COSMIC CENSORSHIP AND OTHER GREAT MATHEMATICAL CHALLENGES OF GENERAL RELATIVITY

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1. Introduction

The cosmic censorhip conjectures are a bold couple of hypothesis, put forth by R. Penrose [Pen1], [Pen2], concerning the nature of gravitational collapse in General Relativity. According to the so called *weak version* of this conjecture, which applies only to isolated physical systems, singularities in GR have no effect on distant observers. In other words asymptotically flat spacetimes in general relativity are free of singularities, in fact free of any possible effects of singularities, outside well specified regions, called black holes, bounded by event horizons. The strong cosmic censorship conjecture, which does not require asymptotic flatness and thus applies also to cosmological spacetimes, asserts that time-like singularities cannot occur under any circumstances, so even observers falling into a black hole will not register their effects. Both conjectures can be violated by specific, yet non-generic, examples thus one can only hope that they are true *generically* in a sense which is best left open until a satisfactory solution will be found.

The conjectures are fundamental from the point of view of Physics in so far as their violation might[1] create serious problems, of failure of predictability, to General Relativity as a classical theory. One would be obliged to find a solution to such difficulties by appealing to a more complete theory, such as Quantum Gravity, yet with all the efforts made by so many brilliant physicists, such a theory is no way in sight. If both conjectures are true, however, then a new theory will only be needed to deal with the non- timelike curvature singularities predicted to exist in the interior of black holes.

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[1]It is entirely possible, however, that some weak violations could be treated in the framework of the classical theory.

The conjecture also provide pure mathematicians, both geometers and analysts, with great challenges. Indeed they fit perfectly the definition given by Hilbert to good mathematical problems. *A good problem*, he advised, *should be clear and easy to comprehend, difficult yet not completely inaccessible lest it mocks at our efforts. It should provide a landmark on our way through the confusing maze and thus guide us towards hidden truth.* A good problem, I would accentuate, should provide us with a strategic height in our quest towards a higher goal, one whose solution leads unequivocally to an important milestone in our understanding of the main issues facing our subject. There is no doubt that the cosmic censorship conjectures verifies this last criterion, its solution will be a a great advance in our understanding of general solutions to the Einstein field equations. There is also no doubt that they are very difficult. Though young in comparison to the other big challenges in mathematics, such as the *magnificent seven* millennium problems, they have resisted so far all our efforts and it is obvious to all concerned that a solution is nowhere in sight. The conjectures are also *clear and easy to comprehend* even though a completely tight formulation could only be given once a solution will be found. It thus only remains to argue, as I will attempt here, that they are not *completely inaccessible* as they have generated, and will continue to generate, reach scientific, activity excellent results and new mathematical techniques which allow us to see a glimmer of light at the end of the tunnel.

In dealing with mathematical problems Hilbert advises us, *specialization plays, I believe, a still more important part then generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler than the one in hand have been either not at all or incompletely solved.* Cosmic censorship offers us a great choice of possible simplifications based on various symmetry assumptions or special choices of matter-fields. Other simplifications can be made by considering special classes of initial conditions, such as perturbations of initial conditions for Minkowski, Kerr or Kerr-Newmann solutions, or by making a-priori assumptions on the space-time under consideration. In this lecture I will give a few examples of such recent results.

2. Initial value problem and basic results

The cosmic censorship conjectures make specific predictions about the global properties of Cauchy developments of general initial data sets.

We recall that an initial data set consists of 3 dimensional manifold Σ, a complete Riemannian metric $g_{(0)}$, a symmetric 2-tensor $k_{(0)}$, and a well specified set of initial conditions corresponding to the matter-fields under consideration. These have to be restricted to a well known set of constraint equations. A Cauchy development of an initial data set is a globally hyperbolic spacetime $(\mathcal{M}, g)$, verifying the Einstein field equations,

EFE $$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta},$$

and an embedding $i : \Sigma \longrightarrow \mathcal{M}$ such that $i_*(g_{(0)}), i_*(k_{(0)})$ are the first and second fundamental forms of $i(\Sigma_{(0)})$ in $\mathcal{M}$. In what follows I will mostly restrict the discussion to the Einstein vacuum equations, i.e. the case when the energy momentum tensor vanishes identically and the equations take the purely geometric form,

EVE $$R_{\alpha\beta} = 0.$$

Most results I will mention can be extended to the case when matterfields are present, yet often including them will only encumber the presentation. In what follows I will restrict myself to asymptotically flat initial data sets, i.e. I assume that outside a sufficiently large compact set K, $\Sigma_{(0)} \setminus K$ is diffeomorphic to the complement of the unit ball in $\mathbb{R}^3$ and admits a system of coordinates in which $g_{(0)}$ is asymptotically euclidean and $k_{(0)}$ vanishes at appropriate order.

