\rho + {}^{\star}\eta\sigma)\right] \qquad (6.13)$$

which can be rewritten as

$$\frac{\partial \beta_a}{\partial \lambda} + \Omega \text{tr}\underline{\chi}\beta_a = 2\Omega\underline{\omega}\beta_a + \Omega\left[\nabla\!\!\!/_a\rho + {}^\star\nabla\!\!\!/_a\sigma + 2(\hat{\chi}\cdot\underline{\beta})_a + 3(\eta\rho + {}^\star\eta\sigma)_a\right] \quad (6.14)$$

where[36] $\beta_a = \beta(e_a)$ and, on scalar functions, $\Omega \mathbf{D}\!\!\!\!/_3 = \frac{\partial}{\partial\lambda}$. From this equation, see Chapter 4 of [Kl-Ni], we obtain the following inequality, whose derivation is in the appendix,

$$\begin{aligned}\frac{d}{d\lambda}|r^{(2-\frac{2}{p})}\beta|_{p,S} &\leq ||2\Omega\underline{\omega} - (1-1/p)(\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}})||_\infty |r^{(2-\frac{2}{p})}\beta|_{p,S} \quad (6.15)\\ &+ \|\Omega\|_\infty\left(|r^{(2-\frac{2}{p})}\nabla\!\!\!/\rho|_{p,S} + 3|r^{(2-\frac{2}{p})}\eta\rho|_{p,S} + |r^{(2-\frac{2}{p})}\tilde{F}|_{p,S}\right)\end{aligned}$$

where $\tilde{F}(\cdot) = 2\hat{\chi}\cdot\underline{\beta} + ({}^\star\nabla\!\!\!/\sigma + 3{}^\star\eta\sigma)$.[37] Integrating along $\underline{C}(\nu)$ we obtain

$$\begin{aligned}&|r^{(2-\frac{2}{p})}\beta|_{p,S}(\lambda,\nu) \leq |r^{(2-\frac{2}{p})}\beta|_{p,S}(\lambda_1)\\ &+\int_{\lambda_1}^{\lambda}||2\Omega\underline{\omega} - (1-1/p)(\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}})||_\infty |r^{(2-\frac{2}{p})}\beta|_{p,S}(\lambda',\nu)\\ &+\|\Omega\|_\infty\left(\int_{\lambda_1}^{\lambda}|r^{(2-\frac{2}{p})}\nabla\!\!\!/\rho|_{p,S} + 3\int_{\lambda_1}^{\lambda}|r^{(2-\frac{2}{p})}\eta\rho|_{p,S} + \frac{1}{2}\int_{\lambda_1}^{\lambda}|r^{(2-\frac{2}{p})}\tilde{F}|_{p,S}\right)\end{aligned}$$

where $\lambda_1 = u|_{\underline{C}(\nu)\cap\Sigma_0}$. In $\widetilde{\mathcal{M}}$ we have at least the following behaviour [38]

$$\|\Omega\underline{\omega}\|_\infty = O(r^{-1}|\lambda|^{-1}) \text{ and } \|\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}}\|_\infty = O(r^{-2}|\lambda|^{-\frac{1}{2}}) .$$

Therefore we can apply the Gronwall's Lemma obtaining:

$$\begin{aligned}|r^{2-\frac{2}{p}}\beta|_{p,S}(\lambda,\nu) &\leq c\Bigg(|r^{2-\frac{2}{p}}\beta|_{p,S}(\lambda_1) + \|\Omega\|_\infty\int_{\lambda_1}^{\lambda}|r^{2-\frac{2}{p}}\nabla\!\!\!/\rho|_{p,S}\\ &+ 3\int_{\lambda_1}^{\lambda}|r^{2-\frac{2}{p}}\eta\rho|_{p,S} + \frac{1}{2}\int_{\lambda_1}^{\lambda}|r^{2-\frac{2}{p}}\tilde{F}|_{p,S}\Bigg) \quad (6.16)\end{aligned}$$

and recalling that, due to Theorem 2.2, see also Chapter 4 of [Kl-Ni], $\|\Omega\|_\infty \leq C$, we will inglobe, hereafter, this factor in the constant $c$. Multiplying both sides by $r^2|\lambda|^{1+\epsilon'}$, with $\epsilon' > 0$, we obtain

$$|r^{4-\frac{2}{p}}|\lambda|^{1+\epsilon'}\beta|_{p,S}(\lambda,\nu) \leq c\left(|r^{4-\frac{2}{p}}|\lambda|^{1+\epsilon'}\beta|_{p,S}(\lambda_1)\right. \quad (6.17)$$

[36] All the notations used in this paper without an explicit definition are those already introduced in [Kl-Ni].

[37] The term ${}^\star\nabla\!\!\!/\sigma + 3{}^\star\eta\sigma$ behaves better than the term $\nabla\!\!\!/_a\rho + 3\eta_a\rho$ and therefore we do no write them.

[38] Consistent with the decay properties proved in [Kl-Ni]. In fact the estimate for the second term are better concerning $|\lambda|$.

$$+\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\nabla\!\!\!/\rho|_{p,S}+3\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\eta\rho|_{p,S}+\frac{1}{2}\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\tilde{F}|_{p,S}\Big)\ .$$

We examine the integral terms in 6.17 and prove that they are bounded (uniformily in $\nu$).

a) $\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\nabla\!\!\!/\rho|_{p,S}$:

Inequality $\tilde{\mathcal{R}}\leq\epsilon_0$ proved in Theorem 2.3, [39] implies the inequality

$$|r^{4-\frac{2}{p}}|\lambda|^{\frac{1}{2}+\gamma}\nabla\!\!\!/(\rho,\sigma)|_{p,S}\leq c\epsilon_0\ .$$

Substituting this inequality in the previous integral we obtain [40]

$$\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\nabla\!\!\!/\rho|_{p,S}\leq c\epsilon_0\int_{\lambda_1}^{\lambda}\frac{1}{|\lambda'|^{1+\delta}}\leq c\epsilon_0 \tag{6.18}$$

where $\delta=\epsilon-\epsilon'>0$, recalling that $\gamma=\frac{3}{2}+\epsilon$.

b) $\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\eta\rho|_{S,p}$:

In this case the decay of $\rho$ is not improved by $\tilde{\mathcal{R}}\leq\epsilon_0$ due to its connection with the $ADM$ mass. On the other side, in view of the estimate $\tilde{\mathcal{O}}\leq c\varepsilon\leq c\epsilon_0$ proved in Theorem 2.2, $\eta$ satisfies the following inequality, see 5.2,

$$|r^{2-2/p}|\lambda|^{\frac{1}{2}+\gamma}\eta|_{p,S}(\lambda,\nu)\leq c\epsilon_0\quad,\ p\in[2,\infty]\ . \tag{6.19}$$

Using 6.19 it follows immediately

$$\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\eta\rho|_{p,S}\leq c\epsilon_0\int_{\lambda_1}^{\lambda}\frac{1}{r|\lambda'|^{1+\delta}}\leq c\epsilon_0\ . \tag{6.20}$$

c) $\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\tilde{F}|_{S,p}$:

From the expression $\tilde{F}(\cdot)={}^\star\nabla\!\!\!/\sigma+3{}^\star\eta\sigma+2\hat{\chi}\cdot\underline{\beta}$ and the previous remark concerning ${}^\star\nabla\!\!\!/\sigma+3{}^\star\eta\sigma$, we have only to prove that

$$\int_{\lambda_1}^{\lambda}|r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\hat{\chi}\underline{\beta}|_{S,p}\leq c\epsilon_0 \tag{6.21}$$

This is easy, as, from Theorems 2.3 and 2.4, we have

$$\sup_{\widetilde{\mathcal{M}}}|r^2|\lambda|^{\frac{3}{2}+\gamma}\underline{\beta}|\leq c\epsilon_0\ ,\ \ ||\lambda|^{\frac{1}{2}+\gamma}r^{2-2/p}\hat{\chi}|_{p,S}\leq c\epsilon_0\ .$$

[39] Hereafter we always assume $\epsilon_0$ small, but larger than $c\varepsilon$.
[40] In this case $\delta>0$ is needed.

