# Formation of Trapped Surfaces in Geodesic Foliation

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**Abstract**

We revisit the classical results of the formation of trapped surfaces for the Einstein vacuum equation relying on the geodesic foliation, rather than the double null foliation used in all previous results, starting with the seminal work of Christodoulou [12] and continued in [21], [2], [4], [16], [3]. The main advantage of the method is that it only requires information on the incoming curvature along the incoming initial null hypersurface, which is more along the lines of [10] on the formation of trapped surfaces in spherical symmetry. Therefore, the methods used here may be better suited for studying the Weak Cosmic Censorship conjecture in the spirit of [11]. Another important advantage, which we plan to bring to fruition in a forthcoming paper, is that it is appropriate to the study of the formation of trapped surfaces from more general Cauchy data than treated in [24]. Our paper is based on a version of the non-integrable PT frame introduced in [20] and [15], associated to the geodesic foliation.

## 1 Introduction

All known results on the formation of trapped surfaces for the Einstein vacuum equation starting with the seminal work of Christodoulou [12] and continued in [21], [4], [16], [3], [2], make use of an adapted double null foliation. The goal of this paper is to show that similar results can be derived using instead a simple geodesic foliation and an associated, non-integrable, PT frame first introduced in [20], [15]. The method only requires information on the incoming curvature along the incoming initial null hypersurface and appears better fit to study the formation of trapped surfaces from Cauchy data, something which we plan to discuss in a forthcoming paper. The result is based on a version of the non-integrable PT (Principal Temporal) gauge introduced in [20] and [15] and uses the $(a, \delta)$ version of the short pulse method introduced in [4].

### 1.1 Set-Up

Consider a spacetime $\mathcal{M} = \mathcal{M}(\delta, a; \tau^*)$ with past null boundaries $\underline{H}_0 \cup H_{-1}$ and future boundaries $\underline{H}_\delta \cup \Sigma_{\tau^*}$, where $\underline{H}_\delta$ is null incoming and $\Sigma_\tau$ is a spacelike level hypersurface of a time function $\tau$ to be specified (See Figure 1). Here $\delta$ is a small constant and, following [4], we introduce another large constant $a$ which satisfies $\delta a \lesssim 1$. The spacetime $\mathcal{M}$ is foliated by the level surfaces of an ingoing optical function $\underline{u}$ such that $\underline{u} = 0$ on $\underline{H}_0$ and $\underline{u} = \delta$ on $\underline{H}_\delta$.

**Geodesic foliation on $H_{-1}$.** The restriction of $\underline{u}$ to $H_{-1}$ coincides with the affine parameter of a null geodesic generator of $H_{-1}$, denoted by $e_4$, normalized on the sphere $S_{-1,0} := H_{-1} \cap \underline{H}_0$. We let $\underline{u} = 0$ on $S_{-1,0}$. This gives a geodesic foliation on $H_{-1}$, and the level surfaces of $\underline{u}$ are 2-spheres. We then have, for $\omega = \frac{1}{4} g(D_4 e_4, e_3)$, $\xi_a = \frac{1}{2} g(D_4 e_4, e_a)$, $\underline{\eta}_a = \frac{1}{2} g(D_4 e_3, e_a)$, $\zeta_a = \frac{1}{2} g(D_a e_4, e_3)$,

$$\omega = 0, \quad \xi = 0, \quad \underline{\eta} = -\zeta. \tag{1.1}$$

We can also derive the bounds of other Ricci coefficients, see Proposition 3.1.

**Geodesic foliation on $\mathcal{M}$.** Using the incoming optical function $\underline{u}$ we define[1] ${}^{(g)}e_3 := -2\,\mathrm{grad}\, \underline{u}$, such that ${}^{(g)}e_3$ is geodesic. We also define $s$ to be the affine parameter of $e_3$, i.e. ${}^{(g)}e_3(s) = 1$ with $s = -1$ on

[1] Recall that, given a function $f$, $(\mathrm{grad}\, f)^\mu := g^{\mu\nu}\partial_\nu f$.

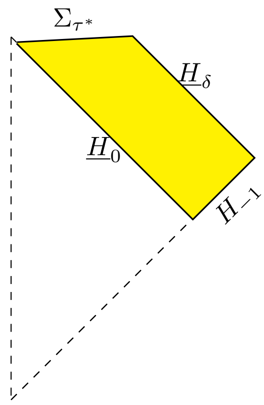


Figure 1: The spacetime $\mathcal{M}$

$H_{-1}$. We then define ${}^{(g)}e_4$ so that $\{{}^{(g)}e_3, {}^{(g)}e_4\}$ forms a null pair orthogonal to the sphere $S_{\underline{u},s}$, defined as the intersection of level hypersurfaces of $\underline{u}$ as $s$, and denote by $\mathcal{S}$ the horizontal structure perpendicular to ${}^{(g)}e_3$, ${}^{(g)}e_4$, tangent to the spheres $S_{\underline{u},s}$. We also denote ${}^{(g)}\nabla$, ${}^{(g)}\nabla_3$, ${}^{(g)}\nabla_4$ the corresponding horizontal derivative operators (see Section 2.1) and by $({}^{(g)}e_a)_{a=1,2}$ an arbitrary orthonormal frame of $\mathcal{S}$. Note that we have

$$ {}^{(g)}e_4(\underline{u}) = g(\mathrm{grad}\,\underline{u}, {}^{(g)}e_4) = -\frac{1}{2}g\big({}^{(g)}e_3, {}^{(g)}e_4\big) = 1. \tag{1.2} $$

In particular, restricted to $H_{-1}$, ${}^{(g)}e_4$ coincides with the $e_4$ defined above on $H_{-1}$.

**Remark 1.1.** *The geodesic foliation and its associated geodesic horizontal null structure defined above are a simple example of a Principal Geodesic (PG) structure, as introduced in [17]. We will thus refer to* ${}^{(g)}e_3$, ${}^{(g)}e_4$, ${}^{(g)}e_1$, ${}^{(g)}e_2$ *as a PG frame.*

We denote by ${}^{(g)}\Gamma$ the corresponding Ricci coefficients and by ${}^{(g)}R$ the null curvature components with respect to the geodesic frame. Thus (see[2] e.g. [20]),

$$ {}^{(g)}\underline{\omega} = 0, \quad {}^{(g)}\underline{\xi} = 0, \quad {}^{(g)}\eta = {}^{(g)}\zeta = -{}^{(g)}\underline{\eta}, \quad {}^{(g)}e_3\big({}^{(g)}e_4(s)\big) = -2\,{}^{(g)}\omega. \tag{1.3} $$

We associate to the geodesic frame a system of angular coordinates $\theta^A$, $A = 1,2$ as follows:

- On $H_{-1}$ we set ${}^{(g)}e_4(\theta^A) = 0$ with $\theta^A$ specified on $S_{0,-1} := \underline{H}_0 \cap H_{-1}$;
- Using the values of $\theta^A$ on $H_{-1}$ we extend them to $\mathcal{M}$ by ${}^{(g)}e_3(\theta^A) = 0$.

We define the time[3] function $\tau := \frac{1}{10}a\underline{u} + s$.

**PT frame**. In the geodesic frame, each non-vanishing Ricci coefficient satisfies a transport equation along the integral curve of ${}^{(g)}e_3$. Some of these equations, however, contain transversal derivatives, leading to a loss of derivatives. We deal with the issue by considering another frame $\{{}^{(T)}e_3, {}^{(T)}e_4, {}^{(T)}e_a\}$ which verifies[4]

$$ {}^{(T)}e_3 = {}^{(g)}e_3, \qquad {}^{(T)}\eta = 0. \tag{1.4} $$

[2]The relations (1.3) follow easily from $D_{{}^{(g)}e_3}{}^{(g)}e_3 = 0$ and by applying the commutation relations (see formula 2.2.3 in [15] for an easy derivation) below to the functions $\underline{u}, s$

$$ \begin{aligned} [{}^{(g)}e_3, {}^{(g)}e_a] &= {}^{(g)}\underline{\xi}_a\,{}^{(g)}e_4 + ({}^{(g)}\eta - {}^{(g)}\zeta)_a\,{}^{(g)}e_3 - {}^{(g)}\underline{\chi}_{ab}\,{}^{(g)}e_b, \\ [{}^{(g)}e_4, {}^{(g)}e_a] &= {}^{(g)}\xi_a\,{}^{(g)}e_3 + ({}^{(g)}\underline{\eta} + {}^{(g)}\zeta)_a\,{}^{(g)}e_4 - {}^{(g)}\chi_{ab}\,{}^{(g)}e_b, \\ [{}^{(g)}e_4, {}^{(g)}e_3] &= 2({}^{(g)}\underline{\eta} - {}^{(g)}\eta)_a\,{}^{(g)}e_a + 2\,{}^{(g)}\omega\,{}^{(g)}e_3 - 2\,{}^{(g)}\underline{\omega}\,{}^{(g)}e_4. \end{aligned} $$

[3]We show later in Section 6.1 that $\tau$ is indeed a time function.

[4]The existence of such a gauge can be easily justified in view of the transformation formula (2.7).

The remarkable feature of the frame, called PT frame in [20], is that the loss of derivative issue disappears once we set up this gauge condition, see Proposition 2.5. This positive feature is however compensated by a negative one, that is the fact that the horizontal structure associated to the null pair $({}^{(T)}e_3, {}^{(T)}e_4)$ is not integrable, see Section 2.1 and the more detailed discussion in Chapter 2 of [15]. This problem can however be resolved by relying on both frames, the non-integrable PT frame to deal with the $e_3$-transport equations and the integrable PG frame for dealing with elliptic and Sobolev type estimates.

**Remark 1.2.** *In what follows, as there is no danger of confusion, we drop the prefix* ${}^{(T)}$ *for the PT foliation. We thus denote* ${}^{(T)}e_3 = e_3$, ${}^{(T)}e_4 = e_4$, *by* $\mathcal{H}$ *the horizontal structure perpendicular to* $e_3, e_4$ *and by* $\nabla, \nabla_3, \nabla_4$ *the corresponding derivative operators. We denote by* $\Gamma = \left\{\mathrm{tr}\,\chi, \mathrm{tr}\,\underline{\chi}, {}^{(a)}\mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\underline{\chi}, \widehat{\chi}, \underline{\widehat{\chi}}, \eta, \underline{\eta}, \zeta, \omega, \underline{\omega}, \xi, \underline{\xi}\right\}$ *the set of all PT-Ricci coefficients.*

**Remark 1.3.** *The PT frame we work with coincides with the geodesic frame on* $H_{-1}$ *and verifies, see Definition 2.3 and Proposition 2.4,*

$$\underline{\omega} = 0, \qquad \underline{\xi} = 0, \qquad \underline{\eta} = -\zeta, \qquad {}^{(a)}\mathrm{tr}\,\underline{\chi} = 0. \tag{1.5}$$

We introduce the renormalized quantity[5]

$$\widetilde{\mathrm{tr}\,\underline{\chi}} := \mathrm{tr}\,\underline{\chi} + \frac{2}{|s|} \tag{1.6}$$

and denote by $\check{\Gamma}$ the set of non-vanishing Ricci coefficients

$$\check{\Gamma} = \left\{\mathrm{tr}\,\chi, \widetilde{\mathrm{tr}\,\underline{\chi}}, {}^{(a)}\mathrm{tr}\,\chi, \widehat{\chi}, \underline{\widehat{\chi}}, \zeta, \omega, \xi\right\}.$$

## 1.2 Initial conditions

**Initial Data on $\underline{H}_0$.** Following the results of [12], [21], [4], [16], we start by assuming that the incoming data on $\underline{H}_0$ is Minkowskian.[6] We note however that we can significantly relax this assumption by only requiring information on the incoming curvature. Indeed, unlike the case of the double null foliation used in these above mentioned works, all Ricci coefficients in the PT frame can be determined by integration along the $e_3$ direction. The incoming data on $\underline{H}_0$ is only used in the derivation of the curvature components by energy estimates.

**Initial Data on $H_{-1}$.** Our data on $H_{-1}$ verifies the An-Luk [4] short pulse assumption on $\widehat{\chi}_0 = \widehat{\chi}|_{H_{-1}}$, which can roughly be thought of as the free data that can be prescribed on $H_{-1}$:

$$\sum_{i \le N_0, j \le 1} ||(\delta\nabla_4)^j \nabla^i \widehat{\chi}_0||_{L^\infty_{\underline{u}} L^2(S_{-1,\underline{u}})} \le C_0\, a^{\frac{1}{2}}, \quad N_0 \ge 9, \tag{1.7}$$

as well as

$$\inf_\theta \int_0^\delta |\widehat{\chi}_0(\underline{u}, \theta)|^2 d\underline{u} \ge \delta a. \tag{1.8}$$

**Remark 1.4.** *Using the argument in [16] (see also [6]), one can relax* (1.8) *by replacing the* inf *over* $\theta$ *by* sup. *See Remark 7.2.*

**Remark 1.5.** *Note that the* $S$*-foliation on* $H_{-1}$ *is that induced by the geodesic foliation and that both the PT and geodesic frames discussed above coincide with double null frame on* $H_{-1}$ *used in [4] and all the other above mentioned works. We point out that in [4] the assumption is weaker as there is no requirement on the* $\nabla_4$ *derivative of* $\widehat{\chi}_0$ *in* (1.7)*. This is achieved by a renormalization of curvature components such that the contribution from* $\nabla_4\widehat{\chi}_0$ *completely decouples from the system. This can, in principle, be also achieved in our framework but we do not pursue this here.*

[5] In contrast, since $\mathrm{tr}\,\chi$ presents a worse behavior similar to $1/|s|$, one does not need to renormalize it by subtracting its Minkowskian value.

[6] One can also study other type of incoming data. In [22] and [2], the incoming data corresponds to Christodoulou's naked singularity solution in [13].

## 1.3 Main result

Here is a short version of our main result.

**Theorem 1.6.** *Consider the characteristic initial value problem described above. If* (1.7) *holds, then the spacetime can be extended to* $\mathcal{M}(\delta, a; -\frac{1}{8}a\delta)$*, together with its incoming geodesic foliation. Moreover, if* (1.8) *also holds, then* $S_{\delta,-\frac{1}{4}a\delta}$ *is a trapped surface.*[7]

We later provide (see Section 3) a more precise version of Theorem 1.6 which also extends to more general incoming initial data on $\underline{H}_0$.

**Previous results.** Christodoulou's pioneering work [12] is the first result on the formation of trapped surfaces in the Einstein-vacuum spacetime. Klainerman–Rodnianski [21] then adopted a systematic approach by scale invariant estimates to simplify the proof of [12]. This idea was then further generalized by An [1]. Li–Yu [24] showed that there exists Cauchy initial data corresponding to Christodoulou's spacetime. The result was later strengthened by Li–Mei [23], who proved that a black hole can indeed form. Along a different line, Klainerman–Luk–Rodnianski [16] significantly relaxed the lower bound in (1.8) by developing a fully anisotropic mechanism for the formation of trapped surfaces.

The first scale-critical result was established by An–Luk [4], which led to the further study of the apparent horizon [5], [6]. Later An [3] gave a simplified proof of the scale-critical result in the far-field regime by designing a scale-invariant norm based on the signature and decay rates. Our work provides proof of a similar result in the finite region using the incoming geodesic foliation instead of the double null foliation.

We also refer the readers to the results generalized to the Einstein equation coupled with matter fields [27], [7], [8], [28], [26], [9].

## 1.4 Main features in the proof of Theorem 1.6

**1.** As mentioned earlier, we make essential use of PT frame (1.4) in order to avoid the loss of derivatives intrinsic to the geodesic foliation. Note that the horizontal structure spanned by $(e_1, e_2)$ is non-integrable[8] with respect to $e_4$, i.e. ${}^{(a)}\mathrm{tr}\,\chi \neq 0$. The use of non-integrable structures was pioneered in the proof of Kerr stability [20], [15]. To compensate for the lack of integrability of the main horizontal structures used in these works, one needs to consider associated integrable structures for which one can derive elliptic (Hodge estimates) and Sobolev inequalities. In our case this role is played by the geodesic frame $\{{}^{(g)}e_\mu\}$.

**2.** The typical transport equation verified by all Ricci coefficients in the PT frame, is of the form

$$\nabla_3\psi + \lambda \mathrm{tr}\,\underline{\chi}\psi = F. \tag{1.9}$$

The main contribution of $\mathrm{tr}\,\underline{\chi}$ is $-2/|s|$, so neglecting $F$ (which we expect to control by a bootstrap assumption), we infer that the quantity $|s|^{2\lambda}\psi$ is conserved. It helps to divide all Ricci coefficients $\check{\Gamma}$ as follows:

i. Those that are of size 1 on $H_{-1}$, and satisfy the transport equation (1.9) with $\lambda = \frac{1}{2}$. They behave like $1/|s|$ on $\mathcal{M}$. We denote these by $\check{\Gamma}_b$ (stands for "bad");

ii. Those that are of size $\delta a^{\frac{1}{2}}$ on $H_{-1}$, and satisfy the transport equation with $\lambda = 1$. They behave like $\delta a^{\frac{1}{2}}/|s|^2$ on $\mathcal{M}$ and are denoted by $\check{\Gamma}_g$ (stands for "good").

iii. The outgoing shear $\widehat{\chi}$ for which we have only the bounds $\widehat{\chi} \sim a^{\frac{1}{2}}/|s|$. In addition, in contrast to the case of the double null foliation (used in [21], [4], [16], [3], [22], [2]), we also have present the signature[9] $+2$ quantity $\xi$ which behaves similarly to $\widehat{\chi}$. Though $\xi$, like $\widehat{\chi}$, is a large quantity, due to signature consideration, it gets paired with better behaved quantities in nonlinear terms.

[7] Note that $\tau(\delta, -\frac{1}{4}a\delta) = \frac{1}{10}a\delta - \frac{1}{4}a\delta < -\frac{1}{8}a\delta$, so $S_{\delta,-\frac{1}{4}a\delta}$ indeed lies in $\mathcal{M}(\delta, a; -\frac{1}{8}a\delta)$.

[8] It is however integrable in $e_3$, i.e. ${}^{(a)}\mathrm{tr}\,\underline{\chi} = 0$. This is due to the fact that the corresponding horizontal structure is tangent to the $\underline{u}$ hypersurfaces, see Proposition 2.4.

[9] The signature of a quantity is basically the number of $e_4$ minus the number of $e_3$ in its expression.

**3.** The right-hand side of (1.9) denoted by $F$ contains linear curvature components and nonlinear terms relative to the Ricci coefficients $(\check{\Gamma}_g, \check{\Gamma}_b)$. As usual, the curvature components are controlled by energy type estimates using the null Bianchi equations. The nonlinear quadratic terms for (i) are of the form $\check{\Gamma}_g \cdot \check{\Gamma}_b$. Those for (ii) are of the form $\check{\Gamma}_b \cdot \check{\Gamma}_b$. Discounting the anomalous behavior discussed below, both of these will result in the gain of at least an extra $a^{-\frac{1}{2}}$ factor in their estimates[10]. In our case the terms in $\check{\Gamma}_b$ have signature $+1$ and those in $\check{\Gamma}_g$ have signature 0 or $-1$. By simple signature considerations, it is easy to see that when (1.9) is applied to $\check{\Gamma}_g$, we cannot have $\check{\Gamma}_g \cdot \check{\Gamma}_b$ terms in $F$. Similarly, when (1.9) is applied to $\check{\Gamma}_b$, one cannot have terms of the type $\check{\Gamma}_b \cdot \check{\Gamma}_b$. The absence of a worse term is crucial to close our estimates. For example, suppose $\psi \in \check{\Gamma}_b$ and we have the equation

$$\nabla_3 \psi + \frac{1}{2} \mathrm{tr}\, \underline{\chi} \psi = \check{\Gamma}_b \cdot \check{\Gamma}_b + \cdots .$$

Since $\check{\Gamma}_b \sim 1/|s|$, we would end up integrating $1/|s|$ which gives an additional logarithmic growth in $|s|$.

**4.** Anomalies: The key quantity $\widehat{\chi}$ is large even compared with $\check{\Gamma}_b$, with an extra $a^{\frac{1}{2}}$ factor. This is a crucial feature for the mechanism of the formation of trapped surfaces. Its presence makes some nonlinear terms become borderline. To overcome this difficulty one needs to make use of the triangular structure of the main $e_3$ transport equations, that is to follow a specified, correct, order in doing the estimates.

**5.** As already mentioned we need to work with both the geodesic and PT frames. The passage from the ${}^{(g)}e$-frame to the $e$ frame is made using the frame transformation formulas (see (2.11))

$$e_4 = {}^{(g)}e_4 - f^a\, {}^{(g)}e_a + \frac{1}{4}|f|^2\, {}^{(g)}e_3, \quad e_a = {}^{(g)}e_a - \frac{1}{2} f_a\, {}^{(g)}e_3, \quad e_3 = {}^{(g)}e_3,$$

where $f$ verifies[11]

$$\nabla_3 f + \frac{1}{2} \mathrm{tr}\, \underline{\chi}\, f = 2\zeta - \widehat{\underline{\chi}} \cdot f. \tag{1.10}$$

The Ricci coefficients in the $e$ and ${}^{(g)}e$ frames are related by Lemma 2.1. The ${}^{(g)}e$ frame is used to derive Hodge elliptic estimates and Sobolev inequalities. More precisely, whenever we need to make use of these, we pass from the PT frame $e$ to the ${}^{(g)}e$ frame and then transform the result back to the PT-frame.

**6.** The ansatz from the bootstrap assumption (see (3.6)) $\zeta \in \check{\Gamma}_g \sim \delta a^{\frac{1}{2}} |s|^{-2}$ (which is true in the double null frame[12]) leads to a logarithmic loss in the $|s|$-weighted estimate when we integrate the equation (1.10). To avoid this problem we show that in fact, in the PT frame, $\zeta$ satisfies a slightly improved estimate of the form $\zeta \sim \delta a^{\frac{1}{2}} |s|^{-1} + \delta^{\frac{3}{2}} a |s|^{-\frac{5}{2}}$ that circumvents the problem.

**7.** Apart from the transport equations of type (1.9) verified by the Ricci coefficients, we also need to control the curvature components by using energy type estimates.[13] This is a standard procedure, see for example Section 8.7 in [17] or Chapter 16 in [15]. A typical pair of null Bianchi equations can, in our case, be written in the form

$$\begin{aligned} \nabla_3 \psi_1 + \lambda \mathrm{tr}\, \underline{\chi} \psi_1 &= \mathcal{D}^* \psi_2 + F_1, \\ \nabla_4 \psi_2 &= \mathcal{D} \psi_1 + F_2, \end{aligned} \tag{1.11}$$

where $\mathcal{D}$, $\mathcal{D}^*$ represent horizontal Hodge operators (defined in Section 6.3) that are formal adjoint of each other. The corresponding energy estimates for $(\psi_1, \psi_2)$ is derived by integrating the divergence identity

$$\mathrm{Div}\,(|s|^{2(2\lambda-1)} |\psi_1|^2 e_3) + \mathrm{Div}\,(|s|^{2(2\lambda-1)} |\psi_2|^2 e_4) = \cdots$$

on the causal region enclosed by the boundaries $\underline{H}_0$, $H_{-1}$, $\underline{H}_\delta$, $\Sigma_\tau$.

