A GEOMETRIC THEORY OF LITTLEWOOD-PALEY THEORY

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Abstract. We develop a geometric invariant Littlewood-Paley theory for arbitrary tensors of a compact 2 dimensional manifold. We show that all the important features of the classical LP theory survive with estimates which depend only on very limited regularity assumptions on the metric. We give invariant descriptions of Sobolev and Besov spaces and prove some sharp product inequalities. This theory has being developed in connection to the work of the authors on the geometry of null hypersurfaces with a finite curvature flux condition, see [Kl-Rodn1], [Kl-Rodn2]. We are confident however that it can be applied, and extended, to many different situations.

1. introduction

In its simplest manifestation Littlewood-Paley theory is a systematic method to understand various properties of functions f, defined on $\mathbb{R}^n$, by decomposing them in infinite dydic sums $f = \sum_{k\in\mathbb{Z}} f_k$, with frequency localized components f_k, i.e. $\widehat{f_k}(\xi) = 0$ for all values of ξ outside the annulus $2^{k-1} \le |\xi| \le 2^{k+1}$. Such a decomposition can be easily achieved by chosing a test function $\chi = \chi(|\xi|)$ in Fourier space, supported in $\frac{1}{2} \le |\xi| \le 2$, and such that, for all $\xi \neq 0$, $\sum_{k\in\mathbb{Z}} \chi(2^{-k}\xi) = 1$. Then set $\widehat{f_k}(\xi) = \chi(2^k\xi)\hat{f}(\xi)$ or , in physical space,

$$P_k f = f_k = m_k * f$$

where $m_k(x) = 2^{nk} m(2^k x)$ and $m(x)$ the inverse Fourier transform of χ. The operators P_k are called cut-off operators or, improperly, LP projections. We denote $P_J = \sum_{k\in J} P_k$ for all intervals $J \subset \mathbb{Z}$.

The following properties of these LP projections are very easy to verify and lie at the heart of the classical LP theory:

LP 1. *Almost Orthogonality:* The operators P_k are selfadjoint and verify $P_{k_1} P_{k_2} = 0$ for all pairs of integers such that $|k_1 - k_2| \ge 2$. In particular,

$$\|F\|_{L^2}^2 \approx \sum_k \|P_k F\|_{L^2}^2$$

LP 2. *L^p-boundedness:* For any $1 \le p \le \infty$, and any interval $J \subset \mathbb{Z}$,

$$\|P_J F\|_{L^p} \lesssim \|F\|_{L^p} \tag{1}$$

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LP 3. *Finite band property:* We can write any partial derivative $\nabla P_k f$ in the form $\nabla P_k f = 2^k \tilde{P}_k f$ where $\tilde{P}_k$ are the LP-projections associated with a slightly different test function $\tilde{\chi}$ and verify the property **LP2**. Thus, in particular, for any $1 \le p \le \infty$

$$\begin{aligned} \|\nabla P_k F\|_{L^p} &\lesssim 2^k \|F\|_{L^p} \\ 2^k \|P_k F\|_{L^p} &\lesssim \|\nabla F\|_{L^p} \end{aligned}$$

LP 4. *Bernstein inequalities.* For any $2 \le p \le \infty$ we have the Bernstein inequality and its dual,

$$\|P_k F\|_{L^\infty} \lesssim 2^{k(1-\frac{2}{p})} \|F\|_{L^2}, \qquad \|P_k F\|_{L^2} \lesssim 2^{k(1-\frac{2}{p})} \|F\|_{L^{p'}}$$

The last two properties go a long way to explain why LP theory is such a useful tool for partial differential equations. The finite band property allows us to replace derivatives of the dydic components f_k by multiplication with 2^k. The $L^2 \to L^\infty$ Bernstein inequality is a dyadic remedy for the failure of the embedding of the Sobolev space $H^{\frac{n}{2}}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. Indeed, in view of the finite band property, the Bernstein inequality does actually imply the desired Sobolev inequality for each LP component f_k, the failure of the Sobolev inequality for f is due to the summation $f = \sum_k f_k$.

Just like Fourier transform, Littlewood-Paley theory allows one to separate waves of various frequencies for linear partial differential equations with constant coefficients and therefore its usefulness in this context is not that surprising. It took longer to realize that it is helpful, in fact even more helful, for the analysis of nonlinear equations. It turns out that multiplication properties of various classical spaces of functions are best understood by decomposing the corresponding functions in dyadic LP components. This allows one to isolate and treat differently interactions of various components of the functions. Moreover the LP calculus allows one to manipulate a nonlinear PDE to derive coupled equations for each particular frequency. A first systematic application[1] of LP theory to nonlinear PDE's was developed by Bony in the form of what is called the paradifferential calculus [B]. Notable applications of LP theory include recent advanced in fluid dynamics, nonlinear dispersive as well as nonlinear wave equations(both semilinear and quasilinear), see e.g. [Ch], [Ba-Ch], [Bour], [Tat], [Tao], [Sm-Ta].

In this paper we develop an invariant LP theory for compact 2-surfaces. Our immediate goal is to apply this theory to study the geometric properties of null hypersurfaces, in Einstein-vacuum manifolds, with a finite curvature flux condition, see [Kl-Rodn1]-[Kl-Rodn2]. We believe however that the theory we develop can have far wider applications.

Following a well-known procedure (see Stein [Stein1]) we base our approach on heat flow,

$$\partial_\tau U(\tau) F - \Delta U(\tau) F = 0, \; U(0)F = F \tag{2}$$

[1] The first manifestation of these type of ideas can be traced to the work of J. Nash on the isometric embedding problem [?]

with $\Delta = g^{ij}\nabla_i\nabla_j$ the usual Laplace-Beltrami operator defined on the space of smooth tensorfields of order $m \geq 0$.

We then define LP projections P_k according to the formula,

$$P_k F = \int_0^\infty m_k(\tau) U(\tau) F d\tau \tag{3}$$

where $m_k(\tau) = 2^{2k} m(2^{2k}\tau)$ and $m(\tau)$ is a Schwartz function with a finite number of vanishing moments.

Under some primitive assumptions on the geometry of our compact 2-dimensional manifold S we prove a sequence of properties for these geometric LP projections, similar to **LP1**– **LP4**. Some of our results are necessarily weaker[2]. For example the pointwise version of the almost orthogonality property **LP1** does not hold. We can replace it however by its sufficiently robust L^p analogue. We also find satisfactory analogues for **LP2-LP4**. However we discover that the minimal geometric assumptions, we impose, restrict the range of p in **LP 3** to $p = 2$ and $p \neq \infty$ in **LP 4**. Moreover, the $L^2 \to L^\infty$ Bernstein inequality requires additional geometric assumptions which differ dependent on whether F is a scalar or a tensor.

In section 2 we state our main regularity assumptions on a 2-D manifold S and establish some basic calculus inequalities. This is the only place in the paper where we make use of special coordinates. Our assumption of *weak regularity* is meant to guarantee the existence of such coordinates.

Section 3 discusses the Böchner identities for scalar functions and general tensorfields. Note that the Bochner identity for tensorfields has an additional term, not present for scalars, which requires stronger assumptions on the Gauss curvature K of our manifold.

In section 4 we define the heat flow generated by the Laplace-Beltrami operator Δ on tensorfields of arbitrary order. The properties of the heat equation derived in that section requires no regularity assumptions on S beyong the fact that the metric must be Riemannan.

In section 5 we use the heat flow to develop an invariant, tensorial, Littlewood-Paley theory on manifolds. We prove analogues of the **LP1–LP4** properties of the classical LP theory. Once more, for most properties of our LP projections, we need no regularity assumptions on the metric, beyond the fact that it is Riemannan. We do however make use of the weak regularity assumption on our manifold S in the proof of the weak Bernstein inequality and its consequences.

In sections 7 and 8 we define fractional Sobolev and Besov spaces.

In Section 9 we show how to use the geometric LP theory developed so far to prove some(non sharp) product estimates in fracational Sobolev and Besov spaces.

[2]Indeed, even in Euclidean space the LP projections constructed by the heat flow do not possess sharp localization properties in Fourier space

In section 10 we discuss the sharp $L^2 \to L^\infty$ Bernstein inequality. In addition to the main weak regularity assumptions on the 2-D manifold S we have to impose conditions on its Gauss curvature K. We detect a sharp difference in the requirements imposed on K dependent on wheteher we consider the scalar or the the general tensorial case.

In section 11 we return to the earlier product estimates and prove their sharp versions under the additional conditions needed for the sharp Bernstein inequality.

In section 12. we consider the mapping property of the covariant differentiation ∇ on the Besov space $B^1_{2,1}$.

2. Calculus inequalities

In this section we establish some basic calculus inequalities on a smooth, compact, 2 -D maniflold S. We say that a coordinate chart $U \subset M$ with coordinates x^1, x^2 is admissible if, relative to these coordinates, there exists a constant $c > 0$ such that,

$$c^{-1}|\xi|^2 \le \gamma_{ab}(p)\xi^a\xi^b \le c|\xi|^2, \qquad \text{uniformly for all } \; p \in U \tag{4}$$

We also assume that the Christoffel symbols Γ^a_{bc} verify,

$$\sum_{a,b,c} \int_U |\Gamma^a_{bc}|^2 dx^1 dx^2 \le c^{-1} \tag{5}$$

Definition 2.1. We say that a a smooth 2-d manifold S is *weakly regular* (**WR**) if can be covered by a finite number of admissible coordinate charts, i.e., charts satisfying the conditions (4), (5).

Remark 2.2. Although we assume that our manifold S is smooth our results below depend only on the constants in (4) and (5). The notion of weak regularity is introduced to emphasize this fact.

Whenever we have inequalities of the type $A \le C \cdot B$, with C a constsnt which depends only on c above, we write $A \lesssim B$.

Under the **WR** assumption a it is easy to prove the following calculus inequalities:

Proposition 2.3. *Let f be a real scalar function on a 2-d weakly regular manifold S. Then,*

$$\|f\|_{L^2} \lesssim \|\nabla f\|_{L^1} + \|f\|_{L^1} \tag{6}$$

$$\|f\|_{L^\infty} \lesssim \|\nabla^2 f\|_{L^1} + \|f\|_{L^2} \tag{7}$$

Proof : Both statements can be reduced, by a partition of unity, to the case when the function f has compact support in an admissible local chart $U \subset S$. Let x^1, x^2

be an admissible system of coordinates in U. Then,

$$
\begin{aligned}
|f(x^1,x^2)^2| &= \Big|\int_{-\infty}^{x^1}\partial_1 f(y,x^2)dy\cdot\int_{-\infty}^{x^2}\partial_2 f(x^1,y)dy\Big|\\
&\lesssim \int_{-\infty}^{\infty}|\partial_1 f(y,x^2)dy|\cdot\int_{-\infty}^{\infty}|\partial_2 f(x^1,y)|dy
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{\mathbb{R}^2}|f(x^1,x^2)|^2dx^1dx^2 &\lesssim \int_{\mathbb{R}^2}|\partial_1 f(x^1,x^2)|dx^1dx^2\cdot\int_{\mathbb{R}^2}|\partial_2 f(x^1,x^2)|dx^1dx^2\\
&\lesssim \int_{\mathbb{R}^2}|\nabla f(x^1,x^2)|dx^1dx^2.
\end{aligned}
$$

Thus, since in view of (4) $c\le\sqrt{|g|}\le c^{-1}$,

$$
\Big(\int_U|f(x)|^2\sqrt{|g|}dx^1dx^2\Big)^{\frac12}\lesssim\Big(\int_U|\nabla f(x)|^2\sqrt{|g|}dx^1dx^2\Big)^{\frac12}.
$$

as desired. Simmilarly,

$$
f(x^1,x^2)=\int_{-\infty}^{x^1}\int_{-\infty}^{x^2}\partial_1\partial_2 f(y^1,y^2)dy^1dy^2.
$$

Hence,

$$
\begin{aligned}
|f(x^1,x^2)| &\le \int_{\mathbb{R}^2}\big(|\nabla^2 f(y^1,y^2)|+|\Gamma||\nabla f(y^1,y^2)|\big)\\
&\lesssim \int_S|\nabla^2 f|+\Big(\int_U|\Gamma|^2\Big)^{\frac12}\|\nabla f\|_{L^2(S)}\\
&\lesssim \|\nabla^2 f\|_{L^1(S)}+\|\nabla f\|_{L^2(S)}
\end{aligned}
$$

■

As a corollary of the estimate (6) we can derive the following Gagliardo-Nirenberg inequality:

Corollary 2.4. *Given an arbitrary tensorfield F on M and any $2\le p<\infty$ we have,*

$$
\|F\|_{L^p}\lesssim\|\nabla F\|_{L^2}^{1-\frac{2}{p}}\|F\|_{L^2}^{\frac{2}{p}}+\|F\|_{L^2} \tag{8}
$$

Proof : For any $p\ge 2$ we can write,

$$
\begin{aligned}
\|F\|_{L^p}^{p/2} &= \||F|^{p/2}\|_{L^2}\lesssim\|\nabla|F|^{p/2}\|_{L^1}+\||F|^{p/2}\|_{L^1}\\
&\lesssim \big(\|\nabla F\|_{L^2}+\|F\|_{L^2}\big)\cdot\|F\|_{L^{p-2}}^{\frac{p-2}{2}}
\end{aligned}
$$

Thus, inductively, for all $p=2k$, $k=1,2,\dots$

$$
\|F\|_{L^{2k}}\lesssim\big(\|\nabla F\|_{L^2}+\|F\|_{L^2}\big)^{1-\frac1k}\cdot\|F\|_{L^2}^{\frac1k}
$$

The result for general p now follows by interpolation in the scale of L^p spaces. ■

As a Corollary to (7) we also derive

Corollary 2.5. *For any tensorfield F on S,*

$$\|F\|_{L^\infty} \lesssim \|\nabla^2 F\|_{L^2}^{\frac{1}{2}} \cdot \|F\|_{L^2}^{\frac{1}{2}} + \|F\|_{L^2} \tag{9}$$

Moreover, we have a more precise estimate for any $2 \le p < \infty$,

$$\|F\|_{L^\infty} \lesssim \|\nabla^2 F\|_{L^2}^{\frac{1}{p}} \big(\|\nabla F\|_{L^2}^{\frac{p-2}{p}} \|F\|_{L^2}^{\frac{1}{p}} + \|F\|_{L^2}^{\frac{p-1}{p}} \big) + \|\nabla F\|_{L^2}. \tag{10}$$

Proof : We apply the estimate (7) to the scalar $|F|^2$ as follows,

$$\begin{aligned} \|F\|^2_{L^\infty} &\lesssim \|\nabla^2 |F|^2\|_{L^1} + \||F|^2\|_{L^2} \\ &\lesssim \|\nabla^2 F\|_{L^2}\|F\|_{L^2} + \|\nabla F\|^2_{L^2} + \|F\|^2_{L^4} \end{aligned}$$

In view of (8),

$$\|F\|^2_{L^4} \lesssim \|\nabla F\|_{L^2}\|F\|_{L^2} + \|F\|^2_{L^2}$$

Hence,

$$\|F\|^2_{L^\infty} \lesssim \|\nabla^2 F\|_{L^2}\|F\|_{L^2} + \|\nabla F\|^2_{L^2} + \|\nabla F\|_{L^2}\|F\|_{L^2} + \|F\|^2_{L^2}$$

The desired estimate now follows by Cauchy-Schwartz. To prove the estimate (10) we observe that applying (7) to $|F|^p$ we obtain

$$\|F\|_{L^\infty} \lesssim \|\nabla^2 F\|_{L^2}^{\frac{1}{p}} \|F\|_{L^{2(p-1)}}^{\frac{p-1}{p}} + \|\nabla F\|_{L^2}$$

By the Galgiardo-Nirenberg inequality (8) we have that

$$\|F\|_{L^{2(p-1)}} \lesssim \|\nabla F\|_{L^2}^{\frac{p-2}{p-1}} \|F\|_{L^2}^{\frac{1}{p-1}} + \|F\|_{L^2}$$

Thus, finally

$$\|F\|_{L^\infty} \lesssim \|\nabla^2 F\|_{L^2}^{\frac{1}{p}} \big(\|\nabla F\|_{L^2}^{\frac{p-2}{p}} \|F\|_{L^2}^{\frac{1}{p}} + \|F\|_{L^2}^{\frac{p-1}{p}} \big) + \|\nabla F\|_{L^2}$$

as desired. ∎

3. Böchner identity

In this section we recall the Böchner identity on a 2-D manifold. This allows us to control the L^2 norm of the second derivatives of a tensorfield in terms of the L^2 norm of the laplacian and geometric quantities associated with a given 2-surface.