2.1. **Special solutions.** We recall that EVE admits a remarkable family of explicit, stationary, solutions are given by the two parameter family of Kerr solutions among which one distinguishes the Schwarzschild family of solutions, of mass $m > 0$,

$$g_S = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} \tag{1}$$

Though the metric seems singular at $r = 2m$ it turns out that one can glue together two regions $r > 2m$ and two regions $r < 2m$ of the Schwarzschild metric to obtain a metric which is smooth along $\mathcal{H} = \{r = 2m\}$, see [H-E], called the Schwarzschild horizon. The portion of $r < 2m$ to the future of the hypersurface $t = 0$ is a black hole whose future boundary $r = 0$ is singular. The region $r > 2m$, called the domain of outer communication, is free if singularities.

The Schwarzschild family is included in a larger two parameter family of solutions $\mathcal{K}(a, m)$ discovered by Kerr. A given Kerr space-time, with $0 \leq a < m$ has a well defined domain of outer communication $r > r_+ := m + (m^2 - a^2)^{1/2}$. In Boyer-Lindquist coordinates, well adapted to $r > r_+$ the Kerr metric has the form,

$$\begin{aligned} g_K &= -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{2a\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi + \\ &+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2 d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \end{aligned}$$

with $\Sigma = r^2 + a^2\cos^2\theta, \Delta = r^2 + a^2 - 2mr$. As in the Schwarzschild case, the exterior Kerr metric extends smoothly across the Kerr event horizon, $\mathcal{H} = \{r = r_+\}$. It can be shown that the future and past sets of any point in the domain of outer communication intersects any timelike curve, passing through points of arbitrary large values of r, in finite time as measured relative to proper time along the curve. This fact is violated by points in the region $r \leq r_+$, which defines the *black hole* region of the space-time. Thus physical signals which initiate at points in $r \leq r_+$ cannot be registered by far away observers. The extended Kerr is singular only at $r = 0$. Thus the singularities in Kerr cannot have any effect on the domain of outer communication which is completely smooth.

The exterior Kerr metrics are *stationary*, which means, roughly, that the coefficients of the metric are independent of the time variable t. One can reformulate this by saying that the vectorfield $T = \partial_t$ is Killing[2] and time-like at points with r large. One can also easily check that T is tangent to the horizon $\mathcal{H}$, which is itself a null hypersurface, i.e. the restriction of the metric to the tangent space to $\mathcal{H}$ is degenerate. In addition to being stationary the coefficients of the Kerr metric are independent of the circular variable ϕ. Thus Kerr is stationary and *axially symmetric*. The Schwarzschild metrics, corresponding to $a = 0$, are not just axially symmetric but spherically symmetric, which means that the metric is left invariant by the whole rotation group of the standard sphere $\mathbb{S}^2$. A well known theorem of Birkhoff, shows that they are the only such solutions of the vacuum Einstein equations. Another peculiarity of a Schwarzschild metric, not true in the case of Kerr, is that the stationary Killing vectorfield $T = \partial_t$ is orthogonal to the hypersurface $t = 0$. A stationary spacetime which has this

[2]A vectorfield X is said to be Killing if its locally induced one parameter flow consists of isometries of g, i.e. the Lie derivative of the metric g with respect to X vanishes, $\mathcal{L}_X g = 0$.

property is called *static*. Moreover T is timelike for all $r > 2m$ and null along the Schwarzschild horizon $\mathcal{H} = \{r = 2m\}$. This is not the case for Kerr solutions in which case $T = \partial_t$ is only time-like for $r > m + (m^2 - a^2 \cos^2\theta)^{1/2}$, null for $r = m + (m^2 - a^2 \cos^2\theta)^{1/2}$ and space-like in the region between r_+ and $r = m + (m^2 - a^2 \cos^2\theta)^{1/2}$, called the *ergosphere*.

2.2. **General results.** The most primitive question asked about the initial value problem, solved in a satisfactory way, for very large classes of evolution equations, is that of local existence and uniqueness of solutions. For the Einstein equations this type of result was first established by Y.C. Bruhat[Br] with the help of wave coordinates which allowed her to cast the Einstein vacuum equations in the form of a system of nonlinear wave equations to which one can apply[3] the standard theory of symmetric hyperbolic systems. The optimal, classical[4], result, see [HKM], states the following,

Theorem 2.3 (Local existence)**.** *Let $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ be an initial data set for the Einstein vacuum equations (EVE). Assume that $\Sigma_{(0)}$ can be covered by a locally finite system of coordinate charts U_α related to each other by C^1 diffeomorphisms, such that $(g_{(0)}, k_{(0)}) \in H^s_{loc}(U_\alpha) \times H^{s-1}_{loc}(U_\alpha)$ with $s > \frac{5}{2}$. Then there exists a unique(up to an isometry) globally hyperbolic, development $(\mathcal{M}, g)$, verifying EVE, for which $\Sigma_{(0)}$ is a Cauchy hypersurface*[5]*.*

In the case of nonlinear systems of differential equations the local existence and uniqueness result leads, through a straightforward extension argument, to a global result concerning the *maximal time interval of existence*. If this interval is bounded the solution must become infinite at its upper boundary. The formulation of the same type of result for the Einstein equations is a little more subtle; something similar was achieved in [Br-Ge].

