

Solving the constraint equation for general free data

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Abstract

We revisit the problem of solving the Einstein constraint equations in vacuum by a new method which allows us to prescribe four scalar quantities, representing the full dynamical degrees of freedom of the constraint system. We show that once appropriate gauge conditions have been chosen and four scalars freely specified (modulo $\ell \leq 1$ modes), we can rewrite the constraint equations as a well-posed system of coupled transport and elliptic equations on 2-spheres, which we solve by an iteration procedure. Our method provides a large class of exterior solutions of the constraint equations that can be matched to given interior solutions, according to the existing gluing techniques. As such it can be applied to provide a large class of initial Cauchy data sets evolving to black holes, generalizing the well-known result of the formation of trapped surfaces due to Li and Yu [36]. Though in our Main Theorem 2.30 we only specify conditions consistent with $g - g_{Schw} = O(r^{-1-\delta})$, $k = O(r^{-2-\delta})$, the method is flexible enough to be applied in many other situations. It can, in particular, be easily adapted to construct arbitrarily fast decaying data. We expect, moreover, that our method can also be applied to construct data with slower decay, such as used¹ by Shen in [49]. In fact, an important motivation for developing our method is to show that the result of [49] is sharp, i.e. construct small, smooth initial data sets which violate Shen's decay conditions, and for which stability of the Minkowski space result is wrong.

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¹In that case, however, one needs to integrate from a compact domain towards infinity, rather than from infinity as we do here.

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1 Introduction

1.1 The Einstein constraint equation

Despite the fundamental role of the (local) well-posedness result [10], [11] for the Einstein vacuum equation, it remains a challenge to construct the *full set* of initial conditions² (Σ, g, k) , verifying the constraint equations

$$\begin{aligned} \operatorname{div} k - \nabla \operatorname{tr} k &= 0, \\ R_g + (\operatorname{tr} k)^2 - |k|^2 &= 0. \end{aligned} \tag{1.1}$$

We start by recalling below some of the main methods to construct solutions to (1.1).

1.1.1 Solving (1.1) as an underdetermined 3D elliptic system

Given the Riemannian character of the metric g , it is tempting to interpret (1.1) as an underdetermined 3D elliptic system. The best known method fitting this description, which we briefly review below, is the *conformal method* of Lichnerowicz [37], Choquet-Bruhat–York [12], Isenberg [25], Maxwell [41, 42]. The idea is that we specify a given choice of a Riemannian metric g_0 on the 3-manifold Σ , and a transverse-traceless (TT) symmetric 2-tensor σ_0 , i.e., $(\operatorname{div}_{g_0} \sigma_0)_i = 0$, $\operatorname{tr}_{g_0} \sigma_0 = 0$. We then seek the solution to the constraint equation of the form

$$g = \phi^4 g_0, \quad k^{ij} = \phi^{-10} (\sigma_0^{ij} + L[W]^{ij}) + \frac{1}{3} \phi^{-4} g_0^{ij} H,$$

where W is a vector field, $L[W]_{ij} := {}^{(g_0)}\nabla_i W_j + {}^{(g_0)}\nabla_j W_i - \frac{2}{3} (g_0)_{ij} \operatorname{div}_{g_0} W$, and H is a scalar field. The constraint equation then becomes

$$\begin{aligned} \operatorname{div}_{g_0} (L[W])_i &= \frac{2}{3} \phi^6 (dH)_i, \\ \Delta \phi &= \frac{1}{8} R_{g_0} \phi - \frac{1}{8} |\sigma_0 + L[W]|^2 \phi^{-7} + \frac{1}{12} H^2 \phi^5, \end{aligned}$$

which is a determined 3D elliptic system, and can thus be solved by standard elliptic methods. By construction, the scalar field H represents the mean curvature $\operatorname{tr}_g k$ of Σ . In the case when Σ is a closed (compact without boundary) manifold, taking $H = \operatorname{const}$ and $\sigma_0 = 0$ reduces the equation to a scalar equations which can be solved by standard elliptic methods. The method also extends to the asymptotically flat case, see for example [4], which uses the fact that Δ_g is an isomorphism between the spaces of fields decaying like $r^{-\delta}$ and $r^{-2-\delta}$ ($0 < \delta < 1$). The conformal method also allows one to construct faster decaying initial data, as considered in the proof of the nonlinear stability of Minkowski space in [9]. In their recent work [20, 21], Fang–Szeftel–Touati have extended the method to construct even more general initial data. Their result treats arbitrary fast decay and, as such, provides in particular nontrivial examples for the initial data sets in [28].

Another well-known method, known under the name of *gluing method*, initiated in the works [17], [18], [14], constructs nontrivial initial data which are precisely Kerr outside a compact region.³ A key observation in that regard, which dates back to Moncrief [44], is that the linearized constraint equations around the trivial Minkowskian data set is uniquely determined by a 10-dimensional cokernel space. The gluing method resolves the obstruction by connecting this freedom to the 10-charge family⁴ associated to Kerr solution, thus matching data given on a compact set to a specified Kerr solution. The gluing method has been used

²Here g denotes the Riemannian metric on the initial hypersurface Σ , with scalar curvature R_g , and k corresponds to the second fundamental form of Σ , as embedded in the spacetime.

³Note that the existence of such solutions is forbidden for purely elliptic systems which have unique continuation properties.

⁴These are the parameters a, m , the linear momentum and center of mass.

to prove the formation of trapped surfaces from Cauchy initial data [36]. Another important extension of the gluing method, due to Corlotto–Schoen [6], constructs localized-in-angle initial data. We also refer to the further developments in [15], [13], [16], [5], [1, 2, 3], [43], [24]. The gluing method was further extended in the work of Czimek–Rodnianski [19] which derived more flexible matching solutions. More precisely, they show that matching can be done provided that a specific condition, related to the positive mass theorem, is verified.⁵ A different, more direct approach, to the obstruction free gluing results of [19] was developed by Mao–Oh–Tao, see [40] and further developed in [26]. The result in [40] have been recently used in the construction of Cauchy data that evolves into multiple trapped surfaces [50], [23].

In this paper, we revisit the problem by introducing a new method which allows us to prescribe four scalar quantities, representing the full dynamical degrees of freedom of the constraint system. We show that once appropriate gauge conditions has been chosen and four scalars freely specified (modulo $\ell \leq 1$ modes), we can rewrite the constraint equations as a well-posed system of coupled transport and elliptic equations on 2-spheres, which we solve by an iteration procedure, similar in spirit to the one used in the construction of GCM spheres and hypersurfaces in [30], [31], [48]. In particular, our results provide a large family of exterior solutions of the constraint equations which can be matched to given interior solutions according to the existing gluing techniques.⁶

1.1.2 The Horizontal Constraint System

Though these various versions of the gluing method have provided a great number of interesting solutions to the constraint equations, they typically produce solutions which are exactly Kerr outside a compact set. The stability results in general relativity study much more general perturbations, and it is thus an important to construct initial data with a lot more flexibility. Ideally, one would like to have a method which takes into account the full degrees of freedom in (1.1).

The goal of this paper is to propose such a method and use it to describe initial data sets with more flexible properties. We divide the degrees of freedom of (1.1) into *gauge* and *free* scalars and show that for a given choice of the former, we have the freedom to fully prescribe, up to $\ell \leq 1$ modes, the remaining four defining scalars. The constraint equations can then be solved as a system of transport and 2D elliptic equations, which we call the Horizontal Constraint System (HCS), similar to the way one constructs solutions to the characteristic initial value problem [8]. In particular, this produces a fully general set of exterior solutions which can be matched to prescribed data on a compact set.

Connections with the free data. An initial data set (Σ, g, k) , with Σ a 3-manifold and g, k symmetric 2-tensors, is formally specified by 12 functions. The constraint equations (1.1) impose 4 conditions, leaving formally 8 degrees of freedom. Three of these are to be accounted by the coordinate covariance of (1.1) on Σ . In our work, we fix a *radial* function r whose level surfaces are 2-dimensional spheres. The other two coordinates ϑ^1, ϑ^2 can be chosen in a canonical way by transporting them from a given sphere S_0 , where $r = r_0$, along the integral curves normal to the r -foliation. Beside these three coordinate conditions, one can identify a fourth which corresponds to the embedding of Σ into the induced Einstein vacuum spacetime.

The remaining four degrees of freedom represent the true dynamical degrees of freedom. We identify them here in terms of four scalars obtained from the Ricci and curvature coefficients associated to the r -foliation. Remarkably, they happen to provide the only obstructions to showing that the structure equations induced by the constraints, expressed as a system of transport equations in the direction normal to the foliation, is

⁵The condition can be written as $|\Delta \mathbf{E}| > C|\Delta \mathbf{P}|$ for some (potentially large) $C > 0$, where $\Delta \mathbf{E}$ and $\Delta \mathbf{P}$ are respectively the differences of the energy and linear momentum between the two spheres considered for gluing.

⁶The main result, stated first in Theorem 1.3, see also Theorem 2.30, constructs solutions with prescribed four scalars and specified asymptotic behavior at space-like infinity. The method can however be also be applied in reverse, by integrating towards space-like infinity, from prescribed data in a compact region of Σ .

well-posed. Thus, once prescribed, modulo $\ell \leq 1$ modes, one can derive a unique solution to (1.1).

Remark 1.1. *It helps to compare this with the characteristic initial data, that is data prescribed on two transversal null hypersurfaces C and $\underline{C}$. In that case, the free data is simply given by the shear tensors on each hypersurface. The characteristic constraint equations have a simple reductive structure that allows one to solve various quantities one-by-one, avoiding loss of derivatives; see Chapter 2 of [8] for details. In contrast, the Cauchy constraint equations are more heavily coupled and yet, once the defining scalars⁷ are identified, we can recover a similar reductive structure.*

1.2 Main ideas and first statement of the main theorem

Given a sphere foliation on Σ with outward unit normal N and compatible⁸ orthonormal frame $\{N, e_a\}_{a=1,2}$, we define the quantities

$$\theta_{ab} := g(\nabla_a N, e_b), \quad \Theta_{ab} := k(e_a, e_b), \quad \Pi := k(N, N), \quad \Xi_a := k(N, e_a), \quad Y_a := R(N, e_b, e_b, e_a),$$

where R denotes the 3-dim Riemann curvature tensor. We also define the lapse function $\hat{a} := (N(r))^{-1}$, and denote the Gauss curvature of the r -spheres by K .

Loss of derivatives. In Section 2.3, we give the version of constraint equations decomposed with respect to the triad $\{N, e_a\}_{a=1,2}$, called the *Horizontal Constraint System* (HCS). In Section 2.3.2, see also Section 2.4.1 in the spacetime language, we find the following six scalars that appear to be responsible for a loss of derivatives in HCS:

$$\mu := -\Delta(\log \hat{a}) + K - \frac{1}{4}(\text{tr} \theta)^2, \quad \nu := \text{div} \Xi, \quad \Pi, \quad \text{curl} \Xi, \quad \text{div} Y, \quad \text{curl} Y. \quad (1.2)$$

Here Δ , div , and curl are horizontal Laplacian, divergence, and curl operators defined in Section 2.1.

Gauge scalars. Among the scalars in (1.2), the one that determines a sphere foliation on Σ is the scalar μ . This has been referred in [9] as the mass aspect function and used there to determine the sphere foliation on the last slice Σ_{t^*} . We can prescribe μ to address the coordinate freedom regarding r . Yet, even when the coordinates on Σ are fixed, we can have different initial data sets that evolve to the same Einstein-vacuum spacetime (see details in Section 2.3.3). In our work, we resolve this ambiguity by prescribing freely the scalar field ν .

Remark 1.2. *The traditional way to deal with this spacetime ambiguity is to impose the maximal foliation condition $\text{tr}_g k = 0$, a condition which is more aligned with the 3D elliptic character of the constraints and is independent of the choice of a foliation on Σ . In contrast, our condition on ν works in tandem with the one on μ . Indeed, as stated in Proposition 2.20, given an initial data set, one can always, at least locally, construct another spacelike hypersurface, embedded in the same vacuum spacetime and with a specific sphere foliation, such that $\mu_{\ell \geq 1} = \nu = 0$.⁹*

Free scalars. Given a gauge choice, specified by the gauge scalars (μ, ν) , we show that the remaining degrees of freedom correspond precisely to the remaining four scalars in (1.2). Our main result is as follows.

Theorem 1.3 (Main Theorem, rough version). *Prescribe four scalars in a given exterior region in $\mathbb{R}^3$, denoted $(\mathcal{B}, {}^*\mathcal{B}, \mathcal{K}, {}^*\mathcal{K})$, supported on spherical modes $\ell \geq 2$ (see Section 2.2.2 for the precise definition), and satisfying certain decaying conditions (to be later specified) as $r \rightarrow \infty$. Then, provided certain $\ell \leq 1$ conditions at spatial infinity, corresponding to a specification of the ADM charges (see Definition 2.1), there exists a solution to the constraint equation (1.1) such that $\mu_{\ell \geq 1} = \nu = 0$, and*

$$(\text{div} Y - \mathcal{B})_{\ell \geq 2} = 0, \quad (\text{curl} Y - {}^*\mathcal{B})_{\ell \geq 2} = 0, \quad (\Delta(\hat{a}\Pi) - \mathcal{K})_{\ell \geq 2} = 0, \quad (r^{-4}\partial_r(r^4 \text{curl} \Xi) - {}^*\mathcal{K})_{\ell \geq 2} = 0.$$

⁷We use the term *defining scalars* to represent the union of gauge and free scalars.

⁸i.e. with $\{e_a\}$ tangent to level surfaces of r .

⁹In fact, we need to impose additional $\ell = 0$ conditions to determine a unique gauge; see Section 2.5.1. For simplicity, we proceed with the vague assertion that $\mu_{\ell \geq 1} = \nu = 0$ determines the gauge.

The precise statement is given in Theorem 2.30.

Remark 1.4. *Though in Theorem 2.30 we give conditions consistent¹⁰ with $g - g_{Schw} = O(r^{-1-\delta})$, $k = O(r^{-2-\delta})$, the method can be easily adapted to construct arbitrarily fast decaying data used in [28]. We expect that our method can also be applied to construct data with slower decay, such as used in [49]. In that case, however, one needs to integrate from a compact domain towards infinity rather than from infinity as we do here. In fact, an important motivation for developing our method here is to show that the result of [49] is sharp, i.e. construct small, smooth initial data sets which violate Shen's decay conditions, and for which stability of the Minkowski space result is wrong.*

The statement of the theorem implies the existence of a rich family of vacuum exterior data with prescribed mass and angular momentum, as well as the center of mass. Combining with Lorentz boosts, which generate nonzero linear momentum, we obtain a generalized 10-charge family of exterior solutions to (1.1) compared with the 10-charge Kerr family constructed in [14]. As a consequence, we obtain a much larger class of exterior solutions that can be used for the gluing method. As mentioned in the abstract, our result can be applied to provide a large class of initial Cauchy data sets evolving to black holes, significantly extending the well-known result of the formation of trapped surfaces of Li and Yu [36].

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2 Set-up and precise statement of the main theorem

2.1 Metrics, connections, and curvature tensors

We adopt the following notations:

- The spacetime metric, connection, Riemann curvature tensor, Ricci tensor, and scalar curvature are denoted respectively by $\mathbf{g}$, $\mathbf{D}$, $\mathbf{R}$, $\mathbf{Ric}$, and $\mathbf{R}_g$. The spacetime coordinate indices are denoted by the Greek letters α, β , etc.
- The metric, connection, Riemann curvature tensor, Ricci tensor, and scalar curvature on 3-dim Riemannian manifolds are denoted respectively by g , ∇ , R , Ric , and R_g . The corresponding divergence, curl, and trace operators are denoted by div , curl , and tr . The spatial coordinate indices are denoted by the Latin letters i, j , etc.
- The connection with respect to the horizontal structure induced by an r -foliation¹¹ is denoted by ∇ . The corresponding divergence, curl, and trace operators are denoted respectively by div , curl , and tr . We always take an orthonormal frame $\{e_a\}_{a=1,2}$ adapted to the horizontal structure. The letters a, b , etc will be used for such frame indices. When the horizontal structure is integrable, we also denote the induced metric by γ , and the Gauss curvature by $K = K_\gamma$, and the coordinate indices by A, B , etc.

Throughout the work, we use the Einstein summation convention on repeated indices. To avoid confusion regarding sign conventions, we remark that the Riemann curvature tensor R here is defined through the

¹⁰Such initial data sets were considered in [38] and are more general than those of [9].

¹¹Or, more generally, orthogonal to a given vectorfield N . See also Section 2.2.1.

relation

$$\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = -R_{ijl}{}^k X^l,$$

and one can also lower the upper index to make it a $(0, 4)$ -tensor. It satisfies

$$g(\nabla_X \nabla_Y Z, W) = g(\nabla_Y \nabla_X Z, W) + R(X, Y, W, Z).$$

Here $\nabla_X \nabla_Y$ means $X^i Y^j \nabla_i \nabla_j$. The Ricci curvature and scalar curvature are then defined as

$$\text{Ric}(X, Y) := g^{ij} R_{X^i Y^j}, \quad R_g = \text{tr}_g \text{Ric}.$$

The spacetime Riemann curvature tensor, Ricci tensor, and scalar curvature **Riem**, **Ric**, **R_g** are defined similarly. This aligns with the convention in [9].¹² The second fundamental form k is defined by

$$k_{ij} = g(\nabla_{\partial_i} T, \partial_j).$$

2.2 Asymptotically flat data

Definition 2.1. An initial data set (Σ, g, k) is said to be asymptotically flat, if there exists a coordinate system (x^1, x^2, x^3) defined in a neighborhood of infinity, such that as $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \rightarrow \infty$, it holds that

$$g_{ij} = \delta_{ij} + o(1), \quad k_{ij} = o(1).$$

Given an r -foliation $\{S_r\}$ with the outward normal N_0 and induced area element dA with respect to the Euclidean metric $(\delta_{ij}, 0)$, the following quantities are defined, if the limits exist:

$$\begin{aligned} \mathbf{E} &:= \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) N_0^j dA, \\ \mathbf{P}_i &:= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} \sum_j (k_{ij} - \text{tr}_\delta k g_{ij}) N_0^j dA, \\ \mathbf{J}_i &:= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} \sum_{j,l,m} \epsilon_{ilm} x^l (k_{mj} - \text{tr}_\delta k g_{mj}) N_0^j dA, \\ \mathbf{C}_i &:= \frac{1}{16\pi m} \lim_{r \rightarrow \infty} \int_{S_r} \sum_{j,k} \left(x^i (\partial_k g_{kj} - \partial_j g_{kk}) - ((g_{ij} - \delta_{ij}) - \delta_{ij} (g_{kk} - \delta_{kk})) \right) N_0^j dA, \end{aligned} \tag{2.1}$$

see, e.g., Section 1.2 of [40]. The quantities **E**, **P**, **J**, **C** are called respectively the ADM energy, linear momentum, angular momentum, and center of mass.

Throughout this work, we consider $\Sigma := (r_0, \infty) \times \mathbb{S}^2$, which can be embedded into the Euclidean space $\mathbb{R}^3$ as an exterior region. This endows Σ with a natural r -function.

2.2.1 Horizontal decomposition on Σ

Assume that $\Sigma := (r_0, \infty) \times \mathbb{S}^2$ is equipped with a metric g . A specification of a unit vector field N determines a horizontal structure $H = N^\perp$, defined through the metric g . We then take an orthonormal frame $\{e_1, e_2\}$ spanning H so that the triad $\{N, e_1, e_2\}$ is an orthonormal frame on Σ . We consider mostly the case of

¹²Note however that in [9], the second fundamental form k is the opposite of the one defined here.

integrable horizontal structures when N is orthogonal to an r -foliation. See Section 2 in [22] for a detailed discussion of horizontal structures.

Ricci coefficients. The corresponding Ricci (rotation) coefficients on the 3-Riemannian manifold are denoted as follows:

$$p_a := g(\nabla_N N, e_a), \quad \theta_{ab} := g(\nabla_a N, e_b). \quad (2.2)$$

The trace and the traceless part of θ are denoted respectively by $\text{tr} \theta$ and $\hat{\theta}$.

Curvature components. The curvature components are denoted as follows:

$$\mathcal{R}_{ab} := R(N, e_a, N, e_b), \quad Y_a := R(N, e_b, e_b, e_a). \quad (2.3)$$

The trace and the traceless part of $\mathcal{R}$ are denoted respectively by $\text{tr} \mathcal{R}$ and $\hat{\mathcal{R}}$.

Denoting the horizontal volume form by ϵ_{ab} , and define the dual ${}^*Y_a := \epsilon_{ab} Y_b$. Then one directly verifies the relation

$$R_{Nabc} = \epsilon_{bc} {}^*Y_a. \quad (2.4)$$

Components of k . Given initial data (g, k) and the triad $\{N, e_1, e_2\}$ on Σ , we define

$$\Theta_{ab}^{(N)} := k(e_a, e_b), \quad \Xi_a^{(N)} := k(N, e_a), \quad \Pi^{(N)} := k(N, N).$$

They are well-defined scalars or horizontal tensors once N is specified. In what follows, when there is no danger of confusion, we simply denote $\Theta = \Theta^{(N)}$, $\Xi = \Xi^{(N)}$, $\Pi = \Pi^{(N)}$. The trace and the traceless part of Θ are denoted respectively by $\text{tr} \Theta$ and $\hat{\Theta}$.

The lapse function. The lapse function of a given r -foliation is defined to be

$$\hat{a} := (Nr)^{-1}. \quad (2.5)$$

Given the r -foliation, one can always write the metric g in the following form

$$g = \hat{a}^2 dr^2 + \gamma, \quad \gamma = \gamma_{AB} d\vartheta^A d\vartheta^B, \quad (2.6)$$

and note that $\hat{a} := (Nr)^{-1} = |\nabla r|_g^{-1}$ are independent of the choice of $(\vartheta^1, \vartheta^2)$.

The Gauss curvature of the r -surfaces is denoted by $K = K_\gamma$, where γ denotes the induced metric. We define

$$\mu := -\Delta(\log \hat{a}) + K - \frac{1}{4}(\text{tr} \theta)^2, \quad (2.7)$$

We often denote $\check{K} := K - r^{-2}$. We also define the following scalar field

$$\nu := \text{div} \Xi = \delta^{ab} \nabla_a \Xi_b. \quad (2.8)$$

We have the following simple relation regarding the radial acceleration 1-form p , defined by equation (2.2).

Lemma 2.2. *For a given r -foliation, we have*

$$p = -\nabla(\log \hat{a}). \quad (2.9)$$

Proof. We write the metric g in the form (2.6), and in addition choose an orthonormal frame $\{e_a\}$ tangent to the r -constant spheres. Then since, with respect to the coordinates $r, \vartheta^1, \vartheta^2$, $\Gamma_{rr}^A = \frac{1}{2}g^{AB}(-\partial_{\vartheta^B}g_{rr}) = -\frac{1}{2}g^{AB}\partial_{\vartheta^B}(\hat{a}^2)$, we have

$$\begin{aligned} p_a &= g(\nabla_N N, e_a) = \hat{a}^{-2}g(\Gamma_{rr}^A \partial_{\vartheta^A}, e_a) = \hat{a}^{-2}\left(-\frac{1}{2}\right)g^{AB}\partial_{\vartheta^B}(\hat{a}^2)g(\partial_{\vartheta^A}, e_a) \\ &= -\frac{1}{2}\hat{a}^{-2}(e_a)^B\partial_{\vartheta^B}(\hat{a}^2) = -\frac{1}{2}\hat{a}^{-2}e_a(\hat{a}^2) = -\nabla_a(\log \hat{a}), \end{aligned} \quad (2.10)$$

as required. Note that the conclusion itself does not depend on the coordinate choice. $\square$

2.2.2 Hodge operators, Spherical harmonics

We adopt the following standard notation of horizontal operators for a horizontal 1-form ψ :

$$\mathrm{d}\!\!\!\!/\psi := \delta^{ab}\nabla_a\psi_b, \quad \mathrm{curl}\psi := \epsilon^{ab}\nabla_a\psi_b, \quad (\nabla\widehat{\otimes}\psi)_{ab} := \nabla_a\psi_b + \nabla_b\psi_a - \delta_{ab}\mathrm{d}\!\!\!\!/\psi.$$

We now recall the Hodge operators defined in [9] and extended to the non-integrable cases in [22].

Definition 2.3. *Given a horizontal structure H , we denote by $\mathfrak{s}_0$ the set of scalar fields in the spacetime, by $\mathfrak{s}_1$ the set of H -horizontal 1-forms, and by $\mathfrak{s}_2$ the set of symmetric traceless H -horizontal covariant 2-tensors.*

Definition 2.4. *We consider the following Hodge operators:*

- $\mathcal{P}_1$ takes $\mathfrak{s}_1$ into $\mathfrak{s}_0$: $\mathcal{P}_1\xi = (\mathrm{d}\!\!\!\!/\xi, \mathrm{curl}\xi),$
- $\mathcal{P}_2$ takes $\mathfrak{s}_2$ into $\mathfrak{s}_1$: $(\mathcal{P}_2h)_a = \nabla^b h_{ab},$
- $\mathcal{P}_1^*$ takes $\mathfrak{s}_0$ into $\mathfrak{s}_1$: $(\mathcal{P}_1^*(f, *f))_a = -\nabla_a f + \epsilon_{ab}\nabla_b *f,$
- $\mathcal{P}_2^*$ takes $\mathfrak{s}_1$ into $\mathfrak{s}_2$: $\mathcal{P}_2^*\xi = -\frac{1}{2}\nabla\widehat{\otimes}\xi.$

Whenever we need to be more precise, we will use the notations $\mathcal{P}_1^\gamma, \mathcal{P}_2^\gamma, (\mathcal{P}_1^\gamma)^*, (\mathcal{P}_2^\gamma)^*$ to specify the dependence of these operators on the horizontal metric γ .

We focus on the integrable case where H is the tangent bundle of a sphere (S, γ) . The operators $\mathcal{P}_1^*, \mathcal{P}_2^*$ are the formal adjoints $\mathcal{P}_1, \mathcal{P}_2$, i.e.,

$$\langle \mathcal{P}_1\xi, (f, *f) \rangle = \langle \xi, \mathcal{P}_1^*(f, *f) \rangle, \quad \langle \mathcal{P}_2h, \xi \rangle = \langle h, \mathcal{P}_2^*\xi \rangle. \quad (2.11)$$

Here $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(S, \gamma)$.

We also recall the following identities in [9]:

$$\begin{aligned} \mathcal{P}_1^*\mathcal{P}_1 &= -\Delta_1 + K, & \mathcal{P}_1\mathcal{P}_1^* &= -\Delta, \\ \mathcal{P}_2^*\mathcal{P}_2 &= -\frac{1}{2}\Delta_2 + K, & \mathcal{P}_2\mathcal{P}_2^* &= -\frac{1}{2}(\Delta_1 + K), \end{aligned} \quad (2.12)$$

where K denotes the Gauss curvature of the sphere.

Spherical harmonics. We fix a choice of the standard spherical coordinates $(\vartheta^1, \vartheta^2)$ on $\mathbb{S}^2$, complemented with (x^1, x^2) near $\vartheta^1 = 0, \pi$. This allows us to define the standard spherical harmonics $\{J_{\ell, \mathfrak{m}}\}$, where the integers $\ell, \mathfrak{m}$ satisfy $\ell \geq 0, -\ell \leq \mathfrak{m} \leq \ell$. They form a complete orthonormal basis of the space $L^2(\mathbb{S}^2)$, where $\mathbb{S}^2$ is equipped with the unit round metric

$$\mathbb{S}_\gamma^2 = (d\vartheta^1)^2 + \sin^2(\vartheta^1)(d\vartheta^2)^2.$$

We also denote the r -weighted round metric

$$\gamma^{(0)} := r^2(\mathbb{S}^2\gamma) = r^2((d\vartheta^1)^2 + \sin^2(\vartheta^1)(d\vartheta^2)^2). \quad (2.13)$$

We denote the following $\ell = 1$ basis, which plays a special role as in [30], [31]:

$$J_0 := J_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \vartheta^1, \quad J_+ := J_{1,1} = \sqrt{\frac{3}{4\pi}} \sin \vartheta^1 \cos \vartheta^2, \quad J_- := J_{1,-1} = \sqrt{\frac{3}{4\pi}} \sin \vartheta^1 \sin \vartheta^2.$$

For any scalar field ϕ on the sphere, one can uniquely decompose

$$\phi = \phi_{\ell \leq 1} + \phi_{\ell \geq 2}, \quad (2.14)$$

where $\phi_{\ell \leq 1}$ is spanned by $\{1, J_0, J_+, J_-\}$, and is orthogonal to $\phi_{\ell \geq 2}$ with respect to the measure induced by $r^{-2}\gamma^{(0)} = \mathbb{S}^2\gamma$.

Remark 2.5. Note that J_+ , J_- , J_0 in fact correspond to the restriction of x^1 , x^2 , x^3 to the unit sphere modulo a constant factor:

$$J_+ = \sqrt{\frac{3}{4\pi}}\omega_1, \quad J_- = \sqrt{\frac{3}{4\pi}}\omega_2, \quad J_0 = \sqrt{\frac{3}{4\pi}}\omega_3, \quad \omega_i := x_i/|x|.$$

While ω_i are not normalized, to have cleaner constant factors in expressing ADM charges in the $\ell = 1$ components (see Appendix C), we also introduce the components under ω_i

$$\phi_{\ell=1,i} := \int_{S_r} \phi \omega_i d\text{vol}_{\mathbb{S}^2\gamma}.$$

In contrast, the components $\phi_{\ell,m}$ to be introduced in (2.16) are defined with respect to the orthonormal basis $\{J_{\ell,m}\}$.

Lemma 2.6. The functions J_p ($p = 0, +, -$) verify the following properties on $(S, \gamma^{(0)} = r^2(\mathbb{S}^2\gamma))$:

$$\begin{aligned} \int_S J_p &= 0, \quad \frac{1}{r^2} \int_S J_p J_q = \delta_{pq}, \quad (r^2 \Delta^{(0)} + 2) J_p = 0, \\ (\mathcal{P}_2^{(0)})^* (\mathcal{P}_1^{(0)})^* (J_p, 0) &= (\mathcal{P}_2^{(0)})^* (\mathcal{P}_1^{(0)})^* (0, J_p) = (0, 0). \end{aligned} \quad (2.15)$$

Proof. This is a special case of Lemma 5.2.8 in [30]. For the benefit of the reader, we repeat the proof of the last statement. Let $F := (\mathcal{P}_2^{(0)})^* (\mathcal{P}_1^{(0)})^* J_p$, where by $(\mathcal{P}_1^{(0)})^* J_p$, we mean either $(\mathcal{P}_1^{(0)})^* (J_p, 0)$ or $(\mathcal{P}_1^{(0)})^* (0, J_p)$. Using the identity $2\mathcal{P}_2^{(0)} (\mathcal{P}_2^{(0)})^* = (\mathcal{P}_1^{(0)})^* \mathcal{P}_1^{(0)} - 2K_{\gamma^{(0)}} = (\mathcal{P}_1^{(0)})^* \mathcal{P}_1^{(0)} - 2r^{-2}$, we deduce¹³

$$\begin{aligned} 2\mathcal{P}_1^{(0)} \mathcal{P}_2^{(0)} F &= \mathcal{P}_1^{(0)} ((\mathcal{P}_1^{(0)})^* \mathcal{P}_1^{(0)} - 2r^{-2}) (\mathcal{P}_1^{(0)})^* (J_p) \\ &= \mathcal{P}_1^{(0)} (\mathcal{P}_1^{(0)})^* \mathcal{P}_1^{(0)} (\mathcal{P}_1^{(0)})^* J_p - 2r^{-2} \mathcal{P}_1^{(0)} (\mathcal{P}_1^{(0)})^* (J_p) \\ &= (\Delta^{(0)})^2 J_p + 2r^{-2} \Delta^{(0)} J_p = 0, \end{aligned}$$

as required. $\square$

2.2.3 Norms

Note that Σ is foliated by a family of spheres $S_r := \{r\} \times \mathbb{S}^2$ by definition. We now define the L^2 , L^∞ , and weighted Sobolev spaces over S_r .

¹³In fact, ΔJ_p should be replaced by either $(\Delta J_p, 0)$ or $(0, \Delta J_p)$ depending whether we consider $(\mathcal{P}_1^{(0)})^* (J_p, 0)$ or $(\mathcal{P}_1^{(0)})^* (0, J_p)$.

Definition 2.7. For horizontal covariant rank- k tensors $U_{a_1 \dots a_k}$, we denote by $L^2(S_r)$ the L^2 space through the metric $\gamma^{(0)}$ defined in (2.13), and by $\mathfrak{h}^s(S_r)$ the Sobolev spaces for positive integers s , defined through $r\nabla^{(0)}$ where $\nabla^{(0)}$ is the covariant derivative with respect to $\gamma^{(0)}$, i.e., through the norm

$$\|U\|_{\mathfrak{h}^s(S_r)} := \sum_{i \leq s} \|(r\nabla^{(0)})^i U\|_{L^2(S_r)}.$$

The $L^\infty(S_r)$ space is defined through the norm

$$\|U\|_{L^\infty(S_r)} := \text{ess sup}_{S_r} |\langle U, U \rangle_{\gamma^{(0)}}|^{\frac{1}{2}}.$$

In this work, whenever we write an $\mathfrak{h}^s$, L^2 , or L^∞ space without specification, we refer to the one over S_r defined here.

Remark 2.8. Given an initial data set (Σ, g, k) and an r -foliation, we can also define similar norms with respect to γ , i.e. the metric induced on the foliation. These can be related to the norms defined through $\gamma^{(0)}$ in Definition 2.7, see Lemma 3.1. Consequently, in the iteration scheme, we shall always refer to the norms defined in Definition 2.7.

Remark 2.9. Since the $\mathfrak{h}^s(S_r)$ norms, in view of the area element of $\gamma^{(0)}$, provides an additional r factor, throughout the paper, we will frequently write our estimates for a quantity ψ in the form $r^{-1}\|\psi\|_{\mathfrak{h}^s}$, reflecting the true L^∞ size of ψ in line with the Sobolev inequality on the sphere.

Remark 2.10. Throughout the paper, we often encounter the difference between ∇ and $\nabla^{(0)}$ on various quantities, which yields the Christoffel symbol of γ with respect to $\gamma^{(0)}$

$$\Gamma_{ab}^c(\gamma; \gamma^{(0)}) = \frac{1}{2}(\gamma^{-1})^{cd}(\nabla_a^{(0)}\gamma_{bd} + \nabla_b^{(0)}\gamma_{cd} - \nabla_d^{(0)}\gamma_{ab}),$$

under a choice of the horizontal orthonormal frame of $\gamma^{(0)}$, denoted by $\{e_a^{(0)}\}_{a=1,2}$. Our assumption always ensures that γ is close to $\gamma^{(0)}$ in terms of the components in $\{e_a^{(0)}\}_{a=1,2}$. Therefore, the inverse of γ with respect to $\gamma^{(0)}$ stays bounded, and hence we have

$$\Gamma_{ab}^c(\gamma; \gamma^{(0)}) = O(\nabla^{(0)}(\gamma - \gamma^{(0)})),$$

where the size is defined through the components in $\{e_a^{(0)}\}_{a=1,2}$.

Sobolev norms in the frequency space. For a scalar field ϕ , we denote its $(\ell, \mathfrak{m})$ -modes by

$$\phi_{\ell, \mathfrak{m}} := \langle \phi, J_{\ell, \mathfrak{m}} \rangle_{\mathbb{S}^2_\gamma} = r^{-2} \langle \phi, J_{\ell, \mathfrak{m}} \rangle_{\gamma^{(0)}}. \quad (2.16)$$

It is well-known that the Sobolev space $H^s(S_r, \mathbb{S}^2_\gamma)$ can be alternatively characterized by

$$\|\phi\|_{H^s(S_r, \mathbb{S}^2_\gamma)}^2 = \sum_{\ell=0}^{\infty} \sum_{\mathfrak{m}=-\ell}^{\ell} (1 + \ell^2)^s |\phi_{\ell, \mathfrak{m}}|^2. \quad (2.17)$$

By simple rescaling, $\|\phi\|_{H^s(S_r, \mathbb{S}^2_\gamma)} = r^{-1} \|\phi\|_{\mathfrak{h}^s(S_r)}$. Therefore, one has

$$\|\phi\|_{\mathfrak{h}^s(S_r)}^2 = r^2 \sum_{\ell=0}^{\infty} \sum_{\mathfrak{m}=-\ell}^{\ell} (1 + \ell^2)^s |\phi_{\ell, \mathfrak{m}}|^2. \quad (2.18)$$

In particular, if ϕ is supported on $\ell \leq 1$, we have

$$\|\phi\|_{\mathfrak{h}^s(S_r)} \approx r \left(|\phi_{\ell=0}| + \sum_{\mathfrak{m}=-1}^1 |\phi_{1, \mathfrak{m}}| \right) \lesssim r \|\phi\|_{L^\infty(S_r)}. \quad (2.19)$$

Integral Minkowski inequality. We recall the standard integral Minkowski inequality applied to $L^1(I)$ and sequence- l^2 spaces, where I is any interval:

$$\left\| \int_I |a_n(r)| dr \right\|_{\ell_n^2} \leq \int_I \|a_n(r)\|_{\ell_n^2} dr. \quad (2.20)$$

2.3 Horizontal Constraint System

2.3.1 Unconditional equations

In what follows, we restrict our attention to the case of $\Sigma = (r_0, \infty) \times \mathbb{S}^2$, where N is the outward unit normal to the r -foliation $\{S_r\}$. The horizontal structure $H = N^\perp$ is then automatically integrable. Recall that $\mathcal{R}_{ab}$, defined in (2.3), can be viewed as an horizontal symmetric 2-tensor, and, as such, it can be decomposed as

$$\mathcal{R}_{ab} = \frac{1}{2} \mathfrak{t} \mathcal{R} \gamma_{ab} + \widehat{\mathcal{R}}_{ab},$$

where $\widehat{\mathcal{R}}$ is traceless. The scalar field $\mathfrak{t} \mathcal{R}$ is, by definition, related to the scalar curvature R_g through the following identity

$$R_g = \text{Ric}_{aa} + \text{Ric}_{NN} = R_{abab} + 2R_{NaNa} = R_{abab} + 2\mathfrak{t} \mathcal{R}. \quad (2.21)$$

Also recall the horizontal 1-form $Y_a := R_{Nba}$.

We have the following equations, which hold regardless of whether (g, k) solves the constraint equations.

Proposition 2.11 (Unconditional equations I). *The following equations hold true:*

$$\nabla_N \mathfrak{t} \theta = \mathfrak{d} \mathfrak{t} p - |\widehat{\theta}|^2 - \frac{1}{2} (\mathfrak{t} \theta)^2 - |p|^2 - \mathfrak{t} \mathcal{R}, \quad (2.22)$$

$$R_{abab} = 2K - \frac{1}{2} (\mathfrak{t} \theta)^2 + |\widehat{\theta}|^2, \quad (2.23)$$

$$\mathfrak{d} \mathfrak{t} \widehat{\theta} = \frac{1}{2} \nabla \mathfrak{t} \theta - Y, \quad (2.24)$$

$$\nabla_N \widehat{\theta} = \nabla \widehat{\theta} p - \mathfrak{t} \theta \widehat{\theta} - p \widehat{\theta} p - \widehat{\mathcal{R}}, \quad (2.25)$$

$$\nabla_N K = -\mathfrak{d} \mathfrak{t} Y - \mathfrak{t} \theta K + 2p \cdot Y - \widehat{\theta} \cdot (\nabla \widehat{\theta} p - p \widehat{\theta} p) + \frac{1}{2} \mathfrak{t} \theta (\mathfrak{d} \mathfrak{t} p - |p|^2). \quad (2.26)$$

Proof. See Appendix A.1. □

Constraint quantities on Σ . We define the momentum and Hamiltonian constraint quantities

$$\mathcal{C}_{Mom}(g, k) := \text{div } k - \nabla \text{tr } k, \quad (2.27)$$

$$\mathcal{C}_{Ham}(g, k) := R_g + (\text{tr } k)^2 - |k|^2. \quad (2.28)$$

Expanding $(\mathcal{C}_{Mom})_N$, $(\mathcal{C}_{Mom})_a := (\mathcal{C}_{Mom})_a$, and $\mathcal{C}_{Ham}$ under the frame $\{N, e_a\}$ to obtain

Proposition 2.12 (Unconditional equations II). *The following equations hold:*

$$\nabla_N \mathfrak{t} \Theta = \mathfrak{d} \mathfrak{t} \Xi + \mathfrak{t} \theta \Pi - \widehat{\theta} \cdot \widehat{\Theta} - \frac{1}{2} \mathfrak{t} \theta \mathfrak{t} \Theta - 2p \cdot \Xi - (\mathcal{C}_{Mom})_N, \quad (2.29)$$

$$\nabla_N \Xi = -\mathfrak{d} \mathfrak{t} \widehat{\Theta} + p \cdot \Theta - \Pi p - \frac{3}{2} \mathfrak{t} \theta \Xi - \widehat{\theta} \cdot \Xi + \frac{1}{2} \nabla \mathfrak{t} \Theta + \nabla \Pi + \mathcal{C}_{Mom}, \quad (2.30)$$

$$\nabla_N \mathfrak{t} \theta = \mathfrak{d} \mathfrak{t} p - \frac{1}{2} |\widehat{\theta}|^2 - \frac{3}{4} (\mathfrak{t} \theta)^2 - |p|^2 + K + \Pi \mathfrak{t} \Theta + \frac{1}{4} (\mathfrak{t} \Theta)^2 - |\Xi|^2 - \frac{1}{2} |\widehat{\Theta}|^2 - \frac{1}{2} \mathcal{C}_{Ham}. \quad (2.31)$$

Proof. See Appendix A.2. □

Definition 2.13. *We call the unconditional equations (2.22)-(2.26), (2.29)-(2.31) with $(\mathcal{C}_{Mom})_N = 0$, $\mathcal{C}_{Mom} = 0$, and $\mathcal{C}_{Ham} = 0$ the Horizontal Constraint System (HCS).*

2.3.2 Loss of derivatives

At first glance, HCS appears to be ill-posed, i.e., it appears to lose derivatives. For example, compared with the Raychaudhuri equation on a null hypersurface (relative to the geodesic foliation)

$$\nabla_4 \text{tr} \chi = -\frac{1}{2}(\text{tr} \chi)^2 - |\hat{\chi}|^2,$$

the HCS equation for $\text{tr} \theta$ (equation (2.31) with $\mathcal{C}_{Ham} = 0$) reads

$$\nabla_N \text{tr} \theta = \text{div} p + K + \dots, \quad (2.32)$$

and the equation for $\text{tr} \Theta$ (equation (2.29) with $(\mathcal{C}_{Mom})_N = 0$) reads

$$\nabla_N \text{tr} \Theta = \text{div} \Xi + \dots, \quad (2.33)$$

with loss of one derivative for p and Ξ .

Note that there are no N -transport equations of p . A simple way to avoid the loss of derivative for (2.32) is to prescribe, as a gauge condition, the scalar field

$$\mu = \text{div} p + K - \frac{1}{4}(\text{tr} \theta)^2.$$

For (2.33), we consider it together with equation (2.30) with $\mathcal{C}_{Mom} = 0$:

$$\nabla_N \Xi = -\text{div} \hat{\Theta} + p \cdot \Theta - \Pi p - \frac{3}{2} \text{tr} \theta \Xi - \hat{\theta} \cdot \Xi + \frac{1}{2} \nabla \text{tr} \Theta + \nabla \Pi.$$

There are several terms on the right that lose derivatives. To deal with this, we first prescribe the scalar field Π , so that the term $\nabla \Pi$ is no longer an issue. The equation then reads

$$\nabla_N \Xi = -\text{div} \hat{\Theta} + \frac{1}{2} \nabla \text{tr} \Theta + \dots. \quad (2.34)$$

Commuting the equation with div and curl respectively, we derive

$$\begin{aligned} \nabla_N \text{div} \Xi &= -\text{div} \text{div} \hat{\Theta} + \frac{1}{2} \Delta \text{tr} \Theta + \dots, \\ \nabla_N \text{curl} \Xi &= -\text{curl} \text{div} \hat{\Theta} + \dots. \end{aligned}$$

This motivates us to also interpret $\text{div} \Xi$, $\text{curl} \Xi$ as scalars to be prescribed. Indeed, prescribing $\nu = \text{div} \Xi$ yields an estimate of $-\text{div} \text{div} \hat{\Theta} + \frac{1}{2} \Delta \text{tr} \Theta$. This also deals with the loss of derivatives in (2.33), providing an estimate for $\text{tr} \Theta$. As a result, one can obtain the estimate of $\text{div} \text{div} \hat{\Theta}$. Also, prescribing $\text{curl} \Xi$ clearly provides the control of $\text{curl} \text{div} \hat{\Theta}$. Since the operator that maps $\hat{\Theta}$ to $\mathcal{P}_1 \mathcal{P}_2 \hat{\Theta} = (\text{div} \text{div} \hat{\Theta}, \text{curl} \text{div} \hat{\Theta})$ is an elliptic Hodge operator with no kernel, we can determine $\hat{\Theta}$.

We also need to estimate the Gauss curvature K_γ ; the estimate of p , which is curl-free in view of (2.9), can then be retrieved from the definition of μ , using the Hodge estimates for $\mathcal{P}_1$. The transport equation of K_γ , (2.26), again contains a term $\text{div} Y$ that loses derivatives. It is hence natural, in fact necessary, to also prescribe the scalar field $\text{div} Y$. In order to fully determine Y , we also prescribe the scalar field $\text{curl} Y$. Recall that with μ prescribed, $\text{tr} \theta$ can be determined from equation (2.32). As a consequence, $\hat{\theta}$ can also be determined from (2.24) using the Hodge estimates for $\mathcal{P}_2$.

Remark 2.14. *We note that $\hat{\mathcal{R}}$ is in fact decoupled from the system and can be retrieved from (2.25) after all other quantities are determined.*

To summarize, we were led to prescribe the following six scalar fields:

$$\Pi, \quad \mu, \quad \nu, \quad \text{curl} \Xi, \quad \text{div} Y, \quad \text{curl} Y. \quad (2.35)$$

As we have argued heuristically above, once these 6 scalars are prescribed, there are no other losses of derivatives for the HCS.

2.3.3 Connection with free data

For a 3-manifold Σ , the initial data $(g, k) \in \Gamma(S_+^2 T^* \Sigma) \times \Gamma(S^2 T^* \Sigma)$ for the Einstein vacuum equations consist of a pair of sections satisfying the Einstein constraint equations (1.1). In local coordinates, since both g and k are symmetric, we have 12 unknowns. The constraint equations (1.1) impose 4 conditions, leaving formally 8 degrees of freedom. Three of these are to be accounted by the coordinate covariance of (1.1) on Σ , which consist of the following:

- The choice of the sphere foliation, i.e., a specification of a coordinate function r whose level set gives a foliation.¹⁴ We expect to prescribe a scalar field to fix this gauge choice.
- The choices of the angular variables $(\vartheta^1, \vartheta^2)$. We have chosen to write our metric in the form (2.6). Provided with the boundary condition, i.e., an initial choice of $(\vartheta^1, \vartheta^2)$ on a given sphere, this corresponds to the coordinate conditions $N(\vartheta^1) = N(\vartheta^2) = 0$, where N is the unit normal of the r -foliation, in the increasing direction of r .

Therefore, excluding the three coordinate ones, we are left with five degrees of freedom. Among the scalar fields we identified in (2.35), the scalar field μ plays the role of choosing the r -foliation, and the remaining five read ν , Π , $\text{curl} \Xi$, $\text{div} Y$, and $\text{curl} Y$. For a given coordinate $(r, \vartheta^1, \vartheta^2)$, these five scalars reflect, at least formally, the freedom of the initial data (g, k) on Σ that solves (1.1). However, not all of them represent the “physical” degrees of freedom, as there is an additional coordinate choice to be made that corresponds to the embedding of Σ into the spacetime. As we show below in Section 2.4, this corresponds to the scalar $\nu = \text{div} \Xi$. We therefore interpret the scalar ν as a spacetime coordinate choice, and accordingly, call ν and μ the *gauge* scalars. Together with the implicit choice $N(\vartheta^1) = N(\vartheta^2) = 0$, this exhausts the four degrees of freedom of solutions to the Einstein-vacuum equations in four spacetime dimensions.

Definition 2.15. *Among the six scalars in (2.35), $\nu = \text{div} \Xi$ and μ are called gauge scalars. The remaining four*

$$\Pi, \quad \text{curl} \Xi, \quad \text{div} Y, \quad \text{curl} Y, \quad (2.36)$$

are called free scalars, indicating that they represent the true dynamical degrees of freedom of the Einstein-vacuum equations.

While the free scalars describe the dynamical degrees of freedom, as we will see heuristically in (2.6), the $\ell \leq 1$ parts of the scalars are subject to much more rigid conditions directly related to the ADM charges (2.1). Therefore, it is in fact the $\ell \geq 2$ part of the free scalars

$$^{(\Sigma)} \mathcal{B} := (\text{div} Y)_{\ell \geq 2}, \quad ^{(\Sigma)} * \mathcal{B} := (\text{curl} Y)_{\ell \geq 2}, \quad ^{(\Sigma)} \mathcal{K} := (\Delta(\hat{\alpha} \Pi))_{\ell \geq 2}, \quad ^{(\Sigma)} * \mathcal{K} := (r^{-4} \partial_r (r^4 \text{curl} \Xi))_{\ell \geq 2},$$

that, as stated in the main theorem (Theorem 2.30), are free to prescribe.

2.4 Spacetime perspective

2.4.1 The null frame formalism

We now discuss the constraint equations from the spacetime perspective.¹⁵ Indeed, the first and second fundamental forms of any spacelike hypersurface in an Einstein-vacuum spacetime solves the constraint

¹⁴There is apparently an ambiguity on $r \mapsto F(r)$ with F an increasing function. We will later eliminate this ambiguity in Section 2.5.1.

¹⁵The spacetime perspective helps to provide additional motivation for the two gauge scalars, but will not be needed in the rest of the paper.

equations (1.1), and according to [10], [11], the converse is also true, i.e. regular initial data solving (1.1) is uniquely embedded in its maximal globally hyperbolic development.

When Σ is an embedded spacelike hypersurface in a spacetime $(\mathcal{M}, \mathbf{g})$, one can define the future unit timelike normal vector field T on Σ , and the following null pair

$$e_3 := T - N, \quad e_4 := T + N, \quad \text{on } \Sigma. \quad (2.37)$$

Here N , as before, is the outward normal vector field to r -spheres S_r on Σ . With such a choice of the null pair, we immediately obtain the following relations of the Ricci coefficients and quantities defined on Σ :¹⁶

$$\chi = \Theta + \theta, \quad \underline{\chi} = \Theta - \theta, \quad \zeta = -\Xi. \quad (2.38)$$

Note that, in contrast to what we discuss below, they do not rely on the extension of the frame beyond Σ .

In a spacetime slab containing Σ , S_r determines a family of incoming null hypersurfaces, which are the constant leaves of some optical function $\underline{u}$, denoted by $\underline{H}_{\underline{u}}$. We extend e_3 so that it is the null geodesic vector on each $\underline{H}_{\underline{u}}$. Regarding the extensions of e_a and e_4 beyond Σ , we recall the following two choices, both exploited in [32]:

- The Principal Geodesic (PG) frame: Each $\underline{H}_{\underline{u}}$ is foliated by spheres given as the constant leaves of the affine parameter of e_3 , and the horizontal space $\{e_a\}_{a=1,2}$ tangent to the corresponding spheres. This determines a null frame¹⁷ $\{e_3, e_4, e_a\}$.
- The Principal Temporal (PT) frame: We extend e_4 by the condition

$$\mathbf{D}_{e_3} e_4 = 0.$$

The null pair $\{e_3, e_4\}$ determines the horizontal structure spanned by $\{e_a\}_{a=1,2}$, which may, in general, be non-integrable beyond Σ .