Proposition 3.1. *Let K denote the Gauss curvature of our 2-D riemannian manifold M. Then*

i) *For a scalar function f*

$$\int_S |\nabla^2 f|^2 = \int_S |\Delta f|^2 - \int_S K |\nabla f|^2 \tag{11}$$

ii) *For a vectorfield* F_a

$$\int_S |\nabla^2 F|^2 = \int_S |\Delta F|^2 - \int_S K(2\,|\nabla F|^2 - |\ div\ F|^2 - |\ curl\ F|^2) + \int_S K^2|F|^2 \tag{12}$$

where $div\ F = \gamma^{ab}\nabla_b F_a,\ \ curl\ F =\ div\ ({}^*F) = \in_{ab} \nabla_a F_b$

Proof : Recall that on a 2-surface the Riemann tensor

$$R_{abcd} = (\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc})K, \qquad R_{ab} = \gamma_{ab}K, \tag{13}$$

To prove **i)** observe that, relative to an arbitrary orthonormal frame $(e_a)_{a=1,2}$,

$$\begin{aligned} \nabla_a(\Delta f) &= \nabla_a(\nabla_c\nabla_c f) = \nabla_c\nabla_a\nabla_c f + [\nabla_a, \nabla_c]\nabla_c f \\ &= \nabla_c\nabla_c\nabla_a f + R_{cdac}\nabla_d f \\ &= \Delta(\nabla_a f) - R_{ad}\nabla_d f \end{aligned}$$

Thus,

$$\begin{aligned} \int_S |\Delta f|^2 &= -\int_S \nabla_a(\Delta f)\cdot\nabla_a f = \int_S \Delta\nabla_a f\cdot\nabla_a f - R_{ab}\nabla_a f\nabla_b f \\ &= \int_S |\nabla^2 f|^2 - \int_S K|\nabla f|^2 \end{aligned}$$

as desired.

Similarily for a vector F_i,

$$\begin{aligned} \nabla_a(\Delta F_i) &= \nabla_a(\nabla_c\nabla_c F_i) = \nabla_c\nabla_a\nabla_c F_i + [\nabla_a, \nabla_c]\nabla_c F_i \\ &= \nabla_c\nabla_c\nabla_a f_i + \nabla_c([\nabla_a, \nabla_c]F_i) + R_{cdac}\nabla_d F_i + R_{idac}\nabla_c F_d \\ &= \Delta(\nabla_a F_i) + \nabla_c\big(R_{idac}F_d\big) + R_{cdac}\nabla_d F_i + R_{idac}\nabla_c F_d \\ &= \Delta(\nabla_a F_i) + \nabla_c\big(R_{idac}F_d\big) - R_{da}\nabla_d F_i + R_{idac}\nabla_c F_d \end{aligned}$$

Hence,

$$\begin{aligned} -\int_S |\Delta F|^2 &= \int_S \nabla_a(\Delta F_a) \\ &= -\int_S |\nabla^2 F|^2 - \int_S R_{idac}F_d\nabla_c\nabla_a F_i \\ &- \int_S R_{da}\nabla_d F_i\nabla_a F_i + \int_S R_{idac}\nabla_c F_d\nabla_a F_i \end{aligned}$$

Now observe that,

$$\int_S R_{idac}F_d\nabla_c\nabla_a F_i = \frac{1}{2}\int_S R_{idac}F_d(\nabla_c\nabla_a F_i - \nabla_a\nabla_c F_i) = \frac{1}{2}\int_S R_{idac}R_{imca}F_d F_m$$

Therefore,

$$\begin{aligned} \int_S |\Delta F|^2 &= \int_S |\nabla^2 F|^2 + \frac{1}{2}\int_S R_{diac}R_{miac}F_d F_m + \int_S R_{da}\nabla_d F_i\nabla_a F_i \\ &- \int_S R_{idac}\nabla_c F_d\nabla_a F_i \end{aligned}$$

Using the formulas (13) and observing that $\nabla_b F_a - \nabla_a F_b = \in_{ba}$ curl F we find,

$$\begin{aligned}
R_{diac}R_{miac}F_dF_m &= 2K^2\delta_{dm}F_dF_m = 2K^2|F|^2\\
R_{da}\nabla_dF_i\nabla_aF_i &= K|\nabla F|^2\\
R_{idac}\nabla_cF_d\nabla_aF_i &= K\big(|\text{ div } F|^2 - \nabla_aF_b\nabla_bF_a\big)\\
&= K\big(|\text{ div } F|^2 - |\nabla F|^2 - \nabla_aF_b(\nabla_bF_a - \nabla_aF_b)\big)\\
&= K\big(|\text{ div } F|^2 + |\text{ curl } F|^2 - |\nabla F|^2\big)
\end{aligned}$$

Therefore,

$$\int_S |\Delta F|^2 = \int_S |\nabla^2 F|^2 - \int_S |K|^2|F|^2 + \int_S K\big((2\,|\nabla F|^2 - (|\text{ div } F|^2 + |\text{ curl } F|^2)\big)$$

as desired. ■

Corollary 3.2 (Böchner inequality). *For any tensorfield F and an arbitrary $2 \le p < \infty$*

$$\begin{aligned}
\|\nabla^2 F\|_{L^2} &\lesssim \|\Delta F\|_{L^2} + (\|K\|_{L^2} + \|K\|_{L^2}^{\frac12})\|\nabla F\|_{L^2} &(14)\\
&+ \|K\|_{L^2}^{\frac{p}{p-1}}\big(\|\nabla F\|_{L^2}^{\frac{p-2}{p-1}}\|F\|_{L^2}^{\frac{1}{p-1}} + \|F\|_{L^2}\big) &(15)
\end{aligned}$$

Proof : The Böchner identity (12) implies that

$$\|\nabla^2 F\|_{L^2} \lesssim \|\Delta F\|_{L^2} + \|K\|_{L^2}^{\frac12}\|\nabla F\|_{L^4} + \|K\|_{L^2}\|F\|_{L^\infty} \tag{16}$$

Using the Gagliardo-Nirenberg inequality (8) and the estimate (10) we infer that for any $2 \le p < \infty$

$$\begin{aligned}
&\|\nabla F\|_{L^4} \lesssim \|\nabla^2 F\|_{L^2}^{\frac12}\|\nabla F\|_{L^2}^{\frac12} + \|\nabla F\|_{L^2},\\
&\|F\|_{L^\infty} \lesssim \|\nabla^2 F\|_{L^2}^{\frac1p}\big(\|\nabla F\|_{L^2}^{\frac{p-2}{p}}\|F\|_{L^2}^{\frac1p} + \|F\|_{L^2}^{\frac{p-1}{p}}\big) + \|\nabla F\|_{L^2}
\end{aligned}$$

Substituting this into (16) we obtain

$$\begin{aligned}
\|\nabla^2 F\|_{L^2} &\lesssim \|\Delta F\|_{L^2} + \|K\|_{L^2}^{\frac12}\Big(\|\nabla^2 F\|_{L^2}^{\frac12}\|\nabla F\|_{L^2}^{\frac12} + \|\nabla F\|_{L^2}\Big)\\
&+ \|K\|_{L^2}\Big(\|\nabla^2 F\|_{L^2}^{\frac1p}\big(\|\nabla F\|_{L^2}^{\frac{p-2}{p}}\|F\|_{L^2}^{\frac1p} + \|F\|_{L^2}^{\frac{p-1}{p}}\big) + \|\nabla F\|_{L^2}\Big)
\end{aligned}$$

This, in turn, implies that

$$\|\nabla^2 F\|_{L^2} \lesssim \|\Delta F\|_{L^2} + (\|K\|_{L^2} + \|K\|_{L^2}^{\frac12})\|\nabla F\|_{L^2} + \|K\|_{L^2}^{\frac{p}{p-1}}\big(\|\nabla F\|_{L^2}^{\frac{p-2}{p-1}}\|F\|_{L^2}^{\frac{1}{p-1}} + \|F\|_{L^2}\big)$$

as desired. ■

4. Heat equation on S

In this section we study the properties of the heat equation for arbitrary tensorfields F on S.

$$\partial_\tau U(\tau)F - \Delta U(\tau)F = 0,\ U(0)F = F,$$

with $\Delta = \Delta_\gamma$ the usual Laplace-Beltrami operator on S. Observe that the operators $U(\tau)$ are selfadjoint[3] and form a semigroup for $\tau > 0$. In other words for all, real valued, smooth tensorfields F, G,

$$\int_S U(\tau)F \cdot G = \int_S F \cdot U(\tau)G, \qquad U(\tau_1)U(\tau_2) = U(\tau_1 + \tau_2) \tag{17}$$

We shall prove the following L^2 estimates for the operator $U(\tau)$.

Proposition 4.1. *We have the following estimates for the operator $U(\tau)$:*

$$\|U(\tau)F\|_{L^2(S)} \le \|F\|_{L^2(S)} \tag{18}$$

$$\|\nabla U(\tau)F\|_{L^2(S)} \le \|\nabla F\|_{L^2(S)} \tag{19}$$

$$\|\nabla U(\tau)F\|_{L^2(S)} \le \frac{\sqrt{2}}{2}\tau^{-\frac{1}{2}}\|F\|_{L^2(S)} \tag{20}$$

$$\|\Delta U(\tau)F\|_{L^2(S)} \le \frac{\sqrt{2}}{2}\tau^{-1}\|F\|_{L^2(S)} \tag{21}$$

We also have,

$$\|U(\tau)\nabla F\|_{L^2(S)} \le \frac{\sqrt{2}}{2}\tau^{-\frac{1}{2}}\|F\|_{L^2(S)} \tag{22}$$

Proof : To prove (18) we multiply the equation

$$\partial_\tau U(\tau)F - \Delta U(\tau)F = 0$$

by $U(\tau)F$ and integrate over S.

$$\frac{1}{2}\frac{d}{d\tau}\|U(\tau)F\|^2_{L^2(S)} + \|\nabla U(\tau)F\|^2_{L^2(S)} = 0$$

Therefore,

$$\frac{1}{2}\|U(\tau)F\|^2_{L^2(S)} + \int_0^\tau \|\nabla U(\tau')F\|^2_{L^2(S)}d\tau' = \frac{1}{2}\|F\|^2_{L^2(S)} \tag{23}$$

and (18) follows. On the other hand, multiplying the equation by $\tau\Delta U(\tau)F$, we similarly obtain the identity

$$\frac{1}{2}\frac{d}{d\tau}\tau\|\nabla U(\tau)F\|^2_{L^2(S)} + \tau\|\Delta U(\tau)F\|^2_{L^2(S)} = \frac{1}{2}\|\nabla U(\tau)F\|^2_{L^2(S)}$$

Integrating this in τ, with the help of (23),

$$\frac{\tau}{2}\|\nabla U(\tau)F\|^2_{L^2(S)} + \int_0^\tau \tau'\|\Delta U(\tau')F\|^2_{L^2(S)}d\tau' \le \frac{1}{2}\int_0^\tau \|\nabla U(\tau)F\|_{L^2(S)} \le \frac{1}{4}\|F\|^2_{L^2(S)} \tag{24}$$

which implies (20). Proceeding in exactly the same way with the multiplier $\tau\Delta U(\tau)F$ replaced by $\Delta U(\tau)F$ yields (19). Furthermore, multiplying the equation by $\tau^2\Delta^2 U(\tau)f$, we have

$$\frac{1}{2}\frac{d}{d\tau}\tau^2\|\Delta U(\tau)F\|^2_{L^2(S)} + \tau^2\|\nabla\Delta U(\tau)F\|^2_{L^2(S)} = \tau\|\Delta U(\tau)F\|^2_{L^2(S)}$$

Integrating in τ and using (24), we obtain

$$\frac{\tau^2}{2}\|\Delta U(\tau)F\|^2_{L^2(S)} + \int_0^\tau (\tau')^2\|\nabla\Delta U(\tau')F\|^2_{L^2(S)}d\tau' = \int_0^\tau \tau'\|\Delta U(\tau')F\|^2_{L^2(S)}d\tau' \le \frac{1}{4}\|F\|^2_{L^2(S)}$$

[3] Indeed observe that Δ is selfadjoint and formally $U(\tau)f = \sum_n \frac{1}{n!}t^n\Delta^n$.

This immediately yields (21).

To prove (22) we observe that

$$\|U(\tau)\nabla F\|_{L^2}^2 =< U(\tau)\nabla F\,,\,U(\tau)\nabla F >=< \text{ div } U(\tau)U(\tau)\nabla F\,,\,F >$$

Therefore,

$$\begin{aligned}\|U(\tau)\nabla F\|_{L^2}^2 &\le \|\nabla(U(\tau)U(\tau)\nabla F)\|_{L^2}\|F\|_{L^2}\\ &\le \frac{\sqrt{2}}{2}\tau^{-\frac12}\|U(\tau)\nabla F\|_{L^2}\|F\|_{L^2}\end{aligned}$$

whence $\|U(\tau)\nabla F\|_{L^2} \lesssim \frac{\sqrt{2}}{2}\tau^{-\frac12}\|F\|_{L^2}$ as desired. ■

In the next proposition we establish a simple L^p estimate for $U(\tau)$.