Therefore

$$\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}}|\lambda'|^{1+\epsilon'}\tilde{F}|_{p,S} \leq c\epsilon_0 \ . \tag{6.22}$$

Collecting all these estimates for the integrals in 6.17 we infer that

$$|r^{4-\frac{2}{p}}|\lambda|^{1+\epsilon'}\beta|_{p,S}(\lambda,\nu) \leq c\left(|r^{4-\frac{2}{p}}|\lambda|^{1+\epsilon'}\beta|_{S,p}(\lambda_1)+\epsilon_0\right) \tag{6.23}$$

and using the initial data assumptions,

$$|r^{4-\frac{2}{p}}|\lambda|^{1+\epsilon'}\beta|_{p,S}(\lambda,\nu) \leq c\epsilon_0 \ , \quad p\in[2,4] \ . \tag{6.24}$$

To estimate $\alpha$ we need to estimate the $|\cdot|_{S,p=2}$ norm of $\nabla\!\!\!/\beta$. [41] This is the content of the following lemma:

**Lemma 6.1** *Assume the initial data are such that* $\tilde{J}_K^{(q)}(\Sigma_0,g,k)<\varepsilon^2$, $q=3$, *and that the* $\tilde{\mathcal{R}},\tilde{\mathcal{O}}$ *norms satisfy*

$$\tilde{\mathcal{R}}\leq\epsilon_0 \ , \ \tilde{\mathcal{O}}\leq\epsilon_0 \ . \tag{6.25}$$

*Then* $\nabla\!\!\!/\beta$ *satisfies the following inequality:*

$$\sup_{\widetilde{\mathcal{M}}} |r^{5-\frac{2}{p}}|\lambda|^{1+\epsilon'}\nabla\!\!\!/\beta|_{p=2,S} \leq c\epsilon_0 \tag{6.26}$$

**Proof:** See appendix.

The evolution equation for $\alpha$ on the incoming null hypersurface $\underline{C}(\nu)$ is [42]

$$D\!\!\!\!/_3\alpha+\frac{1}{2}\text{tr}\underline{\chi}\alpha=\nabla\!\!\!/\widehat{\otimes}\beta+\left[4\underline{\omega}\alpha-3(\hat{\chi}\rho+{}^\star\hat{\chi}\sigma)+(\zeta+4\eta)\widehat{\otimes}\beta\right] \tag{6.27}$$

which can be rewritten as

$$\frac{\partial\alpha}{\partial\lambda}+\frac{1}{2}\Omega\text{tr}\underline{\chi}\alpha=4\Omega\underline{\omega}\alpha+\Omega\left[\nabla\!\!\!/\widehat{\otimes}\beta+\left(-3(\hat{\chi}\rho+{}^\star\hat{\chi}\sigma)+(\zeta+4\eta)\widehat{\otimes}\beta\right)\right] \tag{6.28}$$

From this evolution equation we obtain, as shown in the appendix, the inequality:

$$\frac{d}{d\lambda}|\alpha|_{2,S}\leq 4\|\Omega\|_\infty|\underline{\omega}|_\infty|\alpha|_{2,S}+\|\Omega\|_\infty|\nabla\!\!\!/\widehat{\otimes}\beta|_{2,S}+\frac{1}{2}|F|_{2,S} \tag{6.29}$$

[41] It will be clear during the proof that we cannot obtain an estimate for $|r^{4-\frac{2}{p}}\alpha|_{p,S}$, with $p>2$, under the limited regularity assumption $q=3$.

[42] See equation (3.2.8) of [Kl-Ni].

where $|\omega|_\infty \equiv \sup_{S(\lambda,\nu)} |\omega|$ , $\|\Omega\|_\infty = \sup_{\tilde{\mathcal{M}}} |\Omega|$ and

$$F = -3(\hat{\chi}\rho + {}^\star\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta \ . \tag{6.30}$$

Integrating along $\underline{C}(\nu)$ we obtain

$$|\alpha|_{2,S}(\lambda,\nu) \le |\alpha|_{2,S}(\lambda_1) + 4\|\Omega\|_\infty \int_{\lambda_1}^{\lambda} |\underline{\omega}|_\infty |\alpha|_{2,S} + \|\Omega\|_\infty \int_{\lambda_1}^{\lambda} |\nabla\!\!\!/\widehat{\otimes}\beta|_{2,S} + \frac{1}{2}\int_{\lambda_1}^{\lambda} |F|_{2,S}$$

Multiplying by $r^{5-\frac{2}{p}}|_{p=2} = r^4$ we obtain

$$\begin{aligned}
|r^4\alpha|_{2,S}(\lambda,\nu) \ &\le \ |r^4\alpha|_{2,S}(\lambda_1) + 4\|\Omega\|_\infty \int_{\lambda_1}^{\lambda} |\underline{\omega}|_\infty |r^4\alpha|_{2,S} \\
&+ \ \|\Omega\|_\infty \int_{\lambda_1}^{\lambda} |r^4\nabla\!\!\!/\widehat{\otimes}\beta|_{2,S} + \frac{1}{2}\int_{\lambda_1}^{\lambda} |r^4 F|_{2,S} \\
&\le \ |r^4\alpha|_{2,\underline{C}(\nu)\cap\Sigma_0}(\lambda_1) + 4\|\Omega\|_\infty \int_{\lambda_1}^{\lambda} |\underline{\omega}|_\infty |r^4\alpha|_{2,S} \qquad (6.31)\\
&+ \ \left[\|\Omega\|_\infty \int_{\lambda_1}^{\lambda} |r^4\nabla\!\!\!/\widehat{\otimes}\beta|_{2,S} + \frac{3}{2}\|\Omega\|_\infty \int_{\lambda_1}^{\lambda} |r^4\hat{\chi}\rho|_{2,S}\right] + \frac{1}{2}\int_{\lambda_1}^{\lambda} |r^4\tilde{F}|_{2,S}
\end{aligned}$$

where $\tilde{F} = {}^\star\hat{\chi}\sigma + (\zeta + 4\eta)\widehat{\otimes}\beta$ .

An application of Gronwall's lemma gives, recalling that, see Theorem 2.4, $|\underline{\omega}|_\infty = O(r^{-1}|\lambda|^{-1})$,

$$\begin{aligned}
|r^4\alpha|_{2,S}(\lambda,\nu) \ &\le \ c\Bigg(|r^4\alpha|_{2,S}(\lambda_1) + \int_{\lambda_1}^{\lambda} |r^4\nabla\!\!\!/\widehat{\otimes}\beta|_{2,S} \\
&+ \ \frac{3}{2}\int_{\lambda_1}^{\lambda} |r^4\hat{\chi}\rho|_{2,S} + \frac{1}{2}\int_{\lambda_1}^{\lambda} |r^4\tilde{F}|_{2,S}\Bigg) \qquad (6.32)
\end{aligned}$$

This inequality requires to control $\int_{\lambda_1}^{\lambda} |r^4\nabla\!\!\!/\widehat{\otimes}\beta|_{S,2}$, $\int_{\lambda_1}^{\lambda} |r^4\hat{\chi}\rho|_{S,2}$ and the integral $\int_{\lambda_1}^{\lambda} |r^4\tilde{F}|_{S,2}$. Let us examine them separately. The first integral is bounded using the estimate 6.26 of lemma 6.1. In fact

$$\int_{\lambda_1}^{\lambda} |r^4\nabla\!\!\!/\widehat{\otimes}\beta|_{S,2} \le c\epsilon_0 \int_{\lambda_1}^{\lambda} \frac{1}{|\lambda'|^{1+\epsilon'}} d\lambda' \le c\epsilon_0 \frac{1}{|\lambda'|^{\epsilon'}} \tag{6.33}$$

To control the second integral we use the estimate for $\hat{\chi}$ which follows from the estimate $\tilde{\mathcal{O}} \le \epsilon_0$, in Theorem 2.4,

$$||\lambda|^{\frac{1}{2}+\gamma} r^{2-2/p}\hat{\chi}|_{p,S}(\lambda,\nu) \le \epsilon_0 \ ,$$

obtaining

$$\int_{\lambda_1}^{\lambda}|r^4\hat{\chi}\rho|_{2,S}\leq c\epsilon_0\int_{\lambda_1}^{\lambda}\frac{1}{|\lambda'|^{\frac{1}{2}+\gamma}}\leq c\epsilon_0\frac{1}{|\lambda|^{1+\epsilon}}\ . \tag{6.34}$$

The integral $\frac{1}{2}\int_{\lambda_1}^{\lambda}|r^4\tilde{F}|_{S,2}$ is easier to control [43] and we do not report it here. The conclusion is, therefore that the following inequality holds

$$\sup_{\widetilde{\mathcal{M}}}|r^{5-\frac{2}{p}}|\lambda|^{\epsilon'}\alpha|_{p,S}\leq C_0,\quad p=2\ . \tag{6.35}$$

**Proof of Theorem 6.2 (Weak peeling properties):**
We shall only give the proof in the simplest case $q=5$. In this case the initial data are sufficiently regular to allow us to control the pointwise norms for the Riemann tensor components up to second order derivatives in the whole spacetime. The extension to less regular initial data is immediate and goes on the same lines of the proof of Theorem 6.1.