[10] The absence of worse nonlinear terms is related to the "signature conservation" pointed out in [14].
[11] This follows by using the condition ${}^{(g)}\eta = {}^{(g)}\zeta$ and the transformation formulas of Lemma 2.1.
[12] and in fact, as one can later verify, also in the integrable geodesic frame
[13] This is in fact the only place where we need to take into account the incoming data on $\underline{H}_0$.

**8.** To estimate higher derivatives we need to commute both the transport equations of type (1.9) and the null Bianchi pairs with $\nabla$, more precisely[14] with $|s|\nabla$. A small difficulty appear when we commute the second equation of the Bianchi pair (1.11), applied to $\psi_1 = \underline{\beta}, \psi_2 = \underline{\alpha}$, with $\nabla$ due to the commutator $[\nabla_4, \nabla]\underline{\alpha} = \xi \nabla_3 \underline{\alpha} + \cdots$ which contains the term $\nabla_3 \underline{\alpha}$ for which we do not have an equation.

It turns out that, with very little additional work, we can also commute equation (1.9) with $|s|\nabla_3$ just as with $|s|\nabla$. As a result, $|s|\nabla\psi$ and $|s|\nabla_3\psi$ both behave similarly with $\psi$. We note however that the signature of $\nabla$ and $\nabla_3$ are different, and this is the reason why we do not pursue the strict hierarchy according to the signatures, as in [21], [3], but only distinguish $\nabla_4$ with $\mathfrak{d} = (\nabla, \nabla_3)$, and, by a similar spirit, distinguish $\check{\Gamma}_b$, which is of signature $+1$, with $\check{\Gamma}_g$, of signature 0 or $-1$.

We also note that the analogous problem of the commutator between $\nabla_3$ and $\nabla$ is not present in view of the fact $\underline{\xi} = 0$ in our PT gauge. We rely very little on the $\nabla_4$ transport equations for the Ricci coefficients—they are in fact only needed on $H_{-1}$.

# 2 Preliminaries

## 2.1 Horizontal structures

We review below some basic facts about non-integrable horizontal structures discussed in Chapter 2 of [15].

Given a pair of null vectors $\{e_3, e_4\}$ satisfying $g(e_3, e_4) = -2$, we consider the horizontal structure associated to it given by the distribution $\mathcal{H} = \{e_3, e_4\}^\perp$. With a choice of an orthonormal basis $\{e_1, e_2\}$ of this horizontal structure, we obtain a null frame $\{e_3, e_4, e_a\}$ $(a = 1, 2)$. When the horizontal structure is integrable, i.e. the distribution $\mathcal{H}$ is involutive, we also say that the null frame is integrable (which is not the case for the principal null pair in Kerr spacetime).

The Ricci coefficients and curvature components are defined by[15]

$$\begin{aligned}
\chi_{ab} &= g(D_a e_4, e_b), \quad \underline{\chi}_{ab} = g(D_a e_3, e_b), \quad \xi_a = \frac{1}{2} g(D_4 e_4, e_a), \quad \underline{\xi}_a = \frac{1}{2} g(D_3 e_3, e_a), \\
\omega &= \frac{1}{4} g(D_4 e_4, e_3), \quad \underline{\omega} = \frac{1}{4} g(D_3 e_3, e_4), \quad \eta_a = \frac{1}{2} g(D_3 e_4, e_a), \quad \underline{\eta}_a = \frac{1}{2} g(D_4 e_3, e_a), \\
\zeta_a &= \frac{1}{2} g(D_a e_4, e_3),
\end{aligned}$$

$$\begin{aligned}
\alpha_{ab} &= R_{a4b4}, \quad \beta_a = \frac{1}{2} R_{a434}, \quad \rho = \frac{1}{4} R_{3434}, \quad {}^*\rho = \frac{1}{4} {}^*R_{3434}, \quad \underline{\beta}_a = \frac{1}{2} R_{a334}, \\
\underline{\alpha}_{ab} &= R_{a3b3}.
\end{aligned}$$

For a vector field $X$, we define its projection onto the horizontal structure $\mathcal{H}$ by

$$ {}^{(h)}X := X + \frac{1}{2} g(X, e_3) e_4 + \frac{1}{2} g(X, e_4) e_3. $$

This also defines the projection operator $\Pi$. A $k$-covariant tensor field $U$ is called horizontal, if

$$ U(X_1, \cdots, X_k) = U({}^{(h)}X_1, \cdots, {}^{(h)}X_1). $$

The horizontal covariant derivative operator $\nabla$ is defined by

$$ \nabla_X Y := {}^{(h)}(D_X Y) = D_X Y - \frac{1}{2} \underline{\chi}(X, Y) e_4 - \frac{1}{2} \chi(X, Y) e_3 $$

using the definition of $\chi, \underline{\chi}$. Similarly, one can define $\nabla_3 X$ and $\nabla_4 X$ as the projections of $D_3 X$ and $D_4 X$. Then the horizontal covariant derivative can be generalized for tensors in the standard way

$$ \nabla_Z U(X_1, \cdots, X_k) = Z(U(X_1, \cdots X_k)) - U(\nabla_Z X_1, \cdots, X_k) - \cdots - U(X_1, \cdots, \nabla_Z X_k), $$

[14] This latter can be thought of as the "rotation operator" which is commonly used in the analysis of wave equations.

[15] Here $D$ denotes the spacetime Levi–Civita connection associated to $g$, and $R$ denotes the spacetime Riemann curvature tensor. We also use the shorthand notation $D_\mu = D_{e_\mu}$. The Hodge dual ${}^*R$ is defined by ${}^*R_{\alpha\beta\mu\nu} = \frac{1}{2} \in_{\mu\nu}{}^{\rho\sigma} R_{\alpha\beta\rho\sigma}$.

and similarly for $\nabla_3 U$ and $\nabla_4 U$.

In the non-integrable case, the null second fundamental forms are decomposed as

$$\chi_{ab} = \widehat{\chi}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\chi + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\,\chi,$$

$$\underline{\chi}_{ab} = \underline{\widehat{\chi}}_{ab} + \frac{1}{2}\delta_{ab}\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}\,\underline{\chi}.$$

where the trace and anti-trace are defined by

$$\mathrm{tr}\,\chi = \delta^{ab}\chi_{ab}, \quad \mathrm{tr}\,\underline{\chi} = \delta^{ab}\underline{\chi}_{ab}, \quad {}^{(a)}\mathrm{tr}\,\chi := \in^{ab}\chi_{ab}, \quad {}^{(a)}\mathrm{tr}\,\underline{\chi} := \in^{ab}\underline{\chi}_{ab},$$

and the horizontal volume form $\in_{ab}$ is defined by

$$\in(X,Y) := \frac{1}{2}\in(X,Y,e_3,e_4),$$

and we choose the orientation such that $\in_{12}= 1$. The horizontal structure $\mathcal{H}$ is integrable if and only if ${}^{(a)}\mathrm{tr}\,\chi = {}^{(a)}\mathrm{tr}\,\underline{\chi} = 0$; see Chapter 2 in [15].

The left dual of a horizontal 1-form $\psi$ and a horizontal covariant 2-tensor $U$ are defined by

$${}^*\psi_a := \in_{ab}\ \psi_b, \quad ({}^*U)_{ab} := \in_{ac}\ U_{cb}.$$

For two horizontal 1-forms $\psi$, $\phi$, we also define

$$\psi\cdot\phi := \delta^{ab}\psi_a\phi_b, \quad \psi\wedge\phi := \in^{ab}\psi_a\phi_b, \quad (\psi\widehat{\otimes}\phi)_{ab} = \psi_a\phi_b + \psi_b\phi_a - \delta_{ab}\psi\cdot\phi.$$

In particular $|\psi| := (\psi\cdot\psi)^{\frac{1}{2}}$ with the straightforward generalization to general horizontal covariant tensors. This will be used to define $L^p$-type norms of $\psi$. Similarly we define the derivative operators

$$\mathrm{div}\,\psi := \delta^{ab}\,\nabla_a\psi_b, \quad \mathrm{curl}\,\psi := \in^{ab}\ \nabla_a\psi_b, \quad (\nabla\widehat{\otimes}\psi)_{ab} := \nabla_a\psi_b + \nabla_b\psi_a - \delta_{ab}\,\mathrm{div}\,\psi.$$

## 2.2 Frame transformations

To pass from the geodesic frame ${}^{(g)}e$ to the PT frame we need to appeal to the transformation formulas for the corresponding Ricci coefficients given in Section 2.2 of [20]. The general formula of a transformation between two null frames $e$ and $e'$ was given in Lemma 2.2.1 of that section. In our context we only need transformations that preserve $e_3$:

**Lemma 2.1.** *A general null transformation between two null frames $(e_3, e_4, e_1, e_2)$ and $(e'_3, e'_4, e'_1, e'_2)$ which preserves $e_3$ has the form*

$$e'_3 = e_3, \qquad e'_a = e_a + \frac{1}{2}f_a e_3, \qquad e'_4 = e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3 \tag{2.1}$$

*The inverse transformation which takes the $e$ frame to the $e'$ frame is given by replacing $f$ with $-f$, i.e.*

$$e_3 = e'_3, \qquad e_a = e'_a - \frac{1}{2}f_a e'_3, \qquad e_4 = e'_4 - f^b e'_b + \frac{1}{4}|f|^2 e'_3.$$

**Lemma 2.2.** *Under a null frame transformation* (2.1) *the Ricci coefficients transform as follows:*

- *The transformation formula for $\xi$ is given by*

$$\begin{aligned}\xi' = \xi &+ \frac{1}{2}\nabla'_4 f + \frac{1}{4}(\mathrm{tr}\,\chi f - {}^{(a)}\mathrm{tr}\,\chi\,{}^*f) + \omega f + \frac{1}{2}f\cdot\widehat{\chi} + \frac{1}{4}|f|^2\eta \\ &+ \frac{1}{2}(f\cdot\zeta)\,f + \frac{1}{4}|f|^2\underline{\eta} - \frac{1}{4}|f|^2\omega f + \frac{1}{16}|f|^2 f\cdot\underline{\chi} + \frac{1}{16}|f|^4\underline{\xi}.\end{aligned} \tag{2.2}$$

- *The transformation formula for $\underline{\xi}$ is given by*

$$\underline{\xi}' = \underline{\xi}, \tag{2.3}$$

- *The transformation formulas for $\chi$ are given by*

$$\begin{aligned}\chi'_{ab} &= \chi_{ab} + f_a\eta_b + \nabla_{e'_a} f_b + \frac{1}{4}|f|^2\underline{\chi}_{ab} + \frac{1}{4}|f|^2 f_a\underline{\xi}_b + f_b\zeta_a \\ &\quad - f_a f_b\underline{\omega} - \frac{1}{2} f_b f_c \underline{\chi}_{ac} - \frac{1}{2} f_a f_b f_c \underline{\xi}_c.\end{aligned} \tag{2.4}$$

- *The transformation formulas for $\underline{\chi}$ are given by*

$$\underline{\chi}'_{ab} = \underline{\chi}_{ab} + f_a\underline{\xi}_b. \tag{2.5}$$

- *The transformation formula for $\zeta$ is given by*

$$\zeta' = \zeta - \frac{1}{4}\mathrm{tr}\,\underline{\chi} f - \frac{1}{4}{}^{(a)}\mathrm{tr}\,\underline{\chi}\,{}^* f - \underline{\omega} f - \frac{1}{2}\widehat{\underline{\chi}}\cdot f - \frac{1}{2}(f\cdot\underline{\xi})f. \tag{2.6}$$

- *The transformation formula for $\eta$ is given by*

$$\eta' = \eta + \frac{1}{2}\nabla_3 f - \underline{\omega}\, f + \frac{1}{4}|f|^2\underline{\xi} - \frac{1}{2}(f\cdot\underline{\xi})f. \tag{2.7}$$

- *The transformation formula for $\underline{\eta}$ is given by*

$$\underline{\eta}' = \underline{\eta} + \frac{1}{4}\mathrm{tr}\,\underline{\chi} f - \frac{1}{4}{}^{(a)}\mathrm{tr}\,\underline{\chi}\,{}^* f + \frac{1}{2} f\cdot\widehat{\underline{\chi}} + \frac{1}{4}|f|^2\underline{\xi}. \tag{2.8}$$

- *The transformation formula for $\omega$ is given by*

$$\omega' = \omega + \frac{1}{2} f^a(\zeta - \underline{\eta})_a - \frac{1}{4}|f|^2\underline{\omega} - \frac{1}{4} f^a f^b \underline{\chi}_{ab} - \frac{1}{8}|f|^2 f^a \underline{\xi}_a. \tag{2.9}$$

- *The transformation formula for $\underline{\omega}$ is given by*

$$\underline{\omega}' = \underline{\omega} + \frac{1}{2} f\cdot\underline{\xi}. \tag{2.10}$$

The proof follows from a direct calculation. See [18] for detailed derivations in full generality. Note that, unlike the version in [20], we keep track of all error terms.[16]

## 2.3 Passage from the PG to the PT frame

Consider the transformation formula from the PG frame ${}^{(g)}e$ to a new frame $e$ for which $\eta = 0$. In view of Lemma 2.2, with $f$ replaced[17] by $-f$, we must have

$$\eta = {}^{(g)}\eta - \frac{1}{2}{}^{(g)}\nabla_3 f + {}^{(g)}\underline{\omega}\, f + \frac{1}{4}|f|^2\,{}^{(g)}\underline{\xi} + \frac{1}{2}(f\cdot{}^{(g)}\underline{\xi})f.$$

Note that one can easily verify $\nabla_3 f = {}^{(g)}\nabla_3 f$, as $e_3$ is geodesic. Since ${}^{(g)}\underline{\omega}$, ${}^{(g)}\underline{\xi}$ vanish we deduce that $f$ must verify the equation $0 = {}^{(g)}\eta - \frac{1}{2}\nabla_3 f$.

**Definition 2.3.** *The PT frame ${}^{(T)}e = e$ is defined by the transformation formula*

$$e_4 = {}^{(g)}e_4 - f^a\,{}^{(g)}e_a + \frac{1}{4}|f|^2\,{}^{(g)}e_3, \quad e_a = {}^{(g)}e_a - \frac{1}{2} f_a\,{}^{(g)}e_3, \quad e_3 = {}^{(g)}e_3 \tag{2.11}$$

*with $f$ the unique solution of the equation*

$$\nabla_3 f = 2\,{}^{(g)}\eta, \qquad f\Big|_{H_{-1}} = 0. \tag{2.12}$$

[16] This is needed as our situation here is non-perturbative.

[17] Thus $f$ corresponds to the inverse transformation formula from the $e$-frame to the ${}^{(g)}e$-frame.

**Proposition 2.4.** *The PT frame defined above verifies the following properties:*

1. *We have*

$$\underline{\omega} = 0, \quad \underline{\xi} = 0, \quad \underline{\eta} = -\zeta, \quad {}^{(a)}\mathrm{tr}\,\underline{\chi} = 0. \tag{2.13}$$

2. *We have*

$$\nabla_3 f = -\frac{1}{2}\mathrm{tr}\,\underline{\chi} f + 2\zeta - \widehat{\underline{\chi}} \cdot f. \tag{2.14}$$

*Proof.* To check (2.13) we start from the fact that $e_3 = {}^{(g)}e_3$ is geodesic, i.e. $\underline{\omega} = 0$, $\underline{\xi} = 0$. Note that $e_a(\underline{u}) = {}^{(g)}e_a(\underline{u}) - \frac{1}{2} f_a e_3(\underline{u}) = 0$. Hence $e_a$ are tangent to the level surfaces of $\underline{u}$ and so is the commutator $[e_1, e_2]$. Thus ${}^{(a)}\mathrm{tr}\,\underline{\chi} = \in_{ab} g(D_a e_3, e_b) = -\frac{1}{2} \in_{ab} g(e_3, [e_a, e_b]) = 0$. In view of the transformation formulas for $\zeta$ and $\underline{\eta}$ in Lemma 2.2 we easily check that we also have $\zeta + \underline{\eta} = {}^{(g)}\zeta + {}^{(g)}\underline{\eta} = 0$.

To check (2.14) we use the inverse transformation formulas, corresponding to $e \to {}^{(g)}e$. Thus ${}^{(g)}\zeta = \zeta - \frac{1}{4}\mathrm{tr}\,\underline{\chi} f - \frac{1}{2}\widehat{\underline{\chi}} \cdot f$. Since ${}^{(g)}\zeta = {}^{(g)}\eta$ and ${}^{(g)}\eta = \frac{1}{2}\nabla_3 f$ we deduce that $\frac{1}{2}\nabla_3 f = \zeta - \frac{1}{4}\mathrm{tr}\,\underline{\chi} f - \frac{1}{2}\widehat{\underline{\chi}} \cdot f$ as stated. $\square$

## 2.4 Null structure and Bianchi equations in PT frame

**Proposition 2.5.** *Under the ingoing PT frame the null structure equations in the incoming direction $e_3$ take the form:*

$$\begin{aligned}
\nabla_3 \mathrm{tr}\,\underline{\chi} &= -|\widehat{\underline{\chi}}|^2 - \frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\big)^2,\\
\nabla_3 \widehat{\underline{\chi}} &= -\mathrm{tr}\,\underline{\chi}\,\widehat{\underline{\chi}} - \underline{\alpha},\\
\nabla_3 \mathrm{tr}\,\chi &= -\widehat{\underline{\chi}} \cdot \widehat{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi + 2\rho,\\
\nabla_3 {}^{(a)}\mathrm{tr}\,\chi &= -\widehat{\underline{\chi}} \wedge \widehat{\chi} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\,\chi - 2\,{}^*\rho,\\
\nabla_3 \widehat{\chi} &= -\frac{1}{2}\big(\mathrm{tr}\,\chi\widehat{\underline{\chi}} + \mathrm{tr}\,\underline{\chi}\widehat{\chi}\big) + \frac{1}{2}\,{}^*\widehat{\underline{\chi}}\,{}^{(a)}\mathrm{tr}\,\chi,\\
\nabla_3 \zeta &= -\widehat{\underline{\chi}} \cdot \zeta - \frac{1}{2}\mathrm{tr}\underline{\chi}\zeta - \underline{\beta},\\
\nabla_3 \omega &= |\zeta|^2 + \rho,\\
\nabla_3 \xi &= \widehat{\chi} \cdot \zeta + \frac{1}{2}\mathrm{tr}\chi\zeta - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\chi\,{}^*\zeta + \beta.
\end{aligned}$$

*We also have the equation of* $\widetilde{\mathrm{tr}\,\underline{\chi}} := \mathrm{tr}\,\underline{\chi} + \frac{2}{|s|}$

$$\nabla_3 \widetilde{\mathrm{tr}\,\underline{\chi}} + \mathrm{tr}\,\underline{\chi}\widetilde{\mathrm{tr}\,\underline{\chi}} = \frac{1}{2}(\widetilde{\mathrm{tr}\,\underline{\chi}})^2 - |\widehat{\underline{\chi}}|^2.$$

*The Bianchi equations take the form*

$$
\begin{aligned}
\nabla_3\alpha-\nabla\widehat{\otimes}\beta &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\alpha+\zeta\widehat{\otimes}\beta-3(\rho\widehat{\chi}+\ {}^*\rho\,{}^*\widehat{\chi}),\\
\nabla_4\beta-\mathrm{div}\,\alpha &= -2(\mathrm{tr}\,\chi\beta-\ {}^{(a)}\mathrm{tr}\,\chi\,{}^*\beta)-2\omega\beta+\alpha\cdot\zeta+3(\xi\rho+\ {}^*\xi\ {}^*\rho),\\
\nabla_3\beta+\mathrm{div}\,\varrho &= -\mathrm{tr}\,\underline{\chi}\beta+2\underline{\beta}\cdot\widehat{\chi},\\
\nabla_4\rho-\mathrm{div}\,\beta &= -\frac{3}{2}(\mathrm{tr}\,\chi\rho+\ {}^{(a)}\mathrm{tr}\,\chi\ {}^*\rho)-\zeta\cdot\beta-2\xi\cdot\underline{\beta}-\frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha,\\
\nabla_4\ {}^*\rho+\mathrm{curl}\,\beta &= -\frac{3}{2}(\mathrm{tr}\,\chi\ {}^*\rho-\ {}^{(a)}\mathrm{tr}\,\chi\rho)+\zeta\cdot\ {}^*\beta-2\xi\cdot\ {}^*\underline{\beta}+\frac{1}{2}\widehat{\underline{\chi}}\cdot\ {}^*\alpha,\\
\nabla_3\rho+\mathrm{div}\,\underline{\beta} &= -\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho+\zeta\cdot\underline{\beta}-\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha},\\
\nabla_3\ {}^*\rho+\mathrm{curl}\,\underline{\beta} &= -\frac{3}{2}\mathrm{tr}\,\underline{\chi}\ {}^*\rho+\zeta\cdot\ {}^*\underline{\beta}-\frac{1}{2}\widehat{\chi}\cdot\ {}^*\underline{\alpha},\\
\nabla_4\underline{\beta}-\mathrm{div}\,\check{\varrho} &= -(\mathrm{tr}\,\chi\underline{\beta}+\ {}^{(a)}\mathrm{tr}\,\chi\,{}^*\underline{\beta})+2\omega\,\underline{\beta}+2\beta\cdot\widehat{\underline{\chi}}+3(\rho\zeta-\ {}^*\rho\,{}^*\zeta)-\underline{\alpha}\cdot\xi,\\
\nabla_3\underline{\beta}+\mathrm{div}\,\underline{\alpha} &= -2\mathrm{tr}\,\underline{\chi}\,\underline{\beta}+2\underline{\alpha}\cdot\zeta,\\
\nabla_4\underline{\alpha}+\nabla\widehat{\otimes}\underline{\beta} &= -\frac{1}{2}\big(\mathrm{tr}\,\chi\underline{\alpha}+\ {}^{(a)}\mathrm{tr}\,\chi\,{}^*\underline{\alpha})+4\omega\underline{\alpha}+5\zeta\widehat{\otimes}\underline{\beta}-3(\rho\widehat{\underline{\chi}}-\ {}^*\rho\,{}^*\widehat{\underline{\chi}}).
\end{aligned}
$$

*Here,*

$$
\begin{aligned}
\mathrm{div}\,\varrho &= -(\nabla\rho+\ {}^*\nabla\ {}^*\rho),\\
\mathrm{div}\,\check{\varrho} &= -(\nabla\rho-\ {}^*\nabla\ {}^*\rho).
\end{aligned}
$$

*Proof.* Immediate consequence of Propositions 2.2.5 and 2.2.6 in [15] by using the vanishing of $\underline{\xi}, \underline{\omega}, \eta$, ${}^{(a)}\mathrm{tr}\,\underline{\chi}$, $\underline{\eta}+\zeta$. The equation of $\widetilde{\mathrm{tr}\,\underline{\chi}}$ follows from the one for $\mathrm{tr}\,\underline{\chi}$, $e_3(s)=1$, $s<0$ and direct computations. ☐

## 2.5 Commutation lemma

We rely on the general commutation Lemma, see Section 2.2.7 in [15], to derive the following.