Theorem 2.4 (Bruhat-Geroch)**.** *For each smooth initial data set there exists a unique maximal future global hyperbolic development (MFGHD).*

Thus any construction, obtained by an evolutionary approach from a specific initial data set, must be necessarily contained in its maximal

[3]The original proof in [Br] relied instead on representation formulas.

[4]Based on only energy estimates and classical Sobolev inequalities.

[5]That is any past directed, in-extendable causal curve in $\mathcal{M}$ intersects Σ_0

development MFGHD. This may be said to solve the problem of global[6] existence and uniqueness in General Relativity; all further questions, one could say, concern the qualitative properties of these maximal developments. The central issue becomes that of existence and character of singularities. We can start by defining a regular MFGHD as one which is future geodesically complete, i.e. all future time-like and null geodesics are complete. Roughly speaking this means that any freely moving observer in M can be extended indefinitely, as measured relative to its proper time. It turns out that any initial data set, which is sufficiently close to the flat one, admits a regular MFGHD, see [Ch-Kl] and the resent result in [Lind-Rodn]. This result stated below is a rough version of the global stability of Minkowski, the complete result also provides very precise information about the decay of the curvature tensor along null and timelike directions as well as many other geometric information concerning the causal structure of the corresponding spacetime. Of particular interest are *peeling properties* i.e. the precise decay rates of various components of the curvature tensor along future null geodesics.

Theorem 2.5 (Global Stability of Minkowski)**.** *Any asymptotically flat initial data set which is sufficiently close to the trivial one has a regular MFGHD.*

A related result, see [Kl-Ni] solves the problem of radiation for arbitrary AF initial data sets. In particular the result establishes the following.

Theorem 2.6 (Klainerman-Nicolo)**.** *For any, suitably defined, asymptotically flat initial data set with maximal future development* $(\mathcal{M}, g)$ *one can find a suitable domain* Ω_0 *with compact closure in the initial hypersurface* $\Sigma_{(0)}$ *such that the boundary* $\mathcal{N}_0^+$ *of* $\mathcal{J}^+(\Omega_0)$*, in* $\mathcal{M}$ *has complete null geodesic generators (with respect to the corresponding affine parameter).*

One can in fact foliate $\Sigma_0 \setminus \Omega_0$ by by $2-$surfaces Ω_r such that the boundaries $\mathcal{N}_r^+$ of all future domains $\mathcal{J}^+(\Omega_r)$ have complete null geodesic generators. As in the case of the global stability of Minkowski this result provides a wealth of additional informations. In particular

[6]This is of course misleading, for equations defined in a fixed background global is a solution which exists for all time. In general relativity, however, we have no such background as the spacetime itself is the unknown. A proper definition of global solutions in GR requires a special discussion concerning the proper time of timelike geodesics.

peeling properties, similar to the ones mentioned above hold in the complement of $\mathcal{J}^+(\Omega_0)$. At the opposite end of these results, when the initial data set is very far from flat, we have the following singularity theorem of Penrose, see [Pen1], [Pen2].

Theorem 2.7 (Penrose)**.** *If the manifold support of an initial data set is non-compact and contains a closed trapped surface the corresponding maximal development is incomplete. The results holds true in the presence of any matterfields which verify the positive energy conditions, i.e. if for any null vector L,*

$$Ric(L, L) \geq 0. \tag{2}$$

The notion of a trapped surface $S \subset \Sigma$, can be rigorously defined in terms of a local condition on S. The flat initial data set (whose development is the Minkowski space) have, of course, no such surfaces. On the other hand, for the Schwarzschild initial data set, i.e. the one whose development is Schwarzschild, any surface $r = r_0$, with $r_0 < 2m$ is trapped. Of course, the Schwarzschild metric has a genuine singularity at $r = 0$, where the curvature tensor becomes infinite. This is a lot stronger than just saying that space-time is incomplete. All Kerr solutions, with the exception of the flat Minkowski space itself, have trapped surfaces. Unlike Schwarzschild, Kerr spacetimes have Cauchy horizons, which belong to the boundaries of the MFGHD, of the initial data set at $t = 0$, thus contradicting the non-generic form of strong cosmic censorship.