Recall that if we assume $q=5$ the boundedness of the $\tilde{\mathcal{Q}}$ norms imply the following sup norm estimates for the various null components of the Riemann tensor:

$$\begin{aligned}
&\sup_{\widetilde{\mathcal{M}}}r^{\frac{7}{2}}|\lambda|^{\gamma}|\alpha|\leq C_0\ ,\ \sup_{\widetilde{\mathcal{M}}}r|\lambda|^{\frac{5}{2}+\gamma}|\underline{\alpha}|\leq C_0\\
&\sup_{\widetilde{\mathcal{M}}}r^{\frac{7}{2}}|\lambda|^{\gamma}|\beta|\leq C_0\ ,\ \sup_{\widetilde{\mathcal{M}}}r^2|\lambda|^{\frac{3}{2}+\gamma}|\underline{\beta}|\leq C_0\\
&\sup_{\widetilde{\mathcal{M}}}r^3|\rho|\leq C_0\ ,\ \sup_{\widetilde{\mathcal{M}}}r^3|\lambda|^{\frac{1}{2}+\gamma}|(\rho-\overline{\rho},\sigma)|\leq C_0
\end{aligned} \tag{6.36}$$

Moreover they are supplemented with analogous pointwise estimates for the first and second derivatives of the Riemann tensor obtained by the previous ones just adding $r^k\nabla\!\!\!/^k$ in front of each null component with $k\in[1,2]$. In particular we obtain the following estimates for $\nabla\!\!\!/^k\rho$: [44]

$$\sup_{\widetilde{\mathcal{M}}}|r^{3+k}|\lambda|^{\frac{1}{2}+\gamma}\nabla\!\!\!/^k\rho|\leq C_0 \tag{6.37}$$

[43] Only its first term $\frac{1}{2}\int_{\lambda_1}^{\lambda}|r^4\hat{\chi}\sigma|_{S,2}$ requires the estimate $\tilde{\mathcal{O}}\leq\epsilon_0$, the second one can be estimated using the results in [Kl-Ni].

[44] $C_0^2$ is a constant which bounds the $\tilde{\mathcal{Q}}$ norms. We can pose $C_0=c\varepsilon$.

Using $\beta$ and $\alpha$ evolution equations along an incoming null hypersurface $\underline{C}(\nu)$, see 6.13, 6.27, we obtain, proceeding as before, with the help of Gronwall's lemma, the following inequalities:

$$|r^2\beta|(\lambda,\nu) \leq c|r^2\beta|_{\underline{C}(\nu)\cap\Sigma_0} + c\int_{\lambda_1(\nu)}^{\lambda} |r^2\nabla\!\!\!/\rho|(\lambda')d\lambda' \tag{6.38}$$

$$|r\alpha|(\lambda,\nu) \leq c|r\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + c\int_{\lambda_1(\nu)}^{\lambda} |r\nabla\!\!\!/\beta|(\lambda')d\lambda' \tag{6.39}$$

where all the norms are pointwise norms and we neglected all the (non linear) terms with a better decay factor. Moreover, as done before in the proof of the "Stronger peeling Theorem", see the appendix, we have also the following estimate for $\nabla\!\!\!/\beta$:

$$|r^3\nabla\!\!\!/\beta|(\lambda,\nu) \leq c|r^3\nabla\!\!\!/\beta|_{\underline{C}(\nu)\cap\Sigma_0} + c\int_{\lambda_1(\nu)}^{\lambda} |r^3\nabla\!\!\!/^2\rho|(\lambda')d\lambda' \tag{6.40}$$

The proof of the result is based on a systematic use of equations 6.38, 6.39, 6.40 for the various values of $\gamma = \frac{3}{2} - \delta$, with $\delta \in [0, \frac{3}{2})$.

**i)** $(\delta = 0)$**:** From 6.37 we have $|r^4|\lambda|^2\nabla\!\!\!/\rho| \leq C_0$ which used in 6.38 implies

$$\begin{aligned}
|r^2\beta|(\lambda,\nu) &\leq c|r^2\beta|_{\underline{C}(\nu)\cap\Sigma_0} + c\Big(\sup_{\tilde{\mathcal{M}}} |r^4|\lambda|^2\nabla\!\!\!/\rho|\Big)\int_{\lambda_1(\nu)}^{\lambda} \frac{1}{r^2|\lambda'|^2}d\lambda' \\
&\leq c|r^2\beta|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{1}{r(\lambda,\nu)^2}\left(\frac{1}{|\lambda|} - \frac{1}{|\lambda_1(\nu)|}\right) \\
&\leq c|r^2\beta|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{1}{r(\lambda,\nu)^2|\lambda|}\left(1 - \frac{|\lambda|}{|\lambda_1(\nu)|}\right)
\end{aligned} \tag{6.41}$$

and, multiplying by $r^2|\lambda|$,

$$|r^4|\lambda|\beta|(\lambda,\nu) \leq c|r^4|\lambda|\beta|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\Big(1 - \frac{|\lambda|}{|\lambda_1(\nu)|}\Big) \ . \tag{6.42}$$

From it we immediately infer, recalling the initial data assumptions that [45]

$$|r^4|\lambda|\beta|(\lambda,\nu) \leq cC_0 \ , \ |r^5|\lambda|\nabla\!\!\!/\beta|(\lambda,\nu) \leq cC_0 \ . \tag{6.43}$$

[45] The second inequality is obtained repeating the same argument starting from 6.40.

Given these estimates we apply them to the estimate for $\alpha$, in inequality 6.39 which we multiply on both sides by $r^4(\log r)^{-(1+\epsilon)}$ obtaining, [46]

$$\begin{aligned}
&|\frac{r^5}{(\log r)^{1+\epsilon}}\alpha|(\lambda,\nu) \leq c|\frac{r^5}{(\log r)^{1+\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + c\int_{\lambda_1(\nu)}^{\lambda}|\frac{r^5}{(\log r)^{1+\epsilon}}\nabla\!\!\!/\beta|d\lambda' \\
&\leq c|\frac{r^5}{(\log r)^{1+\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + c\Big(\sup_{\tilde{\mathcal{M}}}|r^5|\lambda|\nabla\!\!\!/\beta|\Big)\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{|\lambda'|(\log r)^{1+\epsilon}}d\lambda' \\
&\leq c|\frac{r^5}{(\log r)^{1+\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{|\lambda'|(\log|\lambda'|)^{1+\epsilon}}d\lambda' \qquad (6.44)\\
&\leq c|\frac{r^5}{(\log r)^{1+\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{1}{(\log|\lambda|)^{\epsilon}}\left(1-\frac{(\log|\lambda|)^{\epsilon}}{(\log|\lambda_1|)^{\epsilon}}\right)
\end{aligned}$$

Multiplying both sides by $(\log|\lambda|)^{\epsilon}$ and observing that $(\log|\lambda_1|)^{\epsilon} \geq (\log|\lambda|)^{\epsilon}$ we infer

$$|\frac{r^5(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha|(\lambda,\nu) \leq c|\frac{r^5}{\log r}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\left(1-\frac{(\log|\lambda|)^{\epsilon}}{(\log|\lambda_1|)^{\epsilon}}\right) \leq cC_0\ , \quad (6.45)$$

the last inequality following from the initial data assumptions.
**Remark:** Observe that from inequality 6.45

$$|r^5\alpha|(\lambda,\nu) \leq c|\frac{r^5(\log r)^{\epsilon}}{(\log|\lambda|)^{\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{(\log r)^{1+\epsilon}}{(\log|\lambda|)^{\epsilon}}\left(1-\frac{(\log|\lambda|)^{\epsilon}}{(\log|\lambda_1|)^{\epsilon}}\right)\ , \quad (6.46)$$

therefore, for any fixed $\nu$,

$$\lim_{\lambda\to\lambda_1}|r^5\alpha|(\lambda,\nu) \leq c|r^5\alpha|_{\underline{C}(\nu)\cap\Sigma_0} \leq cC_0 \qquad (6.47)$$

showing that the values on $\underline{C}(\nu)$ approach continuosly the inital values on $\Sigma_0$.

**ii)** $(\delta\in(0,1))$**:** In this case the analogous of 6.41 and 6.42 are

$$\begin{aligned}
|r^2\beta|(\lambda,\nu) &\leq c|r^2\beta|_{\underline{C}(\nu)\cap\Sigma_0} + c\Big(\sup_{\tilde{\mathcal{M}}}|r^4|\lambda|^{2-\delta}\nabla\!\!\!/\rho|\Big)\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{r^2|\lambda'|^{2-\delta}}d\lambda' \\
&\leq c|r^2\beta|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{1}{r^2|\lambda|^{1-\delta}}\left(1-\frac{|\lambda|^{1-\delta}}{|\lambda_1(\nu)|^{1-\delta}}\right) \qquad (6.48)
\end{aligned}$$

and

$$|r^4|\lambda|^{1-\delta}|\beta|(\lambda,\nu) \leq c|r^4|\lambda|^{1-\delta}\beta|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\left(1-\frac{|\lambda|^{1-\delta}}{|\lambda_1(\nu)|^{1-\delta}}\right) \qquad (6.49)$$

[46]Observe that $\frac{r^4}{(\log r)^{1+\epsilon}}$ is a decreasing function moving along $\underline{C}(\nu)$ toward the future.