In both cases, the null frame $\{e_3, e_4, e_a\}_{a=1,2}$ is determined in a spacetime slab, thereby defining the Ricci coefficients and curvature components:

$$\begin{aligned} \chi_{ab} &= \mathbf{g}(\mathbf{D}_a e_4, e_b), \quad \underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_a e_3, e_b), \quad \eta_a = \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_a), \quad \underline{\eta}_a = \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_a), \quad \zeta_a = \frac{1}{2} \mathbf{g}(\mathbf{D}_a e_4, e_3), \\ \omega &= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3), \quad \underline{\omega} = \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4), \quad \xi_a = \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, e_a), \quad \underline{\xi}_a = \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, e_a), \\ \alpha_{ab} &= \mathbf{W}_{a4b4}, \quad \beta_a = \frac{1}{2} \mathbf{W}_{a434}, \quad \rho = \frac{1}{4} \mathbf{W}_{3434}, \quad {}^* \rho = \frac{1}{4} {}^* \mathbf{W}_{3434}, \quad \beta_a = \frac{1}{2} \mathbf{W}_{a334}, \quad \underline{\alpha}_{ab} = \mathbf{W}_{a3b3}. \end{aligned}$$

Here $\mathbf{W}$ is the Weyl tensor that can be expressed as

$$\mathbf{W}_{\rho\sigma\mu\delta} = \mathbf{R}_{\rho\sigma\mu\delta} + \frac{1}{2} (\mathbf{g}_{\rho\mu} \mathbf{Ric}_{\delta\sigma} - \mathbf{g}_{\rho\delta} \mathbf{Ric}_{\mu\sigma} - \mathbf{g}_{\sigma\mu} \mathbf{Ric}_{\delta\rho} + \mathbf{g}_{\sigma\delta} \mathbf{Ric}_{\mu\rho}) + \frac{1}{6} \mathbf{R}_g (\mathbf{g}_{\rho\mu} \mathbf{g}_{\delta\sigma} - \mathbf{g}_{\rho\delta} \mathbf{g}_{\mu\sigma}). \quad (2.39)$$

Remark 2.16. Note that the Ricci coefficients $\underline{\omega}$, $\underline{\xi}$, η , ω , ξ , $\underline{\eta}$ are not well-defined¹⁸ on Σ . They are however well defined for the PG or PT extension considered above. In particular, given that e_3 is geodesic in both case (in particular $\underline{\omega} = 0$, $\underline{\xi} = 0$), the choice of the PT frame is equivalent to the condition $\eta = 0$.

¹⁶The first two relations are trivial, and the third also follows easily from the calculation $-k(e_a, N) = \mathbf{g}(\mathbf{D}_a N, T) = \mathbf{g}(\mathbf{D}_a (\frac{1}{2} e_4 - \frac{1}{2} e_3), \frac{1}{2} e_4 + \frac{1}{2} e_3) = \frac{1}{4} \mathbf{g}(\mathbf{D}_a e_4, e_3) - \frac{1}{4} \mathbf{g}(\mathbf{D}_a e_3, e_4) = \zeta_a$.

¹⁷In [32], the corresponding hypersurface $\underline{u} = \text{const}$ are in fact not exactly null, and the definition of the PG structure is more general.

¹⁸They cannot be defined by the choice, $\{e_3, e_4, e_a\}$ on Σ , as their definitions contain e_3 or e_4 derivatives of the frame.

Proposition 2.17. *With the choice of $\{e_3, e_4\}$ given by (2.37) on Σ , and its extension to the spacetime via the PT condition, we have the following relation on Σ :*

$$\omega = -\Pi, \quad (2.40)$$

$$\xi = \Xi + p, \quad (2.41)$$

$$\underline{\eta} = \Xi - p. \quad (2.42)$$

Proof. See Appendix B.2. □

Proposition 2.18. *For Σ embedded in a spacetime $(\mathcal{M}, \mathbf{g})$ with a specified r -foliation, the following relations hold true between the intrinsic quantities defined in Section 2.2.1 and the spacetime quantities defined above:*

$$(\beta + \underline{\beta})_a = 2(Y + \Xi \cdot \Theta - \mathfrak{t}\Theta \Xi)_a + 3\mathbf{Ric}_{Na}, \quad (2.43)$$

$$(\beta - \underline{\beta})_a = -2(\nabla \Pi - \nabla_N \Xi - 2\theta \cdot \Xi + p \cdot \Theta - \Pi p)_a + (\mathcal{C}_{Mom})_a, \quad (2.44)$$

$$\rho = -K_\gamma - \frac{1}{4}(\mathfrak{t}\Theta)^2 + \frac{1}{4}(\mathfrak{t}\theta)^2 + \frac{1}{2}|\widehat{\Theta}|^2 - \frac{1}{2}|\widehat{\theta}|^2 \quad (2.45)$$

$$+ \frac{1}{2}\mathcal{C}_{Ham} - \left(\mathbf{Ric} - \frac{1}{2}(\mathbf{R}_g)\mathbf{g}\right)_{NN} + \frac{1}{2}\left(\mathbf{Ric} - \frac{1}{2}(\mathbf{R}_g)\mathbf{g}\right)_{aa} - \frac{2}{3}\mathbf{R}_g, \\ {}^*\rho = -\text{curl}\Xi - \widehat{\Theta} \wedge \widehat{\theta}. \quad (2.46)$$

Proof. See Appendix B.3. □

Loss of derivatives. With the help of Proposition 2.18, the HCS system can be re-expressed in terms of the spacetime quantities. These can also be derived directly from the null structure and Bianchi equations, recorded in full detail in Appendix B.1. Below, we only refer to them schematically.

Remark 2.19. *The spacetime version of HCS consists of the following types of equations:*

- *The structure equations that only involve derivatives tangent to Σ , e.g., the Codazzi equation*

$$\text{div}\widehat{\chi} = \frac{1}{2}\nabla\mathfrak{t}\chi - \zeta \cdot \widehat{\chi} + \frac{1}{2}\mathfrak{t}\chi\zeta - \beta.$$

- *The transport-type equation in the N -direction obtained by combining the e_3, e_4 transport type equations from the null structure and Bianchi equations. Indeed, suppose that we have $\nabla_3\psi = \underline{F}$, $\nabla_4\psi = F$, we can use the formula $N = \frac{1}{2}e_4 - \frac{1}{2}e_3$ to get*

$$\nabla_N\psi = \frac{1}{2}F - \frac{1}{2}\underline{F}.$$

Note that not all quantities have both e_3 and e_4 transport equations; this is true only if ψ belongs to $\{\beta, \rho, {}^\rho, \underline{\beta}, \chi, \underline{\chi}, \zeta\}$ or a combination of these.*

The loss of derivatives manifest in the following spacetime HCS equations:

$$\begin{aligned} \nabla_N\mathfrak{t}\chi &= \frac{1}{4}\mathfrak{t}\chi\mathfrak{t}\underline{\chi} - \frac{1}{4}(\mathfrak{t}\chi)^2 - \omega\mathfrak{t}\chi + \text{div}\xi - \left(\rho - \frac{1}{2}\widehat{\chi} \cdot \widehat{\chi}\right) + \dots, \\ \nabla_N\mathfrak{t}\underline{\chi} &= -\frac{1}{4}\mathfrak{t}\chi\mathfrak{t}\underline{\chi} + \frac{1}{4}(\mathfrak{t}\underline{\chi})^2 + \omega\mathfrak{t}\underline{\chi} + \text{div}\underline{\eta} + \left(\rho - \frac{1}{2}\widehat{\chi} \cdot \widehat{\chi}\right) + \dots, \\ \nabla_N\left(\rho - \frac{1}{2}\widehat{\chi} \cdot \widehat{\chi}\right) &= \frac{1}{2}\text{div}(\beta + \underline{\beta}) + \dots, \\ \nabla_N\zeta &= \nabla\omega - \frac{1}{2}\beta + \frac{1}{2}\underline{\beta} + \dots. \end{aligned}$$

In the first two equations, the expressions $\mathring{d}\mathring{v}\xi - (\rho - \frac{1}{2}\widehat{\chi} \cdot \widehat{\chi})$, $-\mathring{d}\mathring{v}\eta + (\rho - \frac{1}{2}\widehat{\chi} \cdot \widehat{\chi})$ are, in view of the relations (2.41), (2.42), (2.45), equivalent, modulo lower order terms, to the scalars $\mu = -\mathring{\Delta}(\log \hat{a}) + K_\gamma - \frac{1}{4}(\mathring{\chi} \theta)^2$ and $\mathring{d}\mathring{v}\Xi$, which were prescribed in Section 2.3.2. Similarly, in view of (2.43), the right-hand side of the third equation, $\mathring{d}\mathring{v}(\beta + \underline{\beta})$, is equivalent to $\mathring{d}\mathring{v}Y$. Moreover, by the same relation, the scalar $\mathring{c}\mathring{u}\mathring{r}\mathring{l}(\beta + \underline{\beta})$ is equivalent to $\mathring{c}\mathring{u}\mathring{r}\mathring{l}Y$, which is also among the prescribed scalars in the list (2.35). By elliptic estimates $\mathring{d}\mathring{v}(\beta + \underline{\beta})$ and $\mathring{c}\mathring{u}\mathring{r}\mathring{l}(\beta + \underline{\beta})$ determines $\beta + \underline{\beta}$. Finally, to resolve the loss of derivatives in the last equation, we commute with $\mathring{d}\mathring{v}$ and $\mathring{c}\mathring{u}\mathring{r}\mathring{l}$ to derive

$$\begin{aligned}\nabla_N \mathring{d}\mathring{v}\zeta &= \mathring{\Delta}\omega - \frac{1}{2}\mathring{d}\mathring{v}(\beta - \underline{\beta}) + \dots \\ \nabla_N \mathring{c}\mathring{u}\mathring{r}\mathring{l}\zeta &= -\frac{1}{2}\mathring{c}\mathring{u}\mathring{r}\mathring{l}(\beta - \underline{\beta}) + \dots\end{aligned}$$

Note that $\zeta = -\Xi$ and, under the PT condition, $\omega = -\Pi$. Then, using the relation (2.44) and the fact that $\mathring{d}\mathring{v}\Xi$, $\mathring{c}\mathring{u}\mathring{r}\mathring{l}\Xi$ and Π are all prescribed in (2.35), we deduce that both $\mathring{d}\mathring{v}(\beta - \underline{\beta})$ and $\mathring{c}\mathring{u}\mathring{r}\mathring{l}(\beta - \underline{\beta})$ are determined, hence so is $\beta - \underline{\beta}$.

2.4.2 Degrees of freedom revisited

Using the spacetime formalism, we revisit the discussion on degrees of freedom in Section 2.3.3 and explain the role of the gauge scalar $\nu = \mathring{d}\mathring{v}\Xi$.

As mentioned already in Section 2.3.3, even when the coordinates on Σ are fixed, we can have different initial data sets that evolve to the same Einstein-vacuum spacetime. The ambiguity is due to the different ways of embedding Σ into the spacetime or, in other words, the choice of time function t that defines Σ .

To explain the relation between this freedom and the gauge scalars, we consider a sphere $S_0 \subset \Sigma$ that is ε -close to the unit sphere, with Σ embedded in a spacetime $(\mathcal{M}, \mathbf{g})$ and ε -close to the constant time slice in Minkowski. By extending the null frame using the PT condition as explained above, we obtain a null frame in a spacetime neighborhood of S_0 in $\mathcal{M}$. Now we consider another spacelike hypersurface Σ' satisfying $S_0 \subset \Sigma'$. Given a sphere foliation on Σ' , passing through S_0 , one can also define the outward unit normal N' on Σ' , thereby also defining the corresponding primed horizontal operators ∇' , $\mathring{d}\mathring{v}'$, $\mathring{c}\mathring{u}\mathring{r}\mathring{l}'$, and quantities p' , θ' , Ξ' , Π' , μ' , ν' as in Section 2.2.1.

Proposition 2.20. *There exists an embedded spacelike hypersurface Σ' in a neighborhood of S_0 in $(\mathcal{M}, \mathbf{g})$ and a vectorfield N' on Σ' with $(N')^\perp \subset T\Sigma'$ integrable such that, for the integral sphere S' of $(N')^\perp$ foliated by some function r' ,¹⁹*

$$\mu'_{\ell \geq 1} = 0, \quad \nu' = 0, \quad \int_{S'} \Pi' = 0. \quad (2.47)$$

Note that here all quantities with $'$ are well-defined on Σ' as the r' -foliation is determined. The $\ell \geq 1$ modes are suitably defined by deforming the background spherical coordinates.

The proposition is purely motivational and plays no role in the proof of the main results; we postpone its proof to a forthcoming work [7].

To conclude, from the spacetime perspective, there are in fact four coordinate degrees of freedom, and the gauge scalars μ and $\nu = \mathring{d}\mathring{v}\Xi$ account for such coordinate ambiguities for those corresponding to t and r . The remaining four scalars

$$\Pi, \quad \mathring{c}\mathring{u}\mathring{r}\mathring{l}\Xi, \quad \mathring{d}\mathring{v}Y, \quad \mathring{c}\mathring{u}\mathring{r}\mathring{l}Y,$$

¹⁹The last condition will be explained in Section 2.5.1. From the perspective of the null frame transformation $(f, \bar{f}, \lambda)$, this condition in (2.47) fixes the $\ell = 0$ part of λ , a part that is constant on a sphere and reflects isotropic change in the choice of the embedding of Σ into the spacetime.

i.e., the free scalars, correspond to the true dynamical degrees of freedom.

As mentioned already in Remark 1.1, one can compare the situation described above with the case of the null characteristic data on $C \cup \underline{C}$, as analyzed in [8]. In that case also, to specify the free data one needs to rely on a specific gauge choice, for example the corresponding two geodesic foliations on $C, \underline{C}$. These can be thought as playing a role similar to that of μ in our case, while the role of ν is replaced by the simple requirement that $C, \underline{C}$ are null. The dynamical degrees of freedom for the bifurcate characteristic problem are then given by the shear tensors $\widehat{\chi}, \widehat{\underline{\chi}}$ of the null hypersurfaces $C, \underline{C}$, expressed relative to the geodesic foliations, each of which contributes 2 degrees of freedom.

2.5 Linearization of HCS near Schwarzschild

According to Proposition 2.20, we only impose the $\ell \geq 1$ part of the gauge scalar μ . This leaves the $\ell = 0$ part undetermined. The other gauge scalar ν is also, by definition, without a spherical mean. Therefore, we need to impose two additional $\ell = 0$ conditions in Section 2.5.1. We then give the full system in terms of quantities with their Schwarzschildian values subtracted in Section 2.5.2.

2.5.1 Additional $\ell = 0$ conditions

We now impose two additional conditions that eliminate the $\ell = 0$ ambiguities.

The average of $\check{a}$. As remarked in footnote 14, we need to eliminate the ambiguity of the relabeling of the r -spheres. We impose the condition

$$\check{a} = -\frac{1}{2} \Upsilon^{-1} r \overline{\check{\chi} \theta}. \quad (2.48)$$

Remark 2.21. *In fact, if r is the area radius, then (2.48) is approximately verified. Indeed, we have the relation*

$$1 = \partial_r(\sqrt{r^2}) = \frac{1}{2} \frac{1}{\sqrt{r^2}} \partial_r(r^2) = \frac{1}{8\pi r} \partial_r(\text{Area}(S_r)) = \frac{1}{8\pi r} \int_{S_r} \hat{a} \check{\chi} \theta.$$

However, due to the slow decay we consider and the fact that we are constructing from spatial infinity, it is impossible to show the converse. Therefore, we relax the requirement that by simply imposing an approximate condition (2.48) without claiming r to be the area radius.

The average of Π . At a heuristic level, taking the $\ell = 0$ part of (the linearization of) the equation (2.29) of $\check{\chi} \check{\Theta}$ (with $(\mathcal{C}_{Mom})_N = 0$) gives

$$\partial_r(\check{\chi} \check{\Theta})_{\ell=0} = 2r^{-1}(\check{\Pi})_{\ell=0} - r^{-1}(\check{\chi} \check{\Theta})_{\ell=0}. \quad (2.49)$$

There are no other HCS equations that can be used to determine $(\check{\chi} \check{\Theta})_{\ell=0}$ or $(\check{\Pi})_{\ell=0}$. Therefore, we impose an additional condition on the spherical mean of Π :

$$\overline{\Pi} = 0. \quad (2.50)$$

In our context, see Remark 2.31, $\check{\chi} \check{\Theta}$ decays like $r^{-2-\delta}$, hence (2.49) then implies $(\check{\chi} \check{\Theta})_{\ell=0} = 0$.

2.5.2 The HCS in perturbative form

It is well-known that the presence of mass, which is positive for nontrivial complete asymptotically flat data in view of [46], [47], [51], causes an r^{-1} tail. Such a slow decaying tail would be disastrous when treated as a perturbation, and, as a consequence, it is necessary to linearize around the Schwarzschild data rather than the Minkowski one, even when the mass m is small.²⁰ Recall that for the standard Schwarzschild data, we have

$$\mathfrak{t}\theta^{(0)} = 2\Upsilon^{\frac{1}{2}}r^{-1}, \quad \hat{a}^{(0)} = \Upsilon^{-\frac{1}{2}}, \quad N^{(0)} = \Upsilon^{\frac{1}{2}}\partial_r, \quad K^{(0)} = r^{-2},$$

where $\Upsilon = 1 - 2m/r$, and $\psi^{(0)}$ refers to the value of the quantity ψ in Schwarzschild. We denote

$$\widetilde{\mathfrak{t}\theta} := \mathfrak{t}\theta - 2\Upsilon^{\frac{1}{2}}r^{-1}, \quad \check{a} := \hat{a} - \Upsilon^{-\frac{1}{2}}, \quad \check{K} := K - r^{-2}, \quad \check{\mu} := \mu - 2mr^{-3}. \quad (2.51)$$

Definition 2.22 (Schematic notations). *We use the following notations for the appropriately weighted perturbed quantities*

$$\Gamma_0 = \{\check{a}\}, \quad \Gamma_1 = \{\widetilde{\mathfrak{t}\theta}, \hat{\theta}, p, r^{-1}\Gamma_0, \mathfrak{t}\Theta, \hat{\Theta}, \Xi, \Pi\}, \quad \Gamma_2 = \{Y, \check{K}, r^{-1}\Gamma_1, \nabla\Gamma_1\},$$

where k indicates the maximal order of differentiation of the metric.

Remark 2.23. *In the context of the proof of the main theorem, quantities in Γ_k are expected to have the decay rate of $r^{-1-k-\delta}$.*

Proposition 2.24. *The HCS system, along with the conditions (2.48), (2.50), can be expressed in the following form, using the schematic notation in (2.51):*

$$\partial_r \widetilde{\mathfrak{t}\theta} = \Upsilon^{-\frac{1}{2}}\check{\mu} + \check{a}\check{\mu} - 2r^{-1}\widetilde{\mathfrak{t}\theta} - 2(1 - 3mr^{-1})r^{-2}\check{a} + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham}, \quad (2.52)$$

$$\partial_r \check{K} = r^{-1}\check{\mu} - \hat{a}\mathfrak{d}\mathfrak{v}Y - 3r^{-1}\check{K} - 2\Upsilon^{\frac{1}{2}}r^{-3}\check{a} + \Gamma_1 \cdot \Gamma_2, \quad (2.53)$$

$$\Upsilon^{\frac{1}{2}}\check{\Delta}\check{a} = \check{K} - \Upsilon^{\frac{1}{2}}r^{-1}\widetilde{\mathfrak{t}\theta} - \check{\mu} - \check{\Delta}(\Gamma_0 \cdot \Gamma_0) + \Gamma_1 \cdot \Gamma_1, \quad (2.54)$$

$$\bar{\check{a}} = -\frac{1}{2}\Upsilon^{-1}r\widetilde{\mathfrak{t}\theta}, \quad (2.55)$$

$$\mathcal{P}_1\mathcal{P}_2\hat{\theta} = (\frac{1}{2}\check{\Delta}\mathfrak{t}\theta, 0) - (\mathfrak{d}\mathfrak{v}Y, \mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}Y), \quad (2.56)$$

$$\mathcal{P}_1p = (-\Upsilon^{\frac{1}{2}}\check{\Delta}\check{a} + \check{\Delta}(\Gamma_0 \cdot \Gamma_0), 0), \quad (2.57)$$

$$\mathcal{L}_{\partial_r}(r^{-2}\gamma) = 2r^{-2}\hat{a}\hat{\theta} + \hat{a}\widetilde{\mathfrak{t}\theta}(r^{-2}\gamma) + 2\Upsilon^{\frac{1}{2}}\check{a}r^{-1}(r^{-2}\gamma), \quad (2.58)$$

$$\partial_r \mathfrak{t}\Theta = \hat{a}\mathfrak{d}\mathfrak{v}\Xi + 2r^{-1}\Pi - r^{-1}\mathfrak{t}\Theta + \Gamma_1 \cdot \Gamma_1 - \hat{a}(\mathcal{C}_{Mom})_N, \quad (2.59)$$

$$\partial_r \mathfrak{d}\mathfrak{v}\Xi = -\mathfrak{d}\mathfrak{v}\mathfrak{d}\mathfrak{v}(\hat{a}\hat{\Theta}) - 4r^{-1}\mathfrak{d}\mathfrak{v}\Xi + \frac{1}{2}\hat{a}\check{\Delta}\mathfrak{t}\Theta + \check{\Delta}(\hat{a}\Pi) + \Gamma_1 \cdot \Gamma_2 + \mathfrak{d}\mathfrak{v}(\hat{a}\mathcal{C}_{Mom}), \quad (2.60)$$

$$\partial_r \mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}\Xi = -\mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}\mathfrak{d}\mathfrak{v}(\hat{a}\hat{\Theta}) - 4r^{-1}\mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}\Xi + \Gamma_1 \cdot \Gamma_2 + \mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}(\hat{a}\mathcal{C}_{Mom}), \quad (2.61)$$

$$\bar{\Pi} = 0. \quad (2.62)$$

Here $\bar{f}$ denotes the spherical mean of a scalar field f with respect to the metric γ .

Proof. The proof is done by simply subtracting the equations in Propositions 2.11 and 2.12 by the corresponding ones in Schwarzschild. For the equation of Ξ , we commute it with $\mathfrak{d}\mathfrak{v}$ and $\mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}$ respectively. See Appendix A.3 for details. $\square$

²⁰Our analysis does not in fact requires that m is small.

2.5.3 The prescribed conditions for the defining scalars

In view of the discussion above, we seek solutions of HCS satisfying

$$\mu_{\ell \geq 1} = 0, \quad \nu = 0, \quad (2.63)$$

and

$$\begin{aligned} (\mathring{d}\mathring{v}Y)_{\ell \geq 2} &= \mathcal{B}, & (\mathring{c}\mathring{u}\mathring{r}lY)_{\ell \geq 2} &= {}^*\mathcal{B}, \\ (\mathring{\Delta}(\hat{a}\Pi))_{\ell \geq 2} &= \mathcal{K}, & r^{-4}\partial_r(r^4\mathring{c}\mathring{u}\mathring{r}l\Xi)_{\ell \geq 2} &= {}^*\mathcal{K}. \end{aligned} \quad (2.64)$$

We then write

$$\begin{aligned} \mathring{d}\mathring{v}Y &= \mathcal{B} + \mathcal{B}_{\ell \leq 1}, & \mathring{c}\mathring{u}\mathring{r}lY &= {}^*\mathcal{B} + {}^*\mathcal{B}_{\ell \leq 1}, \\ \mathring{\Delta}(\hat{a}\Pi) &= \mathcal{K} + \mathcal{K}_{\ell \leq 1}, & r^{-4}\partial_r(r^4\mathring{c}\mathring{u}\mathring{r}l\Xi) &= {}^*\mathcal{K} - {}^*\mathcal{K}_{\ell \leq 1}. \end{aligned} \quad (2.65)$$

where $\mathcal{B}_{\ell \leq 1} := (\mathring{d}\mathring{v}Y)_{\ell \leq 1}$, ${}^*\mathcal{B}_{\ell \leq 1} := (\mathring{c}\mathring{u}\mathring{r}lY)_{\ell \leq 1}$, $\mathcal{K}_{\ell \leq 1} := (\mathring{\Delta}(\hat{a}\Pi))_{\ell \leq 1}$, ${}^*\mathcal{K}_{\ell \leq 1} := -(r^{-4}\partial_r(r^4\mathring{c}\mathring{u}\mathring{r}l\Xi))_{\ell \leq 1}$.

2.5.4 Triangular block structure of the perturbative form of HCS

In order to illustrate the structure of the system, it helps to introduce the following notation:

$$\begin{aligned} \Psi_1 &= \widetilde{\mathring{\mathfrak{t}}\mathring{\mathfrak{t}}\theta}, & \Psi_2 &= \check{K}, & \Psi_3 &= \check{a}, & \Psi_4 &= \widehat{\theta}, & \Psi_5 &= p, & \Psi_6 &= Y, \\ \Psi_7 &= \mathring{\mathfrak{t}}\mathring{\mathfrak{t}}\Theta, & \Psi_8 &= \widehat{\Theta}, & \Psi_9 &= \Xi, & \Psi_{10} &= \Pi, \\ \Psi_{11} &= (\mathcal{B}_{\ell \leq 1}, {}^*\mathcal{B}_{\ell \leq 1}), & \Psi_{12} &= (\mathcal{K}_{\ell \leq 1}, {}^*\mathcal{K}_{\ell \leq 1}). \end{aligned} \quad (2.66)$$

Before writing the HCS system in terms of these new variables, we make the following substitutions. Projecting (2.56) to $\ell \leq 1$, we obtain

$$\mathcal{B}_{\ell \leq 1} := (\mathring{d}\mathring{v}Y)_{\ell \leq 1} = \frac{1}{2}(\mathring{\Delta}\widetilde{\mathring{\mathfrak{t}}\mathring{\mathfrak{t}}\theta})_{\ell=1} + err, \quad (2.67)$$

where err contains nonlinear error terms.²¹ Similarly, projecting (2.60) to $\ell \leq 1$, using also the gauge condition $\nu = \mathring{d}\mathring{v}\Xi = 0$, we deduce

$$\mathcal{K}_{\ell \leq 1} := (\mathring{\Delta}(\hat{a}\Pi))_{\ell \leq 1} = -\frac{1}{2}(\mathring{\Delta}\mathring{\mathfrak{t}}\mathring{\mathfrak{t}}\Theta)_{\ell=1} + err. \quad (2.68)$$

In addition, using the condition $\check{\mu}_{\ell \geq 1} = 0$, we can also write, according to the definitions (2.7) and (2.51),

$$\check{\mu} = \check{\mu}_{\ell=0} = \check{K}_{\ell=0} - \Upsilon^{\frac{1}{2}}r^{-1}(\widetilde{\mathring{\mathfrak{t}}\mathring{\mathfrak{t}}\theta})_{\ell=0} + err, \quad (2.69)$$

where err is quadratic in $\widetilde{\mathring{\mathfrak{t}}\mathring{\mathfrak{t}}\theta}$.

Combining these substitutions with (2.65), we can now write the HCS system as

$${}^{(\gamma)}L[\Psi] = \begin{pmatrix} 0 \\ -\Upsilon^{-\frac{1}{2}}\mathcal{B} \\ (0, 0) \\ -(\mathcal{B}, {}^*\mathcal{B}) \\ 0 \\ (\mathcal{B}, {}^*\mathcal{B}) \\ 0 \\ (\Upsilon^{-\frac{1}{2}}\mathcal{K}, -\Upsilon^{\frac{1}{2}}{}^*\mathcal{K}) \\ {}^*\mathcal{K} \\ (\Upsilon^{\frac{1}{2}}\mathcal{K}, 0) \end{pmatrix} + err, \quad (2.70)$$

²¹Indeed, in view of (2.56), the terms are the $\ell \leq 1$ parts of $\mathcal{D}_1\mathcal{D}_2\widehat{\theta}$ and the $\ell = 0$ part of $\mathring{\Delta}\mathring{\mathfrak{t}}\mathring{\mathfrak{t}}\theta$, which are both zero at the linear level.

where, for a given perturbed horizontal metric $\tilde{\gamma}$, the linear operator $^{(\tilde{\gamma})}L$ is defined as

$$^{(\tilde{\gamma})}L[\Psi] := \begin{pmatrix} (\partial_r + 2r^{-1})\Psi_1 + 2(1 - 3mr^{-1})r^{-2}\Psi_3 - \Upsilon^{-\frac{1}{2}}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1)_{\ell=0} \\ (\partial_r + 3r^{-1})\Psi_2 + 2\Upsilon^{\frac{1}{2}}r^{-3}\Psi_3 + \frac{1}{2}\Upsilon^{-\frac{1}{2}}(\Delta\Psi_1)_{\ell=1} - r^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1)_{\ell=0} \\ (\Upsilon^{\frac{1}{2}}\Delta\Psi_3, \bar{\Psi}_3) - (\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1 - \Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1, -\frac{1}{2}\Upsilon^{-1}r\bar{\Psi}_1) \\ \mathcal{P}_1\mathcal{P}_2\Psi_4 - (\frac{1}{2}\Delta\Psi_1, 0) + \Psi_{11} \\ \mathcal{P}_1\Psi_5 + (\Upsilon^{\frac{1}{2}}\Delta\Psi_3, 0) \\ \mathcal{P}_1\Psi_6 - \Psi_{11} \\ (\partial_r + r^{-1})\Psi_7 - 2r^{-1}\Psi_{10} \\ \mathcal{P}_1\mathcal{P}_2\Psi_8 - (\frac{1}{2}\Delta\Psi_7, 0)_{\ell \geq 2} - \Upsilon^{-\frac{1}{2}}\Psi_{12} \\ (r^{-4}\partial_r(\text{curl}\Xi))_{\ell=1} + \mathcal{P}_2\Psi_{12} \\ (\Delta\Psi_{10}, \bar{\Psi}_{10}) + \frac{1}{2}((\Delta\Psi_7)_{\ell=1}, 0) \end{pmatrix}. \quad (2.71)$$

Here, all the horizontal operators are defined relative to $\tilde{\gamma}$, and Ψ_4 and Ψ_8 are traceless with respect to $\tilde{\gamma}$. The notation $\mathcal{P}_2$ denotes the projection into the second component, i.e., $\mathcal{P}_2\Psi_{12} = {}^*\mathcal{K}_{\ell \leq 1}$.

Remark 2.25 (Block-triangular structure). *Notice that apart from Ψ_1, Ψ_2, Ψ_3 , other quantities do not enter the first three rows in the expression of $^{(\tilde{\gamma})}L[\Psi]$. In other words, denoting $\Psi_{\text{main}} = (\Psi_1, \Psi_2, \Psi_3)$, the linear operator splits into two parts*

$$^{(\tilde{\gamma})}L[\Psi] = \begin{pmatrix} ^{(\tilde{\gamma})}L_{\text{main}}[\Psi_{\text{main}}] \\ ^{(\tilde{\gamma})}L_{\text{rem}}[\Psi] \end{pmatrix}.$$

Equivalently, if we write $^{(\tilde{\gamma})}L$ in the matrix form, we have a block-triangular structure with respect to the first 3×3 block. Therefore, we can determine $\Psi_{\text{main}} = (\Psi_1, \Psi_2, \Psi_3)$ first, independently of other quantities. Once they are determined, taking into account the fact that $\Delta\Psi_{10}$ is part of the input (corresponding to the free scalar K), the second block itself also has a triangular structure.²²

The metric γ in (2.70) satisfies (2.58), i.e., γ is in turn determined by Ψ . Therefore, we construct the solution through an iteration argument in Section 4. In the iteration scheme, the system is solved as if γ is fixed at each step. The block triangular structure pointed out in Remark 2.25 allows us, in solving the linear system at each step, to invert the main part $^{(\tilde{\gamma})}L_{\text{main}}[\Psi_{\text{main}}]$ first, as we will carry out in Section 4.2.

Definition 2.26 (Linearized system around Schwarzschild). *We call the system $^{(\gamma^{(0)})}L[\tilde{\Psi}] = 0$ the $\gamma^{(0)}$ -linearized system.*

Remark 2.27. *Note that in the definition of the $^{(\tilde{\gamma})}L$ operator, we have already taken the gauge conditions $\mu_{\ell \geq 1} = 0$, $\nu = 0$ into account, and hence the corresponding terms are not included in the expression of $^{(\tilde{\gamma})}L[\Psi]$.*

2.6 The $\ell = 1$ constraints

In this section, we perform the analysis of $\ell = 1$ modes for the $\gamma^{(0)}$ -linearized system $^{(\gamma^{(0)})}L[\tilde{\Psi}] = 0$, with $\gamma^{(0)}$ the round metric defined in (2.13). Therefore, in this section, all horizontal operators $\text{div}, \text{curl}, \dots$ are defined through $\gamma^{(0)}$. Since $\tilde{\Psi}_4 = \hat{\theta}$ and $\tilde{\Psi}_8 = \hat{\Theta}$ are traceless with respect to $\gamma^{(0)}$, hence fully supported on $\ell \geq 2$ modes, they can be disregarded in the analysis below. As we shall see in the following proposition, the $\ell = 1$ modes are completely determined by the conditions at spatial infinity. This is unlike the $\ell \geq 2$ modes, where we have to take into account the additional freedom given by the four free scalars.²³

Proposition 2.28. *Consider the $\gamma^{(0)}$ -linearized system $^{(\gamma^{(0)})}L[\tilde{\Psi}] = 0$.*

²²Here we mainly refer to the $\ell \geq 2$ parts. The structure of the $\ell = 1$ parts of the system is different, as will be discussed in Section 2.6 just below.

²³This is, of course, under the condition that the gauge scalars are specified.

(i) If we impose the conditions

$$\lim_{r \rightarrow \infty} r^3(\dot{\Psi}_1)_{\ell=1,i} = \dot{\mathbf{c}}_i, \quad \lim_{r \rightarrow \infty} r^4(\dot{\Psi}_2)_{\ell=1} = 0, \quad (2.72)$$

then we have

$$(\dot{\Psi}_1)_{\ell=1,i} = \dot{\mathbf{c}}_i r^{-3} + O(|\dot{\mathbf{c}}| r^{-4}), \quad (\dot{\Psi}_2)_{\ell=1} = O(|\dot{\mathbf{c}}| r^{-5}), \quad (\dot{\Psi}_3)_{\ell=1,i} = \frac{1}{2} \dot{\mathbf{c}}_i r^{-2} + O(\dot{\mathbf{c}}_i r^{-3}). \quad (2.73)$$

(ii) If, in addition, we impose the conditions

$$\lim_{r \rightarrow \infty} r^2(\dot{\Psi}_7)_{\ell=1} = 0, \quad \lim_{r \rightarrow \infty} r^4(\text{curl} \dot{\Psi}_9)_{\ell=1,i} = \dot{\mathbf{a}}_i, \quad i = -1, 0, 1, \quad (2.74)$$

then we have

$$(\dot{\Psi}_{10})_{\ell=1} = (\dot{\Psi}_7)_{\ell=1} = 0, \quad (\text{curl} \dot{\Psi}_9)_{\ell=1,i} = r^{-4} \dot{\mathbf{a}}_i, \quad i = -1, 0, 1.$$

Remark 2.29. We will show in Appendix C that the conditions (2.72) and (2.74) are the linearized version of the conditions $\mathbf{C}_i = \mathbf{c}_i$, $\mathbf{P}_i = 0$, $\mathbf{J}_i = \mathbf{a}_i$ with $\mathbf{C}_i$, $\mathbf{P}_i$, $\mathbf{J}_i$ defined in (2.1).

Proof of (i). The corresponding rows of $\dot{\Psi}_1 = \widetilde{\text{tr} \theta}$, $\dot{\Psi}_2 = \widetilde{K}$, and $\dot{\Psi}_3 = \dot{\mathbf{a}}$ in $(\gamma^{(0)})^L[\dot{\Psi}] = 0$, see (2.71), when projected to $\ell = 1$, read

$$\partial_r(\dot{\Psi}_1)_{\ell=1} = -2r^{-1}(\dot{\Psi}_1)_{\ell=1} - 2(1 - 3mr^{-1})r^{-2}(\dot{\Psi}_3)_{\ell=1}, \quad (2.75)$$

$$\partial_r(\dot{\Psi}_2)_{\ell=1} = -3r^{-1}(\dot{\Psi}_2)_{\ell=1} - 2\Upsilon^{\frac{1}{2}}r^{-3}(\dot{\Psi}_3)_{\ell=1} - \frac{1}{2}\Upsilon^{-\frac{1}{2}}(\Delta \dot{\Psi}_1)_{\ell=1}, \quad (2.76)$$

$$-2\Upsilon^{\frac{1}{2}}r^{-2}(\dot{\Psi}_3)_{\ell=1} = (\dot{\Psi}_2)_{\ell=1} - \Upsilon^{\frac{1}{2}}r^{-1}(\dot{\Psi}_1)_{\ell=1}, \quad (2.77)$$

which, by eliminating $(\dot{\Psi}_3)_{\ell=1}$, can be reduced to

$$\begin{aligned} \partial_r(\dot{\Psi}_1)_{\ell=1} + 3r^{-1}(\dot{\Psi}_1)_{\ell=1} &= O(mr^{-2})(\dot{\Psi}_1)_{\ell=1} + (1 + O(mr^{-1}))(\dot{\Psi}_2)_{\ell=1}, \\ \partial_r(\dot{\Psi}_2)_{\ell=1} + 2r^{-1}(\dot{\Psi}_2)_{\ell=1} &= \Upsilon^{-\frac{1}{2}}(1 - \Upsilon)r^{-2}(\dot{\Psi}_1)_{\ell=1}, \end{aligned}$$

or, in the matrix form,

$$\partial_r \begin{pmatrix} (\dot{\Psi}_1)_{\ell=1} \\ (\dot{\Psi}_2)_{\ell=1} \end{pmatrix} = \begin{pmatrix} -3r^{-1} & 1 \\ 0 & -2r^{-1} \end{pmatrix} \begin{pmatrix} (\dot{\Psi}_1)_{\ell=1} \\ (\dot{\Psi}_2)_{\ell=1} \end{pmatrix} + \begin{pmatrix} O(mr^{-1})(\dot{\Psi}_2)_{\ell=1} + O(mr^{-2})(\dot{\Psi}_1)_{\ell=1} \\ O(mr^{-2})(\dot{\Psi}_2)_{\ell=1} \end{pmatrix},$$

and hence,

$$\partial_r \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1} \\ r^4(\dot{\Psi}_2)_{\ell=1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} r^{-1} \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1} \\ r^4(\dot{\Psi}_2)_{\ell=1} \end{pmatrix} + O(mr^{-2}) \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1} \\ r^4(\dot{\Psi}_2)_{\ell=1} \end{pmatrix},$$

or, with the $\dot{\mathbf{c}}$ -part subtracted,

$$\begin{aligned} \partial_r \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1,i} - \dot{\mathbf{c}}_i \\ r^4(\dot{\Psi}_2)_{\ell=1,i} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} r^{-1} \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1,i} \\ r^4(\dot{\Psi}_2)_{\ell=1,i} \end{pmatrix} + O(mr^{-2}) \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1,i} \\ r^4(\dot{\Psi}_2)_{\ell=1,i} \end{pmatrix} + O(|\dot{\mathbf{c}}| r^{-2}) \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} r^{-1} \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1,i} - \dot{\mathbf{c}}_i \\ r^4(\dot{\Psi}_2)_{\ell=1,i} \end{pmatrix} + O(mr^{-2}) \begin{pmatrix} r^3(\dot{\Psi}_1)_{\ell=1,i} - \dot{\mathbf{c}}_i \\ r^4(\dot{\Psi}_2)_{\ell=1,i} \end{pmatrix} + O(|\dot{\mathbf{c}}| r^{-2}), \end{aligned}$$

where, for the second equality, we use that the first column of $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ is zero. The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ is not symmetric, hence not non-negative definite; however, it is accretive with respect to some modified inner product over $\mathbb{R}^2$, see Lemma 5.5. This allows us to integrate the equation from $r = \infty$, using the condition (2.72), and obtain

$$r^3(\dot{\Psi}_1)_{\ell=1,i} - \dot{\mathbf{c}}_i = O(|\dot{\mathbf{c}}| r^{-1}), \quad r^4(\dot{\Psi}_2)_{\ell=1} = O(|\dot{\mathbf{c}}| r^{-1}),$$

as required. The expansion of $(\dot{\Psi}_3)_{\ell=1}$ then follows from (2.77), which we used to eliminate $(\dot{\Psi}_3)_{\ell=1}$. $\square$

Proof of (ii). The corresponding rows of $(\gamma^{(0)})L[\dot{\Psi}] = 0$ in fact come from projecting the linearized version of equations (2.59)-(2.61) into $\ell = 1$ modes, with the condition $(\text{div } \dot{\Psi}_9)_{\ell=1} = 0$. We have²⁴

$$\begin{aligned}\partial_r(\dot{\Psi}_7)_{\ell=1} &= 2r^{-1}(\dot{\Psi}_{10})_{\ell=1} - r^{-1}(\dot{\Psi}_7)_{\ell=1}, \\ \partial_r(\text{curl } \dot{\Psi}_9)_{\ell=1} &= -4r^{-1}(\text{curl } \dot{\Psi}_9)_{\ell=1}, \\ (\Delta \dot{\Psi}_{10})_{\ell=1} &= -\frac{1}{2}(\Delta \dot{\Psi}_7)_{\ell=1}.\end{aligned}$$

The third equation simply gives $(\dot{\Psi}_7)_{\ell=1} = -2(\dot{\Psi}_{10})_{\ell=1}$. Combining this with the first equation gives

$$\begin{aligned}\partial_r(\dot{\Psi}_7)_{\ell=1} &= -2r^{-1}(\dot{\Psi}_7)_{\ell=1}, \\ \partial_r(\text{curl } \dot{\Psi}_9)_{\ell=1} &= -4r^{-1}(\text{curl } \dot{\Psi}_9)_{\ell=1}.\end{aligned}$$

Therefore, the solution is completely determined from the condition at infinity, which we impose in (2.74). Hence, we obtain $(\dot{\Psi}_7)_{\ell=1} = 0$ and $(\text{curl } \dot{\Psi}_9)_{\ell=1,i} = r^{-4}\mathbf{a}_i$. $\square$

2.7 Precise statement of the main theorem

We now state the precise form of the main theorem.

Theorem 2.30 (Main Theorem). *There exists a sufficiently small constant $\varepsilon > 0$, such that given $m > 0$, $r_0 > 2m$, two constant triplets $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{c} = (c_1, c_2, c_3)$ that are ε -close to zero in $\mathbb{R}^3$, and four scalar functions $\mathcal{B}$, ${}^*\mathcal{B}$, $\mathcal{K}$, ${}^*\mathcal{K}$, supported on $\ell \geq 2$ in the sense of (2.14), satisfying*

$$\sup_{r \in [r_0, \infty)} r^{3+\delta} \|(\mathcal{B}, {}^*\mathcal{B}, \mathcal{K})\|_{\mathfrak{h}^s(S_r)} \leq \varepsilon, \quad \sup_{r \in [r_0, \infty)} r^{4+\delta} \|({}^*\mathcal{K})\|_{\mathfrak{h}^s(S_r)} \leq \varepsilon, \quad \text{for some integer } s \geq 3, \quad (2.78)$$

then there exists a metric g and a symmetric 2-tensor k on $\Sigma = (r_0, \infty) \times \mathbb{S}^2$ solving the constraint equation (1.1) such that, under our choice of the frame, for which $\mu_{\ell \geq 1} = \nu = 0$, we have

$$(\text{div } Y - \mathcal{B})_{\ell \geq 2} = 0, \quad (\text{curl } Y - {}^*\mathcal{B})_{\ell \geq 2} = 0, \quad (\Delta(\hat{\alpha}\Pi) - \mathcal{K})_{\ell \geq 2} = 0, \quad (r^{-4}(\partial_r(r^4 \text{curl } \Xi)) - {}^*\mathcal{K})_{\ell \geq 2} = 0.$$

Moreover, the ADM charges defined in (2.1) satisfy

$$\mathbf{E} = m, \quad \mathbf{J}_i = \frac{1}{8\pi}\mathbf{a}_i, \quad \mathbf{P}_i = 0, \quad \mathbf{C}_i = -\frac{1}{8\pi m}\mathbf{c}_i. \quad (2.79)$$

Remark 2.31. *Note that the four scalars $(\mathcal{B}, {}^*\mathcal{B}, \mathcal{K}, {}^*\mathcal{K})$ are all at level of one derivative of curvature (two derivatives of the components of k). The theorem, therefore, asserts that we can produce general perturbed initial data with decay rate $O(r^{-1-\delta})$ at the metric level. However, compared with $(\mathcal{B}, {}^*\mathcal{B}, \mathcal{K})$ that is allowed to decay at $O(r^{-4-\delta})$, ${}^*\mathcal{K}$ must decay one order faster, as is manifest by its alignment with $r^{-4}\partial_r(r^4 \text{curl } \Xi)$, an expression naturally comes from (2.61). This is in fact related to the remark in [9, Page 11] on the existence of the angular momentum: While the metric is allowed to decay at the $r^{-\frac{3}{2}}$ level in [9], it is shown through the momentum constraint $\mathcal{C}_{Ham} = 0$ that the angular momentum exists despite the lack of decay at first glance. The equation (2.61) in fact comes from the same momentum constraint, see Appendix A.2.*

2.8 List of notations and conventions

For the benefit of the reader, we recall below the main notations we have introduced:

$$\begin{aligned}p_a &:= g(\nabla_N N, e_a), \quad \theta_{ab} := g(\nabla_a N, e_b), \quad \hat{R}_{ab} := R(N, e_a, N, e_b), \quad Y_a := R(N, e_b, e_b, e_a), \\ \Theta_{ab} &:= k(e_a, e_b), \quad \Xi_a := k(N, e_a), \quad \Pi := k(N, N), \quad \hat{\alpha} = (Nr)^{-1}, \\ \mu &= -\Delta(\log \hat{\alpha}) + K - \frac{1}{4}(\text{tr } \theta)^2, \quad \nu = \text{div } \Xi, \\ g &= \hat{\alpha}^2 dr^2 + \gamma, \quad \gamma^{(0)} = r^2(\mathbb{S}^2_\gamma) = r^2((d\vartheta^1)^2 + \sin^2(\vartheta^1)(d\vartheta^2)^2).\end{aligned} \quad (2.80)$$

²⁴Note that in view of (2.71), the equation of $\dot{\Psi}_8$ implies that $\dot{\Psi}_{12} = 0$.

We wish to construct solutions such that

$$\mathcal{B} := (\mathrm{div} Y)_{\ell \geq 2}, \quad {}^*\mathcal{B} := (\mathrm{curl} Y)_{\ell \geq 2}, \quad \mathcal{K} := (\Delta \Pi)_{\ell \geq 2}, \quad {}^*\mathcal{K} := (r^{-4} \partial_r (r^4 \mathrm{curl} \Xi))_{\ell \geq 2}.$$

For a general metric γ , we use the notation $\bar{f}^\gamma$ for the spherical mean with respect to γ . We drop the γ when there is no danger of confusion. We use the notation $\tilde{\psi}$ for the quantity ψ subtracted by its value in Schwarzschild.

3 Technical lemmas

3.1 Equivalent norms

We have the following equivalence of the Sobolev norms defined through $\gamma^{(0)}$ and γ .

Lemma 3.1. *Consider a 2-sphere S_r equipped with standard spherical coordinates and the associated rescaled round metric $\gamma^{(0)} = r^2(\mathbb{S}^2_\gamma)$. Suppose that another metric γ on S satisfies, for some integer $s \geq 3$,*

$$r^{-1} \|\gamma - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \leq \varepsilon \ll 1. \quad (3.1)$$

Then, denoting by ∇ the covariant derivative of γ , we have,

- For all $i \leq s + 2$ and scalar field ϕ , we have

$$\|(r\nabla)^{\leq i} \phi\|_{L^2(S_r, \gamma)} \sim \|\phi\|_{\mathfrak{h}^i(S_r)};$$

- For all $i \leq s + 1$ and rank- k covariant tensor $U = U_{a_1 \dots a_k}$,

$$\|(r\nabla)^{\leq i} \phi\|_{L^2(S_r, \gamma)} \sim \|\phi\|_{\mathfrak{h}^i(S_r)}.$$

In both cases, the two-sided implicit constant can be taken to be $(1 + C\varepsilon)$ for some constant $C > 0$.

Proof. By standard Sobolev embeddings, we have $\|(r\nabla^{(0)})^i U\|_{L^\infty(S_r)} \lesssim r^{-1} \|U\|_{\mathfrak{h}^s(S_r)}$ for $i \leq s - 1$. In particular, by (3.1), we infer $\|(r\nabla^{(0)})^i (\gamma - \gamma^{(0)})\|_{L^\infty(S_r)} \lesssim \varepsilon$ for $i \leq s - 1$.

Recall that the covariant derivative of γ (resp. $\gamma^{(0)}$) is ∇ (resp. $\nabla^{(0)}$). In view of Remark 2.10, for a scalar field ϕ , we have, schematically, $\nabla \phi = \nabla^{(0)} \phi$, $\nabla^2 \phi = (\nabla^{(0)})^2 \phi + \nabla^{(0)}(\gamma - \gamma^{(0)}) \cdot \nabla^{(0)} \phi$, and, inductively,

$$\nabla^i \phi = (\nabla^{(0)})^i \phi + \sum_{\substack{i_1 + i_2 = i, \\ i_1 \leq i-1, i_2 \leq i-1}} (\nabla^{(0)})^{i_1} (\gamma - \gamma^{(0)}) \cdot (\nabla^{(0)})^{i_2} \phi. \quad (3.2)$$

For a general horizontal covariant tensor U , we have, schematically, $\nabla U = \nabla^{(0)} U + \nabla^{(0)}(\gamma - \gamma^{(0)}) \cdot U$, $\nabla^2 U = (\nabla^{(0)})^2 U + \nabla^{(0)}(\gamma - \gamma^{(0)}) \cdot \nabla^{(0)} U + (\nabla^{(0)})^2 (\gamma - \gamma^{(0)}) \cdot U$, and, inductively,

$$\nabla^i U = (\nabla^{(0)})^i U + \sum_{i_1 + i_2 = i, i_2 \leq i-1} (\nabla^{(0)})^{i_1} (\gamma - \gamma^{(0)}) \cdot (\nabla^{(0)})^{i_2} U. \quad (3.3)$$

Using (3.2),

$$\|\nabla^i \phi - (\nabla^{(0)})^i \phi\|_{L^2(S_r)} \lesssim \sum_{\substack{i_1 + i_2 = i, \\ i_1 \leq i-1, i_2 \leq i-1}} \|(\nabla^{(0)})^{i_1} (\gamma - \gamma^{(0)}) \cdot (\nabla^{(0)})^{i_2} \phi\|_{L^2(S_r)}.$$

Since $s + 2 \geq 5$, for $i \leq s + 2$, either i_1 or i_2 in the sum is no greater than $s - 1$, for which we can apply the L^∞ estimate, leaving the other controlled by the L^2 -type norms. The estimate then easily follows. The case for covariant tensor follows similarly using (3.3). $\square$

3.2 Hodge estimates

Lemma 3.2. *Consider a 2-sphere S_r equipped with standard spherical coordinates and the associated rescaled round metric $\gamma^{(0)} = r^2(\mathbb{S}^2_\gamma)$, and another metric γ on S which satisfies $r^{-1}\|\gamma - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \leq \tilde{\varepsilon} \ll 1$ for some $s \geq 3$. Suppose for $\xi \in \mathfrak{s}_1$ and $h \in \mathfrak{s}_2(S, \gamma)$ we have*

$$\mathcal{P}_1 \xi = (f, {}^*f), \quad \mathcal{P}_2 h = F.$$

Then the following estimates hold for all $i \leq s$:

$$\|\xi\|_{\mathfrak{h}^{i+1}(S_r)} \lesssim r\|(f, {}^*f)\|_{\mathfrak{h}^i(S_r)}, \quad \|h\|_{\mathfrak{h}^{i+1}(S_r)} \lesssim r\|F\|_{\mathfrak{h}^i(S_r)}. \quad (3.4)$$

Proof. We only prove the first inequality, as the second is similar. Commuting the equations with ∇^i , we have, schematically,

$$\mathcal{P}_1 \nabla^i \xi = \nabla^i (f, {}^*f) + \nabla^{i-1} (K_\gamma \cdot \xi).$$

Here we adopt the convention that $\nabla^{-1}\psi = 0$ for any quantity ψ . In view of the assumption for γ and standard Sobolev embeddings, we have $r^{-1}\|(K_\gamma - r^{-2})\|_{L^\infty(S_r)} \lesssim \tilde{\varepsilon}$, i.e., $r^2 K$ is uniformly close to 1. The standard Hodge estimate ([9, Lemma 2.2.2]) is then applicable and implies

$$\|\nabla^{i+1} \xi\|_{L^2(S, \gamma)} + r^{-1}\|\nabla^i \xi\|_{L^2(S, \gamma)} \lesssim \|\nabla^i (f, {}^*f) + \nabla^{i-1} (K_\gamma \cdot \xi)\|_{L^2(S, \gamma)}.$$

Therefore, applying Lemma 3.1 to ξ , $(f, {}^*f)$ and the schematic 1-form $K_\gamma \cdot \xi$, we obtain, for $i \leq s$,

$$\begin{aligned} \|\xi\|_{\mathfrak{h}^{i+1}(S_r)} &\lesssim \|(r\nabla)^{\leq i+1} \xi\|_{L^2(S, \gamma)} \lesssim \sum_{j \leq i} \|(r\nabla)^{j+1} \xi\|_{L^2(S, \gamma)} + \|\xi\|_{L^2(S, \gamma)} \\ &\lesssim \sum_{j \leq i} \|r(r\nabla)^j (f, {}^*f) + r^2(r\nabla)^{j-1} (K_\gamma \cdot \xi)\|_{L^2(S, \gamma)} + r\|(f, {}^*f)\|_{L^2(S, \gamma)} \\ &\lesssim r\|(f, {}^*f)\|_{\mathfrak{h}^i(S_r)} + r^2\|(K_\gamma \cdot \xi)\|_{\mathfrak{h}^{i-1}(S_r)} + r\|(f, {}^*f)\|_{L^2(S_r)} \\ &\lesssim r\|(f, {}^*f)\|_{\mathfrak{h}^i(S_r)} + \|\xi\|_{\mathfrak{h}^{i-1}(S_r)}, \end{aligned}$$

where for the nonlinear term $K_\gamma \cdot \xi$, we used the standard L^2 - L^∞ type estimates, with L^∞ applied to the factor with less derivatives. The estimate for ξ then follows by induction. The estimate for h follows in a similar way. $\square$

We also have the following estimate regarding the $\ell \leq 1$ part of $\mathcal{P}_1^\gamma \mathcal{P}_2^\gamma h$, which is heuristically mostly supported on $\ell \geq 2$.