Proposition 4.2. *For every* $2 \le p \le \infty$

$$\|U(\tau)F\|_{L^p} \le \|F\|_{L^p}$$

Proof : We shall first prove the Lemma for scalar functions f. We multiply the equation $\partial_\tau U(\tau)f - \Delta U(\tau)f = 0$ by $\big(U(\tau)f\big)^{2p-1}$ and integrate by parts. We get,

$$\frac{1}{2p}\frac{d}{d\tau}\|U(\tau)F\|_{L^{2p}}^{2p} + (2p-1)\int |\nabla U(\tau)f|^2|U(\tau)f|^{2p-2} = 0$$

Therefore,

$$\|U(\tau)F\|_{L^{2p}} \le \|F\|_{L^{2p}}$$

The case when F is a tensorfield can be treated in the same manner with multiplier $\big(|U(\tau)F|^2\big)^{p-1}U(\tau)F$. ■

5. Invariant Littlewood-Paley theory

In this section we shall use the heat flow discussed in the previous section to develop an invariant, fully tensorial, Littlewood-Paley theory on manifolds. Though we restrict ourselves here to two dimensional compact manifolds it is clear that our theory can be extended to arbitrary dimensions and noncompact manifolds.

Definition 5.1. Consider the class $\mathcal{M}$ of smooth functions m on $[0,\infty)$, vanishing sufficiently fast at ∞, verifying the vanishing moments property:

$$\int_0^\infty \tau^{k_1}\partial_\tau^{k_2}m(\tau)d\tau = 0, \quad |k_1| + |k_2| \le N \tag{25}$$

We set, $m_k(\tau) = 2^{2k}m(2^{2k}\tau)$ and define the geometric Littlewood -Paley (LP) projections P_k, associated to the LP- representative function $m \in \mathcal{M}$, for arbitrary tensorfields F on S to be

$$P_kF = \int_0^\infty m_k(\tau)U(\tau)F d\tau \tag{26}$$

Given an interval $I \subset \mathbb{Z}$ we define

$$P_I = \sum_{k \in I} P_k F.$$

In particular we shall use the notation $P_{<k}, P_{\le k}, P_{>k}, P_{\ge k}$.

Observe that P_k are selfadjoint[4], i.e., $P_k = P_k^*$, in the sense,

$$< P_k F, G > = < F, P_k G >,$$

where, for any given m-tensors F, G

$$< F, G > = \int_S \gamma^{i_1 j_1} \ldots \gamma^{i_m j_m} F_{i_1 \ldots i_m} G_{j_1 \ldots j_m} d\mathrm{vol}_\gamma$$

denotes the usual L^2 scalar product.

Consider two LP projections associated to a, b

$$\begin{aligned} P_a P_b F &= \int_0^\infty \int_0^\infty d\tau_1 d\tau_2 a(\tau_1) b(\tau_2) U(\tau_1 + \tau_2) F \\ &= \int_0^\infty d\tau_1 \int_{\tau_1}^\infty d\tau \; a(\tau_1) b(\tau - \tau_1) U(\tau) F \\ &= \int_0^\infty d\tau U(\tau) f \int_0^\tau d\tau_1 \; a(\tau_1) b(\tau - \tau_1) \\ &= \int_0^\infty d\tau \, a \star b(\tau) \, U(\tau) F \end{aligned}$$

where

$$a \star b(\tau) = \int_0^\tau d\tau_1 \; a(\tau_1) b(\tau - \tau_1) \tag{27}$$

Lemma 5.2. *If $a, b \in \mathcal{M}$ so does $a \star b$. Also, $(a \star b)_k = a_k \star b_k$. In particular if we denote by ${}^{(a)}P_k$ and ${}^{(b)}P_k$ the LP projections associated to a, b then,*

$${}^{(a)}P_k \cdot {}^{(b)} P_k = {}^{(a \star b)} P_k$$

Proof : We only need to show that $\int (a \star b)(\tau) d\tau = 0$. Then, we can easily check that $\tau \cdot (a \star b)(\tau)$ and $\frac{d}{d\tau}(a \star b)$ also verify the same property as well as any combination of these. Clearly $\int_0^\infty a \star b \, d\tau = \int_0^\infty a(\tau_1) d\tau_1 \cdot \int_0^\infty b(\tau_2) d\tau_2 = 0$. ∎

Motivated by this Lemma we define:

Definition 5.3. Given a positive integer ℓ we define the class $\mathcal{M}_\ell \subset \mathcal{M}$ of LP-representatives to consist of functions of the form

$$\bar{m} = m \star m \star \ldots \star m = (m\star)^\ell,$$

for some $m \in \mathcal{M}$.

[4]This follows easily in view of the selfadjoint properties of Δ and $U(\tau)$.

Lemma 5.4. *For any $\ell \ge 1$ there exists an element $\bar{m} \in \mathcal{M}_\ell$ such that the LP-projections associated to $\bar{m}$ verify:*

$$\sum_k P_k = I \tag{28}$$

Proof : See [Stein1]. ■

Theorem 5.5. *The LP-projections P_k associated to an arbitrary $m \in \mathcal{M}$ verify the following properties:*

i) L^p-boundedness *For any $1 \le p \le \infty$, and any interval $I \subset \mathbb{Z}$,*

$$\|P_I F\|_{L^p} \lesssim \|F\|_{L^p} \tag{29}$$

ii) L^p- *Almost Orthogonality* *Consider two families of LP-projections $P_k, \tilde{P}_k$ associated to m and respectively $\tilde{m}$, both in $\mathcal{M}$. For any $1 \le p \le \infty$:*

$$\|P_k \tilde{P}_{k'} F\|_{L^p} \lesssim 2^{-2|k-k'|} \|F\|_{L^p} \tag{30}$$

iii) *Bessel inequality*

$$\sum_k \|P_k F\|_{L^2}^2 \lesssim \|F\|_{L^2}^2$$

iv) Reproducing Property *Given any integer $\ell \ge 2$ and $\bar{m} \in \mathcal{M}_\ell$ there exists $m \in \mathcal{M}$ such that such that $\bar{m} = m \star m$. Thus,*

$$^{(\bar{m})}P_k =^{(m)} P_k \cdot^{(m)} P_k.$$

Whenever there is no danger of confusion we shall simply write $P_k = P_k \cdot P_k$.

v) Finite band property *For any $1 \le p \le \infty$.*

$$\begin{aligned} \|\Delta P_k F\|_{L^p} &\lesssim 2^{2k} \|F\|_{L^p} \\ \|P_k F\|_{L^p} &\lesssim 2^{-2k} \|\Delta F\|_{L^p} \end{aligned}$$

Morever give $m \in \mathcal{M}$ we can find $\bar{m} \in \mathcal{M}$ such that $\Delta P_k = 2^{2k} \bar{P}_k$ with $\bar{P}_k$ the LP projections associated to $\bar{m}$.

In addition, the L^2 estimates

$$\begin{aligned} \|\nabla P_k F\|_{L^2} &\lesssim 2^k \|F\|_{L^2} \\ \|P_k F\|_{L^2} &\lesssim 2^{-k} \|\nabla F\|_{L^2} \end{aligned}$$

hold together with the dual estimate

$$\|P_k \nabla F\|_{L^2} \lesssim 2^k \|F\|_{L^2}$$

vi) Weak Bernstein inequality *For any $2 \le p < \infty$*

$$\begin{aligned} &\|P_k F\|_{L^p} \lesssim (2^{(1-\frac{2}{p})k} + 1) \|F\|_{L^2}, \\ &\|P_{<0} F\|_{L^p} \lesssim \|F\|_{L^2} \end{aligned}$$

together with the dual estimates

$$\|P_k F\|_{L^2} \lesssim (2^{(1-\frac{2}{p})k} + 1)\|F\|_{L^{p'}},$$
$$\|P_{<0} F\|_{L^2} \lesssim \|F\|_{L^{p'}}$$

vii) Commutator Estimate *Given two tensorfields F, G and $F \cdot G$ any contraction of the tensor product $F \otimes G$ we have the following estimate for the commutator $[P_k\,,\, F]\cdot G = P_k(F\cdot G) - F\cdot P_k G$*

$$\|\,[P_k\,,\,F]\cdot G\|_{L^2} \lesssim 2^{-k}\|\nabla F\|_{L^\infty}\|G\|_{L^2}.$$

We also have the estimate of the form

$$\|\,[P_k\,,\,F]\cdot G\|_{L^2} \lesssim \Big(2^{-2k}\|\Delta F\|_{L^\infty} + 2^{-k}\|\nabla F\|_{L^\infty}\Big)\|G\|_{L^2}.$$

Proof :

i) The L^p boundedness of P_k follows from the L^p mapping properties of the heat flow $U(\tau)$.

ii) Assume that $k_2 \geq k_1$. By definition and in view of the semigroup property of $U(\tau)$ we write,

$$P_{k_1}\tilde{P}_{k_2} f = \int_0^\infty \int_0^\infty U(\tau_1+\tau_2) f \;\, m_{k_1}(\tau_1)\tilde{m}_{k_2}(\tau_2) d\tau_1 d\tau_2$$

Writing $U(\tau_1+\tau_2) = U(\tau_1) + \int_0^1 \frac{d}{ds} U(\tau_1 + s\tau_2) ds$ and then using the vanishing of $\int_0^\infty \tilde{m}_{k_2}$ we infer that,

$$\begin{aligned} P_{k_1}\tilde{P}_{k_2} f &= \int_0^\infty \int_0^\infty \frac{d}{d\tau_1}\int_0^1 U(\tau_1 + s\tau_2) f \;\, m_{k_1}(\tau_1)\, \tau_2 \tilde{m}_{k_2}(\tau_2) d\tau_1 d\tau_2 \\ &= -\int_0^\infty \int_0^\infty \int_0^1 U(\tau_1+s\tau_2) f \;\, \frac{d}{d\tau_1} m_{k_1}(\tau_1)\, \tau_2 \tilde{m}_{k_2}(\tau_2) d\tau_1 d\tau_2 \\ &- \;\; m_{k_1}(0)\int_0^\infty d\tau_2\, \tau_2 \tilde{m}_{k_2}(\tau_2)\int_0^1 U(s\tau_2) f ds \end{aligned}$$

Now setting $\tilde{n}(\tau) = \tau\tilde{m}(\tau)$, and $n(\tau) = m'(\tau)$ we infer that,

$$\begin{aligned} P_{k_1}\tilde{P}_{k_2} f &= -2^{2(k_1-k_2)} \int_0^\infty \int_0^\infty \int_0^1 U(\tau_1 + s\tau_2) f \;\, n_{k_1}(\tau_1)\, \tilde{n}_{k_2}(\tau_2) d\tau_1 d\tau_2 \\ &- \;\; 2^{2(k_1-k_2)} m(0) \int_0^\infty d\tau_2 \tilde{n}_{k_2}(\tau_2)\int_0^1 U(s\tau_2) f ds \end{aligned}$$

Therefore, using the L^p mapping properties of U,

$$\begin{aligned} \|P_{k_1}\tilde{P}_{k_2} F\|_{L^p} &= 2^{-2|k_1-k_2|}\|F\|_{L^p}\int_0^\infty \int_0^\infty |n_{k_1}(\tau_1)|\, |\tilde{n}_{k_2}(\tau_2)| d\tau_1 d\tau_2 \\ &+ \;\; 2^{-2|k_1-k_2|}|m(0)|\|F\|_{L^p}\int_0^\infty |\tilde{n}_{k_2}(\tau_2)| d\tau_2 \\ &\lesssim \;\; 2^{-2|k_1-k_2|}\|F\|_{L^p} \end{aligned}$$

Remark 5.6. One can give a slicker proof of the almost orthogonality properties of LP projections by using the algebraic formula $2^{2k}P_k f = \Delta \bar{P}_k f$, see (35) below. Moreover, if sufficiently many moments of m are zero, s.t $\tau^{2j}m, \tau^{2j}\tilde{m}$ are good symbols, then in fact,

$$\|P_{k_1}\tilde{P}_{k_2}F\|_{L^2} \lesssim 2^{-2j|k_1-k_2|}\|F\|_{L^2} \tag{31}$$

iii) To prove the Bessel type inequality we write,

$$\sum_k \|P_k F\|_{L^2}^2 = \sum_k < P_k P_k f, f > \le \|(\sum_k P_k^2)F\|_{L^2}\|F\|_{L^2}$$

To show that the operator $P = \sum_k P_k^2$ is bounded on L^2 we appeal to the Cotlar-Stein Lemma, see [Stein2]. Observe first that, in view of Lemma 5.2, $P_k^2 = {}^{(m\star m)}\, P_k$. Since $m \star m \in \mathcal{M}$ we can, without loss of generality, simply write $P_k^2 = P_k$. The conditions of applicability of the Cotlar-Stein Lemma[5] are satisfied in view of the almost orthogonality established in part ii) as well as $P_k = P_k^*$.

iv) The proof is immediate in view of the definition 5.3.

v) According to the definition of $P_k f$ we have

$$\begin{aligned}
\Delta P_k f &= \int_0^\infty m_k(\tau)\Delta U(\tau) f = \int_0^\infty m_k(\tau)\frac{d}{d\tau}U(\tau)f \\
&= -m_k(0)U(0)f - \int_0^\infty \frac{d}{d\tau}m_k(\tau)U(\tau)f \\
&= -2^{2k}\Big(m(0)f + \int_0^\infty (m')_k(\tau)U(\tau)f\Big)
\end{aligned}$$

In view of the L^p properties of $U(\tau)f$ and the obvious bound $\int_0^\infty |(m')_k(\tau)|d\tau \lesssim 1$,

$$\|\Delta P_k F\|_{L^p} \lesssim 2^{2k}\|F\|_{L^p} \tag{32}$$

To prove the second estimate we introduce $\bar{m}(\tau) = -\int_\tau^\infty m(\tau)$ such that $\frac{d}{d\tau}\bar{m} = m(\tau)$ and $\int_0^\infty |\bar{m}(\tau)|d\tau < \infty$. Observe also that $\bar{m}(0) = 0$. Set also,

$$\bar{m}_k(\tau) = 2^{2k}\bar{m}(2^{2k}\tau)$$

$$\begin{aligned}
2^{2k}P_k f &= \int_0^\infty 2^{2k}m_k(\tau)U(\tau)f = \int_0^\infty \frac{d}{d\tau}\bar{m}_k(\tau)U(\tau)f \\
&= -\int_0^\infty \bar{m}_k(\tau)\frac{d}{d\tau}U(\tau)f = -\int_0^\infty \bar{m}_k(\tau)\Delta U(\tau)f \\
&= -\int_0^\infty \bar{m}_k(\tau)U(\tau)\Delta f
\end{aligned} \tag{33}$$

Therefore, using the estimate $\|U(\tau)\Delta F\|_{L^p} \lesssim \|\Delta F\|_{L^p}$, we infer that,

$$2^{2k}\|P_k F\|_{L^p} \lesssim \|\Delta F\|_{L^p}\int_0^\infty |\bar{m}_k(\tau)|d\tau \lesssim \|\Delta F\|_{L^p} \tag{34}$$

[5]Notice that we are in the special case of commuting selfadjoint operators.