From it we infer immediately, recalling the initial data assumptions, that

$$|r^4|\lambda|^{1-\delta}\beta|(\lambda,\nu) \leq cC_0 \ , \ |r^5|\lambda|^{1-\delta}\nabla\!\!\!/\beta|(\lambda,\nu) \leq cC_0 \ . \tag{6.50}$$

We use these inequalities to estimate $\alpha$ multiplying 6.39 by $r^{4-\delta}(\log r)^{-(1+\epsilon)}$,

$$\begin{aligned}
&|\frac{r^{5-\delta}}{(\log r)^{1+\epsilon}}\alpha|(\lambda,\nu) \leq c|\frac{r^{5-\delta}}{(\log r)^{1+\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + c\int_{\lambda_1(\nu)}^{\lambda}|\frac{r^{5-\delta}}{(\log r)^{1+\epsilon}}\nabla\!\!\!/\beta|(\lambda')d\lambda' \\
&\leq c|\frac{r^{5-\delta}}{(\log r)^{1+\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + c\Big(\sup_{\tilde{\mathcal{M}}}|r^5|\lambda|^{1-\delta}\nabla\!\!\!/\beta|(\lambda,\nu)\Big)\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{r^{\delta}|\lambda'|^{1-\delta}(\log r)^{1+\epsilon}}d\lambda' \\
&\leq c|\frac{r^{5-\delta}}{(\log r)^{1+\epsilon}}\alpha|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{1}{(\log|\lambda|)^{\epsilon}}\left(1-\frac{(\log|\lambda|)^{\epsilon}}{(\log|\lambda_1|)^{\epsilon}}\right) \leq cC_0 \ ,
\end{aligned} \tag{6.51}$$

using in the last inequality the initial data assumptions. Observe that, as before, the estimates on $\underline{C}(\nu)$ are consistent with the inital values on $\Sigma_0$. In conclusion,

$$|r^4|\lambda|^{1-\delta}\beta|(\lambda,\nu) \leq cC_0 \quad , \quad |\frac{r^{5-\delta}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\alpha|(\lambda,\nu) \leq cC_0 \tag{6.52}$$

**iii)** ($\delta = 1$)**:** In this case an estimate analogous to 6.49 would be divergent. Therefore we multiply 6.38 by $r^2(\log r)^{1+\epsilon}$ obtaining

$$\begin{aligned}
\left|\frac{r^4}{(\log r)^{1+\epsilon}}\beta\right|(\lambda,\nu) \ &\leq \ c\left|\frac{r^4}{(\log r)^{1+\epsilon}}\beta\right|_{\underline{C}(\nu)\cap\Sigma_0} + c\Big(\sup_{\tilde{\mathcal{M}}}|r^4|\lambda|\nabla\!\!\!/\rho|\Big)\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{(\log r)^{1+\epsilon}|\lambda'|}d\lambda' \\
&\leq \ c\left|\frac{r^4}{(\log r)^{1+\epsilon}}\beta\right|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{1}{(\log|\lambda|)^{\epsilon}}\left(1-\frac{(\log|\lambda|)^{\epsilon}}{(\log|\lambda_1|)^{\epsilon}}\right)
\end{aligned} \tag{6.53}$$

which implies

$$\left|\frac{r^4(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right|(\lambda,\nu) \leq cC_0 \quad , \quad \left|\frac{r^5(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\nabla\!\!\!/\beta\right|(\lambda,\nu) \leq cC_0 \tag{6.54}$$

Proceeding in the same way as before, we infer, multiplying 6.39 by $r^3(\log r)^{-(2+\epsilon)}$,

$$\begin{aligned}
&\left|\frac{r^4}{(\log r)^{2+\epsilon}}\alpha\right|(\lambda,\nu) \leq c\left|\frac{r^4}{(\log r)^{2+\epsilon}}\alpha\right|_{\underline{C}(\nu)\cap\Sigma_0} + c\int_{\lambda_1(\nu)}^{\lambda}|\frac{r^4}{(\log r)^{2+\epsilon}}\nabla\!\!\!/\beta|d\lambda' \\
&\leq c\left|\frac{r^4}{(\log r)^{2+\epsilon}}\alpha\right|_{\underline{C}(\nu)\cap\Sigma_0} + c\Big(\sup_{\tilde{\mathcal{M}}}\left|\frac{r^5(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\nabla\!\!\!/\beta\right|\Big)\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{r\log r(\log|\lambda'|)^{\epsilon}}d\lambda' \\
&\leq c\left|\frac{r^4}{(\log r)^{2+\epsilon}}\alpha\right|_{\underline{C}(\nu)\cap\Sigma_0} + cC_0\frac{1}{(\log|\lambda|)^{\epsilon}}\left(1-\frac{(\log|\lambda|)^{\epsilon}}{(\log|\lambda_1|)^{\epsilon}}\right) \ .
\end{aligned} \tag{6.55}$$

Therefore

$$\left|\frac{r^4(\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}}\beta\right|(\lambda,\nu)\leq cC_0 \quad , \quad \left|\frac{r^4(\log|\lambda|)^\epsilon}{(\log r)^{2+\epsilon}}\alpha\right|(\lambda,\nu)\leq cC_0 \tag{6.56}$$

**iv)** $(\delta\in(1,\frac{3}{2}))$**:** The analogous of 6.42 is obtained multiplying 6.38 by $r^{2-\tau}(\log r)^{-(1+\epsilon)}$

$$\begin{aligned}\left|\frac{r^{4-\tau}}{(\log r)^{1+\epsilon}}\beta\right|(\lambda,\nu) &\leq c\left|\frac{r^{4-\tau}}{(\log r)^{1+\epsilon}}\beta\right|_{\underline{C}(\nu)\cap\Sigma_0}\\ &+c\Big(\sup_{\tilde{\mathcal{M}}}|r^4|\lambda|^{2-\delta}\nabla\!\!\!/\rho|\Big)\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{r^\tau|\lambda'|^{2-\delta}(\log r)^{1+\epsilon}}d\lambda'\\ &\leq c\left|\frac{r^{4-\tau}}{(\log r)^{1+\epsilon}}\beta\right|_{\underline{C}(\nu)\cap\Sigma_0}+cC_0\frac{1}{(\log|\lambda|)^\epsilon}\left(1-\frac{(\log|\lambda|)^\epsilon}{(\log|\lambda_1|)^\epsilon}\right)\ ,\end{aligned} \tag{6.57}$$

choosing $\tau=\delta-1$. Therefore this implies

$$\left|\frac{r^{5-\delta}(\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}}\beta\right|(\lambda,\nu)\leq cC_0 \quad , \quad \left|\frac{r^{6-\delta}(\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}}\nabla\!\!\!/\beta\right|(\lambda,\nu)\leq cC_0 \tag{6.58}$$

Proceeding for $\alpha$ as before, we multiply 6.39 by $r^{4-\sigma}(\log r)^{-\tau}$, obtaining

$$\begin{aligned}\left|\frac{r^{5-\sigma}}{(\log r)^\tau}\alpha\right|(\lambda,\nu) &\leq c\left|\frac{r^{5-\sigma}}{(\log r)^\tau}\alpha\right|_{\underline{C}(\nu)\cap\Sigma_0}+c\int_{\lambda_1(\nu)}^{\lambda}|\frac{r^{5-\sigma}}{(\log r)^\tau}\nabla\!\!\!/\beta|d\lambda'\\ &\leq c\left|\frac{r^{5-\sigma}}{(\log r)^\tau}\alpha\right|_{\underline{C}(\nu)\cap\Sigma_0}+c\Big(\sup_{\tilde{\mathcal{M}}}\left|\frac{r^{5-(\delta-1)}(\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}}\nabla\!\!\!/\beta\right|(\lambda,\nu)\Big)\cdot\\ &\qquad\cdot\int_{\lambda_1(\nu)}^{\lambda}\frac{1}{r^{\sigma-(\delta-1)}(\log r)^{\tau-(1+\epsilon)}(\log|\lambda'|)^\epsilon}d\lambda'\\ &\leq c\left|\frac{r^{5-\sigma}}{(\log r)^\tau}\alpha\right|_{\underline{C}(\nu)\cap\Sigma_0}+cC_0\frac{1}{(\log|\lambda|)^\epsilon}\left(1-\frac{(\log|\lambda|)^\epsilon}{(\log|\lambda_1|)^\epsilon}\right)\ .\end{aligned} \tag{6.59}$$

choosing $\sigma=\delta$ , $\tau=2+\epsilon$. In conclusion we obtain

$$\left|\frac{r^{5-\delta}(\log|\lambda|)^\epsilon}{(\log r)^{1+\epsilon}}\beta\right|(\lambda,\nu)\leq cC_0 \quad ; \quad \left|\frac{r^{5-\delta}(\log|\lambda|)^\epsilon}{(\log r)^{2+\epsilon}}\alpha\right|(\lambda,\nu)\leq cC_0 \tag{6.60}$$

and, as before, the estimates on $\underline{C}(\nu)$ are consistent with the inital values on $\Sigma_0$.