Lemma 3.3. *Suppose h is a symmetric 2-tensor on (S, γ) . Then,*²⁵

$$r^{-1}\|(\mathcal{P}_1^\gamma \mathcal{P}_2^\gamma h)_{\ell \leq 1}\|_{\mathfrak{h}^s(S_r)} \lesssim \|\Delta^{(0)} \mathfrak{t}^{(0)} h\|_{L^\infty(S_r)} + r^{-2}\|(r\nabla^{(0)})^{\leq 2} h\|_{L^\infty(S_r)} \|(r\nabla^{(0)})^{\leq 2} (\gamma - \gamma^{(0)})\|_{L^\infty(S_r)}. \quad (3.5)$$

Proof. Recall from (2.19) that $r^{-1}\|(\mathcal{P}_1^\gamma \mathcal{P}_2^\gamma h)_{\ell \leq 1}\|_{\mathfrak{h}^s(S_r)} \lesssim \|(\mathcal{P}_1^\gamma \mathcal{P}_2^\gamma h)_{\ell \leq 1}\|_{L^\infty(S_r)}$. We have, schematically,

$$\mathcal{P}_1^\gamma \mathcal{P}_2^\gamma h = \mathcal{P}_1^{(0)} \mathcal{P}_2^{(0)} h + (\gamma - \gamma^{(0)}) \cdot (\nabla^{(0)})^2 h + \nabla^{(0)}(\gamma - \gamma^{(0)}) \cdot \nabla^{(0)} h + (\nabla^{(0)})^2 (\gamma - \gamma^{(0)}) \cdot h.$$

Note that $\widehat{h} := h - \frac{1}{2}(\mathfrak{t}^{(0)} h)\gamma^{(0)}$ is traceless, and hence we have $(\mathcal{P}_1^{(0)} \mathcal{P}_2^{(0)} \widehat{h})_{\ell \leq 1} = 0$ by (2.15). Therefore,

$$(\mathcal{P}_1^{(0)} \mathcal{P}_2^{(0)} h)_{\ell \leq 1} = \left(\mathcal{P}_1^{(0)} \mathcal{P}_2^{(0)} \left(\frac{1}{2}(\mathfrak{t}^{(0)} h)\gamma^{(0)} \right) \right)_{\ell \leq 1} = \left(\frac{1}{2} \Delta^{(0)} \mathfrak{t}^{(0)} h, 0 \right)_{\ell \leq 1}.$$

The estimate (3.5) then follows by combining these relations. $\square$

²⁵Here we do not assume that h is traceless with respect to γ , but we extend the definition of $\mathcal{P}_2^\gamma$ trivially, to all symmetric 2-tensors, by $\mathcal{P}_2^\gamma h := \text{div}^\gamma h$.

3.3 Commutation formulas

We first give the commutation formula between ∇_N and ∇_a on horizontal covariant tensors.

Lemma 3.4. *We have*

$$[\nabla_N, \nabla_a]U_{b_1 \dots b_k} = -\theta_{ac}\nabla_c U_{b_1 \dots b_k} - p_a \nabla_N U_{b_1 \dots b_k} + (p_{b_i}\theta_{ac} + p_c\theta_{ab_i} + \epsilon_{b_i c} {}^*Y_a)U_{b_1 \dots c \dots b_k}. \quad (3.6)$$

Proof. We only prove the case $k = 1$ for simplicity, and the higher rank cases are similar. For a horizontal 1-form ξ , we have $\nabla_a \xi_b = \nabla_a \xi_b$, $\nabla_N \xi_b = \nabla_N \xi_b$. Therefore, we have, see (A.1) for the calculation rules,

$$\begin{aligned} \nabla_N \nabla_a \xi_b &= \nabla_N (\nabla \xi)_{ab} = \nabla_N (\nabla \xi)_{ab} + p_a (\nabla \xi)_{Nb} + p_b (\nabla \xi)_{aN} = \nabla_N \nabla_a \xi_b + p_a (\nabla_N \xi_b) + p_b (\nabla_a \xi_N) \\ &= \nabla_N \nabla_a \xi_b + p_a \nabla_N \xi_b - p_b \theta_{ac} \xi_c, \end{aligned}$$

and

$$\begin{aligned} \nabla_a \nabla_N \xi_b &= \nabla_a (\nabla \xi)_{Nb} = \nabla_a (\nabla \xi)_{Nb} - \theta_{ac} (\nabla \xi)_{cb} + \theta_{ab} (\nabla \xi)_{NN} \\ &= \nabla_a \nabla_N \xi_b - \theta_{ac} \nabla_c \xi_b + \theta_{ab} (p_c \xi_c). \end{aligned}$$

On the other hand, we have, using (2.4),

$$\nabla_N \nabla_a \xi_b - \nabla_a \nabla_N \xi_b = R_{Nabc} \xi_c = \epsilon_{bc} \xi_c {}^*Y_a.$$

Therefore,

$$\begin{aligned} \nabla_N \nabla_a \xi_b - \nabla_a \nabla_N \xi_b &= \nabla_a \nabla_N \xi_b - p_a \nabla_N \xi_b + p_b \theta_{ac} \xi_c - (\nabla_a \nabla_N \xi_b + \theta_{ac} \nabla_c \xi_b - \theta_{ab} p_c \xi_c) \\ &= \epsilon_{bc} \xi_c {}^*Y_a - p_a \nabla_N \xi_b + p_b \theta_{ac} \xi_c - \theta_{ac} \nabla_c \xi_b + \theta_{ab} p_c \xi_c, \end{aligned}$$

as required. $\square$

Note that in most situations, we will commute with $\nabla^{(0)}$ rather than ∇ ; When the metric $g = g^{(0)} = \Upsilon^{-1}dr^2 + \gamma^{(0)}$, (3.6) simplifies to $[\nabla_{\partial_r}^{(0)}, \nabla^{(0)}]U = -r^{-1}\nabla^{(0)}U$, or equivalently,

$$[\nabla_{\partial_r}^{(0)}, r\nabla^{(0)}]U = 0. \quad (3.7)$$

Lie derivatives. Recall the usual definition for Lie derivatives on k -covariant tensor on Σ

$$\mathcal{L}_X T_{i_1 \dots i_k} = \nabla_X T + \nabla_{i_1} X^j T_{j \dots i_k} + \dots + \nabla_{i_k} X^j T_{i_1 \dots j}.$$

Such a definition, as is well-known, is in fact independent of the metric. When $T = U_{a_1 \dots a_k}$ is a horizontal tensor, the Lie derivative $\mathcal{L}_X U$ is not necessarily a horizontal tensor. Following [8], see also [22]²⁶, we can instead define the projected Lie derivative

$$\mathcal{L}_X U_{a_1 \dots a_k} := \nabla_X U_{a_1 \dots a_k} + \nabla_{a_1} X^b U_{b \dots a_k} + \dots + \nabla_{a_k} X^b U_{a_1 \dots b}. \quad (3.8)$$

In particular, we have, for $X = f\partial_r$,

$$\begin{aligned} \mathcal{L}_{f\partial_r} U_{a_1 \dots a_k} &= \nabla_{f\partial_r} U_{a_1 \dots a_k} + \nabla_{a_1} (f\partial_r)^b U_{b \dots a_k} + \dots + \nabla_{a_k} (f\partial_r)^b U_{a_1 \dots b} \\ &= f\nabla_{\partial_r} U_{a_1 \dots a_k} + f\nabla_{a_1} (\partial_r)^b U_{b \dots a_k} + \dots + f\nabla_{a_k} (\partial_r)^b U_{a_1 \dots b} \\ &= f\mathcal{L}_{\partial_r} U_{a_1 \dots a_k}. \end{aligned} \quad (3.9)$$

This is independent of the metric as long as ∂_r is orthogonal to the r -spheres. Therefore, we compute using the metric $g^{(0)} = \Upsilon^{-1}dr^2 + \gamma^{(0)}$, for which we have $(\nabla_A^{(0)} \partial_r)_B = r^{-1}\gamma_{AB}^{(0)}$. Therefore, using the definition (3.8), we have

$$\mathcal{L}_{\partial_r} U_{a_1 \dots a_k} = \nabla_{\partial_r}^{(0)} U_{a_1 \dots a_k} + kr^{-1}U_{a_1 \dots a_k}. \quad (3.10)$$

²⁶[22] extends the definition of [8] to non-integrable structures.

3.4 Transport lemma

Lemma 3.5. *Suppose a scalar or horizontal covariant tensor ψ satisfies, for some nonnegative integer i ,*

$$\nabla_{\partial_r}^{(0)} \psi + \lambda r^{-1} \psi = F, \quad \text{and } r^{-1} \|r^\lambda \psi\|_{\mathfrak{h}^i(S_r)} \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

Then we have

$$r^{-1} \|r^\lambda \psi\|_{\mathfrak{h}^i(S_r)} \lesssim \int_r^\infty r'^{-1} \|r'^\lambda F\|_{\mathfrak{h}^i(S_{r'})} dr'.$$

Proof. The equation can be written as $\partial_r(r^\lambda \psi) = r^\lambda F$. Since $d\text{vol}_{r-2\gamma(0)}$ is independent of r , we have

$$\begin{aligned} \left| \partial_r \int_{S_r} |r^{\lambda-1} \psi|^2 d\text{vol}_{\gamma(0)} \right| &= \left| \partial_r \int_{S_r} |r^\lambda \psi|^2 d\text{vol}_{r-2\gamma(0)} \right| \lesssim \left| 2 \int_{S_r} (\partial_r(r^\lambda \psi) \cdot r^\lambda \psi) d\text{vol}_{r-2\gamma(0)} \right| \\ &= \left| 2 \int_{S_r} r^\lambda F \cdot r^\lambda \psi d\text{vol}_{r-2\gamma(0)} \right| = \left| 2 \int_{S_r} r^{\lambda-1} F \cdot r^{\lambda-1} \psi d\text{vol}_{\gamma(0)} \right| \\ &\lesssim \|r^{\lambda-1} F\|_{L^2(S_r)} \|r^{\lambda-1} \psi\|_{L^2(S_r)}, \end{aligned}$$

i.e., $|\partial_r(\|r^{\lambda-1} \psi\|_{L^2(S_r)}^2)| \lesssim \|r^{\lambda-1} F\|_{L^2(S_r)} \|r^{\lambda-1} \psi\|_{L^2(S_r)}$. Therefore, either $\psi = 0$, in which case the lemma automatically holds, or we can divide both sides by $\|r^{\lambda-1} \psi\|_{L^2(S_r)}$ to infer that $|\partial_r(\|r^{\lambda-1} \psi\|_{L^2(S_r)})| \lesssim \|r^{\lambda-1} F\|_{L^2(S_r)}$. Therefore, the estimate follows for $i = 0$. For positive integers i , it follows similarly by commuting the equation with $r\nabla^{(0)}$ using the commutation formula (3.7). $\square$

3.5 Solvability lemma for the operator $\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_2^* \mathcal{P}_1^*$

We study the solvability of the following equation on (S, γ) :

$$\mathcal{P}_1 \mathcal{P}_2 h = (F, {}^*F), \quad h \in \mathfrak{s}_2.$$

Recall that, see Section 2.2 in [9], as the formal adjoints of the injective elliptic operators $\mathcal{P}_2$ and $\mathcal{P}_1$ on 2-spheres, $\mathcal{P}_2^*$ and $\mathcal{P}_1^*$ are surjective. Therefore, for each h , there exists $(f, {}^*f) \in \mathfrak{s}_0$ such that $\mathcal{P}_2^* \mathcal{P}_1^*(f, {}^*f) = h$. The equation then becomes

$$\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_2^* \mathcal{P}_1^*(f, {}^*f) = (F, {}^*F).$$

Recall that these operators are defined in Definition 2.4.

We now prove the following lemma, which is a slight generalization of Lemma 2.19 in [30].

Lemma 3.6. *Consider the operator $L := \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_2^* \mathcal{P}_1^*$ on (S, γ) , where S is equipped with a standard spherical coordinate and a constant $r > 0$, and hence admits the metric $\gamma^{(0)}$. Suppose that the metric satisfies the estimate $\|(r\nabla^{(0)})^{\leq 4}(\gamma - \gamma^{(0)})\|_{L^\infty(S)} \leq \varepsilon \ll 1$. Then the following statements hold:*

- *The operator $L = \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_2^* \mathcal{P}_1^*$ is a densely-defined self-adjoint operator on $L^2(S, \gamma) \times L^2(S, \gamma)$. In addition to the zero eigenvalue corresponding to two trivial kernel elements $(1, 0)$ and $(0, 1)$, there exist six eigenvalues of L , denoted by $\lambda_p, {}^*\lambda_p$ with $p = 0, +, -$, satisfying $|\lambda_p|, |{}^*\lambda_p| \lesssim \varepsilon r^{-4}$, with real-valued eigenfunction pairs*

$$(j, {}^*j)_{\lambda_p} = (J_p, 0) + O(\varepsilon), \quad (j, {}^*j)_{{}^*\lambda_p} = (0, J_p) + O(\varepsilon).$$

Any other eigenvalue λ of L satisfies $|\lambda| \gtrsim r^{-4}$.

- For the equation

$$\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_2^* \mathcal{P}_1^*(f, {}^*f) = \sum_{p=0,+,-} (F + c_p J_p + c_0, {}^*F + {}^*c_p J_p + {}^*c_0), \quad (3.11)$$

there exist unique constants $c_0, {}^*c_0, c_p, {}^*c_p$, for which (3.11) has a unique solution $(f, {}^*f)$ orthogonal to $(1, 0), (0, 1), (j, {}^*j)_{\lambda_p}, (j, {}^*j)^*_{\lambda_p}$ in $L^2(S, \gamma)$. Moreover, the constants satisfy the estimate

$$|(c_0, {}^*c_0)| \lesssim |\overline{(F, {}^*F)}|, \quad |c_p + \langle F, J_p \rangle_{r-2\gamma(0)}| + |{}^*c_p + \langle {}^*F, J_p \rangle_{r-2\gamma(0)}| \lesssim \hat{\varepsilon} r^{-1} \|(F, {}^*F)\|_{L^2(S, \gamma)}.$$

Proof. In view of (2.11), it is clear that L is symmetric on $C^\infty(S) \times C^\infty(S)$ with respect to the inner product of $L^2(S, \gamma) \times L^2(S, \gamma)$. Since L is also clearly non-negative, there exists a Friedrichs extension, still denoted by L , that is densely defined in $L^2(S, \gamma) \times L^2(S, \gamma)$ and self-adjoint. Note that when $r^{-2}\gamma = \mathbb{S}^2_\gamma$, the operator reads $\mathbb{A}^{(0)}(\mathbb{A}^{(0)} + 2r^{-2})$ that acts on scalar pairs, which, in addition to the two constant kernels, has a 6-dimensional kernel spanned by $(J_p, 0)$ and $(0, J_p)$. The first part of the lemma then follows from the fact that $r^{-2}\gamma$ is a perturbation of $\mathbb{S}^2_\gamma$. Note also that since constant function pairs lie in the kernel of L , we have $\langle (j, {}^*j)_{\lambda_p}, (c_1, c_2) \rangle_\gamma = \langle (j, {}^*j)^*_{\lambda_p}, (c_1, c_2) \rangle_\gamma = 0$ for any constants c_1, c_2 .

For the equation (3.11), we take its inner product with the eigenfunction pair and obtain

$$\langle \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_2^* \mathcal{P}_1^*(f, {}^*f), (j, {}^*j)_{\lambda_p} \rangle_\gamma = \langle (F, {}^*F), (j, {}^*j)_{\lambda_p} \rangle_\gamma + \sum_{q=0,+,-} \langle (c_q J_q + c_0, {}^*c_q J_q + {}^*c_0), (j, {}^*j)_{\lambda_p} \rangle_\gamma$$

Recall that we require that the solution $(f, {}^*f)$ is orthogonal to $(j, {}^*j)_{\lambda_p}$. Therefore, since L is self-adjoint and $(j, {}^*j)_{\lambda_p}$ are eigenfunction pairs, the left-hand side is zero, and so are the terms with c_0 and *c_0 as we just remarked. Therefore, we deduce

$$\sum_{q=0,+,-} \langle (c_q J_q, {}^*c_q J_q), (j, {}^*j)_{\lambda_p} \rangle_{r-2\gamma} = - \langle (F, {}^*F), (j, {}^*j)_{\lambda_p} \rangle_{r-2\gamma}.$$

Now using the fact that $(j, {}^*j)_{\lambda_p} = (J_p, 0) + O(\hat{\varepsilon})$, the left hand side equals $(\delta_{pq} + O(\hat{\varepsilon}))c_q$. We then also take the inner product of (3.11) with $(j, {}^*j)^*_{\lambda_p}$. This gives a linear system of $c_p, {}^*c_p$ whose coefficient matrix is $O(\hat{\varepsilon})$ -close to the identity matrix, and hence we obtain the unique existence of $(c_p, {}^*c_p)$. The uniqueness of $(c_0, {}^*c_0)$ is then also clear by taking the spherical mean with respect to γ for (3.11). The bounds for the constants also follow directly from the relations

$$\begin{aligned} \langle (F, {}^*F), (j, {}^*j)_{\lambda_p} \rangle_{r-2\gamma} &= \langle F, J_p \rangle_{r-2\gamma} + \langle (F, {}^*F), O(\hat{\varepsilon}) \rangle_{r-2\gamma}, \\ \langle (F, {}^*F), (j, {}^*j)^*_{\lambda_p} \rangle_{r-2\gamma} &= \langle {}^*F, J_p \rangle_{r-2\gamma} + \langle (F, {}^*F), O(\hat{\varepsilon}) \rangle_{r-2\gamma}, \end{aligned}$$

the bound for $\gamma - \gamma^{(0)}$, and Hölder's inequality. The existence and uniqueness of $(f, {}^*f)$ also follows easily from the fact that L is invertible on the orthogonal complement of $\text{span}\{(1, 0), (0, 1), (j, {}^*j)_{\lambda_p}, (j, {}^*j)^*_{\lambda_p}\}$. $\square$

Corollary 3.7. For the equation

$$\mathcal{P}_1 \mathcal{P}_2 h = \sum_{p=0,+,-} (F + c_p J_p + c_0, {}^*F + {}^*c_p J_p + {}^*c_0), \quad (3.12)$$

there exist unique constants $c_0, {}^*c_0, c_p, {}^*c_p$ for which (3.12) has a unique solution $h \in \mathfrak{s}_2(S, \gamma)$. Moreover, the constants satisfy the estimate

$$|(c_0, {}^*c_0)| \lesssim |\overline{(F, {}^*F)}|, \quad |c_p + \langle F, J_p \rangle_{r-2\gamma(0)}| + |{}^*c_p + \langle {}^*F, J_p \rangle_{r-2\gamma(0)}| \lesssim \hat{\varepsilon} r^{-1} \|(F, {}^*F)\|_{L^2(S, \gamma)}. \quad (3.13)$$

Proof. The uniqueness of $c_0, {}^*c_0, c_p, {}^*c_p$ follows from that h can be expressed in the form $\mathcal{P}_2^* \mathcal{P}_1^*(f, {}^*f)$. The uniqueness of h follows from the fact that $\mathcal{P}_1 \mathcal{P}_2$ has no kernel. $\square$

4 Sketch of the proof of the main theorem

4.1 The linear iteration system

Recall that we are solving the equations on a base manifold $\Sigma := (r_0, \infty) \times \mathbb{S}^2$, and the spherical modes are accordingly defined in Section 2.2.2. According to the statement of Theorem 2.30, at the level of $\ell \geq 2$ modes, we prescribe $(\mathcal{B}, {}^*\mathcal{B}, \mathcal{K}, {}^*\mathcal{K})$, and we iteratively find the data such that

$$(\mathrm{div} Y - \mathcal{B})_{\ell \geq 2} = 0, \quad (\mathrm{curl} Y - {}^*\mathcal{B})_{\ell \geq 2} = 0, \quad (\Delta(\hat{\alpha}\Pi) - \mathcal{K})_{\ell \geq 2} = 0, \quad (r^{-4}(\partial_r(r^4 \mathrm{curl} \Xi)) - {}^*\mathcal{K})_{\ell \geq 2} = 0.$$

More precisely, we show that the sequence of iterates $\Psi^{(n)}$ of the system (2.70) converge to the desired solution. Motivated by the equation (2.58) in Proposition 2.24, starting with $\gamma^{(0)}$ defined in (2.13), $\gamma^{(n)}$ is determined iteratively by solving the transport equation²⁷

$$\mathcal{L}_{\partial_r}(r^{-2}\gamma^{(n+1)}) = 2r^{-2}\hat{\alpha}^{(n)}\hat{\theta}^{(n+1)} + \hat{\alpha}^{(n)}\widetilde{\mathrm{tr}\theta}^{(n+1)}(r^{-2}\gamma^{(n)}) + 2\Upsilon^{\frac{1}{2}}\check{\alpha}^{(n+1)}r^{-1}(r^{-2}\gamma^{(n)}). \quad (4.1)$$

Given $\gamma^{(n)}$, we can define the horizontal operators $\nabla^{(n)}$, $\mathrm{div}^{(n)}$, $\mathrm{curl}^{(n)}$, $\Delta^{(n)}$, $\mathcal{P}_1^{(n)}$, $\mathcal{P}_2^{(n)}$, $\dots$, as well as the spherical mean $\bar{\phi}^{(n)}$ of a scalar field ϕ with respect to $\gamma^{(n)}$. Recalling the definition of Ψ in (2.66), the iterate $\Psi^{(n)}$ reads

$$\begin{aligned} \Psi_1^{(n)} &= \widetilde{\mathrm{tr}\theta}^{(n)}, & \Psi_2^{(n)} &= \check{K}^{(n)}, & \Psi_3^{(n)} &= \check{\alpha}^{(n)}, & \Psi_4^{(n)} &= \hat{\theta}^{(n)}, & \Psi_5^{(n)} &= p^{(n)}, & \Psi_6^{(n)} &= Y^{(n)}, \\ \Psi_7^{(n)} &= \mathrm{tr}\Theta^{(n)}, & \Psi_8^{(n)} &= \hat{\Theta}^{(n)}, & \Psi_9^{(n)} &= \Xi^{(n)}, & \Psi_{10}^{(n)} &= \Pi^{(n)}, \\ \Psi_{11}^{(n)} &= (\mathcal{B}_{\ell \leq 1}^{(n)}, {}^*\mathcal{B}_{\ell \leq 1}^{(n)}), & \Psi_{12}^{(n)} &= (\mathcal{K}_{\ell \leq 1}^{(n)}, {}^*\mathcal{K}_{\ell \leq 1}^{(n)}). \end{aligned} \quad (4.2)$$

We introduce the following norm

$$\begin{aligned} \|(\Psi^{(n)}, \gamma^{(n)})\|_s &:= \sup_{r \in [r_0, \infty)} \left(r^{1+\delta} \|(\Psi_1^{(n)}, \Psi_4^{(n)}, \Psi_5^{(n)}, \Psi_7^{(n)}, \Psi_8^{(n)}, \Psi_9^{(n)}, \Psi_{10}^{(n)})\|_{\mathfrak{h}^{s+1}(S_r)} \right. \\ &\quad + r^\delta \|\Psi_3^{(n)}\|_{\mathfrak{h}^{s+2}(S_r)} + r^{2+\delta} \|(\Psi_2^{(n)}, \Psi_6^{(n)})\|_{\mathfrak{h}^s(S_r)} + r^\delta \|\gamma^{(n)} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \\ &\quad \left. + r^{4+\delta} |(\Psi_{11}^{(n)}, \Psi_{12}^{(n)})| + r^\delta |r^3(\Psi_1^{(n)})_{\ell=1,i} - \mathbf{c}_i| \right), \end{aligned} \quad (4.3)$$

where the $\mathfrak{h}^s$ norms are defined in Definition 2.7.

Remark 4.1. Note that the weights in (4.3) are consistent with the differentiability order of the corresponding quantities, as pointed out in Remark 2.23.

We consider the following iteration system, motivated by Proposition 2.24:

$$(\partial_r + 2r^{-1})\Psi_1^{(n+1)} = \Upsilon^{-\frac{1}{2}}\check{\mu}_{\ell=0}^{(n+1)} + \Psi_3^{(n)}\check{\mu}_{\ell=0}^{(n)} - 2(1 - 3mr^{-1})r^{-2}\Psi_3^{(n+1)} + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \quad (4.4)$$

$$\begin{aligned} (\partial_r + 3r^{-1})\Psi_2^{(n+1)} &= r^{-1}\check{\mu}_{\ell=0}^{(n+1)} - 2\Upsilon^{\frac{1}{2}}r^{-3}\Psi_3^{(n+1)} - \Upsilon^{-\frac{1}{2}}(\mathcal{B} + \widetilde{\mathcal{B}}_{\ell \leq 1}^{(n+1)}) \\ &\quad - \Psi_3^{(n)}(\mathcal{B} + \widetilde{\mathcal{B}}_{\ell \leq 1, aux}^{(n)}) + \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}, \end{aligned} \quad (4.5)$$

$$\begin{aligned} \Upsilon^{\frac{1}{2}}\Delta^{(n)}\Psi_3^{(n+1)} &= \Psi_2^{(n+1)} - \overline{\Psi_2^{(n+1)}}^{(n)} - \Upsilon^{\frac{1}{2}}r^{-1}(\Psi_1^{(n+1)} - \overline{\Psi_1^{(n+1)}}^{(n)}) \\ &\quad + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} - \overline{\Gamma_1^{(n)}} \cdot \Gamma_1^{(n)} - \Delta^{(n)}(\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}), \end{aligned} \quad (4.6)$$

$$\overline{\Psi_3^{(n+1)}}^{(n)} = -\frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1^{(n+1)}}^{(n)}, \quad (4.7)$$

²⁷with the boundary condition at infinity given by $\|\gamma^{(n+1)} - \gamma^{(0)}\|_{\mathfrak{h}^s} \rightarrow 0$. This is ensured in the space where we seek solutions, see (4.3).

$$\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} \Psi_4^{(n+1)} = \frac{1}{2} \left(\mathcal{A}^{(n)} \Psi_1^{(n+1)}, 0 \right) - (\mathcal{B}, * \mathcal{B}) - \Psi_{11}^{(n+1)}, \quad (4.8)$$

$$\mathcal{P}_1^{(n)} \Psi_5^{(n+1)} = \left(-\Upsilon^{\frac{1}{2}} \mathcal{A}^{(n)} \Psi_3^{(n+1)} + \mathcal{A}^{(n)} (\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}), 0 \right), \quad (4.9)$$

$$\mathcal{P}_1^{(n)} \Psi_6^{(n+1)} = (\mathcal{B}, * \mathcal{B}) + \Psi_{11}^{(n+1)} - \overline{(\mathcal{B}, * \mathcal{B}) + \Psi_{11}^{(n+1)}}^{(n)}, \quad (4.10)$$

$$(\partial_r + r^{-1}) \Psi_7^{(n+1)} = 2r^{-1} \Psi_{10}^{(n+1)} + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \quad (4.11)$$

$$\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} \left(\hat{a}^{(n)} \Psi_8^{(n+1)} \right) = \frac{1}{2} \left(\hat{a}^{(n)} \mathcal{A}^{(n)} \Psi_7^{(n+1)}, 0 \right) + (\mathcal{K}, - * \mathcal{K}) + \Psi_{12}^{(n+1)} + \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}, \quad (4.12)$$

$$\begin{aligned} \mathcal{P}_1^{(n)} \Psi_9^{(n+1)} &= \left(0, \frac{3}{4\pi} r^{-4} \sum_i \mathbf{a}_i \omega_i + r^{-4} \int_r^\infty r'^4 (*\mathcal{K} - *\mathcal{K}_{\ell \leq 1}^{(n+1)}) dr' \right) \\ &\quad - \left(0, \frac{3}{4\pi} r^{-4} \sum_i \mathbf{a}_i \omega_i + r^{-4} \int_r^\infty r'^4 (*\mathcal{K} - *\mathcal{K}_{\ell \leq 1}^{(n+1)}) dr' \right)^{(n)}, \end{aligned} \quad (4.13)$$

$$\mathcal{A}^{(n)} \left(\hat{a}^{(n)} \Psi_{10}^{(n+1)} \right) = \mathcal{K} + \widetilde{\mathcal{K}_{\ell \leq 1}^{(n+1)}} - \overline{\mathcal{K} + \widetilde{\mathcal{K}_{\ell \leq 1}^{(n+1)}}}^{(n)}, \quad (4.14)$$

$$\overline{\hat{a}^{(n)} \Psi_{10}^{(n+1)}}^{(n)} = \overline{\Psi_3^{(n)} \Psi_{10}^{(n)}}^{(n)}, \quad (4.15)$$

along with the metric iterates introduced in (4.1):

$$\mathcal{L}_{\partial_r}(r^{-2} \gamma^{(n+1)}) = 2r^{-2} \hat{a}^{(n)} \Psi_4^{(n+1)} + \hat{a}^{(n)} \Psi_1^{(n+1)} (r^{-2} \gamma^{(n)}) + 2\Upsilon^{\frac{1}{2}} \Psi_3^{(n+1)} r^{-1} (r^{-2} \gamma^{(n)}). \quad (4.16)$$

We explain the notations used here:

- The sets $\Psi^{(n)}$ are iterates of the set Ψ introduced in (4.2). For simplicity, in various places, we still denote $\hat{a}^{(n)} = 1 + \Psi_3^{(n)}$.
- For a scalar field f , we use $\bar{f}^{(n)}$ to denote the spherical mean of f with respect to the metric $\gamma^{(n)}$.
- The schematic notations $\Gamma_i^{(n)}$ for $i = 0, 1, 2$ are defined as in Definition 2.22 labeled with $^{(n)}$. The dot products in terms like $\Gamma_1^{(n)} \cdot \Gamma_1^{(n)}$ are defined with respect to $\gamma^{(n)}$.
- The expression $\widetilde{\mathcal{B}_{\ell \leq 1}^{(n+1)}}$ stands for

$$\widetilde{\mathcal{B}_{\ell \leq 1}^{(n+1)}} := \frac{1}{2} (\mathcal{A}^{(n)} \Psi_1^{(n+1)})_{\ell=1} + \frac{1}{2} (\mathcal{A}^{(n)} \Psi_1^{(n)})_{\ell=0} - \left(\mathcal{P}_1(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} \Psi_4^{(n)}) \right)_{\ell \leq 1}. \quad (4.17)$$

We shall also make use of the auxiliary notation

$$\widetilde{\mathcal{B}_{\ell \leq 1, aux}^{(n)}} := \frac{1}{2} (\mathcal{A}^{(n)} \Psi_1^{(n)})_{\ell=1} + \frac{1}{2} (\mathcal{A}^{(n)} \Psi_1^{(n)})_{\ell=0} - \left(\mathcal{P}_1(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} \Psi_4^{(n)}) \right)_{\ell \leq 1}. \quad (4.18)$$

Here $\mathcal{P}_1$ denotes the trivial projection to the first component of a pair of scalars $(\cdot, \cdot) \in \mathfrak{s}_0$. Both $\widetilde{\mathcal{B}_{\ell \leq 1}^{(n+1)}}$ and $\widetilde{\mathcal{B}_{\ell \leq 1, aux}^{(n)}}$ behave like²⁸ $\mathcal{B}_{\ell \leq 1}^{(n+1)}$ in the limit as $n \rightarrow \infty$. In particular, in (4.17), we distinguish linear and nonlinear terms using $^{(n+1)}$ and $^{(n)}$, see heuristics already in (2.67).

- For a similar reason, we introduce the expression $\widetilde{\mathcal{K}_{\ell \leq 1}^{(n+1)}}$:

$$\widetilde{\mathcal{K}_{\ell \leq 1}^{(n+1)}} := -\frac{1}{2} (\hat{a}^{(n)} \mathcal{A}^{(n)} \Psi_7^{(n+1)})_{\ell=1} + \mathcal{P}_1 \left(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}) \right)_{\ell \leq 1} - \frac{1}{2} (\hat{a}^{(n)} \mathcal{A}^{(n)} \Psi_7^{(n)})_{\ell=0} + (\Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1}, \quad (4.19)$$

where the $\Gamma_1^{(n)} \cdot \Gamma_2^{(n)}$ takes the same precise form as the one in (4.12). See heuristics already in (2.68).

²⁸Recall that $\mathcal{B}_{\ell \leq 1}^{(n+1)}$ is the first component of $\Psi_{11}^{(n+1)}$ (defined in (4.2)).

- Consistent with the definition of $\check{\mu}$ in (2.51), $\check{\mu}^{(n)}$ denotes

$$\begin{aligned}\check{\mu}^{(n)} &:= \mu^{(n)} - 2mr^{-3} = -\Delta^{(n)}(\log \hat{a}^{(n)}) + K^{(n)} - \frac{1}{4}(\theta^{(n)})^2 - 2mr^{-3} \\ &= -\Delta^{(n)}(\log \Psi_3^{(n)}) + \Psi_2^{(n)} - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1^{(n)} - \frac{1}{4}(\Psi_1^{(n)})^2.\end{aligned}\quad (4.20)$$

Similar to (4.17), (4.19), we denote

$$\check{\mu}_{\ell=0}^{(n+1)} := (\Psi_2^{(n+1)} - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1^{(n+1)})_{\ell=0} - (\Delta^{(n)} \log \Psi_3^{(n)})_{\ell=0} - \frac{1}{4}((\Psi_1^{(n)})^2)_{\ell=0}. \quad (4.21)$$

See heuristics already in (2.69).

The $\ell \leq 1$ quantities $\Psi_{11}^{(n+1)}$, $\Psi_{12}^{(n+1)}$ will be determined by equations (4.8), (4.12) using Corollary 3.7, i.e., by projections on the $\ell \leq 1$ modes.

4.2 Solving the main part L_{main}

To solve the iteration system (4.4)-(4.16) at each step, we need to study the linear operator $^{(\tilde{\gamma})}L$ defined in (2.71). As we have pointed out in Section 2.5.3, $^{(\tilde{\gamma})}L$ has a triangular structure, such that we can focus on the main part the HCS system

$$^{(\tilde{\gamma})}L_{main}[\Psi_{main}] = \begin{pmatrix} 0 \\ \mathcal{B} \\ 0 \end{pmatrix} + err, \quad (4.22)$$

where $\Psi_{main} = (\Psi_1, \Psi_2, \Psi_3)$, and

$$^{(\tilde{\gamma})}L_{main}[\Psi_{main}] = \begin{pmatrix} (\partial_r + 2r^{-1})\Psi_1 + 2(1 - 3mr^{-1})r^{-2}\Psi_3 - \Upsilon^{-\frac{1}{2}}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1)_{\ell=0} \\ (\partial_r + 3r^{-1})\Psi_2 + 2\Upsilon^{\frac{1}{2}}r^{-3}\Psi_3 + \frac{1}{2}\Upsilon^{-\frac{1}{2}}(\Delta\Psi_1)_{\ell=1} - r^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1)_{\ell=0} \\ (\Upsilon^{\frac{1}{2}}\Delta\Psi_3, \overline{\Psi_3}) - (\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1 - \Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1, -\frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1}) \end{pmatrix}. \quad (4.23)$$

In (4.22), err denotes lower order terms that only involves nonlinear quantities from the previous step. In view of the third row of (4.22), the scalar $\Psi_3 = \check{a}$ can be written as²⁹

$$\Psi_3 = \Upsilon^{-\frac{1}{2}}\Delta_{\tilde{\gamma}}^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1) - \frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1}^{\tilde{\gamma}} + err.$$

The system (4.22) is then reduced to the following system for Ψ_1 and Ψ_2 :

$$\begin{aligned}(\partial_r + 2r^{-1})\Psi_1 &= -2(1 - 3mr^{-1})r^{-2} \left(\Upsilon^{-\frac{1}{2}}\Delta_{\tilde{\gamma}}^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1) - \frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1}^{\tilde{\gamma}} \right) \\ &\quad + \Upsilon^{-\frac{1}{2}}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1)_{\ell=0} + F_1, \\ (\partial_r + 3r^{-1})\Psi_2 &= -2\Upsilon^{\frac{1}{2}}r^{-3} \left(\Upsilon^{-\frac{1}{2}}\Delta_{\tilde{\gamma}}^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1) - \frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1}^{\tilde{\gamma}} \right) - \frac{1}{2}\Upsilon^{-\frac{1}{2}}(\Delta_{\tilde{\gamma}}\Psi_1)_{\ell=1} \\ &\quad + r^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1)_{\ell=0} + F_2,\end{aligned}\quad (4.24)$$

with F_1 and F_2 inhomogeneous terms depending the right-hand side of (4.22), i.e. free scalars or nonlinear terms. In particular, F_2 contains the free scalar $\mathcal{B}$.

We prove the following proposition regarding the system (4.24):

²⁹For given $\tilde{\gamma}$, we extend the definition of $\Delta_{\tilde{\gamma}}^{-1}$ by defining $\Delta_{\tilde{\gamma}}^{-1}\phi := \Delta_{\tilde{\gamma}}^{-1}(\phi - \overline{\phi}^{\tilde{\gamma}})$.

Proposition 4.2. Consider the system (4.24) with a given metric $\tilde{\gamma}$ satisfying

$$\sup_{r \in [r_0, \infty)} r^\delta \|\tilde{\gamma} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon_1. \quad (4.25)$$

There exist constants $\varepsilon_0, \varepsilon_1 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $\mathbf{c} \in \mathbb{R}^3$ with $|\mathbf{c}| \leq \varepsilon$, if the following bounds hold true:

$$\sup_{r \in [r_0, \infty)} r^{-1} \|r^{3+\delta} F_1, r^{4+\delta} F_2, r^{4+\delta} (F_1)_{\ell=1}, r^{5+\delta} (F_2)_{\ell=1}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon, \quad (4.26)$$

then for some suitable constant $C > 0$, there exists a unique solution to the system (4.24) satisfying

$$\sup_{r \in [r_0, \infty)} r^{-1} \|r^{2+\delta} \Psi_1, r^{3+\delta} \Psi_2\|_{\mathfrak{h}^s(S_r)} \leq C\varepsilon, \quad \sup_{r \in [r_0, \infty)} r^\delta |r^3 (\Psi_1)_{\ell=1,i} - \mathbf{c}_i, r^4 (\Psi_2)_{\ell=1,i}| \leq C\varepsilon. \quad (4.27)$$

Proof. See Section 5.1. □

4.3 Boundedness estimates

We are now ready to prove the boundedness result of the iterates.

Proposition 4.3. There exists $\varepsilon > 0$ such that for given $m > 0$, $\mathbf{c}, \mathbf{a} \in \mathbb{R}^3$, and $(\mathcal{B}, * \mathcal{B}, \mathcal{K}, * \mathcal{K})$ as considered in the statement of Theorem 2.30, there exists a constant $C_b > 0$ and a positive integer s , such that for each nonnegative integer n , if $(\Psi^{(n)}, \gamma^{(n)})$ satisfies $\|(\Psi^{(n)}, \gamma^{(n)})\|_s \leq C_b \varepsilon$, then there exists a unique solution $(\Psi^{(n+1)}, \gamma^{(n+1)})$ to the system (4.4)-(4.16) verifying $\|(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s \leq C_b \varepsilon$.

Proof. The proposition relies on the triangular structure, discussed in Remark 2.25, and follows from the following steps below. □

Step 1. We first apply Proposition 4.2 to obtain $\Psi_1^{(n+1)}$ and $\Psi_2^{(n+1)}$ by verifying the requirement (4.26). We can then obtain the estimate for $\Psi_1^{(n+1)}$ with one additional derivative. We then also retrieve the estimate for $\Psi_3^{(n+1)}$.

Proposition 4.4. We have

$$\begin{aligned} & \sup_{r \in [r_0, \infty)} r^{-1} \|r^{2+\delta} \Psi_1^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} + r^{-1} \|r^{3+\delta} \Psi_2^{(n+1)}\|_{\mathfrak{h}^s(S_r)} + r^{-1} \|r^{1+\delta} \Psi_3^{(n+1)}\|_{\mathfrak{h}^{s+2}(S_r)} \\ & + r^\delta |r^3 (\Psi_1^{(n+1)})_{\ell=1,i} - \mathbf{c}_i| \lesssim \varepsilon. \end{aligned} \quad (4.28)$$

Proof. See Section 5.2.1. □

Step 2. Now, since we have obtained $\Psi_1^{(n+1)}$, we can apply the Codazzi equation (4.8) to obtain $\Psi_{11}^{(n+1)}$ and $\Psi_4^{(n+1)}$. Then we also solve for $\Psi_6^{(n+1)}$. The estimate for $\Psi_5^{(n+1)}$ follows from the estimate for $\Psi_3^{(n+1)}$.

Proposition 4.5. We have

$$\sup_{r \in [r_0, \infty)} r^{-1} \|r^{2+\delta} (\Psi_4^{(n+1)}, \Psi_5^{(n+1)})\|_{\mathfrak{h}^{s+1}(S_r)} + r^{-1} \|r^{3+\delta} \Psi_6^{(n+1)}\|_{\mathfrak{h}^s(S_r)} + r^{5+\delta} |\Psi_{11}^{(n+1)}| \lesssim \varepsilon.$$

Proof. See Section 5.2.2. □

Step 3. We solve (4.11) and (4.14) for Ψ_7 and Ψ_{10} . This requires solving a coupled $\ell = 1$ part, which we have analyzed at the linear level in Section 2.6, and the remaining part that is decoupled.

Proposition 4.6. *We have*

$$\sup_{r \in [r_0, \infty)} r^{-1} \|r^{2+\delta}(\Psi_7^{(n+1)}, \Psi_{10}^{(n+1)})\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon.$$

Proof. See Section 5.2.3. □

Step 4. We solve the Codazzi equation (4.12) for $\Psi_8^{(n+1)}$, which also determines $\Psi_{12}^{(n+1)}$, and the div-curl equation (4.13) for $\Psi_9^{(n+1)}$.

Proposition 4.7. *We have*

$$\sup_{r \in [r_0, \infty)} r^{-1} \|r^{2+\delta}(\Psi_8^{(n+1)}, \Psi_9^{(n+1)})\|_{\mathfrak{h}^{s+1}(S_r)} + r^{5+\delta} |\Psi_{12}^{(n+1)}| \lesssim \varepsilon.$$

Proof. See Section 5.2.4. □

Step 5. We derive the estimate for the spherical metric $\gamma^{(n+1)}$ using (4.16).

Proposition 4.8. *We have*

$$\sup_{r \in [r_0, \infty)} r^{-1} \|\gamma^{(n+1)} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon r^{-1-\delta}.$$

Proof. See Section 5.2.5. □

4.4 Contraction estimates

We use the notation $\delta\psi^{(n+1)} := \psi^{(n+1)} - \psi^{(n)}$ for a general quantity ψ . There should be no difficulty in distinguishing this notation with the constant $\delta > 0$ appearing in the r -weights. We show that

$$\|\delta(\Psi^{(n+2)}, \gamma^{(n+2)})\|_s \leq C \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \quad (4.29)$$

for some positive constant $C < 1$. Note that here we define $\|\cdot\|_s$ as in (4.3) but with $\mathbf{c}_i$ and $\gamma^{(0)}$ removed. This is conceptually straightforward and follows in a similar way as the boundedness result, and hence we leave the details to Section 5.3.

4.5 The limit $(g^{(\infty)}, k^{(\infty)})$

The goal of this subsection is to prove $\Gamma_1^{(\infty)} = \Gamma_1(g^{(\infty)}, k^{(\infty)})$, $\Gamma_2^{(\infty)} = \Gamma_2(g^{(\infty)}, k^{(\infty)})$, where $g^{(\infty)}$ and $k^{(\infty)}$ are appropriately identified below. In other words, all limiting quantities are identified with the corresponding geometric quantities associated with $(g^{(\infty)}, k^{(\infty)})$. The fact that $(g^{(\infty)}, k^{(\infty)})$ solves the constraint equation (1.1) will then be an easy corollary.

4.5.1 The limiting equations

In view of (4.29), we see that $\{(\Psi^{(n)}, \gamma^{(n)})\}$ is a Cauchy sequence under the norm $\|\cdot\|_s$. Therefore, we obtain a limit $(\Psi^{(\infty)}, \gamma^{(\infty)})$ satisfying $\|(\Psi^{(\infty)}, \gamma^{(\infty)})\|_s \leq C\varepsilon$ by the boundedness statement in Proposition 4.3. According to our way of introducing the unknowns Ψ_1 , Ψ_2 , and Ψ_3 , we denote $\hat{a}^{(\infty)} := \Upsilon^{-\frac{1}{2}} + \Psi_3^{(\infty)}$, $K^{(\infty)} := \Psi_2^{(\infty)} + r^{-2}$, $\mathfrak{t}\mathfrak{t}\theta^{(\infty)} := \Psi_1^{(\infty)} + 2\Upsilon^{\frac{1}{2}}r^{-1}$, and

$$\mu^{(\infty)} := -\Delta^{(\infty)}(\log \hat{a}^{(\infty)}) + K^{(\infty)} - \frac{1}{4}(\mathfrak{t}\mathfrak{t}\theta^{(\infty)})^2.$$

We expect that these quantities turn out to be precisely those naturally connected to the limiting initial data set.

The limit $(\Psi^{(\infty)}, \gamma^{(\infty)})$ solves the following system, by taking $n \rightarrow \infty$ for the equations (4.4)-(4.14):

$$(\partial_r + 2r^{-1})\Psi_1^{(\infty)} = \Upsilon^{-\frac{1}{2}}\check{\mu}_{\ell=0}^{(\infty)} + \check{a}^{(\infty)}\check{\mu}_{\ell=0}^{(\infty)} - 2(1 - 3mr^{-1})r^{-2}\Psi_3^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)}, \quad (4.30)$$

$$\begin{aligned} (\partial_r + 3r^{-1})\Psi_2^{(\infty)} &= r^{-1}\check{\mu}_{\ell=0}^{(\infty)} - 2\Upsilon^{\frac{1}{2}}r^{-3}\Psi_3^{(\infty)} - \Upsilon^{-\frac{1}{2}}(\mathcal{B} + \tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)}) \\ &\quad - \Psi_3^{(\infty)}(\mathcal{B} + \tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)}) + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)}, \end{aligned} \quad (4.31)$$

$$\begin{aligned} \Upsilon^{\frac{1}{2}}\Delta^{(\infty)}\Psi_3^{(\infty)} &= \Psi_2^{(\infty)} - \overline{\Psi_2^{(\infty)}}^{(\infty)} - \Upsilon^{\frac{1}{2}}r^{-1}(\Psi_1^{(\infty)} - \overline{\Psi_1^{(\infty)}}^{(\infty)}) \\ &\quad + \Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)} - \overline{\Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)}}^{(\infty)} - \Delta^{(\infty)}(\Gamma_0^{(\infty)} \cdot \Gamma_0^{(\infty)}), \end{aligned} \quad (4.32)$$

$$\overline{\Psi_3^{(\infty)}}^{(\infty)} = -\frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1^{(\infty)}}^{(\infty)}, \quad (4.33)$$

$$\mathcal{P}_1^{(\infty)}\mathcal{P}_2^{(\infty)}\Psi_4^{(\infty)} = \frac{1}{2}(\Delta^{(\infty)}\Psi_1^{(\infty)}, 0) - (\mathcal{B}, * \mathcal{B}) - (\mathcal{B}_{\ell \leq 1}^{(\infty)}, * \mathcal{B}_{\ell \leq 1}^{(\infty)}) \quad (4.34)$$

$$\mathcal{P}_1^{(\infty)}\Psi_5^{(\infty)} = \left(-\Upsilon^{\frac{1}{2}}\Delta^{(\infty)}\check{a}^{(\infty)} + \Delta^{(\infty)}(\Gamma_0^{(\infty)} \cdot \Gamma_0^{(\infty)}), 0\right) \quad (4.35)$$

$$\mathcal{P}_1^{(\infty)}\Psi_6^{(\infty)} = (\mathcal{B} + \mathcal{B}_{\ell \leq 1}^{(\infty)}, * \mathcal{B} + * \mathcal{B}_{\ell \leq 1}^{(\infty)}) - \overline{(\mathcal{B} + \mathcal{B}_{\ell \leq 1}^{(\infty)})}^{(\infty)}, * \mathcal{B} + * \mathcal{B}_{\ell \leq 1}^{(\infty)} \quad (4.36)$$

$$(\partial_r + r^{-1})\Psi_7^{(\infty)} = 2r^{-1}\Psi_{10}^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)}, \quad (4.37)$$

$$\mathcal{P}_1^{(\infty)}\mathcal{P}_2^{(\infty)}(\hat{a}^{(\infty)}\Psi_8^{(\infty)}) = \left(\frac{1}{2}\hat{a}^{(\infty)}\Delta^{(\infty)}\Psi_7^{(\infty)} + \mathcal{K}, - * \mathcal{K}\right) + (\mathcal{K}_{\ell \leq 1}^{(\infty)}, * \mathcal{K}_{\ell \leq 1}^{(\infty)}) + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)}, \quad (4.38)$$

$$\begin{aligned} \mathcal{P}_1^{(\infty)}\Psi_9^{(\infty)} &= \left(0, \frac{3}{4\pi}r^{-4}\sum_i \mathbf{a}_i\omega_i + r^{-4}\int_r^\infty r'^4(*\mathcal{K} - * \mathcal{K}_{\ell \leq 1}^{(\infty)})dr'\right) \\ &\quad - \left(0, \frac{3}{4\pi}r^{-4}\sum_i \mathbf{a}_i\omega_i + r^{-4}\int_r^\infty r'^4(*\mathcal{K} - * \mathcal{K}_{\ell \leq 1}^{(\infty)})dr'\right)^{(\infty)}, \end{aligned} \quad (4.39)$$

$$\Delta^{(\infty)}(\hat{a}^{(\infty)}\Psi_{10}^{(\infty)}) = \mathcal{K} + \mathcal{K}_{\ell \leq 1}^{(\infty)} - \overline{\mathcal{K} + \mathcal{K}_{\ell \leq 1}^{(\infty)}}^{(\infty)}. \quad (4.40)$$

$$\overline{\hat{a}^{(\infty)}\Psi_{10}^{(\infty)}}^{(\infty)} = \overline{\Psi_3^{(\infty)}\Psi_{10}^{(\infty)}}^{(\infty)}. \quad (4.41)$$

We note that we have used the observation that by taking the limit of (4.17), (4.18), the quantities $\tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)}$, $\tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(\infty)}$ are the same:

$$\tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)} = \tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(\infty)} = \frac{1}{2}(\Delta^{(\infty)}\Psi_1^{(\infty)})_{\ell=1} + \frac{1}{2}(\Delta^{(\infty)}\Psi_1^{(\infty)})_{\ell=0} - \left(\mathcal{P}_1(\mathcal{P}_1^{(\infty)}\mathcal{P}_2^{(\infty)}\Psi_4^{(\infty)})\right)_{\ell \leq 1}. \quad (4.42)$$

Therefore, we write both of them as $\tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)}$. Similarly, by comparing the limit of (4.21) and (4.20), we see that $\tilde{\mu}_{\ell=0}^{(\infty)} = \check{\mu}_{\ell=0}^{(\infty)}$, and we write both of them as $\check{\mu}_{\ell=0}^{(\infty)}$.