3. Main Conjectures

3.1. **Cosmic Censorship Conjectures.** The unavoidable presence of singularities, for sufficiently large initial data sets, as well as the analysis of explicit examples, mainly Schwarzschild and Kerr, have led Penrose to formulate his two conjectures. To understand the statement of the weak cosmic censorship (WCC) consider the different behavior of null rays in Schwarzschild and Minkowski spacetimes. In Minkowski space light originating at any point $p = (t_0, x_0)$ propagates, towards future, along the null rays of the null cone $t - t_0 = |x - x_0|$. Any free observer in $\mathbb{R}^{1+3}$, following a straight time-like line, will necessarily meet the this light cone in finite time, thus experiencing the event p. On the other hand, any point p in the trapped region $r < 2m$ of the Schwarzschild space, is such that all null rays initiating at p remain trapped in the region $r < 2m$. In particular events connected to the singularity at $r = 0$ cannot influence events in the domain of outer communication $r > 2m$ which is thus entirely free of singularities. The same holds true in any Kerr solution with $0 \leq a < m$. WCC is an optimistic extension of this fact to the future developments of general, asymptotically flat initial data. The desired conclusion of the conjecture is that any such development, with the possible exception of a non-generic set of initial conditions, has the property that any *sufficiently distant observer* will never encounter singularities or any other effects propagating from them. To make this more precise one needs define what a sufficiently distant observer means. This is typically done by introducing the notion of future null infinity $\mathcal{S}^+$ which, roughly speaking, provides end points for the null geodesics which propagate to asymptotically large distances. The future null infinity is formally constructed by conformally embedding the physical spacetime $\mathcal{M}$ under consideration to a larger space-time $\bar{\mathcal{M}}$ with a null boundary $\mathcal{S}^+$.

Definition The future null infinity $\mathcal{S}^+$ is said to be complete if any future null geodesics along it can be indefinitely extended relative an afine parameter.

Given this enlarged space-time, with complete $\mathcal{S}^+$, one defines the black hole region to be

$$\mathcal{B} = \mathcal{M} - \mathcal{I}^-(\mathcal{S}^+) \tag{3}$$

with the chronological past I^- defined relative to the enlarged, non-physical space-time $\bar{\mathcal{M}}$. The event horizon $\mathcal{E}$ of the black hole is defined to be the boundary of $\mathcal{B}$ in $\mathcal{M}$. The requirement that space-time $\mathcal{M}$ has a complete future null infinity can be informally reformulated, by saying that the complement of the black hole region should be free of singularities. Indeed singularities outside the black hole region will necessarily have to affect the completeness of $\mathcal{S}^+$. The black hole region, however, can only be defined a-posterior after the completeness of $\mathcal{S}^+$ has been established.

A more precise definition of complete future null infinity, which avoids the technical and murky issue of the precise degree of smoothness of the conformal compactification was proposed by Christodoulou [Chr1]. It is based on the statement of theorem 2.6.

Definition 3.2. The maximal future development $(\mathcal{M}, g)$ of an AF initial data set possesses as complete future null infinity if, for any $A > 0$, we can find a domain $\Omega \subset \Sigma_0$ containing the set Ω_0 of Theorem 2.6 such that the boundary $\mathcal{N}^-(\Omega)$ of the domain of dependence $\mathcal{D}^-(\Omega)$

of Ω in $\mathcal{M}$ has the property that each of its null geodesic generators has a total affine length, measured from $\mathcal{N}^+(\Omega_0)$, of at least A.

Here is now a precise formulation of the Weak Cosmic Censorship (WCC) conjecture.

Conjecture 1[*WCC Conjecture*] *Generic asymptotically flat initial data have maximal future developments possessing a complete future null infinity.*

The WCC conjecture was formulated in order to guarantee the unique predictability of observations visible from infinity. It does not preclude, however, the possibility that singularities may be visible by local observers inside the black hole region. Since predictability is a fundamental requirement of all classical physics it seems reasonable to want it valid throughout spacetime. Predictability is known to fail, however, within the black hole of a Kerr solution[7] in which case the maximum domain of development of any complete spacelike hypersurface has a future boundary, called a Cauchy horizon, where the Kerr solution is perfectly smooth and yet beyond which there are many possible smooth extensions. This failure of predictability is due to a global pathology of the geometry of characteristics and not to a loss of local regularity. It is to avoid this pathology and ensure uniqueness that we want the maximum domain of development of generic data to be in-extendible. This motivation has led Penrose to introduce the following conjecture, called Strong Cosmic Censorship (SCC). Since Kerr itself, however, violates this requirement we can only hope that the conjecture holds for generic data.

Conjecture 2 [*SCC Conjecture*] *Generic asymptotically flat or compact initial data sets have maximal future developments which are locally in-extendible.*

Thus the MFGHD of generic AF initial data are either complete, as those provided by theorem 2.5 on the global stability of the Minkowski space, or terminate in a singular future boundary. According to this scenario the presence of smooth Cauchy horizons (across which the metric can be smoothly extended) in Kerr spacetimes is accidental, i.e. they cannot persist for arbitrary small perturbations of the Kerr initial conditions. The formulation above leads open the sense in which the

[7]Or the Reissner-Nordstrom solution of the Einstein -Maxwell equations.

maximal future developments are *in-extendible*. The precise notion of extendibility, which is to be avoided by SCC, is a subtle issue which may only be settled together with a complete solution of the conjecture. There have been various proposals among which I will only mention two, see [Chr1] for a more thorough discussion.