**Remarks:**

**a)** Observe that the case $\delta = \frac{3}{2}$ implies, here, an estimate for $\alpha$ and $\beta$ of this kind:

$$\left|\frac{r^{\frac{7}{2}}(\log|\lambda|)^{\epsilon}}{(\log r)^{1+\epsilon}}\beta\right|(\lambda,\nu) \leq cC_0 \quad ; \quad \left|\frac{r^{\frac{7}{2}}(\log|\lambda|)^{\epsilon}}{(\log r)^{2+\epsilon}}\alpha\right|(\lambda,\nu) \leq cC_0 \tag{6.61}$$

different from the one in [Kl-Ni]. The difference is is due to the fact that there we assumed, as initial conditions, that $g$ and $k$ behave as $o(r^{-\frac{3}{2}})$ and $o(r^{-\frac{5}{2}})$ respectively, while, in the present case, $\gamma = \frac{3}{2} - \delta = 0$ implies that the metric and second fundamental form on $\Sigma_0$ behave as $O(r^{-\frac{3}{2}})$ and $O(r^{-\frac{5}{2}})$. With this in mind the case in [Kl-Ni] is easily reobtained.

**b)** Looking at the evolution equation for $\beta$ along the outgoing null hypersurface $C(\lambda)$ it is also easy to realize that, proceeding as in [Kl-Ni], Chapter 8, the sup estimates for $\alpha$ allow to proof the existence of the limit: $\lim_{C(\lambda);\nu\to\infty} r^4\beta$, for $\delta < 1$.

# 7 Appendix

## 7.1 The $\tilde{\mathcal{R}}$ norms.

The $\tilde{\mathcal{R}}$ norms are straightforword modifications of the $\mathcal{R}$ norms introduced in Chapter 3, subsection 3.5.2 of [Kl-Ni].
More precisely all norms defined in subsection 3.5.2 are modified as follows: If the $L^2$ integral norm consists of an integral along a null outgoing hypersurface $C(\lambda)$ we just multiply the norm integrand by the factor $|\lambda|^{2\gamma}$. If the $L^2$ integral norm consists of an integral along a null incoming hypersurface $\underline{C}(\nu)$, the integrand has to be multiplied by the function $|u|^{2\gamma}$. The tangential derivatives present in these norms are up to order $q-1$.
This recipe has to be applied for all the quantities except when the integrand contains the Riemann component $\rho$ without any tangential derivative.

## 7.2 The $\tilde{\mathcal{O}}$ norms.

These are also straightforward modifications of the $\mathcal{O}$ norms introduced in subsection 3.5.3 of [Kl-Ni]. More precisely all the norms which are relative to connection coefficients or their derivatives which are different from zero in the Schwarzschild case are left unchanged. The remaining quantities are modified multiplying them by the factor $|\lambda|^{\gamma}$. The tangential derivatives present in these norms are up to order $q$.

## 7.3 Proof of inequality 6.15

From the equation

$$\frac{\partial\beta_a}{\partial\lambda}+\Omega\mathrm{tr}\underline{\chi}\beta_a=2\Omega\underline{\omega}\beta_a+\Omega\left[\nabla\!\!\!\!/_a\rho+{}^\star\nabla\!\!\!\!/_a\sigma+2(\hat{\chi}\cdot\underline{\beta})_a+3(\eta\rho+{}^\star\eta\sigma)_a\right] \quad (7.1)$$

it follows that

$$\begin{aligned}&\frac{\partial|\beta|^p}{\partial\lambda}=p|\beta|^{p-1}\frac{\partial|\beta|}{\partial\lambda}=p|\beta|^{p-2}\beta\cdot\frac{\partial\beta}{\partial\lambda}\\&=p|\beta|^{p-2}\beta\cdot\left[(-\Omega\mathrm{tr}\underline{\chi}+2\Omega\underline{\omega})\beta+\Omega\nabla\!\!\!\!/\rho+\Omega\left({}^\star\nabla\!\!\!\!/\sigma+2\hat{\chi}\cdot\underline{\beta}+3(\eta\rho+{}^\star\eta\sigma)\right)\right]\\&=p(-\Omega\mathrm{tr}\underline{\chi}+2\Omega\underline{\omega})|\beta|^p+p|\beta|^{p-2}\Omega\beta\cdot\nabla\!\!\!\!/\rho+3p\Omega|\beta|^{p-2}\beta\cdot\eta\rho+p\Omega|\beta|^{p-2}\beta\cdot\tilde{F}(\cdot)\end{aligned}$$

where

$$\tilde{F}(\cdot)={}^\star\nabla\!\!\!\!/\sigma+2\hat{\chi}\cdot\underline{\beta}+3{}^\star\eta\sigma \quad (7.2)$$

Therefore

$$\frac{\partial|\beta|^p}{\partial\lambda}+p\Omega\mathrm{tr}\underline{\chi}|\beta|^p=2p\Omega\underline{\omega}|\beta|^p+p|\beta|^{p-2}\Omega\beta\cdot\nabla\!\!\!/\rho$$
$$+3p\Omega|\beta|^{p-2}\beta\cdot\eta\rho+p\Omega|\beta|^{p-2}\beta\cdot\tilde{F}(\cdot)\ . \tag{7.3}$$

From the equation

$$\frac{d}{d\lambda}\left(\int_{S(\lambda,\nu)}r^{\sigma}fd\mu_{\gamma}\right)=\int_{S(\lambda,\nu)}r^{\sigma}\left(\frac{df}{d\lambda}+(1+\frac{\sigma}{2})\Omega\mathrm{tr}\underline{\chi}f\right)-\frac{\lambda}{2}\int_{S(\lambda,\nu)}r^{\sigma}f(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\ ,$$

choosing $\sigma=p(2-\frac{2}{p})$, we obtain

$$\frac{d}{d\lambda}\left(\int_{S(\lambda,\nu)}|r^{(2-\frac{2}{p})}\beta|^pd\mu_{\gamma}\right)=\int_{S(\lambda,\nu)}r^{p(2-\frac{2}{p})}\left(\frac{\partial|\beta|^p}{\partial\lambda}+p\Omega\mathrm{tr}\underline{\chi}|\beta|^p\right)$$
$$-\frac{p(2-\frac{2}{p})}{2}\int_{S(\lambda,\nu)}|r^{(2-\frac{2}{p})}\beta|^p(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})$$
$$=\int_{S(\lambda,\nu)}r^{p(2-\frac{2}{p})}\left(2p\Omega\underline{\omega}|\beta|^p+p|\beta|^{p-2}\Omega\beta\cdot\nabla\!\!\!/\rho+3p\Omega|\beta|^{p-2}\beta\cdot\eta\rho+p\Omega|\beta|^{p-2}\beta\cdot\tilde{F}(\cdot)\right)$$
$$-(p-1)\int_{S(\lambda,\nu)}|r^{(2-\frac{2}{p})}\beta|^p(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\ . \tag{7.4}$$

Equation 7.4 can be rewritten as

$$\frac{d}{d\lambda}|r^{(2-\frac{2}{p})}\beta|^p_{p,S}=p|r^{(2-\frac{2}{p})}\beta|^{p-1}_{p,S}\frac{d}{d\lambda}|r^{(2-\frac{2}{p})}\beta|_{p,S}$$
$$=\int_{S(\lambda,\nu)}\left(2p\Omega\underline{\omega}+(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\right)|r^{(2-\frac{2}{p})}\beta|^p$$
$$+p\int_{S(\lambda,\nu)}r^{p(2-\frac{2}{p})}|\beta|^{p-2}\Omega\left(\beta\cdot\nabla\!\!\!/\rho+3\beta\cdot\eta\rho+\beta\cdot\tilde{F}(\cdot)\right)$$

and applying Holder inequality, [47]

$$\frac{d}{d\lambda}|r^{(2-\frac{2}{p})}\beta|_{p,S}\ \leq\ |2\Omega\underline{\omega}+p^{-1}(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})|_{\infty}|r^{(2-\frac{2}{p})}\beta|_{p,S} \tag{7.5}$$
$$+\ \|\Omega\|_{\infty}\left(|r^{(2-\frac{2}{p})}\nabla\!\!\!/\rho|_{p,S}+3|r^{(2-\frac{2}{p})}\eta\rho|_{p,S}+|r^{(2-\frac{2}{p})}\tilde{F}|_{p,S}\right)$$

[47] $|f|_{\infty}\equiv\sup_{S(\lambda,\nu)}|f|\ ;\ \|f\|_{\infty}\equiv\sup_{\tilde{\mathcal{M}}}|f|$.