**Lemma 2.6.** *With respect to the PT frame we have, for a general k-horizontal tensorfield* $\psi_A=\psi_{a_1\cdots a_k}$,

$$
\begin{aligned}
[\nabla_3,\nabla_b]\psi_A &= -\underline{\chi}_{bc}\nabla_c\psi_A-\zeta_b\nabla_3\psi_A-\sum_{i=1}^k \in_{a_i c}\ {}^*\underline{\beta}_b\psi_{a_1\cdots}{}^{c}{}_{\cdots a_k},\\
[\nabla_3,\nabla_4]\psi_A &= -2\underline{\eta}_b\nabla_b\psi_A+2\sum_{i=1}^k\in_{a_i b}\ {}^*\rho\,\psi_{a_1\cdots}{}^{c}{}_{\cdots a_k}-2\omega\nabla_3\psi_A,\\
[\nabla_4,\nabla_b]\psi_A &= -\chi_{bc}\nabla_c\psi_A+\sum_{i=1}^k\Big(\chi_{ba_i}\underline{\eta}_c-\chi_{bc}\underline{\eta}_{a_i}\Big)\psi_{a_1\cdots}{}^{c}{}_{\cdots a_k}+\xi_b\nabla_3\psi_A\\
&\quad+\sum_{i=1}^k\Big(\underline{\chi}_{ba_i}\xi_c-\underline{\chi}_{bc}\xi_{a_i}+\in_{a_i c}\ {}^*\beta_b\Big)\psi_{a_1\cdots}{}^{c}{}_{\cdots a_k}.
\end{aligned}
\tag{2.15}
$$

*Moreover*

$$
\begin{aligned}
[\nabla_a,\nabla_b]\psi_A &= \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\chi\nabla_3\psi_A\in_{ab}+{}^{(h)}K\sum_{i=1}^k\in_{a_i c}(g_{a_i a}g_{cb}-g_{a_i b}g_{ca})\,\psi_{a_1\cdots}{}^{c}{}_{\cdots a_k},\\
{}^{(h)}K &:= -\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}-\rho.
\end{aligned}
\tag{2.16}
$$

*Proof.* The commutation formulas (2.15) follow immediately from Lemma 2.2.7 in [15] while (2.16) follows from Proposition 2.1.45 in [15]. In both cases we take into account the vanishing of the quantities $\underline{\xi}, \underline{\omega}, \eta$, ${}^{(a)}\mathrm{tr}\,\underline{\chi}$, $\underline{\eta}+\zeta$ in our PT frame. ☐

# 3 Precise version of the Main Theorem

Throughout the remaining of the paper, we use $\{e_3, e_4, e_a\}$ to denote the PT frame, and $\{{}^{(g)}e_3, {}^{(g)}e_4, {}^{(g)}e_a\}$ to denote the PG frame. We may also denote the ${}^{(g)}e$ frame simply by $e'$. We denote the corresponding horizontal derivative $\nabla$ and ${}^{(g)}\nabla$ (or $\nabla'$). We shall also denote $\mathfrak{d} = (\nabla, \nabla_3)$.

## 3.1 Main Norms

We introduce our basic integral norms on $\mathcal{M}$. All Ricci and curvature coefficients are defined with respect to the PT frame but may be integrated along the $S(u,s)$ spheres of the associated geodesic foliation. Thus, for example, we define

$$\begin{aligned} \mathcal{R}^S_k :=& \frac{|s|^{\frac{7}{2}}}{\delta^{\frac{5}{2}}a}||s^k\mathfrak{d}^k\underline{\alpha}||_{L^2(S_{\underline{u},s})} + \frac{|s|^3}{\delta^2 a^{\frac{3}{2}}}||s^k\mathfrak{d}^k\underline{\beta}||_{L^2(S_{\underline{u},s})} + \frac{|s|^2}{\delta a}||s^k\mathfrak{d}^k(\rho,\ {}^*\rho)||_{L^2(S_{\underline{u},s})} \\ &+ \frac{|s|}{a^{\frac{1}{2}}}||s^k\mathfrak{d}^k\beta||_{L^2(S_{\underline{u},s})} + \frac{1}{\delta^{-1}a^{\frac{1}{2}}}||s^k\mathfrak{d}^k\alpha||_{L^2(S_{\underline{u},s})}. \end{aligned} \tag{3.1}$$

Also, with $\widetilde{\mathrm{tr}\,\underline{\chi}} := \mathrm{tr}\,\underline{\chi} + \frac{2}{|s|}$,

$$\begin{aligned} \mathcal{O}^S_k :=& \frac{1}{a^{\frac{1}{2}}}||s^k\mathfrak{d}^k\widehat{\chi}||_{L^2(S_{\underline{u},s})} + ||s^k\mathfrak{d}^k\mathrm{tr}\,\chi||_{L^2(S_{\underline{u},s})} + \frac{|s|^{\frac{1}{2}}}{\delta^{\frac{1}{2}}a^{\frac{1}{2}}}||s^k\mathfrak{d}^k\omega||_{L^2(S_{\underline{u},s})} \\ &+ \frac{|s|^{-\frac{1}{2}}}{\delta^{-\frac{1}{2}}a^{\frac{1}{2}}}||s^k\mathfrak{d}^k\xi||_{L^2(S_{\underline{u},s})} + \frac{|s|}{\delta a^{\frac{1}{2}}}||s^k\mathfrak{d}^k(\zeta,\widehat{\underline{\chi}},\widetilde{\mathrm{tr}\,\underline{\chi}})||_{L^2(S_{\underline{u},s})} + \frac{1}{\delta a^{\frac{1}{2}}}||s^k\mathfrak{d}^k f||_{L^2(S_{\underline{u},s})} \end{aligned} \tag{3.2}$$

along with a few $L^\infty$ norms

$$\mathcal{O}^S_{k,\infty} := \frac{|s|^2}{\delta a^{\frac{1}{2}}}||s^k\mathfrak{d}^k(\widehat{\underline{\chi}},\widetilde{\mathrm{tr}\,\underline{\chi}}, s^{-1}f)||_{L^\infty(S_{\underline{u},s})}.$$

We also define the energy type norms ($\Sigma_{\tau;\underline{u}} := \Sigma_\tau \cap \{0 \le \underline{u}' \le \underline{u}\}$)

$$\begin{aligned} \mathcal{R}_{k,2} =& \delta^{\frac{1}{2}}a^{-\frac{1}{2}}\left(a^{-\frac{1}{4}}||s^k\mathfrak{d}^k\alpha||_{L^2(\Sigma_{\tau;\underline{u}})} + ||s^k\mathfrak{d}^k\beta||_{L^2(\underline{H}_{\underline{u}})}\right) \\ &+ \delta^{-\frac{1}{2}}a^{-\frac{1}{2}}\left(a^{-\frac{1}{4}}||s(s^k\mathfrak{d}^k)\beta||_{L^2(\Sigma_{\tau;\underline{u}})} + ||s(s^k\mathfrak{d}^k)(\rho,{}^*\rho)||_{L^2(\underline{H}_{\underline{u}})}\right) \\ &+ \delta^{-\frac{3}{2}}a^{-1}||s^2s^k\mathfrak{d}^k\underline{\beta}||_{L^2(\underline{H}_{\underline{u}})} + \delta^{-\frac{5}{2}}a^{-\frac{3}{2}}||s^3s^k\mathfrak{d}^k\underline{\alpha}||_{L^2(\underline{H}_{\underline{u}})}. \end{aligned}$$

We also use $\mathcal{R}_k[\psi]$ to denote the $\psi$-part of the $\mathcal{R}_{k,2}$ norm, e.g., we denote $\mathcal{R}_k[\underline{\beta}] := \delta^{-\frac{3}{2}}a^{-1}||s^2s^k\nabla^k\underline{\beta}||_{L^2(\underline{H}_{\underline{u}})}$.

We also make use of the compound norms

$$\mathcal{O}_{\le N} := \sup_{k\le N}\mathcal{O}^S_k, \qquad \mathcal{R}_{\le N} := \sup_{k\le N}\mathcal{R}_{k,2}, \qquad \mathcal{R}^S_{\le N-1} = \sup_{k\le N-1}\mathcal{R}^S_k, \tag{3.3}$$

or simply $\mathcal{O}$ and $\mathcal{R}$ when there is no possible confusion.

## 3.2 Initial data on $H_{-1}$

First, we note that the PT frame[18] on $H_{-1}$ coincides with the double null frame used in previous works. One can thus easily compare our conditions with those of [12] and [4].

**Proposition 3.1.** *Assume that the short pulse condition* (1.7) *holds true for some* $N_0 \ge 9$ *and that the data on* $H_{-1} \cap \underline{H}_0$ *is Minkowskian. Then on* $H_{-1}$*, as well as in a local existence region,*[19] *for all* $i \le N$*, with* $\mathfrak{d} = (\nabla, \nabla_3)$.

$$\begin{aligned} &\mathfrak{d}^i(\xi,\widehat{\chi}) \sim a^{\frac{1}{2}}, \quad \mathfrak{d}^i(\mathrm{tr}\,\chi,\ {}^{(a)}\mathrm{tr}\,\chi) \sim 1, \quad \mathfrak{d}^i\omega \sim \delta^{\frac{1}{2}}a^{\frac{1}{2}}, \quad \mathfrak{d}^i(\zeta,\widehat{\underline{\chi}},\widetilde{\mathrm{tr}\,\underline{\chi}}) \sim \delta a^{\frac{1}{2}}, \\ &\mathfrak{d}^i\alpha \sim \delta^{-1}a^{\frac{1}{2}}, \quad \mathfrak{d}^i\beta \sim a^{\frac{1}{2}}, \quad \mathfrak{d}^i(\rho,\ {}^*\rho) \sim \delta a, \quad \mathfrak{d}^i\underline{\beta} \sim \delta^2 a, \quad \mathfrak{d}^i\underline{\alpha} \sim \delta^3 a^{\frac{3}{2}}, \end{aligned} \tag{3.4}$$

[18]Recall that the PT and the PG frames coincide on $H_{-1}$, see Remark 1.5.

[19]Note that ${}^{(a)}\mathrm{tr}\,\chi = 0$ initially on $H_{-1}$ but not in a non-trivial local existence region in the PT frame.

*Proof.* To start with one can deduce, using the analogues of Proposition 2.5 in the $e_4$ direction,[20] the bounds

$$\widehat{\chi} \sim a^{\frac{1}{2}}, \quad \mathrm{tr}\,\chi \sim 1, \quad \zeta \sim \delta a^{\frac{1}{2}}, \quad \widehat{\underline{\chi}} \sim \delta a^{\frac{1}{2}}, \quad \mathrm{tr}\,\underline{\chi} + 2 \sim \delta a^{\frac{1}{2}},$$
$$\alpha \sim \delta^{-1} a^{\frac{1}{2}}, \quad \beta \sim a^{\frac{1}{2}}, \quad (\rho, \ {}^*\rho) \sim \delta a, \quad \underline{\beta} \sim \delta^2 a, \quad \underline{\alpha} \sim \delta^3 a^{\frac{3}{2}}.$$

We can then show, using the commutation Lemma 2.6 that the same asymptotic conditions hold true for the angular derivatives $\nabla$ of these components. The same bounds for the $\nabla_3$ derivatives hold also true—they can be easily deduced from transport equations in Proposition 2.5. Indeed, all quantities except $\underline{\alpha}$, verify a $\nabla_3$ equation. Estimates for $\nabla_3\underline{\alpha}$ can be derived by integrating on $H_{-1}$ of the equation for $\nabla_4(\nabla_3\underline{\alpha})$ obtained by commuting the $\nabla_4\underline{\alpha}$ Bianchi identity with $\nabla_3$. Schematically,

$$\nabla_4\nabla_3\underline{\alpha} + \nabla\widehat{\otimes}\nabla_3\underline{\beta} = [\nabla_4, \nabla_3]\underline{\alpha} + [\nabla\widehat{\otimes}, \nabla_3]\underline{\beta} + \nabla_3(\check{\Gamma}_b \cdot \underline{\alpha}) + \nabla_3(\check{\Gamma}_g \cdot (\underline{\beta}, \rho, \ {}^*\rho)).$$

For details we refer the reader to [12]. □

## 3.3 Main Theorem (second version)

**Theorem 3.2.** *There exist a small constant $\delta_0 > 0$ and a large constant $a_0 \gg 1$ such that for any $0 < \delta < \delta_0$ and $a > a_0$ with $\delta a \leq 1$, for the characteristic initial value problem described above, with the data on $\underline{H}_0$ Minkowskian or a perturbation satisfying $\mathcal{R}_{k,2}|_{\underline{u}=0} \lesssim 1$ for $k \leq N$, where $N \geq 5$ is a positive integer,*

1. *If (1.7) holds, then the spacetime can be extended to $\mathcal{M}(\delta, a; -\frac{1}{8}a\delta)$ such that the following estimate hold true for all $k \leq N$, with a sufficiently large constant $C = C(N) > 0$,*
$$\mathcal{R}_{k,2} \leq C, \qquad \mathcal{O}^S_k \leq C. \tag{3.5}$$
2. *Moreover, if (1.8) also holds, then $S_{\delta, -\frac{1}{4}a\delta}$ is a trapped surface.*

We prove the theorem by a continuity argument based on the following bootstrap assumption.
**Bootstrap assumptions.** Assume that for some $\tau^*$ with $-1 < \tau^* \leq -\frac{1}{8}a\delta$, the following bounds hold true for all $\underline{u}$ and $s$ satisfying $\tau = \frac{1}{10}a\underline{u} + s \leq \tau^*$, and for a sufficiently large constant $C_b$ to be chosen,

$$\begin{aligned} \mathcal{O}^S_{k,\infty}(\underline{u}, s) &\leq C_b, && \text{for all } k \leq [N/2] + 1, \\ \mathcal{R}^S_k(\underline{u}, s) &\leq C_b, && \text{for all } k \leq N - 1, \\ \mathcal{O}^S_k(\underline{u}, s) + \mathcal{R}_{k,2}(\underline{u}) &\leq C_b, && \text{for all } k \leq N. \end{aligned} \tag{3.6}$$

The existence of such $\tau^*$ is ensured by a standard, characteristic local wellposedness result, see for example [25]. We improve the bootstrap assumption, summarized in the sequence of steps below, showing that the constant $C_b$ can be replaced by a universal constant depending only on the initial data and the size of $-\frac{1}{8}a\delta$. The local existence result can then be invoked to extend the existence region to the whole $\mathcal{M}(a, \delta; -\frac{1}{8}a\delta)$ while preserving the same bounds.

### 3.3.1 Main Steps

1. In Section 5, we integrate the $e_3$-transport equations in Proposition 2.5, including the equation (2.14) for $f$, and derive $L^2$-estimates on the spheres $S_{\underline{u},s}$.
2. To pass from $L^2$ to $L^\infty$ estimates we need to rely on a version of the Sobolev inequalities which holds true for the non-integrable PT frame. This is done in Section 4 by going back and forth between the PT frame and the integrable PG frame.[21]
3. In Section 6, we derive the spacetime energy estimate for the null curvature components and close the bootstrap argument.

[20] The $e_4$-equations on $H_{-1}$, where the foliation is geodesic, are similar to the ones in Proposition 2.5 by the substitution $e_3 \to e_4$, $\chi \to \underline{\chi}$, $\underline{\chi} \to \chi$, $\xi \to \underline{\xi}$, $\omega \to \underline{\omega}$, $\zeta \to -\zeta$ with potential loss of derivatives (this is like a PG frame rather than a PT frame), which does not matter on $H_{-1}$. Similar for curvature components and note that ${}^*\rho \to -{}^*\rho$.

[21] Where we can rely on the standard Sobolev inequalities in the geodesic frame.

**Remark 3.3.** *We note that since $0 \le \underline{u} \le \delta$, the size of $|\tau|$ and $|s|$ are always comparable. In particular, we always have $|s| \ge \frac{1}{8}a\delta$. The reason to introduce $\tau$ is to give an achronal boundary of the spacetime to derive the energy estimates.*

# 4 Sobolev estimates in a non-integrable frame

Recall that we denote by $\mathcal{S}$ the horizontal structure given by the PG frame and by $\mathcal{H}$ the one of the PT frame. For simplicity of notation we use $'$ rather than $^{(g)}$ to denote the quantities associated to the PG frame $\mathcal{S}$.

**Lemma 4.1.** *Suppose that the bootstrap assumption* (3.6) *holds. Then for an $\mathcal{H}$-horizontal covariant tensor $\psi$, we have the estimate*[22]

$$|s|\,||\psi||_{L^\infty(S_{\underline{u},s})} \lesssim \sum_{i\le 2} ||s^i\nabla^i\psi||_{L^2(S_{\underline{u},s})} + a^{-\frac{1}{2}}||(s\mathfrak{d})^{\le 2}\psi||_{L^2(S_{\underline{u},s})}. \tag{4.1}$$

*for any $S_{\underline{u},s}$ in $\mathcal{M}(\delta,a;\tau^*)$. The right-hand side can also simply be replaced by $\sum_{i\le 2}||s^i\mathfrak{d}^i\psi||_{L^2(S_{\underline{u},s})}$.*

*Proof.* Given an $\mathcal{H}$-horizontal covariant tensor $\psi_A = \psi_{a_1\cdots a_k}$, we define the $\mathcal{S}$-horizontal tensor $\tilde{\psi}_A = \tilde{\psi}_{a_1\cdots a_k}$ so that

$$\tilde{\psi}_{a_1\cdots a_k} := \tilde{\psi}(e'_{a_1},\cdots,e'_{a_k}) = \psi_{a_1\cdots a_k}$$

for any $a_1,\cdots,a_k$. To apply the Sobolev estimate, we wish to control $\nabla'\nabla'\tilde{\psi}$. We first compute $\nabla'\psi$, which is a $\mathcal{S}$-horizontal covariant $(k+1)$-tensor:

$$\begin{aligned}
\nabla'_b\tilde{\psi}_{a_1\cdots a_k} &= e'_b(\tilde{\psi}_{a_1\cdots a_k}) - \tilde{\psi}(D_{e'_b}e'_{a_1},\cdots,e'_{a_k}) - \cdots - \tilde{\psi}(e'_{a_1},\cdots,D_{e'_b}e'_{a_k})\\
&= e'_b(\psi_{a_1\cdots a_k}) - \tilde{\psi}(D_{e'_b}e'_{a_1},\cdots,e'_{a_k}) - \cdots - \tilde{\psi}(e'_{a_1},\cdots,D_{e'_b}e'_{a_k})\\
&= (D_{e'_b}\psi)(e_{a_1},\cdots,e_{a_k}) + \psi(D_{e'_b}e_{a_1},\cdots,e_{a_k}) + \cdots + \psi(e_{a_1},\cdots,D_{e'_b}e_{a_k})\\
&\quad - \tilde{\psi}(D_{e'_b}e'_{a_1},\cdots,e'_{a_k}) - \cdots - \tilde{\psi}(e'_{a_1},\cdots,D_{e'_b}e'_{a_k})\\
&= \nabla_{e'_b}\psi_{a_1\cdots a_k} + \sum_{i=1}^k\left(g(D_{e'_b}e_{a_i},e_c) - g(D_{e'_b}e_{a'_i},e'_c)\right)\psi_{a_1\cdots c\cdots a_k}\\
&= \nabla_{e'_b}\psi_{a_1\cdots a_k} + \sum_{i=1}^k\left(-\frac{1}{2}f_{a_i}\underline{\chi}_{bc}\psi_{a_1\cdots c\cdots a_k} + \frac{1}{2}f_c\underline{\chi}_{ba_i}\psi_{a_1\cdots c\cdots a_k}\right) =: (\nabla\tilde{\psi})_{ba_1\cdots a_k},
\end{aligned}$$

where we used $\underline{\xi} = 0$ in the last step. Here we define $\nabla\tilde{\psi}$ as a $\mathcal{H}$-horizontal tensor (this is simply a notation). Then, applying the calculation above with $\tilde{\psi}$ replaced $\nabla'\tilde{\psi}$, we obtain

$$\nabla'_b\nabla'_c\tilde{\psi}_{a_1\cdots a_k} = \nabla_{e'_b}(\nabla\tilde{\psi})_{ca_1\cdots a_k} + \sum_{i=0}^k(\nabla\tilde{\psi})_{a_0a_1\cdots d\cdots a_k}\cdot\left(-\frac{1}{2}f_{a_i}\underline{\chi}_{bd} + \frac{1}{2}f_d\underline{\chi}_{ba_i}\right),$$

where $a_0 := c$. For the first term, by the definition of $\nabla\tilde{\psi}$, we have

$$\begin{aligned}
\nabla_{e'_b}(\nabla\psi)_{ca_1\cdots a_k} &= \nabla_{e'_b}\nabla_{e'_c}\psi_{a_1\cdots a_k} + \sum_{i=1}^k\nabla_{e'_b}\left(\left(-\frac{1}{2}f_{a_i}\underline{\chi}_{cd} + \frac{1}{2}f_d\underline{\chi}_{ca_i}\right)\psi_{a_1\cdots d\cdots a_k}\right)\\
&= (\nabla_b + \frac{1}{2}f_b\nabla_3)(\nabla_c + \frac{1}{2}f_c\nabla_3)\psi_{a_1\cdots a_k} + \mathfrak{d}(f\cdot\underline{\chi}\cdot\psi)\\
&= \nabla_b\nabla_c\psi_{a_1\cdots a_k} + \mathfrak{d}f\cdot\mathfrak{d}\psi + f\cdot\mathfrak{d}^{\le 2}\psi + \mathfrak{d}(\check{\Gamma}_g\cdot\psi)
\end{aligned}$$

using that $|s|^{-1}f \in \check{\Gamma}_g^1 := \{\widehat{\underline{\chi}}, \widetilde{\mathrm{tr}\,\underline{\chi}}, |s|^{-1}f\}$, and hence $f\cdot\underline{\chi} = \check{\Gamma}_g^1\cdot f + |s|^{-1}f \in \check{\Gamma}_g^1$.

[22]Throughout this work, the implicit constant implied by the symbol "$\lesssim$" is independent of bootstrap constants $C_b$ and $\mathcal{O}$, $\mathcal{R}$.

Therefore, using the bootstrap assumption $|s\nabla\check{\Gamma}_g^1|, |\check{\Gamma}_g^1| \lesssim \mathcal{O}\delta a^{\frac{1}{2}}|s|^{-2}$ for $\check{\Gamma}_g^1$ and the Sobolev estimate on the sphere $S_{\underline{u},s}$,[23] we derive, using $|s| \geq \frac{1}{8}a\delta$,

$$
\begin{aligned}
|s|\,||\psi||_{L^\infty(S_{\underline{u},s})} = |s|\,||\tilde{\psi}||_{L^\infty(S_{\underline{u},s})} &\lesssim ||(s\nabla')^{\leq 2}\tilde{\psi}||_{L^2(S_{\underline{u},s})} \\
&\lesssim ||s^2\nabla_{e_b'}(\nabla\tilde{\psi})||_{L^2(S_{\underline{u},s})} + ||s\nabla\tilde{\psi}||_{L^2(S_{\underline{u},s})}||sf\cdot\underline{\chi}||_{L^\infty(S_{\underline{u},s})} \\
&\lesssim ||s^2\nabla^2\psi||_{L^2(S_{\underline{u},s})} + ||(s\nabla)^{\leq 1}\psi||_{L^2(S_{\underline{u},s})}||s(s\nabla)^{\leq 1}\check{\Gamma}_g||_{L^\infty(S_{\underline{u},s})} \\
&\quad + ||f||_{L^\infty(S_{\underline{u},s})}||(s\mathfrak{d})^{\leq 2}\psi||_{L^2(S_{\underline{u},s})} + ||s\mathfrak{d}f||_{L^\infty(S_{\underline{u},s})}||s\mathfrak{d}\psi||_{L^2(S_{\underline{u},s})} \\
&\lesssim \sum_{i=0}^{2}||s^i\nabla^i\psi||_{L^2(S_{\underline{u},s})}(1+\mathcal{O}\delta a^{\frac{1}{2}}|s|^{-2}) + \mathcal{O}\delta a^{\frac{1}{2}}|s|^{-2}||(s\mathfrak{d})^{\leq 2}\psi||_{L^2(S_{\underline{u},s})} \\
&\lesssim \sum_{i=0}^{2}||s^i\nabla^i\psi||_{L^2(S_{\underline{u},s})} + a^{-\frac{1}{2}}||(s\mathfrak{d})^{\leq 2}\psi||_{L^2(S_{\underline{u},s})},
\end{aligned}
$$

and the result follows. □

With this estimate, each bootstrap bound on $L^2(S_{\underline{u},s})$ norm implies a lower-order $L^\infty(S_{\underline{u},s})$ bound of the same quantity. We will make use of these $L^\infty$ bounds without mentioning the use of the Sobolev lemma.