Moreover, taking the limit of (4.16) gives

$$\mathcal{L}_{\partial_r}(r^{-2}\gamma^{(\infty)}) = 2r^{-2}\hat{a}^{(\infty)}\Psi_4^{(\infty)} + \hat{a}^{(\infty)}\Psi_1^{(\infty)}(r^{-2}\gamma^{(\infty)}) + 2\Upsilon^{\frac{1}{2}}\Psi_3^{(\infty)}r^{-1}(r^{-2}\gamma^{(\infty)}). \quad (4.43)$$

We now define the metric

$$g^{(\infty)} := (\hat{a}^{(\infty)})^2 dr^2 + \gamma^{(\infty)} \text{ on } \Sigma = (r_0, \infty) \times \mathbb{S}^2. \quad (4.44)$$

This provides a choice of the triad $\{N^{(\infty)}, e_a^{(\infty)}\}_{a=1,2}$. We then define the “second fundamental form” $k^{(\infty)}$ through its components:

$$k^{(\infty)}(e_a^{(\infty)}, e_b^{(\infty)}) := (\Psi_8^{(\infty)})_{ab} + \frac{1}{2}\Psi_7^{(\infty)}\delta_{ab}, \quad k^{(\infty)}(N^{(\infty)}, e_a^{(\infty)}) := (\Psi_9^{(\infty)})_a, \quad k^{(\infty)}(N^{(\infty)}, N^{(\infty)}) := \Psi_{10}^{(\infty)}. \quad (4.45)$$

4.5.2 The limit $(g^{(\infty)}, k^{(\infty)})$ verifies the constraint equation

It remains to show that $(g^{(\infty)}, k^{(\infty)})$ solves the constraint equation (1.1). To prove this, we first need several observations listed in the following lemma:

Lemma 4.9. *The following statements hold true:*

1. The horizontal tensors $\Psi_4^{(\infty)}$ and $\Psi_8^{(\infty)}$ are traceless with respect to $\gamma^{(\infty)}$.
2. With respect to the metric $g^{(\infty)} := (\hat{a}^{(\infty)})^2 dr^2 + \gamma^{(\infty)}$, the quantities $\Psi_4^{(\infty)}$ and $\Psi_1^{(\infty)} + 2\Upsilon^{\frac{1}{2}}r^{-1}$ are exactly the traceless part and the trace of the second fundamental form of the r -spheres. We hence denote $\hat{\theta}^{(\infty)} = \Psi_4^{(\infty)}$ and $\mathfrak{t}\theta^{(\infty)} = \Psi_1^{(\infty)} + 2\Upsilon^{\frac{1}{2}}r^{-1}$ without ambiguity.
3. We have $\Psi_5^{(\infty)} = -\nabla^{(\infty)}(\log \hat{a}^{(\infty)})$, $\mu_{\ell \geq 1}^{(\infty)} = 0$, and the average of $\Psi_3^{(\infty)} + \frac{1}{2}\Upsilon^{-1}r\Psi_1^{(\infty)}$ vanishes with respect to $\gamma^{(\infty)}$.
4. For the quantity defined in (4.42), we have $\tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)} = \mathcal{B}_{\ell \leq 1}^{(\infty)}$, the latter being the first component of $\Psi_{11}^{(\infty)} \in \mathfrak{s}_0$.
5. Denote by $Y(g^{(\infty)})$ the horizontal tensor Y with respect to $g^{(\infty)}$ defined through (2.3). Then we have $\Psi_6^{(\infty)} = Y(g^{(\infty)})$.
6. Denote the Gauss curvature of $\gamma^{(\infty)}$ by $K(\gamma^{(\infty)})$. Then we have $\Psi_2^{(\infty)} = K(\gamma^{(\infty)}) - r^{-2}$.

Proof. See Section 5.4.1. □

Proposition 4.10. *The data $(g^{(\infty)}, k^{(\infty)})$ solves the Einstein constraint equation (1.1).*

Proof. This follows from comparing the equations (4.30), (4.37), (4.38) with the unconditional equation (2.52), (2.59), (2.60), (2.61), along with the statements in Lemma 4.9. We leave the details to Section 5.4.2.

4.6 Conclusions

We have proved that $(g^{(\infty)}, k^{(\infty)})$ solves the Einstein constraint equation (1.1), and under the ambient r -foliation, the corresponding geometric quantities satisfy the following estimate:

$$\begin{aligned} r^{-1}\|\hat{a}^{(\infty)} - \Upsilon^{-\frac{1}{2}}\|_{\mathfrak{h}^{s+2}(S_r)} &\lesssim \varepsilon r^{-1-\delta}, \quad r^{-1}\|\gamma^{(\infty)} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon r^{-1-\delta}, \\ r^{-1}\|\widetilde{\mathfrak{t}\theta}^{(\infty)}, \hat{\theta}^{(\infty)}, p^{(\infty)}, \mathfrak{t}\theta^{(\infty)}, \hat{\Theta}^{(\infty)}, \Xi^{(\infty)}, \Pi^{(\infty)}\|_{\mathfrak{h}^{s+1}(S_r)} &\lesssim \varepsilon r^{-2-\delta}, \\ r^{-1}\|\tilde{K}^{(\infty)}, Y^{(\infty)}\|_{\mathfrak{h}^s(S_r)} &\lesssim \varepsilon r^{-3-\delta}. \end{aligned}$$

Moreover, in the proof of Lemma 4.9 and Proposition 4.10, we obtain the relations

$$\begin{aligned} (\mathrm{div}^{(\infty)} Y^{(\infty)})_{\ell \geq 2} &= \mathcal{B}, \quad (\mathrm{curl}^{(\infty)} Y^{(\infty)})_{\ell \geq 2} = {}^* \mathcal{B}, \\ (\mathbb{A}^{(\infty)}(\hat{a}^{(\infty)} \Pi^{(\infty)}))_{\ell \geq 2} &= \mathcal{K}, \quad r^{-4} \partial_r (r^4 (\mathrm{curl}^{(\infty)} \Xi^{(\infty)}))_{\ell \geq 2} = {}^* \mathcal{K}, \\ \mu_{\ell \geq 1}^{(\infty)} &= 0, \quad \mathrm{div}^{(\infty)} \Xi^{(\infty)} = 0. \end{aligned}$$

We also have the following limits

$$\lim_{r \rightarrow \infty} r^3 (\widetilde{\mathrm{tr} \theta}^{(\infty)})_{\ell=1,i} = \mathbf{c}_i, \quad \lim_{r \rightarrow \infty} r^2 (\mathrm{tr} \Theta^{(\infty)})_{\ell=1,i} = 0, \quad \lim_{r \rightarrow \infty} r^4 (\mathrm{curl}^{(\infty)} \Xi^{(\infty)})_{\ell=1,i} = \mathbf{a}_i,$$

and hence, in view of Proposition C.1, proves (2.79) regarding the ADM charges.

5 Details in the proof of the main theorem

5.1 Proof of Proposition 4.2

We first outline the main ideas in the proof of the Proposition.

1. Since $\tilde{\gamma}$ is in general not round, $\mathbb{A}_{\tilde{\gamma}}^{-1}$ mixes the different modes. In Section 5.1.1, we show that, for any scalar field ϕ ,

$$(\mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}} = -\frac{r^2}{\ell(\ell+1)} (r^{-2} \mathbb{A}^{(0)} \mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}} = -\frac{r^2}{\ell(\ell+1)} (\phi_{\ell, \mathbf{m}} + \mathcal{R}(\phi)_{\ell, \mathbf{m}}),$$

where $\mathcal{R}: \mathfrak{h}^s \rightarrow \mathfrak{h}^s$ is a linear operator satisfying $\|\mathcal{R}(\phi)\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon_1 r^{-1-\delta} \|\phi\|_{\mathfrak{h}^s(S_r)}$.

2. We study the projection into spherical harmonic modes $J_{\ell, \mathbf{m}}$ based on the background coordinates. Each mode satisfies the system of the form, for $\mathbf{u} \in \mathbb{R}^2$,

$$\partial_r \mathbf{u} = r^{-1} A(r) \mathbf{u} + r^{-2} B(r) \mathbf{u} + \mathbf{F},$$

with a vanishing condition at infinity. It is crucial for the first matrix $A(r)$ on the right to be accretive³⁰ for some inner product over $\mathbb{R}^2$. Under this assumption, we first provide a version of Duhamel formula in Lemma 5.5 in Section 5.1.2. The equations written in modes are derived in Section 5.1.3.

3. In Section 5.1.4, we study the equations projected into different modes.

- For $\ell \geq 2$, the equation reads

$$\partial_r \begin{pmatrix} r^2 (\Psi_1)_{\ell, \mathbf{m}} \\ r^3 (\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} = r^{-1} \begin{pmatrix} -\frac{2}{\ell(\ell+1)} & \frac{2}{\ell(\ell+1)} \\ -\frac{2}{\ell(\ell+1)} & \frac{2}{\ell(\ell+1)} \end{pmatrix} \begin{pmatrix} r^2 (\Psi_1)_{\ell, \mathbf{m}} \\ r^3 (\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} + \begin{pmatrix} r^2 (F_1)_{\ell, \mathbf{m}} \\ r^3 (F_2)_{\ell, \mathbf{m}} \end{pmatrix} + l.o.t.,$$

where the first matrix on the right is a nilpotent matrix, in particular, not accretive. To deal with this, we consider instead the unknown $\begin{pmatrix} r^{2+\delta'} (\Psi_1)_{\ell, \mathbf{m}} \\ r^{3+\delta'} (\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix}$ with $0 < \delta' < \delta$, so that the first matrix of the new system becomes, as is shown in Lemma 5.6, positive definite under a certain inner product over $\mathbb{R}^2$ for all $\ell \geq 2$. This verifies the condition of Lemma 5.5 and allows us to construct the solution to such a system.

- The corresponding analysis of the matrix for $\ell \leq 1$ parts is easier by incorporating appropriate r -weights. Note that, however, as has already appeared in Section 2.6, the $\ell = 1$ part contains a non-zero center-of-mass tail $\mathbf{c}_m J_{1, \mathbf{m}} r^{-3}$ that has to be subtracted from $(\Psi_1)_{1, \mathbf{m}}$.

³⁰i.e., $\langle A\mathbf{v}, \mathbf{v} \rangle_H + \langle \mathbf{v}, A\mathbf{v} \rangle_H \geq 0$ for all $\mathbf{v} \in \mathbb{R}^2$ for some inner product H .

4. The system (4.24) can now be rewritten in the form

$$\partial_r \mathbf{v}_{\ell, \mathbf{m}} = r^{-1} A_\ell \mathbf{v}_{\ell, \mathbf{m}} + r^{-2} B_\ell(r) \mathbf{v}_{\ell, \mathbf{m}} + \mathbf{F}_{\ell, \mathbf{m}} + \mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v}), \quad (5.1)$$

where $\mathcal{R}^{new}$ is an appropriate weighted³¹ version of $\mathcal{R}$. Due to different weights for different ℓ , one can only expect $\mathcal{R}^{new}$ to satisfy a relaxed uniform estimate, and it is important that this still provides enough r -decaying weights. In Section 5.1.6, we use (5.1) to prove the existence and uniqueness of the solution by the contraction argument.

5.1.1 The perturbed metric and the $\mathcal{R}$ operator

Recall that the assumption on the given perturbed metric $\tilde{\gamma}$ in (4.25) reads

$$\sup_{r \in [r_0, \infty)} r^\delta \|\tilde{\gamma} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon_1, \quad (5.2)$$

where ε_1 is a small constant to be determined.

We need to deal with the fact that the operator $\mathbb{A}_{\tilde{\gamma}}^{-1}$ mixes different modes. For $\ell \geq 1$, we write

$$(r^{-2} \mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}} = -\frac{1}{\ell(\ell+1)} (\mathbb{A}^{(0)} \mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}} = -\frac{1}{\ell(\ell+1)} (\phi_{\ell, \mathbf{m}} + \mathcal{R}_{\ell, \mathbf{m}}(\phi)), \quad (5.3)$$

and for $\ell = 0$, we write, schematically,

$$\left((r^{-2} \mathbb{A}^{-1} \phi)_{\ell=0}, \bar{\phi}^{\tilde{\gamma}} - \bar{\phi}^{\gamma^{(0)}} \right) = \mathcal{R}_{\ell=0}(\phi). \quad (5.4)$$

Definition 5.1. The linear operator $\mathcal{R}$ is defined by $\mathcal{R}(\phi) := \sum_{\ell=0}^{\infty} \sum_{\mathbf{m}=-\ell}^{\ell} \mathcal{R}_{\ell, \mathbf{m}}(\phi) J_{\ell, \mathbf{m}}$.

Proposition 5.2. The linear operator $\mathcal{R}$ satisfies the bound

$$\|\mathcal{R}(\phi)\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon_1 r^{-1-\delta} \|\phi\|_{\mathfrak{h}^s(S_r)}.$$

Proof. Since $J_{\ell, \mathbf{m}}$ and $r^{-2} \gamma^{(0)}$ are independent of r , by the definition (2.16) of the modes, it remains true that $(\partial_r \phi)_{\ell, \mathbf{m}} = \partial_r (\phi_{\ell, \mathbf{m}})$. Since for $\ell \geq 1$, $\mathbb{A}^{(0)} J_{\ell, \mathbf{m}} = -\frac{\ell(\ell+1)}{r^2} J_{\ell, \mathbf{m}}$, we have

$$\begin{aligned} (\mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}} &= \int_{S_r} (\mathbb{A}_{\tilde{\gamma}}^{-1} \phi) J_{\ell, \mathbf{m}} d\text{vol}_{S_r} = \int_{S_r} (\mathbb{A}_{\tilde{\gamma}}^{-1} \phi) \left(-\frac{r^2}{\ell(\ell+1)} \right) \mathbb{A}^{(0)} J_{\ell, \mathbf{m}} d\text{vol}_{S_r} \\ &= -\frac{r^2}{\ell(\ell+1)} \int_{S_r} (\mathbb{A}^{(0)} \mathbb{A}_{\tilde{\gamma}}^{-1} \phi) J_{\ell, \mathbf{m}} d\text{vol}_{S_r} = -\frac{r^2}{\ell(\ell+1)} (\mathbb{A}^{(0)} \mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}}. \end{aligned} \quad (5.5)$$

Then, combining (5.5) with the definition (5.3), we have

$$\mathcal{R}(\phi)_{\ell, \mathbf{m}} = -\phi_{\ell, \mathbf{m}} - \frac{\ell(\ell+1)}{r^2} (\mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}} = (\mathbb{A}^{(0)} \mathbb{A}_{\tilde{\gamma}}^{-1} \phi)_{\ell, \mathbf{m}} - \phi_{\ell, \mathbf{m}}.$$

We then apply Lemma 5.3, as well as Lemma 5.4 for the $\ell = 0$ part defined in (5.4), to obtain, using (2.18),

$$\|\mathcal{R}(\phi)\|_{\mathfrak{h}^s}^2 = r^2 \left(|\mathcal{R}_{\ell=0}(\phi)| + \sum_{\ell=1}^{\infty} \sum_{\mathbf{m}=-\ell}^{\ell} (1 + \ell^2)^s |(\mathcal{R}(\phi))_{\ell, \mathbf{m}}|^2 \right) \lesssim (\varepsilon_1 r^{-1-\delta})^2 (\|\phi\|_{L^2(S_r)}^2 + \|\phi\|_{\mathfrak{h}^s(S_r)}^2).$$

Therefore, it remains to prove Lemma 5.3 and Lemma 5.4. □

³¹with the weight depending on ℓ .

Lemma 5.3. *Assume (5.2) holds. For any integer $s \geq 0$ and scalar field ϕ , we have*

$$\|\Delta^{(0)} \Delta_{\tilde{\gamma}}^{-1} \phi - \phi_{\ell \geq 1}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon_1 r^{-1-\delta} \|\phi\|_{\mathfrak{h}^s(S_r)}. \quad (5.6)$$

Note from footnote 29 that the domain of $\Delta_{\tilde{\gamma}}^{-1}$ is extended through $\Delta_{\tilde{\gamma}}^{-1} \phi := \Delta_{\tilde{\gamma}}^{-1}(\phi - \bar{\phi}^{\tilde{\gamma}})$.

Proof. We write $r^2 \Delta_{\tilde{\gamma}} = r^2 \Delta^{(0)} + H$. In view of Remark 2.10, H is of the form

$$H\phi = O(\tilde{\gamma} - \gamma^{(0)}) \cdot (r\nabla^{(0)})^2 \phi + O(r\nabla^{(0)}\tilde{\gamma}) \cdot r\nabla^{(0)}\phi.$$

Therefore, applying the $(r\nabla^{(0)})$ derivatives s times, using Definition 2.7, we obtain, by standard L^2 - L^∞ estimates, for $s \geq 3$,

$$\begin{aligned} \|H\phi\|_{\mathfrak{h}^s(S_r)} &\lesssim \|(r\nabla^{(0)})^{\leq s} \left(O(\tilde{\gamma} - \gamma^{(0)}) \cdot (r\nabla^{(0)})^2 \phi + O(r\nabla^{(0)}\tilde{\gamma}) \cdot r\nabla^{(0)}\phi \right)\|_{L^2(S_r)} \\ &\lesssim r^{-1} \|\tilde{\gamma} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \|\phi\|_{\mathfrak{h}^{s+2}(S_r)} \\ &\lesssim \varepsilon_1 r^{-1-\delta} \|\phi\|_{\mathfrak{h}^{s+2}(S_r)}. \end{aligned}$$

We have the identity

$$\begin{aligned} \Delta^{(0)} \Delta_{\tilde{\gamma}}^{-1} \phi - \phi_{\ell \geq 1} &= (r^2 \Delta^{(0)})(r^2 \Delta_{\tilde{\gamma}})^{-1} \phi - \phi_{\ell \geq 1} = (r^2 \Delta_{\tilde{\gamma}} - H)(r^2 \Delta_{\tilde{\gamma}})^{-1} \phi - \phi + \bar{\phi}^{\gamma^{(0)}} \\ &= (\phi - \bar{\phi}^{\tilde{\gamma}}) - H(r^2 \Delta_{\tilde{\gamma}})^{-1} \phi - \phi + \bar{\phi}^{\gamma^{(0)}} \\ &= (\bar{\phi}^{\tilde{\gamma}} - \bar{\phi}^{\gamma^{(0)}}) - H(r^2 \Delta_{\tilde{\gamma}})^{-1} \phi. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\Delta^{(0)} \Delta_{\tilde{\gamma}}^{-1} \phi - \phi_{\ell \geq 1}\|_{\mathfrak{h}^s(S_r)} &\lesssim \|\bar{\phi}^{\tilde{\gamma}} - \bar{\phi}^{\gamma^{(0)}}\|_{L^2(S_r)} + \|H(r^2 \Delta_{\tilde{\gamma}})^{-1} \phi\|_{\mathfrak{h}^s(S_r)} \\ &\lesssim \|\tilde{\gamma} - \gamma^{(0)}\|_{L^\infty(S_r)} \|\phi\|_{L^2(S_r)} + \varepsilon_1 r^{-1-\delta} \|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^{s+2}(S_r)} \\ &\lesssim \varepsilon_1 r^{-1-\delta} \left(\|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^{s+2}(S_r)} + \|\phi\|_{L^2(S_r)} \right). \end{aligned} \quad (5.7)$$

It then remains to estimate $\|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^{s+2}(S_r)}$. Notice that the estimate (5.7) in fact implies

$$\|\Delta^{(0)} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^s(S_r)} \lesssim \|\phi_{\ell \geq 1}\|_{\mathfrak{h}^s(S_r)} + \varepsilon_1 r^{-1-\delta} \left(\|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^{s+2}(S_r)} + \|\phi\|_{L^2(S_r)} \right).$$

Standard elliptic estimates for $r^2 \Delta^{(0)}$ imply $\|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^{s+2}(S_r)} \lesssim \|\phi\|_{\mathfrak{h}^s(S_r)} + \varepsilon_1 r^{-1-\delta} \|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^{s+2}(S_r)}$, hence $\|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{\mathfrak{h}^{s+2}(S_r)} \lesssim \|\phi\|_{\mathfrak{h}^s(S_r)}$. Plugging this back to (5.7), we obtain the desired estimate. $\square$

Lemma 5.4. *Suppose that (5.2) holds. We have*

$$|(r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi)_{\ell=0}|, |\bar{\phi}^{\tilde{\gamma}} - \bar{\phi}^{\gamma^{(0)}}| \lesssim \varepsilon_1 r^{-1-\delta} (r^{-1} \|\phi\|_{L^2(S_r)}).$$

Proof. We have

$$\begin{aligned} r^{-2} \int_{S_r} (r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi) d\text{vol}_{\gamma^{(0)}} &= r^{-2} \int_{S_r} (r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi) (\text{dvol}_{\gamma^{(0)}} - \text{dvol}_{\tilde{\gamma}}) \\ &\lesssim \|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{L^\infty(S_r)} \left| r^{-2} \int_{S_r} (\sqrt{\det(\tilde{\gamma}_{ab})} - 1) \in_{ab}^{(0)} \right| \\ &\lesssim \|\tilde{\gamma} - \gamma^{(0)}\|_{L^\infty(S_r)} \|r^{-2} \Delta_{\tilde{\gamma}}^{-1} \phi\|_{L^\infty(S_r)} \lesssim \varepsilon_1 r^{-1-\delta} (r^{-1} \|\phi\|_{L^2(S_r)}), \end{aligned}$$

where $\in_{ab} = (\text{dvol}_{\gamma^{(0)}})_{ab}$ denotes the volume form of $\gamma^{(0)}$, which satisfies $\int_{S_r} \in_{ab}^{(0)} = 4\pi r^2$. The estimate for $|\bar{\phi}^{\tilde{\gamma}} - \bar{\phi}^{\gamma^{(0)}}|$ is similar, and in fact we have already used it in the proof of Lemma 5.3. $\square$

5.1.2 Duhamel's formula, accretiveness of matrices

The following Lemma establishes a Duhamel type representation formula³² for systems of the type (5.10).

Lemma 5.5. *Take an inner product $\langle \cdot, \cdot \rangle_H$ over $\mathbb{R}^2$ independent of r . Consider the equation*

$$\partial_r \mathbf{u} = r^{-1} A(r) \mathbf{u} + r^{-2} B(r) \mathbf{u} \quad (5.8)$$

for $\mathbb{R}^2$ -valued vector $\mathbf{u} = \mathbf{u}(r)$. If $N(r)$ is accretive with respect to H , i.e., $\langle A\mathbf{v}, \mathbf{v} \rangle_H + \langle \mathbf{v}, A\mathbf{v} \rangle_H \geq 0$ for all r and $\mathbf{v} \in \mathbb{R}^2$, and $B(r) = O(1)$, then the solution operator $U(r, r^*)$ for $r < r^*$, defined through

$$\partial_r U(r, r^*) = (r^{-1} A(r) + r^{-2} B(r)) U(r, r^*), \quad U(r^*, r^*) = I, \quad (5.9)$$

satisfies $\|U(r, r^*)\| \leq C$ uniformly for all r, r^* with $r_0 \leq r < r^*$.

Moreover, for the inhomogeneous equation

$$\partial_r \mathbf{u} = r^{-1} A(r) \mathbf{u} + r^{-2} B(r) \mathbf{u} + \mathbf{N} \quad (5.10)$$

with $\mathbf{N} \in L^1((r_0, \infty), \mathbb{R}^2)$ and the condition

$$\lim_{r \rightarrow \infty} \|\mathbf{u}(r)\|_H = 0, \quad (5.11)$$

there exists a unique solution $\mathbf{u} \in C^1((r_0, \infty), \mathbb{R}^2)$ to (5.10) satisfying (5.11). In fact, $\mathbf{u}$ can be expressed as

$$\mathbf{u} = - \int_r^\infty U(r, r') \mathbf{N}(r') dr'. \quad (5.12)$$

Proof. We first derive the following boundedness estimate

$$\begin{aligned} \frac{d}{dr} \|U(r, r^*) \mathbf{u}\|_H^2 &= \left\langle U(r, r^*) \mathbf{u}, r^{-1} A U(r, r^*) \mathbf{u} \right\rangle_H + \left\langle U(r, r^*) \mathbf{u}, r^{-1} A U(r, r^*) \mathbf{u} \right\rangle_H + O(r^{-2}) \|U(r, r^*) \mathbf{u}\|_H^2 \\ &\geq -O(r^{-2}) \|U(r, r^*) \mathbf{u}\|_H^2, \end{aligned}$$

where the accretiveness of A is crucially used. Hence, we have $\frac{d}{dr} \left(\exp \left(- \int_r^{r^*} O(r'^{-2}) dr' \right) \|U(r, r^*) \mathbf{u}\|_H^2 \right) \geq 0$, i.e.,

$$\|U(r, r^*) \mathbf{u}\|_H^2 \lesssim \|\mathbf{u}\|_H^2 \exp \left(\int_r^{r^*} O(r'^{-2}) dr' \right) \lesssim \|\mathbf{u}\|_H^2. \quad (5.13)$$

The formula (5.12) itself proves the existence of the solution to the inhomogeneous equation (5.10), for which the condition (5.11) is verified using the boundedness (5.13) and the integrability of $\mathbf{N}$. To see the uniqueness, suppose there are two solutions $\mathbf{u}_1, \mathbf{u}_2$. Then $\mathbf{u}_1 - \mathbf{u}_2$ solves the homogeneous equation, and hence for each r, r' with $r < r'$, $(\mathbf{u}_1 - \mathbf{u}_2)(r) = U(r, r')(\mathbf{u}_1(r') - \mathbf{u}_2(r'))$. If $\|\mathbf{u}_1(r) - \mathbf{u}_2(r)\| = c \neq 0$ for some r , then for each $r' > r$, $\|\mathbf{u}_1(r') - \mathbf{u}_2(r')\| \gtrsim \|\mathbf{u}_1(r) - \mathbf{u}_2(r)\| = c > 0$, contradicting the convergence $\lim_{s \rightarrow \infty} \|\mathbf{u}_j(s)\| = 0$, $j = 1, 2$. This proves the uniqueness. $\square$

We will use below the following property of a nilpotent matrix $Q = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$.

³²This will be applied to the specific modes of the system (4.24).

Lemma 5.6. Let $Q = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$. Then, for any given $\delta' > 0$ and the matrix $A = \delta'I + xQ$, there exists a positive-definite matrix $G_{\delta'}$ such that the matrix

$$G_{\delta'}A + A^T G_{\delta'}$$

is positive-definite for all $x \in (0, 1)$. In other words, there exists a positive-definite inner product $\langle \cdot, \cdot \rangle_{G_{\delta'}}$ on $\mathbb{R}^2$ such that the matrix $\frac{1}{2}(A + A^*)$ is positive definite with respect to $\langle \cdot, \cdot \rangle_{G_{\delta'}}$ for all $x \in (0, 1)$, where A^* is the adjoint also with respect to this inner product. In particular, A verifies the accretiveness required in Lemma 5.5.

Proof. The matrix $A := \delta'I + xQ$ is not symmetric, and its symmetrized matrix is not always positive definite for all $x \in (0, 1)$. To deal with this, we consider the following inner product in $\mathbb{R}^2$:

$$\langle \mathbf{v}, \mathbf{w} \rangle_{G_{\delta'}} := \mathbf{v}^T G_{\delta'} \mathbf{w}, \quad G_{\delta'} := \begin{pmatrix} 1 & -1 + (\delta')^2 \\ -1 + (\delta')^2 & 1 \end{pmatrix}.$$

We compute

$$Q^T \begin{pmatrix} 1 & -1 + (\delta')^2 \\ -1 + (\delta')^2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 + (\delta')^2 \\ -1 + (\delta')^2 & 1 \end{pmatrix} Q = \begin{pmatrix} -2(\delta')^2 & 0 \\ 0 & 2(\delta')^2 \end{pmatrix}.$$

Therefore, we have, using $A = \delta'I + xQ$,

$$\begin{aligned} \frac{1}{2}(G_{\delta'}A + A^T G_{\delta'}) &= \delta' G_{\delta'} + \frac{1}{2}x(G_{\delta'}Q + Q^T G_{\delta'}) \\ &= \begin{pmatrix} \delta' - x(\delta')^2 & -\delta'(1 - (\delta')^2) \\ -\delta'(1 - (\delta')^2) & \delta' + x(\delta')^2 \end{pmatrix} = \delta' \begin{pmatrix} 1 - x\delta' & -1 + (\delta')^2 \\ -1 + (\delta')^2 & 1 + x\delta' \end{pmatrix}. \end{aligned}$$

The last matrix is positive-definite since it is symmetric, $1 \pm x\delta' > 0$, and its determinant is $1 - x^2(\delta')^2 - 1 + 2(\delta')^2 - (\delta')^4 = (2 - x^2)(\delta')^2 - (\delta')^4 > 0$ for all $x \in (0, 1)$ and $\delta \leq 1$. For $\delta' > 1$ one can simply take $G_{\delta'} = I$. This concludes the proof. $\square$

Remark 5.7. The proof is an explicit construction of the solution to the Lyapunov matrix equation; see e.g. [45] for a historical review.

5.1.3 Derivation of the projected equation in modes

We now derive the equations projected into modes. We introduce the notation, with $\mathbf{c}$, the prescribed center-of-mass parameter appeared in (4.27).

$$\check{\Psi}_1 := \Psi_1 - \frac{3}{4\pi} \sum_{i=1}^3 \mathbf{c}_i \omega_i r^{-3}, \quad (5.14)$$

where we recall from Remark 2.5 that the functions ω_i only differ from $J_{1,\mathbf{m}}$ by a constant factor $\sqrt{4\pi/3}$. According to this notation, we have $(\check{\Psi}_1)_{\ell \neq 1} = (\Psi_1)_{\ell \neq 1}$, and the last condition in (4.27) reads $\lim_{r \rightarrow \infty} (\check{\Psi}_1)_{\ell=1} = 0$.

Proposition 5.8. For the system (4.24), we denote its components in spherical harmonic modes:

$$\mathbf{v}_{\ell,\mathbf{m}} = \begin{pmatrix} r^{2+\delta'} (\Psi_1)_{\ell,\mathbf{m}} \\ r^{3+\delta'} (\Psi_2)_{\ell,\mathbf{m}} \end{pmatrix}, \quad \ell = 0 \text{ or } \ell \geq 2, \quad \mathbf{v}_{\ell,\mathbf{m}} = \begin{pmatrix} r^3 (\check{\Psi}_1)_{\ell,\mathbf{m}} \\ r^4 (\Psi_2)_{\ell,\mathbf{m}} \end{pmatrix}, \quad \ell = 1, \quad (5.15)$$

where $0 < \delta' < \delta$. Then, the system (4.24) is equivalent to the following projected equations into the spherical harmonic modes defined in (2.16):

$$\partial_r \mathbf{v}_{\ell, \mathbf{m}} = r^{-1} A_\ell \mathbf{v}_{\ell, \mathbf{m}} + r^{-2} B_\ell(r) \mathbf{v}_{\ell, \mathbf{m}} + \mathbf{F}_{\ell, \mathbf{m}} + O(\ell^{-2}) \mathcal{R}_{\ell, \mathbf{m}} \left(r^{2+\delta'} \Psi_2, r^{1+\delta'} \check{\Psi}_1 \right), \quad \ell \geq 2, \quad (5.16)$$

$$\partial_r \mathbf{v}_{1, \mathbf{m}} = r^{-1} A_1 \mathbf{v}_{1, \mathbf{m}} + r^{-2} B_1(r) \mathbf{v}_{1, \mathbf{m}} + \mathbf{F}_{1, \mathbf{m}} + O(r^3) \mathcal{R}_{1, \mathbf{m}} \left(\Psi_2, r^{-1} \check{\Psi}_1 \right), \quad (5.17)$$

$$\partial_r \mathbf{v}_{\ell=0} = r^{-1} A_0 \mathbf{v}_{\ell=0} + r^{-2} B_0(r) \mathbf{v}_{\ell=0} + \mathbf{F}_{\ell=0} + O(r^{2+\delta'}) \mathcal{R}_{\ell=0} \left(\Psi_2, r^{-1} \check{\Psi}_1 \right). \quad (5.18)$$

Here,

$$A_\ell = \begin{pmatrix} \delta' - \frac{2}{\ell(\ell+1)} & \frac{2}{\ell(\ell+1)} \\ -\frac{2}{\ell(\ell+1)} & \delta' + \frac{2}{\ell(\ell+1)} \end{pmatrix} \text{ for } \ell \geq 2, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \delta' & 1 \\ 0 & 1 + \delta' \end{pmatrix},$$

the matrices $B_\ell(r)$ have all their entries bounded uniformly in r and ℓ , and the inhomogeneous terms read

$$\mathbf{F}_{\ell, \mathbf{m}} = \begin{pmatrix} r^{2+\delta'} (F_1)_{\ell, \mathbf{m}} \\ r^{3+\delta'} (F_2)_{\ell, \mathbf{m}} \end{pmatrix} + O(r^{-2+\delta'}) \mathcal{R}_{\ell, \mathbf{m}} (\sum_i \mathbf{c}_i \omega_i), \quad \ell = 0 \text{ or } \ell \geq 2, \quad (5.19)$$

and

$$\mathbf{F}_{1, \mathbf{m}} := r^{-2} (\sum_i \mathbf{c}_i \omega_i) B_1(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} r^3 (F_1)_{1, \mathbf{m}} \\ r^4 (F_2)_{1, \mathbf{m}} \end{pmatrix} + O(r^{-1}) \mathcal{R}_{1, \mathbf{m}} (\sum_i \mathbf{c}_i \omega_i), \quad \ell = 1. \quad (5.20)$$

Moreover, the following bounds hold true for $\mathbf{F} := \sum_{\ell=0}^{\infty} \sum_{\mathbf{m}=-\ell}^{\ell} \mathbf{F}_{\ell, \mathbf{m}} J_{\ell, \mathbf{m}}$:

$$r^{-1} \|\mathbf{F}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon r^{-1-(\delta-\delta')}, \quad |\mathbf{F}_{1, \mathbf{m}}| \lesssim \varepsilon r^{-1-\delta}. \quad (5.21)$$

Proof. We proceed as follows:

Case $\ell \geq 2$. Projecting (4.24) to modes with $\ell \geq 2$ and using (5.3), we obtain

$$\begin{aligned} (\partial_r + 2r^{-1})(\Psi_1)_{\ell, \mathbf{m}} &= -2(1 - 3mr^{-1})r^{-2} \cdot \Upsilon^{-\frac{1}{2}} \left(-\frac{r^2}{\ell(\ell+1)} \right) (\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1)_{\ell, \mathbf{m}} + (F_1)_{\ell, \mathbf{m}} \\ &\quad + O(\ell^{-2}) \mathcal{R}_{\ell, \mathbf{m}} (\Psi_2, r^{-1} \Psi_1), \\ (\partial_r + 3r^{-1})(\Psi_2)_{\ell, \mathbf{m}} &= -2\Upsilon^{\frac{1}{2}} r^{-3} \cdot \Upsilon^{-\frac{1}{2}} \left(-\frac{r^2}{\ell(\ell+1)} \right) (\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1)_{\ell, \mathbf{m}} + (F_2)_{\ell, \mathbf{m}} \\ &\quad + O(\ell^{-2}) \mathcal{R}_{\ell, \mathbf{m}} (r^{-1} \Psi_2, r^{-2} \Psi_1), \end{aligned} \quad (5.22)$$

or, in the matrix form for Ψ_1 and $r\Psi_2$, using that $\Upsilon = 1 + O(mr^{-1})$,

$$\begin{aligned} \partial_r \begin{pmatrix} (\Psi_1)_{\ell, \mathbf{m}} \\ r(\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} &= \begin{pmatrix} -2 - \frac{2}{\ell(\ell+1)} & \frac{2}{\ell(\ell+1)} \\ -\frac{2}{\ell(\ell+1)} & -2 + \frac{2}{\ell(\ell+1)} \end{pmatrix} r^{-1} \begin{pmatrix} (\Psi_1)_{\ell, \mathbf{m}} \\ r(\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} + r^{-2} B_\ell(r) \begin{pmatrix} (\Psi_1)_{\ell, \mathbf{m}} \\ r(\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} + \begin{pmatrix} (F_1)_{\ell, \mathbf{m}} \\ r(F_2)_{\ell, \mathbf{m}} \end{pmatrix} \\ &\quad + O(\ell^{-2}) \mathcal{R}_{\ell, \mathbf{m}} (\Psi_2, r^{-1} \Psi_1), \end{aligned}$$

where $B_\ell(r)$ is a matrix whose entries are bounded uniformly in r and ℓ . Mutiplying each row by $r^{2+\delta'}$ for some positive $\delta' < \delta$, we have

$$\begin{aligned} \partial_r \begin{pmatrix} r^{2+\delta'} (\Psi_1)_{\ell, \mathbf{m}} \\ r^{3+\delta'} (\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} &= \begin{pmatrix} \delta' - \frac{2}{\ell(\ell+1)} & \frac{2}{\ell(\ell+1)} \\ -\frac{2}{\ell(\ell+1)} & \delta' + \frac{2}{\ell(\ell+1)} \end{pmatrix} r^{-1} \begin{pmatrix} r^{2+\delta'} (\Psi_1)_{\ell, \mathbf{m}} \\ r^{3+\delta'} (\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} + r^{-2} B_\ell(r) \begin{pmatrix} r^{2+\delta'} (\Psi_1)_{\ell, \mathbf{m}} \\ r^{3+\delta'} (\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix} \\ &\quad + \begin{pmatrix} r^{2+\delta'} (F_1)_{\ell, \mathbf{m}} \\ r^{3+\delta'} (F_2)_{\ell, \mathbf{m}} \end{pmatrix} + O(\ell^{-2}) \mathcal{R}_{\ell, \mathbf{m}} (r^{2+\delta'} \Psi_2, r^{1+\delta'} \Psi_1), \end{aligned}$$

and the last term can be further decomposed as, using (5.14),

$$\mathcal{R}_{\ell, \mathbf{m}}(r^{2+\delta'} \Psi_2, r^{1+\delta'} \Psi_1) = O(r^{-2+\delta'}) \mathcal{R}_{\ell, \mathbf{m}}(\sum_i \mathbf{c}_i \omega_i) + \mathcal{R}_{\ell, \mathbf{m}}(r^{2+\delta'} \Psi_2, r^{1+\delta'} \check{\Psi}_1).$$

This proves the expression (5.16).

Case $\ell = 0$. Projecting the system (4.24) to the $\ell = 0$ mode, we obtain

$$\begin{aligned} (\partial_r + 2r^{-1})(\Psi_1)_{\ell=0} &= -2(1 - 3mr^{-1})r^{-2} \left(\Upsilon^{-\frac{1}{2}} \mathbb{A}_{\tilde{\gamma}}^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1) \right)_{\ell=0} + r^{-1} \overline{\Psi_1}^{\tilde{\gamma}} \\ &\quad + \Upsilon^{-\frac{1}{2}}(\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1)_{\ell=0} + (F_1)_{\ell=0}, \\ (\partial_r + 3r^{-1})(\Psi_2)_{\ell=0} &= -2\Upsilon^{\frac{1}{2}} r^{-3} \left(\Upsilon^{-\frac{1}{2}} \mathbb{A}_{\tilde{\gamma}}^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1) \right)_{\ell=0} + \Upsilon^{-\frac{1}{2}} r^{-2} \overline{\Psi_1}^{\tilde{\gamma}} \\ &\quad + r^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1)_{\ell=0} + (F_2)_{\ell=0}, \end{aligned} \quad (5.23)$$

where we again recall that we extend the definition of $\mathbb{A}_{\tilde{\gamma}}^{-1}$ using footnote 29. Using (5.4), we write

$$\partial_r \begin{pmatrix} (\Psi_1)_{\ell=0} \\ (\Psi_2)_{\ell=0} \end{pmatrix} = \begin{pmatrix} -2r^{-1} & \Upsilon^{\frac{1}{2}} \\ (\Upsilon^{-\frac{1}{2}} - \Upsilon^{\frac{1}{2}})r^{-2} & -2r^{-1} \end{pmatrix} \begin{pmatrix} (\Psi_1)_{\ell=0} \\ (\Psi_2)_{\ell=0} \end{pmatrix} + \begin{pmatrix} (F_1)_{\ell=0} \\ (F_2)_{\ell=0} \end{pmatrix} + \mathcal{R}_{\ell=0}(\Psi_2, r^{-1} \Psi_1) \begin{pmatrix} 1 \\ O(r^{-1}) \end{pmatrix},$$

Denoting $\mathbf{v}_{\ell=0} = \begin{pmatrix} r^{2+\delta'}(\Psi_1)_{\ell=0} \\ r^{3+\delta'}(\Psi_2)_{\ell=0} \end{pmatrix}$ and $\mathbf{F}_{\ell=0} = \begin{pmatrix} r^{2+\delta'}(F_1)_{\ell=0} \\ r^{3+\delta'}(F_2)_{\ell=0} \end{pmatrix}$, we have

$$\partial_r \mathbf{v}_{\ell=0} = r^{-1} \begin{pmatrix} \delta' & 1 \\ 0 & 1 + \delta' \end{pmatrix} \mathbf{v}_{\ell=0} + r^{-2} B_0(r) \mathbf{v}_{\ell=0} + \mathbf{F}_{\ell=0} + O(r^{2+\delta'}) \mathcal{R}_{\ell=0}(\Psi_2, r^{-1} \Psi_1),$$

for some matrix $B_0(r)$ with all its entries uniformly bounded in r . The last term can be further decomposed as, using (5.14),

$$\mathcal{R}_{\ell=0}(r^{2+\delta'} \Psi_2, r^{1+\delta'} \Psi_1) = O(r^{-2+\delta'}) \mathcal{R}_{\ell=0}(\sum_i \mathbf{c}_i \omega_i) + \mathcal{R}_{\ell=0}(r^{2+\delta'} \Psi_2, r^{1+\delta'} \check{\Psi}_1).$$

This proves the expression (5.18).

Case $\ell = 1$. Projecting the system (4.24) to the $\ell = 1$ modes, we obtain

$$\begin{aligned} (\partial_r + 2r^{-1})(\Psi_1)_{1, \mathbf{m}} &= -2(1 - 3mr^{-1})r^{-2} \Upsilon^{-\frac{1}{2}} \left(\mathbb{A}_{\tilde{\gamma}}^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1) \right)_{1, \mathbf{m}} + (F_1)_{\ell, \mathbf{m}}, \\ (\partial_r + 3r^{-1})(\Psi_2)_{1, \mathbf{m}} &= -2\Upsilon^{\frac{1}{2}} r^{-3} \Upsilon^{-\frac{1}{2}} \left(\mathbb{A}_{\tilde{\gamma}}^{-1}(\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1) \right)_{1, \mathbf{m}} - \frac{1}{2} \Upsilon^{-\frac{1}{2}} (\mathbb{A}_{\tilde{\gamma}} \Psi_1)_{1, \mathbf{m}} + (F_2)_{\ell, \mathbf{m}}. \end{aligned} \quad (5.24)$$

Using (5.3), we can rewrite the system as

$$\begin{aligned} (\partial_r + 2r^{-1})(\Psi_1)_{1, \mathbf{m}} &= (1 - 3mr^{-1}) \Upsilon^{-\frac{1}{2}} (\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1)_{\ell, \mathbf{m}} + O(1) \mathcal{R}_{1, \mathbf{m}}(\Psi_2, r^{-1} \Psi_1) + (F_1)_{\ell, \mathbf{m}}, \\ (\partial_r + 3r^{-1})(\Psi_2)_{1, \mathbf{m}} &= r^{-1} (\Psi_2 - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1)_{\ell, \mathbf{m}} + r^{-2} \Upsilon^{-\frac{1}{2}} (\Psi_1)_{1, \mathbf{m}} + O(r^{-1}) \mathcal{R}_{1, \mathbf{m}}(\Psi_2, r^{-1} \Psi_1) + (F_2)_{\ell, \mathbf{m}}, \end{aligned} \quad (5.25)$$

or, in the matrix form,

$$\partial_r \begin{pmatrix} (\Psi_1)_{1, \mathbf{m}} \\ (\Psi_2)_{1, \mathbf{m}} \end{pmatrix} = \begin{pmatrix} -3r^{-1} + O(mr^{-2}) & 1 + O(mr^{-1}) \\ (\Upsilon^{\frac{1}{2}} - \Upsilon^{-\frac{1}{2}})r^{-2} & -2r^{-1} \end{pmatrix} \begin{pmatrix} (\Psi_1)_{1, \mathbf{m}} \\ (\Psi_2)_{1, \mathbf{m}} \end{pmatrix} + \begin{pmatrix} (F_1)_{1, \mathbf{m}} \\ (F_2)_{1, \mathbf{m}} \end{pmatrix} + \mathcal{R}_{1, \mathbf{m}}(\Psi_2, r^{-1} \Psi_1) \begin{pmatrix} O(1) \\ O(r^{-1}) \end{pmatrix}.$$

This can be further written as

$$\partial_r \begin{pmatrix} r^3(\Psi_1)_{1, \mathbf{m}} \\ r^4(\Psi_2)_{1, \mathbf{m}} \end{pmatrix} = r^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} r^3(\Psi_1)_{1, \mathbf{m}} \\ r^4(\Psi_2)_{1, \mathbf{m}} \end{pmatrix} + r^{-2} B_1(r) \begin{pmatrix} r^3(\Psi_1)_{1, \mathbf{m}} \\ r^4(\Psi_2)_{1, \mathbf{m}} \end{pmatrix} + \begin{pmatrix} r^3(F_1)_{1, \mathbf{m}} \\ r^4(F_2)_{1, \mathbf{m}} \end{pmatrix} + r^3 \mathcal{R}_{1, \mathbf{m}}(\Psi_2, r^{-1} \Psi_1),$$

for some matrix $B_1(r)$ with all its entries uniformly bounded in r .

Recall that in view of (4.27), $r^3(\Psi_1)_{1,m}$ does not vanish at infinity. However, as remarked after (5.14), $r^3(\check{\Psi}_1)_{1,m}$ does, and therefore, we consider $\mathbf{v}_{1,m} = \begin{pmatrix} r^3(\check{\Psi}_1)_{1,m} \\ r^4(\Psi_2)_{1,m} \end{pmatrix}$. Since the first column of $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ is zero, we have

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} r^3(\Psi_1)_{1,m} \\ r^4(\Psi_2)_{1,m} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{v}_{1,m}.$$

The system then reads,

$$\begin{aligned} \partial_r \mathbf{v}_{1,m} &= \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} r^{-1} \mathbf{v}_{1,m} + r^{-2} B_1(r) \mathbf{v}_{1,m} + r^{-2} (\sum_i \mathbf{c}_i \omega_i) B_1(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} r^3(F_1)_{1,m} \\ r^4(F_2)_{1,m} \end{pmatrix} \\ &\quad + O(r^3) \mathcal{R}_{1,m}(\Psi_2, r^{-1} \Psi_1) \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} r^{-1} \mathbf{v}_{1,m} + r^{-2} B_1(r) \mathbf{v}_{1,m} + r^{-2} (\sum_i \mathbf{c}_i \omega_i) B_1(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} r^3(F_1)_{1,m} \\ r^4(F_2)_{1,m} \end{pmatrix} \\ &\quad + O(r^{-1}) \mathcal{R}_{1,m}(\sum_i \mathbf{c}_i \omega_i) + O(r^3) \mathcal{R}_{1,m}(\Psi_2, r^{-1} \check{\Psi}_1), \end{aligned}$$

Therefore, denoting

$$\mathbf{F}_{1,m} := r^{-2} \mathbf{c}_m B_1(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} r^3(F_1)_{1,m} \\ r^4(F_2)_{1,m} \end{pmatrix} + O(r^{-1}) \mathcal{R}_{1,m}(\sum_i \mathbf{c}_i \omega_i), \quad (5.26)$$

we obtain the expression (5.18) as required. The equivalence of the equations in modes and the original system (4.24) is also clear since $\{J_{\ell,m}\}$ is a complete orthonormal basis over $L^2(S_r)$.

It remains to verify the bounds for $\mathbf{F}$. Recall the condition (4.26)

$$\sup_{r \in [r_0, \infty)} r^{-1} \|r^{3+\delta} F_1, r^{4+\delta} F_2, r^{4+\delta} (F_1)_{\ell=1}, r^{5+\delta} (F_2)_{\ell=1}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon.$$

As a result, by definition (5.19),

$$\begin{aligned} r^{-1} \|\mathbf{F}_{\ell \neq 1}\|_{\mathfrak{h}^s(S_r)} &\lesssim \varepsilon r^{-1-(\delta-\delta')} + r^{-2+\delta'} \cdot r^{-1} \|\mathcal{R}_{\ell \neq 1}(\sum_i \mathbf{c}_i \omega_i)\|_{\mathfrak{h}^s} \\ &\lesssim \varepsilon r^{-1-(\delta-\delta')} + \varepsilon_1 |\mathbf{c}| r^{-3-\delta+\delta'} \\ &\lesssim \varepsilon r^{-1-(\delta-\delta')}, \end{aligned}$$

and, by (5.20),

$$\begin{aligned} |\mathbf{F}_{1,m}| &\lesssim |r^{-2} \mathbf{c}_m| + |r^3(F_1)_{1,m}| + |r^4(F_2)_{1,m}| + r^{-1} |\mathcal{R}_{1,m}(\sum_i \mathbf{c}_i \omega_i)| \\ &\lesssim |\mathbf{c}| r^{-2} + \varepsilon r^{-1-\delta} + r^{-1} |\mathbf{c}| \varepsilon_1 r^{-1-\delta} \\ &\lesssim \varepsilon r^{-1-\delta}, \end{aligned}$$

where we used that, in view of the bound for $\mathcal{R}$ established in Proposition 5.2,

$$|\mathcal{R}_{1,m}(\sum_i \mathbf{c}_i \omega_i)| \lesssim r^{-1} \|\mathcal{R}(\sum_i \mathbf{c}_i \omega_i)\|_{\mathfrak{h}^s(S_r)} \lesssim r^{-1} \cdot \varepsilon_1 r^{-1-\delta} \|\sum_i \mathbf{c}_i \omega_i\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon_1 |\mathbf{c}| r^{-1-\delta}.$$

Moreover, the bound for $|\mathbf{F}_{1,m}|$ means that we can in fact replace $\mathbf{F}_{\ell \neq 1}$ with $\mathbf{F}$ for the first estimate. This concludes the proof of Proposition 5.8. $\square$

The combined expression. Since the r -weights we put in for different modes are different, we need to derive a uniform bound for the perturbative $\mathcal{R}$ terms. This is done through the lemma below.