### 7.4 Proof of Lemma 6.1

**Lemma 6.1** *Assume the initial data are such that* $\tilde{J}_K^{(q)}(\Sigma_0,g,k)<\varepsilon^2$, $q=3$, *and that the* $\tilde{\mathcal{R}},\tilde{\mathcal{O}}$ *norms satisfy*

$$\tilde{\mathcal{R}}\le\epsilon_0\ ,\ \ \tilde{\mathcal{O}}\le\epsilon_0\ . \tag{7.6}$$

*Then* $\nabla\!\!\!/\beta$ *satisfies the following inequality:*

$$\sup_{\widetilde{\mathcal{M}}}|r^{5-\frac{2}{p}}|\lambda|^{1+\epsilon'}\nabla\!\!\!/\beta|_{p=2,S}\le c\epsilon_0 \tag{7.7}$$

**Proof:** We derive the evolution equation for $\nabla\!\!\!/\beta$ starting from the one for $\beta$, see 6.13,

$$\begin{aligned}\mathbf{D}\!\!\!\!/_3\nabla\!\!\!/\beta &= [\mathbf{D}\!\!\!\!/_3,\nabla\!\!\!/]\beta+\nabla\!\!\!/\mathbf{D}\!\!\!\!/_3\beta=[\mathbf{D}\!\!\!\!/_3,\nabla\!\!\!/]\beta-\mathrm{tr}\underline{\chi}\nabla\!\!\!/\beta\\ &- (\nabla\!\!\!/\mathrm{tr}\underline{\chi})\beta+\nabla\!\!\!/\left[\nabla\!\!\!/\rho+2\underline{\omega}\beta+{}^\star\nabla\!\!\!/\sigma+2\hat{\chi}\cdot\underline{\beta}+3(\eta\rho+{}^\star\eta\sigma)\right]\end{aligned}$$

which we rewrite as

$$\begin{aligned}\mathbf{D}\!\!\!\!/_3\nabla\!\!\!/\beta+\mathrm{tr}\underline{\chi}\nabla\!\!\!/\beta &= [\mathbf{D}\!\!\!\!/_3,\nabla\!\!\!/]\beta+2\underline{\omega}\nabla\!\!\!/\beta+\nabla\!\!\!/\nabla\!\!\!/\rho-(\nabla\!\!\!/\mathrm{tr}\underline{\chi})\beta+2(\nabla\!\!\!/\underline{\omega})\beta\\ &+ \nabla\!\!\!/\left[{}^\star\nabla\!\!\!/\sigma+2\hat{\chi}\cdot\underline{\beta}+3(\eta\rho+{}^\star\eta\sigma)\right]\end{aligned} \tag{7.8}$$

From the relation, proved in the appendix to Chapter 4 of [Kl-Ni],

$$[\mathbf{D}\!\!\!\!/_3,\nabla\!\!\!/]\beta=-\frac{1}{2}\mathrm{tr}\underline{\chi}\nabla\!\!\!/\beta-\underline{\hat{\chi}}\nabla\!\!\!/\beta-\eta\cdot\underline{\chi}\cdot\beta+\underline{\chi}(\eta\cdot\beta)+(\nabla\!\!\!/\log\Omega)\mathbf{D}\!\!\!\!/_3\beta+c\ {}^*\underline{\beta}\cdot{}^*\beta$$

where $|c|=1$, we obtain

$$\begin{aligned}&\mathbf{D}\!\!\!\!/_3\nabla\!\!\!/\beta+\frac{3}{2}\mathrm{tr}\underline{\chi}\nabla\!\!\!/\beta=2\underline{\omega}\nabla\!\!\!/\beta-\underline{\hat{\chi}}\nabla\!\!\!/\beta+\nabla\!\!\!/\nabla\!\!\!/\rho+\nabla\!\!\!/{}^\star\nabla\!\!\!/\sigma\\ &\quad+\left[-(\nabla\!\!\!/\mathrm{tr}\underline{\chi})\beta+2(\nabla\!\!\!/\underline{\omega})\beta-\eta\cdot\underline{\chi}\cdot\beta+\underline{\chi}(\eta\cdot\beta)\right]\\ &\quad+\nabla\!\!\!/\left[2\hat{\chi}\cdot\underline{\beta}+3(\eta\rho+{}^\star\eta\sigma)\right]+c\ {}^*\underline{\beta}\cdot{}^*\beta\\ &\quad+\nabla\!\!\!/\log\Omega\cdot\left[-\mathrm{tr}\underline{\chi}\beta+\nabla\!\!\!/\rho+\left(2\underline{\omega}\beta+{}^\star\nabla\!\!\!/\sigma+2\hat{\chi}\cdot\underline{\beta}+3(\eta\rho+{}^\star\eta\sigma)\right)\right]\end{aligned}$$

which we rewrite as

$$\mathbf{D}\!\!\!\!/_3\nabla\!\!\!/\beta+\frac{3}{2}\mathrm{tr}\underline{\chi}\nabla\!\!\!/\beta=(2\underline{\omega}-\underline{\hat{\chi}}\cdot)\nabla\!\!\!/\beta+\underline{F} \tag{7.9}$$

where

$$
\begin{aligned}
F &= \nabla\!\!\!/\nabla\!\!\!/\rho+\nabla\!\!\!/^{\star}\nabla\!\!\!/\sigma \\
&+ \left[2\hat{\chi}\nabla\!\!\!/\underline{\beta}+3(\eta\nabla\!\!\!/\rho+{}^{\star}\eta\nabla\!\!\!/\sigma)+(\nabla\!\!\!/\log\Omega)(\nabla\!\!\!/\rho+{}^{\star}\nabla\!\!\!/\sigma)\right] \\
&+ \left[3\nabla\!\!\!/\eta+3(\nabla\!\!\!/\log\Omega)\eta\right]\rho \\
&+ \left\{\left[2(\nabla\!\!\!/\hat{\chi})\underline{\beta}+3(\nabla\!\!\!/^{\star}\eta)\sigma+(\nabla\!\!\!/\log\Omega)\left(-\mathrm{tr}\underline{\chi}\beta+2\underline{\omega}\beta+2\hat{\chi}\cdot\underline{\beta}+{}^{\star}\eta\sigma\right)\right]\right. \\
&+ \left.\left[-(\nabla\!\!\!/\mathrm{tr}\chi)\beta+2(\nabla\!\!\!/\underline{\omega})\beta-\eta(\underline{\chi}\cdot\beta)+\underline{\chi}(\eta\cdot\beta)\right]+c\ {}^{*}\underline{\beta}\ {}^{*}\beta\right\}
\end{aligned}
\tag{7.10}
$$

We apply to 7.9 Lemma 4.1.5 of [Kl-Ni] with $p=2$ and obtain:

$$
|r^{3-\frac{2}{p}}\nabla\!\!\!/\beta|_{S,p=2}(\lambda,\nu)\le c\left(|r^{3-\frac{2}{p}}\nabla\!\!\!/\beta|_{S,p=2}(\lambda_1)+\int_{\lambda_1}^{\lambda}|r^2\underline{F}|_{S,2}d\lambda'\right) \tag{7.11}
$$

where $\underline{F}$ is defined in 7.10. Multiplying both sides of this inequality by $r^2|\lambda|^{1+\epsilon'}$ we obtain

$$
\begin{aligned}
|r^{5-\frac{2}{p}}|\lambda|^{1+\epsilon'}\nabla\!\!\!/\beta|_{S,p=2}(\lambda,\nu) &\le c|r^{5-\frac{2}{p}}|\lambda|^{1+\epsilon'}\nabla\!\!\!/\beta|_{S,p=2}(\lambda_1) \\
&+ c\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{1+\epsilon'}\underline{F}|_{S,2}d\lambda'
\end{aligned}
\tag{7.12}
$$

To prove that the integral in the right hand side is bounded we have to examine the various terms in which it can be decomposed. They can be collected in different groups.

a) The integrals with second partial derivatives of the Riemann tensor

$$
\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}\nabla\!\!\!/^2\rho|_{S,2}d\lambda'\ ,\ \int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}\nabla\!\!\!/^2\sigma|_{S,2}d\lambda' \tag{7.13}
$$

b) The integrals with first partial derivatives of the Riemann tensor

$$
\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}\hat{\chi}\nabla\!\!\!/\underline{\beta}|_{S,2}d\lambda'\ ,\ \int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}(\eta,\underline{\eta})\nabla\!\!\!/(\rho,\sigma)|_{S,2}d\lambda' \tag{7.14}
$$

c) The integrals with no partial dervatives on the Riemann components. Between these ones the integrals which depend on $\rho$ require a specific attention as $\rho$ is the only term whose decay cannot be improved in any way, due to its connection to the $ADM$ mass of $\Sigma_0$.

$$
\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}(\nabla\!\!\!/\eta)\rho|_{S,2}d\lambda'\ ,\ \int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}(\nabla\!\!\!/\log\Omega)\eta\rho|_{S,2}d\lambda' \tag{7.15}
$$

The remaining ones are easier; we collect them here and do not report their estimates.