# 5 Estimate of Ricci coefficients in PT frame

We denote $\Gamma$ as all possible Ricci coefficients in the PT frame. We also set, with $f$ introduced in Definition 2.3 and verifying (2.14),

$$\check{\Gamma} := \{\xi, \widehat{\chi}, \mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\chi, \omega, \zeta, \underline{\widehat{\chi}}, \widetilde{\mathrm{tr}\,\underline{\chi}}, |s|^{-1}f\} = \check{\Gamma}_a \cup \check{\Gamma}_b \cup \check{\Gamma}_g,$$

where

$$\check{\Gamma}_a = \{\xi, \widehat{\chi}\}, \quad \check{\Gamma}_b = \{\mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\chi, \omega\}, \quad \check{\Gamma}_g = \{\zeta, \underline{\widehat{\chi}}, \widetilde{\mathrm{tr}\,\underline{\chi}}, |s|^{-1}f\}. \tag{5.1}$$

**Remark 5.1.** *Note that, using (2.11), which implies $e_a(s) = -\frac{1}{2}f^{(g)}e_3(s) = -\frac{1}{2}f$, we have $\nabla\mathrm{tr}\,\underline{\chi} = \nabla\widetilde{\mathrm{tr}\,\underline{\chi}} + \nabla(2s^{-1}) = \nabla\widetilde{\mathrm{tr}\,\underline{\chi}} + 2s^{-2}f = s^{-1}(s\nabla\widetilde{\mathrm{tr}\,\underline{\chi}} + 2s^{-1}f)$, so we see that while $\mathrm{tr}\,\underline{\chi}$ is not in $\check{\Gamma}_g$, $s\nabla\mathrm{tr}\,\underline{\chi}$ is schematically $s\nabla\check{\Gamma}_g + \check{\Gamma}_g$ (higher orders are also similar).*

## 5.1 Integrating the model transport equation

We rely on the following weighted integration lemma. This is similar to Proposition 5.5 in [4].

**Lemma 5.2.** *Suppose that the bootstrap assumptions* (3.6) *holds true in $\mathcal{M}(\delta, a, \tau_*)$. Then for a $\mathcal{H}$-horizontal covariant tensor field satisfying the equation*

$$\nabla_3\psi + \lambda\mathrm{tr}\,\underline{\chi}\psi = F,$$

*we have, for $\lambda_1 = 2(\lambda - \frac{1}{2})$ and $s \leq -\frac{1}{8}a\delta$,*

$$|s|^{\lambda_1}||\psi||_{L^2(S_{\underline{u},s})} \lesssim ||\psi||_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s}|s'|^{\lambda_1}||F||_{L^2(S_{\underline{u},s'})}\,ds'. \tag{5.2}$$

*We also have the higher-order estimates for $k \leq N$:*

$$||s^{\lambda_1+k}\mathfrak{d}^k\psi||_{L^2(S_{\underline{u},s})} \lesssim ||s^{\lambda_1+k}\mathfrak{d}^k\psi||_{L^2(S_{\underline{u},-1})} + \sum_{i\leq k}\int_{-1}^{s}|s'|^{\lambda_1+i}||\mathfrak{d}^iF||_{L^2(S_{\underline{u},s'})}\,ds'. \tag{5.3}$$

[23] which follows from the proof in [12] with the $e_4$ direction replaced by the $e_3$ direction.

*Proof.* Note that $e_3(|\psi|^2) = 2\psi\cdot\nabla_3\psi$. We can thus make use of the following variation formula for scalar functions $\phi$

$$\partial_s \int_{S_{\underline{u},s}} \phi = \int_{S_{\underline{u},s}} e_3\phi + \operatorname{tr}\underline{\chi}\phi$$

since $e_3 = \partial_s$. Letting $\phi = |s|^{2\lambda_1}|\psi|^2$, we have (note that $s<0$)

$$\begin{aligned}
\left|\partial_s \int_{S_{\underline{u},s}} |s|^{2\lambda_1}|\psi|^2\right| &= \left|\int_{S_{\underline{u},s}} -2\lambda_1|s|^{2\lambda_1-1}|\psi|^2 + 2|s|^{2\lambda_1}\psi\cdot\nabla_3\psi + |s|^{2\lambda_1}\operatorname{tr}\underline{\chi}|\psi|^2\right|\\
&= \left|\int_{S_{\underline{u},s}} 2|s|^{2\lambda_1}\psi\cdot(-\lambda_1|s|^{-1}\psi+\nabla_3\psi) + |s|^{2\lambda_1}\operatorname{tr}\underline{\chi}|\psi|^2\right|\\
&= \left|\int_{S_{\underline{u},s}} 2|s|^{2\lambda_1}\psi\cdot(F-\lambda\operatorname{tr}\underline{\chi}\psi-\lambda_1|s|^{-1}\psi) + |s|^{2\lambda_1}\operatorname{tr}\underline{\chi}|\psi|^2\right|\\
&= \left|\int_{S_{\underline{u},s}} 2|s|^{2\lambda_1}\psi\cdot F - \lambda_1|s|^{2\lambda_1}\psi\cdot\widetilde{\operatorname{tr}\underline{\chi}}\psi\right|\\
&\le 2\left(\int_{S_{\underline{u},s}} |s|^{2\lambda_1}|\psi|^2\right)^{\frac12}\left(\int_{S_{\underline{u},s}} |s|^{2\lambda_1}|F|^2\right)^{\frac12} + \lambda_1\mathcal{O}\delta a^{\frac12}|s|^{-2}\int_{S_{\underline{u},s}}|s|^{2\lambda_1}|\psi|^2,
\end{aligned}$$

where we made use of the bootstrap assumption in the last step. Now, since[24] $|s|\ge\frac18 a\delta$, we have $\int_{-1}^{s}\mathcal{O}\delta a^{\frac12}|s'|^{-2}ds' \lesssim \mathcal{O}a^{-\frac12}\ll 1$, and the estimate follows by integration using the Grönwall inequality. This finishes the proof of (5.2).

For the higher-order version, we need to commute the equation with $\mathfrak{d} = (\nabla,\nabla_3)$. When we commute the equation with $\nabla_3$, we have

$$\nabla_3\nabla_3\psi + \lambda\operatorname{tr}\underline{\chi}\nabla_3\psi + \lambda\nabla_3(\operatorname{tr}\underline{\chi})\psi = \nabla_3 F,$$

so using $\nabla_3\operatorname{tr}\underline{\chi} = -|\widehat{\underline{\chi}}|^2 - \frac12(\operatorname{tr}\underline{\chi})^2$ and the original equation $\lambda\operatorname{tr}\underline{\chi}\psi = F-\nabla_3\psi$, we get

$$\nabla_3(\nabla_3\psi) + \lambda\operatorname{tr}\underline{\chi}\nabla_3\psi - \frac12\lambda|\widehat{\underline{\chi}}|^2\psi + \frac12\operatorname{tr}\underline{\chi}(\nabla_3\psi - F) = \nabla_3 F,$$

i.e.,

$$\nabla_3(\nabla_3\psi) + \Big(\lambda+\frac12\Big)\operatorname{tr}\underline{\chi}\nabla_3\psi = \nabla_3 F + \frac12\operatorname{tr}\underline{\chi}F + \frac12\lambda|\widehat{\underline{\chi}}|^2\psi.$$

The term $|\widehat{\underline{\chi}}|^2\psi = s^{-1}(s|\widehat{\underline{\chi}}|^2)\psi$ and is of the type (in fact better) $s^{-1}\check{\Gamma}_g\psi$. Inductively, we get the schematic expression

$$\nabla_3\nabla_3^i\psi + \Big(\lambda+\frac{i}{2}\Big)\operatorname{tr}\underline{\chi}\nabla_3^i\psi = \nabla_3^i F + \sum_{j=1}^{i}|s|^{-j}\nabla_3^{i-j}(F+\check{\Gamma}_g\cdot\psi).$$

The commutation with $\nabla^i$, by the formula (2.15), gives

$$[\nabla_3,\nabla^i]\psi = -\frac{i}{2}\operatorname{tr}\underline{\chi}\nabla^i\psi + \nabla^{i-1}(\check{\Gamma}_g\cdot\mathfrak{d}\psi) + \nabla^{i-1}(\underline{\beta}^*\cdot\psi)$$

hence

$$\nabla_3\nabla^i\psi + \Big(\lambda+\frac{i}{2}\Big)\operatorname{tr}\underline{\chi}\nabla^i\psi = \nabla^i F + \nabla^{i-1}(\check{\Gamma}_g\cdot\mathfrak{d}\psi) + \nabla^{i-1}(\underline{\beta}^*\cdot\psi).$$

Therefore, to commute with $\mathfrak{d}^i$, we can commute in either way for finite times, and get

$$\nabla_3\mathfrak{d}^i\psi + \Big(\lambda+\frac{i}{2}\Big)\operatorname{tr}\underline{\chi}\mathfrak{d}^i\psi = \sum_{j=0}^{i}|s|^{-j}\mathfrak{d}^{i-j}F + \mathfrak{d}^{\le i-1}(\check{\Gamma}_g\cdot\mathfrak{d}\psi) + \mathfrak{d}^{\le i-1}(\underline{\beta}^*\cdot\psi).$$

[24] See Remark 3.3 .

Then, applying the integration lemma (Lemma 5.2), we get (for simplicity, we denote here $S = S_{\underline{u},s'}$),

$$
\begin{aligned}
||s^{2\lambda+i-1}\mathfrak{d}^i\psi||_{L^2(S_{\underline{u},s})} &\lesssim ||s^{2\lambda+i-1}\mathfrak{d}^i\psi||_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^{2\lambda+i-1}||\mathfrak{d}^i F||_{L^2(S)}\, ds' \\
&+ \int_{-1}^{s} |s'|^{2\lambda+i-1}||\mathfrak{d}^i(\check{\Gamma}_g\cdot\psi)||_{L^2(S)} + ||\mathfrak{d}^{i-1}(\underline{\beta}^*\cdot\psi)||_{L^2(S)}\, ds' \\
&\lesssim ||s^{2\lambda+i-1}\mathfrak{d}^i\psi||_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^{2\lambda+i-1}||\mathfrak{d}^i F||_{L^2(S)} ds' \\
&+ \int_{-1}^{s} |s'|^{2\lambda-1} \sum_{\substack{i_1+i_2=i\\ i_2\le i/2}} ||s'^{i_1}\mathfrak{d}^{i_1}\check{\Gamma}_g||_{L^2(S)}||s'^{i_2}\mathfrak{d}^{i_2}\psi||_{L^\infty(S)} + ||s'^{i_2}\mathfrak{d}^{i_2}\check{\Gamma}_g||_{L^\infty(S)}||s'^{i_1}\mathfrak{d}^{i_1}\psi||_{L^2(S)} \\
&+ \int_{-1}^{s} |s'|^{2\lambda} \sum_{\substack{i_1+i_2=i-1\\ i_2\le i/2}} ||s'^{i_1}\mathfrak{d}^{i_1}\underline{\beta}||_{L^2(S)}||s'^{i_2}\mathfrak{d}^{i_2}\psi||_{L^\infty(S)} + ||s'^{i_2}\mathfrak{d}^{i_2}\underline{\beta}||_{L^\infty(S)}||s'^{i_1}\mathfrak{d}^{i_1}\psi||_{L^2(S)} \\
&\lesssim ||s^{2\lambda+i-1}\mathfrak{d}^i\psi||_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^{2\lambda+i-1}||\mathfrak{d}^i F||_{L^2(S)}\, ds' \\
&+ \int_{-1}^{s} \mathcal{O}\frac{\delta a^{\frac12}}{|s'|}\cdot \sum_{i_2\le i/2} ||s'^{2\lambda-1}s'^{i_2}\mathfrak{d}^{i_2}\psi||_{L^\infty(S_{\underline{u},s'})} + \mathcal{O}\frac{\delta a^{\frac12}}{|s'|^2}\sum_{i_1\le i}||s'^{2\lambda-1+i_1}\mathfrak{d}^{i_1}\psi||_{L^2(S)}\, ds' \\
&+ \int_{-1}^{s} \mathcal{R}^S_{\le i-1}[\underline{\beta}]\delta^2 a^{\frac32}|s'|^{-3}\cdot|s'|\sum_{i_2\le i/2}||s'^{2\lambda-1+i_2}\mathfrak{d}^{i_2}\psi||_{L^\infty(S_{\underline{u},s'})}ds' \\
&+ \int_{-1}^{s} \mathcal{R}^S_{\le i-1}[\underline{\beta}]\delta^2 a^{\frac32}|s|^{-4}\cdot|s'|\sum_{i_1\le i-1}||s'^{2\lambda-1}s^{i_1}\mathfrak{d}^{i_1}\psi||_{L^2(S_{\underline{u},s'})}\, ds'.
\end{aligned}
$$

We note that the argument here in fact requires $i/2 \geq 2$ in view of the Sobolev estimates we employed; however, if $i/2 < 2$, we can directly estimate $\check{\Gamma}_g$ and $\beta$ in $L^\infty$ using the bootstrap bounds and Sobolev estimates, hence controlling $\psi$ in $L^2(S)$ and resulting in the same estimate.

Then, using the non-integrable Sobolev estimate, together with the bootstrap assumption (3.6), we have

$$
\begin{aligned}
||s^{2\lambda+i-1}\mathfrak{d}^i\psi||_{L^2(S_{\underline{u},s})} &\lesssim ||s^{2\lambda+i-1}\mathfrak{d}^i\psi||_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^{2\lambda+i-1}||\mathfrak{d}^i F||_{L^2(S_{\underline{u},s'})}\, ds' \\
&+ \int_{-1}^{s} \Big(C_b\frac{\delta a^{\frac12}}{|s'|^2} + C_b\delta^2 a^{\frac32}|s'|^{-3}\Big)\sum_{j\le i}||s'^{2\lambda-1+j}\mathfrak{d}^j\psi||_{L^2(S_{\underline{u},s'})}\, ds'.
\end{aligned}
$$

Since

$$
\int_{-1}^{-a\delta} C_b\frac{\delta a^{\frac12}}{|s'|^2} + C_b\delta^2 a^{\frac32}|s'|^{-3}ds' \lesssim C_b\, a^{-\frac12} \ll 1,
$$

we can sum up $i = 1, \cdots, k$ and use Grönwall's lemma to get

$$
||s^{2\lambda-1+k}\mathfrak{d}^k\psi||_{L^2(S_{\underline{u},s})} \lesssim ||s^{2\lambda-1+k}\mathfrak{d}^k\psi||_{L^2(S_{\underline{u},-1})} + \sum_{i\le k}\int_{-1}^{s} |s'|^{2\lambda-1+i}||\mathfrak{d}^i F||_{L^2(S_{\underline{u},s'})}\, ds',
$$

which finishes the proof of (5.3). ☐

**Remark 5.3.** *From the proof, it is clear that the same estimate holds when $F$ is replaced by $F + \check{\Gamma}_g\cdot\psi$ in view of the bootstrap bounds of $\check{\Gamma}_g$.*

## 5.2 Estimate of Ricci coefficients

In this subsection, we derive the estimates for all non-vanishing Ricci coefficients through the integration along the direction of $e_3$. As emphasized in the introduction, there is no loss of derivatives due to our choice of the PT gauge.

**Remark 5.4.** *We make systematic use of the $e_3$ transport equations of Proposition 2.5 to which we apply the transport Lemma 5.2. Without further notice we estimate weighted $L^2(S_{\underline{u},s})$ expressions of the form $||s^i \mathfrak{d}^i(\psi_1 \cdot \psi_2)||_{L^2(S_{\underline{u},s})}$ by*

$$\sum_{\substack{i_1+i_2=i\\ i_2\le i/2}} ||s^{i_1}\mathfrak{d}^{i_1}\psi_1||_{L^2(S_{\underline{u},s})}||s^{i_2}\mathfrak{d}^{i_2}\psi_2||_{L^\infty(S_{\underline{u},s})} + ||s^{i_2}\mathfrak{d}^{i_2}\psi_1||_{L^\infty(S_{\underline{u},s})}||s^{i_1}\mathfrak{d}^{i_1}\psi_2||_{L^2(S_{\underline{u},s})}.$$

**Proposition 5.5.** *We have $||s^k\mathfrak{d}^k\omega||_{L^2(S_{\underline{u},s})} \lesssim (\mathcal{R}_k[\rho]+1)\,\delta^{\frac12}a^{\frac12}|s|^{-\frac12}$, $k\le N$.*

*Proof.* Recall from (1.1) and (2.12) that $\omega = 0$ on $H_{-1}$. We apply Lemma 5.2 to the equation $\nabla_3\,\omega = |\zeta|^2+\rho$ and derive

$$\begin{aligned}
||\,|s|^{i-1}\mathfrak{d}^i\omega||_{L^2(S_{\underline{u},s})} &\lesssim ||\,|s|^{i-1}\mathfrak{d}^i\omega||_{L^2(S_{\underline{u},-1})} + \int_{-1}^s |s'|^{i-1}||\mathfrak{d}^i(|\zeta|)^2||_{L^2(S_{\underline{u},s'})} + |s'|^{i-1}||\mathfrak{d}^i\rho||_{L^2(S_{\underline{u},s'})}ds'\\
&\lesssim 0 + \int_{-1}^s |s'|^{-1}\mathcal{O}\delta a^{\frac12}|s'|^{-2}\cdot\mathcal{O}\delta a^{\frac12}|s'|^{-1}\,ds' + \Big(\int_{-1}^s |s'|^{-4}ds'\Big)^{\frac12}||s(s^i\mathfrak{d}^i\rho)||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \mathcal{O}^2\delta^2 a|s|^{-3} + \mathcal{R}_i[\rho]\delta^{\frac12}a^{\frac12}|s|^{-\frac32}\\
&\lesssim (\mathcal{R}_i[\rho]+\mathcal{O}^2a^{-\frac12})\delta^{\frac12}a^{\frac12}|s|^{-\frac32} \lesssim (\mathcal{R}_i[\rho]+1)\,\delta^{\frac12}a^{\frac12}|s|^{-\frac32}.
\end{aligned}$$ □

**Proposition 5.6.** *We have $||s^k\mathfrak{d}^k\xi||_{L^2(S_{\underline{u},s})} \lesssim (\mathcal{R}_k[\beta]+1)\,\delta^{-\frac12}a^{\frac12}|s|^{\frac12}$, $k\le N$.*

*Proof.* Recall from (1.1) and (2.12) that $\xi = 0$ on $H_{-1}$. We apply Lemma 5.2 to the equation for $\nabla_3\xi$ written schematically in the form $\nabla_3\xi = \widehat{\chi}\cdot\zeta + \check{\Gamma}_b\cdot\zeta + \beta$ to derive

$$\begin{aligned}
||\,|s|^{i-1}\mathfrak{d}^i\xi||_{L^2(S_{\underline{u},s})} &\lesssim ||\,|s|^{i-1}\mathfrak{d}^i\xi||_{L^2(S_{\underline{u},-1})} + \int_{-1}^s |s'|^{i-1}||\mathfrak{d}^i((\widehat{\chi},\check{\Gamma}_b)\cdot\zeta)||_{L^2(S_{\underline{u},s'})} + |s'|^{i-1}||\mathfrak{d}^i\beta||_{L^2(S_{\underline{u},s'})}ds'\\
&\lesssim 0 + \int_{-1}^s |s'|^{-1}\mathcal{O}a^{\frac12}|s'|^{-1}\cdot\mathcal{O}\delta a^{\frac12}|s'|^{-1}\,ds' + \Big(\int_{-1}^s |s'|^{-2}ds'\Big)^{\frac12}||s^i\mathfrak{d}^i\beta||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \mathcal{O}^2\delta a|s|^{-2} + \mathcal{R}_i[\beta]\,\delta^{-\frac12}a^{\frac12}|s|^{-\frac12}\\
&\lesssim (\mathcal{R}_i[\beta]+\mathcal{O}^2a^{-\frac12})\,\delta^{-\frac12}a^{\frac12}|s|^{-\frac12} \lesssim (\mathcal{R}_i[\beta]+1)\,\delta^{-\frac12}a^{\frac12}|s|^{-\frac12}.
\end{aligned}$$ □

**Proposition 5.7.** *We have $||s^k\mathfrak{d}^k\zeta||_{L^2(S_{\underline{u},s})} \lesssim \delta a^{\frac12} + (\mathcal{R}_k[\underline{\beta}]+1)\,\delta^{\frac32}a|s|^{-\frac32}$, $k\le N$.*

*Proof.* We apply Lemma 5.2 (with $\lambda=\frac12$) to the equation for $\nabla_3\zeta$, written schematically in the form $\nabla_3\zeta + \frac12\mathrm{tr}\,\underline{\chi}\,\zeta = \check{\Gamma}_g\cdot\zeta - \underline{\beta}$,

$$\begin{aligned}
||\,|s|^{i}\mathfrak{d}^i\zeta||_{L^2(S_{\underline{u},s})} &\lesssim ||\,|s|^{i}\mathfrak{d}^i\zeta||_{L^2(S_{\underline{u},-1})} + \int_{-1}^s |s'|^{i}||\mathfrak{d}^i(\check{\Gamma}_g\cdot\zeta)||_{L^2(S_{\underline{u},s'})} + |s'|^{i}||\mathfrak{d}^i\underline{\beta}||_{L^2(S_{\underline{u},s'})}ds'\\
&\lesssim \delta a^{\frac12} + \int_{-1}^s \mathcal{O}\delta a^{\frac12}|s'|^{-2}\cdot\mathcal{O}\delta a^{\frac12}|s'|^{-1}\,ds' + \Big(\int_{-1}^s |s'|^{-4}ds'\Big)^{\frac12}||s^2s^i\mathfrak{d}^i\underline{\beta}||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \delta a^{\frac12} + \mathcal{O}^2\delta^2a|s|^{-2} + \mathcal{R}_i[\underline{\beta}]\delta^{\frac32}a|s|^{-\frac32}\\
&\lesssim \delta a^{\frac12} + (\mathcal{R}_i[\underline{\beta}]+\mathcal{O}^2a^{-\frac12})\delta^{\frac32}a|s|^{-\frac32} \lesssim \delta a^{\frac12} + (\mathcal{R}_i[\underline{\beta}]+1)\,\delta^{\frac32}a|s|^{-\frac32}.
\end{aligned}$$ □

**Remark 5.8.** *This is slightly better than the $\delta a^{\frac12}|s|^{-2}$ size of the bootstrap assumption. This turns out to be useful to avoid a logarithmic loss in the estimate of the frame transformation $f$.*

**Proposition 5.9.** *We have $||s^k\mathfrak{d}^k(\widetilde{\mathrm{tr}\,\underline{\chi}},\widehat{\underline{\chi}})||_{L^2(S_{\underline{u},s})} \lesssim \delta a^{\frac12}|s|^{-1}$, $k\le N$.*

*Proof.* We apply Lemma 5.2 (with $\lambda=1$) to the equations

$$\nabla_3\widetilde{\mathrm{tr}\,\underline{\chi}} + \mathrm{tr}\,\underline{\chi}\widetilde{\mathrm{tr}\,\underline{\chi}} = \frac12(\widetilde{\mathrm{tr}\,\underline{\chi}})^2 - |\widehat{\underline{\chi}}|^2, \qquad \nabla_3\widehat{\underline{\chi}} + \mathrm{tr}\,\underline{\chi}\,\widehat{\underline{\chi}} = -\underline{\alpha}.$$

$$
\begin{aligned}
\|s^{1+i}\mathfrak{d}^i(\widehat{\underline{\chi}}, \widetilde{\mathrm{tr}\,\underline{\chi}})\|_{L^2(S_{\underline{u},s})} &\lesssim \|s^{1+i}\mathfrak{d}^i(\widehat{\underline{\chi}}, \widetilde{\mathrm{tr}\,\underline{\chi}})\|_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^{1+i}\|\mathfrak{d}^i(\check{\Gamma}_g\cdot\check{\Gamma}_g), \mathfrak{d}^i\underline{\alpha}\|_{L^2(S_{\underline{u},s'})} \\
&\lesssim \delta a^{\frac12} + \int_{-1}^{s} \mathcal{O}^2\delta^2 a|s'|^{-2}ds' + \Big(\int_{-1}^{s}|s'|^{-4}ds'\Big)^{\frac12}\|s^3 s^i\mathfrak{d}^i\underline{\alpha}\|_{L^2(\underline{H}_{\underline{u}})} \\
&\lesssim \delta a^{\frac12} + \mathcal{O}^2\delta^2 a|s|^{-1} + \mathcal{R}[\underline{\alpha}]\,\delta^{\frac52}a^{\frac32}|s|^{-\frac32} \\
&\lesssim (1+\mathcal{O}^2 a^{-\frac12} + \mathcal{R}[\underline{\alpha}]\,a^{-\frac12})\,\delta a^{\frac12} \lesssim \delta a^{\frac12}. \qquad \square
\end{aligned}
$$

**Proposition 5.10.** *We have* $\|s^k\mathfrak{d}^k\widehat{\chi}\|_{L^2(S_{\underline{u},s})} \lesssim a^{\frac12}$, $k\le N$.