Observe also that, according to (33) we have

$$2^{2k}P_kF = \Delta\bar{P}_kF \tag{35}$$

where $\bar{P}_k$ is defined by the symbol $\bar{m}(\tau) = -\int_\tau^\infty m(\tau') \in \mathcal{M}$.

To prove the L^2 estimates involving one derivative we observe that

$$\begin{aligned} \|\nabla P_kF\|_{L^2}^2 &= <\nabla P_kF, \nabla P_kF> = -<\Delta P_kF, P_kF> \\ &\le \|\Delta P_kF\|_{L^2}\|P_kF\|_{L^2} \lesssim 2^{2k}\|F\|_{L^2}^2 \end{aligned}$$

On the other hand, using (33)

$$\begin{aligned} &2^{2k}\|P_kF\|_{L^2}^2 = 2^{2k} <P_kF, P_kF> = -\int_0^\infty \bar{m}_k(\tau) <\Delta U(\tau)F, P_kF> \\ =\ &\int_0^\infty \bar{m}_k(\tau) <\nabla U(\tau)F, \nabla P_kF> \le \int_0^\infty |\bar{m}_k(\tau)| \cdot \|\nabla U(\tau)F\|_{L^2}\|\nabla P_kF\|_{L^2} \lesssim \|\nabla F\|_{L^2}^2, \end{aligned}$$

where we used the inequality (19), $\|\nabla U(\tau)F\|_{L^2} \le \|\nabla F\|_{L^2}$ together with the bound $\|\nabla P_kF\|_{L^2} \lesssim \|\nabla F\|_{L^2}$, which follows from it.

vi) The proof of the L^p Bernstein inequality is an easy consequence of the Gagliardo-Nirenberg inequality (8):

$$\|P_kF\|_{L^p} \lesssim \|\nabla P_kF\|_{L^2}^{1-\frac{2}{p}}\|P_kF\|_{L^2}^{\frac{2}{p}} + \|P_kF\|_{L^2} \tag{36}$$

for $2 \le p < \infty$ and the finite band property.

vii) By definition

$$[P_k\,,\,F]G = \int_0^\infty \Big(U(\tau)(F\cdot G) - F\cdot U(\tau)G\Big)m_k(\tau)d\tau$$

Let $w = U(\tau)(F\cdot G) - F\cdot U(\tau)G$. Clearly,

$$\begin{aligned} \partial_\tau w - \Delta w &= \nabla\big(\nabla F\cdot U(\tau)G\big) + \nabla F\cdot\nabla U(\tau)G \\ &= \Delta F\cdot U(\tau)G + 2\nabla F\cdot\nabla U(\tau)G \end{aligned}$$

Consequently, since $w(0) = 0$,

$$\begin{aligned} w &= w_1 + w_2 \\ w_1(\tau) &= \int_0^\tau U(\tau-\tau')\big(\Delta F\cdot U(\tau')G\big)d\tau' \\ w_2(\tau) &= \int_0^\tau U(\tau-\tau')\big(\nabla F\cdot\nabla U(\tau')G\big)d\tau' \end{aligned}$$

and,

$$\begin{aligned} \|w_1(\tau)\|_{L^2} &\lesssim \int_0^\tau \|U(\tau-\tau')\big(\Delta F\cdot U(\tau')G\big)\|_{L^2}d\tau' \lesssim \int_0^\tau \|(\Delta F\cdot U(\tau')g\|_{L^2}d\tau' \\ &\lesssim \|\Delta F\|_{L^\infty}\int_0^\tau \|U(\tau')G\|_{L^2} \lesssim \tau\|\Delta F\|_{L^\infty}\cdot\|G\|_{L^2} \end{aligned}$$

$$\begin{aligned}
\|w_2(\tau)\|_{L^2} &\lesssim \int_0^\tau \|U(\tau-\tau')\big(\nabla F\cdot\nabla U(\tau')G\big)\|_{L^2}d\tau' \lesssim \int_0^\tau \|(\nabla F\cdot\nabla U(\tau')G\|_{L^2}d\tau' \\
&\lesssim \|\nabla F\|_{L^\infty}\int_0^\tau \|\nabla U(\tau')G\|_{L^2} \lesssim \|\nabla F\|_{L^\infty}\cdot\|G\|_{L^2}\int_0^\tau \tau^{-\frac12} \\
&\lesssim \tau^{\frac12}\|\nabla F\|_{L^\infty}\cdot\|G\|_{L^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|[P_k\,,\,F]G\| &\lesssim \int_0^\infty \|w(\tau)\|_{L^2}|m_k(\tau)|d\tau \\
&\lesssim \Big(2^{-2k}\|\Delta F\|_{L^\infty}+2^{-k}\|\nabla F\|_{L^\infty}\Big)\cdot\|G\|_{L^2}
\end{aligned}$$

Remark 5.7. To get the inequality

$$\|[P_k\,,\,F]G\| \lesssim 2^{-k}\|\nabla F\|_{L^\infty}\|G\|_{L^2}$$

we need the L^2 estimate $\|U(\tau)\nabla F\|_{L^2}\lesssim \tau^{-\frac12}\|F\|_{L^2}$ established in (22). We rewrite

$$\begin{aligned}
w_1(\tau) &= w_{11}(\tau)-w_{12}(\tau), \\
w_{11}(\tau) &= \int_0^\tau U(\tau-\tau')\nabla\big(\nabla F\cdot U(\tau')G\big)d\tau', \\
w_{12}(\tau) &= \int_0^\tau U(\tau-\tau')\big(\nabla F\cdot\nabla U(\tau')G\big)d\tau'
\end{aligned}$$

The term w_{12} is exactly the same as $w_2(\tau)$ and gives rise to the desired estimate. To estimate w_{11} we use (22) and write

$$\|w_{11}(\tau)\|_{L^2} \lesssim \int_0^\tau {\tau'}^{-\frac12}\|\nabla F\cdot U(\tau')G\|_{L^2}d\tau' \lesssim \tau^{\frac12}\|\nabla F\|_{L^\infty}\|G\|_{L^2}$$

which again leads to the desired estimate.

■

6. Sobolev space $H^1(S)$.

Before discussing the general, fractional, Sobolev spaces in the next section it is instructive to see how the the standard Sobolev space $H^1(S)$ can be characterized by our LP projections. We prove the following:

Proposition 6.1.
i.) *Consider the LP projections P_k associated to an arbitrary $m\in\mathcal{M}_2$. Then,*

$$\sum_k \|P_kF\|_{L^2}^2 \lesssim \|F\|_{L^2}^2 \tag{37}$$

$$\sum_k 2^{2k}\|P_kF\|_{L^2}^2 \lesssim \|\nabla F\|_{L^2}^2 \tag{38}$$

ii.) *If in addition the LP-projections P_k verify:*

$$\sum_k P_k^2 = I \tag{39}$$

Then,

$$\|F\|_{L^2}^2 = \sum_k \|P_kF\|_{L^2}^2 \tag{40}$$

$$\|\nabla F\|_{L^2}^2 \lesssim \sum_k 2^{2k}\|P_kF\|_{L^2}^2 \tag{41}$$

Proof : The first statement of part i) is nothing else but the Bessel inequality established above. To prove the second statement of i) we write $P_k = \tilde{P}_k^2$ and make use of of the L^2-finite band properties of the $\tilde{P}_k$'s, as well as the L^2- boundedness of the operator $\sum_k P_k = \sum_k \tilde{P}_k^2$. We shall also make use of the following simple formula based on the standard definition of $(-\Delta)^{\frac{1}{2}}$,

$$\|\nabla G\|_{L^2} =< \nabla G, \nabla G >= - < \Delta G, G >=< (-\Delta)^{\frac{1}{2}}G, (-\Delta)^{\frac{1}{2}}G >= \|(-\Delta)^{\frac{1}{2}}G\|_{L^2}$$

Therefore,

$$\begin{aligned}\sum_k 2^{2k}\|P_kF\|_{L^2}^2 &\lesssim \sum_k \|\nabla\tilde{P}_kF\|_{L^2}^2 = \sum_k \|(-\Delta)^{\frac{1}{2}}\tilde{P}_kF\|_{L^2}^2 = \sum_k \|\tilde{P}_k(-\Delta)^{\frac{1}{2}}F\|_{L^2}^2\\ &= \sum_k < \tilde{P}_k(-\Delta)^{\frac{1}{2}}f \,,\, \tilde{P}_k(-\Delta)^{\frac{1}{2}}f >=< \sum_k \tilde{P}_k^2(-\Delta)^{\frac{1}{2}}f \,,\, (-\Delta)^{\frac{1}{2}}f >\\ &\lesssim \|\sum_k \tilde{P}_k^2(-\Delta)^{\frac{1}{2}}F\|_{L^2}\|(-\Delta)^{\frac{1}{2}}F\|_{L^2} \lesssim \|(-\Delta)^{\frac{1}{2}}F\|_{L^2}^2 = \|\nabla F\|_{L^2}^2\end{aligned}$$

as desired.

The first identity of part ii) is trivial,

$$\|F\|_{L^2}^2 =< \sum_k P_k^2 f, f >= \sum_k \|P_kF\|_{L^2}^2$$

To prove the second inequality of part ii) we introduce $P_k = \tilde{P}_k^2$ and and make use of $\sum_k P_k^2 = I$, the L^2-finite band inequality $\|\Delta\tilde{P}_k g\|_{L^2} \lesssim 2^{2k}\|g\|_{L^2}$, the inequality (51), as well as as the commutation properties of our LP projections with Δ:

$$\begin{aligned}\|\nabla F\|_{L^2}^2 &= < -\Delta f \,,\, f >=< -\Delta\big(\sum_k P_k^2\big)f \,,\, f >= \sum_k < -\Delta P_k f \,,\, P_k f >\\ &\lesssim \sum_k \|\Delta\tilde{P}_k^2F\|_{L^2}\|P_kF\|_{L^2} \le \sum_k 2^{2k}\|\tilde{P}_kF\|_{L^2}\|P_kF\|_{L^2}\\ &\le \big(\sum_k 2^{2k}\|\tilde{P}_kF\|_{L^2}^2\big)^{\frac{1}{2}}\big(\sum_k 2^{2k}\|P_kF\|_{L^2}^2\big)^{\frac{1}{2}} \lesssim \|\nabla F\|_{L^2}\big(\sum_k 2^{2k}\|P_kF\|_{L^2}^2\big)^{\frac{1}{2}}\end{aligned}$$

whence,

$$\|\nabla F\|_{L^2} \lesssim \big(\sum_k 2^{2k}\|P_kF\|_{L^2}^2\big)^{\frac{1}{2}}$$

as desired. ∎

7. Fractional powers of Δ and Sobolev spaces.

We recall the definition of the Gamma function, for $\Re(z) > 0$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{42}$$

as well as the beta function,

$$B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1} ds \tag{43}$$

Recall that

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} \tag{44}$$

Let $j_a(\lambda)$, $\Re(a) < 0$, denote the function which is identically zero for $\lambda < 0$ and

$$j_a(\lambda) = \frac{1}{\Gamma(-a)} \lambda^{-a-1}, \qquad \lambda > 0. \tag{45}$$

The followwing proposition is well known, see e.g. [],

Proposition 7.1. *For all a, b, $\Re(a)$, $\Re(b) < 0$,*

$$j_a * j_b = j_{a+b}$$

Moreover there exists a family of distribution j_a, defined for all $a \in \mathbb{C}$, such that, $j_a * j_b = j_{a+b}$ and $j_0 = \delta_0$, the Dirac delta function at the origin.

Proof : We only recall the formula $j_a * j_b = j_{a+b}$ for $\Re(a)$, $\Re(b) < 0$

$$\begin{aligned}
j_a * j_b(\lambda) &= \frac{1}{\Gamma(-a)} \frac{1}{\Gamma(-b)} \int_0^\lambda \mu^{-a-1}(\lambda-\mu)^{-b-1} d\mu \\
&= \frac{1}{\Gamma(-a)} \frac{1}{\Gamma(-b)} \lambda^{-a-b-1} \int_0^1 s^{-a-1}(1-s)^{-b-1} \\
&= \frac{B(-a,-b)}{\Gamma(-a)\cdot\Gamma(-b)} \lambda^{-a-b-1} = \frac{1}{\Gamma(-a-b)} \lambda^{-a-b-1} = j_{a+b}(\lambda)
\end{aligned}$$

$\blacksquare$

Definition 7.2. We define the negative fractional powers of $\Lambda^2 = I - \Delta$ on any smooth tensorfield F on S according to the formula

$$\Lambda^a F = \frac{1}{\Gamma(-a/2)} \int_0^\infty \tau^{-\frac{a}{2}-1} e^{-\tau} U(\tau) F d\tau \tag{46}$$

where a is an arbitrary complex number with $\Re(a) < 0$.