$$
\begin{aligned}
&\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}(\nabla\!\!\!/\hat{\chi}+(\nabla\!\!\!/\log\Omega)\hat{\chi})\cdot\underline{\beta}|_{S,2}d\lambda'\ ,\\
&\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}(\nabla\!\!\!/\eta+(\nabla\!\!\!/\log\Omega)\eta)\cdot\sigma|_{S,2}d\lambda'\ ,\ \int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}({}^{\star}\underline{\beta}\cdot{}^{\star}\beta)|_{S,2}d\lambda'\\
&\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}((\nabla\!\!\!/\log\Omega)2\underline{\omega}+\nabla\!\!\!/\mathrm{tr}\chi)\cdot\beta|_{S,2}d\lambda'\ ,\ \int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}(\nabla\!\!\!/\underline{\omega})\beta|_{S,2}d\lambda'\\
&\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}\eta(\underline{\chi}\cdot\beta)|_{S,2}d\lambda'\ ,\ \int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}\underline{\chi}(\eta\cdot\beta)|_{S,2}d\lambda' \qquad (7.16)
\end{aligned}
$$

The two integrals in 7.14 are the more delicate. To be estimated, they have to be transformed to $L^2(\underline{C}_*)$ integrals and then bounded using the $\widetilde{\mathcal{Q}}$ norms.[48]

$$
\begin{aligned}
\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon')}\nabla\!\!\!/^2\rho|_{S,2}d\lambda' &\leq \left(\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{2\gamma}\nabla\!\!\!/^2\rho|^2_{S,2}\right)^{\frac{1}{2}}\left(\int_{\lambda_1}^{\lambda}\frac{1}{|\lambda'|^{1+2\delta}}\right)^{\frac{1}{2}}\\
&\leq c\left(\int_{\underline{C}(\nu)}|r^4|\lambda'|^{2\gamma}\nabla\!\!\!/^2\rho|^2\right)^{\frac{1}{2}}\leq c\epsilon_0 \qquad (7.17)
\end{aligned}
$$

The last term in the right hand side is controlled from the assumption $\tilde{\mathcal{R}}\leq\epsilon_0$ of Lemma 6.1. Exactly the same estimate holds for the integral involving the second tangential derivative of $\sigma$ as at the tangential derivative level $\rho$ and $\sigma$ behave in the same way.
Let us now examine the integrals with first derivatives of the null components of the Riemann tensor. The first one, can be estimated as follows

$$
\begin{aligned}
&\int_{\lambda_1}^{\lambda}|r^4|\lambda'|^{(1+\epsilon)}\hat{\chi}\nabla\!\!\!/\underline{\beta}|_{S,2}d\lambda'\leq\int_{\lambda_1}^{\lambda}|r^2|\lambda|^{\frac{1}{2}}\hat{\chi}||r^2|\lambda'|^{\frac{3}{2}+\gamma}\nabla\!\!\!/\underline{\beta}|_{S,2}\frac{1}{|\lambda|^{2+\delta}}d\lambda'\\
&\leq c\|r^2|\lambda|^{\frac{1}{2}}\hat{\chi}\|_{\infty}\sup_{\widetilde{\mathcal{M}}}|r^2|\lambda|^{\frac{3}{2}+\gamma}\nabla\!\!\!/\underline{\beta}|_{S,2}\leq c\epsilon_0^2 \qquad (7.18)
\end{aligned}
$$

where in the last inequality we used the inequalities $\tilde{\mathcal{O}}\leq\epsilon_0$ and $\tilde{\mathcal{R}}\leq\epsilon_0$.

**Remark:** This term behaves even better than we need. In fact to control it we used the estimate $|r^2\lambda^{(\frac{3}{2}+\gamma)}\nabla\!\!\!/\underline{\beta}|_{S,2}\leq c\epsilon_0$ following from Theorem 2.3,

[48]It is important to note that this cannot be done for $\int_{\underline{C}(\underline{u})\cap V(u,\underline{u})}|r^4|\lambda'|^{(1+\epsilon')}\beta|^2$. In fact this would require some extra power of $r$ in the $\widetilde{\mathcal{Q}}$ norms which is not allowed as these norms would not be bounded. Viceversa this can be done for the integrals involving $\rho$ and $\sigma$ as the power in $r$ are already the correct ones.

but we did not use the inequality $\tilde{\mathcal{O}} \leq \epsilon_0$ to control $|r^2|\lambda|^{\frac{1}{2}}\hat{\chi}|_\infty$ as the estimates in [Kl-Ni] are sufficient. The same happens for the other integral where we use the [Kl-Ni] estimates for $\eta$ and $\underline{\eta}$.

$$\begin{aligned} &\int_{\lambda_1}^{\lambda} |r^4|\lambda'|^{(1+\epsilon')}(\eta,\underline{\eta})\nabla\!\!\!\!/(\rho,\sigma)|_{S,2}d\lambda' \\ &\leq \int_{\lambda_1}^{\lambda} |r^2|\lambda'|^{\frac{1}{2}}(\eta,\underline{\eta})||r^3|\lambda'|^{\frac{1}{2}+\gamma}\nabla\!\!\!\!/(\rho,\sigma)|_{S,2}\frac{1}{r|\lambda'|^{1+\delta}}d\lambda' \\ &\leq c|r^2|\lambda|^{\frac{1}{2}}(\eta,\underline{\eta})|_\infty \sup_{\lambda}|r^3|\lambda|^{\frac{1}{2}+\gamma}\nabla\!\!\!\!/(\rho,\sigma)|_{S,2} \leq c\epsilon_0^2 \end{aligned} \tag{7.19}$$

In the third group integrals we consider the integral involving $\rho$:

$$\begin{aligned} &\int_{\lambda_1}^{\lambda} |r^4|\lambda'|^{1+\epsilon'}[3\nabla\!\!\!\!/\eta + 3(\nabla\!\!\!\!/\log\Omega)\eta]\rho|_{S,2} \\ &\leq c\left(\int_{\lambda_1}^{\lambda} |r^4|\lambda'|^{1+\epsilon'}(\nabla\!\!\!\!/\eta)\rho|_{S,2}d\lambda' + \int_{\lambda_1}^{\lambda} |r^4|\lambda'|^{1+\epsilon'}(\nabla\!\!\!\!/\log\Omega)\eta\rho|_{S,2}d\lambda'\right) \end{aligned}$$

To prove that the first integral in the right-hand side is bounded, the [Kl-Ni] estimates for $\nabla\!\!\!\!/\eta$ are not enough and we have to use the estimate $\tilde{\mathcal{O}} \leq \epsilon_0$ and, more specifically, the estimate for $\nabla\!\!\!\!/\eta$ provided by Theorem 2.4. We have, for the first integral,

$$\begin{aligned} \int_{\lambda_1}^{\lambda} |r^4|\lambda'|^{1+\epsilon'}(\nabla\!\!\!\!/\eta)\rho|_{S,2}d\lambda' &\leq \int_{\lambda_1}^{\lambda} |r^{3-\frac{2}{p}}|\lambda'|^{\frac{1}{2}+\gamma}(\nabla\!\!\!\!/\eta)|_{S,p=2}|r^3\rho|_\infty\frac{1}{r|\lambda'|^{1+\delta}}d\lambda' \\ &\leq c\epsilon_0\int_{\lambda_1}^{\lambda}\frac{1}{|\lambda'|^{1+\delta}}d\lambda' \leq c\epsilon_0 \end{aligned}$$

For the second integral, viceversa, the [Kl-Ni] estimates for the connection coefficients are sufficient. The remaining terms without derivatives for the null components of the Riemann tensor are easier to estimate and we do not report them here.

## 7.5 Proof of inequality 6.29

From the evolution equation 6.28

$$\frac{\partial\alpha}{\partial\lambda} + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\alpha = 4\Omega\underline{\omega}\alpha + \Omega\left[\nabla\!\!\!\!/\widehat{\otimes}\beta + (-3(\hat{\chi}\rho + {}^\star\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta)\right] \ ,$$

it follows that

$$\frac{\partial|\alpha|^p}{\partial\lambda} \quad = \quad p|\alpha|^{p-1}\frac{\partial|\alpha|}{\partial\lambda} = p|\alpha|^{p-2}\alpha\cdot\frac{\partial\alpha}{\partial\lambda}$$

$$= p|\alpha|^{p-2}\alpha\cdot(-\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}+4\Omega\underline{\omega})\alpha+\Omega\nabla\!\!\!/\widehat{\otimes}\beta+F(\cdot)) \tag{7.20}$$

$$= p(-\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}+4\Omega\underline{\omega})|\alpha|^p+p|\alpha|^{p-2}\Omega\alpha\cdot(\nabla\!\!\!/\widehat{\otimes}\beta)+p|\alpha|^{p-2}\alpha\cdot F(\cdot)\ .$$

and, immediately,

$$\frac{\partial|\alpha|^p}{\partial\lambda}+\frac{p}{2}\Omega\mathrm{tr}\underline{\chi}|\alpha|^p=4p\Omega\underline{\omega}|\alpha|^p+p|\alpha|^{p-2}\Omega\alpha\cdot(\nabla\!\!\!/\widehat{\otimes}\beta)+p|\alpha|^{p-2}\alpha\cdot F(\cdot) \tag{7.21}$$