(1) The maximal future development is in-extendible as a $C^{1,1}$ Lorentzian manifold. This means, in particular, that some components of the curvature tensor must become infinite[8]
(2) The maximal future development is in-extendible as a continuous Lorentzian manifold.

The maximalist requirement of the second proposal was shown to be deficient in the particular case of spherically symmetric solutions of the Einstein equations coupled with both the Maxwell equations and a scalar field, see [D1], [D2]. M. Dafermos has shown in fact that in fact Cauchy horizons persist, through which the metric can be continued in a C^0 manner but such there exist no local coordinate systems, around any point of these horizons, in which the Christoffel symbols are square integrable. Though a precise version of extendibility, in connection to SCC, is not available one can nevertheless hope to gain some insight about it by considering the related question of optimal well-posedness, i.e. the minimum regularity for which predictability of solutions can be maintained. We will discuss this issue in the next section.

As alluded in the introduction cosmic censorship offers important simplifications based on various symmetry assumptions, special choices of matter-fields, special classes of initial conditions or by making a-priori assumptions. Additional symmetry assumptions greatly reduce the mathematical difficulties of the conjectures and, for some choices of matterfields, allow us to actually settle the conjectures. In the asymptotically flat case the two relevant symmetry simplifications are those corresponding to a $SO(3)$ action, i.e. spherical symmetry, and $SO(2)$ action corresponding to axial symmetry. A lot of progress has been made in the case of spherical symmetry in the case of specific matter-fields, such as a scalar field (for a comprehensive discussion and complete set of references see the prologue in [Chr2]), scalar and Maxwell

[8]More precisely, along any future, in-extendible, timelike geodesic of finite length the some components of the Riemann curvature tensor, expressed relative to a parallel transported orthonormal frame along the geodesic, become infinite as the value of the arc-length approaches its limiting value.

field (see [D1], [D2]) and colisionless plasma (see [D-R] and the references therein). The case of axial symmetry is far less studied. In what follows I will restrict the discussion to some recent results in connection to the other type of simplification mentioned above, namely special classes of initial conditions and a-priori assumptions.

3.3. **Other Conjectures.** Though general, asymptotically flat, solutions of the Einstein vacuum equations are exceedingly complicated we expect that their asymptotic behavior is quite simple and is dictated in fact by the two parameter family of explicit Kerr solutions, corresponding to axially symmetric, rotating black holes. Here is a rough version of the final state conjecture.

Conjecture 3 [*Final State Conjecture*] *Generic asymptotically flat initial data sets have maximal future developments which can be described, asymptotically, as a finite number of black holes, Kerr solutions, moving away from each other.*

The simple motivation behind this conjecture is that one expects, due to gravitational radiation, that general, dynamic, solutions of the Einstein field equations settle down, asymptotically, into a stationary regime. A spacetime is said to be stationary if it admits a Killing vectorfield which is timelike in the asymptotic region, i.e. at space-like infinity. The Kerr family of solutions are obvious examples of regular[9], stationary solutions but are they unique ? Can there be, in other words, other stationary solutions of the Einstein vacuum equations? It has been shown, under very general conditions, that if the Killing vectorfield is also static, i.e. hypersurface orthogonal, than the spacetime must be Schwarzschild, see discussion and references in [Be-Chr]. A less satisfactory uniqueness result holds true for stationary, real analytic space-times, see discussion and references in [Be-Chr]. The condition of real analyticity is however very unnatural in General Relativity and ought to be removed.

Conjecture 4[Uniqueness of Kerr] *Remove the analyticity assumption in the Hawking-Carter-Robinson proof of uniqueness of the Kerr space-time among stationary solutions.*

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[9]The space-time is assumed smooth, asymptotically flat and satisfies an appropriate causality condition

Recent progress in this direction was made in [I-Kl] and [Al-Io-Kl1], [Al-Io-Kl2]. In [I-Kl], based on a characterization of the Kerr solution (see [Ma]) by the vanishing of four covariant complex valued tensor[10] $\mathcal{S}$, we were able to remove the analyticity assumption by replacing it with a complex scalar condition to be satisfied on the bifurcate sphere of the horizon. The main idea of the proof was to derive a covariant wave equation for $\mathcal{S}$, show that $\mathcal{S}$ vanishes on the bifurcate event horizon of the stationary solution, and then use Carleman estimates to deduce that $\mathcal{S}$ must vanish in the entire domain of outer communication of the space-time. In [Al-Io-Kl2] we also remove the additional complex scalar condition but require instead that the Mars-Simon tensor is sufficiently small. In other words we prove that any regular, non-degenerate stationary, asymptotically flat solution which is a small perturbation of a given Kerr solution $\mathcal{K}(a, m)$, $0 \leq a < m$, is in fact a Kerr solution. The proof is based on Hawking's original idea of showing that, under suitable assumptions, stationary, real analytic solutions have to be axially symmetric. Uniqueness would then follow from a previous work of Carter-Robinson concerning stationary, axially symmetric space-times, not necessarily analytic. In his work, see [H-E], Hawking proves that the event horizon of these stationary solutions[11] must support a vectorfield, tangent to the generators of the horizon, which is Killing up to any order along the horizon. This vectorfield can then be extended by a Cauchy-Kowalewsky type argument[12]. In the absence of analyticity one would like to extend this Hawking vectorfield K by solving a covariant wave equation $\square_g K = 0$ with prescribed data on the horizon. This problem is however ill-posed, that is one cannot prove existence in the non analytic, smooth, category. Our work in [Al-Io-Kl2] circumvents this difficulty by starting with some natural, geometric, extension of K consider its associated flow Ψ_t, and show that, for small $|t|$, the pull back metric $\Psi_t^* g$ must coincide with g, in view of the fact they they are both solutions of the Einstein vacuum equations which coincide, by construction, on the horizon. We are then reduced to prove a uniqueness result for two Einstein vacuum metrics g, g' which coincide on the horizon. This strategy is illustrated in [Al-Io-Kl1] where we prove a local version of Hawking's result which does not rely on analyticity.