Proposition 7.3. *The operators Λ^a is symmetric and verify the group property,*

$$\Lambda^a \cdot \Lambda^b = \Lambda^{(a+b)}$$

Proof : According to the definition of Λ^a and the semigroup properties of U we have, for any tensorfield F,

$$\begin{aligned}
\Lambda^a\cdot\Lambda^b F &= \frac{1}{\Gamma(-a/2)}\frac{1}{\Gamma(-b/2)}\int_0^\infty\int_0^\infty \tau_1^{-a/2-1}\tau_2^{-b/2-1}U(\tau_1+\tau_2)e^{-\tau_1-\tau_2}Fd\tau_1 d\tau_2\\
&= \frac{1}{\Gamma(-a/2)}\frac{1}{\Gamma(-b/2)}\int_0^\infty e^{-\lambda}U(\lambda)F\int_0^\lambda \tau_1^{-a/2-1}(\lambda-\tau_1)^{-b/2-1}d\tau_1\\
&= \int_0^\infty j_{a/2}*j_{b/2}(\lambda)e^{-\lambda}U(\lambda)F = \int_0^\infty j_{a/2+b/2}(\lambda)e^{-\lambda}U(\lambda)F\\
&= \Lambda^{a+b}F
\end{aligned}$$

as desired. ■

We extend the definition of fractional powers of Λ to the range of a with $\Re(a)>0$, on smooth tensorfields F, by defining first

$$\Lambda^a F = \Lambda^{a-2}\cdot(I-\Delta)F$$

for $0<\Re(a)\le 2$ and then, in general, for $0<\Re(a)\le 2m$, with an arbitrary positive integer m, according to the formula

$$\Lambda^a F = \Lambda^{a-2m}\cdot(I-\Delta)^m F.$$

Observe that for $0<\Re(a)<2$,

$$\begin{aligned}
\Lambda^a F = \Lambda^{a-2}(I-\Delta)F &= \frac{1}{\Gamma(-a/2+1)}\int_0^\infty \tau^{-a/2}e^{-\tau}U(\tau)(I-\Delta)Fd\tau\\
&= \frac{1}{\Gamma(-a/2+1)}\int_0^\infty (\frac{d}{d\tau}\tau^{-a/2})e^{-\tau}U(\tau)Fd\tau\\
&= \frac{1}{\Gamma(-a/2)}\int_0^\infty \tau^{-a/2-1}e^{-\tau}U(\tau)Fd\tau
\end{aligned}$$

Moreover, for $a=0$, the integration by parts we have performed above yields also a boundary term.

$$\Lambda^0 F = \Lambda^{-2}(I-\Delta)F \quad = \quad \frac{1}{\Gamma(1)}\int_0^\infty e^{-\tau}U(\tau)(I-\Delta)Fd\tau = -\int_0^\infty e^{-\tau}(I-\frac{d}{d\tau})U(\tau)Fd\tau = F$$

i.e. Λ^{-2} is trully the inverse of $I-\Delta$.

Remark 7.4. In a similar fashion, we can introduce the family of operators $\mathcal{D}^a = (-\Delta)^{\frac{a}{2}}$ for all $a\in\mathbb{C}$. As before, we start by defining formally, for $\Re(a)<0$,

$$\mathcal{D}^a F = \frac{1}{\Gamma(-a/2)}\int_0^\infty \tau^{-\frac{a}{2}-1}U(\tau)Fd\tau. \tag{47}$$

However, unlike Λ^a, this formula makes sense only for smooth tensors F which verify the additional property that F is orthogonal to the kernel of the tensor laplacean Δ. In view of our smoothness assumption on the manifold S and the ellipticity of Δ, the above kernel is finite dimensional. We can also extend the definition of $\mathcal{D}^a$ to the range of $a\in\mathbb{C}$ with $\Re(a)>0$ according to

$$\mathcal{D}^a = \mathcal{D}^{a-2m}(-\Delta)^m$$

with an integer m such that $2m-2<\Re(a)<2m$. It follows that the operators $\mathcal{D}^a$ with $\Re(a)>0$ can naturally be extended to the space of all smooth tensors. We also check, as before, that $\mathcal{D}^0=I$.

We have thus proved the following:

Theorem 7.5. *There exist two family of operators* $(\mathcal{D}^a)_{a\in\mathbb{C}}\,,(\Lambda^a)_{a\in\mathbb{C}}$ *such that,*

$$\Lambda^0=I,\qquad \Lambda^a\cdot\Lambda^b=\Lambda^{(a+b)},\qquad \Lambda^{2k}=(I-\Delta)^k,\qquad k=0,1,2\dots,$$
$$\mathcal{D}^0=I,\qquad \mathcal{D}^{2k}=(-\Delta)^k,\qquad k=0,1,2\dots$$

on the space of all smooth tensorfields. In addition, the identity

$$\mathcal{D}^a\cdot\mathcal{D}^b=\mathcal{D}^{(a+b)}$$

holds on the space of all smooth tensorfields provided that $\Re(b)>0$ *and* $\Re(a+b)>0$. *For the remaining values of* $a,b\in\mathbb{C}$ *the above identity holds only on the orthogonal complement of the kernel of* Δ.

For $\Re(a)<-2$, *and any tensorfield* F, Λ^aF *can be defined by the formula* (46), *while* $\mathcal{D}^aF$ *is defined in* (47) *for* F *in the orthogonal complement of the kernel of* Δ.

Proposition 7.6. *The following estimates hold true, for any* $\Re(a)<0$.

$$\|\Lambda^aF\|_{L^2}\quad\lesssim\quad\|F\|_{L^2}.$$

Moreover, for $2k\le\Re(a)<2k+2$, $k\in\mathbb{N}$,

$$\|(\Lambda^a-\mathcal{D}^a-c_1\mathcal{D}^{a-2}-c_2\mathcal{D}^{a-4}-\dots-c_kD^{a-2k})F\|_{L^2}\lesssim\|F\|_{L^2}\tag{48}$$

where $c_i=(-1)^i\frac{1}{i!}\frac{\Gamma(-a/2+i)}{\Gamma(-a/2)}$.

Proof : To show the boundedness of Λ^a, $\Re(a)<0$, we only have to use the L^2 boundedness of the heat flow, $\|U(\tau)F\|_{L^2(S)}\le\|F\|_{L^2(S)}$. Thus,

$$\|F\|_{L^2}^{-1}\cdot\|\Lambda^{-a}F\|_{L^2}\quad\le\quad\frac{1}{\Gamma(-a/2)}\int_0^\infty\tau^{-a/2-1}e^{-\tau}d\tau\le C_a$$

To prove (48) we expand $e^{-\tau}$ in the formula defining Λ^aF,

$$e^{-\tau}=1-\tau+\frac{1}{2!}\tau^2+\dots+(-1)^k\frac{1}{k!}\tau^k+O(\tau^{k+1}e^{-\tau}).$$

Hence,

$$\begin{aligned}\Lambda^aF\quad&=\quad\frac{1}{\Gamma(-a/2)}\int_0^\infty\tau^{-a/2-1}e^{-\tau}U(\tau)F=\Lambda^aF-\frac{\Gamma(-a/2+1)}{\Gamma(-a/2)}\Lambda^{a-2}F\\&+\quad+\frac{1}{2!}\frac{\Gamma(-a/2+2)}{\Gamma(-a/2)}\Lambda^{a-4}F+\cdots(-1)^k\frac{1}{k!}\frac{\Gamma(-a/2+k)}{\Gamma(-a/2)}\Lambda^{a-2k}+E_k(F)\end{aligned}$$

where, in view of the L^2 boundedness of $U(\tau)$ and the integrability of $\tau^{-a/2+k}e^{-\tau}$ for $\Re(-a/2)+k>-1$, we have $\|E_k(F)\|_{L^2}\lesssim\|F\|_{L^2}$ as desired. ∎

The following proposition follows easily by standard complex interpolation.

Proposition 7.7. *For every smooth tensorfield F and any $b \ge a \ge 0$,*

$$\|\Lambda^a F\|_{L_2} \lesssim \|\Lambda^b F\|_{L^2}^{a/b} \|F\|_{L^2}^{1-a/b} \tag{49}$$

$$\|\mathcal{D}^a F\|_{L_2} \lesssim \|\mathcal{D}^b F\|_{L^2}^{a/b} \|F\|_{L^2}^{1-a/b} \tag{50}$$

We next estblish a comparison between $\|\mathcal{D}^a F\|_{L^2}$ and $\|\Lambda^a F\|_{L^2}$.

Proposition 7.8. *For every $a \ge 0$ and every smooth tensorfield F we have,*

$$\|\mathcal{D}^a F\|_{L^2} \lesssim \|\Lambda^a F\|_{L^2} \lesssim \|\mathcal{D}^a F\|_{L^2} + \|F\|_{L^2}$$

Proof : Indeed, according to the expansion (48), we have for $k \in \mathbb{N}$ for which $2k \le \Re(a) < 2k+2$,

$$\|(\Lambda^a - \mathcal{D}^a)F\|_{L^2} \lesssim \sum_{i=1}^{k} \|\mathcal{D}^{a-2i} F\|_{L^2}$$

Thus, in view of the interpolation formulas of proposition 7.7,

$$\|\Lambda^a F\|_{L^2} \quad \le \quad \|\mathcal{D}^a F\|_{L^2} + \sum_{i=1}^{k} \|\mathcal{D}^{a-2i} F\|_{L^2} \lesssim \|\mathcal{D}^a F\|_{L^2} + \|F\|_{L^2}$$

To prove the remaining estimate, $\|\mathcal{D}^a F\|_{L^2} \lesssim \|\Lambda^a F\|_{L^2}$ it suffices to prove that, the operators $\Lambda^{-a}\mathcal{D}^a$ are bounded in L^2. Observe that $\Lambda^{-2}\mathcal{D}^2 = I - \Lambda^{-2}$. Thus, $\Lambda^{-2}\mathcal{D}^2$ is bounded. On the other hand, since the operators Λ^a and $\mathcal{D}^a$ are selfadjoint and commute with each other,

$$\|\Lambda^{-a} \cdot \mathcal{D}^a F\|_{L^2}^2 = < \Lambda^{-2a} \cdot \mathcal{D}^{2a} F, F > \lesssim \|\Lambda^{-2a} \cdot \mathcal{D}^{2a} F\|_{L^2} \cdot \|F\|_{L^2}$$

Thus $\Lambda^{-a}\mathcal{D}^a$ is bounded in L^2 if $\Lambda^{-2a}\mathcal{D}^{2a}$ is. On the other hand if $\Lambda^{-a} \cdot \mathcal{D}^a$, $\Lambda^{-b} \cdot \mathcal{D}^b$ are bounded in L^2 so is $\Lambda^{-a-b} \cdot \mathcal{D}^{a+b}$. Thus, since we already know that $\Lambda^{-2}\mathcal{D}^2$ is L^2 bounded, we easily infer that $\Lambda^{-a}\mathcal{D}^a$ are all bounded for all positive numbers of the form $m2^{-k}$, $m, k \in \mathbb{Z}$. The general statement follows now by a limiting argument. ■

We are now ready to define Sobolev norms as follows.

Definition 7.9. For positive values of a we set,

$$\|F\|_{H^a(S)} = \|\Lambda^a F\|_{L^2(S)} \approx \left(\|\mathcal{D}^a F\|_{L^2(S)}^2 + \|F\|_{L^2(S)}^2\right)^{\frac{1}{2}}$$

In the next theorem we give a characterization of the Sobolev norm defined above with the help of LP projections. The proof depends heavily on the following lemma:

Lemma 7.10. *For all values of $a \in \mathbb{C}$ and any family of LP projections P_k with symbol m there exists another family of LP projection $\bar{P}_k$, with symbol $\bar{m} = m * j_{a/2}$, such that,*

$$P_k \mathcal{D}^a F = \mathcal{D}^a P_k F = 2^{2ak} \bar{P}_k F.$$

Proof : Since the statement is clearly true for even positive integers it suffices to check it for $\Re(a) < 0$. In this case,

$$\begin{aligned}
\mathcal{D}^a P_k F &= \frac{1}{\Gamma(-a/2)} \int_0^\infty \tau^{-a/2-1} U(\tau) P_k F d\tau \\
&= \frac{1}{\Gamma(-a/2)} \int_0^\infty \int_0^\infty \tau_1^{-a/2-1} m_k(\tau_2) U(\tau_1+\tau_2) F d\tau_1 d\tau_2 \\
&= \int_0^\infty J_k(\lambda) U(\lambda) F d\lambda
\end{aligned}$$

where

$$\begin{aligned}
J_k(\lambda) &= \frac{1}{\Gamma(-a/2)} \int_0^\lambda m_k(\tau)(\lambda-\tau)^{-a/2-1} d\tau \\
&= \frac{1}{\Gamma(-a/2)} 2^{2k} \int_0^\lambda m(2^{2k}\tau)(\lambda-\tau)^{-a/2-1} d\tau \\
&= \frac{1}{\Gamma(-a/2)} \int_0^{2^{2k}\lambda} m(x)(\lambda - 2^{-2k}x)^{-a/2-1} dx \\
&= 2^{ak} 2^{2k} \frac{1}{\Gamma(-a/2)} \int_0^{2^{2k}\lambda} m(x)(2^{2k}\lambda - x)^{-a/2-1} dx = 2^{ak} 2^{2k} \bar{m}(2^{2k}\lambda)
\end{aligned}$$

and

$$\bar{m}(\lambda) = \frac{1}{\Gamma(-a/2)} \int_0^\lambda m(x)(\lambda - x)^{-a/2-1} dx = m * j_{a/2}(\lambda),$$

is clearly a symbol in $\mathcal{M}$. Therefore,

$$\mathcal{D} P_k F = 2^{2ak} \bar{P}_k F$$

as desired. ■

Theorem 7.11.
i.) *Consider the LP projections P_k associated to an arbitrary $m \in \mathcal{M}$. Then, for any $a \geq 0$ and any smooth tensorfield F,*

$$\sum_k 2^{2ak} \|P_k F\|_{L^2}^2 \quad \lesssim \quad \|\mathcal{D}^a F\|_{L^2}^2 \tag{51}$$

ii.) *If in addition the LP-projections P_k verify:*

$$\sum_k P_k^2 = I \tag{52}$$

then, for[6] $0 \leq a < 2$,

$$\|\mathcal{D}^a F\|_{L^2}^2 \quad \lesssim \quad \sum_k 2^{2ak} \|P_k F\|_{L^2}^2 \tag{53}$$

Proof : For $a = 0$ part i) is nothing else but the Bessel inequality established earlier. To prove (51) for all $a > 0$. we make use of lemma 7.10. Let P_k an arbitrary family of LP projections acccording with symbol $m \in \mathcal{M}$. Let $\tilde{P}_k$ be the LP-family

[6]In fact the estimate holds true for large a provided that sufficiently many moments of the symbol m of P_k's vanish, see remark 5.6.

defined by the symbol $\bar{m} = m * j_{-a/2}$. In view of lemma 7.10 $\tilde{P}_k \mathcal{D}^a F = 2^{ak} P'_k F$ with the corresponding symbols $\tilde{m}$ and m' verifying:

$$m' = \tilde{m} * j_{a/2} = (m * j_{-a/2}) * j_{a/2} = m * (j_{-a/2} * j_{a/2}) = m * \delta = m.$$

Therefore $\tilde{P}_k \mathcal{D}^a F = 2^{ak} P_k F$ and consequently, using Stein-Cotlar lemma as in the proof of part iii) of theorem 5.5,

$$\begin{aligned}
\sum_k \|2^{ak} P_k F\|_{L^2}^2 &= \sum_k \|\tilde{P}_k \mathcal{D}^a F\|_{L^2}^2 =< (\sum_k \tilde{P}_k^2)\, \mathcal{D}^a F\, ,\, \mathcal{D}^a F > \\
&\le \|(\sum_k \tilde{P}_k^2)\, \mathcal{D}^a F\|_{L^2} \|\mathcal{D}^a F\|_{L^2} \lesssim \|\mathcal{D}^a F\|_{L^2}^2
\end{aligned}$$

as desired.