*Proof.* We apply Lemma 5.2 (with $\lambda=\frac12$) to the equation for $\nabla_3\widehat{\chi}$ written schematically $\nabla_3\widehat{\chi} = -\frac12\mathrm{tr}\,\underline{\chi}\widehat{\chi} + \check{\Gamma}_g\cdot\check{\Gamma}_b$

$$
\begin{aligned}
\|s^i\mathfrak{d}^i\widehat{\chi}\|_{L^2(S_{\underline{u},s})} &\lesssim \|s^i\mathfrak{d}^i\widehat{\chi}\|_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s}|s'|^i\|\mathfrak{d}^i(\check{\Gamma}_g\cdot\check{\Gamma}_b)\|_{L^2(S_{\underline{u},s'})}\,ds' \lesssim a^{\frac12} + \int_{-1}^{s}\mathcal{O}\delta a^{\frac12}|s'|^{-2}\cdot\mathcal{O}\,ds' \\
&\lesssim (1+\mathcal{O}^2\delta|s|^{-1})a^{\frac12} \lesssim a^{\frac12}. \qquad \square
\end{aligned}
$$

**Proposition 5.11.** *We have* $\|s^k\mathfrak{d}^k(\mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\chi)\|_{L^2(S_{\underline{u},s})} \lesssim 1 + \mathcal{R}_k[\rho, {}^*\rho]$, $k\le N$.

*Proof.* We apply Lemma 5.2 (with $\lambda=\frac12$) to the $\nabla_3$ equations for $\mathrm{tr}\,\chi$ and ${}^{(a)}\mathrm{tr}\,\chi$ of the form

$$
\nabla_3(\mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\chi) + \frac12\mathrm{tr}\,\underline{\chi}(\mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\chi) = -(\widehat{\underline{\chi}}\cdot\widehat{\chi}, \widehat{\underline{\chi}}\wedge\widehat{\chi}) + 2(\rho, -{}^*\rho).
$$

Making use of the $L^2$ bounds of $\widehat{\underline{\chi}}$ and $\widehat{\chi}$ and the corresponding $L^\infty$ bounds, derived using Lemma 4.1, we obtain

$$
\begin{aligned}
\|s^i\mathfrak{d}^i(\mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\chi)\|_{L^2(S_{\underline{u},s})} &\lesssim \|s^i\mathfrak{d}^i(\mathrm{tr}\,\chi, {}^{(a)}\mathrm{tr}\,\chi)\|_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s}|s'|^i\|\mathfrak{d}^i(\widehat{\underline{\chi}}\cdot\widehat{\chi})\|_{L^2(S_{\underline{u},s'})} \\
&\quad + \int_{-1}^{s}|s'|^i\|\mathfrak{d}^i(\rho, {}^*\rho)\|_{L^2(S_{\underline{u},s'})} \\
&\lesssim 1 + \sum_{j\le i/2}\int_{-1}^{s} a^{\frac12}\cdot\|s^j\mathfrak{d}^j\widehat{\underline{\chi}}\|_{L^\infty(S_{\underline{u},s'})} + \frac{\delta a^{\frac12}}{|s'|}\cdot\|s^j\mathfrak{d}^j\widehat{\chi}\|_{L^\infty(S_{\underline{u},s'})}ds' \\
&\quad + \Big(\int_{-1}^{s}|s'|^{-2}ds'\Big)^{\frac12}\|s(s^i\mathfrak{d}^i(\rho, {}^*\rho))\|_{L^2(\underline{H}_{\underline{u}})} \\
&\lesssim 1 + \int_{-1}^{s}\delta a|s'|^{-2}ds' + \mathcal{R}_i[\rho, {}^*\rho]\,\delta^{\frac12}a^{\frac12}|s|^{-\frac12} \\
&\lesssim 1 + \mathcal{R}_i[\rho, {}^*\rho] + \delta a|s|^{-1} \lesssim 1 + \mathcal{R}_i[\rho, {}^*\rho]. \qquad \square
\end{aligned}
\tag{5.4}
$$

## 5.3 $L^2$ Estimate of $f$

To derive the estimate of $f$, we make use of the equation, see (2.14),

$$
\nabla_3 f + \frac12\mathrm{tr}\,\underline{\chi}f = 2\zeta - \widehat{\underline{\chi}}\cdot f, \quad f\big|_{H_{-1}} = 0.
$$

**Proposition 5.12.** *We have* $\|s^k\mathfrak{d}^k f\|_{L^2(S_{\underline{u},s})} \lesssim \mathcal{R}_k[\underline{\beta}]\,\delta a^{\frac12}$, $k\le N$.

*Proof.* Applying Lemma 5.2 (with $\lambda=\frac12$) and using the $L^2(S)$ estimate of $\zeta$ obtained[25] in Proposition 5.7,

[25]It is essential here to use the better result of Proposition 5.7 rather than $\zeta\in\check{\Gamma}_g\sim\delta a^{\frac12}|s|^{-2}$.

we derive

$$\begin{aligned}
\|s^i\mathfrak{d}^i f\|_{L^2(S_{\underline{u},s})} &\lesssim \|s^i\mathfrak{d}^i f\|_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^i \|\mathfrak{d}^i\zeta, \mathfrak{d}^i(\check{\Gamma}_g\cdot f)\|_{L^2(S_{\underline{u},s'})}\, ds' \\
&\lesssim 0 + \int_{-1}^{s} \delta a^{\frac12} + \mathcal{R}_i[\underline{\beta}]\,\delta^{\frac32} a|s'|^{-\frac32} + \mathcal{O}^2\delta^2 a|s'|^{-2}\, ds' \\
&\lesssim \delta a^{\frac12} + \mathcal{R}_i[\underline{\beta}]\,\delta^{\frac32} a|s|^{-\frac12} + \mathcal{O}^2\delta^2 a|s|^{-1} \\
&\lesssim \delta a^{\frac12}(1 + \mathcal{R}_i[\underline{\beta}]\,\delta^{\frac12} a^{\frac12}|s|^{-\frac12} + \mathcal{O}^2\delta a^{\frac12}|s|^{-1}) \lesssim (1+\mathcal{R}_i[\underline{\beta}])\,\delta a^{\frac12}. \qquad \square
\end{aligned}$$

## 5.4 $L^2(S)$-estimates for the curvature components

In what follows we derive non-top $L^2(S_{\underline{u},s})$ estimates[26] of the curvature components. We start with the following.

**Proposition 5.13.** *We have* $\|s^k\mathfrak{d}^k\underline{\alpha}\|_{L^2(S_{\underline{u},s})} \lesssim \mathcal{R}[\underline{\alpha}]\,\delta^{\frac52}a^{\frac32}|s|^{-\frac72}$, $k \le N-1$.

*Proof.* Applying Lemma 5.2 to "$\nabla_3\underline{\alpha} = \mathfrak{d}\underline{\alpha}$", we have

$$\begin{aligned}
\|s^{-1+i}\mathfrak{d}^i\underline{\alpha}\|_{L^2(S_{\underline{u},s})} &\lesssim \|s^{-1+i}\mathfrak{d}^i\underline{\alpha}\|_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^{-5}\|s'^3 s'^{i+1}\mathfrak{d}^{i+1}\underline{\alpha}\|_{L^2(S_{\underline{u},s'})}\, ds' \\
&\lesssim \delta^3 a^2 + \Big(\int_{-1}^{s}|s'|^{-10}ds'\Big)^{\frac12}\|s^3 s^{i+1}\mathfrak{d}^{i+1}\underline{\alpha}\|_{L^2(\underline{H}_{\underline{u}})} \\
&\lesssim \delta^3 a^2 + \mathcal{R}[\underline{\alpha}]\,\delta^{\frac52}a^{\frac32}|s|^{-\frac92} \lesssim \mathcal{R}[\underline{\alpha}]\,\delta^{\frac52}a^{\frac32}|s|^{-\frac92},
\end{aligned}$$

so multiplying both sides by $|s|$ we obtain the estimate. $\square$

**Proposition 5.14.** *We have* $\|s^k\mathfrak{d}^k\underline{\beta}\|_{L^2(S_{\underline{u},s})} \lesssim \delta^2 a^{\frac32}|s|^{-3}$, $k \le N-1$.

*Proof.* We apply Lemma 5.2 (with $\lambda = 2$, $i \le N-1$) to the equation $\nabla_3\underline{\beta} + \operatorname{div}\underline{\alpha} = -2\operatorname{tr}\underline{\chi}\,\underline{\beta} + 2\underline{\alpha}\cdot\zeta$. Making also use of the estimates for the Ricci coefficients and $\underline{\alpha}$ already obtained, we derive

$$\begin{aligned}
\|s^{3+i}\mathfrak{d}^i\underline{\beta}\|_{L^2(S_{\underline{u},s})} &\lesssim \|s^{3+i}\mathfrak{d}^i\underline{\beta}\|_{L^2(S_{\underline{u},-1})} + \int_{-1}^{s} |s'|^{3+i}\|\mathfrak{d}^{i+1}\underline{\alpha}\|_{L^2(S_{\underline{u},s'})} + |s'|^{3+i}\|\mathfrak{d}^i(\underline{\alpha}\cdot\zeta)\|_{L^2(S_{\underline{u},s'})}\, ds' \\
&\lesssim \delta^2 a^{\frac32} + \Big(\int_{-1}^{s}|s'|^{-2}ds'\Big)^{\frac12}\|s^3 s^{i+1}\mathfrak{d}^{i+1}\underline{\alpha}\|_{L^2(\underline{H}_{\underline{u}})} + \int_{-1}^{s}|s'|^3\mathcal{O}\mathcal{R}\delta a^{\frac12}|s'|^{-2}\cdot\delta^{\frac52}a^{\frac32}|s'|^{-\frac72}\, ds' \\
&\lesssim \delta^2 a^{\frac32} + \mathcal{R}\delta^{\frac52}a^{\frac32}|s|^{-\frac12} + \mathcal{O}\mathcal{R}[\underline{\alpha}]\,\delta^{\frac72}a^2|s|^{-\frac32} \\
&\lesssim (1 + \mathcal{R}\delta^{\frac12}|s|^{-\frac12} + \mathcal{O}\mathcal{R}\delta^{\frac32}a^{\frac12}|s|^{-\frac32})\,\delta^2 a^{\frac32} \lesssim \delta^2 a^{\frac32}. \qquad \square
\end{aligned}$$

**Proposition 5.15.** *We have* $\|s^k\mathfrak{d}^k(\rho, {}^*\rho)\|_{L^2(S_{\underline{u},s})} \lesssim \delta a|s|^{-2}$, $k \le N-1$.

*Proof.* We apply Lemma 5.2 (with $\lambda = \frac32$, $i \le N-1$) to

$$\begin{aligned}
\nabla_3\rho + \operatorname{div}\underline{\beta} &= -\frac32\operatorname{tr}\underline{\chi}\rho + \zeta\cdot\underline{\beta} - \frac12\widehat{\chi}\cdot\underline{\alpha}, \\
\nabla_3\,{}^*\rho + \operatorname{curl}\underline{\beta} &= -\frac32\operatorname{tr}\underline{\chi}\,{}^*\rho + \zeta\cdot{}^*\underline{\beta} - \frac12\widehat{\chi}\cdot{}^*\underline{\alpha},
\end{aligned}$$

[26] That is ignoring loss of derivatives.

$$||s^{2+i}\mathfrak{d}^i\rho||_{L^2(S_{\underline{u},s})} \lesssim ||s^{2+i}\mathfrak{d}^i\rho||_{L^2(S_{\underline{u},-1})} + \int_{-1}^s |s'|^{2+i}||\mathfrak{d}^{i+1}\underline{\beta}, \mathfrak{d}^i(\zeta\cdot\underline{\beta}), \mathfrak{d}^i(\widehat{\chi}\cdot\underline{\alpha})||_{L^2(S_{\underline{u},s'})}\, ds'$$

$$\lesssim \delta a + \Big(\int_{-1}^s |s'|^{-2}ds'\Big)^{\frac{1}{2}}||s^2 s^{i+1}\mathfrak{d}^{i+1}\underline{\beta}||_{L^2(\underline{H}_{\underline{u}})} + \int_{-1}^s |s|^2\mathcal{O}\mathcal{R}\delta a^{\frac{1}{2}}|s'|^{-2}\cdot\delta^2 a^{\frac{3}{2}}|s'|^{-3}\, ds'$$

$$+ \int_{-1}^s |s'|^2\mathcal{O}\mathcal{R}a^{\frac{1}{2}}|s'|^{-1}\cdot\delta^{\frac{5}{2}}a^{\frac{3}{2}}|s'|^{-\frac{7}{2}}\, ds'$$

$$\lesssim \delta a + \mathcal{R}[\underline{\beta}]\delta^{\frac{3}{2}}a|s|^{-\frac{1}{2}} + \mathcal{O}\mathcal{R}\delta^3 a^2|s|^{-2} + \mathcal{O}\mathcal{R}\delta^{\frac{5}{2}}a^2|s|^{-\frac{3}{2}}$$

$$\lesssim (1 + \mathcal{R}[\underline{\beta}]a^{-\frac{1}{2}} + \mathcal{O}\mathcal{R}a^{-\frac{1}{2}})\delta a \lesssim \delta a.$$

The estimate of ${}^*\rho$ follows in the same way. □

**Proposition 5.16.** *We have* $||s^k\mathfrak{d}^k\beta||_{L^2(S_{\underline{u},s})} \lesssim a^{\frac{1}{2}}|s|^{-1}$, $k \le N-1$.

*Proof.* We apply Lemma 5.2 (with $\lambda = 1$, $i \le N-1$) to $\nabla_3\beta + \text{div}\,\rho = -\text{tr}\,\underline{\chi}\beta + 2\underline{\beta}\cdot\widehat{\chi}$

$$||s^{1+i}\mathfrak{d}^i\beta||_{L^2(S_{\underline{u},s})} \le ||s^{1+i}\mathfrak{d}^i\beta||_{L^2(S_{\underline{u},-1})} + \int_{-1}^s |s'|^{1+i}||\mathfrak{d}^{i+1}\rho, \mathfrak{d}^i(\underline{\beta}\cdot\widehat{\chi})||_{L^2(S_{\underline{u},s'})}\, ds'$$

$$\lesssim a^{\frac{1}{2}} + \Big(\int_{-1}^s |s'|^{-2}ds'\Big)^{\frac{1}{2}}||s(s^{1+i}\mathfrak{d}^{1+i}\rho)||_{L^2(\underline{H}_{\underline{u}})} + \int_{-1}^s |s|\mathcal{O}\mathcal{R}a^{\frac{1}{2}}|s|^{-1}\cdot\delta^2 a^{\frac{3}{2}}|s'|^{-3}\, ds'$$

$$\lesssim a^{\frac{1}{2}} + \mathcal{R}[\rho]\delta^{\frac{1}{2}}a^{\frac{1}{2}}|s|^{-\frac{1}{2}} + \mathcal{O}\mathcal{R}\delta^2 a^2|s|^{-2} \lesssim a^{\frac{1}{2}}.$$ □

**Proposition 5.17.** *We have* $||s^k\mathfrak{d}^k\alpha||_{L^2(S_{\underline{u},s})} \lesssim \delta^{-1}a^{\frac{1}{2}}$, $k \le N-1$.

*Proof.* We apply[27] Lemma 5.2 (with $\lambda = 1/2$, $i \le N-1$) to $\nabla_3\alpha - \nabla\widehat{\otimes}\beta = -\frac{1}{2}\text{tr}\,\underline{\chi}\alpha + \zeta\widehat{\otimes}\beta - 3(\rho\widehat{\chi} + {}^*\rho\,{}^*\widehat{\chi})$

$$||s^i\mathfrak{d}^i\alpha||_{L^2(S_{\underline{u},s})} \lesssim ||s^i\mathfrak{d}^i\alpha||_{L^2(S_{\underline{u},-1})} + \int_{-1}^s |s'|^i||\mathfrak{d}^{i+1}\beta, \mathfrak{d}^i(\rho\cdot\widehat{\chi})||_{L^2(S_{\underline{u},s'})}\, ds'$$

$$\lesssim \delta^{-1}a^{\frac{1}{2}} + \Big(\int_{-1}^s |s'|^{-2}ds'\Big)^{\frac{1}{2}}||s^{i+1}\mathfrak{d}^{i+1}\beta||_{L^2(\underline{H}_{\underline{u}})} + \int_{-1}^s \mathcal{O}\mathcal{R}a^{\frac{1}{2}}|s|^{-1}\cdot\delta a|s'|^{-2}\, ds'$$

$$\lesssim \delta^{-1}a^{\frac{1}{2}} + \mathcal{R}[\beta]\,\delta^{-\frac{1}{2}}a^{\frac{1}{2}}|s|^{-\frac{1}{2}} + \mathcal{O}\mathcal{R}\delta a^{\frac{3}{2}}|s|^{-2}$$

$$\lesssim \delta^{-1}a^{\frac{1}{2}}(1 + \mathcal{R}[\beta]\,\delta^{\frac{1}{2}}|s|^{-\frac{1}{2}} + \mathcal{O}\mathcal{R}\delta^2 a|s|^{-2}) \lesssim \delta^{-1}a^{\frac{1}{2}}.$$ □

## 5.5 Improved estimate for $\xi$

We improve the estimate for $\xi$ obtained in Proposition 5.6 for all but the top derivatives.

**Proposition 5.18.** *We have* $||s^k\mathfrak{d}^k\xi||_{L^2(S_{\underline{u},s})} \lesssim a^{\frac{1}{2}}$, $k \le N-1$.

*Proof.* We proceed as in the proof of Proposition 5.6, starting with $\nabla_3\xi = \widehat{\chi}\cdot\zeta + \check{\Gamma}_b\cdot\zeta + \beta$, by taking into account the new bound for $\beta$ just derived in Proposition 5.16. Thus, for $i \le N-1$,

$$||\,|s|^{i-1}\mathfrak{d}^i\xi||_{L^2(S_{\underline{u},s})} \lesssim ||\,|s|^{i-1}\mathfrak{d}^i\xi||_{L^2(S_{\underline{u},-1})} + \int_{-1}^s |s'|^{i-1}||\mathfrak{d}^i(\widehat{\chi}\cdot\zeta)||_{L^2(S_{\underline{u},s'})}ds'$$

$$+ \int_{-1}^s |s'|^{i-1}||\mathfrak{d}^i\beta||_{L^2(S_{\underline{u},s'})}ds'$$

$$\lesssim 0 + \int_{-1}^s |s'|^{-1}\mathcal{O}a^{\frac{1}{2}}\cdot\delta a^{\frac{1}{2}}|s'|^{-2} + \int_{-1}^s a^{\frac{1}{2}}|s'|^{-2}\,ds'$$

$$\lesssim \mathcal{O}^2\delta a|s|^{-2} + a^{\frac{1}{2}}|s|^{-1} = a^{\frac{1}{2}}|s|^{-1}(1 + \mathcal{O}^2\delta a^{\frac{1}{2}}|s|^{-1}) \lesssim a^{\frac{1}{2}}|s|^{-1}.$$ □

[27] We use $\rho$ in the estimates below to represent both $\rho$ and ${}^*\rho$. We omit the estimate for $\zeta\widehat{\otimes}\beta$ as it is even better than $\check{\Gamma}_g\cdot\alpha$.