[10] which is traceless and possesses all the symmetries of the Riemann curvature tensor

[11] This is based on the fact that the stationary Killing field is tangent to the horizon. I also assume, in this discussion, that it is not tangent to its generators.

[12] A correct extension was in fact

Another important open problem in general relativity, whose solution would have to be understood long before the full Final State conjecture is settled, is that of the nonlinear stability of the exterior Kerr metric.

Conjecture 5 [*Global stability of Kerr*] *Any small perturbation of the initial data set of a Kerr space-time has a global future development with a complete future null infinity which, within its domain of outer communication*[13]*, behaves asymptotically like a (another) Kerr solution.*

Though it is widely expected that the conjecture must be true (if false, black holes would be nothing more than mathematical artifacts) so far only the full, non-linear, stability of the Minkowski space has been established. A first, essential step, in the proof of stability of the Kerr solution is to establish to prove its linear stability, which amounts, roughly, to prove appropriate decay estimates for solutions to linear field equations in a fixed Kerr background. In a somewhat simplified version, one has to show that all solutions of the covariant wave equation $\square_g \phi = 0$, with g the space-time metric of $\mathcal{K}(m, a)$, $0 \leq a < m$, are *well behaved* in the complement of the black hole region of the space-time. By well behaved I mean, in particular, that solutions evolving from smooth, compactly supported initial data, disperse and thus decay at rates similar to those in flat space. A more elementary task, and yet very difficult in the rotating case[14], $a > 0$, is to show that solutions remain bounded in the entire exterior region of the space-time. The difficulty is due to the presence of the ergo-region, in which the stationary Killing vector-field becomes space-like. The presence of an ergo-region is connected, physically, with the so called Penrose process according to which energy can be extracted from a rotating black hole and thus contribute to linear instability. To establish decay one has to overcome another major difficulty, present even in the non-rotating case, due to the presence of trapped null geodesics. In both cases one also has to understand the behavior of solutions near and along the event horizon, i.e. the boundary of the black hole region. It is important to stress here that a satisfactory solution of these problems has to be robust, i.e. it has to be applicable in principle to small perturbations of the background metric. This disqualifies the methods used so far, based on spectral decompositions (such as those of [Ch] or [F-K-S-Y]). Not only

[13]That means, roughly, outside the black hole region.

[14]The much simpler non-rotating case $a = 0$, corresponding to the Schwarzschild space-time, was solved previously in work by Kay and Wald.

do these methods fail to give quantitative results; they do not extend, in principle, to perturbations of the Kerr family.

In a recent series of papers Dafermos-Rodnianski, (see [D-Rodn1] and references therein) give a satisfactory solution to these problems for all Kerr space-times[15] with small angular momentum, based on a far reaching extension of the vectorfield method which takes into account the geometry of the Kerr solution. In fact their proof of boundedness holds not just for Kerr solutions, with small angular momentum, but also small perturbations of these (among solutions of the Einstein vacuum equations). I should remark that the vectorfield method is intrinsically robust and as such has played a crucial role in the proof of nonlinear stability of the Minkowski space-time.

3.4. **Bounded L^2 curvature conjecture.** One of the most interesting phenomenon discovered by mathematicians in connection nonlinear equations is the fact that below a certain regularity threshold, generalized solutions[16] of important nonlinear PDE's lose their predictability. Thus, in the case of the incompressible Euler equations, for example, one can prove (see [DL-Sz] and references to the previous results of V. Schaeffer and A. Schnirelman) the existence of finite energy uniformly bounded solutions, $v \in L^\infty(\mathbb{R} \times \mathbb{R}^3)$, solving the equations in a distributional sense and having compact support in all of spacetime. This raises the obvious question *what is that threshold and what is its physical relevance*. In the case of the Euler equations it is believed that the threshold, with respect to Hölder spaces C^α, is the Onsager exponent $\alpha = 1/3$, which corresponds exactly to the famous Kolmogorov-Obukhov energy spectrum exponent $-5/3$, see [Co].

