Response to the referee reports

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The authors would like to thank the referees for carefully reading the manuscript and providing numerous helpful comments. Based on referees' comments, we have made the changes listed below.

Unless otherwise stated, all the section, equation, theorem, and reference numbers mentioned below coincide with the previous version.

1 To Report Y

1. Added as suggested.
2. Corrected.
3. Thanks also to the referee's comment 47, we have realized the derivative loss is unnecessary and can be avoided. Hence, the regularity has been changed to C^N. We have also added remarks regarding a stronger assumption that leads to a C^∞ event horizon.
4. Changed as suggested.
5. Corrected.
6. The referee is correct. The order of the indices has been corrected.
7. Corrected in this and other places.
8. Explanations have been added.
9. Corrected in this and other places.
10. The referee is correct. The calculation has been corrected.
11. Corrected jointly with the previous item.
12. The referee's suggestion is nice and we have distinguished the notations ω and $\omega_{a,m}$.
13. The referee is correct. To avoid confusion, we have rewritten them as $|(\nabla_4, \nabla)^{\leq N} F| \leq C_N \delta$ and other things bounded by $C_N \max\{\delta, |a|\}$.

14. The referee is correct. We have removed this sentence since it is not too relevant to the current work.

15. Corrected in this and other displayed equations where punctuations are missed.

16. The referee is correct and it has been simplified.

17. Corrected.

18. Explanations have been added. We have removed "and V is future directed" to avoid confusion.

19. The referee is correct. Since the reduction to $\tau = 0$ is not necessary, we have removed it and now directly prove for τ_0.

20. Here we simply used that the pointwise convergence of functions over $\mathbb{S}^2$ is uniform. We have rewritten the sentence.

21. By the Riemannian metric we mean the 4-dim Riemannian metric associated to the $1+3$ spacetime decomposition. We have now added this explanation.

22. The referee is correct. We have removed the word "immersion" to avoid the redundancy.

23. Added as suggested.

24. A sentence has been added to explain the role of R.

25. The referee is correct that a stronger version is needed. We have added the new version in Proposition 2.15.

26. Corrected.

27. Corrected.

28. Corrected.

29. The relation $\delta \leq \delta_{\mathcal{H}}/2 \ll 1$ has been added in these two places. elsewhere?

30. Added as suggested.

31. Added as suggested.

32. Yes; The sentence has been rewritten.

33. Corrected.

34. We have added "with the constant C independent of ...".

35. Here we had a typo; We meant to write "ends the proof of Proposition 4.4" not 4.6, and it has now been corrected.

36. The word "below" has been removed.

37. The referee is correct. It has now been corrected.

38. The referee is correct; In the old version, we made the remark regarding $\lambda = 1$ in a later place. We have now moved this remark (Remark 4.8 in the new version) to earlier parts so that it applies here. As a consequence, we no longer need to consider λ.

39. Corrected.

40. The use of $f = O(\delta)$ has now been explicitly added.

41. Corrected.

42. Corrected.

43. Added as suggested.

44. The referee is correct. The $O(\epsilon)$'s have now been replaced by $O(\delta)$'s.

45. The referee is correct. The missed term has been added, and, to highlight that the estimates higher than order 2 (in terms of the derivatives on f) are linear, we have derived the estimates in more detail.

46. We have added a remark regarding the use of $N\delta \ll 1$. This is also related with the referee's 3rd comment regarding the smoothness, and discussions are included in this remark.

47. Upon check, we do not need to lose one derivative for the estimate of f and $\underline{f}$ but only for χ. The statement and the proof have been changed accordingly.

48. For the current revised version, we do not have the previous λ anymore, and here we denote $\lambda = {}^{(0)}\lambda$. There is a typo λ' in xx which should be λ.

49. The real form has been added.

50. Corrected.

51. Added as suggested.

52. Corrected.

53. Corrected.

54. The causal future $\mathcal{J}^{\pm}(S)$ (and past) and the chronological future $I^{+}(S)$ (and past) are both defined in Definition 2.14. To avoid confusion, we have rewritten the sentence and now directly quote Proposition 2.16.

55. Corrected.

56. In-text citations are vague.

57. The referee is correct. The sentence has been rewritten.

58. Corrected.

59. The referee is correct that the non-emptiness needs to be shown. Since we in fact only need to pick some v_0, we have rewritten the sentences in an easier way.

60. Changed as suggested.

61. Explanations of why $r(p_1) \leq r(p_0)$ have been added.

62. A sentence has been added to remark that 0 can be replaced by this bound.

63. Corrected.

64. Added as suggested. Explanations of how $|V(r)| \leq 2\mathring{\delta}$ have also been added.

65. The authors thank the referee for pointing out this observation. This alternative approach is indeed clean and we have added a remark regarding this. Since the referee also points out that the current proof is interesting, we have kept the original proof.

66. Corrected.

67. We have added a sentence explaining that this follows by a topological contradiction.

68. The center of this disk has now been added.

69. Corrected.

70. The referee is correct. Since changing this will require further changes in **A2**, we have instead added a remark that for G_κ with $0 < \kappa < \pi$, the connectivity is equivalent to path-connectivity. Location of the remark

71. Corrected.

72. We have now added in several places that these curves are in G_{ϑ^*} as suggested.

73. Corrected.

74. Corrected.

75. Corrected.

76. Corrected.

2 To referee U

2.1 Specific Comments

1. The referee is correct that the word may have different meanings in different contexts. We realize that it might be better to directly use the terminology "black hole region", defined as $\mathcal{M} \backslash \mathcal{J}^-(\mathcal{I}^+)$ and we have changed it in this way.

2. The authors agree that this is a very interesting question. A natural (or even canonical) foliation of $\mathcal{H}^+$ may play a fundamental role similar to e.g. the canonical foliation of null infinity studied in [1]. This is definitely something we are planning to address in future work.

3. Since the current proof relies fundamentally on the sign of ω near the horizon, hence on the subextremality, it remains a difficult question to study the extremal case. The authors agree that even formulating some concepts is interesting, as the stability problem of extremal black holes is already much more subtle.

2.2 Minor Corrections and Typos

1. Corrected.
2. Corrected.
3. Corrected.
4. Corrected.
5. Corrected.
6. The text has been added as suggested.

References

[1] S. Klainerman, D. Shen, and J. Wan, *A canonical foliation on null infinity in perturbations of Kerr*, arXiv:2412.20119 (2024).