Lemma 5.9. *The system (4.24) can be written in modes as*

$$\partial_r \mathbf{v}_{\ell,m} = r^{-1} A_\ell \mathbf{v}_{\ell,m} + r^{-2} B_\ell(r) \mathbf{v}_{\ell,m} + \mathbf{F}_{\ell,m} + \mathcal{R}_{\ell,m}^{new}(\mathbf{v}), \quad (5.27)$$

where the linear operator $\mathcal{R}^{new}$ satisfies

$$\|\mathcal{R}^{new}(\mathbf{v})\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon_1 r^{-1-\delta-\delta'} \|\mathbf{v}\|_{\mathfrak{h}^s(S_r)}.$$

Proof. According to Proposition 5.8, the system (4.24) is equivalent to

$$\partial_r \mathbf{v}_{\ell, \mathbf{m}} = r^{-1} A_\ell \mathbf{v}_{\ell, \mathbf{m}} + r^{-2} B_\ell(r) \mathbf{v}_{\ell, \mathbf{m}} + \mathbf{F}_{\ell, \mathbf{m}} + \mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v}), \quad (5.28)$$

where the $\mathcal{R}^{new}$ term reads, schematically, in terms of $\mathcal{R}$ defined in Definition 5.1,

$$\mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v}) = \begin{cases} \mathcal{R}_{1, \mathbf{m}} \left(r^2 \check{\Psi}_1, r^3 \Psi_2 \right) & \ell = 1, \\ \mathcal{R}_{\ell, \mathbf{m}} \left(r^{1+\delta'} \check{\Psi}_1, r^{2+\delta'} \Psi_2 \right), & \ell = 0 \text{ or } \ell \geq 2. \end{cases}$$

Since $\delta' < \delta < 1$, relaxing the r weights for $\ell \neq 1$, we have for each ℓ that

$$\mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v}) = O(1) \mathcal{R}_{\ell, \mathbf{m}}(r^2 \check{\Psi}_1, r^3 \Psi_2).$$

Therefore we have, using the bound for $\mathcal{R}$ in Proposition 5.2,

$$\begin{aligned} \|\mathcal{R}^{new}(\mathbf{v})\|_{\mathfrak{h}^s(S_r)} &\lesssim \|\mathcal{R}(r^2 \check{\Psi}_1, r^3 \Psi_2)\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon_1 r^{-1-\delta} \|r^2 \check{\Psi}_1, r^3 \Psi_2\|_{\mathfrak{h}^s(S_r)} \\ &\lesssim \varepsilon_1 r^{-1-\delta-\delta'} \|\mathbf{v}_{\ell \neq 1}\|_{\mathfrak{h}^s(S_r)} + \varepsilon_1 r^{-2-\delta} \|\mathbf{v}_{\ell=1}\|_{\mathfrak{h}^s(S_r)} \\ &\lesssim \varepsilon_1 r^{-1-\delta-\delta'} \|\mathbf{v}\|_{\mathfrak{h}^s(S_r)}. \end{aligned}$$

where we used that $\mathbf{v}_{\ell, \mathbf{m}} = \begin{pmatrix} r^{2+\delta'} (\Psi_1)_{\ell, \mathbf{m}} \\ r^{3+\delta'} (\Psi_2)_{\ell, \mathbf{m}} \end{pmatrix}$ for $\ell \neq 1$ and $\mathbf{v}_{1, \mathbf{m}} = \begin{pmatrix} r^3 (\check{\Psi}_1)_{1, \mathbf{m}} \\ r^4 (\Psi_2)_{1, \mathbf{m}} \end{pmatrix}$ as defined in (5.15). This concludes the proof of Lemma 5.9. $\square$

5.1.4 The solution operators in modes

In this part, we verify that the matrices A_ℓ satisfy the accretiveness required in Lemma 5.5 for all ℓ , hence giving uniformly bounded backward solution operators introduced in (5.9).

The case $\ell \geq 2$. For simplicity, we denote $x = \frac{1}{\ell(\ell+1)}$ and consider $x \in (0, \frac{1}{6}]$, corresponding to $\ell \geq 2$. Denote the matrix

$$Q := \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad (5.29)$$

as in Lemma 5.6. Then we have $A_\ell = \delta' I + xQ$, which verifies the condition of Lemma 5.6. Therefore, $A_\ell := \delta' I + xQ$ is accretive for some inner product H over $\mathbb{R}^2$, and hence by Lemma 5.5, we obtain a solution operator $U_\ell(r, r^*)$ for all $\ell \leq 2$

$$\partial_r U_\ell(r, r^*) = (r^{-1} A_\ell + r^{-2} B_\ell(r)) U_\ell(r, r^*), \quad U_\ell(r^*, r^*) = I, \quad (5.30)$$

where U_ℓ is uniformly bounded.

The case $\ell \leq 1$. Since the matrices $A_0 = \begin{pmatrix} \delta' & 1 \\ 0 & 1+\delta' \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ can both be diagonalized to a positive definite matrix, they easily verify the accretiveness condition. Hence, we obtain the backward solution operators $U_0(r, r^*)$, $U_1(r, r^*)$ through

$$\partial_r U_0(r, r^*) = (r^{-1} A_0 + r^{-2} B_0(r)) U_0(r, r^*), \quad U_0(r^*, r^*) = I, \quad (5.31)$$

$$\partial_r U_1(r, r^*) = (r^{-1} A_1 + r^{-2} B_1(r)) U_1(r, r^*), \quad U_1(r^*, r^*) = I, \quad (5.32)$$

and they are both uniformly bounded, as stated in Lemma 5.5.

5.1.5 The inhomogeneous solution

Proposition 5.10. *Define $\mathring{\mathbf{v}}$ using*

$$\mathring{\mathbf{v}}_{\ell, \mathbf{m}} = - \int_r^\infty U_\ell(r, r') \mathbf{F}_{\ell, \mathbf{m}}(r') dr'.$$

We have the estimate

$$r^{-1+(\delta-\delta')} \|\mathring{\mathbf{v}}\|_{\mathfrak{h}^s(S_r)} + r^\delta |\mathring{\mathbf{v}}_{1, \mathbf{m}}| \lesssim \varepsilon. \quad (5.33)$$

Proof. For $\ell = 1$, we apply the second bound in (5.21), which yields

$$|\mathring{\mathbf{v}}_{1, \mathbf{m}}| \lesssim \varepsilon r^{-\delta}.$$

Moreover, using the first bound in (5.21)

$$r^{-1} \|\mathbf{F}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon r^{-1-(\delta-\delta')},$$

we obtain

$$\begin{aligned} r^{-1} \|\mathring{\mathbf{v}}\|_{\mathfrak{h}^s(S_r)} &\leq \left(\sum_{\ell=0}^\infty \sum_{\mathbf{m}=-\ell}^\ell (1+\ell^2)^s |\mathring{\mathbf{v}}_{\ell, \mathbf{m}}|^2 \right)^{\frac{1}{2}} \leq C \left(\sum_{\ell=0}^\infty \sum_{\mathbf{m}=-\ell}^\ell \left(\int_r^\infty (1+\ell^2)^s |\mathbf{F}_{\ell, \mathbf{m}}(r')|^2 dr' \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\leq C \int_r^\infty \left(\sum_{\ell=0}^\infty \sum_{\mathbf{m}=-\ell}^\ell (1+\ell^2)^s |\mathbf{F}_{\ell, \mathbf{m}}|^2(r') \right)^{\frac{1}{2}} dr' \\ &\leq C \int_r^\infty r'^{-1} \|\mathbf{F}\|_{\mathfrak{h}^s(S_{r'})} dr' \\ &\lesssim \varepsilon r^{-(\delta-\delta')}, \end{aligned}$$

where we used the integral Minkowski inequality (2.20) from the third inequality. This concludes the proof of Proposition 5.10. $\square$

5.1.6 The contraction argument

Since the $\mathcal{R}_{\ell, \mathbf{m}}^{new}$ terms can involve different modes of $\mathbf{v}$, in order to obtain the solution of (4.24), or equivalently (5.27), we need a physical space norm to estimate $\mathbf{v}$, independent of its modes $\mathbf{v}_{\ell, \mathbf{m}}$. We define

$$\|\mathbf{v}\|_{\mathcal{V}} := \sup_{r \in [r_0, \infty)} \left(r^{-1+(\delta-\delta')} \|\mathbf{v}\|_{\mathfrak{h}^s(S_r)} + r^\delta |\mathbf{v}_{\ell=1}| \right), \quad (5.34)$$

and seek solutions in the following neighborhood of $\mathring{\mathbf{v}}$:

$$\mathcal{V}_{C\varepsilon} := \{\mathbf{v} : \|\mathbf{v}\|_{\mathcal{V}} < C\varepsilon\},$$

where C is a positive constant to be determined. In view of (5.12), solutions to (5.27) satisfy

$$\mathbf{v} = \Phi(\mathbf{v}), \quad (5.35)$$

where the map Φ is defined through $\Phi(\mathbf{v}) := \sum_{\ell=0}^\infty \sum_{\mathbf{m}=-\ell}^\ell \Phi(\mathbf{v})_{\ell, \mathbf{m}} J_{\ell, \mathbf{m}}$, with

$$\Phi(\mathbf{v})_{\ell, \mathbf{m}} := - \int_r^\infty U_\ell(r, r') (\mathbf{F}_{\ell, \mathbf{m}} + \mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v})) dr'.$$

Conversely, any $\mathbf{v}$ satisfying (5.35) gives a solution to the original system (4.24). Recall that for $\ell \geq 2$, U_ℓ is the same solution operator defined in (5.30), and for $\ell = 0, 1$, U_ℓ is defined respectively in (5.31), (5.32).

It suffices to show that $\Phi(\mathcal{V}_{C\varepsilon}) \subset \mathcal{V}_{C\varepsilon}$ and Φ is a contraction in $\mathcal{V}_{C\varepsilon}$ with respect to the norm $\|\cdot\|_{\mathcal{V}}$. We have

$$\begin{aligned}\Phi(\mathbf{v})_{\ell, \mathbf{m}} &= - \int_r^\infty U_\ell(r, r') (\mathbf{F}_{\ell, \mathbf{m}} + \mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v})) dr' \\ &= \hat{\mathbf{v}}_{\ell, \mathbf{m}} - \int_r^\infty U_\ell(r, r') \mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v}) dr' .\end{aligned}$$

Therefore,

$$\begin{aligned}r^{-1} \|\Phi(\mathbf{v}) - \hat{\mathbf{v}}\|_{\mathfrak{h}^s(S_r)} &\leq \left(\sum_{\ell=0}^\infty \sum_{\mathbf{m}=-\ell}^\ell (1 + \ell^2)^s |\Phi(\mathbf{v})_{\ell, \mathbf{m}} - \hat{\mathbf{v}}_{\ell, \mathbf{m}}|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{\ell=0}^\infty \sum_{\mathbf{m}=-\ell}^\ell \left(\int_r^\infty (1 + \ell^2)^s |\mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v})|(r') dr' \right)^2 \right)^{\frac{1}{2}} \\ &\leq C \int_r^\infty \left(\sum_{\ell=0}^\infty \sum_{\mathbf{m}=-\ell}^\ell (1 + \ell^2)^s |\mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v})|^2(r') \right)^{\frac{1}{2}} dr' \\ &\leq C \int_r^\infty r'^{-1} \|\mathcal{R}^{new}(\mathbf{v})\|_{\mathfrak{h}^s(S_{r'})} dr' ,\end{aligned}$$

where we used the integral Minkowski inequality (2.20) from the second line to the third line. Then, using Lemma 5.9 and (5.33),

$$\begin{aligned}r^{-1+(\delta-\delta')} \|\Phi(\mathbf{v}) - \hat{\mathbf{v}}\|_{\mathfrak{h}^s(S_r)} &\leq C r^{\delta-\delta'} \int_r^\infty r'^{-1} \cdot \varepsilon_1 r'^{-1-\delta-\delta'} \|\mathbf{v}\|_{\mathfrak{h}^s(S_{r'})} dr' \\ &\leq C r^{\delta-\delta'} \int_r^\infty C \varepsilon_1 (r'^{-2-\delta-\delta'}) \cdot r'^{1-(\delta-\delta')} \|\mathbf{v}\|_{\mathcal{V}} dr' \\ &\leq C \varepsilon_1 \varepsilon \ll \varepsilon ,\end{aligned}$$

for suitable $C > 0$.³³ We also have

$$\begin{aligned}r^\delta |\Phi(\mathbf{v})_{\ell=1} - \hat{\mathbf{v}}_{\ell=1}| &\leq C r^\delta \int_r^\infty |\mathcal{R}_{1, \mathbf{m}}^{new}(\mathbf{v})| dr' \leq C r^\delta \int_r^\infty r'^{-1} \|\mathcal{R}^{new}(\mathbf{v})\|_{L^2(S_{r'})} dr' \\ &\leq C r^\delta \int_r^\infty \varepsilon_1 r'^{-2-\delta-\delta'} \|\mathbf{v}\|_{\mathfrak{h}^s(S_{r'})} dr' \leq C r^\delta \left(\int_r^\infty \varepsilon_1 r'^{-2-\delta-\delta'} \cdot r'^{1-(\delta-\delta')} dr' \right) \|\mathbf{v}\|_{\mathcal{V}} \\ &\leq C \varepsilon_1 \varepsilon \ll \varepsilon .\end{aligned}$$

Therefore, we obtain $\|\Phi(\mathbf{v}) - \hat{\mathbf{v}}\|_{\mathcal{V}} \ll \varepsilon$. From (5.33) we know that $\|\hat{\mathbf{v}}\|_{\mathcal{V}} \lesssim \varepsilon$, and hence we see that $\Phi(\mathcal{V}_{C\varepsilon}) \subset \mathcal{V}_{C\varepsilon}$ for suitable $C > 0$.

To prove that Φ is a contraction, we note that

$$(\Phi(\mathbf{v}_1) - \Phi(\mathbf{v}_2))_{\ell, \mathbf{m}} = - \int_r^\infty U_\ell(r, r') \mathcal{R}_{\ell, \mathbf{m}}^{new}(\mathbf{v}_1 - \mathbf{v}_2) dr' .$$

³³We omit writing δ^{-1} since $\delta > 0$ is a given constant.

Hence, by similar estimates using Lemma 5.9, we obtain

$$\begin{aligned}
r^{-1+(\delta-\delta')} \|\Phi(\mathbf{v}_1) - \Phi(\mathbf{v}_2)\|_{\mathfrak{h}^s(S_r)} &\leq Cr^\delta \int_r^\infty r'^{-1} \|\mathcal{R}^{new}(\mathbf{v}_1 - \mathbf{v}_2)\|_{\mathfrak{h}^s(S_{r'})} dr' \\
&\leq Cr^\delta \int_r^\infty \varepsilon_1 r'^{-2-\delta-\delta'} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathfrak{h}^s(S_{r'})} dr' \\
&\leq C\varepsilon_1 r^\delta \left(\int_r^\infty r'^{-1-2\delta} dr' \right) \sup_{r \in [r_0, \infty)} r^{-1+(\delta-\delta')} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathfrak{h}^s(S_r)} \\
&\leq C\varepsilon_1 \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}},
\end{aligned}$$

and

$$\begin{aligned}
r^\delta |\Phi(\mathbf{v}_1)_{\ell=1} - \Phi(\mathbf{v}_2)_{\ell=1}| &\leq Cr^\delta \int_r^\infty |\mathcal{R}_{1,m}^{new}(\mathbf{v}_1 - \mathbf{v}_2)| dr' \leq Cr^\delta \int_r^\infty r'^{-1} \|\mathcal{R}^{new}(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^2(S_{r'})} dr' \\
&\leq Cr^\delta \int_r^\infty \varepsilon_1 r'^{-2-\delta-\delta'} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathfrak{h}^s(S_{r'})} dr' \leq Cr^\delta \left(\int_r^\infty \varepsilon_1 r'^{-1-2\delta} dr' \right) \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}} \\
&\leq C\varepsilon_1 \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathcal{V}}.
\end{aligned}$$

Therefore, by the fixed point theorem, we obtain a unique solution $\mathbf{v}$ in $\mathcal{V}_{C\varepsilon}$, which, when expressed in terms of Ψ_1 and Ψ_2 , verifies (4.27) in view of the definition of $\|\cdot\|_{\mathcal{V}}$ in (5.34). This concludes the proof of Proposition 4.2.

5.2 Boundedness estimates: Proof of Proposition 4.3

Remark 5.11. Throughout this proof, the implicit constants in the symbol $\lesssim$ do not include the bootstrap constant C_b stated in the Proposition 4.3.

Remark 5.12. The L^∞ estimates needed in the proof can be easily derived by standard Sobolev embedding from the L^2 estimates:

$$\begin{aligned}
&r^{2+\delta} \|(r\mathcal{V}^{(0)})^{\leq s-1}(\Psi_1^{(n)}, \Psi_4^{(n)}, \Psi_5^{(n)}, \Psi_7^{(n)}, \Psi_8^{(n)}, \Psi_9^{(n)}, \Psi_{10}^{(n)})\|_{L^\infty(S_r)} \\
&+ r^{3+\delta} \|(r\mathcal{V}^{(0)})^{\leq s-2}(\Psi_2^{(n)}, \Psi_6^{(n)})\|_{L^\infty(S_r)} + r^{1+\delta} \|(r\mathcal{V}^{(0)})^{\leq s} \Psi_3^{(n)}\|_{L^\infty(S_r)} \\
&+ r^{1+\delta} \|(r\mathcal{V}^{(0)})^{\leq s}(\gamma^{(n)} - \gamma^{(0)})\|_{L^\infty(S_r)} \lesssim C_b \varepsilon.
\end{aligned} \tag{5.36}$$

Remark 5.13. Throughout the proof, we will use the following bound, ensured by the assumption of Proposition 4.3, without explicit reference:

$$r^\delta \|\tilde{\gamma} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim C_b \varepsilon. \tag{5.37}$$

In particular, this allows us to apply the Hodge estimate in Lemma 3.2.

We now proceed as follows.

5.2.1 Proof of Proposition 4.4

We explicitly write down the expression of $\Psi_3^{(n+1)}$ using (4.6) and (4.7):

$$\begin{aligned}
\Psi_3^{(n+1)} &= \Upsilon^{-\frac{1}{2}} (\mathbb{A}^{(n)})^{-1} \left(\Psi_2^{(n+1)} - \Upsilon^{\frac{1}{2}} r^{-1} \Psi_1^{(n+1)} + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} \right) - \Upsilon^{-\frac{1}{2}} (\mathbb{A}^{(n)})^{-1} \mathbb{A}^{(n)} (\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}) \\
&\quad - \frac{1}{2} \Upsilon^{-1} r \overline{\Psi_1^{(n+1)}}^{(n)}.
\end{aligned} \tag{5.38}$$

Here we again adopt the extended definition of $(\Delta^{(n)})^{-1}$ as in footnote 29.

We first apply Proposition 4.2 to obtain $\Psi_1^{(n+1)}$ and $\Psi_2^{(n+1)}$. Denote the error terms

$$\begin{aligned}\mathcal{N}^{(n)}[\tilde{a}] &:= -\Upsilon^{-\frac{1}{2}}(\Delta^{(n)})^{-1}\Delta^{(n)}(\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}) + \Upsilon^{-\frac{1}{2}}(\Delta^{(n)})^{-1}(\Gamma_1^{(n)} \cdot \Gamma_1^{(n)}) \\ &= -\Upsilon^{-\frac{1}{2}}(\Gamma_0^{(n)} \cdot \Gamma_0^{(n)} - \overline{\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}}^{(n)}) + \Upsilon^{-\frac{1}{2}}(\Delta^{(n)})^{-1}(\Gamma_1^{(n)} \cdot \Gamma_1^{(n)}), \\ \mathcal{N}^{(n)}[\tilde{\mathcal{B}}_{\ell \leq 1}] &:= \frac{1}{2}(\Delta^{(n)}\Psi_1^{(n)})_{\ell=0} - \left(\mathcal{P}_1(\mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)}\Psi_4^{(n)})\right)_{\ell \leq 1}, \\ \mathcal{N}^{(n)}[\mu] &:= -(\Delta^{(n)} \log \Psi_3^{(n)})_{\ell=0} - \frac{1}{4}((\Psi_1)^2)_{\ell=0}.\end{aligned}$$

Then the system of $\Psi_1^{(n+1)}$ and $\Psi_2^{(n+1)}$, originating from (4.4), (4.5), reads, in view of (4.17),

$$\begin{aligned}(\partial_r + 2r^{-1})\Psi_1^{(n+1)} &= -2(1 - 3mr^{-1})r^{-2} \left(\Upsilon^{-\frac{1}{2}}(\Delta^{(n)})^{-1}(\Psi_2^{(n+1)} - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1^{(n+1)}) - \frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1^{(n+1)}}^{(n)} \right) \\ &\quad + \Upsilon^{-\frac{1}{2}}(\Psi_2^{(n+1)} - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1^{(n+1)})_{\ell=0} + \Psi_3^{(n)}\check{\mu}_{\ell=0}^{(n)} + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} \\ &\quad + \Upsilon^{-\frac{1}{2}}\mathcal{N}^{(n)}[\mu] - 2(1 - 3mr^{-1})r^{-2}\mathcal{N}^{(n)}[\tilde{a}], \\ (\partial_r + 3r^{-1})\Psi_2^{(n+1)} &= -2\Upsilon^{\frac{1}{2}}r^{-3} \left(\Upsilon^{-\frac{1}{2}}(\Delta^{(n)})^{-1}(\Psi_2^{(n+1)} - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1^{(n+1)}) - \frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1^{(n+1)}}^{(n)} \right) \\ &\quad + r^{-1}(\Psi_2^{(n+1)} - \Upsilon^{\frac{1}{2}}r^{-1}\Psi_1^{(n+1)})_{\ell=0} + r^{-1}\mathcal{N}^{(n)}[\mu] - 2\Upsilon^{\frac{1}{2}}r^{-3}\mathcal{N}^{(n)}[\tilde{a}] \\ &\quad - \Upsilon^{-\frac{1}{2}}\left(\mathcal{B} + \frac{1}{2}(\Delta^{(n)}\Psi_1^{(n+1)})_{\ell=1} + \mathcal{N}^{(n)}[\tilde{\mathcal{B}}_{\ell \leq 1}]\right) + \Psi_3^{(n)}(\mathcal{B} + \tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(n)}) + \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}.\end{aligned}\tag{5.39}$$

The system (5.39) is of the form (4.24), with $\tilde{\gamma} = \gamma^{(n)}$, and

$$\begin{aligned}F_1 &= -2(1 - 3mr^{-1})r^{-2}\mathcal{N}^{(n)}[\tilde{a}] + \Upsilon^{-\frac{1}{2}}\mathcal{N}^{(n)}[\mu] + \Psi_3^{(n)}\check{\mu}_{\ell=0}^{(n)} + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \\ F_2 &= -\Upsilon^{-\frac{1}{2}}(\mathcal{B} + \mathcal{N}^{(n)}[\tilde{\mathcal{B}}_{\ell \leq 1}]) + \Psi_3^{(n)}(\mathcal{B} + \tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(n)}) + \Gamma_1^{(n)} \cdot \Gamma_2^{(n)} - 2\Upsilon^{\frac{1}{2}}r^{-3}\mathcal{N}^{(n)}[\tilde{a}] + r^{-1}\mathcal{N}^{(n)}[\mu].\end{aligned}$$

We now verify the bounds required in Proposition 4.2. We have

$$\begin{aligned}r^{-1}\|\mathcal{N}^{(n)}[\tilde{a}]\|_{\mathfrak{h}^{s+1}(S_r)} &\lesssim r^{-1}\|\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}\|_{\mathfrak{h}^{s+1}(S_r)} + r^{-1}\|(\Delta^{(n)})^{-1}(\Gamma_1^{(n)} \cdot \Gamma_1^{(n)})\|_{\mathfrak{h}^{s+1}(S_r)} \\ &\lesssim C_b^2\varepsilon^2r^{-2-2\delta}.\end{aligned}$$

Applying (3.5), we obtain

$$\begin{aligned}r^{-1}\|\mathcal{N}^{(n)}[\tilde{\mathcal{B}}_{\ell \leq 1}]\|_{\mathfrak{h}^s(S_r)} &\lesssim r^{-1}\|(\Delta^{(n)}\Psi_1^{(n)})_{\ell=0}\|_{\mathfrak{h}^s(S_r)} + r^{-1}\|(\mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)}\Psi_4^{(n)})_{\ell \leq 1}\|_{\mathfrak{h}^s(S_r)} \\ &\lesssim C_b\varepsilon r^{-1-\delta}r^{-2}\|\Psi_1^{(n)}\|_{L^\infty} + \|\Delta^{(0)}\mathfrak{t}^{(0)}\Psi_4^{(n)}\|_{L^\infty} \\ &\quad + C_b\varepsilon r^{-3-\delta}\|(r\nabla^{(0)})^{\leq 2}\Psi_4^{(n)}\|_{L^\infty} \\ &\lesssim C_b^2\varepsilon^2r^{-5-2\delta},\end{aligned}$$

$$\begin{aligned}r^{-1}\|\mathcal{N}^{(n)}[\mu]\|_{\mathfrak{h}^s(S_r)} &\lesssim |-(\Delta^{(n)} \log \Psi_3^{(n)})_{\ell=0}| + \frac{1}{4}|(\Psi_1)^2| \lesssim C_b\varepsilon^2r^{-2} \cdot r^{-1-\delta} \cdot r^{-1-\delta} + C_b^2\varepsilon^2r^{-4-2\delta} \\ &\lesssim C_b^2\varepsilon^2r^{-4-2\delta}.\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}r^{-1}\|F_1\|_{\mathfrak{h}^{s+1}(S_r)} &\lesssim r^{-1}\|r^{-2}\mathcal{N}^{(n)}[\tilde{a}]\|_{\mathfrak{h}^{s+1}} + r^{-1}\|\mathcal{N}^{(n)}[\mu]\|_{\mathfrak{h}^{s+1}} + r^{-1}\|\Psi_3^{(n)}\check{\mu}_{\ell=0}^{(n)}\|_{\mathfrak{h}^{s+1}} + r^{-1}\|\Gamma_1^{(n)} \cdot \Gamma_1^{(n)}\|_{\mathfrak{h}^{s+1}} \\ &\lesssim C_b^2\varepsilon^2r^{-4-2\delta} + C_b\varepsilon^2r^{-1-\delta} \cdot r^{-3-\delta} \lesssim C_b^2\varepsilon^2r^{-4-2\delta},\end{aligned}$$

$$\begin{aligned}
r^{-1} \|F_2\|_{\mathfrak{h}^s(S_r)} &\lesssim r^{-1} \|\mathcal{B}\|_{\mathfrak{h}^s} + r^{-1} \|\mathcal{N}^{(n)}[\tilde{\mathcal{B}}_{\ell \leq 1}]\|_{\mathfrak{h}^s} + r^{-1} \|\Psi_3^{(n)}(\mathcal{B} + \tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(n)})\|_{\mathfrak{h}^s} \\
&\quad + r^{-1} \|\Gamma_1^{(n)} \cdot \Gamma_2^{(n)}\|_{\mathfrak{h}^s} + r^{-1} \|r^{-3} \mathcal{N}^{(n)}[\tilde{a}]\|_{\mathfrak{h}^s} + r^{-1} \|r^{-1} \mathcal{N}^{(n)}[\mu]\|_{\mathfrak{h}^s} \\
&\lesssim \varepsilon r^{-4-\delta} + C_b^2 \varepsilon^2 r^{-5-2\delta}.
\end{aligned}$$

Moreover, since $\mathcal{B}_{\ell=1} = 0$, we have

$$r^{-1} \|(F_2)_{\ell=1}\|_{\mathfrak{h}^s(S_r)} \lesssim C_b^2 \varepsilon^2 r^{-5-2\delta}.$$

Therefore, for given center of mass value $\mathbf{c} \in \mathbb{R}^3$, applying Proposition 4.2 to (5.39), we obtain the unique solution $(\Psi_1^{(n+1)}, \Psi_2^{(n+1)})$ verifying the bounds

$$\sup_{r \in [r_0, \infty)} r^{-1} \|r^{2+\delta} \Psi_1^{(n+1)}, r^{3+\delta} \Psi_2^{(n+1)}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon, \quad \sup_{r \in [r_0, \infty)} r^\delta |r^3 (\Psi_1^{(n+1)})_{\ell=1, i} - \mathbf{c}_i, r^4 (\Psi_2^{(n+1)})_{\ell=1, i}| \lesssim \varepsilon. \quad (5.40)$$

Note that the right-hand side is ε instead of $C_b \varepsilon$.

To derive the estimate for $\Psi_3^{(n+1)}$, it suffices to recall the expression (5.38), which, again in view of Lemma 5.3, implies

$$r^{-1} \|\Psi_3^{(n+1)}\|_{\mathfrak{h}^{s+2}} \lesssim \varepsilon r^{-1-\delta}. \quad (5.41)$$

Remark 5.14. Using the second bound in (5.40), we also easily deduce the behavior of $(\Psi_3^{(n+1)})_{\ell=1}$ using (5.38):

$$|(\Psi_3^{(n+1)})_{\ell=1, i} - \frac{1}{2} \mathbf{c}_i r^{-2}| \lesssim C_b \varepsilon r^{-1-\delta} \cdot \|r^2 \Psi_2^{(n+1)}, r \Psi_1^{(n+1)}\|_{L^\infty} \lesssim C_b^2 \varepsilon^2 r^{-2-2\delta}. \quad (5.42)$$

Such a more precise estimate will be useful in Appendix C.

We now further derive the $\mathfrak{h}^{s+1}$ estimates of $\Psi_1^{(n+1)}$. Commuting the equation (4.4) with $(r \nabla^{(0)})^{s+1}$ using (3.7), we have

$$(\partial_r + 2r^{-1})(r \nabla^{(0)})^{s+1} \Psi_1^{(n+1)} = -2(1 - 3mr^{-1})r^{-2}(r \nabla^{(0)})^{s+1} \Psi_3^{(n+1)} + (r \nabla^{(0)})^{s+1} (\Gamma_1^{(n)} \cdot \Gamma_1^{(n)}).$$

Directly applying Lemma 3.5 to this equation, using the bound (5.41) we just obtained, we deduce

$$r^{-1} \|r^2 \Psi_1^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \int_r^\infty r'^{-1} \cdot r'^2 \|r'^{-2} \Psi_3^{(n+1)} + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}\|_{\mathfrak{h}^{s+1}(S_{r'})} dr' \lesssim \varepsilon r^{-\delta}. \quad (5.43)$$

This finishes the proof of Proposition 4.4.

5.2.2 Proof of Proposition 4.5

We proceed to determine $\Psi_4^{(n+1)}$ and $\Psi_{11}^{(n+1)}$ (which is supported on $\ell \leq 1$) from the equation (4.8):

$$\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} \Psi_4^{(n+1)} = \frac{1}{2} (\mathcal{A}^{(n)} \Psi_1^{(n+1)}, 0) - (\mathcal{B}, * \mathcal{B}) - \Psi_{11}^{(n+1)}.$$

Taking into account that $\Psi_1^{(n+1)}$ has already been obtained, we can apply Corollary 3.7 with $(S, \gamma) = (S_r, \gamma^{(n)})$ to obtain a unique $\Psi_{11}^{(n+1)}$ for which (4.8) is solvable.

Moreover, using the estimate (3.13) with $\varepsilon = C_b \varepsilon r^{-1-\delta}$, noticing also that $\mathcal{A}^{(n)} \Psi_1^{(n+1)}$ on the right-hand side of (4.8) has zero spherical mean over $\gamma^{(n)}$, we have

$$|(\Psi_{11}^{(n+1)})_{\ell=0}| \lesssim |-(\mathcal{B}, * \mathcal{B})^{(n)}| \lesssim \|(\mathcal{B}, * \mathcal{B})\|_{L^\infty(S_r)} \|\gamma^{(n)} - \gamma^{(0)}\|_{L^\infty(S_r)} \lesssim C_b^2 \varepsilon^2 r^{-5-2\delta},$$

$$\begin{aligned}
|(\Psi_{11}^{(n+1)})_{\ell=1} - \frac{1}{2}(\mathbb{A}^{(n)}\Psi_1^{(n+1)}, 0)_{\ell=1}| &\lesssim C_b \varepsilon r^{-1-\delta} \cdot r^{-1} \|(\mathcal{B}, * \mathcal{B})\|_{L^2(S_r, \gamma^{(n)})} \\
&\lesssim C_b^2 \varepsilon^2 r^{-5-2\delta},
\end{aligned}$$

where we have used the equivalence of norms from Lemma 3.1. Then, using the improved estimate (4.28) for $(\Psi_1^{(n+1)})_{\ell=1}$, we obtain

$$|(\Psi_{11}^{(n+1)})_{\ell=1}| \lesssim |\mathbf{c}| r^{-5} + \varepsilon r^{-5-\delta} + C_b^2 \varepsilon^2 r^{-5-2\delta} \lesssim \varepsilon r^{-5}.$$

Corollary 3.7 also implies that the solution $\Psi_4^{(n+1)}$ to (4.8) exists and, in view of Lemma 3.2 applied to $\mathcal{P}_1 \mathcal{P}_2$,

$$r^{-1} \|\Psi_4^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim r^{-1} \cdot r^2 \|(\mathcal{B}, * \mathcal{B})\|_{\mathfrak{h}^{s-1}(S_r)} + r^{-1} \|\Psi_1^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon r^{-2-\delta}.$$

Similarly, applying Lemma 3.2, we obtain $\Psi_5^{(n+1)}$ from (4.9) and show that it verifies the estimate

$$r^{-1} \|\Psi_5^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \|\mathbb{A}^{(n)}\Psi_3^{(n+1)}\|_{\mathfrak{h}^s(S_r)} + \|\mathbb{A}^{(n)}(\Gamma_0^{(n)} \cdot \Gamma_0^{(n)})\|_{\mathfrak{h}^s(S_r)} \lesssim r^{-2} \cdot \varepsilon r^{-\delta} \lesssim \varepsilon r^{-2-\delta}.$$

To conclude this step, we derive the estimate of $\Psi_6^{(n+1)}$ through the equation (4.10). We obtain, by Lemma 3.2,

$$r^{-1} \|\Psi_6^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \|(\mathcal{B}, * \mathcal{B})\|_{\mathfrak{h}^s(S_r)} + \|\Psi_{11}^{(n+1)}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon r^{-3-\delta}.$$

5.2.3 Proof of Proposition 4.6

We recall the equations (4.11), (4.14), and (4.15):

$$\begin{aligned}
(\partial_r + r^{-1})\Psi_7^{(n+1)} &= 2r^{-1}\Psi_{10}^{(n+1)} + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \\
\mathbb{A}^{(n)} \left(\hat{a}^{(n)} \Psi_{10}^{(n+1)} \right) &= \mathcal{K} + \widetilde{\mathcal{K}}_{\ell \leq 1}^{(n+1)} - \overline{\mathcal{K} + \widetilde{\mathcal{K}}_{\ell \leq 1}^{(n+1)}}^{(n)}, \\
\overline{\hat{a}^{(n)} \Psi_{10}^{(n+1)}}^{(n)} &= \overline{\Psi_3^{(n)} \Psi_{10}^{(n)}}^{(n)}.
\end{aligned}$$

The equations (4.14) and (4.15) imply the following expression of $\Psi_{10}^{(n+1)}$:

$$\hat{a}^{(n)} \Psi_{10}^{(n+1)} = (\mathbb{A}^{(n)})^{-1} (\mathcal{K} + \widetilde{\mathcal{K}}_{\ell \leq 1}^{(n+1)}) + \overline{\Psi_3^{(n)} \Psi_{10}^{(n)}}^{(n)}. \quad (5.44)$$

Plugging this into the equation of $\Psi_7^{(n+1)}$, we obtain

$$(\partial_r + r^{-1})\Psi_7^{(n+1)} = 2r^{-1}(\hat{a}^{(n)})^{-1}(\mathbb{A}^{(n)})^{-1}(\mathcal{K} + \widetilde{\mathcal{K}}_{\ell \leq 1}^{(n+1)}) + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} + 2r^{-1} \overline{\Psi_3^{(n)} \Psi_{10}^{(n)}}^{(n)}, \quad (5.45)$$

where we recall (4.19)

$$\widetilde{\mathcal{K}}_{\ell \leq 1}^{(n+1)} := \mathcal{P}_1 \left(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}) \right)_{\ell \leq 1} - \frac{1}{2} (\hat{a}^{(n)} \mathbb{A}^{(n)} \Psi_7^{(n+1)})_{\ell=1} - \frac{1}{2} (\hat{a}^{(n)} \mathbb{A}^{(n)} \Psi_7^{(n)})_{\ell=0} + (\Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1}.$$

Therefore, with all quantities labeled with $^{(n)}$ viewed as known quantities, (5.45) is an equation of $\Psi_7^{(n+1)}$:

$$\begin{aligned}
(\partial_r + r^{-1})\Psi_7^{(n+1)} &= 2r^{-1}(\hat{a}^{(n)})^{-1}(\mathbb{A}^{(n)})^{-1} \left(-\frac{1}{2} (\hat{a}^{(n)} \mathbb{A}^{(n)} \Psi_7^{(n+1)})_{\ell=1} \right) + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} + 2r^{-1} \overline{\Psi_3^{(n)} \Psi_{10}^{(n)}}^{(n)} \\
&\quad + 2r^{-1}(\hat{a}^{(n)})^{-1}(\mathbb{A}^{(n)})^{-1} \left(\mathcal{K} + \mathcal{P}_1 \left(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}) \right)_{\ell \leq 1} + (\Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1} \right).
\end{aligned} \quad (5.46)$$

Lemma 5.15. *There exists a constant $C > 0$ such that the equation (5.46) has a unique solution $\Psi_7^{(n+1)}$ verifying $\|\Psi_7^{(n+1)}\|_s \leq C\varepsilon$. More precisely, the solution satisfies*

$$r^{-1}\|\Psi_7^{(n+1)}\|_{\mathfrak{h}^{s+1}} \lesssim \varepsilon r^{-2-\delta}, \quad r^{-1}\|(\Psi_7^{(n+1)})_{\ell=1}\|_{\mathfrak{h}^{s+1}} \lesssim C_b^2 \varepsilon^2 r^{-3-2\delta}. \quad (5.47)$$

Proof. This is a situation similar to, but much simpler than, the one we dealt with in Section 5.1, and hence we only provide a sketch.³⁴ Applying Lemma 5.3, we can write equation (5.46) in the form

$$\begin{aligned} (\partial_r + r^{-1})\Psi_7^{(n+1)} &= -r^{-1}(\Psi_7^{(n+1)})_{\ell=1} + \mathcal{R}(r^{-1}\Psi_7^{(n+1)}) + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} + r^{-1}\Gamma_0^{(n)} \cdot \Gamma_1^{(n)} \\ &\quad + 2r^{-1}(\hat{a}^{(n)})^{-1}(\Delta^{(n)})^{-1} \left(\mathcal{K} + \mathcal{P}_1 \left(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}) \right)_{\ell \leq 1} + (\Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1} \right), \end{aligned}$$

for some error linear operator $\mathcal{R}$ that has similar properties³⁵ as the $\mathcal{R}$ introduced in Definition 5.1. Alternatively, the equation can be written as

$$\begin{aligned} (\partial_r + 2r^{-1})\Psi_7^{(n+1)} &= r^{-1}(\Psi_7^{(n+1)})_{\ell \neq 1} + \mathcal{R}(r^{-1}\Psi_7^{(n+1)}) + \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} + r^{-1}\Gamma_0^{(n)} \cdot \Gamma_1^{(n)} \\ &\quad + 2r^{-1}(\hat{a}^{(n)})^{-1}(\Delta^{(n)})^{-1} \left(\mathcal{K} + \mathcal{P}_1 \left(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}) \right)_{\ell \leq 1} + (\Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1} \right). \end{aligned}$$

In the latter form, the first term on the right is a positive term, i.e., a special case of the positive definite matrix studied in Section 5.1, and hence can be neglected. We then repeat the contraction argument in Section 5.1.6, in an easier situation, to obtain the existence of $\Psi_7^{(n+1)}$ in the space consistent with the estimate

$$r^{-1}\|\Psi_7^{(n+1)}\|_{\mathfrak{h}^{s+1}} \lesssim \varepsilon r^{-2-\delta}.$$

We then project the equation to $\ell = 1$ to obtain an improved estimate for $\ell = 1$. The main reason for the improvement is that the free scalar $\mathcal{K}$, while only decaying at the rate $r^{-4-\delta}$, is not supported on $\ell = 1$. Therefore, the $\ell = 1$ part of the right-hand side consists of only nonlinear terms. Since the existence of $\Psi_7^{(n+1)}$ and its $\mathfrak{h}^{s+1}$ bound have been obtained, such an improved estimate for $(\Psi_7^{(n+1)})_{\ell=1}$ is straightforward using the bound for the error operator $\mathcal{R}$. $\square$

To conclude the proof of Proposition 4.6, we apply the bound (5.47) for $\Psi_7^{(n+1)}$ we just obtained to (5.44) and derive the estimate for $\Psi_{10}^{(n+1)}$:

$$\begin{aligned} r^{-1}\|\Psi_{10}^{(n+1)}\|_{\mathfrak{h}^{s+1}} &\lesssim r^{-1} \cdot r^2 \|\mathcal{K}\|_{\mathfrak{h}^{s-1}} + r^{-1}\|\Psi_7^{(n+1)}\|_{\mathfrak{h}^{s+1}} + r^{-1} \cdot r^2 \|(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}))_{\ell \leq 1}\|_{\mathfrak{h}^{s+1}} \\ &\quad + r^2 |(\Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1}| + |\Gamma_0^{(n)} \cdot \Gamma_1^{(n)}| \\ &\lesssim \varepsilon r^{-2-\delta} + C_b^2 \varepsilon^2 r^{-3-2\delta} \lesssim \varepsilon r^{-2-\delta}. \end{aligned}$$

Note that we used that the term $(\mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}))_{\ell \leq 1}$, in view of (3.5), is in fact nonlinear.

³⁴In particular, here we only have a single equation (5.46) rather than a system, and the $\ell = 1$ condition is zero at infinity, in contrast to the nonzero $\mathbf{c}$ in Section 5.1.

³⁵More precisely, the bound in Proposition 5.2. At a heuristic level, $\mathcal{R}$ provides an additional $\varepsilon r^{-1-\delta}$ factor.

5.2.4 Proof of Proposition 4.7

We recall the equations (4.12) and (4.13):

$$\begin{aligned}\mathcal{D}_1^{(n)} \mathcal{D}_2^{(n)} \left(\hat{a}^{(n)} \Psi_8^{(n+1)} \right) &= \frac{1}{2} \left(\hat{a}^{(n)} \Delta^{(n)} \Psi_7^{(n+1)}, 0 \right) + (\mathcal{K}, -^* \mathcal{K}) + \Psi_{12}^{(n+1)} + \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}, \\ \mathcal{D}_1^{(n)} \Psi_9^{(n+1)} &= \left(0, \frac{3}{4\pi} r^{-4} \sum_i \mathbf{a}_i \omega_i + r^{-4} \int_r^\infty r'^4 ({}^* \mathcal{K} - {}^* \mathcal{K}_{\ell \leq 1}^{(n+1)}) dr' \right) \\ &\quad - \overline{\left(0, \frac{3}{4\pi} r^{-4} \sum_i \mathbf{a}_i \omega_i + r^{-4} \int_r^\infty r'^4 ({}^* \mathcal{K} - {}^* \mathcal{K}_{\ell \leq 1}^{(n+1)}) dr' \right)}^{(n)}.\end{aligned}$$

Since we have determined $\Psi_7^{(n+1)}$, we can apply Corollary 3.7 to the first equation with $\varepsilon = C_b \varepsilon r^{-1-\delta}$ to obtain

$$\begin{aligned} |(\Psi_{12}^{(n+1)})_{\ell=0}| &\lesssim |(\mathcal{K}, -^* \mathcal{K})^{(n)}| \lesssim \|(\mathcal{K}, -^* \mathcal{K})\|_{L^\infty(S_r)} \|\gamma^{(n)} - \gamma^{(0)}\|_{L^\infty(S_r)} \lesssim C_b \varepsilon^2 r^{-5-2\delta}, \\ |(\Psi_{12}^{(n+1)})_{\ell=1} + \frac{1}{2} \Upsilon^{-\frac{1}{2}} (\Delta^{(n)} \Psi_7^{(n+1)})_{\ell=1}| &\lesssim C_b \varepsilon r^{-1-\delta} \cdot r^{-1} \|(\mathcal{K}, -^* \mathcal{K}) + \frac{1}{2} \Psi_3^{(n)} (\Delta^{(n)} \Psi_7^{(n+1)}, 0) + \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}\|_{L^2(S_r, \gamma^{(n)})} \\ &\lesssim C_b \varepsilon^2 r^{-5-2\delta}, \end{aligned}$$

and the second estimate implies, in view of the improved $\ell = 1$ bound for $\Psi_7^{(n+1)}$ obtained in Lemma 5.15,

$$|(\Psi_{12}^{(n+1)})_{\ell=1}| \lesssim C_b^2 \varepsilon^2 r^{-5-2\delta} + C_b \varepsilon^2 r^{-5-2\delta} \lesssim \varepsilon r^{-5-2\delta}.$$

Corollary 3.7 then, in addition, implies that the solution $\Psi_8^{(n+1)}$ to (4.12) exists and, in view of Lemma 3.2 applied to $\mathcal{D}_1 \mathcal{D}_2$,

$$r^{-1} \|\Psi_8^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim r^{-1} \cdot r^2 \|(\mathcal{K}, -^* \mathcal{K})\|_{\mathfrak{h}^{s-1}(S_r)} + r^{-1} \|\Psi_7^{(n+1)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon r^{-2-\delta}.$$

This proves the estimate for $\Psi_8^{(n+1)}$. In view of the assumption on ${}^* \mathcal{K}$ in (2.78), we have

$$r^{-1} \left\| r^{-4} \int_r^\infty r'^4 ({}^* \mathcal{K}) dr' \right\|_{\mathfrak{h}^s} \lesssim \varepsilon r^{-4-\delta},$$

and hence we obtain, by the Hodge estimate in Lemma 3.2 to (4.13),

$$r^{-1} \|\Psi_9^{(n+1)}\|_{\mathfrak{h}^{s+1}} \lesssim |\mathbf{a}| r^{-3} + \varepsilon r^{-3-\delta} \lesssim \varepsilon r^{-2-\delta}.$$

5.2.5 Proof of Proposition 4.8

We now derive the estimate for the spherical metric $\gamma^{(n+1)}$. Since $\mathcal{L}_{\partial_r}(r^{-2} \gamma^{(0)}) = 0$, the left-hand side of (4.16) can be rewritten as $\mathcal{L}_{\partial_r}(r^{-2} \gamma^{(n+1)} - r^{-2} \gamma^{(0)})$. Then, using (3.10), the equation (4.16) is equivalent to

$$\nabla_{\partial_r}^{(0)}(\gamma^{(n+1)} - \gamma^{(0)}) = 2\hat{a}^{(n)} \Psi_4^{(n+1)} + \hat{a}^{(n)} \Psi_1^{(n+1)} \gamma^{(n)} + 2\Upsilon^{\frac{1}{2}} \Psi_3^{(n+1)} r^{-1} \gamma^{(n)},$$

where we recall the notations

$$\hat{a}^{(n)} = \Upsilon^{-\frac{1}{2}} + \check{a}^{(n)} = \Upsilon^{-\frac{1}{2}} + \Psi_3^{(n)}, \quad \widetilde{\not{x}\theta}^{(n)} = \Psi_1^{(n)}, \quad \hat{\theta}^{(n)} = \Psi_4^{(n)}.$$

Since we seek solution with $\|(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s < \infty$, we have $\lim_{r \rightarrow \infty} r^{-1} \|r(\gamma^{(n+1)} - \gamma^{(0)})\|_{\mathfrak{h}^{s+1}(S_r)} = 0$. This is already stronger than what we need for applying Lemma 3.5 with $\lambda = 0$, and hence, using the improved $\mathfrak{h}^{s+1}$ bounds for $\Psi_1^{(n+1)}$, $\Psi_3^{(n+1)}$, $\Psi_4^{(n+1)}$ obtained in previous steps, we obtain

$$r^{-1} \|\gamma^{(n+1)} - \gamma^{(0)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon r^{-1-\delta}.$$

This proves Proposition 4.8.

5.3 Contraction estimates

We use the notation $\delta\psi^{(n+1)} := \psi^{(n+1)} - \psi^{(n)}$ for a general quantity ψ . We aim to show the contraction estimate $\|\delta(\Psi^{(n+2)}, \gamma^{(n+2)})\|_s \leq C\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s$ for some positive constant $C < 1$. Note again that here we define $\|\cdot\|_s$ as in (4.3) but with $\mathbf{c}_i$ and $\gamma^{(0)}$ removed.

5.3.1 The main part

We first analyze the main part regarding $(\delta\Psi_1^{(n+2)}, \delta\Psi_2^{(n+2)}, \delta\Psi_3^{(n+2)})$.