To prove part ii) we observe that, if

$$\|G\|_{L^2}^2 =< \sum_k P_k^2 G, G >= \sum_k \|P_k G\|_{L^2}^2$$

Thus, using lemma 7.10 once more,

$$\|\mathcal{D}^a F\|_{L^2}^2 = \sum_k \|P_k \mathcal{D}^a F\|_{L^2}^2 = \sum_k 2^{2ak} \|\bar{P}_k F\|_{L^2}^2$$

It remains to prove that,

$$\sum_k 2^{2ak} \|\bar{P}_k F\|_{L^2}^2 \lesssim \sum_k 2^{2ak} \|P_k F\|_{L^2}^2 \tag{54}$$

To show this we proceed as follows, with the help of the almost orthogonality estimate $\|P_{k'} \bar{P}_k G\|_{L^2} \lesssim 2^{-2|k-k'|} \|G\|_{L^2}$. Thus setting $J^2 = \sum_k 2^{2ak} \|\bar{P}_k F\|_{L^2}^2$

$$\begin{aligned}
J^2 &= \sum_k 2^{2ak} < \bar{P}_k^2 F, F >= \sum_{k,k'} 2^{2ak} < \bar{P}_k^2 F\, ,\, P_{k'}^2 F > \\
&= \sum_{k,k'} 2^{2ak} < P_{k'} \bar{P}_k P_{k'} F\, ,\, \bar{P}_k F > \lesssim \sum_{k,k'} 2^{2ak} \|P_{k'} \bar{P}_k P_{k'} F\|_{L^2} \cdot \|\bar{P}_k F\|_{L^2} \\
&\lesssim \sum_{k,k'} 2^{2ak} 2^{-2|k-k'|} \|P_{k'} F\|_{L^2} \cdot \|\bar{P}_k F\|_{L^2} \\
&\lesssim \sum_{k,k'} 2^{a(k-k')} 2^{-2|k-k'|} (2^{ak'} \|P_{k'} F\|_{L^2}) \cdot (2^{ak} \|\bar{P}_k F\|_{L^2}) \\
&\lesssim \big(\sum_{k'} 2^{2ak'} \|P_{k'} F\|_{L^2}^2\big)^{\frac12} \big(\sum_k 2^{2ak} \|\bar{P}_k F\|_{L^2}^2\big)^{\frac12} = J \cdot \big(\sum_{k'} 2^{2ak'} \|P_{k'} F\|_{L^2}^2\big)^{\frac12}
\end{aligned}$$

and thus,

$$J \lesssim \big(\sum_{k'} 2^{2ak'} \|P_{k'} F\|_{L^2}^2\big)^{\frac12}$$

as desired. ∎

As a corollary to theorem 7.11 and proposition 7.8 we derive:

Corollary 7.12. *For an arbitrary LP projection, $a \ge 0$ and any smooth tensor F we have,*

$$\sum_{k\ge 0} 2^{2ak}\|P_kF\|_{L^2}^2 \le \|\Lambda^a F\|_{L^2}^2$$

Moreover, if $\sum_k P_k^2 = I$,

$$\|\Lambda^a F\|_{L^2}^2 \lesssim \sum_{k\ge 0} 2^{2ak}\|P_kF\|_{L^2}^2 + \|F\|_{L^2}^2$$

8. Besov spaces

In the last section we have defined invariant Sobolev norms using the fractional integral operators $\mathcal{D}^a, \Lambda^a$ and then characterized them with the help of the LP projections. In this section we define invariant Besov spaces using directly the LP projections P_k.

Definition 8.1. Consider the LP projections associated to a fixed $m \in \mathcal{M}$ such that, $\sum_k P_k^2 = I$ and define the Besov norms, for $0 \le a < 2$,

$$\|F\|_{B^a_{p,q}} = \big(\sum_{k\ge 0} 2^{aqk}\|P_kF\|_{L^p}^q\big)^{\frac{1}{q}} + \|F\|_{L^p} \tag{55}$$

Proposition 8.2. *Let the LP projections P_k verify $\sum_k P_k^2 = I$ and consider the* (55) *defined relative to them. Let $\tilde{P}_k$ any family of LP-projections associated to an arbitrary $\tilde{m} \in \mathcal{M}$. Then, for every $0 \le a \le 1$,*

$$\sum_{k\ge 0} 2^{ak}\|\tilde{P}_kF\|_{L^2} \quad \lesssim \quad \|F\|_{B^a_{2,1}} \tag{56}$$

Proof : We shall use the fact that, in view of the almost orthogonality property iii) of Theorem 5.5 of the P_k's we have $\|P_{k'}\tilde{P}_kG\|_{L^2} \lesssim 2^{-2|k-k'|}\|G\|_{L^2}$. In particular,

$$\|P_k\tilde{P}_{<0}G\|_{L^2} \lesssim 2^{-2k}\|G\|_{L^2},$$

Now,

$$\begin{aligned}
\sum_{k\ge 0} 2^{k\alpha}\|\tilde{P}_kF\|_{L^2} \quad &\le \quad \sum_{k,k'\ge 0} 2^{k\alpha}\|\tilde{P}_kP_{k'}^2F\|_{L^2} + \sum_{k\ge 0} 2^{k\alpha}\|\tilde{P}_kP_{<0}^2F\|_{L^2}\\
&= \quad \sum_{k,k'\ge 0} 2^{k\alpha}\|P_{k'}\tilde{P}_kP_{k'}F\|_{L^2} + \sum_{k\ge 0} 2^{k\alpha}\|P_{<0}\tilde{P}_kP_{<0}F\|_{L^2}\\
&\lesssim \quad \sum_{k,k'\ge 0} 2^{k\alpha}2^{-2|k-k'|}\|P_{k'}F\|_{L^2} + \sum_{k\ge 0} 2^{k(\alpha-1)}\|P_{<0}F\|_{L^2}\\
&\lesssim \quad \sum_{k'\ge 0} 2^{k'\alpha}\|P_{k'}F\|_{L^2} + \|F\|_{L^2} = \|F\|_{B^\alpha_{2,1}}
\end{aligned}$$

as desired. ■

According to corollary 7.12 the norms $B^a_{2,2}$ are equivalent to the Sobolev norms H^a, for $0 \le a < 2$. For the Besov index 1 we have the obvious inequalities,

Proposition 8.3. *For any smooth tensorfield F,*

$$\begin{aligned} \|F\|_{B^a_{2,1}} &\lesssim \|F\|_{B^b_{2,1}}, \qquad a\le b &(57)\\ \|F\|_{H^a} &\lesssim \|F\|_{B^a_{2,1}}, \qquad 0\le a &(58)\\ \|F\|_{B^a_{2,1}} &\lesssim \|F\|_{H^b}, \qquad 0\le a<b. &(59)\end{aligned}$$

Proposition 8.4. *The following, non-sharp, Sobolev inequality holds true with $2<p<\infty$, $a=1-\frac{2}{p}$ and any tensorfield F,*

$$\|F\|_{L^p}\lesssim \|F\|_{B^a_{2,1}} \tag{60}$$

Proof : We write $F=\sum_{k\ge 0}P_kF+P_{<0}F$. Thus, in view of the L^p Bernstein inequality,

$$\|F\|_{L^p} \le \sum_{k\ge 0}\|P_kF\|_{L^p}+\|P_{<0}F\|_{L^p}\lesssim \sum_{k\ge0}2^{k(1-\frac{2}{p})}\|P_kF\|_{L^2}+\|F\|_{L^2}\lesssim\|F\|_{B^\alpha_{2,1}}$$

■

9. LP - decompositions and product estimates

Let P_k the geometric LP projections associate to an $m\in\mathcal{M}_\ell$. We also assume that $\sum_k P_k=I$. Given a tensorfield F we write, for a given $k\in\mathbb{Z}$

$$F=P_{<k}F+P_{\ge k}F \tag{61}$$

where $P_{<k}=\sum_{l<k}P_l$, $P_{\ge k}=\sum_{l\ge k}P_l$. Given two tensors F,g and $F\cdot g$ some geometric product between them we decompose,

$$F\cdot g = P_{\ge k}F\cdot P_{\ge k}G+P_{<k}F\cdot P_{<k}G+P_{<k}F\cdot P_{\ge k}G+P_{\ge k}F\cdot P_{<k}G$$

Thus,

$$\begin{aligned} P_k(F\cdot G) &= \pi_k(F,G)+\sigma_k(F,G)+\rho_k(F,G) &(62)\\ \pi_k(F,G) &= P_k\big(P_{\ge k}F\cdot P_{\ge k}G\big)\\ \sigma_k(F,G) &= P_k\big(P_{<k}F\cdot P_{<k}G\big)\\ \rho_k(F,G) &= P_k\big(P_{<k}F\cdot P_{\ge k}G\big)+P_k\big(P_{\ge k}F\cdot P_{<k}G\big)\end{aligned}$$

Observe that for the classical LP theory, based on the Fourier transform, the terms σ_k and ρ_k are absent. Unfortunately this is not the case for our definition of geometric LP-projections. We shall see however that the presence of such terms does not in any way affect the main results that can be obtained by the standard LP-theory. In what follows we shall apply the decomposition (62) to prove a geometric version of the classical Sobolev and Besov norm multiplication estimates. We start with the following

Lemma 9.1. *Let $F,G\in H^1$ and consider* (62)*. Then, the high-high interaction term $\pi_k(F,G)$ verifies,*

$$\sum_{k\ge0}2^k\|\pi_k\|_{L^2}\lesssim\|F\|_{H^1}\|G\|_{H^1}$$

Proof : For $k\ge 0$ we write, $\pi_k=\pi_k^1+\pi_k^2$ where,

$$\pi_k^1=\sum_{k<m'<m}P_k\big(P_{m'}F\cdot P_mG\big),\qquad \pi_k^2=\sum_{k<m'<m}P_k\big(P_mF\cdot P_{m'}G\big)$$

By symmetry it suffices to estimate π_k^1. Using first the dual weak Bernstein inequality for some sufficiently large $p<\infty$, followed by Cauchy -Schwartz and then again the direct weak Bernstein, we obtain for any $k\ge 0$, with $q^{-1}+2^{-1}={p'}^{-1}=1-p^{-1}$,

$$\begin{aligned}
\|\pi_k^1\|_{L^2} &\lesssim \sum_{k<m'<m}\|P_k\big(P_{m'}F\cdot P_mG\big)\|_{L^2}\\
&\lesssim 2^{\frac{2k}{q}}\sum_{k<m'<m}\|P_{m'}F\cdot P_mG\|_{L^{p'}}\\
&\lesssim \sum_{k<m'<m}2^{-m-m'}2^{\frac{2k}{q}}\|2^{m'}P_{m'}F\|_{L^q}\|2^mP_mG\|_{L^2}\\
&\lesssim \sum_{k<m'<m}2^{-m}2^{\frac{2k}{q}}2^{-m'\frac{2}{q}}\|2^{m'}P_{m'}F\|_{L^2}\|2^mP_mG\|_{L^2}
\end{aligned}$$

Thus, in view of the proposition 6.1

$$\begin{aligned}
\sum_{k\ge 0}2^k\|\pi_k^1\|_{L^2} &\lesssim \sum_k\sum_{k<m'<m}2^{-m+m'}2^{(k-m')(1+\frac{2}{q})}\|2^{m'}P_{m'}F\|_{L^2}\|2^mP_mG\|_{L^2}\\
&\lesssim \sum_{0\le m'<m}2^{-|m-m'|}\|2^{m'}P_{m'}F\|_{L^2}\|2^mP_mG\|_{L^2}\\
&\lesssim \|F\|_{H^1}\|G\|_{H^1}
\end{aligned}$$

∎

We are now ready to prove the following product estimates.

Proposition 9.2. *Let $\alpha,\alpha',\beta,\beta'\in(0,1)$ such that $\alpha+\beta=\alpha'+\beta'=1$. Then for all tensorfields f,g and any $0\le\gamma<1$,*

$$\|F\cdot G\|_{B^\gamma_{2,1}}\lesssim \|\Lambda^{\alpha+\gamma}F\|_{L^2}\|\Lambda^\beta G\|_{L^2}+\|\Lambda^{\alpha'}F\|_{L^2}\|\Lambda^{\beta'+\gamma}G\|_{L^2}\tag{63}$$

Proof : Observe that the low frequency part $\|P_{<0}(F\cdot G)\|_{L^2}$ can be trivially estimated in view of the dual version of the weak Bernstein inequality with $q^{-1}+2^{-1}={p'}^{-1}=1-p^{-1}$ for some sufficiently large p,

$$\|P_{<0}(F\cdot G)\|_{L^2}\lesssim\|F\cdot G\|_{L^{p'}}\lesssim\|F\|_{L^q}\|G\|_{L^2}$$

followed by the Sobolev embedding (60) with $\alpha>\frac{2}{p}$,

$$\|F\|_{L^q}\lesssim\|F\|_{B^{\frac{2}{p}}_{2,1}}\lesssim\|\Lambda^\alpha F\|_{L^2}$$

Consider now the high frequency part $\sum_{k\ge0}\|P_k(F\cdot G)\|_{L^2}$. Decomposing as in (62) we write

$$P_k(F\cdot g)=\pi_k(F,G)+\sigma_k(F,G)+\rho_k(F,G)$$

The estimates for the high-high interaction term $\pi_k=\pi_k^1+\pi_k^2$, $k\ge0$ are as follows: For $k\ge0$ we write, $\pi_k=\pi_k^1+\pi_k^2$; by symmetry it suffices to estimate π_k^1. Using first the dual weak Bernstein inequality for some sufficiently large $p<\infty$, followed by Cauchy -Schwartz and the the direct L^p Bernstein, we obtain for any $k \geq 0$, with $q^{-1} + 2^{-1} = p'^{-1} = 1 - p^{-1}$,

$$
\begin{aligned}
\|\pi_k^1\|_{L^2} &\lesssim \sum_{k<m'<m} \|P_k\big(P_{m'}F \cdot P_m G\big)\|_{L^2} \\
&\lesssim 2^{\frac{2k}{q}} \sum_{k<m'<m} \|P_{m'}F \cdot P_m G\|_{L^{p'}} \\
&\lesssim \sum_{k<m'<m} 2^{-\alpha m' - \beta m} 2^{\frac{2k}{q}} \|2^{m'\alpha} P_{m'}F\|_{L^q} \|2^{m\beta} P_m G\|_{L^2} \\
&\lesssim \sum_{k<m'<m} 2^{-\alpha m' - \beta m} 2^{\frac{2k}{q}} 2^{m'(1-\frac{2}{q})} \|2^{m'\alpha} P_{m'}F\|_{L^2} \|2^{m\beta} P_m G\|_{L^2} \\
&\lesssim \sum_{k<m'<m} 2^{-\beta(m-m')} 2^{\frac{2(k-m')}{q}} \|2^{m'\alpha} P_{m'}F\|_{L^2} \|2^{m\beta} P_m G\|_{L^2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{k\geq 0} 2^{k\gamma} \|\pi_k^1\|_{L^2} &\lesssim \sum_{k<m'<m} 2^{-\beta(m-m')} 2^{k\gamma} 2^{\frac{2(k-m')}{q}} \|2^{m'\alpha} P_{m'}F\|_{L^2} \|2^{m\beta} P_m G\|_{L^2} \\
&\lesssim \sum_{m'<m} 2^{-\beta(m-m')} \|2^{m'(\alpha+\gamma)} P_{m'}F\|_{L^2} \|2^{m\beta} P_m G\|_{L^2} \\
&\lesssim \|\Lambda^{a+\gamma} F\|_{L^2} \|\Lambda^\beta G\|_{L^2}
\end{aligned}
$$

since $\beta > 0$.