One can approach this issue from the point of view of optimal well-posedness of the Einstein equations. According to the local existence theorem 2.3 the initial value problem is well posed for exponents $s > 5/2$. Clearly this cannot be optimal. By only scaling considerations one might expect to make sense of the initial value problem for $s \geq s_c = 3/2$. A result of well posedness for the Einstein equations for the critical regularity, $s = s_c$, is not only completely out of reach but it is

[15]Previous results in the non-rotating case of Schwarzschild were established in works by Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski, Alinhac and see precise references in [D-Rodn1].

[16]These are non-smooth solutions of the equations which can be defined, typically, in a distributional sense.

probably wrong, as we shall argue below. A far more realistic goal at the present time is the following:

Conjecture[L^2-Bounded Curvature Conjecture BCC.] *The Einstein Vacuum equations are strongly, locally*[17]*, well posed for initial data sets* $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ *with locally finite* L^2 *curvature and locally finite* L^2 *norm of the covariant derivatives of* $k_{(0)}$.

It is important to emphasize here that the conjecture can be interpreted as a continuation argument for the Einstein equations; that is the spacetime constructed by evolution from smooth data can be smoothly continued, together with a time foliation, as long as the curvature of the foliation and covariant derivatives of its second fundamental form remain L^2- bounded on the leaves of the foliation. The following, loosely formulated, *removal of singularities* result may be viewed as a possible corollary of the bounded L^2 curvature conjecture:

Corollary. *Consider a future Cauchy development, for the Einstein-vacuum equations, of smooth, regular, initial data set. If the curvature flux along any backward null cone initiating in the past of a point* p *is uniformly bounded, the solution cannot have a singularity at* p.

Clearly, such a result would be an important step in understanding formation and structure of singularities for the $3+1$ Einstein equations. The description of a local continuation criterion stated above in terms of a curvature flux is natural from both geometric and physical point of view. On the other hand, an even more ambitious goal is to find geometrically meaningful dimensionless quantities whose boundedness ensures a unique local extension of the corresponding spacetime.

Problem: *Is there dimensionless local extension criteria for solutions of the* $3+1$ *Einstein vacuum equations ?*

3.5. **Strategy for BCC.** The conjecture, first proposed in [Kl1], requires one to significantly improve the classical local existence theorem

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[17]The size of the extension may also depend on the radius of injectivity of the original manifold.

2.3 mentioned above from $s > 5/2$ to $s = 2$. The conjecture was motivated by the progress[18] made earlier on geometric semilinear wave equations such as Wave Maps and Yang Mills.

To improve the exponent $s > 5/2$ one needs to abandon the naive use of Sobolev inequalities, used in the classical existence and uniqueness argument, and rely instead on the more sophisticated Strichartz and bilinear type estimates. The difficulty lies in that one needs to extend these estimates from the well known case of the standard wave equation in flat space to wave operators on rough background metrics. Using such generalized Strichartz estimates we were able to reduce the $s > 5/2$ condition to $s > 2$ (see [Kl-R1] and the references therein).

The case $s = 2$ is far more difficult. First of all such a result cannot hold for general quasilinear wave equations of the type used in the proof of the classical local existence result. As the experience with semilinear wave equations demonstrates, see discussion in [Kl1], to prove such a result we need the following ingredients:

A. Provide a system of coordinates relative to which (EV) verifies an appropriate version of the null condition, similar to the one of [Kl-Ma2].

B. Construct a suitable parametrix for solutions to the homogeneous wave equation $\square_{\mathbf{g}}\phi = 0$ on a fixed vacuum Einstein background and provide control of the error term, relying only on the limited regularity of the space-time metric.

C. Prove appropriate bilinear estimates for solutions to homogeneous wave equations, of the type $\square_{\mathbf{g}}\phi = 0$, on a fixed Einstein Vacuum background (endowed with the coordinate system indicated in step 1. with bounded L^2 curvature. In the flat case such estimates were first proved in [Kl-M1] and used to prove global existence for the Yang Mills equations in the energy norm, see [Kl-Ma2]

An approximate solution for the homogenous wave equation $\square_g\phi = 0$ was constructed in [Kl-R6], in the form,

$$Tf(t,x) \;=\; \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u_\omega(t,x)} f\lambda\omega\lambda^2 d\lambda d\omega \tag{4}$$

[18] In that case, however, one expects (in the case of Wave Maps and higher dimensional Yang -Mills it has actually been proved) well posedness for the corresponding critical Sobolev exponents

where u_ω is a solution to the eikonal equation

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 \tag{5}$$

such that $u_\omega = x \cdot \omega$ when $|x| \to \infty$. To control the error term,

$$\Box_g Tf(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u_\omega(t,x)} \Box u_\omega f(\lambda\omega)\lambda^2 d\lambda d\omega$$

one needs to control the geometry of the level hypersurfaces of the optical function u_ω. These are null hypersurfaces, characteristic hypersurfaces of the wave operator $\Box_g$. We need, in particular, to derive uniform bounds for $\Box u_\omega$. This leads one to the fourth ingredient in a proof of the bounded L^2 conjecture.