Proposition 5.16. *The quantities $\delta\Psi_1^{(n+2)}, \delta\Psi_2^{(n+2)}, \delta\Psi_3^{(n+2)}$ satisfy the following system*

$$(\partial_r + 3r^{-1})(\delta\Psi_1^{(n+2)}) = \Upsilon^{-\frac{1}{2}}(\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)})_{\ell=0} - 2(1 - 3mr^{-1})r^{-2}(\delta\Psi_3^{(n+2)}) + \mathcal{N}[\delta\Psi_1], \quad (5.48)$$

$$(\partial_r + 3r^{-1})(\delta\Psi_2^{(n+2)}) = r^{-1}(\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)})_{\ell=0} - 2\Upsilon^{\frac{1}{2}}r^{-3}(\delta\Psi_3^{(n+2)}) - \frac{1}{2}\Upsilon^{-\frac{1}{2}}(\mathcal{A}^{(n+1)}\delta\Psi_1^{(n+2)})_{\ell=1} + \mathcal{N}[\delta\Psi_2], \quad (5.49)$$

$$\Upsilon^{\frac{1}{2}}\mathcal{A}^{(n+1)}(\delta\Psi_3^{(n+2)}) = (\delta\Psi_2^{(n+2)}) - \overline{(\delta\Psi_2^{(n+2)})}^{(n+1)} - \Upsilon^{\frac{1}{2}}r^{-1}(\delta\Psi_1^{(n+2)} - \overline{\delta\Psi_1^{(n+2)}}^{(n+1)}) + \mathcal{N}[\delta\Psi_3], \quad (5.50)$$

$$\overline{\delta\Psi_3^{(n+2)}}^{(n+1)} = -\frac{1}{2}\Upsilon^{-1}r\delta\Psi_1^{(n+2)(n+1)} + \mathcal{N}_{av}[\delta\Psi_3], \quad (5.51)$$

where the remainders satisfy the bounds

$$\begin{aligned} r^{-1}\|\mathcal{N}[\delta\Psi_1]\|_{\mathfrak{h}^{s+1}(S_r)} &\lesssim \varepsilon r^{-4-2\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1}\|\mathcal{N}[\delta\Psi_3], r\mathcal{N}[\delta\Psi_2]\|_{\mathfrak{h}^s(S_r)} &\lesssim \varepsilon r^{-4-2\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1}\|\mathcal{N}_{av}[\delta\Psi_3]\|_{\mathfrak{h}^{s+1}(S_r)} &\lesssim \varepsilon r^{-2-2\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \end{aligned}$$

Proof. See Appendix D.1. □

We can then write

$$\delta\Psi_3^{(n+2)} = \Upsilon^{-\frac{1}{2}}(\mathcal{A}^{(n+1)})^{-1} \left(\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)} + \mathcal{N}[\delta\Psi_3] \right) - \frac{1}{2}\Upsilon^{-1}r\overline{\delta\Psi_1^{(n+2)}}^{(n+1)} + \mathcal{N}_{av}[\delta\Psi_3]. \quad (5.52)$$

This reduces the system to the following one for $(\delta\Psi_1^{(n+2)}, \delta\Psi_2^{(n+2)})$:

$$\begin{aligned} (\partial_r + 3r^{-1})(\delta\Psi_1^{(n+2)}) &= \Upsilon^{-\frac{1}{2}}(\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)})_{\ell=0} + (1 - 3mr^{-1})r^{-1}\Upsilon^{-1}\overline{\delta\Psi_1^{(n+2)}}^{(n+1)} \\ &\quad - 2(1 - 3mr^{-1})r^{-2}\Upsilon^{-\frac{1}{2}}(\mathcal{A}^{(n+1)})^{-1} \left(\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)} \right) \\ &\quad + \mathcal{N}[\delta\Psi_1] - 2(1 - 3mr^{-1})r^{-2} \left(\Upsilon^{-\frac{1}{2}}(\mathcal{A}^{(n+1)})^{-1}(\mathcal{N}[\delta\Psi_3]) + \mathcal{N}_{av}[\delta\Psi_3] \right), \\ (\partial_r + 3r^{-1})(\delta\Psi_2^{(n+2)}) &= r^{-1}(\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)})_{\ell=0} + \Upsilon^{-\frac{1}{2}}r^{-2}\overline{\delta\Psi_1^{(n+2)}}^{(n+1)} \\ &\quad - 2r^{-3}(\mathcal{A}^{(n+1)})^{-1} \left(\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)} \right) \\ &\quad - \frac{1}{2}\Upsilon^{-\frac{1}{2}}(\mathcal{A}^{(n+1)}\delta\Psi_1^{(n+2)})_{\ell=1} + \mathcal{N}[\delta\Psi_2] - 2r^{-3}(\mathcal{A}^{(n+1)})^{-1}(\mathcal{N}[\delta\Psi_3]) \\ &\quad - 2\Upsilon^{\frac{1}{2}}r^{-3}\mathcal{N}_{av}[\delta\Psi_3], \end{aligned}$$

which is already of the form (4.24). Moreover, we have the bounds

$$\begin{aligned} & r^{-1} \|\mathcal{N}[\delta\Psi_1] - 2(1 - 3mr^{-1})r^{-2} \left(\Upsilon^{-\frac{1}{2}} (\mathbb{A}^{(n+1)})^{-1} (\mathcal{N}[\delta\Psi_3]) + \mathcal{N}_{av}[\delta\Psi_3] \right) \|_{\mathfrak{h}^{s+1}} \\ & \lesssim \varepsilon r^{-4-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \end{aligned} \quad (5.53)$$

and

$$\begin{aligned} & r^{-1} \|\mathcal{N}[\delta\Psi_2] - 2r^{-3} (\mathbb{A}^{(n+1)})^{-1} (\mathcal{N}[\delta\Psi_3]) - 2\Upsilon^{\frac{1}{2}} r^{-3} \mathcal{N}_{av}[\delta\Psi_3]\|_{\mathfrak{h}^s} \\ & \lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \end{aligned} \quad (5.54)$$

Since $\gamma^{(n+1)}$ satisfies the first condition for $\tilde{\gamma}$ in (4.25), and $(\delta\Psi_1^{(n+2)}, \delta\Psi_2^{(n+2)})$ satisfies the condition (4.27) with $\mathbf{c}_i$ replaced by 0 in view of the boundedness result, applying Proposition 4.2 to this system, we see that $(\delta\Psi_1^{(n+2)}, \delta\Psi_2^{(n+2)})$ must coincide with the unique solution given by the proposition. Moreover, in terms of the resulting estimates for $(\delta\Psi_1^{(n+2)}, \delta\Psi_2^{(n+2)})$, since the estimates in (5.53), (5.54) have for each an additional factor of r decay compared with what is needed in Proposition 4.2, it is in fact obvious from the proof of Proposition 4.2 that one obtains a corresponding improvement for the solution:³⁶

$$r^{-1} \|r^{3+\delta} \delta\Psi_1^{(n+2)}, r^{4+\delta} \delta\Psi_2^{(n+2)}\|_{\mathfrak{h}^s(S_r)} \lesssim \varepsilon \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \quad (5.55)$$

Plugging back to (5.52), we also deduce

$$r^{-1} \|r^{2+\delta} \delta\Psi_3^{(n+1)}\|_{\mathfrak{h}^{s+2}(S_r)} \lesssim \varepsilon \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \quad (5.56)$$

We also obtain, similar to (5.43), the $\mathfrak{h}^{s+1}$ estimate for $\delta\Psi_1^{(n+1)}$ by commuting the equation (5.48) with $r(\nabla^{(0)})^{s+1}$:

$$r^{-1} \|r^{3+\delta} \delta\Psi_1^{(n+2)}\|_{\mathfrak{h}^{s+1}(S_r)} \lesssim \varepsilon \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \quad (5.57)$$

5.3.2 The remaining spatial part

Proposition 5.17. *The quantities $\delta\Psi_4^{(n+2)}$, $\delta\Psi_5^{(n+2)}$, $\delta\Psi_6^{(n+2)}$ satisfy the following system*

$$\mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} \delta\Psi_4^{(n+2)} = \frac{1}{2} (\mathbb{A}^{(n+1)} \delta\Psi_1^{(n+2)}, 0) - \delta\Psi_{11}^{(n+2)} + \mathcal{N}[\delta\Psi_4], \quad (5.58)$$

$$\mathcal{P}_1^{(n+1)} \delta\Psi_5^{(n+2)} = -(\Upsilon^{\frac{1}{2}} \mathbb{A}^{(n+1)} \delta\Psi_3^{(n+2)}, 0) + \mathcal{N}[\delta\Psi_5], \quad (5.59)$$

$$\mathcal{P}_1^{(n+1)} \delta\Psi_6^{(n+2)} = \delta\Psi_{11}^{(n+2)} - \overline{\delta\Psi_{11}^{(n+2)}}^{(n+1)} + \mathcal{N}[\delta\Psi_6]. \quad (5.60)$$

where the following bounds hold

$$r^{-1} \|\mathcal{N}[\delta\Psi_4], r^{-1} \mathcal{N}[\delta\Psi_5], \mathcal{N}[\delta\Psi_6]\|_{\mathfrak{h}^s} \lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.$$

Proof. See Appendix D.2. □

We note that the horizontal tensor $\delta\Psi_4^{(n+2)} = \Psi_4^{(n+2)} - \Psi_4^{(n+1)}$ is not strictly traceless with respect to $\gamma^{(n+1)}$. We can rewrite (5.58) as

$$\begin{aligned} & \mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} \left(\delta\Psi_4^{(n+2)} - \frac{1}{2} (\mathfrak{t}^{(n+1)} \delta\Psi_4^{(n+2)}) \gamma^{(n+1)} \right) \\ & = \frac{1}{2} (\mathbb{A}^{(n+1)} (\mathfrak{t}^{(n+1)} \Psi_4^{(n+1)}), 0) + \frac{1}{2} \mathbb{A}^{(n+1)} (\delta\Psi_1^{(n+2)}, 0) - \delta\Psi_{11}^{(n+2)} + \mathcal{N}[\delta\Psi_4], \end{aligned} \quad (5.61)$$

³⁶Or, instead, one could stay content with the improvement for the $\ell = 1$ part, which is also enough for the contraction estimates.

where we used that

$$\mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} ((\mathfrak{t}^{(n+1)} \delta \Psi_4^{(n+2)}) \gamma^{(n+1)}) = (\mathfrak{A}^{(n+1)} (\mathfrak{t}^{(n+1)} \delta \Psi_4^{(n+2)}), 0) = -(\mathfrak{A}^{(n+1)} (\mathfrak{t}^{(n+1)} \Psi_4^{(n+1)}), 0).$$

We then also have $\mathfrak{t}^{(n+1)} \Psi_4^{(n+1)} = O(\delta \gamma^{(n+1)} \cdot \Psi_4^{(n+1)})$ since $\Psi_4^{(n+1)}$ is traceless with respect to $\gamma^{(n)}$. The fact that the solution exists for (5.61) implies, using Corollary 3.7, that the $\ell \leq 1$ coefficients $\delta \Psi_{11}^{(n+2)}$ satisfy the estimate

$$\begin{aligned} |\delta \Psi_{11}^{(n+2)}| &\lesssim r^{-1} \|\mathfrak{A}^{(n+1)} (\delta \gamma^{(n+1)} \cdot \Psi_4^{(n+1)})\|_{L^2(S_r, \gamma^{(n+1)})} + r^{-1} \|\mathfrak{A}^{(n+1)} (\delta \Psi_1^{(n+2)})\|_{L^2(S_r, \gamma^{(n+1)})} \\ &\quad + r^{-1} \|\mathcal{N}[\delta \Psi_4]\|_{L^2(S_r, \gamma^{(n+1)})} \\ &\lesssim r^{-1} \|\mathfrak{A}^{(n+1)} (\delta \gamma^{(n+1)} \cdot \Psi_4^{(n+1)})\|_{L^2(S_r)} + r^{-1} \|\mathfrak{A}^{(n+1)} (\delta \Psi_1^{(n+2)})\|_{L^2(S_r)} + r^{-1} \|\mathcal{N}[\delta \Psi_4]\|_{L^2(S_r)} \\ &\lesssim \varepsilon r^{-5-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \end{aligned}$$

where we used the estimate for $\delta \Psi_1^{(n+1)}$ obtained in (5.55).

To estimate $\delta \Psi_4^{(n+2)}$, we now apply the Hodge estimate (3.4) to (5.61) and obtain

$$\begin{aligned} r^{-1} \|\delta \Psi_4^{(n+2)} - \frac{1}{2} (\mathfrak{t}^{(n+1)} \delta \Psi_4^{(n+2)}) \gamma^{(n+1)}\|_{\mathfrak{h}^{s+1}} &\lesssim \|r^2 \mathfrak{A}^{(n+1)} (\delta \gamma^{(n+1)} \cdot \Psi_4^{(n+1)})\|_{\mathfrak{h}^{s-1}} \\ &\quad + r^{-1} \|r^2 \mathfrak{A}^{(n+1)} (\delta \Psi_1^{(n+2)})\|_{\mathfrak{h}^{s-1}} + r^2 |\delta \Psi_{11}^{(n+2)}| + r^{-1} \|r^2 \mathcal{N}[\delta \Psi_4]\|_{\mathfrak{h}^{s-1}} \\ &\lesssim \varepsilon r^{-3-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \end{aligned}$$

This implies

$$r^{-1} \|\delta \Psi_4^{(n+2)}\|_{\mathfrak{h}^{s+1}} \lesssim \varepsilon r^{-3-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.$$

We then apply the Hodge estimates (3.4) to (5.59) and (5.60) to obtain, using the improved estimates for $\delta \Psi_3^{(n+2)}$ and $\delta \Psi_{11}^{(n+2)}$,

$$\begin{aligned} r^{-1} \|\delta \Psi_5^{(n+2)}\|_{\mathfrak{h}^{s+1}} &\lesssim r^{-1} \|r \mathfrak{A}^{(n+1)} \delta \Psi_3^{(n+2)}\|_{\mathfrak{h}^s} + r^{-1} \|r \mathcal{N}[\delta \Psi_5]\|_{\mathfrak{h}^s} \lesssim \varepsilon r^{-3-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1} \|\delta \Psi_6^{(n+2)}\|_{\mathfrak{h}^s} &\lesssim r |\delta \Psi_{11}^{(n+2)}| + r^{-1} \|r \mathcal{N}[\delta \Psi_6]\|_{\mathfrak{h}^{s-1}} \lesssim \varepsilon r^{-4-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \end{aligned}$$

5.3.3 The k part

Proposition 5.18. *The quantities $\delta \Psi_7^{(n+2)}$, $\delta \Psi_8^{(n+2)}$, $\delta \Psi_9^{(n+2)}$, $\delta \Psi_{10}^{(n+2)}$ satisfy the following system*

$$(\partial_r + r^{-1}) \delta \Psi_7^{(n+2)} = 2r^{-1} \delta \Psi_{10}^{(n+2)} + \mathcal{N}[\delta \Psi_7], \quad (5.62)$$

$$\mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} \left(\delta(\hat{a}^{(n+1)} \Psi_8^{(n+2)}) \right) = \frac{1}{2} (\hat{a}^{(n+1)} \mathfrak{A}^{(n+1)} \delta \Psi_7^{(n+2)}, 0) + \delta \Psi_{12}^{(n+2)} + \mathcal{N}[\delta \Psi_8], \quad (5.63)$$

$$\begin{aligned} \mathcal{P}_1^{(n+1)} \delta \Psi_9^{(n+2)} &= - \left(0, r^{-4} \int_r^\infty r'^4 \delta^* \mathcal{K}_{\ell \leq 1}^{(n+2)} dr' \right) \\ &\quad + \left(0, r^{-4} \int_r^\infty r'^4 (\delta^* \mathcal{K}_{\ell \leq 1}^{(n+2)}) dr' \right)^{(n+1)} + \mathcal{N}[\delta \Psi_9], \end{aligned} \quad (5.64)$$

$$\begin{aligned} \mathfrak{A}^{(n+1)} \left(\delta(\hat{a}^{(n+1)} \Psi_{10}^{(n+2)}) \right) &= -\frac{1}{2} \left(\hat{a}^{(n+1)} \mathfrak{A}^{(n+1)} \delta \Psi_7^{(n+2)} \right)_{\ell=1} \\ &\quad + \frac{1}{2} \left(\hat{a}^{(n+1)} \mathfrak{A}^{(n+1)} \delta \Psi_7^{(n+2)} \right)_{\ell=1}^{(n+1)} + \mathcal{N}[\delta \Psi_{10}], \end{aligned} \quad (5.65)$$

$$\overline{\delta(\hat{a}^{(n+1)} \Psi_{10}^{(n+2)})}^{(n+1)} = \mathcal{N}_{av}[\delta \Psi_{10}], \quad (5.66)$$

where the following bounds hold

$$\begin{aligned}
r^{-1} \|\mathcal{N}[\delta\Psi_7]\|_{\mathfrak{h}^{s+1}} &\lesssim \varepsilon r^{-4-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\
r^{-1} \|\mathcal{N}[\delta\Psi_8]\|_{\mathfrak{h}^{s-1}} &\lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\
r^{-1} \|\mathcal{N}[\delta\Psi_9]\|_{\mathfrak{h}^s} &\lesssim \varepsilon r^{-4-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\
r^{-1} \|\mathcal{N}[\delta\Psi_{10}]\|_{\mathfrak{h}^{s-1}} &\lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\
r^{-1} \|\mathcal{N}_{av}[\delta\Psi_{10}]\|_{\mathfrak{h}^{s+1}} &\lesssim \varepsilon r^{-3-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.
\end{aligned}$$

Proof. See Appendix D.3. $\square$

The equations (5.65), (5.66) implies

$$\delta(\hat{a}^{(n+1)}\Psi_{10}^{(n+2)}) = (\mathbb{A}^{(n+1)})^{-1} \left(-\frac{1}{2} \left(\hat{a}^{(n+1)} \mathbb{A}^{(n+1)} \delta\Psi_7^{(n+2)} \right)_{\ell=1} + \mathcal{N}[\delta\Psi_{10}] \right) + \mathcal{N}_{av}[\delta\Psi_{10}].$$

Note that $\delta(\hat{a}^{(n+1)}\Psi_{10}^{(n+2)}) = \hat{a}^{(n+1)}\delta\Psi_{10}^{(n+2)} + \delta\hat{a}^{(n+1)}\Psi_{10}^{(n+1)}$. Therefore, we obtain an expression of $\delta\Psi_{10}^{(n+2)}$. Plugging it into (5.62), we derive the equation

$$\begin{aligned}
(\partial_r + r^{-1})\delta\Psi_7^{(n+2)} &= 2r^{-1}(\hat{a}^{(n+1)})^{-1}(\mathbb{A}^{(n+1)})^{-1} \left(-\frac{1}{2} \left(\hat{a}^{(n+1)} \mathbb{A}^{(n+1)} \delta\Psi_7^{(n+2)} \right)_{\ell=1} + \mathcal{N}[\delta\Psi_{10}] \right) \\
&\quad + 2r^{-1}(\hat{a}^{(n+1)})^{-1} \left(\mathcal{N}_{av}[\delta\Psi_{10}] - \delta\hat{a}^{(n+1)}\Psi_{10}^{(n+1)} \right) + \mathcal{N}[\delta\Psi_7]
\end{aligned}$$

Recall that the boundedness result implies $r^{-1}\|\delta\Psi_7^{(n+2)}\|_{\mathfrak{h}^{s+1}} \lesssim \varepsilon r^{-2-\delta}$. This provides the vanishing condition we need, and we can proceed as in Lemma 5.15 to obtain

$$r^{-1}\|\delta\Psi_7^{(n+2)}\|_{\mathfrak{h}^{s+1}} \lesssim \varepsilon r^{-3-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \quad (5.67)$$

where we note that compared with the first estimate in (5.47), the improvement on the decay rate arises from the fact that, unlike for the equation of $\Psi_7^{(n+1)}$, here the leading contribution from the free scalar in $\mathcal{K}$ is cancelled.

Then, plugging back to the expression of $\delta\Psi_{10}^{(n+2)}$, we obtain

$$r^{-1}\|\delta(\hat{a}^{(n+1)}\Psi_{10}^{(n+2)})\|_{\mathfrak{h}^{s+1}} \lesssim \varepsilon r^{-2-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.$$

We now analyze the equation of $\delta(\hat{a}^{(n+1)}\Psi_8^{(n+2)})$. As in (5.61), since $\delta(\hat{a}^{(n+1)}\Psi_8^{(n+2)})$ is not necessarily traceless with respect to $\gamma^{(n+1)}$, we write

$$\begin{aligned}
&\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)} \left(\delta(\hat{a}^{(n+1)}\Psi_8^{(n+2)}) - \frac{1}{2}(\text{tr}^{(n+1)}\delta(\hat{a}^{(n+1)}\Psi_8^{(n+2)}))\gamma^{(n+1)} \right) \\
&= \frac{1}{2}(\mathbb{A}^{(n+1)}(\text{tr}^{(n+1)}(\hat{a}^{(n)}\Psi_8^{(n+1)})), 0) + \frac{1}{2}(\mathbb{A}^{(n+1)}\delta\Psi_7^{(n+2)}, 0) - \delta\Psi_{12}^{(n+2)} + \mathcal{N}[\delta\Psi_8],
\end{aligned} \quad (5.68)$$

and the first term on the right can be further written in the form $O(\delta\gamma^{(n+1)} \cdot \hat{a}^{(n)}\Psi_8^{(n+1)})$, using that $\Psi_8^{(n+1)}$ is traceless with respect to $\gamma^{(n)}$. Then, the fact that the solution exists for (5.68) implies the following estimate for $\delta\Psi_{12}^{(n+2)}$, using Corollary 3.7:

$$\begin{aligned}
|\delta\Psi_{12}^{(n+2)}| &\lesssim r^{-1}\|\mathbb{A}^{(n+1)}(\delta\gamma^{(n+1)} \cdot \hat{a}^{(n)}\Psi_8^{(n+1)})\|_{L^2(S, \gamma^{(n+1)})} + r^{-1}\|\mathbb{A}^{(n+1)}(\delta\Psi_7^{(n+2)})\|_{L^2(S, \gamma^{(n+1)})} \\
&\quad + r^{-1}\|\mathcal{N}[\delta\Psi_8]\|_{L^2(S, \gamma^{(n+1)})} \\
&\lesssim r^{-1}\|\mathbb{A}^{(n+1)}(\delta\gamma^{(n+1)} \cdot \hat{a}^{(n)}\Psi_8^{(n+1)})\|_{L^2} + r^{-1}\|\delta\Psi_7^{(n+2)}\|_{L^2} + r^{-1}\|\mathcal{N}[\delta\Psi_8]\|_{L^2} \\
&\lesssim \varepsilon r^{-5-\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s,
\end{aligned}$$

where we used the estimate for $\delta\Psi_7^{(n+1)}$ we just obtained in (5.67).

We now apply the Hodge estimate (3.4) to (5.61) and obtain

$$\begin{aligned}
& r^{-1} \|\delta(\hat{a}^{(n+1)}\Psi_8^{(n+2)}) - \frac{1}{2}(\mathfrak{H}^{(n+1)}\delta(\hat{a}^{(n+1)}\Psi_8^{(n+2)}))\gamma^{(n+1)}\|_{\mathfrak{h}^{s+1}} \\
& \lesssim \|r^2\mathfrak{A}^{(n+1)}(\delta\gamma^{(n+1)} \cdot \hat{a}^{(n)}\Psi_8^{(n+1)})\|_{\mathfrak{h}^{s-1}} \\
& \quad + r^{-1}\|r^2\mathfrak{A}^{(n+1)}(\delta\Psi_7^{(n+2)})\|_{\mathfrak{h}^{s-1}} + r^2\|\delta\Psi_{12}^{(n+2)}\| + r^{-1}\|r^2\mathcal{N}[\delta\Psi_8]\|_{\mathfrak{h}^{s-1}} \\
& \lesssim \varepsilon r^{-3-\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.
\end{aligned}$$

This yields the estimate

$$r^{-1}\|\delta\Psi_8^{(n+2)}\|_{\mathfrak{h}^{s+1}} \lesssim \varepsilon r^{-3-\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.$$

To conclude, we apply the Hodge estimate (3.4) to (5.64) to deduce

$$\begin{aligned}
r^{-1}\|\delta\Psi_9^{(n+2)}\|_{\mathfrak{h}^{s+1}} & \lesssim r \cdot r^{-4} \int_r^\infty r'^4 |\delta\Psi_{12}^{(n+2)}| dr' + \varepsilon r^{-3-\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s \\
& \lesssim \varepsilon r^{-3-\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.
\end{aligned}$$

5.3.4 The horizontal metric

Finally, we derive the equation of $\delta\gamma^{(n+2)}$ using (4.16)

$$\begin{aligned}
\mathcal{L}_{\partial_r}(r^{-2}\delta\gamma^{(n+2)}) & = 2r^{-2}(\hat{a}^{(n+1)})^{-1}\delta\Psi_4^{(n+2)} + (\hat{a}^{(n+1)})^{-1}\delta\Psi_1^{(n+2)}(r^{-2}\gamma^{(n+1)}) \\
& \quad + 2\Upsilon^{\frac{1}{2}}\delta\Psi_3^{(n+2)}r^{-1}(r^{-2}\gamma^{(n+1)}) + \mathcal{N}[\delta\gamma],
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}[\delta\gamma] & := 2r^{-2}\left((\hat{a}^{(n+1)})^{-1} - (\hat{a}^{(n)})^{-1}\right)\Psi_4^{(n+1)} + r^{-2}\left((\hat{a}^{(n+1)})^{-1}\gamma^{(n+1)} - (\hat{a}^{(n)})^{-1}\gamma^{(n)}\right)\Psi_1^{(n+1)} \\
& \quad + 2r^{-3}\Upsilon^{\frac{1}{2}}(\gamma^{(n+1)} - \gamma^{(n)})\Psi_3^{(n+1)}.
\end{aligned}$$

Using (3.10), the equation is equivalent to

$$\nabla_{\partial_r}^{(0)}(\delta\gamma^{(n+2)}) = 2(\hat{a}^{(n+1)})^{-1}\delta\Psi_4^{(n+2)} + (\hat{a}^{(n+1)})^{-1}\delta\Psi_1^{(n+2)}\gamma^{(n+1)} + 2\Upsilon^{\frac{1}{2}}\delta\Psi_3^{(n+2)}r^{-1}\gamma^{(n+1)} + r^2\mathcal{N}[\delta\gamma].$$

We omit the estimate of $\mathcal{N}[\delta\gamma]$ since it contains additional small and decaying factors. Integrating in the r -direction from infinity using Lemma 3.5, we obtain

$$\begin{aligned}
r^{-1}\|\delta\gamma^{(n+2)}\|_{\mathfrak{h}^{s+1}(S_r)} & \lesssim \int_r^\infty r'^{-1}\|\delta\Psi_4^{(n+2)}\|_{\mathfrak{h}^{s+1}(S_{r'})} + r'^{-1}\|\delta\Psi_1^{(n+2)}\|_{\mathfrak{h}^{s+1}(S_{r'})} \\
& \quad + r'^{-2}\|\delta\Psi_3^{(n+2)}\|_{\mathfrak{h}^{s+1}(S_{r'})} + r'^{-1}\|r^2\mathcal{N}[\delta\gamma]\|_{\mathfrak{h}^{s+1}(S_{r'})} dr' \\
& \lesssim \varepsilon r^{-2-\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.
\end{aligned}$$

5.4 The limit $(g^{(\infty)}, k^{(\infty)})$

5.4.1 Proof of Lemma 4.9

According to the lemma, we need to verify the following statements:

1. The horizontal tensors $\Psi_4^{(\infty)}$ and $\Psi_8^{(\infty)}$ are traceless with respect to $\gamma^{(\infty)}$.
2. With respect to the metric $g^{(\infty)} := (\hat{a}^{(\infty)})^2 dr^2 + \gamma^{(\infty)}$, see (4.44), the quantities $\Psi_4^{(\infty)}$ and $\Psi_1^{(\infty)} + 2\Upsilon^{\frac{1}{2}} r^{-1}$ are exactly the traceless part and the trace of the second fundamental form of the r -spheres. We hence denote $\hat{\theta}^{(\infty)} = \Psi_4^{(\infty)}$ and $\mathfrak{t}\mathfrak{t}\theta^{(\infty)} = \Psi_1^{(\infty)} + 2\Upsilon^{\frac{1}{2}} r^{-1}$ without ambiguity.
3. We have $\Psi_5^{(\infty)} = -\nabla^{(\infty)}(\log \hat{a}^{(\infty)})$, $\mu_{\ell \geq 1}^{(\infty)} = 0$, and the average of $\Psi_3^{(\infty)} + \frac{1}{2}\Upsilon^{-1} r \Psi_1^{(\infty)}$ vanishes with respect to $\gamma^{(\infty)}$.
4. For the quantity defined in (4.42), we have $\widetilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)} = \mathcal{B}_{\ell \leq 1}^{(\infty)}$, the latter being the first component of $\Psi_{11}^{(\infty)} \in \mathfrak{s}_0$.
5. Denote by $Y(g^{(\infty)})$ the horizontal tensor Y with respect to $g^{(\infty)}$ defined through (2.3). Then we have $\Psi_6^{(\infty)} = Y(g^{(\infty)})$.
6. Denote the Gauss curvature of $\gamma^{(\infty)}$ by $K(\gamma^{(\infty)})$. Then we have $\Psi_2^{(\infty)} = K(\gamma^{(\infty)}) - r^{-2}$.

Recall that, see equation (4.2),

$$\begin{aligned} \Psi_1^{(n)} &= \widetilde{\mathfrak{t}\mathfrak{t}\theta}^{(n)}, & \Psi_2^{(n)} &= \check{K}^{(n)}, & \Psi_3^{(n)} &= \check{a}^{(n)}, & \Psi_4^{(n)} &= \hat{\theta}^{(n)}, & \Psi_5^{(n)} &= p^{(n)}, & \Psi_6^{(n)} &= Y^{(n)}, \\ \Psi_7^{(n)} &= \mathfrak{t}\mathfrak{t}\Theta^{(n)}, & \Psi_8^{(n)} &= \hat{\Theta}^{(n)}, & \Psi_9^{(n)} &= \Xi^{(n)}, & \Psi_{10}^{(n)} &= \Pi^{(n)}, \\ \Psi_{11}^{(n)} &= (\mathcal{B}_{\ell \leq 1}^{(n)}, * \mathcal{B}_{\ell \leq 1}^{(n)}), & \Psi_{12}^{(n)} &= (\mathcal{K}_{\ell \leq 1}^{(n)}, * \mathcal{K}_{\ell \leq 1}^{(n)}). \end{aligned}$$

Proof. We proceed as follows:

1st statement: Since, by construction, we have $(\gamma^{(n)})^{AB}(\Psi_4^{(n+1)})_{AB} = (\gamma^{(n)})^{AB}(\Psi_8^{(n+1)})_{AB} = 0$, the first statement follows by taking the limit $n \rightarrow \infty$.

2nd statement: Note that the equation (4.43) implies the following reversed derivation of the identity (A.10) in the proof of Proposition 2.24:

$$\begin{aligned} \mathfrak{L}_{\partial_r}(r^{-2}\gamma^{(\infty)}) &= 2r^{-2}\hat{a}^{(\infty)}\Psi_4^{(\infty)} + \hat{a}^{(\infty)}\Psi_1^{(\infty)}(r^{-2}\gamma^{(\infty)}) + 2\Upsilon^{\frac{1}{2}}\Psi_3^{(\infty)}r^{-1}(r^{-2}\gamma^{(\infty)}) \\ &= 2r^{-2}\hat{a}^{(\infty)}\Psi_4^{(\infty)} + \hat{a}^{(\infty)}\Psi_1^{(\infty)}(r^{-2}\gamma^{(\infty)}) + 2\Upsilon^{\frac{1}{2}}(\hat{a}^{(\infty)} - \Upsilon^{-\frac{1}{2}})r^{-1}(r^{-2}\gamma^{(\infty)}) \\ &= 2r^{-2}\hat{a}^{(\infty)}\Psi_4^{(\infty)} + \hat{a}^{(\infty)}(\Psi_1^{(\infty)} + 2\Upsilon^{\frac{1}{2}}r^{-1})(r^{-2}\gamma^{(\infty)}) - 2r^{-1}(r^{-2}\gamma^{(\infty)}). \end{aligned} \quad (5.69)$$

Therefore, using that $\mathfrak{L}_{(\hat{a}^{(\infty)})^{-1}\partial_r}\gamma^{(\infty)} = (\hat{a}^{(\infty)})^{-1}\mathfrak{L}_{\partial_r}(r^{-2}\gamma^{(\infty)})$ by the form of $g^{(\infty)}$ and (3.9), we deduce

$$\mathfrak{L}_{(\hat{a}^{(\infty)})^{-1}\partial_r}\gamma^{(\infty)} = 2\Psi_4^{(\infty)} + (\Psi_1^{(\infty)} + 2\Upsilon^{\frac{1}{2}}r^{-1})\gamma^{(\infty)},$$

and the second statement follows.

3rd statement: To prove the third statement, note that using the precise structure of the nonlinear term, (4.35) implies $\mathcal{P}_1^{(\infty)}\Psi_5^{(\infty)} = -\mathfrak{A}^{(\infty)}(\log \hat{a}^{(\infty)})$. Since $\nabla^{(\infty)}(\log \hat{a}^{(\infty)}) = \mathfrak{d}\log(\hat{a}^{(\infty)})$ is, with respect to $\gamma^{(\infty)}$, the only curl-free 1-form whose divergence equals $\mathfrak{A}^{(\infty)}(\log \hat{a}^{(\infty)})$, we have $\Psi_5^{(\infty)} = -\nabla^{(\infty)}(\log \hat{a}^{(\infty)})$. Similarly, using the precise structure of the nonlinear term in (4.32), in particular Remark A.3 and the fact that the $\mathfrak{A}^{(\infty)}(\Gamma_0^{(\infty)} \cdot \Gamma_0^{(\infty)})$ term turns $\Upsilon^{\frac{1}{2}}\mathfrak{A}^{(\infty)}\hat{a}^{(\infty)}$ to $\mathfrak{A}^{(\infty)}(\log \hat{a}^{(\infty)})$, we have

$$\mathfrak{A}^{(\infty)}(\log \hat{a}^{(\infty)}) = \Psi_2^{(\infty)} - \overline{\Psi_2^{(\infty)}}^{(\infty)} - \frac{1}{4}(\mathfrak{t}\mathfrak{t}\theta^{(\infty)})^2 + \frac{1}{4}(\overline{\mathfrak{t}\mathfrak{t}\theta^{(\infty)}})^{2(\infty)}.$$

Therefore, we have

$$\mu_{\ell \geq 1}^{(\infty)} = \left(-\mathfrak{A}^{(\infty)}(\log \hat{a}^{(\infty)}) + \Psi_2^{(\infty)} - \frac{1}{4}(\mathfrak{t}\mathfrak{t}\theta^{(\infty)})^2 \right)_{\ell \geq 1} = 0.$$

The relation $\overline{\Psi_3^{(\infty)}}^{(\infty)} = -\frac{1}{2}\Upsilon^{-1}r\overline{\Psi_1^{(\infty)}}^{(\infty)}$ is also justified by taking the limit of the equation (4.7).

4th statement: We have pointed out in (4.42) that

$$\begin{aligned}\tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)} &= \tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(\infty)} = \frac{1}{2}(\mathbb{A}^{(\infty)} \widetilde{\mathfrak{t}\mathfrak{t}\theta}^{(\infty)})_{\ell \leq 1} - \left(\mathcal{P}_1(\mathcal{P}_1^{(\infty)} \mathcal{P}_2^{(\infty)} \widehat{\theta}^{(\infty)}) \right)_{\ell \leq 1} \\ &= \left(\frac{1}{2} \mathbb{A}^{(\infty)} \widetilde{\mathfrak{t}\mathfrak{t}\theta}^{(\infty)} - \mathfrak{d}\mathfrak{I}^{(\infty)} \mathfrak{d}\mathfrak{I}^{(\infty)} \widehat{\theta}^{(\infty)} \right)_{\ell \leq 1}.\end{aligned}$$

Comparing this with (4.34) projected to $\ell \leq 1$ and using that $\mathcal{B}_{\ell \geq 1} = 0$, we see that $\tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)} = \mathcal{B}_{\ell \leq 1}^{(\infty)}$. This proves the fourth statement.

5th statement: In view of the 2nd statement, we have established that the limit $\theta^{(\infty)} = \widehat{\theta}^{(\infty)} + \frac{1}{2}\mathfrak{t}\mathfrak{t}\theta^{(\infty)}\gamma^{(\infty)}$ is in fact the second fundamental form of the r -foliation with respect to the metric $g^{(\infty)}$. We can therefore make use of the unconditional equation (2.24) of Proposition 2.11, according to which,

$$\mathfrak{d}\mathfrak{I}^{(\infty)} \widehat{\theta}^{(\infty)} = \frac{1}{2} \nabla^{(\infty)} \mathfrak{t}\mathfrak{t}\theta^{(\infty)} - Y(g^{(\infty)}). \quad (5.70)$$

Taking $\mathcal{P}_1^{(\infty)}$ of (5.70) and comparing it with (4.34), we deduce that

$$\mathcal{P}_1^{(\infty)} Y(g^{(\infty)}) = (\mathcal{B}, * \mathcal{B}) + (\mathcal{B}_{\ell \leq 1}^{(\infty)}, * \mathcal{B}_{\ell \leq 1}^{(\infty)}).$$

Comparing it with (4.36), we have

$$\mathcal{P}_1^{(\infty)} (\Psi_6^{(\infty)} - Y(g^{(\infty)})) = \overline{(\mathcal{B}, * \mathcal{B}) + (\mathcal{B}_{\ell \leq 1}^{(\infty)}, * \mathcal{B}_{\ell \leq 1}^{(\infty)})}^{(\infty)}.$$

Taking the spherical mean over $\gamma^{(\infty)}$, we see that the right-hand side is in fact zero. Therefore, we obtain that $\Psi_6^{(\infty)} = Y(g^{(\infty)})$ using the injectivity of $\mathcal{P}_1^{(\infty)}$. This proves the fifth statement.

6th statement: Using the previous statements, we appeal to the unconditional equation (2.53) for $\check{K}(\gamma^{(\infty)}) := K(\gamma^{(\infty)}) - r^{-2}$, applied to the metric $g^{(\infty)}$,³⁷ as

$$\partial_r \check{K}(\gamma^{(\infty)}) = r^{-1} \check{\mu}(g^{(\infty)}) - \hat{a}^{(\infty)} \mathfrak{d}\mathfrak{I}^{(\infty)} Y^{(\infty)} - 3r^{-1} \check{K}(\gamma^{(\infty)}) - 2\Upsilon^{\frac{1}{2}} r^{-3} \check{a}^{(\infty)} + \Gamma_1(g^{(\infty)}) \cdot \Gamma_2(g^{(\infty)}). \quad (5.71)$$

Here

$$\check{\mu}(g^{(\infty)}) := -\mathbb{A}^{(\infty)} (\log \hat{a}^{(\infty)}) + K(\gamma^{(\infty)}) - \frac{1}{4} (\mathfrak{t}\mathfrak{t}\theta^{(\infty)})^2 - 2mr^{-3},$$

and, due to our previous statements as well as Remark A.2, we have $\Gamma_1(g^{(\infty)}) = \Gamma_1^{(\infty)}$ and $\Gamma_2(g^{(\infty)}) = \Gamma_2^{(\infty)}$, with the exception that, whenever $\check{K}^{(\infty)}$ appears, it is replaced by $\check{K}(\gamma^{(\infty)})$. On the other hand, since we have proved that $\mu_{\ell \geq 1}^{(\infty)} = 0$, $\tilde{\mathcal{B}}_{\ell \leq 1}^{(\infty)} = \mathcal{B}_{\ell \leq 1}^{(\infty)}$, and $\mathfrak{d}\mathfrak{I}^{(\infty)} Y^{(\infty)} = \mathcal{B} + \mathcal{B}_{\ell \leq 1}^{(\infty)}$, the equation (4.31) reads

$$(\partial_r + 3r^{-1}) \Psi_2^{(\infty)} = r^{-1} \check{\mu}^{(\infty)} - 2\Upsilon^{\frac{1}{2}} r^{-3} \Psi_3^{(\infty)} - \hat{a}^{(\infty)} \mathfrak{d}\mathfrak{I}^{(\infty)} Y^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)}. \quad (5.72)$$

Note that since they both originate from (2.53), the schematic forms in (5.71) and (5.72) have the same expression,³⁸ apart from the difference between $\check{K}(\gamma^{(\infty)})$ and $\check{K}^{(\infty)}$. Therefore, taking the difference between (5.72) and (5.71), we obtain

$$\partial_r \left(\check{K}^{(\infty)} - \check{K}(\gamma^{(\infty)}) \right) = -2r^{-1} \left(\check{K}^{(\infty)} - \check{K}(\gamma^{(\infty)}) \right) + \Gamma_1^{(\infty)} \cdot \left(\check{K}^{(\infty)} - \check{K}(\gamma^{(\infty)}) \right).$$

³⁷The derivation of unconditional equation (2.53) is independent of $k^{(\infty)}$, see Remark A.2.

³⁸The precise expressions of the schematic terms $\Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)}$, $\Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)}$, etc. can be tracked down from the corresponding terms in the derivation of the equations in Proposition 2.24.

We already know that $r^3 \check{K}^{(\infty)} = r^3 \Psi_2^{(\infty)} \rightarrow 0$ by the boundedness of $\Psi^{(\infty)}$ in $\|\cdot\|_s$. Moreover, since $\gamma^{(\infty)}$ and its $r \nabla^{(0)}$ derivatives decay at the rate $r^{-1-\delta}$, we deduce that $\check{K}(\gamma^{(\infty)}) = K(\gamma^{(\infty)}) - r^{-2}$ decays at the rate $r^{-3-\delta}$, so in particular $\lim_{r \rightarrow \infty} r^3 \check{K}(\gamma^{(\infty)}) \rightarrow 0$. This is stronger than the condition $\lim_{r \rightarrow \infty} r^2 (\check{K}^{(\infty)} - \check{K}(\gamma^{(\infty)})) = 0$ needed here, and hence, using that $\Gamma_1^{(\infty)} = O(\varepsilon r^{-2-\delta})$, we integrate from infinity to obtain $\check{K}^{(\infty)} = \check{K}(\gamma^{(\infty)})$, i.e., $K^{(\infty)} = K(\gamma^{(\infty)})$. $\square$

5.4.2 Proof of Proposition 4.10

The goal is to prove that $(g^{(\infty)}, k^{(\infty)})$ solves the Einstein constraint equation (1.1), where $g^{(\infty)}$ and $k^{(\infty)}$ are defined respectively in (4.44) and (4.45).

Throughout this proof, we use the shorthand notation

$$\mathcal{C}_{Ham}^{(\infty)} := \mathcal{C}_{Ham}(g^{(\infty)}, k^{(\infty)}), \quad \mathcal{C}_{Mom}^{(\infty)} := \mathcal{C}_{Mom}(g^{(\infty)}, k^{(\infty)}), \quad \mathcal{C}_{Mom}^{(\infty)} := \mathcal{C}_{Mom}(g^{(\infty)}, k^{(\infty)}).$$

The way of defining $k^{(\infty)}$ in (4.45) implies

$$\Psi_7^{(\infty)} = \not\!t \Theta(g^{(\infty)}, k^{(\infty)}), \quad \Psi_8^{(\infty)} = \widehat{\Theta}(g^{(\infty)}, k^{(\infty)}), \quad \Psi_9 = \Xi(g^{(\infty)}, k^{(\infty)}), \quad \Psi_{10} = \Pi(g^{(\infty)}, k^{(\infty)}).$$

Therefore we will denote them by $\not\!t \Theta^{(\infty)}$, $\widehat{\Theta}^{(\infty)}$, $\Xi^{(\infty)}$, and $\Pi^{(\infty)}$ without ambiguity. Together with the statements in Lemma 4.9, we see that now all quantities in $\Gamma_1^{(\infty)}$ and $\Gamma_2^{(\infty)}$ have no ambiguities.

Using $\mu_{\ell \geq 1}^{(\infty)} = 0$, and $K^{(\infty)} = K(\gamma^{(\infty)})$ from Lemma 4.9, the equation (4.30) implies

$$(\partial_r + 2r^{-1}) \widetilde{\not\!t \theta}^{(\infty)} = \hat{a}^{(\infty)} \check{\mu}^{(\infty)} - 2(1 - 3mr^{-1}) r^{-2} \check{a}^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)},$$

We now apply the unconditional equation (2.52), noting that we have shown in Lemma 4.9 that $\not\!t \theta^{(\infty)}$ is the $N^{(\infty)}$ -expansion with respect to $g^{(\infty)}$ and $\check{\mu}(g^{(\infty)}) = \check{\mu}^{(\infty)}$,

$$(\partial_r + 2r^{-1}) \widetilde{\not\!t \theta}^{(\infty)} = \hat{a}^{(\infty)} \check{\mu}^{(\infty)} - 2(1 - 3mr^{-2}) r^{-2} \check{a}^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)} - \frac{1}{2} \hat{a}^{(\infty)} \mathcal{C}_{Ham}^{(\infty)}.$$

Comparing the two equations, using that the schematic terms in fact have the same algebraic expression, we deduce $\mathcal{C}_{Ham}^{(\infty)} = 0$.

We now prove that the momentum constraint also vanishes. The first component of the equation (4.39) reads

$$\not\!d \not\!t v^{(\infty)} \Xi^{(\infty)} = 0.$$

Using this, the unconditional equation (2.59) applied to $(g^{(\infty)}, k^{(\infty)})$ reads

$$\partial_r \not\!t \Theta^{(\infty)} = 2r^{-1} \Pi^{(\infty)} - r^{-1} \not\!t \Theta^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)} - \hat{a}^{(\infty)} (\mathcal{C}_{Mom}^{(\infty)})_{N^{(\infty)}}, \quad (5.73)$$

Comparing (5.73) with (4.37) and using that $\Gamma_1^{(\infty)} \cdot \Gamma_1^{(\infty)}$ in the two equations have the same algebraic expression, we obtain $(\mathcal{C}_{Mom}^{(\infty)})_{N^{(\infty)}} = 0$.

The unconditional equations (2.60)-(2.61) read

$$\begin{aligned} (\partial_r + 4r^{-1}) \not\!d \not\!t v^{(\infty)} \Xi^{(\infty)} &= -\not\!d \not\!t v^{(\infty)} \not\!d \not\!t v^{(\infty)} (\hat{a}^{(\infty)} \widehat{\Theta}^{(\infty)}) + \frac{1}{2} \hat{a}^{(\infty)} \not\!d \not\!t v^{(\infty)} \not\!d \not\!t v^{(\infty)} \not\!t \Theta^{(\infty)} + \frac{1}{2} \not\!d \not\!t v^{(\infty)} (\hat{a}^{(\infty)} \Pi^{(\infty)}) \\ &\quad + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)} + \not\!d \not\!t v^{(\infty)} (\hat{a}^{(\infty)} \mathcal{C}_{Mom}^{(\infty)}), \end{aligned} \quad (5.74)$$

and

$$r^{-4} \partial_r (r^4 \not\!c \not\!t r l^{(\infty)} \Xi^{(\infty)}) = -\not\!c \not\!t r l^{(\infty)} \not\!d \not\!t v^{(\infty)} (\hat{a}^{(\infty)} \widehat{\Theta}^{(\infty)}) + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)} + \not\!c \not\!t r l^{(\infty)} (\hat{a}^{(\infty)} \mathcal{C}_{Mom}^{(\infty)}). \quad (5.75)$$

On the other hand, the two components of (4.38) read, respectively,

$$\mathbf{d}\mathbf{I}_V^{(\infty)} \mathbf{d}\mathbf{I}_V^{(\infty)} (\hat{a}^{(\infty)} \Psi_8^{(\infty)}) = \frac{1}{2} \hat{a}^{(\infty)} \mathbb{A}^{(\infty)} \Psi_7^{(\infty)} + \mathcal{K} + \mathcal{K}_{\ell \leq 1}^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)}, \quad (5.76)$$

$$\mathbf{c}\mathbf{I}_V^{(\infty)} \mathbf{d}\mathbf{I}_V^{(\infty)} (\hat{a}^{(\infty)} \Psi_8^{(\infty)}) = -{}^* \mathcal{K} + {}^* \mathcal{K}_{\ell \leq 1}^{(\infty)} + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)}, \quad (5.77)$$

Note that (4.40) implies $\mathbb{A}^{(\infty)} (\hat{a}^{(\infty)} \Pi^{(\infty)}) = \mathcal{K} + \mathcal{K}_{\ell \leq 1}^{(\infty)} - \overline{\mathcal{K} + \mathcal{K}_{\ell \leq 1}^{(\infty)}}^{(\infty)}$. Therefore, compare (5.76) with (5.74), we obtain

$$\mathbf{d}\mathbf{I}_V^{(\infty)} (\hat{a}^{(\infty)} \mathcal{C}_{Mom}^{(\infty)}) = \overline{\mathcal{K} + \mathcal{K}_{\ell \leq 1}^{(\infty)}}^{(\infty)}. \quad (5.78)$$

The second component of (4.39) implies

$$r^{-4} \partial_r (r^4 \mathbf{c}\mathbf{I}_V^{(\infty)} \Xi^{(\infty)}) = {}^* \mathcal{K} - {}^* \mathcal{K}_{\ell \leq 1}^{(\infty)} - \partial_r \left(\int_r^\infty r'^4 ({}^* \mathcal{K} - {}^* \mathcal{K}_{\ell \leq 1}^{(\infty)}) dr' \right)^{(\infty)},$$

where the last term on the right is only dependent on r , and we denote it by $F(r)$. Plugging this into (5.75), we obtain

$${}^* \mathcal{K} - {}^* \mathcal{K}_{\ell \leq 1}^{(\infty)} - F(r) = -\mathbf{c}\mathbf{I}_V^{(\infty)} \mathbf{d}\mathbf{I}_V^{(\infty)} (\hat{a}^{(\infty)} \hat{\Theta}^{(\infty)}) + \Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)} + \mathbf{c}\mathbf{I}_V^{(\infty)} (\hat{a}^{(\infty)} \mathcal{C}_{Mom}^{(\infty)}).$$

Comparing this with (5.77) and noting that the $\Gamma_1^{(\infty)} \cdot \Gamma_2^{(\infty)}$ terms of (5.75) and (5.77) come from the same equation, hence have identical algebraic expressions, we deduce

$$\mathbf{c}\mathbf{I}_V^{(\infty)} (\hat{a}^{(\infty)} \mathcal{C}_{Mom}^{(\infty)}) = F(r). \quad (5.79)$$

Therefore, taking the spherical averages of (5.78) and (5.79) over $\gamma^{(\infty)}$, we see that $\overline{\mathcal{K} + \mathcal{K}_{\ell \leq 1}^{(\infty)}}^{(\infty)} = F(r) = 0$. Then (5.78) and (5.79) together read $\mathcal{P}_1^{(\infty)} \mathcal{C}_{Mom}^{(\infty)} = 0$, and hence we obtain $\mathcal{C}_{Mom}^{(\infty)} = 0$. $\square$

A Derivation of Horizontal Constraint System

A.1 Proof of Proposition 2.11

We have

$$\nabla_a N = \theta_{ab} e_b, \quad \nabla_a e_b = \nabla_a e_b - \theta_{ab} N, \quad \nabla_N e_a = \nabla_N e_a - p_a N, \quad \nabla_N N = p_a e_a.$$

Therefore, for a 1-form w on Σ , we have

$$\begin{aligned} \nabla_N w_N &= N(w_N) - w(\nabla_N N) = \nabla_N(w_N) - p_a w_a, \\ \nabla_N w_a &= N(w_a) - w(\nabla_N e_a) = \nabla_N w_a - w(\nabla_N e_a - \nabla_N e_a) = \nabla_N w_a + p_a w_N, \\ \nabla_a w_N &= e_a(w_N) - w(\nabla_a N) = \nabla_a(w_N) - \theta_{ab} w_b, \\ \nabla_a w_b &= e_a(w_b) - w(\nabla_a e_b) = \nabla_a w_b - w(\nabla_a e_b - \nabla_a e_b) = \nabla_a w_b + \theta_{ab} w_N, \end{aligned} \quad (A.1)$$

and similar rules apply for tensors of higher ranks.

We now derive³⁹

$$\begin{aligned}
\nabla_a p_b &= e_a(p_b) - p(\nabla_a e_b) = e_a(g(\nabla_N N, e_b)) - g(\nabla_N N, \nabla_a e_b) \\
&= e_a(g(\nabla_N N, e_b)) - g(\nabla_N N, \nabla_a e_b) = g(\nabla_a(\nabla_N N), e_b) \\
&= g(\nabla_N(\nabla_a N), e_b) + g(\nabla_{[e_a, N]} N, e_b) + R(e_a, N, e_b, N) \\
&= g(\nabla_N(\theta_{ac} e_c), e_b) + g(\nabla_{\nabla_a N} N, e_b) - g(\nabla_{\nabla_N e_a} N, e_b) + R(e_a, N, e_b, N) \\
&= g(\nabla_N(\theta_{ac} e_c), e_b) + \theta_{ac} g(\nabla_c N, e_b) + p_a g(\nabla_N N, e_a) - g(\nabla_{\nabla_N e_a} N, e_b) + R_{aNbN} \\
&= g(N(\theta_{ac}) e_c + \theta_{ac} \nabla_N e_c, e_b) + \theta_{ac} g(\nabla_c N, e_b) + p_a g(\nabla_N N, e_a) - g(\nabla_{\nabla_N e_a} N, e_b) + R_{aNbN} \\
&= \nabla_N \theta_{ab} + \theta_{ac} \theta_{cb} + p_a p_b + R_{aNbN} + cov,
\end{aligned}$$

where, for the term cov that contains $\nabla_N e_a$ type terms,

$$\begin{aligned}
cov &= g(\theta_{ad} g(\nabla_N e_c, e_d) e_c + \theta_{dc} g(\nabla_N e_a, e_d) e_c + \theta_{ac} \nabla_N e_c, e_b) - g(\nabla_N e_a, e_c) g(\nabla_c N, e_b) \\
&= \theta_{ad} g(\nabla_N e_b, e_d) + \theta_{db} g(\nabla_N e_a, e_d) + \theta_{ac} g(\nabla_N e_c, e_b) - \theta_{cb} g(\nabla_N e_a, e_c) \\
&= \theta_{ac} g(\nabla_N e_b, e_c) + \theta_{cb} g(\nabla_N e_a, e_c) - \theta_{ac} g(\nabla_N e_b, e_c) - \theta_{cb} g(\nabla_N e_a, e_c) \\
&= 0.
\end{aligned}$$

Therefore, we obtain

$$\nabla_N \theta_{ab} = \nabla_a p_b - \theta_{ac} \theta_{cb} - p_a p_b - R_{aNbN}. \quad (\text{A.2})$$

Note that $\theta_{ac} \theta_{cb} = (\hat{\theta}_{ac} + \frac{1}{2} \text{tr} \theta \delta_{ac})(\hat{\theta}_{cb} + \frac{1}{2} \text{tr} \theta \delta_{cb}) = \hat{\theta}_{ac} \hat{\theta}_{cb} + \frac{1}{4} (\text{tr} \theta)^2 \delta_{ab}$. The trace part of the equation (A.2) reads

$$\nabla_N \text{tr} \theta = \text{div} p - |\hat{\theta}|^2 - \frac{1}{2} (\text{tr} \theta)^2 - |p|^2 - \text{tr} R. \quad (\text{A.3})$$

This proves (2.22). Also note that $\hat{\theta}_{ac} \hat{\theta}_{cb}$ only has trace part (equation (2.2.3) in [9]), i.e., $\hat{\theta}_{ac} \hat{\theta}_{cb} = \frac{1}{2} |\hat{\theta}|^2 \delta_{ab}$. Therefore the traceless part of (A.2) reads

$$\nabla_N \hat{\theta} = \nabla \hat{\otimes} p - \text{tr} \theta \hat{\theta} - p \hat{\otimes} p - \hat{R}. \quad (\text{A.4})$$

The Gauss curvature. The Gauss equation implies

$$\begin{aligned}
R_{abab} &= \hat{R}_{abab} - \theta_{aa} \theta_{bb} + \theta_{ab} \theta_{ba} \\
&= 2K - (\text{tr} \theta)^2 + \theta \cdot \theta \\
&= 2K - \frac{1}{2} (\text{tr} \theta)^2 + |\hat{\theta}|^2,
\end{aligned}$$

i.e.,

$$2K = R_{abab} - |\hat{\theta}|^2 + \frac{1}{2} (\text{tr} \theta)^2. \quad (\text{A.5})$$

This proves (2.23).