### 5.6 Summary of the results proved in this section

**Proposition 5.19.** *The following estimates hold true:*[28]

$$\mathcal{O}_{\le N} + \mathcal{R}^S_{\le N-1} \lesssim \mathcal{R}_{\le N}.$$

*We also have the improved estimates*

$$\mathcal{O}_{\le N}[\widehat{\chi}, \widetilde{\mathrm{tr}\,\underline{\chi}}, \widehat{\underline{\chi}}, \zeta] + \mathcal{R}^S_{\le N-1}[\underline{\beta}, \rho, {}^*\rho, \beta, \alpha] \lesssim 1.$$

With the help of the non-integrable Sobolev estimates of Lemma 4.1 we also obtain $O_{\le N-3,\infty} \lesssim \mathcal{R}_{\le N}$. We state the precise estimates:

**Corollary 5.20.** *For $k \le N-3$, we have the estimates:*

$$||s^k\mathfrak{d}^k\widehat{\chi}||_{L^\infty(S_{\underline{u},s})} \le \frac{a^{\frac{1}{2}}}{|s|}, \quad ||s^k\mathfrak{d}^k\omega||_{L^\infty(S_{\underline{u},s})} \le \mathcal{R}[\rho]\frac{\delta^{\frac{1}{2}}a^{\frac{1}{2}}}{|s|^{\frac{3}{2}}}, \quad ||s^k\mathfrak{d}^k\mathrm{tr}\,\chi||_{L^\infty(S_{\underline{u},s})} \lesssim \mathcal{R}[\rho]\frac{1}{|s|},$$

$$||s^k\mathfrak{d}^k(\widetilde{\mathrm{tr}\,\underline{\chi}}, \widehat{\underline{\chi}})||_{L^\infty(S_{\underline{u},s})} \le \frac{\delta a^{\frac{1}{2}}}{|s|^2}, \quad ||s^k\mathfrak{d}^k\zeta||_{L^\infty(S_{\underline{u},s})} \le \mathcal{R}[\beta]\frac{\delta a^{\frac{1}{2}}}{|s|^2}, \quad ||s^k\mathfrak{d}^k\xi||_{L^\infty(S_{\underline{u},s})} \lesssim \frac{a^{\frac{1}{2}}}{|s|},$$

$$||s^k\mathfrak{d}^k f||_{L^\infty(S_{\underline{u},s})} \lesssim \mathcal{R}[\underline{\beta}]\frac{\delta a^{\frac{1}{2}}}{|s|}, \quad ||s^k\mathfrak{d}^k\underline{\alpha}||_{L^\infty(S_{\underline{u},s})} \lesssim \mathcal{R}[\underline{\alpha}]\frac{\delta^{\frac{5}{2}}a^{\frac{3}{2}}}{|s|^{\frac{9}{2}}}, \quad ||s^k\mathfrak{d}^k\underline{\beta}||_{L^\infty(S_{\underline{u},s})} \lesssim \frac{\delta^2a^{\frac{3}{2}}}{|s|^4},$$

$$||s^k\mathfrak{d}^k(\rho, {}^*\rho)||_{L^\infty(S_{\underline{u},s})} \lesssim \frac{\delta a}{|s|^3}, \quad ||s^k\mathfrak{d}^k\beta||_{L^\infty(S_{\underline{u},s})} \lesssim \frac{a^{\frac{1}{2}}}{|s|^2}, \quad ||s^k\mathfrak{d}^k\alpha||_{L^\infty(S_{\underline{u},s})} \lesssim \frac{\delta^{-1}a^{\frac{1}{2}}}{|s|}.$$

Note that for $\xi$, we have used the improved bound obtained in Proposition 5.18.

## 6 Energy Estimates

**Notations.** We make the following notational conventions to be used throughout this section.

1. Whenever we use the index $i_1$, we mean summation over $i_1 = 0, 1, \cdots, i$ (with the covention that replaces $i_1 - 1$ by 0 if $i_1 = 0$); whenever we use the index $i_2$, we mean summation over $i_2 = 0, 1, \cdots, [i/2]$. In these situations we drop the summation symbol.
2. We use $S$, when there is no ambiguity, to denote the spheres $S_{\underline{u},s}$.
3. We use the double integral sign $\iint$ to denote either a full spacetime integral or the integral over the $\underline{u}$, $s$ variable (i.e. the non-angular variables).

### 6.1 Integrating region

Recall $\tau = \frac{1}{10}a\underline{u} + s$. In the region $\{\tau \le \tau^*\}$ with $\tau^* \le -\frac{1}{8}a\delta$, we have, using (1.2),

$${}^{(g)}e_3(\tau) = {}^{(g)}e_3(s) = 1, \quad {}^{(g)}e_4(\tau) = \frac{1}{10}a + {}^{(g)}e_4(s). \tag{6.1}$$

We need to estimate ${}^{(g)}e_4(s)$ for which we use the formula, see (1.3), $-2\,{}^{(g)}\omega = {}^{(g)}e_3({}^{(g)}e_4(s))$. Moreover, the estimates in the PT frame, together with the transformation formula for $\omega$ (see Lemma 2.2) imply that

$$|{}^{(g)}\omega| \lesssim |\omega| + |f| \cdot |(\zeta, \underline{\eta})| + |s|^{-1}|f|^2 \lesssim \mathcal{O}\delta^{\frac{1}{2}}a^{\frac{1}{2}}|s|^{-\frac{3}{2}} + \mathcal{O}^2\delta^2a|s|^{-3}.$$

Then using $e_4(s) = 0$ on $H_{-1}$ and integrating in ${}^{(g)}e_3 = \partial_s$ direction, we obtain

$$|{}^{(g)}e_4(s)| \lesssim \mathcal{O}\delta^{\frac{1}{2}}a^{\frac{1}{2}}|s|^{-\frac{1}{2}} + \mathcal{O}^2\delta^2a|s|^{-2} \lesssim \mathcal{O} \ll a/10. \tag{6.2}$$

[28] See (3.3) for the definition of the norms.

In particular ${}^{(g)}e_4(\tau) > 0$. Let $(\operatorname{grad}\tau)^\mu$ be the vectorfield perpendicular to the level surfaces of $\tau$ defined by $(\operatorname{grad}\tau)^\mu = g^{\mu\nu}\partial_\nu\tau$. We have

$$(\operatorname{grad}\tau)^\mu = g^{\mu\nu}\partial_\nu\tau = -\frac{1}{2}\big({}^{(g)}e_3(\tau)\,{}^{(g)}e_4 + {}^{(g)}e_4(\tau)\,{}^{(g)}e_3\big) = -\frac{1}{2}\big({}^{(g)}e_4 + {}^{(g)}e_4(\tau)\,{}^{(g)}e_3\big),$$

so

$$g(\operatorname{grad}\tau, \operatorname{grad}\tau) \quad = \quad -{}^{(g)}e_4(\tau) = -\Big(\frac{a}{10} + {}^{(g)}e_4(s)\Big),$$

which is strictly negative. This shows, in particular, that $\Sigma_\tau$ is a spacelike hypersurface with future unit normal given by

$$N_\tau = -\frac{\operatorname{grad}\tau}{|\operatorname{grad}\tau|}.$$

## 6.2 Divergence Lemma

We apply the spacetime divergence lemma (see e.g. [14], [15]) to causal domains of the form $\mathcal{M} \subset \mathcal{M}(\delta, a; \tau_*)$ enclosed by $\Sigma_\tau = \{\tau = \text{const}\} \cup \underline{H}_{\underline{u}}$ to the future and $H_{-1} \cup \underline{H}_0$ to the past.

**Lemma 6.1.** *Consider a vectorfield $X$ on a causal domain $\mathcal{M} \subset \mathcal{M}(\delta, a; \tau_*)$ enclosed by $\Sigma_\tau = \{\tau = \text{const}\} \cup \underline{H}_{\underline{u}}$ to the future and $H_{-1} \cup \underline{H}_0$ to the past. Then*

$$\int_{\Sigma_\tau} g(X, N_\tau) + \int_{\underline{H}_{\underline{u}}} g(X, e_3) = \int_{H_{-1}} g(X, e_4) + \int_{\underline{H}_0} g(X, e_3) - \iint_{\mathcal{M}} (\operatorname{Div} X)$$

*where the integrations on $\Sigma_\tau$ and $\mathcal{M}$ are with respect to their standard area and volume forms and $N_\tau = -\frac{\operatorname{grad}\tau}{|\operatorname{grad}\tau|}$ is the future unit normal to $\Sigma_\tau$. The integrations on the null hypersurfaces $\underline{H}_{\underline{u}}$ and $H_{-1}$ of scalar functions $f$ are defined as follows*

$$\int_{\underline{H}_{\underline{u}}} f = \int_s ds \int_{S_{\underline{u},s}} f d\text{vol}_S, \qquad \int_{H_{-1}} f = \int_{\underline{u}} d\underline{u} \int_{S_{\underline{u},-1}} f d\text{vol}_S$$

*Proof.* Immediate application of the Stokes formula applied to the differential form $({}^*X)_{\alpha\beta\gamma} = \in_{\alpha\beta\gamma\mu} X^\mu$ by observing that ${}^*(d\,{}^*X) = \operatorname{Div} X$. See Section 8.1 in [14] for the details. □

**Corollary 6.2.** *Consider the vectorfield $X = \lambda_3 e_3 + \lambda_4 e_4$, where $\lambda_3, \lambda_4$ are given smooth functions. Then, integrating on the same domain $\mathcal{M}$,*

$$\int_{\Sigma_\tau} \frac{1}{|\operatorname{grad}\tau|}\left(\lambda_3 + \lambda_4\left(\frac{a}{10} + e_4(s)\right)\right) + \int_{\underline{H}_{\underline{u}}} 2\lambda_4 \quad = \int_{\underline{H}_0} 2\lambda_4 + \int_{H_{-1}} 2\lambda_3 + \iint_{\mathcal{M}} (\operatorname{Div} X) \tag{6.3}$$

*with $|\operatorname{grad}\tau| \approx a^{1/2}$.*

*Proof.* We have

$$\begin{aligned} g(X, N_\tau) &= -\frac{1}{|\operatorname{grad}\tau|}(\lambda_3 e_3(\tau) + \lambda_4 e_4(\tau)) = -\frac{1}{|\operatorname{grad}\tau|}\left(\lambda_3 + \lambda_4\left(\frac{a}{10} + e_4(s)\right)\right), \\ g(X, e_3) &= -2\lambda_4, \qquad g(X, e_4) = -2\lambda_3, \end{aligned}$$

and the result follows from Lemma 6.1. □

## 6.3 Estimates for general Bianchi pairs

**Definition 6.3.** *We denote $\mathfrak{s}_0$ by the set of pairs of scalar fields in the spacetime, $\mathfrak{s}_1$ by the set of $\mathcal{H}$-horizontal 1-forms, and $\mathfrak{s}_2$ by the set of symmetric traceless $\mathcal{H}$-horizontal covariant 2-tensors.*

**Definition 6.4.** *We consider the non-integrable horizontal Hodge-type operators*[29]

- $\not{\mathcal{D}}_1$ *takes* $\mathfrak{s}_1$ *into* $\mathfrak{s}_0$: $\quad \not{\mathcal{D}}_1\xi = (\operatorname{div}\xi, \operatorname{curl}\xi)$,
- $\not{\mathcal{D}}_2$ *takes* $\mathfrak{s}_2$ *into* $\mathfrak{s}_1$: $\quad (\not{\mathcal{D}}_2\xi)_a = \nabla^b\xi_{ab}$,
- $\not{\mathcal{D}}_1^*$ *takes* $\mathfrak{s}_0$ *into* $\mathfrak{s}_1$: $\quad (\not{\mathcal{D}}_1^*(f,f_*))_a = -\nabla_a f + \in_{ab}\nabla_b f_*$,
- $\not{\mathcal{D}}_2^*$ *takes* $\mathfrak{s}_1$ *into* $\mathfrak{s}_2$: $\quad \not{\mathcal{D}}_2^*\xi = -\frac{1}{2}\nabla\widehat{\otimes}\xi$.

**Lemma 6.5.** *The following identities hold:*

$$\begin{aligned}\not{\mathcal{D}}_1^*(f,f_*)\cdot u &= (f,f_*)\cdot\not{\mathcal{D}}_1 u - \nabla_a(fu^a + f_*({}^*u)^a), \quad (f,f_*)\in\mathfrak{s}_0, \quad u\in\mathfrak{s}_1,\\ (\not{\mathcal{D}}_2^* f)\cdot u &= f\cdot(\not{\mathcal{D}}_2 u) - \nabla_a(f_b u^{ab}), \quad f\in\mathfrak{s}_1, \quad u\in\mathfrak{s}_2.\end{aligned} \tag{6.4}$$

*Proof.* Direct calculation. See Lemma 2.1.23 in [15]. □

**Definition 6.6.** *We consider the following two types of abstract Bianchi pairs*[30]*:*

*Type I. These are systems in $\psi_1\in\mathfrak{s}_k$ and $\psi_2\in\mathfrak{s}_{k-1}$ $(k=1,2)$ of the form*

$$\begin{aligned}\nabla_3\psi_1 + \lambda\operatorname{tr}\underline{\chi}\psi_1 &= -k\not{\mathcal{D}}_k^*\psi_2 + F_1,\\ \nabla_4\psi_2 &= \not{\mathcal{D}}_k\psi_1 + F_2,\end{aligned} \tag{6.5}$$

*Type II. These are systems in $\psi_1\in\mathfrak{s}_{k-1}$ and $\psi_2\in\mathfrak{s}_k$ $(k=1,2)$ of the form*

$$\begin{aligned}\nabla_3\psi_1 + \lambda\operatorname{tr}\underline{\chi}\psi_1 &= \not{\mathcal{D}}_k\psi_2 + F_1,\\ \nabla_4\psi_2 &= -k\not{\mathcal{D}}_k^*\psi_1 + F_2,\end{aligned} \tag{6.6}$$

The main goal of this subsection is to prove the following lemma:

**Lemma 6.7.** *Suppose that the bootstrap assumption holds. Then, for both pairs* (6.5), (6.6), *we have the estimate*

$$\begin{aligned}&a^{-\frac{1}{2}}\int_{\Sigma_\tau} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 + \int_{\underline{H}_u} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 \lesssim \int_{H_{-1}} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 + \int_{\underline{H}_0} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2\\ &+\sum_{j=0}^{i}|s|^{-j}\Big|\iint_{\mathcal{M}} s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1\cdot\mathfrak{d}^{i-j}F_1\Big| + \Big|\iint_{\mathcal{M}} s^{2i+4\lambda-2}\mathfrak{d}^i\psi_2\cdot\mathfrak{d}^i F_2\Big|.\end{aligned} \tag{6.7}$$

*Proof.* We prove the estimate for the first type; the second type follows in the same way. Commuting the equation with $\mathfrak{d}^i$, using the commutator Lemma[31] 2.6, we derive[32]

$$\begin{aligned}\nabla_3\mathfrak{d}^i\psi_1 + \Big(\lambda + \frac{i}{2}\Big)\operatorname{tr}\underline{\chi}\,\mathfrak{d}^i\psi_1 &= -k\not{\mathcal{D}}_k^*\mathfrak{d}^i\psi_2 + F_1^i,\\ \nabla_4\mathfrak{d}^i\psi_2 &= \not{\mathcal{D}}_k\mathfrak{d}^i\psi_1 + F_2^i,\end{aligned} \tag{6.8}$$

where $F_1^0 = F_1, F_2^0 = F_2$, as in (6.5)-(6.6), and for $i\geq 1$

$$\begin{aligned}F_1^i &= \sum_{j=0}^{i}|s|^{-j}\mathfrak{d}^{i-j}F_1 + \mathfrak{d}^{i-1}(\check{\Gamma}_g\cdot\mathfrak{d}\psi_1) + \mathfrak{d}^{i-1}(\underline{\beta}^*\cdot\psi_1) + k[\not{\mathcal{D}}_k^*,\mathfrak{d}^i]\psi_2,\\ F_2^i &= \mathfrak{d}^i F_2 + [\nabla_4,\mathfrak{d}^i]\psi_2 - [\not{\mathcal{D}}_k,\mathfrak{d}^i]\psi_1.\end{aligned} \tag{6.9}$$

[29] See [14] for the original definitions and [15] for the extensions to the non-integrable case.

[30] The null Bianchi identities in Proposition 2.5 can be split in the pairs $(\alpha,\beta)$, $\big(\beta,(\rho,\ {}^*\rho)\big)$, $\big((\rho,\ {}^*\rho),\underline{\beta}\big)$ and $(\underline{\beta},\underline{\alpha})$ which fit into one of the two types described here.

[31] We deal with the commutation between $\nabla_3$ and $\mathfrak{d}^i$ in the same way as in Section 5.

[32] Here the Hodge-type operators act with the indices of $\psi_1$, $\psi_2$, e.g., $(\not{\mathcal{D}}_2\mathfrak{d}^i\alpha)_a = \nabla^b\mathfrak{d}^i\alpha_{ab}$, $\quad(\not{\mathcal{D}}_2^*\mathfrak{d}^i\beta)_{ab} = -\frac{1}{2}\nabla_a\widehat{\otimes}(\mathfrak{d}^i\beta)_b$.

We next make use of the formulas

$$\mathrm{Div}\, e_3 = -2\underline{\omega} + \mathrm{tr}\,\underline{\chi} = \mathrm{tr}\,\underline{\chi}, \qquad \mathrm{Div}\, e_4 = -2\omega + \mathrm{tr}\,\chi \tag{6.10}$$

to calculate the divergence of the vectorfield

$$X = s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 e_3 + ks^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 e_4,$$

$$\begin{aligned}
\mathrm{Div}\, X &= s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 \mathrm{Div}\, e_3 + e_3(s^{2i+4\lambda-2}|\mathfrak{d}^i\psi|^2) + ks^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 \mathrm{Div}\, e_4 \\
&\quad + ke_4(s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2) \\
&= s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 \mathrm{tr}\,\underline{\chi} + (2i+4\lambda-2)s^{2i+4\lambda-3}|\mathfrak{d}^i\psi_1|^2 + 2s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot \nabla_3 \mathfrak{d}^i\psi_1 \\
&\quad + ks^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2(-2\omega + \mathrm{tr}\,\chi) + k(2i+4\lambda-2)s^{2i+4\lambda-3} e_4(s)|\mathfrak{d}^i\psi_2|^2 \\
&\quad + 2ks^{2i+4\lambda-2}\mathfrak{d}^i\psi_2 \cdot \nabla_4 \mathfrak{d}^i\psi_2 \\
&= s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 \mathrm{tr}\,\underline{\chi} + (2i+4\lambda-2)s^{2i+4\lambda-3}|\mathfrak{d}^i\psi_1|^2 \\
&\quad + 2s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot \left(-k\mathcal{D}_k^*\psi_2 - \left(\frac{i}{2}+\lambda\right)\mathrm{tr}\,\underline{\chi}\,\mathfrak{d}^i\psi_1 + F_1^i\right) \\
&\quad + ks^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2(-2\omega + \mathrm{tr}\,\chi) + k(2i+4\lambda-2)s^{2i+4\lambda-3} e_4(s)|\mathfrak{d}^i\psi_2|^2 \\
&\quad + 2ks^{2i+4\lambda-2}\mathfrak{d}^i\psi_2 \cdot (\mathcal{D}_k \mathfrak{d}^i\psi_1 + F_2^i) \\
&= (i+2\lambda-1)s^{2i+4\lambda-2}\left(\frac{2}{s} - \mathrm{tr}\,\underline{\chi}\right)|\mathfrak{d}^i\psi_1|^2 + 2s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot F_1^i - 2ks^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot \mathcal{D}_k^*\mathfrak{d}^i\psi_2 \\
&\quad + ks^{2i+4\lambda-2}\left(-2\omega + \mathrm{tr}\,\chi + (2i+4\lambda-2)s^{-1}e_4(s)\right)|\mathfrak{d}^i\psi_2|^2 + 2ks^{2i+4\lambda-2}\mathfrak{d}^i\psi_2 \cdot F_2^i \\
&\quad + 2ks^{2i+4\lambda-2}\mathfrak{d}^i\psi_2 \cdot \mathcal{D}_k \mathfrak{d}^i\psi_1.
\end{aligned}$$

Note that

$$\mathfrak{d}^i\psi_2 \cdot \mathcal{D}_k \mathfrak{d}^i\psi_1 - \mathfrak{d}^i\psi_1 \cdot \mathcal{D}_k^* \mathfrak{d}^i\psi_2 = \mathrm{div}\,(\mathfrak{d}^i\psi_1 \cdot \mathfrak{d}^i\psi_2),$$

which is a direct generalization of Lemma 6.5. Therefore we get

$$\begin{aligned}
\mathrm{Div}\, X &= -(i+2\lambda-1)s^{2i+4\lambda-2}\widetilde{\mathrm{tr}\,\underline{\chi}}\,|\mathfrak{d}^i\psi_1|^2 + 2s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot F_1^i \\
&\quad + ks^{2i+4\lambda-2}\left(-2\omega + \mathrm{tr}\,\chi + (2i+4\lambda-2)s^{-1}e_4(s)\right)|\mathfrak{d}^i\psi_2|^2 + 2ks^{2i+4\lambda-2}\mathfrak{d}^i\psi_2 \cdot F_2^i \\
&\quad + 2ks^{2i+4\lambda-2}\mathrm{div}\,(\mathfrak{d}^i\psi_1 \cdot \mathfrak{d}^i\psi_2).
\end{aligned}$$

The last term is equal to

$$2ks^{2i+4\lambda-2}\mathrm{div}\,(\mathfrak{d}^i\psi_1 \cdot \mathfrak{d}^i\psi_2) = 2k\mathrm{div}\,(s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot \mathfrak{d}^i\psi_2) - 2k(2i+4\lambda-2)s^{2i+4\lambda-2}s^{-1}\nabla_a(s)(\mathfrak{d}^i\psi_1 \cdot \mathfrak{d}^i\psi_2)_a.$$

Therefore,

$$\begin{aligned}
\mathrm{Div}\, X &= 2k\mathrm{div}\,(s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot \mathfrak{d}^i\psi_2) - (i+2\lambda-1)s^{2i+4\lambda-2}\widetilde{\mathrm{tr}\,\underline{\chi}}\,|\mathfrak{d}^i\psi_1|^2 \\
&\quad + ks^{2i+4\lambda-2}\Big(-2\omega + \mathrm{tr}\,\chi + (2i+4\lambda-2)s^{-1}e_4(s)\Big)|\mathfrak{d}^i\psi_2|^2 \\
&\quad - 2k(2i+4\lambda-2)s^{2i+4\lambda-2}s^{-1}\nabla_a(s)(\mathfrak{d}^i\psi_1 \cdot \mathfrak{d}^i\psi_2)_a \\
&\quad + 2s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot F_1^i + 2ks^{2i+4\lambda-2}\mathfrak{d}^i\psi_2 \cdot F_2^i.
\end{aligned} \tag{6.11}$$

To derive our final result it remains to integrate (6.11) on $\mathcal{M}$. In view of the lack of integrability of our PT frame, we need however to replace div with Div with the help of the formula, for an arbitrary $\mathcal{H}$-horizontal 1-form $\Psi$, see[33] [15, Lemma 2.40]

$$\mathrm{Div}\,\Psi = \mathrm{div}\,\Psi + \underline{\eta} \cdot \Psi.$$

[33] Note that in our case, we have $\eta = 0$.