D. Make sense of null hypersurfaces, on vacuum Einstein backgrounds with only L^2- bounds on their curvature tensor, and provide estimates on their geometry.

The reason we suspect that the exponent $s = 2$ is optimal is due to the geometry of null hypersurfaces[19]. In the sequence of papers [Kl-R2]–[Kl-R5] we were able to establish, based on an extremely tight argument, lower bounds for the radius of injectivity of null hypersurfaces depending, essentially, only on the boundedness of the flux of curvature across them (this is, roughly, the integral along the hypersurface of the square of the tangential components of the Riemann curvature tensor). Our proof does not seem to allow any room for improvement, and thus we think (though we don't yet have an explicit counterexample) that it is highly improbable that one could control the geometry of null hypersurfaces for regularity lower than $s = 2$. On the other hands all known methods to deal with low regularity, such as Strichartz and bilinear estimates, require regular null hypersurfaces as a starting point for constructing approximate solutions or appropriate vectorfields. It will be extremely interesting if a softness phenomenon (i.e. lack of predictability), similar to that mentioned above for the Euler equation[20], holds for $s < 2$.

The most difficult parts of the program to solve BCC are parts C and D. Part D has, essentially, been resolved in the sequence of papers [Kl-R2] [Kl-R3], [Kl-R4] and [Kl-R5]. In [Kl-R6] we were able to prove bilinear estimates for the approximate solutions (4). The harder task,

[19] as known these play a fundamental role in the geometric optics approximation of solutions to quasilinear hyperbolic equations

[20] A far simpler task would be to show that any notion of weak solution at the level of one derivative of the metric must be soft

still open, is to provide estimates for the error term generated by our parametrix, which depend only on the admissible L^2- bounds for the curvature of the metric g. Substantial progress in this direction has been made in collaboration with I. Rodnianski and J. Szeftel

3.6. **A Break-down criterion.** Consider an asymptotically flat spacetime $(\mathcal{M}, g)$ foliated by the level hypersurfaces Σ_t of a time function t. Let T be the future unit normal and n the lapse of the foliation. Let Σ_0 be a fixed leaf of the t foliation, corresponding to $t = t_0 \in I$, which we consider the initial slice, and assume that it $\mathcal{M}$ is included in the MFGHD of Σ_0. For any coordinate chart $\mathcal{O}$, with coordinates $x = (x^1, x^2, x^3)$, in Σ_0 we denote by $(x^0 = t, x^1, x^2, x^3)$ the transported coordinates on $I \times \mathcal{O}$ obtained by following the integral curves of T. In these coordinates the spacetime metric g takes the form

$$-n^2 dt^2 + g_{ij} dx^i dx^j, \tag{6}$$

The breakdown criterion in [Kl-R7], though related to the L^2-BC conjecture above, circumvents some of its main technical difficulties. It asserts that any spacetime, which admits a regular maximal foliation with lapse n and second fundamental form k can be continued as long as n does not degenerate to zero and we control the uniform norms of k and $\nabla \log n$. No smallness conditions on the data are necessary. This is a significant improvement of the result of L. Anderson [And] which require uniform bounds on the curvature. The proof depends on the results and methods of [Kl-R2]– [Kl-R5][21] which establish a lower bound for the radius of injectivity of null hypersurfaces with finite curvature flux as well as [Kl-R8] in which we construct a Kirchoff-Sobolev type parametrix for solutions to covariant wave equations. Here is the precise result.

Theorem 3.7. *Let $(\mathcal{M}, g)$ be a globally hyperbolic development of an asymptotically flat Σ_0 foliated by maximal hypersurfaces Σ_t (with second fundamental form k and lapse n), of a time function t, such that Σ_0 corresponds to the level surface $t = t_0$. Then, under reasonable assumptions of the initial data at Σ_0, the first time T_* of a breakdown is characterized by the condition*

$$\limsup_{t \to T_*^-} \left(\|k(t)\|_{L^\infty} + \|n^{-1}(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} \right) = \infty. \tag{7}$$

[21] Which were initially developed for the L^2 BC conjecture

More precisely the space-time together with the foliation Σ_t can be extended beyond any value t_* for which,

$$\begin{aligned} \|n^{-1}(t)\|_{L^\infty} &< \infty \qquad (8) \\ \sup_{t\in[t_0,t_*)} \|k(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} &< \infty, \qquad (9) \end{aligned}$$

Recently condition (9) was relaxed by D. Parlongue [Par]. He was able to replace it with the integral condition,

$$\int_0^{t_*} \big(\|k(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} \big)^2 dt < \infty \qquad (10)$$

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