Consider now, $\sigma_k(F, G) = P_k\big(P_{<k}F \cdot P_{<k}G\big) = \sigma_k^1 + \sigma_k^2$,

$$
\sigma_k^1(F, G) = \sum_{k'<k''<k} P_{k'}F \cdot P_{k''}G, \qquad \sigma_k^2(F, G) = \sum_{k''\leq k'<k} P_{k'}F \cdot P_{k''}G.
$$

By symmetry it suffices to estimate σ_k^1. Using the L^2 finite band condition followed by the dual weak Bernstein inequality for $p > 2$ sufficiently close to $p = 2$ and the direct L^p Bernstein, we estimate[7] with $q^{-1} + 2^{-1} = p'^{-1}$ as in the case of π_k,

$$
\begin{aligned}
\sigma_k^1(F, G) &\lesssim \sum_{k'<k''<k} 2^{-k(1-\frac{2}{q})} \|P_{k'}F\|_{L^q} \|\nabla P_{k''}G\|_{L^2} \\
&\lesssim \sum_{k'<k''<k} 2^{k''} 2^{k'(1-\frac{2}{q})} 2^{-k(1-\frac{2}{q})} \|P_{k'}F\|_{L^2} \|P_{k''}G\|_{L^2} \\
&\lesssim \sum_{k'<k''<k} 2^{k''} 2^{(k'-k)(1-\frac{2}{q})} 2^{-\alpha k' - \beta k''} \|2^{k'\alpha} P_{k'}F\|_{L^2} \|2^{k''\beta} P_{k''}G\|_{L^2}
\end{aligned}
$$

[7]We consider only the case when the derivative affects the higher frequency; the other case is simpler.

Summing over k we obtain

$$
\begin{aligned}
\sum_k 2^{k\gamma}\|\sigma_k(F,\,G\,)\|_{L^2} &\lesssim \sum_k \sum_{k'<k''<k} 2^{k\gamma} 2^{(k'-k)(1-\frac{2}{q})} 2^{\alpha(k''-k')} \|2^{k'\alpha} P_{k'}F\|_{L^2} \|2^{k''\beta} P_{k''}G\|_{L^2}\\
&\lesssim \sum_{k'<k''} 2^{k''\gamma} 2^{(k'-k'')(1-\frac{2}{q})} 2^{\alpha(k''-k')} \|2^{k'\alpha} P_{k'}F\|_{L^2} \|2^{k''\beta} P_{k''}G\|_{L^2}\\
&\sum_{k'<k''} 2^{(k'-k'')(\beta-\frac{2}{q})} \|2^{k'\alpha} P_{k'}F\|_{L^2} \|2^{k''(\beta+\gamma)} P_{k''}G\|_{L^2}\\
&\lesssim \|\Lambda^\alpha F\|_{L^2} \|\Lambda^{\beta+\gamma} G\|_{L^2}
\end{aligned}
$$

provided that $\beta > \frac{2}{q}$, which can be ensured by the choice of q, as long as $\beta > 0$.

We now estimate $\rho_k(F,\,G\,) = P_k\big(P_{<k}F \cdot P_{>k}G\big) + P_k\big(P_{>k}F \cdot P_{<k}G\big) = \rho_k^1 + \rho_k^2$. By symmetry it suffices to estimate $\rho_k^1 = \sum_{k'<k<m} P_k\big(P_{k'}F \cdot P_m G\big)$. Arguing as in the estimate for σ_k we use the dual weak Bernstein inequality followed by Cauchy-Schwartz and the L^p Bernstein inequality, we obtain with $q^{-1} + 2^{-1} = {p'}^{-1}$ for a sufficiently large value of q,

$$
\begin{aligned}
\|\rho_k^1\|_{L^2} &\lesssim \sum_{k'<k<m} \|P_k\big(P_{k'}F \,\cdot\, P_m G\big)\|_{L^2} \lesssim \sum_{k'<k<m} 2^{k'} 2^{\frac{2k}{q}} \|P_k\big(P_{k'}F \,\cdot\, P_m G\big)\|_{L^{p'}}\\
&\lesssim 2^{\frac{2k}{q}} 2^{k'(1-\frac{2}{q})} \|P_{k'}F\|_{L^2} \|P_m G\|_{L^2}\\
&\lesssim \sum_{k'<k<m} 2^{(k-k')\frac{2}{q}} 2^{\beta(k'-m)} \|2^{k'\alpha} P_{k'}F\|_{L^2} \|2^{m\beta} P_m G\|_{L^2}
\end{aligned}
$$

Now summing over k,

$$
\begin{aligned}
\sum_k 2^{k\gamma}\|\rho_k^1\|_{L^2} \quad &\lesssim \quad \sum_k \sum_{k'<k<m} 2^{k\gamma} 2^{(k-k')\frac{2}{q}} 2^{\beta(k'-m)} \|2^{k'\alpha} P_{k'}F\|_{L^2} \|2^{m\beta} P_m G\|_{L^2}\\
&\lesssim \quad \sum_{k'<m} 2^{(\beta-\frac{2}{q})(k'-m)} \|2^{k'\alpha} P_{k'}F\|_{L^2} \|2^{m(\beta+\gamma)} P_m G\|_{L^2}\\
&\lesssim \quad \|\Lambda^\alpha F\|_{L^2} \|\Lambda^{\beta+\gamma} G\|_{L^2}
\end{aligned}
$$

we obtain the desired estimate providied that $\beta > \frac{2}{q}$, which can be satisfied by the choice of q, as long as $\beta > 0$. The corresponding estimate for ρ_k^2 requires the condition that $\alpha > 0$.

■

10. The sharp Bernstein inequality

In this section we shall prove the geometric version of the Bernstein inequality for arbitrary tensorfields on M. The inequality requires additional assumptions on the Gauss curvature K of the manifold M. We shall introduce the following L^2- norms depending on K,

$$K_\gamma := \|\Lambda^{-\gamma} K\|_{L^2} \tag{64}$$

with $0 \le \gamma < 1$.

Theorem 10.1. *Let S be a 2-d weakly regular mannifold with Gauss curvature K.* **i.)** *For any scalar function f on S, $0 \le \gamma < 1$, any $k \ge 0$, and an arbitrary $2 \le p < \infty$,*

$$\|P_k f\|_{L^\infty} \lesssim 2^k\big(1 + 2^{-\frac{k}{p}}\big(K_\gamma^{\frac{1}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big) + 1\big)\|f\|_{L^2}, \tag{65}$$

$$\|P_{<0} f\|_{L^\infty} \lesssim \big(1 + K_\gamma^{\frac{2}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big)\|f\|_{L^2} \tag{66}$$

ii.) *For any tensorfield F on S, any $k \ge 0$, and an arbitrary $2 \le p < \infty$,*

$$\|P_k F\|_{L^\infty} \lesssim 2^k\big(1 + 2^{-\frac{k}{p}} K_0^{\frac{1}{p}} + 2^{-k\frac{1}{p-1}} K_0^{\frac{1}{p-1}}\big)\|F\|_{L^2}, \tag{67}$$

$$\|P_{<0} F\|_{L^\infty} \lesssim \big(1 + (K_0^{\frac{1}{p}} + K_0^{\frac{1}{2p}}) + K_0^{\frac{1}{p-1}}\big)\|F\|_{L^2}. \tag{68}$$

Proof : The proof is based on an argument involving the product estimates developed in the previous section.

In view of the estimate (10), we have for $k \ge 0$,

$$\begin{aligned}
\|P_k F\|_{L^\infty} &\lesssim \|\nabla^2 P_k F\|_{L^2}^{\frac{1}{p}}\big(\|\nabla P_k F\|_{L^2}^{\frac{p-2}{p}}\|P_k F\|_{L^2}^{\frac{1}{p}} + \|P_k F\|_{L^2}^{\frac{p-1}{p}}\big) + \|\nabla P_k F\|_{L^2}\\
&\lesssim 2^{k\frac{p-2}{p}}\|\nabla^2 P_k F\|_{L^2}^{\frac{1}{p}}\|F\|_{L^2}^{\frac{p-1}{p}} + 2^k\|F\|_{L^2}
\end{aligned} \tag{69}$$

It remains to estimate the quantity $\|\nabla^2 P_k F\|_{L^2}$. We do this with the help of the Böchner identity,

10.2. **Scalar Case.** Recall that the Bochner identity for scalars has the form,

$$\int_S |\nabla^2 g|^2 = \int_S |\Delta g|^2 - \int_S K|\nabla g|^2.$$

With the help of the product estimates developed in the previous section with the following choice of parameters $\alpha = 1-\gamma$, $\beta = \gamma$ and $\alpha' = \gamma$, $\beta' = 1-\gamma$, we estimate,

$$\begin{aligned}
\int_S K|\nabla g|^2 &= \int_S (\Lambda^{-\gamma}K)(\Lambda^\gamma|\nabla g|^2)\\
&\le K_\gamma\|\Lambda^\gamma|\nabla g|^2\|_{L^2} \lesssim K_\gamma\|\nabla g\cdot\nabla g\|_{B^\gamma_{2,1}}\\
&\lesssim K_\gamma\|\Lambda\nabla g\|_{L^2}\|\Lambda^\gamma\nabla g\|_{L^2} \lesssim K_\gamma\|\Lambda\nabla g\|_{L^2}^{1+\gamma}\|\nabla g\|_{L^2}^{1-\gamma}
\end{aligned}$$

The last inequality follows from the condition that $\gamma \le 1$ and the interpolation inequality (49). Since, $\|\Lambda\nabla g\|_{L^2}^2 \lesssim \int_S |\nabla^2 g|^2 + \int_S |\nabla g|^2$ we infer that

$$\int_S K|\nabla g|^2 \le \frac{1}{2}\int_S |\nabla^2 g|^2 + (K_\gamma^{\frac{2}{1-\gamma}} + K_\gamma)\int_S |\nabla g|^2$$

Therefore,

$$\int_S |\nabla^2 g|^2 \le \int_S |\Delta g|^2 + \frac{1}{2}\int_S |\nabla^2 g|^2 + \big(K_\gamma^{\frac{2}{1-\gamma}} + K_\gamma\big)\int_S |\nabla g|^2$$

This implies

$$\int_S |\nabla^2 g|^2 \lesssim \int_S |\Delta g|^2 + \big(K_\gamma^{\frac{2}{1-\gamma}} + K_\gamma\big)\int_S |\nabla g|^2 \tag{70}$$

Applying (70) to $g = P_k f$ and using the inequalities

$$\|\nabla P_k f\|_{L^2} \le 2^k \|f\|_{L^2}, \qquad \|\Delta P_k f\|_{L^2} \le 2^{2k}\|f\|_{L^2}$$

we obtain

$$\|\nabla^2 P_k f\|_{L^2} \lesssim \big(2^{2k} + 2^k\big(K_\gamma^{\frac{1}{1-\gamma}} + K_\gamma^{\frac12}\big) + 1\big)\|f\|_{L^2} \tag{71}$$

Combining (71) with (69), yields

$$\|P_k f\|_{L^\infty} \lesssim 2^k\big(1 + 2^{-\frac{k}{p}}\big(K_\gamma^{\frac{2}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big) + 1\big)\|f\|_{L^2}, \tag{72}$$

$$\|P_{<0} f\|_{L^\infty} \lesssim \big(1 + K_\gamma^{\frac{2}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big)\|f\|_{L^2} \tag{73}$$

as desired.

10.3. **Tensor case.** We recall the Böchner inequality (14) of Corollary **??**,

$$\|\nabla^2 F\|_{L^2} \lesssim \|\Delta F\|_{L^2} + (\|K\|_{L^2} + \|K\|_{L^2}^{\frac12})\|\nabla F\|_{L^2} + \|K\|_{L^2}^{\frac{p}{p-1}}\big(\|\nabla F\|_{L^2}^{\frac{p-2}{p-1}}\|F\|_{L^2}^{\frac{1}{p-1}} + \|F\|_{L^2}\big)$$

Applying this to $P_k F$ we obtain

$$\|\nabla^2 P_k F\|_{L^2} \lesssim \big(2^{2k} + 2^k(K_0 + K_0^{\frac12}) + 2^{k\frac{p-2}{p-1}}K_0^{\frac{p}{p-1}}\big)\|F\|_{L^2} \tag{74}$$

Combining (74) with (69) we derive

$$\|P_k F\|_{L^\infty} \lesssim 2^k\big(1 + 2^{-\frac{k}{p}}(K_0^{\frac1p} + K_0^{\frac{1}{2p}}) + 2^{-k\frac{1}{p-1}}K_0^{\frac{1}{p-1}}\big)\|F\|_{L^2}$$

as desired. ■

11. Sharp product estimates

In this section we prove the sharp version of the product estimates of Proposition 9.2 invloving Besov spaces. These estimates require an additional curvature assumptions which vary from the scalar to the tensor case. The former only needs the bound on the quantity $\|\Lambda^{-\gamma}K\|_{L^2}$, while the latter requires the finiteness of $\|K\|_{L^2}$.

Let for $0 \le \gamma < 1$

$$A_\gamma := 1 + K_\gamma^{\frac{1}{2(1-\gamma)}} \tag{75}$$

denote the constants appearing in the sharp Bernstein inequalities (65) and (67).