Recalling that

$$\begin{aligned}\frac{d}{d\lambda}\left(\int_{S(\lambda,\nu)}r^\sigma|\alpha|^p d\mu_\gamma\right) &= \int_{S(\lambda,\nu)}r^\sigma\left(\frac{\partial|\alpha|^p}{\partial\lambda}+(1+\frac{\sigma}{2})\Omega\mathrm{tr}\underline{\chi}|\alpha|^p\right)\\ &- \frac{\sigma}{2}\int_{S(\lambda,\nu)}r^\sigma|\alpha|^p(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\end{aligned} \tag{7.22}$$

and choosing $\sigma=p(1-\frac{2}{p})$ we obtain

$$\begin{aligned}&\frac{d}{d\lambda}\left(\int_{S(\lambda,\nu)}|r^{(1-\frac{2}{p})}\alpha|^p d\mu_\gamma\right)=\int_{S(\lambda,\nu)}r^{p(1-\frac{2}{p})}\left(\frac{\partial|\alpha|^p}{\partial\lambda}+\frac{p}{2}\Omega\mathrm{tr}\underline{\chi}|\alpha|^p\right)\\ &-\frac{p(1-\frac{2}{p})}{2}\int_{S(\lambda,\nu)}|r^{(1-\frac{2}{p})}\alpha|^p(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\\ &=\int_{S(\lambda,\nu)}r^{p(1-\frac{2}{p})}\left(4p\Omega\underline{\omega}|\alpha|^p+p|\alpha|^{p-2}\Omega\alpha\cdot(\nabla\!\!\!/\widehat{\otimes}\beta)+p|\alpha|^{p-2}\alpha\cdot F(\cdot)\right)\\ &-\left(\frac{p}{2}-1\right)\int_{S(\lambda,\nu)}|r^{(1-\frac{2}{p})}\alpha|^p(\Omega\mathrm{tr}\underline{\chi}-\overline{\Omega\mathrm{tr}\underline{\chi}})\end{aligned} \tag{7.23}$$

Choosing $p=2$ we obtain

$$\frac{d}{d\lambda}|\alpha|^2_{S,2}=\int_{S(\lambda,\nu)}8\Omega\underline{\omega}|\alpha|^2+\int_{S(\lambda,\nu)}2\Omega\alpha\cdot(\nabla\!\!\!/\widehat{\otimes}\beta)+\int_{S(\lambda,\nu)}\alpha\cdot F(\cdot)$$

and, therefore,

$$\begin{aligned}2|\alpha|_{S,2}\frac{d}{d\lambda}|\alpha|_{S,2} &\leq 8\|\Omega\|_\infty|\underline{\omega}|_\infty\int_{S(\lambda,\nu)}|\alpha|^2+2\|\Omega\|_\infty\Big(\int_{S(\lambda,\nu)}|\alpha|^2\Big)^{\frac{1}{2}}\Big(\int_{S(\lambda,\nu)}|\nabla\!\!\!/\widehat{\otimes}\beta|^2\Big)^{\frac{1}{2}}\\ &+ \Big(\int_{S(\lambda,\nu)}|\alpha|^2\Big)^{\frac{1}{2}}\Big(\int_{S(\lambda,\nu)}|F|^2\Big)^{\frac{1}{2}}\\ &\leq 8\|\Omega\|_\infty|\underline{\omega}|_\infty|\alpha|^2_{S,2}+2\|\Omega\|_\infty|\alpha|_{S,2}|\nabla\!\!\!/\widehat{\otimes}\beta|_{S,2}+|\alpha|_{S,2}|F|_{S,2}\end{aligned} \tag{7.24}$$

and from it

$$\frac{d}{d\lambda}|\alpha|_{S,2}\leq 4\|\Omega\|_\infty|\underline{\omega}|_\infty|\alpha|_{S,2}+\|\Omega\|_\infty|\nabla\!\!\!/\widehat{\otimes}\beta|_{S,2}+\frac{1}{2}|F|_{S,2}\ .$$

## 8 Open questions and developments.

The problem we would like to discuss is if it is possible to prove that the spacetime we have produced and which satisfies the peeling properties is in fact an asimptotically simple spacetime.
Some preliminary remarks have to be done.

First of all we recall that the global spacetime we have built in [Kl-Ni] is not inextendible. In fact is defined outside the region of influence of a compact set contained in the initial hypersurface $\Sigma_0$. Therefore in particular while the null outgoing geodesics are complete this is not true for the timelike null geodesics. this implies that in the language of the conformal compactification we cannot reach the $i_+$ infinity. Therefore to answer to the question if our class of spacetimes (with initial conditions satisfying the peeling theorem) we should, preliminary "extend our spacetime to the internal region" or more precisely extend the spacetime to its maximal development.

This is therefore the first thing to do. Probably this has to be done mimicking the procedure for the interior region developed in [Ch-Kl] with the main difference that in this case we have to define, starting from the time line of the origin two null hypersurfaces foliations, one made by outgoing null hypersurfaces and the other one made by incoming null hypersurfaces. Of course we have not only to provide inside a double null foliation, but we have to prove a global existence for a characteristic problem. Due to the fact that we have to avoid caustics there is a problem in matching the null hypersurfaces with initial data on the (extension) of the last slice and those coming from the time line of the origin. Here we have to mimick the procedure developed in [Ch-Kl].

Assume for the moment that this can be done, with the strongest initial data on the characteristic cone $C_B$, that is those data which are provided from the initial data on $\Sigma_0$ decaying so fast to guarantee the peeling. There is still the problem of proving that also in this interior region we have the peeling decay when we move along the outgoing cones. One would reasonably expect that this is what happens, but nevertheless the technique used in the external region has, at least, be modified. The reason in very simple, when we cross the $C_0$ cone, the one with vertex at the origin inside it the function $u(p)$ is no more negative. Therefore the argument which allowed to add a factor $|u|^\gamma$ to the Bel-Robinson term and prove that the correction to the error term is negative does not work anymore. Nevertheless to cure this problem it seems that the norm relative to the incoming cone $\underline{C}(\nu; [u_{\underline{C}(\nu)\cap C_0}, t_0])$ has to be multiplied by a factor $|u-2t_0|^\gamma$ which is negative and contribute therefore, as before, to an extra error term which can be neglected and to the various Riemann components an extra factor decay of the type $|\lambda - 2t_0|^{-\gamma}$ which should be fine for the purpose to control again the integral on the internal part of $\underline{C}(\nu)$ that is integral of the type $\int_0^{t_0} d\lambda' \cdot \cdots$ . Of course here $t_0$ can be arbitrarily large. If this can be achieved the remaining question is which is the asymptotic behaviour along the timelike geodesics. This is not clear to me. from the initial data one would expect something as $t^{-5}$ or $t^{-5-\epsilon}$ , but how this should be provided is not evident. Once all this has been achieved one could start to try to answer to the original question of the asymptotic simplicity.

Another approach for the interior could be the following one: Instead of trying to solve a characteristic problem, just use the fact that we can extend the solution a little inside $C_B$ and then reach an hyperboloidal hypersurface tending asymptotically to $C_B$. The data on the hyperboloidal are assigned automatically, moreover this hypersurface in spacelike, therefore the local existence is easier (see Friedrich). Then one has to repeat the argument of the double null foliation and of the right factor which multiplies the Bel-Robinson tensor. Again also in this case the existence proof must be of the bootstrap type: one start with a small portion of hyperboloidal obtains a local existence and then prove that it is possible to extend. It is possible that this result can be based on already known results. First one has to prove, or see if it has already been proved, that it is possible to prove the existence of a hyperboloidal hypersurface between, let us say $C_B$ and $C_{B'}$ with $B' \subset B$, second, if this is possible, to show how to extend on it the "initial data" and finally if with this setting a global existence proof can be obtained perhaps using the $CMC$ foliation.

**Another thing to do is** prove that the same method used by Christodoulou for the initial data provides initial data with stronger decay. Check if this is true, if not how can we proceed?

**Another approach to** study the internal problem should be the following one: one starts from the well known fact thatin the "external" part of $\mathcal{M}$ we have also a foliation in terms of space hypersurfaces which are near to maximal ones. Therefore one should envisage in the interior a foliation which has the right boundary properties, which are the two dimensional surfaces which foliate $C(\lambda_0)$. The choice of the foliation implies a choice of the gauge and, therefore we have the Einstein equations on these spacetimes leaves plus some "gauge equations" which make the leaves fitting correctly with the outside spacelike hypersurfaces. then one has to put the Einstein equations in a hyperbolic form and then, probably the approach of Christodoulou and Klainerman could be followed.

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