The Codazzi equation. We have

$$R_{Nabc} = \nabla_c \theta_{ba} - \nabla_b \theta_{ca}.$$

Note that the equation only has two independent components. Contracting a and b we obtain

$$\begin{aligned}
R_{Naac} &= \nabla_c \theta_{aa} - \nabla_a \theta_{ca} = \nabla_c \text{tr} \theta - \nabla_a \left(\hat{\theta}_{ac} + \frac{1}{2} \text{tr} \theta \delta_{ac} \right) \\
&= -(\text{div} \hat{\theta})_c + \frac{1}{2} \nabla_c (\text{tr} \theta),
\end{aligned}$$

³⁹Within the following displayed equation, $\nabla_X \nabla_Y$ means $(X^i \nabla_i)(Y^j \nabla_j)$.

i.e.,

$$\mathrm{d}\hat{\nu}\hat{\theta} = \frac{1}{2}\nabla\hat{\theta}\hat{\theta} - Y. \quad (\text{A.6})$$

This proves (2.24).

We also have the following Bianchi-type equations.

Lemma A.1. *We have*

$$\nabla_N(R_{abab} - |\hat{\theta}|^2) = -\hat{\theta}\theta R_{abab} - 2\mathrm{d}\hat{\nu}Y + 4p \cdot Y + \hat{\theta}\theta\hat{\theta}\hat{\theta} + 2\hat{\theta}\theta|\hat{\theta}|^2 - 2\hat{\theta} \cdot (\nabla\hat{\theta}p - p\hat{\theta}p).$$

Proof. We have the Bianchi identity

$$\nabla_N R_{abcd} + \nabla_b R_{Nacd} + \nabla_a R_{bNcd} = 0$$

i.e., using the rules (A.1),

$$\begin{aligned} & \nabla_N R_{abcd} + p_a R_{Nbcd} + p_b R_{aNcd} + p_c R_{abNd} + p_d R_{abcN} \\ & + \nabla_b R_{Nacd} - \theta_{be} R_{eacd} + \theta_{bc} R_{NaNd} + \theta_{bd} R_{NacN} \\ & + \nabla_a R_{bNcd} - \theta_{ae} R_{becd} + \theta_{ac} R_{bNNd} + \theta_{ad} R_{bNCN} = 0. \end{aligned}$$

Then, contracting with both δ^{ac} and δ^{bd} , using that $\theta_{be} R_{eaaab} = \frac{1}{2}\hat{\theta}\theta R_{baab} = -\frac{1}{2}\hat{\theta}\theta R_{abab}$,⁴⁰ we deduce

$$\begin{aligned} & \nabla_N R_{abab} - 4p_a Y_a + \nabla_b Y_b + \frac{1}{2}\hat{\theta}\theta R_{abab} \\ & + \theta_{ba} \left(\frac{1}{2}\hat{\theta}\theta\delta_{ab} + \hat{\mathcal{R}}_{ab} \right) - \hat{\theta}\theta\hat{\theta}\hat{\theta} + \nabla_a Y_a + \frac{1}{2}\hat{\theta}\theta R_{abab} - \hat{\theta}\theta\hat{\theta}\hat{\theta} + \theta_{ab} \left(\frac{1}{2}\hat{\theta}\theta\delta_{ba} + \hat{\mathcal{R}}_{ba} \right) = 0. \end{aligned}$$

Therefore, we obtain

$$\nabla_N R_{abab} + \hat{\theta}\theta R_{abab} = -2\mathrm{d}\hat{\nu}Y + 4p \cdot Y + \hat{\theta}\theta\hat{\theta}\hat{\theta} - 2\hat{\theta} \cdot \hat{\mathcal{R}}.$$

Using (A.4), we also obtain

$$\begin{aligned} \nabla_N(R_{abab} - |\hat{\theta}|^2) &= -\hat{\theta}\theta R_{abab} - 2\mathrm{d}\hat{\nu}Y + 4p \cdot Y + \hat{\theta}\theta\hat{\theta}\hat{\theta} - 2\hat{\theta} \cdot \hat{\mathcal{R}} \\ &\quad - 2\hat{\theta} \cdot (\nabla\hat{\theta}p - \hat{\theta}\hat{\theta} - p\hat{\theta}p - \hat{\mathcal{R}}) \\ &= -\hat{\theta}\theta R_{abab} - 2\mathrm{d}\hat{\nu}Y + 4p \cdot Y + \hat{\theta}\theta\hat{\theta}\hat{\theta} + 2\hat{\theta}\theta|\hat{\theta}|^2 - 2\hat{\theta} \cdot (\nabla\hat{\theta}p - p\hat{\theta}p), \end{aligned}$$

as required. This concludes the proof of Lemma A.1. $\square$

We then further derive the equation of K using (A.5) and (A.3):

$$\begin{aligned} 2\nabla_N K &= \nabla_N \left(R_{abab} - |\hat{\theta}|^2 + \frac{1}{2}(\hat{\theta}\theta)^2 \right) \\ &= -\hat{\theta}\theta R_{abab} - 2\mathrm{d}\hat{\nu}Y + 4p \cdot Y + \hat{\theta}\theta\hat{\theta}\hat{\theta} + 2\hat{\theta}\theta|\hat{\theta}|^2 - 2\hat{\theta} \cdot (\nabla\hat{\theta}p - p\hat{\theta}p) \\ &\quad + \hat{\theta}\theta \left(\mathrm{d}\hat{\nu}p - |\hat{\theta}|^2 - \frac{1}{2}(\hat{\theta}\theta)^2 - |p|^2 - \hat{\theta}\hat{\theta} \right) \\ &= -\hat{\theta}\theta \left(2K - \frac{1}{2}(\hat{\theta}\theta)^2 + |\hat{\theta}|^2 \right) - 2\mathrm{d}\hat{\nu}Y + 4p \cdot Y + \hat{\theta}\theta\hat{\theta}\hat{\theta} + 2\hat{\theta}\theta|\hat{\theta}|^2 - 2\hat{\theta} \cdot (\nabla\hat{\theta}p - p\hat{\theta}p) \\ &\quad + \hat{\theta}\theta \left(\mathrm{d}\hat{\nu}p - |\hat{\theta}|^2 - \frac{1}{2}(\hat{\theta}\theta)^2 - |p|^2 - \hat{\theta}\hat{\theta} \right) \\ &= -2\mathrm{d}\hat{\nu}Y - 2\hat{\theta}\theta K + 4p \cdot Y - 2\hat{\theta} \cdot (\nabla\hat{\theta}p - p\hat{\theta}p) + \hat{\theta}\theta(\mathrm{d}\hat{\nu}p - |p|^2). \end{aligned}$$

⁴⁰Indeed, we have $R_{1aa1} = R_{2aa2}$ and $R_{1aa2} = R_{2aa1} = 0$, and hence $R_{eaaab} = \frac{1}{2}R_{caac}\delta_{eb}$.

Therefore,

$$\nabla_N K = -\text{div} Y - \text{tr} \theta K + 2p \cdot Y - \widehat{\theta} \cdot (\nabla \widehat{\otimes} p - p \widehat{\otimes} p) + \frac{1}{2} \text{tr} \theta (\text{div} p - |p|^2).$$

This proves (2.26). In particular, the identity does not contain $\text{tr} \mathcal{R}$.

A.2 Proof of Proposition 2.12

We can express the constraint quantities defined in (2.27) and (2.28) as follows, using the rules in (A.1),

$$\begin{aligned} (\mathcal{C}_{Mom})_N &= \nabla_i k_{iN} - N(\text{tr} k) = \nabla_a k_{aN} + \nabla_N k_{NN} - N(\text{tr} \Theta + k_{NN}) \\ &= \nabla_a k_{aN} + \text{tr} \theta k_{NN} - \theta_{ab} k_{ab} + N\Pi - p_a k_{Na} - p_a k_{aN} - N(\text{tr} \Theta) - N\Pi \\ &= \text{div} \Xi + \text{tr} \theta \Pi - \theta \cdot \Theta - 2p \cdot \Xi - \nabla_N \text{tr} \Theta, \\ &= \text{div} \Xi + \text{tr} \theta \Pi - \widehat{\theta} \cdot \widehat{\Theta} - \frac{1}{2} \text{tr} \theta \text{tr} \Theta - 2p \cdot \Xi - \nabla_N \text{tr} \Theta, \end{aligned}$$

and

$$\begin{aligned} (\mathcal{C}_{Mom})_a := (\mathcal{C}_{Mom})_a &= \nabla_i k_{ia} - \nabla_a(\text{tr} k) = \nabla_N k_{Na} + \nabla_b k_{ba} - \nabla_a(\text{tr} k) \\ &= \nabla_N k_{Na} - p_b k_{ba} + p_a k_{NN} + \nabla_b k_{ba} + \text{tr} \theta k_{Na} + \theta_{ba} k_{bN} - \nabla_a \text{tr} \Theta - \nabla_a(k_{NN}) \\ &= (\nabla_N \Xi + \text{div} \Theta - p \cdot \Theta + \Pi p + \text{tr} \theta \Xi + \theta \cdot \Xi - \nabla \text{tr} \Theta - \nabla \Pi)_a \\ &= \left(\nabla_N \Xi + \text{div} \widehat{\Theta} - p \cdot \Theta + \Pi p + \frac{3}{2} \text{tr} \theta \Xi + \widehat{\theta} \cdot \Xi - \frac{1}{2} \nabla \text{tr} \Theta - \nabla \Pi \right)_a. \end{aligned}$$

Then the first two equations in Proposition 2.12 follow. To prove the third one, we first calculate

$$\begin{aligned} \mathcal{C}_{Ham} &= R_g + (k_{NN} + \text{tr} \Theta)^2 - (k_{NN})^2 - 2|\Xi|^2 - |\Theta|^2 \\ &= R_g + 2k_{NN} \text{tr} \Theta + (\text{tr} \Theta)^2 - 2|\Xi|^2 - |\widehat{\Theta}|^2 - \frac{1}{2}(\text{tr} \Theta)^2 \\ &= R_g + 2\Pi \text{tr} \Theta + \frac{1}{2}(\text{tr} \Theta)^2 - 2|\Xi|^2 - |\widehat{\Theta}|^2. \end{aligned}$$

Combining this relation with the unconditional equations (2.22), (2.23), and the identity $\text{tr} \mathcal{R} = -\frac{1}{2} R_{abab} + \frac{1}{2} R_g$ from (2.21), we get

$$\begin{aligned} \nabla_N \text{tr} \theta &= \text{div} p - |\widehat{\theta}|^2 - \frac{1}{2}(\text{tr} \theta)^2 - |p|^2 - \text{tr} \mathcal{R} \\ &= \text{div} p - |\widehat{\theta}|^2 - \frac{1}{2}(\text{tr} \theta)^2 - |p|^2 + \frac{1}{2} R_{abab} - \frac{1}{2} R_g \\ &= \text{div} p - |\widehat{\theta}|^2 - \frac{1}{2}(\text{tr} \theta)^2 - |p|^2 + K - \frac{1}{4}(\text{tr} \theta)^2 + \frac{1}{2} |\widehat{\theta}|^2 - \frac{1}{2} R_g \\ &= \text{div} p - |\widehat{\theta}|^2 - \frac{1}{2}(\text{tr} \theta)^2 - |p|^2 + K - \frac{1}{4}(\text{tr} \theta)^2 + \frac{1}{2} |\widehat{\theta}|^2 \\ &\quad + \frac{1}{2} \left(2\Pi \text{tr} \Theta + \frac{1}{2}(\text{tr} \Theta)^2 - 2|\Xi|^2 - |\widehat{\Theta}|^2 - \mathcal{C}_{Ham} \right) \\ &= \text{div} p - \frac{1}{2} |\widehat{\theta}|^2 - \frac{3}{4}(\text{tr} \theta)^2 - |p|^2 + K + \Pi \text{tr} \Theta + \frac{1}{4}(\text{tr} \Theta)^2 - |\Xi|^2 - \frac{1}{2} |\widehat{\Theta}|^2 - \frac{1}{2} \mathcal{C}_{Ham}. \end{aligned}$$

This proves the third identity in Proposition 2.12.

A.3 Proof of Proposition 2.24

For the standard Schwarzschild, we have, with $\Upsilon := 1 - 2mr^{-1}$,

$$\mathfrak{t}\mathfrak{t}\theta^{(0)} = 2\Upsilon^{\frac{1}{2}}r^{-1}, \quad \hat{a}^{(0)} = \Upsilon^{-\frac{1}{2}}, \quad N^{(0)} = \Upsilon^{\frac{1}{2}}\partial_r, \quad K^{(0)} = r^{-2}.$$

The structure equations hold true:

$$\partial_r \mathfrak{t}\mathfrak{t}\theta^{(0)} = \hat{a}^{(0)}r^{-2} - \frac{3}{4}\hat{a}^{(0)}(\mathfrak{t}\mathfrak{t}\theta^{(0)})^2, \quad (\text{A.7})$$

$$\partial_r K^{(0)} = -\hat{a}^{(0)}\mathfrak{t}\mathfrak{t}\theta^{(0)}r^{-2}. \quad (\text{A.8})$$

In view of the expression of μ in (2.7) and the relation (2.9) between p and $\hat{a}$, the equation (2.31) can be written as

$$\nabla_N \mathfrak{t}\mathfrak{t}\theta = \mu - \frac{1}{2}(\mathfrak{t}\mathfrak{t}\theta)^2 - \frac{1}{2}|\widehat{\theta}|^2 - |p|^2 + \Pi \mathfrak{t}\mathfrak{t}\Theta + \frac{1}{4}(\mathfrak{t}\mathfrak{t}\Theta)^2 - |\Xi|^2 - \frac{1}{2}|\widehat{\Theta}|^2 - \frac{1}{2}\mathcal{C}_{Ham}.$$

Therefore, using $N = \hat{a}^{-1}\partial_r$, subtracting (A.7), we have, recalling the schematic notations introduced in Definition 2.22,

$$\begin{aligned} \partial_r \widetilde{\mathfrak{t}\mathfrak{t}\theta} &= \hat{a}\mu - \frac{1}{2}\hat{a}(\mathfrak{t}\mathfrak{t}\theta)^2 - \hat{a}^{(0)}r^{-2} + \frac{3}{4}\hat{a}^{(0)}(\mathfrak{t}\mathfrak{t}\theta^{(0)})^2 + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} \\ &= \hat{a}\mu + \left(\frac{1}{4}\hat{a}(\mathfrak{t}\mathfrak{t}\theta)^2 - \hat{a}^{(0)}r^{-2}\right) + \left(\frac{3}{4}\hat{a}^{(0)}(\mathfrak{t}\mathfrak{t}\theta^{(0)})^2 - \frac{3}{4}(\hat{a}^{(0)} + \check{a})(\mathfrak{t}\mathfrak{t}\theta)^2\right) + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} \\ &= \hat{a}\mu + \left(\frac{1}{4}\hat{a}^{(0)}(\mathfrak{t}\mathfrak{t}\theta)^2 + \frac{1}{4}\check{a}(\mathfrak{t}\mathfrak{t}\theta)^2 - \hat{a}^{(0)}r^{-2}\right) + \frac{3}{4}\hat{a}^{(0)}\left((\mathfrak{t}\mathfrak{t}\theta^{(0)})^2 - (\mathfrak{t}\mathfrak{t}\theta)^2\right) - \frac{3}{4}\check{a}(\mathfrak{t}\mathfrak{t}\theta)^2 \\ &\quad + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} \\ &= \hat{a}\mu + \hat{a}^{(0)}\left(\frac{1}{4}(\mathfrak{t}\mathfrak{t}\theta)^2 - r^{-2}\right) + \frac{1}{4}\check{a}\left(\widetilde{\mathfrak{t}\mathfrak{t}\theta} + 2\Upsilon^{\frac{1}{2}}r^{-1}\right)^2 - \frac{3}{4}\hat{a}^{(0)}\widetilde{\mathfrak{t}\mathfrak{t}\theta}\left(2\mathfrak{t}\mathfrak{t}\theta^{(0)} + \widetilde{\mathfrak{t}\mathfrak{t}\theta}\right) - \frac{3}{4}\check{a}(\mathfrak{t}\mathfrak{t}\theta)^2 \\ &\quad + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} \\ &= \hat{a}\mu + \hat{a}^{(0)}\left(\Upsilon r^{-2} + \Upsilon^{\frac{1}{2}}r^{-1}\widetilde{\mathfrak{t}\mathfrak{t}\theta} - r^{-2}\right) + \frac{1}{4}\check{a}\left(\widetilde{\mathfrak{t}\mathfrak{t}\theta} + 2\Upsilon^{\frac{1}{2}}r^{-1}\right)^2 - \frac{3}{4}\hat{a}^{(0)}\widetilde{\mathfrak{t}\mathfrak{t}\theta}\left(2\mathfrak{t}\mathfrak{t}\theta^{(0)}\right) \\ &\quad - \frac{3}{4}\check{a}\left(\widetilde{\mathfrak{t}\mathfrak{t}\theta} + 2\Upsilon^{\frac{1}{2}}r^{-1}\right)^2 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} + \Gamma_1 \cdot \Gamma_1 \\ &= \hat{a}\mu + \hat{a}^{(0)}(\Upsilon - 1)r^{-2} + r^{-1}\widetilde{\mathfrak{t}\mathfrak{t}\theta} + \Upsilon r^{-2}\check{a} - 3r^{-1}\widetilde{\mathfrak{t}\mathfrak{t}\theta} - 3\Upsilon r^{-2}\check{a} + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} \\ &= (\Upsilon^{-\frac{1}{2}} + \check{a})(\check{\mu} + 2mr^{-3}) + \Upsilon^{-\frac{1}{2}}(\Upsilon - 1)r^{-2} - 2r^{-1}\widetilde{\mathfrak{t}\mathfrak{t}\theta} - 2\Upsilon r^{-2}\check{a} + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} \\ &= (\Upsilon^{-\frac{1}{2}} + \check{a})\check{\mu} + 2mr^{-3}\check{a} - 2r^{-1}\widetilde{\mathfrak{t}\mathfrak{t}\theta} - 2\Upsilon r^{-2}\check{a} + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham} \\ &= (\Upsilon^{-\frac{1}{2}} + \check{a})\check{\mu} - 2r^{-1}\widetilde{\mathfrak{t}\mathfrak{t}\theta} - 2(1 - 3mr^{-1})r^{-2}\check{a} + \Gamma_1 \cdot \Gamma_1 - \frac{1}{2}\hat{a}\mathcal{C}_{Ham}. \end{aligned}$$

This proves (2.52).

We now derive the equation for $\check{K}$ by subtracting (A.8) from (2.26). Using $\text{div} p = \mu + \frac{1}{4}(\mathfrak{t}\mathfrak{t}\theta)^2 - \check{K} - r^{-2}$,

we have

$$\begin{aligned}
\partial_r \check{K} &= -\hat{a} \text{div} Y - \hat{a} \text{tr} \theta K + \frac{1}{2} \hat{a} \text{tr} \theta \text{div} p + \Gamma_1 \cdot \Gamma_2 + \hat{a}^{(0)} \text{tr} \theta^{(0)} r^{-2} \\
&= -\hat{a} \text{div} Y - \hat{a} \text{tr} \theta \check{K} - (\hat{a} \text{tr} \theta - \hat{a}^{(0)} \text{tr} \theta^{(0)}) r^{-2} + \frac{1}{2} \hat{a} \text{tr} \theta \text{div} p + \Gamma_1 \cdot \Gamma_2 \\
&= -\hat{a} \text{div} Y - \hat{a}^{(0)} \text{tr} \theta^{(0)} \check{K} - \check{\alpha} \text{tr} \theta r^{-2} - \hat{a}^{(0)} \widetilde{\text{tr} \theta} r^{-2} + \frac{1}{2} \hat{a}^{(0)} \text{tr} \theta^{(0)} \text{div} p + \Gamma_1 \cdot \Gamma_2, \\
&= -\hat{a} \text{div} Y - 2r^{-1} \check{K} - 2\Upsilon^{\frac{1}{2}} r^{-3} \check{\alpha} - \Upsilon^{-\frac{1}{2}} r^{-2} \widetilde{\text{tr} \theta} + r^{-1} (\mu + \frac{1}{4} (\text{tr} \theta)^2 - \check{K} - r^{-2}) + \Gamma_1 \cdot \Gamma_2 \\
&= -\hat{a} \text{div} Y - 3r^{-1} \check{K} - 2\Upsilon^{\frac{1}{2}} r^{-3} \check{\alpha} - \Upsilon^{-\frac{1}{2}} r^{-2} \widetilde{\text{tr} \theta} + r^{-1} \mu + \Upsilon^{\frac{1}{2}} r^{-2} \widetilde{\text{tr} \theta} + (\Upsilon - 1) r^{-3} + \Gamma_1 \cdot \Gamma_2 \\
&= -\hat{a} \text{div} Y - 3r^{-1} \check{K} - 2\Upsilon^{\frac{1}{2}} r^{-3} \check{\alpha} + r^{-1} \mu - (2mr^{-1}) r^{-3} + \Gamma_1 \cdot \Gamma_2 \\
&= -\hat{a} \text{div} Y - 3r^{-1} \check{K} - 2\Upsilon^{\frac{1}{2}} r^{-3} \check{\alpha} + r^{-1} \check{\mu} + \Gamma_1 \cdot \Gamma_2.
\end{aligned}$$

This proves (2.53).

Remark A.2. We note that here Γ_1 only involves $r^{-1} \check{\alpha}$, $\widetilde{\text{tr} \theta}$, $\hat{\theta}$, p , and Γ_2 only involves Y , $\check{K}$, ∇p .

To prove (2.54), note that

$$\begin{aligned}
\mathbb{A}(\log(\hat{a})) &= \mathbb{A}(\log(\hat{a}^{(0)} + \check{\alpha})) = \mathbb{A} \left(\log \hat{a}^{(0)} + \log(1 + \frac{\check{\alpha}}{\hat{a}^{(0)}}) \right) = \mathbb{A} \left(\frac{\check{\alpha}}{\hat{a}^{(0)}} + \Gamma_0 \cdot \Gamma_0 \right) \\
&= \Upsilon^{\frac{1}{2}} \mathbb{A} \check{\alpha} + \mathbb{A}(\Gamma_0 \cdot \Gamma_0).
\end{aligned} \tag{A.9}$$

Therefore,

$$\begin{aligned}
\Upsilon^{\frac{1}{2}} \mathbb{A} \check{\alpha} &= -\mathbb{A}(\Gamma_0 \cdot \Gamma_0) + K - \frac{1}{4} (\text{tr} \theta)^2 - \mu = -\mathbb{A}(\Gamma_0 \cdot \Gamma_0) + K - \Upsilon^{\frac{1}{2}} r^{-1} \widetilde{\text{tr} \theta} - \Upsilon r^{-2} + \Gamma_1 \cdot \Gamma_1 - \mu \\
&= \check{K} - \Upsilon^{\frac{1}{2}} r^{-1} \widetilde{\text{tr} \theta} - \mu + (1 - \Upsilon) r^{-2} - \mathbb{A}(\Gamma_0 \cdot \Gamma_0) + \Gamma_1 \cdot \Gamma_1.
\end{aligned}$$

Remark A.3. We note that here the $\Gamma_1 \cdot \Gamma_1$ term in fact only consists of $(\widetilde{\text{tr} \theta})^2$.

To prove (2.58), recall that

$$\mathring{\mathcal{L}}_N \gamma_{AB} = 2\theta_{AB} = 2\hat{\theta}_{AB} + \text{tr} \theta \gamma_{AB}.$$

From (3.9) we know that $\mathring{\mathcal{L}}_N \gamma_{AB} = \hat{a}^{-1} \mathring{\mathcal{L}}_{\partial_r} \gamma_{AB}$. We then have

$$\begin{aligned}
\mathring{\mathcal{L}}_{\partial_r} (r^{-2} \gamma) &= 2r^{-2} \hat{a} \hat{\theta} + \hat{a} (\widetilde{\text{tr} \theta} + 2\Upsilon^{\frac{1}{2}} r^{-1}) (r^{-2} \gamma) - 2r^{-1} (r^{-2} \gamma) \\
&= 2r^{-2} \hat{a} \hat{\theta} + \hat{a} \text{tr} \theta (r^{-2} \gamma) + 2\hat{a} \Upsilon^{\frac{1}{2}} r^{-1} (r^{-2} \gamma) - 2r^{-1} (r^{-2} \gamma) \\
&= 2r^{-2} \hat{a} \hat{\theta} + \hat{a} \widetilde{\text{tr} \theta} (r^{-2} \gamma) + \Upsilon^{\frac{1}{2}} (\hat{a} - \Upsilon^{-\frac{1}{2}}) 2r^{-1} (r^{-2} \gamma) \\
&= 2r^{-2} \hat{a} \hat{\theta} + \hat{a} \text{tr} \theta (r^{-2} \gamma) + 2\Upsilon^{\frac{1}{2}} \check{\alpha} r^{-1} (r^{-2} \gamma).
\end{aligned} \tag{A.10}$$

The equations (2.56) and (2.57) directly follow by taking $\mathcal{P}_1$ of (2.24) and (2.9) respectively.⁴¹ The equation (2.59) is the same as (2.31) with both sides multiplied by $\hat{a}$, except also taking into account that $\text{tr} \theta = 2\Upsilon^{\frac{1}{2}} r^{-1} + \widetilde{\text{tr} \theta}$ and $\hat{a} = \Upsilon^{-\frac{1}{2}} + \check{\alpha}$.

To derive the equations (2.60) and (2.61), we first note that, using $p = -\nabla(\log \hat{a})$ by (2.9),

$$-\text{div} \hat{\Theta} + p \cdot \hat{\Theta} = -\text{div} \hat{\Theta} - \hat{a}^{-1} \nabla(\hat{a}) \cdot \hat{\Theta} = -\hat{a}^{-1} \text{div}(\hat{a} \hat{\Theta}), \quad \nabla \Pi - p \Pi = \nabla \Pi + \hat{a}^{-1} \nabla(\hat{a}) \Pi = \hat{a}^{-1} \nabla(\hat{a} \Pi).$$

⁴¹For the latter, we also use (A.9).

Similarly, we also have $\nabla_a(\mathcal{C}_{Mom})_b - p_a \cdot (\mathcal{C}_{Mom})_b = \hat{a}^{-1} \nabla_a(\hat{a} \mathcal{C}_{Mom})_b$. Then (2.30) can be rewritten as

$$\nabla_N \Xi = -\hat{a}^{-1} \text{div}(\hat{a} \hat{\Theta}) + \frac{1}{2} \text{tr} \Theta p - \frac{3}{2} \text{tr} \theta \Xi - \hat{\theta} \cdot \Xi + \frac{1}{2} \nabla \text{tr} \Theta + \hat{a}^{-1} \nabla(\hat{a} \Pi) + \mathcal{C}_{Mom}. \quad (\text{A.11})$$

Then we recall the commutation formula (3.6), which, when applied to (A.11), gives

$$\begin{aligned} \nabla_N \nabla_a \Xi_b &= -\nabla_a(\hat{a}^{-1} \text{div}(\hat{a} \hat{\Theta}))_b - \frac{3}{2} \text{tr} \theta \nabla_a \Xi_b + \nabla_a(\Gamma_1 \cdot \Gamma_1) + \frac{1}{2} \nabla_a \nabla_b \text{tr} \Theta + \nabla_a(\hat{a}^{-1} \nabla_b(\hat{a} \Pi)) + \nabla_a(\mathcal{C}_{Mom})_b \\ &\quad - \frac{1}{2} \text{tr} \theta \nabla_a \Xi_b + \Gamma_1 \cdot \nabla \Gamma_1 - p_a(\nabla_N \Xi_b) + \Gamma_2 \cdot \Xi \\ &= -\nabla_a(\hat{a}^{-1} \text{div}(\hat{a} \hat{\Theta}))_b - \frac{3}{2} \text{tr} \theta \nabla_a \Xi_b + \nabla_a(\Gamma_1 \cdot \Gamma_1) + \frac{1}{2} \nabla_a \nabla_b \text{tr} \Theta + \nabla_a(\hat{a}^{-1} \nabla_b(\hat{a} \Pi)) + \nabla_a(\mathcal{C}_{Mom})_b \\ &\quad - \frac{1}{2} \text{tr} \theta \nabla_a \Xi_b - \Gamma_1 \cdot (\nabla \Gamma_1, r^{-1} \Gamma_1) + p_a(\hat{a}^{-1} \text{div}(\hat{a} \hat{\Theta}) - \hat{a}^{-1} \nabla(\hat{a} \Pi))_b + \Gamma_2 \cdot \Xi - p_a(\mathcal{C}_{Mom})_b \\ &= -\hat{a}^{-1} \nabla_a(\text{div}(\hat{a} \hat{\Theta}))_b - \frac{3}{2} \text{tr} \theta \nabla_a \Xi_b + \nabla_a(\Gamma_1 \cdot \Gamma_1) + \frac{1}{2} \nabla_a \nabla_b \text{tr} \Theta + \hat{a}^{-1} \nabla_a(\nabla_b(\hat{a} \Pi)) + \hat{a}^{-1} \nabla_a(\hat{a} \mathcal{C}_{Mom})_b \\ &\quad - \frac{1}{2} \text{tr} \theta \nabla_a \Xi_b - \Gamma_1 \cdot (\nabla \Gamma_1, r^{-1} \Gamma_1) + \Gamma_2 \cdot \Xi \end{aligned}$$

where we used $\nabla_N \Xi = -\hat{a}^{-1} \text{div}(\hat{a} \hat{\Theta}) + \hat{a}^{-1} \nabla(\hat{a} \Pi) + (\nabla \Gamma_1, r^{-1} \Gamma_1) + \mathcal{C}_{Mom}$. Hence,

$$\begin{aligned} \nabla_N \text{div} \Xi &= -\hat{a}^{-1} \text{div} \text{div}(\hat{a} \hat{\Theta}) - 2 \text{tr} \theta \text{div} \Xi + \frac{1}{2} \Delta \text{tr} \Theta + \frac{1}{2} \hat{a}^{-1} \Delta(\hat{a} \Pi) + \hat{a}^{-1} \text{div}(\hat{a} \mathcal{C}_{Mom}) + \Gamma_1 \cdot \Gamma_2, \\ \nabla_N \text{curl} \Xi &= -\hat{a}^{-1} \text{curl} \text{div}(\hat{a} \hat{\Theta}) - 2 \text{tr} \theta \text{curl} \Xi + \hat{a}^{-1} \text{curl}(\hat{a} \mathcal{C}_{Mom}) + \Gamma_1 \cdot \Gamma_2. \end{aligned}$$

The equations (2.60) and (2.61) then follow by multiplying both sides by $\hat{a}$, and using that $\text{tr} \theta = 2\Upsilon^{\frac{1}{2}} r^{-1} + \Gamma_1$, $\hat{a} = \Upsilon^{-\frac{1}{2}} + \Gamma_0$.

B Computation in spacetime notations

B.1 The null structure and Bianchi equations

We recall the null structure and Bianchi equations for Einstein-vacuum spacetime, given in full generality in [32], [22].

Proposition B.1 (Null structure equations). *The connection coefficients verify the following equations:*

$$\begin{aligned} \nabla_3 \text{tr} \underline{\chi} &= -|\hat{\underline{\chi}}|^2 - \frac{1}{2} (\text{tr} \underline{\chi}^2 - {}^{(a)} \text{tr} \underline{\chi}^2) + 2 \text{div} \underline{\xi} - 2 \omega \text{tr} \underline{\chi} + 2 \underline{\xi} \cdot (\eta + \underline{\eta} - 2 \zeta), \\ \nabla_3 {}^{(a)} \text{tr} \underline{\chi} &= -\text{tr} \underline{\chi} {}^{(a)} \text{tr} \underline{\chi} + 2 \text{curl} \underline{\xi} - 2 \omega {}^{(a)} \text{tr} \underline{\chi} + 2 \underline{\xi} \wedge (-\eta + \underline{\eta} + 2 \zeta), \\ \nabla_3 \hat{\underline{\chi}} &= -\text{tr} \underline{\chi} \hat{\underline{\chi}} + \nabla \hat{\underline{\xi}} - 2 \omega \hat{\underline{\chi}} + \hat{\underline{\xi}} \wedge (\eta + \underline{\eta} - 2 \zeta) - \underline{\alpha}, \\ \nabla_3 \text{tr} \chi &= -\hat{\chi} \cdot \hat{\chi} - \frac{1}{2} \text{tr} \chi \text{tr} \chi + \frac{1}{2} {}^{(a)} \text{tr} \chi {}^{(a)} \text{tr} \chi + 2 \text{div} \eta + 2 \omega \text{tr} \chi + 2(\xi \cdot \underline{\xi} + |\eta|^2) + 2\rho, \\ \nabla_3 {}^{(a)} \text{tr} \chi &= -\hat{\chi} \wedge \hat{\chi} - \frac{1}{2} ({}^{(a)} \text{tr} \chi \text{tr} \chi + \text{tr} \chi {}^{(a)} \text{tr} \chi) + 2 \text{curl} \eta + 2 \omega {}^{(a)} \text{tr} \chi + 2 \underline{\xi} \wedge \xi - 2^* \rho, \\ \nabla_3 \hat{\chi} &= -\frac{1}{2} (\text{tr} \chi \hat{\chi} + \text{tr} \chi \hat{\chi}) - \frac{1}{2} (-^* \hat{\chi} {}^{(a)} \text{tr} \chi + ^* \hat{\chi} {}^{(a)} \text{tr} \chi) + \nabla \hat{\eta} + 2 \omega \hat{\chi} + \hat{\underline{\xi}} \wedge \xi + \eta \hat{\otimes} \eta, \\ \nabla_4 \text{tr} \underline{\chi} &= -\hat{\chi} \cdot \hat{\chi} - \frac{1}{2} \text{tr} \chi \text{tr} \underline{\chi} + \frac{1}{2} {}^{(a)} \text{tr} \chi {}^{(a)} \text{tr} \underline{\chi} + 2 \text{div} \underline{\eta} + 2 \omega \text{tr} \underline{\chi} + 2(\xi \cdot \underline{\xi} + |\underline{\eta}|^2) + 2\rho, \\ \nabla_4 {}^{(a)} \text{tr} \underline{\chi} &= -\hat{\chi} \wedge \hat{\chi} - \frac{1}{2} ({}^{(a)} \text{tr} \chi \text{tr} \underline{\chi} + \text{tr} \chi {}^{(a)} \text{tr} \underline{\chi}) + 2 \text{curl} \underline{\eta} + 2 \omega {}^{(a)} \text{tr} \underline{\chi} + 2 \underline{\xi} \wedge \underline{\xi} + 2^* \rho, \\ \nabla_4 \hat{\underline{\chi}} &= -\frac{1}{2} (\text{tr} \chi \hat{\chi} + \text{tr} \chi \hat{\chi}) - \frac{1}{2} (-^* \hat{\chi} {}^{(a)} \text{tr} \underline{\chi} + ^* \hat{\chi} {}^{(a)} \text{tr} \underline{\chi}) + \nabla \hat{\underline{\eta}} + 2 \omega \hat{\underline{\chi}} + \hat{\xi} \wedge \underline{\xi} + \underline{\eta} \hat{\otimes} \underline{\eta}, \end{aligned}$$

$$\begin{aligned}
\nabla_4 \mathfrak{t}\chi &= -|\widehat{\chi}|^2 - \frac{1}{2}(\mathfrak{t}\chi^2 - {}^{(a)}\mathfrak{t}\chi^2) + 2\mathfrak{d}\mathfrak{f}\mathfrak{v}\xi - 2\omega\mathfrak{t}\chi + 2\xi \cdot (\underline{\eta} + \eta + 2\zeta), \\
\nabla_4 {}^{(a)}\mathfrak{t}\chi &= -\mathfrak{t}\chi {}^{(a)}\mathfrak{t}\chi + 2\mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}\xi - 2\omega {}^{(a)}\mathfrak{t}\chi + 2\xi \wedge (-\underline{\eta} + \eta - 2\zeta), \\
\nabla_4 \widehat{\chi} &= -\mathfrak{t}\chi \widehat{\chi} + \nabla \widehat{\otimes} \xi - 2\omega \widehat{\chi} + \xi \widehat{\otimes} (\underline{\eta} + \eta + 2\zeta) - \alpha.
\end{aligned}$$

Also,

$$\begin{aligned}
\nabla_3 \zeta + 2\nabla \omega &= -\widehat{\chi} \cdot (\zeta + \eta) - \frac{1}{2}\mathfrak{t}\chi \underline{\chi}(\zeta + \eta) - \frac{1}{2} {}^{(a)}\mathfrak{t}\chi \underline{\chi}({}^*\zeta + {}^*\eta) + 2\omega(\zeta - \eta) \\
&\quad + \widehat{\chi} \cdot \underline{\xi} + \frac{1}{2}\mathfrak{t}\chi \underline{\chi} \underline{\xi} + \frac{1}{2} {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\underline{\xi} + 2\omega \underline{\xi} - \underline{\beta}, \\
\nabla_4 \zeta - 2\nabla \omega &= \widehat{\chi} \cdot (-\zeta + \underline{\eta}) + \frac{1}{2}\mathfrak{t}\chi \underline{\chi}(-\zeta + \underline{\eta}) + \frac{1}{2} {}^{(a)}\mathfrak{t}\chi \underline{\chi}(-{}^*\zeta + {}^*\underline{\eta}) + 2\omega(\zeta + \underline{\eta}) \\
&\quad - \widehat{\chi} \cdot \underline{\xi} - \frac{1}{2}\mathfrak{t}\chi \underline{\chi} \underline{\xi} - \frac{1}{2} {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\underline{\xi} - 2\omega \underline{\xi} - \underline{\beta}, \\
\nabla_3 \underline{\eta} - \nabla_4 \underline{\xi} &= -\widehat{\chi} \cdot (\underline{\eta} - \eta) - \frac{1}{2}\mathfrak{t}\chi \underline{\chi}(\underline{\eta} - \eta) + \frac{1}{2} {}^{(a)}\mathfrak{t}\chi \underline{\chi}({}^*\underline{\eta} - {}^*\eta) - 4\omega \underline{\xi} + \underline{\beta}, \\
\nabla_4 \eta - \nabla_3 \underline{\xi} &= -\widehat{\chi} \cdot (\eta - \underline{\eta}) - \frac{1}{2}\mathfrak{t}\chi \underline{\chi}(\eta - \underline{\eta}) + \frac{1}{2} {}^{(a)}\mathfrak{t}\chi \underline{\chi}({}^*\eta - {}^*\underline{\eta}) - 4\omega \underline{\xi} - \underline{\beta},
\end{aligned}$$

and

$$\nabla_3 \omega + \nabla_4 \omega - 4\omega \omega - \xi \cdot \underline{\xi} - (\eta - \underline{\eta}) \cdot \zeta + \eta \cdot \underline{\eta} = \rho.$$

Also,

$$\begin{aligned}
\mathfrak{d}\mathfrak{f}\mathfrak{v}\widehat{\chi} + \zeta \cdot \widehat{\chi} &= \frac{1}{2}\nabla \mathfrak{t}\chi + \frac{1}{2}\mathfrak{t}\chi \zeta - \frac{1}{2} {}^*\nabla {}^{(a)}\mathfrak{t}\chi - \frac{1}{2} {}^{(a)}\mathfrak{t}\chi {}^*\zeta - {}^{(a)}\mathfrak{t}\chi {}^*\eta - {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\xi - \underline{\beta}, \\
\mathfrak{d}\mathfrak{f}\mathfrak{v}\widehat{\chi} - \zeta \cdot \widehat{\chi} &= \frac{1}{2}\nabla \mathfrak{t}\chi - \frac{1}{2}\mathfrak{t}\chi \zeta - \frac{1}{2} {}^*\nabla {}^{(a)}\mathfrak{t}\chi + \frac{1}{2} {}^{(a)}\mathfrak{t}\chi {}^*\zeta - {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\eta - {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\xi + \underline{\beta},
\end{aligned}$$

and

$$\mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l}\zeta = -\frac{1}{2}\widehat{\chi} \wedge \widehat{\chi} + \frac{1}{4}(\mathfrak{t}\chi {}^{(a)}\mathfrak{t}\chi \underline{\chi} - \mathfrak{t}\chi \underline{\chi} {}^{(a)}\mathfrak{t}\chi) + \omega {}^{(a)}\mathfrak{t}\chi \underline{\chi} - \omega {}^{(a)}\mathfrak{t}\chi + {}^*\rho.$$

Proposition B.2 (Null Bianchi identities). *The curvature components verify the following equations:*

$$\begin{aligned}
\nabla_3 \alpha - \nabla \widehat{\otimes} \beta &= -\frac{1}{2}(\mathfrak{t}\chi \underline{\chi} \alpha + {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\alpha) + 4\omega \alpha + (\zeta + 4\underline{\eta}) \widehat{\otimes} \beta - 3(\rho \widehat{\chi} + {}^*\rho {}^*\widehat{\chi}), \\
\nabla_4 \beta - \mathfrak{d}\mathfrak{f}\mathfrak{v} \alpha &= -2(\mathfrak{t}\chi \beta - {}^{(a)}\mathfrak{t}\chi {}^*\beta) - 2\omega \beta + \alpha \cdot (2\zeta + \underline{\eta}) + 3(\xi \rho + {}^*\xi {}^*\rho), \\
\nabla_3 \beta + \mathfrak{d}\mathfrak{f}\mathfrak{v} \varrho &= -(\mathfrak{t}\chi \underline{\chi} \beta + {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\beta) + 2\omega \beta + 2\underline{\beta} \cdot \widehat{\chi} + 3(\rho \eta + {}^*\rho {}^*\eta) + \alpha \cdot \underline{\xi}, \\
\nabla_4 \rho - \mathfrak{d}\mathfrak{f}\mathfrak{v} \beta &= -\frac{3}{2}(\mathfrak{t}\chi \rho + {}^{(a)}\mathfrak{t}\chi {}^*\rho) + (2\underline{\eta} + \zeta) \cdot \beta - 2\xi \cdot \underline{\beta} - \frac{1}{2}\widehat{\chi} \cdot \alpha, \\
\nabla_4 {}^*\rho + \mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l} \beta &= -\frac{3}{2}(\mathfrak{t}\chi {}^*\rho - {}^{(a)}\mathfrak{t}\chi \chi \rho) - (2\underline{\eta} + \zeta) \cdot {}^*\beta - 2\xi \cdot {}^*\underline{\beta} + \frac{1}{2}\widehat{\chi} \cdot {}^*\alpha, \\
\nabla_3 \rho + \mathfrak{d}\mathfrak{f}\mathfrak{v} \underline{\beta} &= -\frac{3}{2}(\mathfrak{t}\chi \underline{\chi} \rho - {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\rho) - (2\underline{\eta} - \zeta) \cdot \underline{\beta} + 2\xi \cdot \underline{\beta} - \frac{1}{2}\widehat{\chi} \cdot \underline{\alpha}, \\
\nabla_3 {}^*\rho + \mathfrak{c}\mathfrak{u}\mathfrak{r}\mathfrak{l} \underline{\beta} &= -\frac{3}{2}(\mathfrak{t}\chi \underline{\chi} {}^*\rho + {}^{(a)}\mathfrak{t}\chi \underline{\chi} \rho) - (2\underline{\eta} - \zeta) \cdot {}^*\underline{\beta} - 2\xi \cdot {}^*\underline{\beta} - \frac{1}{2}\widehat{\chi} \cdot {}^*\underline{\alpha}, \\
\nabla_4 \underline{\beta} - \mathfrak{d}\mathfrak{f}\mathfrak{v} \underline{\varrho} &= -(\mathfrak{t}\chi \underline{\chi} \underline{\beta} + {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\underline{\beta}) + 2\omega \underline{\beta} + 2\underline{\beta} \cdot \widehat{\chi} - 3(\rho \underline{\eta} - {}^*\rho {}^*\underline{\eta}) - \underline{\alpha} \cdot \underline{\xi}, \\
\nabla_3 \underline{\beta} + \mathfrak{d}\mathfrak{f}\mathfrak{v} \underline{\alpha} &= -2(\mathfrak{t}\chi \underline{\chi} \underline{\beta} - {}^{(a)}\mathfrak{t}\chi \underline{\chi} {}^*\underline{\beta}) - 2\omega \underline{\beta} - \underline{\alpha} \cdot (-2\zeta + \eta) - 3(\underline{\xi} \rho - {}^*\underline{\xi} {}^*\rho), \\
\nabla_4 \underline{\alpha} + \nabla \widehat{\otimes} \underline{\beta} &= -\frac{1}{2}(\mathfrak{t}\chi \chi \underline{\alpha} - {}^{(a)}\mathfrak{t}\chi {}^*\underline{\alpha}) + 4\omega \underline{\alpha} + (\zeta - 4\underline{\eta}) \widehat{\otimes} \underline{\beta} - 3(\rho \widehat{\chi} - {}^*\rho {}^*\widehat{\chi}).
\end{aligned}$$

Here,

$$\begin{aligned}
\mathfrak{d}\mathfrak{f}\mathfrak{v} \varrho &= -(\nabla \rho + {}^*\nabla {}^*\rho), \\
\mathfrak{d}\mathfrak{f}\mathfrak{v} \underline{\varrho} &= -(\nabla \rho - {}^*\nabla {}^*\rho).
\end{aligned}$$

B.2 Proof of Proposition 2.17

We have, using $\mathbf{D}_3 e_3 = \mathbf{D}_3 e_4 = 0$,

$$\begin{aligned}\omega &= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3) = \frac{1}{4} \mathbf{g}(\mathbf{D}_{2N+e_3} e_4, e_3) = \frac{1}{2} \mathbf{g}(\mathbf{D}_N(T+N), T-N) + \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_4, e_3) \\ &= \frac{1}{2} \mathbf{g}(\mathbf{D}_N(T+N), T-N) = \frac{1}{2} \mathbf{g}(\mathbf{D}_N T, -N) + \frac{1}{2} \mathbf{g}(\mathbf{D}_N N, T) \\ &= -k(N, N).\end{aligned}$$

This proves (2.40). We also have

$$\begin{aligned}\Xi_a &= \mathbf{g}(\mathbf{D}_N T, e_a) = \mathbf{g}\left(\mathbf{D}_{\frac{1}{2}e_4 - \frac{1}{2}e_3}\left(\frac{1}{2}e_3 + \frac{1}{2}e_4\right), e_a\right) \\ &= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_3, e_a) + \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_a) \\ &= \frac{1}{2} \xi_a + \frac{1}{2} \eta_a,\end{aligned}\tag{B.1}$$

and

$$\begin{aligned}p_a &= g(\nabla_N N, e_a) = \mathbf{g}(\mathbf{D}_N N, e_a) = \mathbf{g}\left(\mathbf{D}_N\left(\frac{1}{2}e_4 - \frac{1}{2}e_3\right), e_a\right) = \frac{1}{2} \mathbf{g}(\mathbf{D}_N e_4, e_a) - \frac{1}{2} \mathbf{g}(\mathbf{D}_N e_3, e_a) \\ &= \frac{1}{2} \left(\frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, e_a) - \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, e_a)\right) - \frac{1}{2} \left(\frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_a) - \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, e_a)\right) \\ &= \frac{1}{2} \xi_a - \frac{1}{2} \eta_a.\end{aligned}\tag{B.2}$$

Therefore, combining (B.1) and (B.2), we obtain

$$\xi = \Xi + p, \quad \underline{\eta} = \Xi - p,$$

This proves (2.41)-(2.42).

B.3 Proof of Proposition 2.18

We first state the following general relations.

Lemma B.3. *The following equations hold true:*

$$\mathbf{R}_{ijkl} = R_{ijkl} + k_{ik}k_{jl} - k_{il}k_{jk},\tag{B.3}$$

$$\mathbf{R}_{Tabc} = \nabla_c \Theta_{ab} - \nabla_b \Theta_{ac} + \theta_{ca} \Xi_b - \theta_{ba} \Xi_c,\tag{B.4}$$

$$\mathbf{R}_{TaNb} = \nabla_b \Xi_a - \nabla_N \Theta_{ab} + \Pi \theta_{ba} - \theta_{bc} \Theta_{ac} - p_a \Xi_b - p_b \Xi_a,\tag{B.5}$$

$$\mathbf{R}_{TNab} = \nabla_b \Xi_a - \nabla_a \Xi_b - \Theta_{ac} \theta_{bc} + \Theta_{bc} \theta_{ac},\tag{B.6}$$

$$\mathbf{R}_{TNNa} = \nabla_a \Pi - \nabla_N \Xi_a - 2\theta_{ab} \Xi_b + p_b \Theta_{ba} - p_a \Pi.\tag{B.7}$$

Proof. The first equation is the Gauss equation (note the sign flip due to the Lorentzian signature). We have the Codazzi equation⁴²

$$\mathbf{R}_{Tijl} = \nabla_l k_{ij} - \nabla_j k_{il}.\tag{B.8}$$

⁴²Note again our convention $k_{ij} = \mathbf{g}(\mathbf{D}_i T, \partial_j)$.