Proposition 11.1. *Let S be a 2-d weakly regular manifold with Gauss curvature K.*
i.) *For all scalar functions f, g, any $0 \le \alpha < 2$, and an arbitrary $0 \le \gamma < 1$,*

$$\begin{aligned}\|f\cdot g\|_{B^\alpha_{2,1}} \lesssim{}& \|f\|_{B^\alpha_{2,1}}\left(\|g\|_{B^1_{2,1}} + A_\gamma\|g\|_{B^{\frac12}_{2,1}}\right)\\ &+ \|g\|_{B^\alpha_{2,1}}\left(\|f\|_{B^1_{2,1}} + A_\gamma\|f\|_{B^{\frac12}_{2,1}}\right)\end{aligned} \tag{76}$$

ii.) *For all tensorfields F, G, any $0 \le \alpha < 2$, and an arbitrary $2 \le p < \infty$,*

$$\|F \cdot G\|_{B^{\alpha}_{2,1}} \lesssim \|F\|_{B^{\alpha}_{2,1}} \Big(\|G\|_{B^1_{2,1}} + A_0^{\frac{p}{p-1}} \|G\|_{B^{\frac12}_{2,1}} \Big) + \|G\|_{B^{\alpha}_{2,1}} \Big(\|F\|_{B^1_{2,1}} + A_0^{\frac{p}{p-1}} \|F\|_{B^{\frac12}_{2,1}} \Big) \tag{77}$$

Proof : The proof relies on the application of the sharp Bernstein inequalities proved in the previous section. We shall only give the arguments for the scalar inequality (76). The modifications leading to the tensor inequality (77) will be obvious and follow by replacing the scalar Bernstein inequality (65) with its tensorial version (67).

As in the proof of Proposition 9.2 the low frequency part $\|P_{<0}(f \cdot g)\|_{L^2}$ can be trivially estimated by means of the weak Bernstein inequality.

Consider now the high frequency part $\sum_{k \ge 0} \|P_k(f \cdot g)\|_{L^2}$. Decomposing as in (62) we write

$$P_k(f \cdot g) = \pi_k(f, g) + \sigma_k(f, g) + \rho_k(f, g)$$

The estimates for the high-high interaction term $\pi_k = \pi_k^1 + \pi_k^2$, $k \ge 0$ are as follows: For $k \ge 0$ we write, $\pi_k = \pi_k^1 + \pi_k^2$; by symmetry it suffices to estimate π_k^1. Using first the dual weak Bernstein inequality for some sufficiently large $p < \infty$, followed by Cauchy -Schwartz and the the direct L^p Bernstein, we obtain for any $k \ge 0$, with $q^{-1} + 2^{-1} = p'^{-1} = 1 - p^{-1}$,

$$\begin{aligned}
\|\pi_k^1\|_{L^2} &\lesssim \sum_{k < m' < m} \|P_k\big(P_{m'} f \cdot P_m g\big)\|_{L^2} \\
&\lesssim 2^{\frac{2k}{q}} \sum_{k < m' < m} \|P_{m'} f \cdot P_m g\|_{L^{p'}} \\
&\lesssim \sum_{k < m' < m} 2^{\frac{2k}{q}} \|P_{m'} f\|_{L^q} \|P_m g\|_{L^2} \\
&\lesssim \sum_{k < m' < m} 2^{\frac{2k}{q}} 2^{-\frac{2m'}{q})} \|2^{m'} P_{m'} f\|_{L^2} \|P_m g\|_{L^2} \\
&\lesssim \sum_{k < m' < m} 2^{-\frac{2}{q}(m' - k)} \|2^{m'} P_{m'} f\|_{L^2} \|P_m g\|_{L^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k \ge 0} 2^{k\alpha} \|\pi_k^1\|_{L^2} &\lesssim \sum_{k < m' < m} 2^{-\frac{2}{q}(m' - k)} 2^{-\alpha(m-k)} \|2^{m'} P_{m'} f\|_{L^2} \|2^{m\alpha} P_m g\|_{L^2} \\
&\lesssim \sum_{m' < m} 2^{-\alpha(m - m')} \|2^{m'} P_{m'} f\|_{L^2} \|2^{m\alpha} P_m g\|_{L^2} \\
&\lesssim \|f\|_{B^1_{2,1}} \|g\|_{B^{\alpha}_{2,1}}
\end{aligned}$$

Consider now, $\sigma_k(f, g) = P_k\big(P_{<k} f \cdot P_{<k} g\big) = \sigma_k^1 + \sigma_k^2$,

$$\sigma_k^1(f, g) = \sum_{k' < k'' < k} P_{k'} f \cdot P_{k''} g, \qquad \sigma_k^2(f, g) = \sum_{k'' \le k' < k} P_{k'} f \cdot P_{k''} g.$$

By symmetry it suffices to estimate σ_k^1. Using the L^2 finite band condition accordinog to which $\|\sigma_k^1(f,g)\|_{L^2} \lesssim 2^{-2k}\|\Delta\sigma_k^1(f,g)\|_{L^2}$ we decompose

$$\begin{aligned}\Delta\sigma_k^1(f,g) &= P_k \sum_{k'<k''<k} \Big(P_{k'}f\cdot\Delta P_{k''}g + 2\nabla P_{k'}f\cdot\nabla P_{k''}g + \Delta P_{k'}f\cdot P_{k''}g\Big)\\ &= \sigma_k^{11}(f,g)+\sigma_k^{12}(f,g)+\sigma_k^{13}(f,g)\end{aligned}$$

By symmetry it suffices to estimate the terms $\sigma_k^{11},\sigma_k^{12}$. Using the Bernstein inequality we have

$$\begin{aligned}\sigma_k^{11}(f,g) &\lesssim \sum_{k'<k''<k} 2^{-2k}\|P_{k'}f\|_{L^\infty}\|\Delta P_{k''}g\|_{L^2}\\ &\lesssim \sum_{k'<k''<k} 2^{-2k}2^{2k''}\big(2^{k'}+2^{\frac{k'}{2}}A_\gamma\big)\|P_{k'}f\|_{L^2}\|P_{k''}g\|_{L^2}\\ &\lesssim \sum_{k'<k''<k} 2^{-2k}2^{k''(2-\alpha)}\big(\|2^{k'}P_{k'}f\|_{L^2}+A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{k''\alpha}P_{k''}g\|_{L^2}\end{aligned}$$

Summing over k we obtain for $\alpha<2$

$$\begin{aligned}\sum_k 2^{k\alpha}\|\sigma_k^{11}(f,g)\|_{L^2} &\lesssim \sum_k\sum_{k'<k''<k} 2^{-(2-\alpha)(k-k'')}\big(\|2^{k'}P_{k'}f\|_{L^2}+A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{k''\alpha}P_{k''}g\|_{L^2}\\ &\sum_{k'<k''<k}\big(\|2^{k'}P_{k'}f\|_{L^2}+A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{k''\alpha}P_{k''}g\|_{L^2}\\ &\lesssim \big(\|f\|_{B^1_{2,1}}+A_\gamma\|f\|_{B^{\frac12}_{2,1}}\big)\|g\|_{B^\alpha_{2,1}}\end{aligned}$$

To estimate $\sigma_k^{12}(f,g)$ we use the Gagliardo-Nirenberg inequality (8)

$$\|f\|_{L^4}\lesssim \|\nabla f\|_{L^2}^{\frac12}\|f\|_{L^2}^{\frac12}+\|f\|_{L^2}.$$

Using the Gagliardo-Nirenberg estimate[8] followed by the scalar Bochner inequality (70)

$$\begin{aligned}\sigma_k^{12}(f,g) &\lesssim \sum_{k'<k''<k} 2^{-2k}\|\nabla P_{k'}f\|_{L^4}\|\nabla P_{k''}g\|_{L^4}\\ &\lesssim \sum_{k'<k''<k} 2^{-2k}\|\nabla^2 P_{k'}f\|_{L^2}^{\frac12}\|\nabla^2P_{k''}g\|_{L^2}^{\frac12}\|\nabla P_{k'}f\|_{L^2}^{\frac12}\|\nabla P_{k''}g\|_{L^2}^{\frac12}\\ &\lesssim \sum_{k'<k''<k} 2^{-2k}2^{\frac{k'+k''}{2}}\big(2^{k'}+2^{\frac{k'}{2}}A_\gamma^{\frac12}\big)\|P_{k'}f\|_{L^2}\big(2^{k''}+2^{\frac{k''}{2}}A_\gamma^{\frac12}\big)\|P_{k''}g\|_{L^2}\\ &\lesssim \sum_{k'<k''<k} 2^{-2k}2^{\frac{3k''}{2}+\frac{k''}{2}}\big(2^{k'}+2^{\frac{k'}{2}}A_\gamma\big)\|P_{k'}f\|_{L^2}\|P_{k''}g\|_{L^2}\\ &\lesssim \sum_{k'<k''<k} 2^{-2k}2^{k''(2-\alpha)}\big(\|2^{k'}P_{k'}f\|_{L^2}+A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{k''\alpha}P_{k''}g\|_{L^2}\end{aligned}$$

[8]We drop the low order term in the Gagliardo-Nirenberg inequality since we consider the case of high frequencies $k\ge 0$.

As before, summing over k we obtain for $\alpha < 2$

$$\sum_k 2^{k\alpha}\|\sigma_k^{12}(f,g)\|_{L^2} \lesssim \sum_k \sum_{k'<k''<k} 2^{-(2-\alpha)(k-k'')}\big(\|2^{k'}P_{k'}f\|_{L^2} + A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{k''\alpha}P_{k''}g\|_{L^2}$$
$$\sum_{k'<k''<k}\big(\|2^{k'}P_{k'}f\|_{L^2} + A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{k''\alpha}P_{k''}g\|_{L^2}$$
$$\lesssim \big(\|f\|_{B^1_{2,1}} + A_\gamma\|f\|_{B^{\frac12}_{2,1}}\big)\|g\|_{B^\alpha_{2,1}}$$

and the estimate for $\sigma_k(f,g)$ follows.

We now estimate $\rho_k(f,g) = P_k\big(P_{<k}f\cdot P_{>k}g\big) + P_k\big(P_{>k}f\cdot P_{<k}g\big) = \rho_k^1 + \rho_k^2$. By symmetry it suffices to estimate $\rho_k^1 = \sum_{k'<k<m} P_k\big(P_{k'}f\cdot P_m g\big)$.

$$\|\rho_k^1\|_{L^2} \lesssim \sum_{k'<k<m}\|P_{k'}f\|_{L^\infty}\|P_m g\|_{L^2} \lesssim \sum_{k'<k<m}\big(2^{k'} + 2^{\frac{k'}{2}}A_\gamma\big)\|P_k\big(P_{k'}f\cdot P_m g\big)\|_{L^{p'}}$$
$$\lesssim \|P_{k'}f\|_{L^2}\|P_m g\|_{L^2}$$
$$\lesssim \sum_{k'<k<m} 2^{-m\alpha}\big(\|2^{k'}P_{k'}f\|_{L^2} + A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{m\alpha}P_m g\|_{L^2}$$

Now summing over k,

$$\sum_k 2^{k\alpha}\|\rho_k^1\|_{L^2} \lesssim \sum_k\sum_{k'<k<m} 2^{-\alpha(m-k)}\big(\|2^{k'}P_{k'}f\|_{L^2} + A_\gamma\|2^{\frac{k'}{2}}P_{k'}f\|_{L^2}\big)\|2^{m\alpha}P_m g\|_{L^2}$$
$$\lesssim \big(\|f\|_{B^1_{2,1}} + A_\gamma\|f\|_{B^{\frac12}_{2,1}}\big)\|g\|_{B^\alpha_{2,1}}$$

we obtain the desired estimate ■

12. Operator ∇ on $B^1_{2,1}$ space

Motivated by classical considerations we expect the operator of covariant differentiation ∇ to act continuously in the scale of Besov spaces: $\nabla : B^s_{2,1} \to B^{s-1}_{2,1}$ for any $s \ge 1$. The weak regularity assumptions which we impose on the geometry of a surface S gives hope to prove this mapping property only for sufficiently low values of s. In this section we shall show this for the particular lowest value $s = 1$. Moreover, as in the case of the Böchner and sharp Bernstein inequalities, the regularity assumptions needed to prove the result differ drastically dependent on whether ∇ is considered on the space of scalar functions or tensorfields.

Proposition 12.1. *Let S be a 2-d weakly regular surface with Gauss curvature K and let the constants A_γ be as in* (75).

i.) *For all scalar functions f and an arbitrary $0 \le \gamma < 1$*

$$\|\nabla f\|_{B^0_{2,1}} \lesssim \|f\|_{B^1_{2,1}} + A_\gamma^2\|f\|_{B^0_{2,1}}. \tag{78}$$

ii.) *For all tensorfields F and an arbitrary $2 \le p < \infty$*

$$\|\nabla F\|_{B^0_{2,1}} \lesssim \|f\|_{B^1_{2,1}} + A_0^{\frac{2p}{p-1}}\|f\|_{B^0_{2,1}}. \tag{79}$$

Proof : Once again we shall only provide the arguments in the scalar case. The proof of part **ii.)** is similar and relies on the tensor Böchner inequality (14).

We consider

$$\begin{aligned} \|\nabla f\|_{B^0_{2,1}} &= \sum_k \|P_k \nabla f\|_{L^2} \lesssim \sum_\ell \sum_k \|P_k \nabla P_\ell f\|_{L^2} \\ &= \sum_\ell \sum_{k\le\ell} \|P_k \nabla P_\ell f\|_{L^2} + \sum_\ell \sum_{k>\ell} \|P_k \nabla P_\ell f\|_{L^2} \end{aligned}$$

Using the dual finite band property we obtain

$$\begin{aligned} \sum_\ell \sum_{k\le\ell} \|P_k \nabla P_\ell f\|_{L^2} &\lesssim \sum_\ell \sum_{k\le\ell} 2^k \|P_\ell f\|_{L^2} \\ &\lesssim \sum_\ell 2^\ell \|P_\ell f\|_{L^2} \sum_{k\le\ell} 2^{k-\ell} \lesssim \|f\|_{B^1_{2,1}} \end{aligned}$$

It remains to estimate $\sum_\ell \sum_{k>\ell} \|P_k \nabla P_\ell f\|_{L^2}$. Applying the finite band property followed by the scalar Böchner inequality (71) we derive

$$\|P_k \nabla P_\ell f\|_{L^2} \lesssim 2^{-k} \|\nabla^2 P_\ell f\|_{L^2} \lesssim 2^{-k}(2^\ell + A_\gamma^2 2^\ell) \|P_\ell f\|_{L^2}.$$

Summing we infer that

$$\begin{aligned} \sum_\ell \sum_{k>\ell} \|P_k \nabla P_\ell f\|_{L^2} &\lesssim \sum_\ell (2^\ell + A_\gamma) \|P_\ell f\|_{L^2} \sum_{k>\ell} 2^{\ell-k} \\ &\lesssim \|f\|_{B^1_{2,1}} + A_\gamma^2 \|f||_{B^0_{2,1}} \end{aligned}$$

as desired. ■

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