Integrating this relation over $\mathcal{M}$ and using the divergence formula in Lemma 6.1 for $\Psi^{\#}$ (the index raising with respect to $g$), we obtain, in view of the fact that $\Psi_3 = \Psi_4 = 0$,

$$\begin{aligned}\iint_{\mathcal{M}} \operatorname{div}\Psi &= \iint_{\mathcal{M}} \operatorname{Div}\Psi - \underline{\eta}\cdot\Psi = -\int_{\Sigma_\tau} g(e_a, N_\tau)\Psi_a - \iint_{\mathcal{M}} \underline{\eta}\cdot\Psi \\ &= \int_{\Sigma_\tau} |\operatorname{grad}\tau|^{-1}\nabla(\tau)\cdot\Psi - \iint_{\mathcal{M}} \underline{\eta}\cdot\Psi.\end{aligned} \tag{6.12}$$

Note that we have $|\operatorname{grad}\tau|^{-1}e_a(\tau) = |\operatorname{grad}\tau|^{-1}e_a(s) = -|\operatorname{grad}\tau|^{-1}f$. Since $|\operatorname{grad}\tau|^{-1} \approx a^{-\frac{1}{2}}$, applying (6.12) to $\Psi = s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1\cdot\mathfrak{d}^i\psi_2$, we obtain

$$\Big|\iint_{\mathcal{M}} 2k\operatorname{div}(s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1\cdot\mathfrak{d}^i\psi_2)\Big| \lesssim a^{-\frac{1}{2}}\int_{\Sigma_\tau}\Big|f\cdot(s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1\cdot\mathfrak{d}^i\psi_2)\Big| + \Big|\iint_{\mathcal{M}}\underline{\eta}\cdot(s^{2i+4\lambda-2}\mathfrak{d}^i\psi_1\cdot\mathfrak{d}^i\psi_2)\Big|.$$

We then apply Corollary 6.2 to (6.11) with $X = s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2e_3 + ks^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2e_4$ to derive

$$\begin{aligned}&a^{-\frac{1}{2}}\int_{\Sigma_\tau} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 + \Big(\frac{a}{10}+e_4(s)\Big)ks^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 + \int_{\underline{H}_{\underline{u}}} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 \\ &\lesssim \int_{H_{-1}} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 + \int_{\underline{H}_0} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 + a^{-\frac{1}{2}}\int_{\Sigma_\tau} s^{2i+4\lambda-2}\Big|f\cdot\mathfrak{d}^i\psi_1\cdot\mathfrak{d}^i\psi_2\Big| \\ &\quad + \iint_{\mathcal{M}} s^{2i+4\lambda-2}\Big(|(\check{\Gamma}_b + s^{-1}e_4(s))||\mathfrak{d}^i\psi_2|^2 + |\check{\Gamma}_g||\mathfrak{d}^i\psi_1|^2 + |\underline{\eta}||\mathfrak{d}^i\psi_1||\mathfrak{d}^i\psi_2|\Big) \\ &\quad + \Big|\iint_{\mathcal{M}} s^{2i+4\lambda-2}(\mathfrak{d}^i\psi_1\cdot F_1^i + \mathfrak{d}^i\psi_2\cdot F_2^i)\Big|.\end{aligned}$$

Clearly, due to the smallness of $|f|$, the weighted integral of $f\cdot\mathfrak{d}^i\psi_1\cdot\mathfrak{d}^i\psi_2$ can be absorbed by the flux terms over $\Sigma_\tau$ on the left-hand side.

We now estimate the spacetime integrals on the right-hand side. Note that on $\Sigma_\tau$, $|s| = |\tau| + \frac{1}{10}|a\underline{u}| \geq |\tau|$. Also, the bound of ${}^{(g)}e_4(s)$ in (6.2) implies $|e_4(s)| \lesssim \mathcal{O}$. Then,

$$\begin{aligned}\iint_{\mathcal{M}} s^{2i+4\lambda-2}|\check{\Gamma}_g||\mathfrak{d}^i\psi_1|^2\,d\mathrm{vol} &\lesssim \int_{-1}^{-a\delta}\left(\int_{\Sigma_\tau}\mathcal{O}\frac{\delta a^{\frac{1}{2}}}{|s|^2}s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2\,a^{-\frac{1}{2}}\,d\Sigma_\tau\right)d\tau \\ &\lesssim a^{-\frac{1}{2}}\sup_\tau\Big(\int_{\Sigma_\tau} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2\,d\Sigma_\tau\Big)\cdot\int_{-1}^{-a\delta}\mathcal{O}\frac{\delta a^{\frac{1}{2}}}{|\tau|^2}d\tau \\ &\lesssim a^{-\frac{1}{2}}\sup_\tau\Big(\int_{\Sigma_\tau} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2\,d\Sigma_\tau\Big)\cdot\frac{\mathcal{O}}{a^{\frac{1}{2}}},\end{aligned}$$

and

$$\iint_{\mathcal{M}}|(\check{\Gamma}_b + s^{-1}e_4(s))||\mathfrak{d}^i\psi_2|^2\,d\mathrm{vol} \leq \int_0^\delta\Big(\int_{\underline{H}_{\underline{u}}}\mathcal{O}|s|^{-1}|\mathfrak{d}^i\psi_2|^2\Big)d\underline{u} \leq \frac{\mathcal{O}}{a}\cdot\sup_{\underline{u}}\int_{\underline{H}_{\underline{u}}}|\mathfrak{d}^i\psi_2|^2. \tag{6.13}$$

Therefore, taking the supremum over $\tau$ and $\underline{u}$, we can absorb several terms on the right by the left-hand side and obtain

$$\begin{aligned}&a^{-\frac{1}{2}}\sup_\tau\int_{\Sigma_\tau} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 + \sup_{\underline{H}_{\underline{u}}}\int_{\underline{H}_{\underline{u}}} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 \lesssim \int_{H_{-1}} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2 \\ &+ \int_{\underline{H}_0} s^{2i+4\lambda-2}|\mathfrak{d}^i\psi_2|^2 + \Big|\iint_{\mathcal{M}} s^{2i+4\lambda-2}(\mathfrak{d}^i\psi_1\cdot F_1^i + \mathfrak{d}^i\psi_2\cdot F_2^i)\Big|.\end{aligned}$$

It remains to estimate the terms $F_1^i, F_2^i$ in (6.9). We have, schematically, using (2.16) and ${}^{(a)}\mathrm{tr}\,\underline{\chi} = 0$,

$$[\not{\mathcal{D}}, \nabla^i]\psi = \sum_{j=0}^{i-1}\nabla^j\big({}^{(h)}K\psi + {}^{(a)}\mathrm{tr}\,\chi\nabla_3\psi\big)$$

where $\not{\mathcal{D}}$ stands for any of the four Hodge-type operators in Definition 6.4. We also have, by (2.15),

$$[\not{\mathcal{D}}, \nabla_3^i]\psi = \mathfrak{d}^{i-1}(|s|^{-1}\mathfrak{d}\psi) + \mathfrak{d}^{i-1}(\check{\Gamma}_g \cdot \mathfrak{d}\psi) + \mathfrak{d}^{i-1}(\underline{\beta} \cdot \psi),$$

so composing these two formulas we get

$$[\not{\mathcal{D}}, \mathfrak{d}^i]\psi = \mathfrak{d}^{i-1}\left(|s|^{-1}\mathfrak{d}\psi\right) + \mathfrak{d}^{i-1}\left({}^{(a)}\mathrm{tr}\,\chi \cdot \mathfrak{d}\psi\right) + \mathfrak{d}^{i-1}\left(\underline{\beta} \cdot \psi + {}^{(h)}K \cdot \psi\right)$$

Also,

$$[\nabla_4, \mathfrak{d}^i]\psi = \mathfrak{d}^{i-1}\left((\check{\Gamma}_b, \xi) \cdot \mathfrak{d}\psi\right) + \mathfrak{d}^{i-1}\left((|s|^{-1}\xi,\ {}^*\beta,\ {}^*\rho) \cdot \psi\right).$$

Therefore

$$\begin{aligned} F_1^i &= |s|^{-j}\mathfrak{d}^{i-j}F_1 + \mathrm{err}_1^i, \\ F_2^i &= \mathfrak{d}^i F_2 + \mathrm{err}_2^i, \end{aligned} \tag{6.14}$$

where

$$\begin{aligned} \mathrm{err}_1^i &= \mathfrak{d}^{i-1}\left(\check{\Gamma}_g \cdot \mathfrak{d}\psi_1\right) + \mathfrak{d}^{i-1}\left(\underline{\beta}^* \cdot \psi_1\right) \\ &\quad + \mathfrak{d}^{i-1}\left(|s|^{-1}\mathfrak{d}\psi_2\right) + \mathfrak{d}^{i-1}\left({}^{(a)}\mathrm{tr}\,\chi \cdot \mathfrak{d}\psi_2\right) + \mathfrak{d}^{i-1}\left(\underline{\beta} \cdot \psi_2 + {}^{(h)}K \cdot \psi_2\right), \\ \mathrm{err}_2^i &= \mathfrak{d}^{i-1}\left((\check{\Gamma}_b, \xi) \cdot \mathfrak{d}\psi_2\right) + \mathfrak{d}^{i-1}\left((|s|^{-1}\xi,\ {}^*\beta,\ {}^*\rho) \cdot \psi_2\right) \\ &\quad + \mathfrak{d}^{i-1}\left(|s|^{-1}\mathfrak{d}\psi_1\right) + \mathfrak{d}^{i-1}\left({}^{(a)}\mathrm{tr}\,\chi \cdot \mathfrak{d}\psi_1\right) + \mathfrak{d}^{i-1}\left(\underline{\beta} \cdot \psi_1 + {}^{(h)}K \cdot \psi_1\right). \end{aligned}$$

We deal below with the contributions of $\mathrm{err}_1, \mathrm{err}_2$ to (6.13). We first deal with the second line in the expression of $\mathrm{err}_1^i$. The estimate of the second line of $\mathrm{err}_2^i$ is identical. We have

$$\begin{aligned} &\left| \iint_{\mathcal{M}} |s|^{2i+4\lambda-2}\mathfrak{d}^i\psi_1 \cdot \left(\mathfrak{d}^i\left((|s|^{-1}, {}^{(a)}\mathrm{tr}\,\chi, \check{\Gamma}_g)\psi_2\right) + \mathfrak{d}^{i-1}\left(\underline{\beta} \cdot \psi_2 + {}^{(h)}K \cdot \psi_2\right)\right) \right| \\ &\lesssim \iint_{\mathcal{M}} |s|^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|\left(|s|^{-1}|\mathfrak{d}^i\psi_2| + |s|^{-2}|\mathfrak{d}^{i-1}\psi_2|\right) \\ &+ \iint_{\mathcal{M}} \left(1 + |s| \cdot ||s^{i-1}\mathfrak{d}^{i-1}\,{}^{(h)}K||_{L^2(S)}\right) \cdot ||s^{i_2+2\lambda-1}\mathfrak{d}^{i_2}\psi_2||_{L^\infty(S)}||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_1||_{L^2(S)} \\ &+ \iint_{\mathcal{M}} |s| \cdot ||s^{i_2-1}\mathfrak{d}^{i_2-1}\,{}^{(h)}K||_{L^\infty(S)} \cdot ||s^{i+2\lambda-1}\mathfrak{d}^i\psi_2||_{L^2(S)}||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_1||_{L^2(S)} \\ &\lesssim \left(\iint_{\mathcal{M}} |s|^{-2}|s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_1|^2\right)^{\frac{1}{2}} \cdot \left(\iint_{\mathcal{M}} |s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2|^2\right)^{\frac{1}{2}} \\ &\quad + \mathcal{R}[\rho] \iint_{\mathcal{M}} |s|^{-1}||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2||_{L^2(S)}||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_1||_{L^2(S)} \\ &\lesssim C_b\left(\iint_{\mathcal{M}} |s|^{-2}|s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_1|^2\right)^{\frac{1}{2}} \cdot \left(\iint_{\mathcal{M}} |s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2|^2\right)^{\frac{1}{2}} \\ &\lesssim C_b\left(\int_{-1}^{-a\delta} |\tau|^{-2}\sup_\tau\left(a^{-\frac{1}{2}}\int_{\Sigma_\tau} |s|^{2i_1+4\lambda-2}|\mathfrak{d}^{i_1}\psi_1|^2\right)d\tau\right)^{\frac{1}{2}} \cdot \left(\delta \cdot \sup_{\underline{u}} ||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2||^2_{L^2(\underline{H}_{\underline{u}})}\right)^{\frac{1}{2}} \\ &\lesssim C_b\,(a\delta)^{-\frac{1}{2}} \cdot \sup_\tau\left(a^{-\frac{1}{2}}\int_{\Sigma_\tau} |s|^{2i_1+4\lambda-2}|\mathfrak{d}^{i_1}\psi_1|^2\right)^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}} \cdot \sup_{\underline{u}} ||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2||_{L^2(\underline{H}_{\underline{u}})} \\ &\lesssim C_b\,a^{-\frac{1}{2}}\left(\sup_\tau a^{-\frac{1}{2}}\int_{\Sigma_\tau} |s|^{2i_1+4\lambda-2}|\mathfrak{d}^{i_1}\psi_1|^2 + \sup_{\underline{u}} ||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2||^2_{L^2(\underline{H}_{\underline{u}})}\right), \end{aligned}$$

which, after summing up over the index $i$, can be absorbed by the left-hand side of the desired estimate (6.7).

For the first line in the expression of $\mathrm{err}_1^i$, we have

$$
\begin{aligned}
&\Big|\iint_{\mathcal{M}}|s|^{2i+4\lambda-2}\mathfrak{d}^i\psi_1\cdot\Big(\mathfrak{d}^{i-1}(\check{\Gamma}_g\cdot\mathfrak{d}\psi_1+\underline{\beta}\cdot\psi_1)\Big)\Big|\\
&\quad\lesssim\iint_{\mathcal{M}}\Big(\mathcal{O}\delta a^{\frac12}|s|^{-2}+\delta^2a^{\frac32}|s|^{-4}\cdot|s|\Big)\,||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_1||_{L^2(S)}\cdot||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_1||_{L^2(S)}\\
&\quad\lesssim\int_{-1}^{-a\delta}\Big(\mathcal{O}\delta a^{\frac12}|\tau|^{-2}+\delta^2a^{\frac32}|\tau|^{-3}\Big)\sup_\tau\Big(a^{-\frac12}\int_{\Sigma_\tau}|s|^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2d\mathrm{vol}_{\Sigma_\tau}\Big)d\tau\\
&\quad\lesssim(\mathcal{O}a^{-\frac12}+a^{-\frac12})\sup_\tau\Big(a^{-\frac12}\int_{\Sigma_\tau}|s|^{2i+4\lambda-2}|\mathfrak{d}^i\psi_1|^2d\mathrm{vol}_{\Sigma_\tau}\Big).
\end{aligned}
$$

Similarly, for the first line in the expression of $\mathrm{err}_2^i$, we derive

$$
\begin{aligned}
&\Big|\iint_{\mathcal{M}}|s|^{2i+4\lambda-2}\mathfrak{d}^i\psi_2\cdot\Big(\mathfrak{d}^{i-1}\big((\check{\Gamma}_b,\xi)\cdot\mathfrak{d}\psi_2+(|s|^{-1}\xi,\beta,\rho)\cdot\psi_2\big)\Big)\Big|\\
&\quad\lesssim\iint_{\mathcal{M}}\Big(a^{\frac12}|s|^{-1}+\mathcal{R}|s|^{-1}+|s|\cdot\delta^2a^{\frac32}|s|^{-4}+|s|\cdot\delta a|s|^{-3}\Big)\,||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2||^2_{L^2(S)}\\
&\quad\lesssim\delta(a^{\frac12}(a\delta)^{-1})\cdot\sup_{\underline{u}}||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2||^2_{L^2(\underline{H}_{\underline{u}})}\lesssim a^{-\frac12}\sup_{\underline{u}}||s^{i_1+2\lambda-1}\mathfrak{d}^{i_1}\psi_2||^2_{L^2(\underline{H}_{\underline{u}})}.
\end{aligned}
$$

Both terms can be absorbed by the left-hand side. This finishes the proof of the estimate (6.7). □

**Remark 6.8.** *In the estimates below we apply Lemma 6.7 to the actual Bianchi pairs* $(\alpha,\beta)$, $\big(\beta,(\rho,\ {}^*\rho)\big)$, $\big((\rho,\ {}^*\rho),\underline{\beta}\big)$ *and* $(\underline{\beta},\underline{\alpha})$. *In doing this we will ignore nonlinear terms of the form* $\check{\Gamma}_g\cdot\psi_1$ *in* $F_1$ *or* $F_2$, *and* $(\check{\Gamma}_b,\widehat{\chi})\cdot\psi_2$ *in* $F_2$, *as they have already been dealt with in the proof of Lemma 6.7.*

## 6.4 Estimate of the pair $(\alpha,\beta)$

**Proposition 6.9.** *We have* $a^{-\frac12}||s^i\mathfrak{d}^i\alpha||^2_{L^2(\Sigma_{\tau;\underline{u}})}+||s^i\mathfrak{d}^i\beta||^2_{L^2(\underline{H}_{\underline{u}})}\lesssim\delta^{-1}a$, $i\le N$.

*Proof.* Consider the equations

$$
\begin{aligned}
\nabla_3\alpha-\nabla\widehat{\otimes}\beta &= -\frac12\mathrm{tr}\,\underline{\chi}\alpha+F_1,\\
\nabla_4\beta-\mathrm{div}\,\alpha &= F_2,
\end{aligned}
$$

with

$$
\begin{aligned}
F_1 &= \zeta\widehat{\otimes}\beta-3(\rho\widehat{\underline{\chi}}+\ {}^*\rho\ {}^*\widehat{\underline{\chi}}),\\
F_2 &= -2\mathrm{tr}\,\chi\beta+2\,{}^{(a)}\mathrm{tr}\,\chi\,{}^*\beta-2\omega\beta+\alpha\cdot(2\zeta+\underline{\eta})+3(\xi\rho+\ {}^*\xi\ {}^*\rho),
\end{aligned}
$$

of the form (6.5), with $k=2$, $\lambda=\frac12$, $\psi_1=\alpha$, $\psi_2=\beta$. Therefore, applying Lemma 6.7, we have

$$
\begin{aligned}
a^{-\frac12}\sup_\tau\int_{\Sigma_\tau}s^{2i}|\mathfrak{d}^i\alpha|^2+\sup_{\underline{u}}\int_{\underline{H}_{\underline{u}}}s^{2i}|\mathfrak{d}^i\beta|^2\lesssim&\int_{H_{-1}}s^{2i}|\mathfrak{d}^i\alpha|^2+\int_{\underline{H}_0}s^{2i}|\mathfrak{d}^i\beta|^2\\
&+\Big|\iint_{\mathcal{M}}s^{2i}(\mathfrak{d}^i\alpha\cdot\mathfrak{d}^iF_1+\mathfrak{d}^i\beta\cdot\mathfrak{d}^iF_2)\Big|.
\end{aligned}
$$

As before, we only need to estimate the nonlinear terms with one of the factors replaced by its $L^\infty(S)$ norm. Note that at the top order of derivatives, we can only use the weaker estimate of $\xi$ from Proposition 5.6.

We have

$$\begin{aligned}
&\Big|\iint_{\mathcal{M}} s^{2i}\mathfrak{d}^i(\check{\Gamma}_g\cdot\beta,\widehat{\chi}\cdot\rho))\cdot\mathfrak{d}^i\alpha + s^{2i}\mathfrak{d}^i(\check{\Gamma}_g\cdot\alpha)\cdot\mathfrak{d}^i\beta\Big|\\
&\lesssim \iint_{\mathcal{M}}\left(\mathcal{O}\delta a^{\frac{1}{2}}|s|^{-2}||s^{i_1}\mathfrak{d}^{i_1}\beta||_{L^2(S)} + a^{\frac{1}{2}}|s|^{-1}||s^{i_1}\mathfrak{d}^{i_1}\rho||_{L^2(S)}\right)||s^{i_1}\mathfrak{d}^{i_1}\alpha||_{L^2(S)}\\
&\lesssim \mathcal{O}\delta a^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s|^{-4}|s^{i_1}\mathfrak{d}^{i_1}\alpha|^2\right)^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s^{i_1}\mathfrak{d}^{i_1}\beta|^2\right)^{\frac{1}{2}}\\
&\quad + a^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s|^{-4}|s^{i_1}\mathfrak{d}^{i_1}\alpha|^2\right)^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s|^2|s^{i_1}\mathfrak{d}^{i_1}\rho|^2\right)^{\frac{1}{2}}\\
&\lesssim \mathcal{O}\delta a^{\frac{1}{2}}\left(\int_{-1}^{-a\delta}|\tau|^{-4}\sup_{\tau} a^{-\frac{1}{2}}||s^{i_1}\mathfrak{d}^{i_1}\alpha||^2_{L^2(\Sigma_\tau)}\,d\tau\right)^{\frac{1}{2}}\cdot\delta^{\frac{1}{2}}\sup_{\underline{u}}||s^{i_1}\mathfrak{d}^{i_1}\beta||_{L^2(\underline{H}_{\underline{u}})}\\
&\quad + a^{\frac{1}{2}}\left(\int_{-1}^{-a\delta}|\tau|^{-4}\sup_{\tau} a^{-\frac{1}{2}}||s^{i_1}\mathfrak{d}^{i_1}\alpha||^2_{L^2(\Sigma_\tau)}\,d\tau\right)^{\frac{1}{2}}\cdot\delta^{\frac{1}{2}}\sup_{\underline{u}}||ss^{i_1}\mathfrak{d}^{i_1}\rho||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \mathcal{O}\delta a^{\frac{1}{2}}|a\delta|^{-\frac{3}{2}}\mathcal{R}\delta^{-\frac{1}{2}}a^{\frac{1}{2}}\cdot\delta^{\frac{1}{2}}\cdot\mathcal{R}\delta^{-\frac{1}{2}}a^{\frac{1}{2}} + a^{\frac{1}{2}}|a\delta|^{-\frac{3}{2}}\mathcal{R}\delta^{-\frac{1}{2}}a^{\frac{1}{2}}\cdot\delta^{\frac{1}{2}}\mathcal{R}\delta^{\frac{1}{2}}a^{\frac{1}{2}}\\
&\lesssim \mathcal{O}\mathcal{R}^2\delta^{-1} + \mathcal{R}^2\delta^{-1}\lesssim\delta^{-1}a,
\end{aligned}$$

and

$$\begin{aligned}
\Big|\iint_{\mathcal{M}} s^{2i}\mathfrak{d}^i(\xi\cdot\rho)\cdot\mathfrak{d}^i\beta\Big| &\lesssim \iint_{\mathcal{M}}\mathcal{O}\delta^{-\frac{1}{2}}a^{\frac{1}{2}}|s|^{-\frac{1}{2}}||s^{i_1}\mathfrak{d}^{i_1}\rho||_{L^2(S)}||s^{i_1}\mathfrak{d}^{i_1}\beta||_{L^2(S)}\\
&\lesssim \mathcal{O}\delta^{-\frac{1}{2}}a^{\frac{1}{2}}|a\delta|^{-\frac{3}{2}}\delta\sup_{\underline{u}}||ss^{i_1}\mathfrak{d}^{i_1}\rho||_{L^2(\underline{H}_{\underline{u}})}\sup_{\underline{u}}||s^{i_1}\mathfrak{d}^{i_1}\beta||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \mathcal{O}\delta^{-\frac{1}{2}}a^{\frac{1}{2}}|a\delta|^{-\frac{3}{2}}\delta\mathcal{R}\delta^{\frac{1}{2}}a^{\frac{1}{2}}\cdot\mathcal{R}\delta^{-\frac{1}{2}}a^{\frac{1}{2}}\\
&\lesssim \mathcal{O}\mathcal{R}^2\delta^{-1}\lesssim\delta^{-1}a.
\end{aligned}$$

□

## 6.5 Estimate of the pair $(\beta,\rho)$

**Proposition 6.10.** *We have, for all $i\le N$,*

$$a^{-\frac{1}{2}}||s^{i+1}\mathfrak{d}^i\beta||^2_{L^2(\Sigma_{\tau;\underline{u}})} + ||s^{i+1}\mathfrak{d}^i(\rho,\ {}^*\rho)||^2_{L^2(\underline{H}_{\underline{u}})}\lesssim\mathcal{R}[\alpha]^2\delta a.$$

Note that we have shown that $\mathcal{R}[\alpha]\lesssim 1$, so this is an improvement of $C_b^2\delta a$ in the bootstrap assumption, if $C_b$ is sufficiently larger to start with.

*Proof.* We start with

$$\begin{aligned}
\nabla_3\beta &= -\not{\mathcal{D}}_1^*(\rho,-{}^*\rho) - \mathrm{tr}\,\underline{\chi}\beta + F_1,\\
\nabla_4(\rho,-{}^*\rho) &= \not{\mathcal{D}}_1\beta + F_2,
\end{aligned}$$

with

$$\begin{aligned}
F_1 &= 2\underline{\beta}\cdot\widehat{\chi},\\
F_2 &= (2\underline{\eta}+\zeta)\cdot(\beta,\ {}^*\beta) - 2\xi\cdot(\underline{\beta},-{}^*\underline{\beta}) - \frac{1}{2}\widehat{\underline{\chi}}\cdot(\alpha,\ {}^*\alpha)
\end{aligned}$$

of the form (6.5), with $k=1$, $\lambda=1$, $\psi_1=\beta$, $\psi_2=(\rho,-{}^*\rho)$.

We estimate the contribution from $F_1$, in a similar way to Section 6.4:

$$\begin{aligned}
\Big|\iint_{\mathcal{M}}|s|^{2+2i}\mathfrak{d}^i(\widehat{\chi}\cdot\underline{\beta})\cdot\mathfrak{d}^i\beta\Big| &\lesssim \iint_{\mathcal{M}}\left(\mathcal{O}a^{\frac{1}{2}}|s|^{-1}||s^2s^{i_1}\mathfrak{d}^{i_1}\underline{\beta}||_{L^2(S)}\right)||s^i\mathfrak{d}^i\beta||_{L^2(S)}\\
&\lesssim \mathcal{O}a^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s|^4|s^{i_1}\mathfrak{d}^{i_1}\underline{\beta}|^2\right)^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s|^{-2}|s^i\mathfrak{d}^i\beta|^2\right)^{\frac{1}{2}}\\
&\lesssim \mathcal{O}a^{\frac{1}{2}}\left(\int_{-1}^{-a\delta}|\tau|^{-4}\sup_\tau a^{-\frac{1}{2}}||s^{i_1+1}\mathfrak{d}^{i_1}\beta||^2_{L^2(\Sigma_\tau)}\,d\tau\right)^{\frac{1}{2}}\cdot\delta^{\frac{1}{2}}\sup_{\underline{u}}||s^2s^{i_1}\mathfrak{d}^{i_1}\underline{\beta}||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \mathcal{O}a^{\frac{1}{2}}|a\delta|^{-\frac{3}{2}}\cdot\mathcal{R}(\delta a)^{\frac{1}{2}}\cdot\delta^{\frac{1}{2}}\cdot\mathcal{R}\delta^{\frac{3}{2}}a = \mathcal{O}\mathcal{R}^2\delta a^{\frac{1}{2}}\ll\delta a.
\end{aligned}$$

We now estimate the terms $\xi\cdot\underline{\beta}$ and $\widehat{\underline{\chi}}\cdot\alpha$ in $F_2$. We have

$$\begin{aligned}
\Big|\iint_{\mathcal{M}}|s|^{2+2i}\mathfrak{d}^i(\xi\cdot\underline{\beta})\cdot\mathfrak{d}^i(\rho,\ {}^*\rho)\Big| &\lesssim \iint_{\mathcal{M}}|s|^2\mathcal{O}\mathcal{R}\delta^{-\frac{1}{2}}a^{\frac{1}{2}}|s|^{-\frac{1}{2}}\cdot||s^{i_1}\mathfrak{d}^{i_1}\underline{\beta}||_{L^2(S)}\cdot||s^i\mathfrak{d}^i\rho||_{L^2(S)}\\
&\lesssim \mathcal{O}\mathcal{R}\delta\cdot\delta^{-\frac{1}{2}}a^{\frac{1}{2}}|a\delta|^{-\frac{3}{2}}\sup_{\underline{u}}||s^2s^{i_1}\mathfrak{d}^{i_1}\underline{\beta}||_{L^2(\underline{H}_{\underline{u}})}||ss^{i_1}\mathfrak{d}^{i_1}\rho||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \mathcal{O}\mathcal{R}\delta^{-1}a^{-1}\cdot\mathcal{R}\delta^{\frac{3}{2}}a\cdot\mathcal{R}\delta^{\frac{1}{2}}a^{\frac{1}{2}} = \mathcal{O}\mathcal{R}^3\delta a^{\frac{1}{2}}\ll\delta a,
\end{aligned}$$

and, using the improved estimate of $\widehat{\underline{\chi}}$ obtained in Proposition 5.9,

$$\begin{aligned}
\Big|\iint_{\mathcal{M}}|s|^{2+2i}\mathfrak{d}^i(\widehat{\underline{\chi}}\cdot\alpha)\cdot\mathfrak{d}^i(\rho,\ {}^*\rho)\Big| &\le \iint_{\mathcal{M}}|s|^2\delta a^{\frac{1}{2}}|s|^{-2}||s^{i_1}\mathfrak{d}^{i_1}\alpha||_{L^2(S)}\cdot||s^i\mathfrak{d}^i(\rho,\ {}^*\rho)||_{L^2(S)}\\
&\le \delta a^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s|^{-2}|s^{i_1}\mathfrak{d}^{i_1}\alpha|^2\right)^{\frac{1}{2}}\left(\iint_{\mathcal{M}}|s|^2|s^{i_1}\mathfrak{d}^{i_1}(\rho,\ {}^*\rho)|^2\right)^{\frac{1}{2}}\\
&\le \delta a^{\frac{1}{2}}\left(\int_{-1}^{-a\delta}|\tau|^{-2}\left(\sup_\tau a^{-\frac{1}{2}}\int_{\Sigma_\tau}|s^{i_1}\mathfrak{d}^{i_1}\alpha|^2\right)d\tau\right)^{\frac{1}{2}}\left(\delta\cdot\sup_{\underline{u}}||ss^{i_1}\mathfrak{d}^{i_1}\rho||^2_{L^2(\underline{H}_{\underline{u}})}\right)^{\frac{1}{2}}\\
&\le \delta a^{\frac{1}{2}}|a\delta|^{-\frac{1}{2}}\mathcal{R}[\alpha]\delta^{-\frac{1}{2}}a^{\frac{1}{2}}\cdot\mathcal{R}[\rho]\delta a^{\frac{1}{2}}\\
&\le \mathcal{R}[\rho]\mathcal{R}[\alpha]\delta a\le C^{-1}\mathcal{R}[\rho]^2\delta a+C\mathcal{R}[\alpha]^2\delta a,
\end{aligned}$$

so taking a suitable $C>0$, the first term can be absorbed by the left-hand side (which is like $\mathcal{R}[\rho]^2\delta a$), and we obtain the result. □

## 6.6 Estimate of the pair $(\rho,\underline{\beta})$

**Proposition 6.11.** *We have*[34] $||s^{i+2}\mathfrak{d}^i\underline{\beta}||^2_{L^2(\underline{H}_{\underline{u}})}\lesssim\delta^3a^2$, $i\le N$.

*Proof.* Consider the equations

$$\begin{aligned}
\nabla_3(\rho,\ {}^*\rho)+\mathcal{D}_1\underline{\beta} &= -\frac{3}{2}\mathrm{tr}\,\underline{\chi}(\rho,\ {}^*\rho)+F_1,\\
\nabla_4\underline{\beta} &= \mathcal{D}_1^*(\rho,\ {}^*\rho).
\end{aligned}$$

with

$$\begin{aligned}
F_1 &= \zeta\cdot(\underline{\beta},\ {}^*\underline{\beta})-\frac{1}{2}\widehat{\chi}\cdot(\underline{\alpha},\ {}^*\underline{\alpha}),\\
F_2 &= -(\mathrm{tr}\,\chi\underline{\beta}+{}^{(a)}\mathrm{tr}\,\chi\,{}^*\underline{\beta})+2\omega\,\underline{\beta}+2\beta\cdot\widehat{\underline{\chi}}-3(\rho\underline{\eta}-\ {}^*\rho\,{}^*\underline{\eta})-\underline{\alpha}\cdot\xi,
\end{aligned}$$

[34] We omit the estimate of the flux on $\Sigma_\tau$, as it is no longer needed.

of the type (6.6), with $k=1$, $\lambda=\frac{3}{2}$, $\psi_1=(\rho,\ ^*\rho)$, $\psi_2=\underline{\beta}$. We have

$$\Big|\iint_{\mathcal{M}}|s|^{2i+4}\mathfrak{d}^i(\widehat{\chi}\cdot\underline{\alpha})\cdot\mathfrak{d}^i(\rho,\ ^*\rho)\Big|\lesssim\iint_{\mathcal{M}}|s|^4a^{\frac12}|s|^{-1}||s^{i_1}\mathfrak{d}^{i_1}\underline{\alpha}||_{L^2(S)}||s^i\mathfrak{d}^i(\rho,\ ^*\rho)||_{L^2(S)}$$
$$\lesssim\delta\cdot a^{\frac12}|a\delta|^{-1}\cdot\sup_{\underline{u}}||s^3s^{i_1}\mathfrak{d}^{i_1}\underline{\alpha}||_{L^2(\underline{H}_{\underline{u}})}||ss^i\mathfrak{d}^i(\rho,\ ^*\rho)||_{L^2(\underline{H}_{\underline{u}})}$$
$$\lesssim a^{-\frac12}\mathcal{R}[\underline{\alpha}]\delta^{\frac52}a^{\frac32}\cdot\mathcal{R}[\rho,\ ^*\rho]\delta^{\frac12}a^{\frac12}\lesssim\mathcal{R}^2\delta^3a^{\frac32}\ll\delta^3a^2.$$

Recall that, see Proposition 6.10,

$$\sup_{\tau}a^{-\frac12}\int_{\Sigma_\tau}|s^{i+1}\mathfrak{d}^i\beta|^2\lesssim\mathcal{R}[\alpha]^2\delta a,\quad i\le N.$$

The $\mathcal{R}[\alpha]^2$ factor here can in fact be dropped in view of Proposition 6.9. Then we have

$$\Big|\iint_{\mathcal{M}}|s|^{2i+4}\mathfrak{d}^i(\beta\cdot\underline{\widehat{\chi}})\cdot\mathfrak{d}^i\underline{\beta}\Big|\lesssim\iint_{\mathcal{M}}|s|^4\mathcal{O}\delta a^{\frac12}|s|^{-2}||s^{i_1}\mathfrak{d}^{i_1}\beta||_{L^2(S)}||s^i\mathfrak{d}^i\underline{\beta}||_{L^2(S)}$$
$$\lesssim\mathcal{O}\delta a^{\frac12}\cdot\left(\iint_{\mathcal{M}}|s|^{-2}|ss^{i_1}\mathfrak{d}^{i_1}\beta|^2\right)^{\frac12}\left(\iint_{\mathcal{M}}|s|^4|s^i\mathfrak{d}^i\underline{\beta}|^2\right)^{\frac12}$$
$$\lesssim\mathcal{O}\delta a^{\frac12}\left(\int_{-1}^{-a\delta}|\tau|^{-2}\mathcal{R}^2\delta a\,d\tau\right)^{\frac12}\left(\delta\sup_{\underline{u}}||s^2s^i\mathfrak{d}^i\underline{\beta}||^2_{L^2(\underline{H}_{\underline{u}})}\right)^{\frac12}$$
$$\lesssim\mathcal{O}\delta a^{\frac12}\cdot|a\delta|^{-\frac12}\mathcal{R}\delta^{\frac12}a^{\frac12}\cdot\delta^{\frac12}\mathcal{R}[\underline{\beta}]\delta^{\frac32}a$$
$$\lesssim\mathcal{O}\mathcal{R}^2\delta^3a^{\frac32}\ll\delta^3a^2.$$

Also,

$$\begin{aligned}\Big|\iint_{\mathcal{M}}|s|^{2i+4}\mathfrak{d}^i(\underline{\alpha}\cdot\xi)\cdot\mathfrak{d}^i\underline{\beta}\Big| &\lesssim \iint_{\mathcal{M}}|s|^4\mathcal{O}\delta^{-\frac12}a^{\frac12}|s|^{-\frac12}||s^{i_1}\mathfrak{d}^{i_1}\underline{\alpha}||_{L^2(S)}||s^i\mathfrak{d}^i\underline{\beta}||_{L^2(S)}\\ &\lesssim \iint_{\mathcal{M}}\mathcal{O}\delta^{-\frac12}a^{\frac12}|s|^{-\frac32}||s^3s^{i_1}\mathfrak{d}^{i_1}\underline{\alpha}||_{L^2(S)}\cdot||s^2s^i\mathfrak{d}^i\underline{\beta}||_{L^2(S)}\\ &\lesssim \mathcal{O}\delta^{-\frac12}a^{\frac12}\cdot\delta|a\delta|^{-\frac32}\sup_{\underline{u}}||s^3s^{i_1}\mathfrak{d}^{i_1}\underline{\alpha}||_{L^2(\underline{H}_{\underline{u}})}\cdot||s^2s^i\mathfrak{d}^i\underline{\beta}||_{L^2(\underline{H}_{\underline{u}})}\\ &\lesssim \mathcal{O}\delta^{-\frac12}a^{\frac12}\cdot\delta|a\delta|^{-\frac32}\cdot\mathcal{R}\delta^{\frac52}a^{\frac32}\cdot\mathcal{R}\delta^{\frac32}a\\ &\lesssim \mathcal{R}^2\delta^3a^{\frac32}\ll\delta^3a^2.\end{aligned}$$

□

## 6.7 Estimate of the pair $(\underline{\beta},\underline{\alpha})$

**Proposition 6.12.** *We have* $||s^{i+3}\mathfrak{d}^i\underline{\alpha}||_{L^2(\underline{H}_{\underline{u}})}\lesssim\delta^5a^3$, $i\le N$.

*Proof.* Consider the equations

$$\begin{aligned}\nabla_3\underline{\beta} &= -\not{\mathcal{D}}_2\underline{\alpha}-2\mathrm{tr}\,\underline{\chi}\,\underline{\beta}+F_1,\\ \nabla_4\underline{\alpha} &= 2\not{\mathcal{D}}_2^*\underline{\beta}+F_2.\end{aligned}$$

with

$$F_1=2\underline{\alpha}\cdot\zeta,$$
$$F_2=-\frac12\mathrm{tr}\,\chi\underline{\alpha}+\frac12\,{}^{(a)}\mathrm{tr}\,\chi\,{}^*\underline{\alpha}+4\omega\underline{\alpha}+(\zeta-4\underline{\eta})\widehat{\otimes}\underline{\beta}-3(\rho\underline{\widehat{\chi}}-\ ^*\rho\,{}^*\underline{\widehat{\chi}}).$$

This is of the type (6.6), with $k=2$, $\lambda=2$, $\psi_1=\underline{\beta}$, $\psi_2=\underline{\alpha}$.

As above, we only need to deal with the term $\rho\widehat{\underline{\chi}}$ in $F_2$ ( ${}^*\rho\,{}^*\widehat{\underline{\chi}}$ is, of course, similar). We have

$$
\begin{aligned}
\Big|\iint_{\mathcal{M}} s^{2i+6}\mathfrak{d}^i(\widehat{\underline{\chi}}\cdot\rho)\cdot\mathfrak{d}^i\underline{\alpha}\Big| &\lesssim \iint_{\mathcal{M}} |s|^6\mathcal{O}\delta a^{\frac{1}{2}}|s|^{-2}||s^{i_1}\mathfrak{d}^{i_1}\rho||_{L^2(S)}||s^i\mathfrak{d}^i\underline{\alpha}||_{L^2(S)}\\
&\lesssim \mathcal{O}\delta a^{\frac{1}{2}}\cdot\delta\cdot\sup_{\underline{u}}||ss^{i_1}\mathfrak{d}^{i_1}\rho||_{L^2(\underline{H}_{\underline{u}})}||s^3s^i\mathfrak{d}^i\underline{\alpha}||_{L^2(\underline{H}_{\underline{u}})}\\
&\lesssim \mathcal{O}\delta^2a^{\frac{1}{2}}\mathcal{R}\delta^{\frac{1}{2}}a^{\frac{1}{2}}\cdot\mathcal{R}\delta^{\frac{5}{2}}a^{\frac{3}{2}}\\
&\lesssim \mathcal{O}\mathcal{R}^2\delta^5a^{\frac{5}{2}}\ll\delta^5a^3. \qquad\square
\end{aligned}
$$

### 6.8 End of the proof of Part 1 of Theorem 3.2

According to Proposition 5.19 we have the following estimates hold true, see (3.3) for the definition of the norms,

$$\mathcal{O}_{\le N}+\mathcal{R}_{\le N-1}\lesssim\mathcal{R}_{\le N}.$$

According to the results of this section, we also have

$$\mathcal{R}_{\le N}\lesssim 1.$$

Therefore, combining them together and using the non-integrable Sobolev estimate (4.1), we have improved all bootstrap assumptions, and as a result, the spacetime can be extended to $\tau^*=-\frac{1}{8}a\delta$ such that the estimates $\mathcal{O}_{\le N}+\mathcal{R}_{\le N}\lesssim 1$ remain valid. This ends the proof of the first part of the Main Theorem 3.2.

## 7 Formation of trapped surfaces

On $H_{-1}$, in view of the equation

$$
\begin{aligned}
\nabla_4\left(\rho-\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}\right) &= \operatorname{div}\beta-\frac{3}{2}\mathrm{tr}\,\chi\rho-\zeta\cdot\beta-\frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha-\frac{1}{2}(-\mathrm{tr}\,\chi\widehat{\chi}-\alpha)\cdot\widehat{\underline{\chi}}\\
&\quad-\frac{1}{2}\widehat{\chi}\cdot\left(-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\widehat{\chi}+\mathrm{tr}\,\chi\widehat{\underline{\chi}}+\nabla\widehat{\otimes}\underline{\eta}+\underline{\eta}\widehat{\otimes}\underline{\eta}\right)\\
&= \frac{1}{4}\mathrm{tr}\,\underline{\chi}|\widehat{\chi}|^2+O(a^{\frac{1}{2}}),
\end{aligned}
$$

we obtain

$$\left(\rho-\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}\right)\Bigg|_{H_{-1}}=-\frac{1}{2}\int_0^{\underline{u}}|\widehat{\chi}|^2|_{H_{-1}}d\underline{u}'+O(\delta a^{\frac{1}{2}}).$$

We also have

$$
\begin{aligned}
\nabla_3\left(\rho-\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}\right) &= -\operatorname{div}\underline{\beta}-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho+\zeta\cdot\underline{\beta}-\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha}-\frac{1}{2}\left(-\frac{1}{2}(\mathrm{tr}\,\chi\widehat{\underline{\chi}}+\mathrm{tr}\,\underline{\chi}\widehat{\chi})+\frac{1}{2}{}^*\widehat{\underline{\chi}}\,{}^{(a)}\mathrm{tr}\,\chi\right)\cdot\widehat{\underline{\chi}}\\
&\quad-\frac{1}{2}\widehat{\chi}\cdot(-\mathrm{tr}\,\underline{\chi}\widehat{\underline{\chi}}-\underline{\alpha})\\
&= -\operatorname{div}\underline{\beta}-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho+\zeta\cdot\underline{\beta}+\frac{1}{4}\mathrm{tr}\,\chi|\widehat{\underline{\chi}}|^2+\frac{1}{4}\mathrm{tr}\,\underline{\chi}\widehat{\chi}\cdot\widehat{\underline{\chi}}-\frac{1}{4}{}^{(a)}\mathrm{tr}\,\chi\,{}^*\widehat{\underline{\chi}}\cdot\widehat{\underline{\chi}}+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\widehat{\chi}\cdot\widehat{\underline{\chi}}\\
&= -\operatorname{div}\underline{\beta}-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\left(\rho-\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}\right)+\zeta\cdot\underline{\beta}+\frac{1}{4}\mathrm{tr}\,\chi|\widehat{\underline{\chi}}|^2-\frac{1}{4}{}^{(a)}\mathrm{tr}\,\chi\,{}^*\widehat{\underline{\chi}}\cdot\widehat{\chi}.
\end{aligned}
$$

Therefore, it is straightforward to derive

$$\rho-\frac{1}{2}\widehat{\chi}\cdot\widehat{\underline{\chi}}=-\frac{1}{2}\frac{\int_0^{\underline{u}}|\widehat{\chi}|^2|_{H_{-1}}d\underline{u}'}{|s|^3}+O(\delta a^{\frac{1}{2}}|s|^{-3}).$$

Recall that $\mathrm{tr}\,\chi$ satisfies the $e_3$-transport equation

$$\nabla_3 \mathrm{tr}\,\chi + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi = 2\rho - \widehat{\chi}\cdot\underline{\widehat{\chi}}.$$

On $H_{-1}$, the Raychaudhuri equation clearly implies

$$\mathrm{tr}\,\chi = 2 - \int_0^{\underline{u}} |\widehat{\chi}|^2|_{H_{-1}} d\underline{u}' + O(\delta). \tag{7.1}$$

This sets up the initial value. We then obtain, by integrating the $e_3$-transport equation,

$$\mathrm{tr}\,\chi = \frac{2}{|s|} - \frac{\int_0^{\underline{u}} |\widehat{\chi}|^2|_{H_{-1}} d\underline{u}'}{|s|^2} + O(\delta a^{\frac{1}{2}}|s|^{-2}). \tag{7.2}$$

Therefore, by the lower bound assumption (1.8), we have, for $\underline{u} = \delta$,

$$\mathrm{tr}\,\chi \le 2|s|^{-1} - a\delta|s|^{-2} = |s|^{-1}(2 - a\delta|s|^{-1}),$$

so taking $s = -\frac{1}{4}a\delta$ we get $\mathrm{tr}\,\chi < 0$. Note, however, here $\mathrm{tr}\,\chi$ is defined under the PT frame, so in order to show the formation of trapped surfaces, we still need to switch back to the integrable PG frame. This is straightforward since, by (2.4),

$$\mathrm{tr}\,\chi' = \mathrm{tr}\,\chi + \delta^{ab}\nabla_{e'_a} f_b + \frac{1}{4}|f|^2\mathrm{tr}\,\underline{\chi} - \frac{1}{2}f_a f_b \underline{\chi}_{ab} + f\cdot\zeta = \mathrm{tr}\,\chi + O(\delta a^{\frac{1}{2}}|s|^{-2}),$$

and hence we also have $\mathrm{tr}\,\chi' < 0$. According to the bound for $\widecheck{\mathrm{tr}\,\underline{\chi}}$ we obtained (see Corollary 5.20), we also have $\mathrm{tr}\,\underline{\chi}' = \mathrm{tr}\,\underline{\chi} = -\frac{2}{|s|}(1 + O(\delta a^{\frac{1}{2}}|s|^{-1})) = -\frac{2}{|s|}(1 + O(a^{-\frac{1}{2}})) < 0$, using $a \gg 1$. Therefore, we obtain a trapped surface $S_{\delta,-\frac{1}{4}a\delta}$.

**Remark 7.1.** *The equation* (7.1) *also rules out trapped surfaces on the initial outgoing null hypersurface.*

**Remark 7.2.** *This precise upper bound of* $\mathrm{tr}\,\chi$*, which only depends on the initial data, is in fact everything needed to prove formation of trapped surfaces in anisotropic situations, i.e. with the* inf *in* (1.8) *replaced by* sup*, in view of the argument in [16] (see also [6]).*

**Acknowledgements.** Both authors would like to thank IHES for their hospitality during visits. X.C. thanks his advisor, Hans Lindblad, for supporting his many visits to Columbia University through the Simons Collaboration Grant 638955. Klainerman was supported by the NSF grant 2201031.

**Data availability statement** No datasets were generated or analyzed during the current study.

**Conflict of Interest** The authors declare that they have no conflict of interest.

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