This yields the following relations, again using the rules in (A.1),

$$\begin{aligned}
\mathbf{R}_{Tabc} &= \nabla_c k_{ab} - \nabla_b k_{ac} = \nabla_c k_{ab} + \theta_{ca} k_{Nb} + \theta_{cb} k_{aN} - \nabla_b k_{ac} - \theta_{ba} k_{Nc} - \theta_{bc} k_{aN}, \\
&= \nabla_c k_{ab} - \nabla_b k_{ac} + \theta_{ca} k_{Nb} - \theta_{ba} k_{Nc}, \\
\mathbf{R}_{TaNb} &= \nabla_b k_{aN} - \nabla_N k_{ab} = \nabla_b \Xi_a + \theta_{ba} k_{NN} - \theta_{bc} k_{ac} - (\nabla_N k_{ab} + p_a k_{Nb} + p_b k_{Na}), \\
\mathbf{R}_{TNab} &= \nabla_b k_{Na} - \nabla_a k_{Nb} = \nabla_b k_{Na} - \theta_{bc} k_{ca} + \theta_{ba} k_{NN} - (\nabla_a k_{Nb} - \theta_{ac} k_{cb} + \theta_{ab} k_{NN}) \\
&= \nabla_b k_{Na} - \nabla_a k_{Nb} - k_{ac} \theta_{bc} + k_{bc} \theta_{ac}, \\
\mathbf{R}_{TNNa} &= \nabla_a k_{NN} - \nabla_N k_{Na} = \nabla_a k_{NN} - 2\theta_{ab} k_{bN} - (\nabla_N k_{Na} - p_b k_{ba} + p_a k_{NN}) \\
&= \nabla_a k_{NN} - \nabla_N k_{Na} - 2\theta_{ab} k_{bN} + p_b k_{ba} - p_a k_{NN}.
\end{aligned} \tag{B.9}$$

Therefore, the equations (B.3)-(B.7) follow. This concludes the proof of Lemma B.3. $\square$

Remark B.4. In view of the relations (B.3), (B.8), and the definitions (2.27)-(2.28), we have

$$\begin{aligned}
\mathbf{Ric}_{Ti} &= (\operatorname{div} k)_i - \nabla_i (\operatorname{tr} k) = (\mathcal{C}_{Mom})_i, \\
\mathbf{R}_g &= -\mathbf{Ric}_{TT} + \mathbf{Ric}_{ii} = -2\mathbf{R}_{TiTi} + R_{ijij} + (\operatorname{tr} k)^2 - |k|^2 \\
&= -2\mathbf{Ric}_{TT} + \mathcal{C}_{Ham}.
\end{aligned}$$

We now start to prove (2.43)-(2.44). From the spacetime notations, we have

$$\begin{aligned}
\underline{\beta}(\mathbf{R}) &:= \frac{1}{2} \mathbf{R}_{a334} = \frac{1}{2} \mathbf{R}(e_a, T - N, T - N, T + N) = \mathbf{R}_{aTTN} - \mathbf{R}_{aNTN}, \\
\beta(\mathbf{R}) &:= \frac{1}{2} \mathbf{R}_{a434} = \frac{1}{2} \mathbf{R}(e_a, T + N, T - N, T + N) = \mathbf{R}_{aTTN} + \mathbf{R}_{aNTN},
\end{aligned}$$

and hence

$$\begin{aligned}
\beta(\mathbf{R}) + \underline{\beta}(\mathbf{R}) &= -2\mathbf{R}_{aTNT} = 2\mathbf{Ric}_{aN} - 2\mathbf{R}_{abNb}, \\
\beta(\mathbf{R}) - \underline{\beta}(\mathbf{R}) &= 2\mathbf{R}_{TNaN}.
\end{aligned}$$

Using the definition of the Weyl tensor (2.39), we have

$$\begin{aligned}
\underline{\beta} &= \frac{1}{2} \mathbf{W}_{a334} = \underline{\beta}(\mathbf{R}) + \frac{1}{4} (-2) \mathbf{Ric}_{3a} = \underline{\beta}(\mathbf{R}) - \frac{1}{2} \mathbf{Ric}_{3a}, \\
\beta &= \frac{1}{2} \mathbf{W}_{a434} = \beta(\mathbf{R}) - \frac{1}{4} (-2) \mathbf{Ric}_{4a} = \beta(\mathbf{R}) + \frac{1}{2} \mathbf{Ric}_{4a},
\end{aligned}$$

hence

$$\begin{aligned}
\beta + \underline{\beta} &= \beta(\mathbf{R}) + \underline{\beta}(\mathbf{R}) + \mathbf{Ric}_{Na} = 3\mathbf{Ric}_{Na} + 2\mathbf{R}_{Nbba}, \\
\beta - \underline{\beta} &= \beta(\mathbf{R}) - \underline{\beta}(\mathbf{R}) + \mathbf{Ric}_{Ta} = 2\mathbf{R}_{TNaN} + \mathbf{Ric}_{Ta}.
\end{aligned}$$

Using (B.3) and (B.7), along with Remark B.4, we have

$$\begin{aligned}
(\beta + \underline{\beta})_a - 3\mathbf{Ric}_{Na} &= 2\mathbf{R}_{Nbba} = 2(R_{Nbba} + k_{Nb} k_{ba} - k_{Na} k_{bb}) = 2(Y_a + \Xi_b \cdot \Theta_{ba} - \mathfrak{t} \Theta \Xi_a), \\
(\beta - \underline{\beta})_a - (\mathcal{C}_{Mom})_a &= -2\mathbf{R}_{TNNa} = -2(\nabla_a \Pi - \nabla_N \Xi_a - 2\theta_{ab} \Xi_b + p_b \Theta_{ba} - p_a \Pi),
\end{aligned}$$

as required.

For ρ , first note that by (2.39)

$$\rho = \frac{1}{4} \mathbf{W}_{3434} = \frac{1}{4} (\mathbf{R}_{3434} + 2\mathbf{Ric}_{34} - \frac{2}{3} \mathbf{R}_g) = \rho(\mathbf{R}) + \frac{1}{2} \mathbf{Ric}_{34} - \frac{1}{6} \mathbf{R}_g.$$

Using the Gauss equation for codimension 2, we have, again noting the sign flip for the T -direction,

$$\mathbf{R}_{abab} = 2K_\gamma + \frac{1}{2} (\mathfrak{t} \Theta)^2 - \frac{1}{2} (\mathfrak{t} \theta)^2 - |\hat{\Theta}|^2 + |\hat{\theta}|^2.$$

Therefore, using $\mathbf{Ric}_{34} = -\frac{1}{2}\mathbf{R}_{3443} + \mathbf{R}_{3a4a} = 2\rho(\mathbf{R}) + \mathbf{R}_{3a4a}$ and $\mathbf{Ric}_{aa} = -\mathbf{R}_{3a4a} + \mathbf{R}_{abab}$, we have

$$\begin{aligned}
\rho &= \rho(\mathbf{R}) + \frac{1}{2}\mathbf{Ric}_{34} - \frac{1}{6}\mathbf{R}_g = \frac{1}{2}(\mathbf{Ric}_{34} + \mathbf{Ric}_{aa} - \mathbf{R}_{abab}) + \frac{1}{2}\mathbf{Ric}_{34} - \frac{1}{6}\mathbf{R}_g \\
&= -\frac{1}{2}\mathbf{R}_{abab} + \mathbf{Ric}_{TT} - \mathbf{Ric}_{NN} + \frac{1}{2}\mathbf{Ric}_{aa} - \frac{1}{6}\mathbf{R}_g \\
&= -\frac{1}{2}\mathbf{R}_{abab} + \frac{1}{2}(\mathcal{C}_{Ham} - \mathbf{R}_g) - (\mathbf{Ric} - \frac{1}{2}(\mathbf{R}_g)\mathbf{g})_{NN} - \frac{1}{2}\mathbf{R}_g + \frac{1}{2}(\mathbf{Ric} - \frac{1}{2}(\mathbf{R}_g)\mathbf{g})_{aa} + \frac{1}{2}\mathbf{R}_g - \frac{1}{6}\mathbf{R}_g \\
&= -K_\gamma - \frac{1}{4}(\psi\Theta)^2 + \frac{1}{4}(\psi\theta)^2 + \frac{1}{2}|\widehat{\Theta}|^2 - \frac{1}{2}|\widehat{\theta}|^2 \\
&\quad + \frac{1}{2}\mathcal{C}_{Ham} - (\mathbf{Ric} - \frac{1}{2}(\mathbf{R}_g)\mathbf{g})_{NN} + \frac{1}{2}(\mathbf{Ric} - \frac{1}{2}(\mathbf{R}_g)\mathbf{g})_{aa} - \frac{2}{3}\mathbf{R}_g.
\end{aligned}$$

This proves (2.45).

For ${}^*\rho$ we have

$${}^*\rho = \frac{1}{4}{}^*\mathbf{W}_{3434} = \frac{1}{4}(\frac{1}{2}\epsilon_{34ab}\mathbf{W}_{ab34}) = \frac{1}{2}\mathbf{W}_{1234} = \frac{1}{2}\mathbf{R}_{1234} = \mathbf{R}_{12TN} = -\text{curl}\Xi - \widehat{\Theta} \wedge \widehat{\theta}.$$

where we used the relation (B.6) for the last equality. This proves (2.46) and concludes the proof of Proposition 2.18.

C Physical quantities

In this appendix, we prove the following alternative expression of the ADM charges:

Proposition C.1. *Under the class of initial data (g, k) we construct in Theorem 2.30, we have*

$$\mathbf{E} = m, \quad \mathbf{C}_i = -\frac{1}{8\pi m} \lim_{r \rightarrow \infty} r^3 (\widetilde{\psi\theta})_{\ell=1,i}, \quad \mathbf{P}_i = -\frac{1}{8\pi} \lim_{r \rightarrow \infty} r^2 (\psi\Theta)_{\ell=1,i}, \quad \mathbf{J}_i = \frac{1}{8\pi} \lim_{r \rightarrow \infty} r^4 (\text{curl}\Xi)_{\ell=1,i}.$$

We prove the relations for $\mathbf{E}, \mathbf{C}_i$ in the first subsection and for $\mathbf{P}_i, \mathbf{J}_i$ in the next.

In this appendix, we frequently use the vector field $\bar{\partial}_i := \partial_i - \omega_i \partial_r$, where $\omega_i := x^i/r$. We record the well-known relations

$$\gamma^{(0)}(\bar{\partial}_i, \bar{\partial}_j) = \delta_{ij} - \omega_i \omega_j, \quad \partial_i \omega_j = \frac{1}{r}(\delta_{ij} - \omega_i \omega_j).$$

C.1 Energy and center of mass

We relate the ADM energy $\mathbf{E}$ and center of mass $\mathbf{C}_i$ defined in (2.1) with the conditions we impose at infinity. The definitions in (2.1) are written in Cartesian coordinates, and hence we first need the following lemma.

Lemma C.2. *Under the assumption that $g = \hat{a}^2 dr^2 + \gamma_{AB} d\theta^A d\theta^B$, we have*

$$\begin{aligned}
\mathbf{E} &= \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \frac{2}{r} g_{rr} - \frac{1}{r} \gamma_{kk} - \partial_r(\gamma_{kk}) dA, \\
\mathbf{C}_i &= \frac{1}{16\pi m} \lim_{r \rightarrow \infty} \int_{S_r} \omega^i (2(g_{rr} - 1) - r \partial_r(\gamma_{kk})) dA,
\end{aligned}$$

where $\gamma_{kk} := \sum_{k=1}^3 \gamma(\bar{\partial}_k, \bar{\partial}_k)$.

Proof. In Cartesian coordinates, we have, using that $g(\partial_r, \bar{\partial}_i) = 0$,

$$g_{ij} = g(\bar{\partial}_i + \omega_i \partial_r, \bar{\partial}_j + \omega_j \partial_r) = \omega_i \omega_j g_{rr} + \gamma(\bar{\partial}_i, \bar{\partial}_j).$$

We have, recalling that $\omega^k \partial_k = \partial_r$,

$$\begin{aligned} g_{kr} &= \omega_k g_{rr}, \\ (\partial_k g_{kj})(\partial_r)^j &= \partial_k(g_{kr}) - g_{kj} \partial_k(\partial_r)^j = \partial_k(\omega_k g_{rr}) - g_{kj} \frac{1}{r}(\delta_{kj} - \omega_k \omega_j) \\ &= \frac{2}{r} g_{rr} + \partial_r(g_{rr}) - \frac{1}{r} \gamma(\bar{\partial}_k, \bar{\partial}_k), \\ \partial_r g_{kk} &= \partial_r(g_{rr} + \gamma(\bar{\partial}_k, \bar{\partial}_k)). \end{aligned}$$

Therefore,

$$\begin{aligned} (\partial_k g_{kj} - \partial_j g_{kk})(\partial_r)^j &= \frac{2}{r} g_{rr} - \frac{1}{r} \gamma(\bar{\partial}_k, \bar{\partial}_k) - \partial_r(\gamma(\bar{\partial}_k, \bar{\partial}_k)), \\ x^i (\partial_k g_{kj} - \partial_j g_{kk})(\partial_r)^j &= r \omega^i \left(\frac{2}{r} g_{rr} - \frac{1}{r} \gamma(\bar{\partial}_k, \bar{\partial}_k) - \partial_r(\gamma(\bar{\partial}_k, \bar{\partial}_k)) \right), \end{aligned}$$

and

$$\begin{aligned} (g_{ir} - \delta_{ir}) - \delta_{ir}(g_{kk} - \delta_{kk}) &= \omega_i(g_{rr} - 1) - \omega_i \left(g_{rr} - 1 + \gamma(\bar{\partial}_k, \bar{\partial}_k) - \gamma^{(0)}(\bar{\partial}_k, \bar{\partial}_k) \right) \\ &= -\omega_i \left(\gamma(\bar{\partial}_k, \bar{\partial}_k) - \gamma^{(0)}(\bar{\partial}_k, \bar{\partial}_k) \right). \end{aligned}$$

Therefore,

$$\mathbf{E} = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ji})(\partial_r)^j dA = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \frac{2}{r} g_{rr} - \frac{1}{r} \gamma_{kk} - \partial_r(\gamma_{kk}) dA,$$

and, using $\gamma_{kk}^{(0)} = 2$,

$$\begin{aligned} \mathbf{C}_i &= \frac{1}{16\pi m} \lim_{r \rightarrow \infty} \int_{S_r} \left(x^i (\partial_k g_{kj} - \partial_j g_{kk}) - ((g_{ij} - \delta_{ij}) - \delta_{ij}(g_{kk} - \delta_{kk})) \right) (\partial_r)^j dA \\ &= \frac{1}{16\pi m} \lim_{r \rightarrow \infty} \int_{S_r} \left(r \omega^i \left(\frac{2}{r} g_{rr} - \frac{1}{r} \gamma_{kk} - \partial_r(\gamma_{kk}) \right) + \omega_i (\gamma_{kk} - \gamma_{kk}^{(0)}) \right) dA \\ &= \frac{1}{16\pi m} \lim_{r \rightarrow \infty} \int_{S_r} \omega^i (2g_{rr} - 2 - r \partial_r(\gamma_{kk})) dA, \end{aligned}$$

as required. $\square$

We now derive the expansion of γ_{kk} .

Proposition C.3. *We have*

$$\partial_r(\gamma_{kk}) = 2\Upsilon^{-\frac{1}{2}}(\widetilde{\mathfrak{t}\mathfrak{t}}\theta)_{\ell=1} + 4\Upsilon^{\frac{1}{2}}r^{-1}(\check{a})_{\ell=1} + O_{\ell \neq 1}(r^{-2-\delta}) + O(r^{-3-2\delta}). \quad (\text{C.1})$$

Here, the $O_{\ell \neq 1}$ refers to a term supported on $\ell \neq 1$ and bounded by the quantities in the parentheses.

Proof. Recall from (2.58) that

$$\begin{aligned} \mathfrak{L}_{\partial_r}(r^{-2}\gamma) &= 2r^{-2}\hat{a}\hat{\theta} + \hat{a}\widetilde{\mathfrak{t}\mathfrak{t}}\theta(r^{-2}\gamma) + 2\Upsilon^{\frac{1}{2}}\check{a}r^{-1}(r^{-2}\gamma) \\ &= 2r^{-2}\Upsilon^{-\frac{1}{2}}\hat{\theta} + \Upsilon^{-\frac{1}{2}}\widetilde{\mathfrak{t}\mathfrak{t}}\theta(r^{-2}\gamma^{(0)}) + r^{-2}O\left(\left((\gamma - \gamma^{(0)}), \check{a}\right) \cdot (\hat{\theta}, \widetilde{\mathfrak{t}\mathfrak{t}}\theta)\right) \\ &\quad + 2\Upsilon^{\frac{1}{2}}r^{-1}\check{a}(r^{-2}\gamma^{(0)}) + r^{-3}O\left((\gamma - \gamma^{(0)}) \cdot \check{a}\right). \end{aligned}$$

We have

$$\begin{aligned} [\partial_r, \bar{\partial}_i] &= [\partial_r, \partial_i - \omega_i \partial_r] = [\partial_r, \partial_i] - \partial_r(\omega_i) \partial_r = -\partial_i(\omega_j) \partial_j - \omega_j \partial_j(\omega^i) \partial_r \\ &= -\frac{1}{r}(\partial_i - \omega_i \partial_r) + 0 = -\frac{1}{r} \bar{\partial}_i. \end{aligned}$$

Therefore, $\mathcal{L}_{\partial_r} \bar{\partial}_i = -\frac{1}{r} \bar{\partial}_i$, and hence

$$\begin{aligned} \partial_r(\gamma_{kk}) &= \mathcal{L}_{\partial_r} \gamma(\bar{\partial}_k, \bar{\partial}_k) - 2\gamma(\bar{\partial}_k, \frac{1}{r} \bar{\partial}_k) = r^2 \mathcal{L}_{\partial_r}(r^{-2} \gamma)(\bar{\partial}_k, \bar{\partial}_k) \\ &= 2\Upsilon^{-\frac{1}{2}} \widetilde{\mathfrak{t}\mathfrak{t}\theta} + 4\Upsilon^{\frac{1}{2}} r^{-1} \check{a} + 2\Upsilon^{\frac{1}{2}} \widehat{\theta}(\bar{\partial}_k, \bar{\partial}_k) + O(r^{-1-\delta} \cdot r^{-2-\delta}) \\ &= 2\Upsilon^{-\frac{1}{2}} (\widetilde{\mathfrak{t}\mathfrak{t}\theta})_{\ell=1} + 4\Upsilon^{\frac{1}{2}} r^{-1} (\check{a})_{\ell=1} + O_{\ell \neq 1}(r^{-2-\delta}) + O(r^{-3-2\delta}) \end{aligned}$$

where we use that the scalar $\widehat{\theta}(\bar{\partial}_k, \bar{\partial}_k) = \mathfrak{t}\mathfrak{t}^{(0)} \widehat{\theta} = O(\gamma - \gamma^{(0)}) \cdot \widehat{\theta}$, i.e., can be controlled by the size of metric perturbation $O(r^{-1-\delta})$ multiplied by the size of $\widehat{\theta} = O(r^{-2-\delta})$. $\square$

Recall from (4.27) and (5.42) that

$$(\widetilde{\mathfrak{t}\mathfrak{t}\theta})_{\ell=1,i} = \mathbf{c}_i r^{-3} + O(r^{-3-\delta}), \quad (\check{a})_{\ell=1,i} = \frac{1}{2} \mathbf{c}_i r^{-2} + O(r^{-2-\delta}).$$

Therefore, we have, using $\Upsilon = 1 + O(mr^{-1})$,

$$\begin{aligned} (\partial_r(\gamma_{kk}))_{\ell=1,i} &= 2\Upsilon^{-\frac{1}{2}} (\widetilde{\mathfrak{t}\mathfrak{t}\theta})_{\ell=1,i} + 4\Upsilon^{\frac{1}{2}} r^{-1} (\check{a})_{\ell=1,i} + O(r^{-3-2\delta}) \\ &= 4\mathbf{c}_i r^{-3} + O(r^{-3-\delta}). \end{aligned}$$

We also know that $g_{rr} = \hat{a}^2 = \hat{a}_0^2 + 2\hat{a}_0 \check{a} + O(\check{a}^2) = \Upsilon^{-1} + 2\Upsilon^{-\frac{1}{2}} \check{a}_{\ell=1} + O(r^{-2-2\delta})$. Hence, again using $\Upsilon = 1 + O(mr^{-1})$,

$$\begin{aligned} \mathbf{C}_i &= \frac{1}{16\pi m} \lim_{r \rightarrow \infty} \int_{S_r} \omega^i (2(g_{rr} - 1) - r \partial_r(\gamma_{kk})) dA \\ &= \frac{1}{16\pi m} \lim_{r \rightarrow \infty} r^2 (4\check{a}_{\ell=1,i} - r(\partial_r(\gamma_{kk}))_{\ell=1,i}) \\ &= \frac{1}{16\pi m} (2\mathbf{c}_i - 4\mathbf{c}_i) = -\frac{1}{8\pi m} \mathbf{c}_i. \end{aligned}$$

This proves the relation for $\mathbf{C}_i$ with $\mathbf{c}_i = \lim_{r \rightarrow \infty} r^3 (\widetilde{\mathfrak{t}\mathfrak{t}\theta})_{\ell=1,i}$.

The equation (C.1) also implies the rougher form

$$(\partial_r(\gamma_{kk}))_{\ell=0} = O(r^{-2-\delta}).$$

Therefore, using $\gamma_{kk}^{(0)} = 2$, we deduce $(\gamma_{kk})_{\ell=0} = 2 + O(r^{-1-\delta})$, and

$$\mathbf{E} = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \frac{2}{r} g_{rr} - \frac{1}{r} \gamma_{kk} - \partial_r(\gamma_{kk}) dA = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \frac{2}{r} \left(\frac{2m}{r} \right) + O(r^{-2-\delta}) dA = m.$$

This proves the relation for $\mathbf{E}$.

C.2 Linear momentum and angular momentum

Lemma C.4. *We have*

$$(\nabla^{(0)} \omega_i)^\# = \frac{1}{r} \bar{\partial}_i.$$

Here, the index raising $\#$ is defined with respect to $\gamma^{(0)}$.

Proof. By rotational symmetry, we assume i corresponds to the z direction. In this case, $\omega_i = \cos \vartheta$ in the standard spherical coordinates (ϑ, φ) , and $\bar{\partial}_i = \bar{\partial}_z = \partial_z - g(\partial_z, \partial_r) \partial_r = \partial_z - \cos \vartheta \partial_r$. In Cartesian coordinates,

$$\partial_r = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta), \quad \partial_\vartheta = r(\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, -\sin \vartheta).$$

It is then straightforward to verify that $\bar{\partial}_z = -r^{-1} \sin \vartheta \partial_\vartheta$. We compute,

$$(\nabla^{(0)} \cos \vartheta)^\vartheta = r^{-2} \partial_\vartheta \cos \vartheta = -r^{-2} \sin \vartheta, \quad (\nabla^{(0)} \cos \vartheta)^\varphi = 0.$$

Therefore, we obtain $(\nabla^{(0)} \cos \vartheta)^\# = r^{-1} \bar{\partial}_z$. $\square$

For notational simplicity, we define $\Xi^{(0)}$ by $\Xi_a^{(0)} := k(\partial_r, e_a^{(0)})$, where $\{e_a^{(0)}\}$ is a horizontal orthonormal frame with respect to $\gamma^{(0)}$. We have, using the bounds we obtain in Section 4.6,

$$|r \mathrm{d}\nabla \Xi - r \mathrm{d}\nabla^{(0)} \Xi^{(0)}| \lesssim |(r \nabla^{(0)})^{\leq 1} (\gamma - \gamma^{(0)})| |\Xi| + |(r \nabla^{(0)})^{\leq 1} k| |\check{\alpha}, (\gamma - \gamma^{(0)})| \lesssim \varepsilon^2 r^{-3-2\delta}. \quad (\text{C.2})$$

We have, using (C.4),

$$(\mathrm{d}\nabla^{(0)} \Xi^{(0)})_{\ell=1,i} = r^{-2} \int_{S_r} (\mathrm{d}\nabla^{(0)} \Xi^{(0)}) \omega_i dA = -r^{-2} \int_{S_r} \Xi^{(0)} \cdot \nabla^{(0)} \omega_i dA = -r^{-3} \int_{S_r} k(\partial_r, \bar{\partial}_i) dA.$$

Note that we have imposed in Theorem 2.30 that $(\mathrm{d}\nabla \Xi)_{\ell=1} = 0$. Therefore, we have $\int_{S_r} k(\partial_r, \bar{\partial}_i) dA = O(r^2 (\mathrm{d}\nabla \Xi - \mathrm{d}\nabla^{(0)} \Xi^{(0)})_{\ell=1}) = O(r^{-1-2\delta})$. We now calculate

$$\begin{aligned} \mathbf{P}_i &= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} (k_{ij} - \mathrm{tr}_\delta k \delta_{ij}) (\partial_r)^j dA = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} (k(\partial_i, \partial_r) - \mathrm{tr}_\delta k \omega_i) dA \\ &= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} k(\bar{\partial}_i, \partial_r) - (\mathfrak{t}^{(0)} k) \omega_i dA \end{aligned}$$

Here $\mathfrak{t}^{(0)} k$ is the $\gamma^{(0)}$ -spherical trace of k , and hence $\mathfrak{t}^{(0)} k = \mathfrak{t} \Theta + O(r^{-3-2\delta})$. As a result, the spherical integral becomes

$$\mathbf{P}_i = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} -(\mathfrak{t} \Theta) \omega^i dA = -\frac{1}{8\pi} r^2 (\mathfrak{t} \Theta)_{\ell=1,i}.$$

For the angular momentum, we have

$$\mathbf{J}_i := \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} \in_{ilm} x^l (k^{mj} - \delta^{mj} \mathrm{tr}_\delta k) (\partial_r)_j dA = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} k(\partial_r, \in_{ilm} x^l \partial_m) dA,$$

where we use $\in_{ilm} x^\ell (\partial_r)^m = 0$. Now, note that, using Lemma C.4,

$$\in_{ilm} x^l \partial_m = r \in_{ab}^{(0)} (\bar{\partial}_i)^b e_a^{(0)} = r^2 \in_{ab}^{(0)} (\nabla^{(0)} \omega_i)_b e_a^{(0)}.$$

We also have the relation

$$(\mathrm{c}\nabla \mathrm{r} \Xi^{(0)})_{\ell=1,i} = -r^{-2} \int_{S_r} \in_{ab}^{(0)} (\nabla_b^{(0)} \Xi_a^{(0)}) \omega_i dA = r^{-2} \int_{S_r} \in_{ab}^{(0)} \Xi_a^{(0)} (\nabla_b^{(0)} \omega_i) dA,$$

and, similar to (C.2), we have $\mathrm{c}\nabla \mathrm{r} \Xi = \mathrm{c}\nabla \mathrm{r}^{(0)} \Xi^{(0)} + O(r^{-4-2\delta})$. Therefore, we deduce

$$\mathbf{J}_i = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} k(\partial_r, \in_{ab}^{(0)} e_a^{(0)} (\nabla^{(0)} \omega_i)_b) dA = \frac{1}{8\pi} \lim_{r \rightarrow \infty} r^4 (\mathrm{c}\nabla \mathrm{r} \Xi)_{\ell=1,i}.$$

D Contraction estimates

We will frequently use the relation

$$\psi^{(n+1)} \cdot \phi^{(n+1)} - \psi^{(n)} \cdot \phi^{(n)} = \delta\psi^{(n+1)} \cdot \phi^{(n+1)} + \psi^{(n)} \cdot \delta\phi^{(n+1)}.$$

Here, the dots may be with respect to $\gamma^{(n+1)}$ or $\gamma^{(n)}$. The difference, however, generates lower order terms of $\delta\gamma^{(n+1)}$ and hence we omit the estimates.

We also have the following calculation for any sequence of linear operators $L^{(n)}$:

$$\begin{aligned} L^{(n+1)}[\phi^{(n+2)}] - L^{(n)}[\phi^{(n+1)}] &= L^{(n+1)}[\phi^{(n+2)}] - L^{(n+1)}[\phi^{(n+1)}] + L^{(n+1)}[\phi^{(n+1)}] - L^{(n)}[\phi^{(n+1)}] \\ &= L^{(n+1)}\delta\phi^{(n+2)} + (L^{(n+1)} - L^{(n)})[\phi^{(n+1)}]. \end{aligned} \quad (\text{D.1})$$

This applies for $L^{(n)} = \mathbb{A}^{(n)}, \mathbb{P}_1^{(n)}, \mathbb{P}_2^{(n)}$, as well as the spherical mean operator $\phi \mapsto \bar{\phi}^{(n)}$. Moreover, the following schematic relations hold, in view of Remark 2.10,

$$\begin{aligned} (\mathbb{P}^2)^{(n+1)}\psi - (\mathbb{P}^2)^{(n)}\psi &= \delta\gamma^{(n+1)} \cdot \nabla^2\psi + \nabla(\delta\gamma^{(n+1)}) \cdot \nabla\psi, \quad \mathbb{P}^2 = \mathbb{A}, \mathbb{P}_1\mathbb{P}_2, \\ \mathbb{P}^{(n+1)}\psi - \mathbb{P}^{(n)}\psi &= \delta\gamma^{(n)}\psi, \quad \mathbb{P} = \mathbb{P}_1, \mathbb{P}_2, \text{div}, \text{curl}, \\ \bar{\phi}^{(n+1)} - \bar{\phi}^{(n)} &= \delta\gamma^{(n)} \cdot \phi, \\ \mathfrak{t}^{(n+1)}\psi - \mathfrak{t}^{(n)}\psi &= \delta\gamma^{(n)} \cdot \psi, \quad \psi \in \mathfrak{s}_2. \end{aligned}$$

D.1 Proof of Proposition 5.16

Taking the differences of the equations (4.4)-(4.7) between $n \mapsto n+1$ and n , we obtain

$$\begin{aligned} (\partial_r + 2r^{-1})(\delta\Psi_1^{(n+2)}) &= \Upsilon^{-\frac{1}{2}}(\delta\tilde{\mu}^{(n+2)})_{\ell=0} - 2(1 - 3mr^{-1})r^{-2}(\delta\Psi_3^{(n+2)}) + \Gamma_1^{(n+1)} \cdot \Gamma_1^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \\ (\partial_r + 3r^{-1})(\delta\Psi_2^{(n+2)}) &= r^{-1}(\delta\tilde{\mu}^{(n+2)})_{\ell=0} - 2\Upsilon^{\frac{1}{2}}r^{-3}(\delta\Psi_3^{(n+2)}) - \Upsilon^{-\frac{1}{2}}(\tilde{\mathcal{B}}_{\ell \leq 1}^{(n+2)} - \tilde{\mathcal{B}}_{\ell \leq 1}^{(n+1)}) \\ &\quad + \Gamma_1^{(n+1)} \cdot \Gamma_2^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_2^{(n)} - \delta\Psi_3^{(n+1)}\mathcal{B} - \Psi_3^{(n+1)}\tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(n+1)} + \Psi_3^{(n)}\tilde{\mathcal{B}}_{\ell \leq 1, aux}^{(n)}, \\ \Upsilon^{\frac{1}{2}}\mathbb{A}^{(n+1)}(\delta\Psi_3^{(n+2)}) &= (\delta\Psi_2^{(n+2)}) - (\overline{\delta\Psi_2^{(n+2)}}^{(n+1)}) - \Upsilon^{\frac{1}{2}}r^{-1}(\delta\Psi_1^{(n+2)} - \overline{\delta\Psi_1^{(n+2)}}^{(n+1)}) \\ &\quad - \Upsilon^{\frac{1}{2}}(\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)})(\Psi_3^{(n+1)}) - (\overline{\Psi_2^{(n+1)}}^{(n+1)} - \overline{\Psi_2^{(n+1)}}^{(n)}) \\ &\quad + \Upsilon^{\frac{1}{2}}r^{-1}(\overline{\Psi_1^{(n+1)}}^{(n+1)} - \overline{\Psi_1^{(n+1)}}^{(n)}) \\ &\quad - \mathbb{A}^{(n+1)}(\Gamma_0^{(n+1)} \cdot \Gamma_0^{(n+1)}) + \mathbb{A}^{(n)}(\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}) \\ &\quad + \Gamma_1^{(n+1)} \cdot \Gamma_1^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} + \overline{\Gamma_1^{(n+1)}} \cdot \overline{\Gamma_1^{(n+1)}}^{(n+1)} - \overline{\Gamma_1^{(n)}} \cdot \overline{\Gamma_1^{(n)}}^{(n)}, \\ \overline{\delta\Psi_3^{(n+2)}}^{(n+1)} &= -\frac{1}{2}\Upsilon^{-1}r\overline{\delta\Psi_1^{(n+2)}}^{(n+1)} \\ &\quad - (\overline{\Psi_3^{(n+1)}}^{(n+1)} - \overline{\Psi_3^{(n+1)}}^{(n)}) + \frac{1}{2}\Upsilon^{-1}r(\overline{\Psi_1^{(n+1)}}^{(n+1)} - \overline{\Psi_1^{(n+1)}}^{(n)}). \end{aligned}$$

Note that, by (4.17) and (4.21), we have

$$\begin{aligned} \tilde{\mathcal{B}}_{\ell \leq 1}^{(n+2)} - \tilde{\mathcal{B}}_{\ell=1}^{(n+1)} &= \frac{1}{2}(\mathbb{A}^{(n+1)}\widetilde{\mathfrak{t}}\theta^{(n+2)})_{\ell=1} - \frac{1}{2}(\mathbb{A}^{(n)}\widetilde{\mathfrak{t}}\theta^{(n+1)})_{\ell \leq 1} \\ &\quad + \frac{1}{2}(\mathbb{A}^{(n+1)}\widetilde{\mathfrak{t}}\theta^{(n+1)})_{\ell=0} - \frac{1}{2}(\mathbb{A}^{(n)}\widetilde{\mathfrak{t}}\theta^{(n)})_{\ell=0} \\ &\quad - \left(\mathcal{P}_1(\mathbb{P}_1^{(n+1)}\mathbb{P}_2^{(n+1)}\widehat{\theta}^{(n+1)})\right)_{\ell \leq 1} + \left(\mathcal{P}_1(\mathbb{P}_1^{(n)}\mathbb{P}_2^{(n)}\widehat{\theta}^{(n)})\right)_{\ell \leq 1}, \end{aligned}$$

and

$$\begin{aligned}\delta\tilde{\mu}_{\ell=0}^{(n+2)} &= \tilde{\mu}_{\ell=0}^{(n+2)} - \tilde{\mu}_{\ell=0}^{(n+1)} = (\delta\Psi_2^{(n+2)} - \Upsilon^{\frac{1}{2}}r^{-1}\delta\Psi_1^{(n+2)})_{\ell=0} \\ &\quad - (\mathbb{A}^{(n+1)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n+1)}))_{\ell=0} + (\mathbb{A}^{(n)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n)}))_{\ell=0} \\ &\quad - \frac{1}{4}((\Psi_1^{(n+1)})^2)_{\ell=0} + \frac{1}{4}((\Psi_1^{(n)})^2)_{\ell=0}.\end{aligned}$$

Note that the last line is of the form $\Gamma_1^{(n+1)} \cdot \Gamma_1^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}$. Therefore, the system (5.48)-(5.49) holds, with

$$\begin{aligned}\mathcal{N}[\delta\Psi_1] &= -(\mathbb{A}^{(n+1)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n+1)}))_{\ell=0} + (\mathbb{A}^{(n)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n)}))_{\ell=0} \\ &\quad + \Gamma_1^{(n+1)} \cdot \Gamma_1^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \\ \mathcal{N}[\delta\Psi_2] &= -\frac{1}{2}\Upsilon^{-\frac{1}{2}}\left((\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)})\widetilde{\mathfrak{t}\mathfrak{t}\theta}^{(n+1)}\right)_{\ell=1} - \frac{1}{2}\Upsilon^{-\frac{1}{2}}\left(\mathbb{A}^{(n+1)}\widetilde{\mathfrak{t}\mathfrak{t}\theta}^{(n+1)} - \mathbb{A}^{(n)}\widetilde{\mathfrak{t}\mathfrak{t}\theta}^{(n)}\right)_{\ell=0} \\ &\quad - \Upsilon^{-\frac{1}{2}}\left(\mathcal{P}_1(\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)}\widehat{\theta}^{(n+1)} - \mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)}\widehat{\theta}^{(n)})\right)_{\ell\leq 1} \\ &\quad + r^{-1}\left(-(\mathbb{A}^{(n+1)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n+1)}))_{\ell=0} + (\mathbb{A}^{(n)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n)}))_{\ell=0}\right) \\ &\quad + \Gamma_1^{(n+1)} \cdot \Gamma_2^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_2^{(n)} - \delta\Psi_3^{(n+1)}\mathcal{B} - \Psi_3^{(n+1)}\widetilde{\mathcal{B}}_{\ell\leq 1,aux}^{(n+1)} + \Psi_3^{(n)}\widetilde{\mathcal{B}}_{\ell\leq 1,aux}^{(n)}, \\ \mathcal{N}[\delta\Psi_3] &= -\Upsilon^{\frac{1}{2}}(\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)})(\Psi_3^{(n+1)}) - (\overline{\Psi_2^{(n+1)}}^{(n+1)} - \overline{\Psi_2^{(n+1)}}^{(n)}) + \Upsilon^{\frac{1}{2}}r^{-1}(\overline{\Psi_1^{(n+1)}}^{(n+1)} - \overline{\Psi_1^{(n+1)}}^{(n)}) \\ &\quad - \mathbb{A}^{(n+1)}(\Gamma_0^{(n+1)} \cdot \Gamma_0^{(n+1)}) + \mathbb{A}^{(n)}(\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}) \\ &\quad + \Gamma_1^{(n+1)} \cdot \Gamma_1^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_1^{(n)} + \overline{\Gamma_1^{(n+1)}} \cdot \overline{\Gamma_1^{(n+1)}}^{(n+1)} - \overline{\Gamma_1^{(n)}} \cdot \overline{\Gamma_1^{(n)}}^{(n)}, \\ \mathcal{N}_{av}[\delta\Psi_3] &= -(\overline{\Psi_3^{(n+1)}}^{(n+1)} - \overline{\Psi_3^{(n+1)}}^{(n)}) + \frac{1}{2}\Upsilon^{-1}r(\overline{\Psi_1^{(n+1)}}^{(n+1)} - \overline{\Psi_1^{(n+1)}}^{(n)}).\end{aligned}$$

We first note that, since $\widehat{\theta}^{(n+1)}$ and $\widehat{\theta}^{(n)}$ are traceless with respect to $\gamma^{(n+1)}$ and $\gamma^{(n)}$, we have

$$\begin{aligned}\mathfrak{t}\mathfrak{t}^{(0)}(\widehat{\theta}^{(n+1)} - \widehat{\theta}^{(n)}) &= (\mathfrak{t}\mathfrak{t}^{(0)} - \mathfrak{t}\mathfrak{t}^{(n+1)})\widehat{\theta}^{(n+1)} - (\mathfrak{t}\mathfrak{t}^{(0)} - \mathfrak{t}\mathfrak{t}^{(n)})\widehat{\theta}^{(n)} \\ &= (\mathfrak{t}\mathfrak{t}^{(0)} - \mathfrak{t}\mathfrak{t}^{(n+1)})\delta\widehat{\theta}^{(n+1)} - (\mathfrak{t}\mathfrak{t}^{(n+1)} - \mathfrak{t}\mathfrak{t}^{(n)})\widehat{\theta}^{(n)} \\ &= (\gamma^{(n+1)} - \gamma^{(0)}) \cdot \delta\widehat{\theta}^{(n+1)} - \delta\gamma^{(n+1)}\widehat{\theta}^{(n)}.\end{aligned}$$

Therefore, using (3.5),

$$\begin{aligned}&r^{-1}\|(\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)}\widehat{\theta}^{(n+1)} - \mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)}\widehat{\theta}^{(n)})_{\ell\leq 1}\|_{\mathfrak{h}^s} \\ &\lesssim r^{-1}\|(\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)}(\delta\widehat{\theta}^{(n+1)}))_{\ell\leq 1}\|_{\mathfrak{h}^s} + r^{-1}\|(\delta\gamma^{(n+1)} \cdot \nabla^2\widehat{\theta}^{(n)} + \nabla(\delta\gamma^{(n+1)}) \cdot \nabla\widehat{\theta}^{(n)})_{\ell\leq 1}\|_{\mathfrak{h}^s} \\ &\lesssim \|\mathbb{A}^{(0)}\mathfrak{t}\mathfrak{t}^{(0)}\delta\widehat{\theta}^{(n+1)}\|_{L^\infty} + \|(r\nabla^{(0)})^{\leq 2}\delta\widehat{\theta}^{(n+1)}\|_{L^\infty}\|(r\nabla^{(0)})^{\leq 2}(\gamma^{(n+1)} - \gamma^{(0)})\|_{L^\infty} \\ &\quad + r^{-4-\delta}(r^{-1}\|\delta\gamma^{(n+1)}\|_{\mathfrak{h}^s}) \\ &\lesssim \varepsilon r^{-5-2\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.\end{aligned}\tag{D.2}$$

We then have, using the standard L^2 - L^∞ type estimate,

$$\begin{aligned}r^{-1}\|\mathcal{N}[\delta\Psi_1]\|_{\mathfrak{h}^{s+1}} &\lesssim \|\mathbb{A}^{(n+1)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n+1)})_{\ell=0} + (\mathbb{A}^{(n)}\log(\Upsilon^{-\frac{1}{2}} + \Psi_3^{(n)}))_{\ell=0}\|_{L^\infty} \\ &\quad + r^{-1}\|(\Gamma_1^{(n+1)}, \Gamma_1^{(n)}) \cdot \delta\Gamma_1^{(n+1)}\|_{\mathfrak{h}^{s+1}} \\ &\lesssim \varepsilon r^{-3-\delta}\|\delta\gamma^{(n+1)}\|_{L^\infty} + \varepsilon r^{-2-\delta}(r^{-1}\|\delta\Gamma_1^{(n+1)}\|_{\mathfrak{h}^{s+1}}) \\ &\lesssim \varepsilon r^{-4-2\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1}\|\mathcal{N}[\delta\Psi_2]\|_{\mathfrak{h}^s} &\lesssim \|(\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)})\widetilde{\mathfrak{t}\mathfrak{t}\theta}^{(n+1)}\|_{L^\infty} + \varepsilon r^{-5-2\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s \\ &\quad + r^{-1}\|\Gamma_1^{(n+1)} \cdot \Gamma_2^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}\|_{\mathfrak{h}^{s+1}} + \varepsilon r^{-4-\delta}(r^{-1}\|\delta\Psi_3^{(n+1)}\|_{\mathfrak{h}^s}) \\ &\quad + r^{-1}\|\widetilde{\mathcal{B}}_{\ell\leq 1,aux}^{(n+1)}\delta\Psi_3^{(n+1)}\|_{\mathfrak{h}^s} + r^{-1}\|\Psi_3^{(n)}\delta(\widetilde{\mathcal{B}}_{\ell\leq 1,aux}^{(n+1)})\|_{\mathfrak{h}^s} \\ &\lesssim \varepsilon r^{-5-2\delta}\|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s,\end{aligned}$$

Similarly, we obtain

$$\begin{aligned} r^{-1} \|\mathcal{N}[\delta\Psi_3]\|_{\mathfrak{H}^s} &\lesssim \varepsilon r^{-4-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ |\mathcal{N}_{av}[\delta\Psi_3]| &\lesssim \varepsilon r^{-2-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \end{aligned}$$

and we omit the details since the reasoning is totally the same.

D.2 Proof of Proposition 5.17

Taking the differences of the equations (4.8)-(4.10) between $n \mapsto n+1$ and n , we have

$$\begin{aligned} \mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} \Psi_4^{(n+2)} - \mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} \Psi_4^{(n+1)} &= \frac{1}{2} (\mathbb{A}^{(n+1)} \Psi_1^{(n+2)}, 0) - \frac{1}{2} (\mathbb{A}^{(n)} \Psi_1^{(n+1)}, 0) - \delta \Psi_{11}^{(n+2)}, \\ \mathcal{P}_1^{(n+1)} \Psi_5^{(n+2)} - \mathcal{P}_1^{(n)} \Psi_5^{(n+1)} &= -(\Upsilon^{\frac{1}{2}} \mathbb{A}^{(n+1)} \Psi_3^{(n+2)}, 0) + (\Upsilon^{\frac{1}{2}} \mathbb{A}^{(n)} \Psi_3^{(n+1)}, 0) \\ &\quad + \left(\mathbb{A}^{(n+1)} (\Gamma_0^{(n+1)} \cdot \Gamma_0^{(n+1)}), 0 \right) - \left(\mathbb{A}^{(n)} (\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}), 0 \right), \\ \mathcal{P}_1^{(n+1)} \Psi_6^{(n+2)} - \mathcal{P}_1^{(n)} \Psi_6^{(n+1)} &= \delta \Psi_{11}^{(n+2)} - (\overline{\Psi_{11}^{(n+2)}}^{(n+1)} - \overline{\Psi_{11}^{(n+1)}}^{(n)}) \\ &\quad - ((\mathcal{B}, * \mathcal{B})^{(n+1)} - (\mathcal{B}, * \mathcal{B})^{(n)}). \end{aligned}$$

The equations can be rewritten as

$$\begin{aligned} \mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} \delta \Psi_4^{(n+2)} &= \frac{1}{2} (\mathbb{A}^{(n+1)} \delta \Psi_1^{(n+2)}, 0) - \delta \Psi_{11}^{(n+2)} \\ &\quad - (\mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} - \mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)}) \Psi_4^{(n+1)} + \frac{1}{2} ((\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)}) \Psi_1^{(n+1)}, 0), \\ \mathcal{P}_1^{(n+1)} \delta \Psi_5^{(n+2)} &= -(\Upsilon^{\frac{1}{2}} \mathbb{A}^{(n+1)} \delta \Psi_3^{(n+2)}, 0) \\ &\quad - (\mathcal{P}_1^{(n+1)} - \mathcal{P}_1^{(n)}) \Psi_5^{(n+1)} - \Upsilon^{\frac{1}{2}} ((\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)}) \Psi_3^{(n+1)}, 0) \\ &\quad + \left(\mathbb{A}^{(n+1)} (\Gamma_0^{(n+1)} \cdot \Gamma_0^{(n+1)}), 0 \right) - \left(\mathbb{A}^{(n)} (\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}), 0 \right), \\ \mathcal{P}_1^{(n+1)} \delta \Psi_6^{(n+2)} &= \delta \Psi_{11}^{(n+2)} - \overline{\delta \Psi_{11}^{(n+2)}}^{(n+1)} - ((\mathcal{B}, * \mathcal{B})^{(n+1)} - (\mathcal{B}, * \mathcal{B})^{(n)}) \\ &\quad - (\mathcal{P}_1^{(n+1)} - \mathcal{P}_1^{(n)}) \Psi_6^{(n+1)} - (\overline{\Psi_{11}^{(n+1)}}^{(n+1)} - \overline{\Psi_{11}^{(n+1)}}^{(n)}). \end{aligned}$$

This is of the form (5.58)-(5.60) with

$$\begin{aligned} \mathcal{N}[\delta\Psi_4] &= -(\mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} - \mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)}) \Psi_4^{(n+1)} + \frac{1}{2} ((\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)}) \Psi_1^{(n+1)}, 0), \\ \mathcal{N}[\delta\Psi_5] &= -(\mathcal{P}_1^{(n+1)} - \mathcal{P}_1^{(n)}) \Psi_5^{(n+1)} - \Upsilon^{\frac{1}{2}} ((\mathbb{A}^{(n+1)} - \mathbb{A}^{(n)}) \Psi_3^{(n+1)}, 0) \\ &\quad + \left(\mathbb{A}^{(n+1)} (\Gamma_0^{(n+1)} \cdot \Gamma_0^{(n+1)}), 0 \right) - \left(\mathbb{A}^{(n)} (\Gamma_0^{(n)} \cdot \Gamma_0^{(n)}), 0 \right), \\ \mathcal{N}[\delta\Psi_6] &= -(\mathcal{P}_1^{(n+1)} - \mathcal{P}_1^{(n)}) \Psi_6^{(n+1)} - (\overline{\Psi_{11}^{(n+1)}}^{(n+1)} - \overline{\Psi_{11}^{(n+1)}}^{(n)}) - ((\mathcal{B}, * \mathcal{B})^{(n+1)} - (\mathcal{B}, * \mathcal{B})^{(n)}). \end{aligned}$$

We then proceed as in Section D.1 to estimate

$$\begin{aligned} r^{-1} \|\mathcal{N}[\delta\Psi_4]\|_{\mathfrak{H}^s} &\lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1} \|\mathcal{N}[\delta\Psi_5]\|_{\mathfrak{H}^s} &\lesssim \varepsilon r^{-4-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1} \|\mathcal{N}[\delta\Psi_6]\|_{\mathfrak{H}^s} &\lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s. \end{aligned}$$

This concludes the proof of Proposition 5.17.

D.3 Proof of Proposition 5.18

We denote $\delta(\hat{a}^{(n+1)}\psi^{(n+2)}) := \hat{a}^{(n+1)}\psi^{(n+2)} - \hat{a}^{(n)}\psi^{(n+1)}$.

Taking the differences of the equations (4.11)-(4.15) between $n \mapsto n+1$ and n , we have

$$\begin{aligned}
(\partial_r + r^{-1})\delta\Psi_7^{(n+2)} &= 2r^{-1}\delta\Psi_{10}^{(n+2)} + \Gamma_1^{(n+1)} \cdot \Gamma_1^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \\
\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)}(\hat{a}^{(n+1)}\Psi_8^{(n+2)}) - \mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)}(\hat{a}^{(n)}\Psi_8^{(n+1)}) &= \frac{1}{2}(\hat{a}^{(n+1)}\Delta^{(n+1)}\Psi_7^{(n+2)} - \hat{a}^{(n)}\Delta^{(n)}\Psi_7^{(n+1)}, 0) \\
&\quad + \delta\Psi_{12}^{(n+1)} + \Gamma_1^{(n+1)} \cdot \Gamma_2^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}, \\
\mathcal{P}_1^{(n+1)}\Psi_9^{(n+2)} - \mathcal{P}_1^{(n)}\Psi_9^{(n+2)} &= -\left(0, r^{-4} \int_r^\infty r'^4 \delta^* \mathcal{K}_{\ell \leq 1}^{(n+2)} dr'\right) \\
&\quad - \left(0, r^{-4} \int_r^\infty r'^4 (*\mathcal{K}) dr' - r^{-4} \int_r^\infty r'^4 (*\mathcal{K}) dr'\right)^{(n+1)} \\
&\quad + \left(0, r^{-4} \int_r^\infty r'^4 (*\mathcal{K}_{\ell \leq 1}^{(n+2)}) dr' - r^{-4} \int_r^\infty r'^4 (*\mathcal{K}_{\ell \leq 1}^{(n+1)}) dr'\right)^{(n)}, \\
\Delta^{(n+1)}(\hat{a}^{(n+1)}\Psi_{10}^{(n+2)}) - \Delta^{(n)}(\hat{a}^{(n)}\Psi_{10}^{(n+1)}) &= \delta\widehat{\mathcal{K}}_{\ell \leq 1}^{(n+2)} - (\overline{\mathcal{K}}^{(n+1)} - \overline{\mathcal{K}}^{(n)}) - (\widehat{\mathcal{K}}_{\ell \leq 1}^{(n+2)})^{(n+1)} - \widehat{\mathcal{K}}_{\ell \leq 1}^{(n+1)(n)}, \\
\overline{\hat{a}^{(n+1)}\Psi_{10}^{(n+2)}}^{(n+1)} - \overline{\hat{a}^{(n)}\Psi_{10}^{(n+1)}}^{(n)} &= \overline{\Psi_3^{(n+1)}\Psi_{10}^{(n+1)}}^{(n+1)} - \overline{\Psi_3^{(n)}\Psi_{10}^{(n)}}^{(n)}.
\end{aligned}$$

Using the formula (D.1), we have the following relations

$$\begin{aligned}
&\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)}(\hat{a}^{(n+1)}\Psi_8^{(n+2)}) - \mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)}(\hat{a}^{(n)}\Psi_8^{(n+1)}) \\
&= \mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)}\delta(\hat{a}^{(n+1)}\Psi_8^{(n+2)}) + (\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)} - \mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)})(\hat{a}^{(n)}\Psi_8^{(n+1)}), \\
&\quad \hat{a}^{(n+1)}\Delta^{(n+1)}\Psi_7^{(n+2)} - \hat{a}^{(n)}\Delta^{(n)}\Psi_7^{(n+1)} \\
&= \hat{a}^{(n+1)}\Delta^{(n+1)}\delta\Psi_7^{(n+2)} + (\hat{a}^{(n+1)}\Delta^{(n+1)} - \hat{a}^{(n)}\Delta^{(n)})\Psi_7^{(n+1)}, \\
\mathcal{P}_1^{(n+1)}\Psi_9^{(n+2)} - \mathcal{P}_1^{(n)}\Psi_9^{(n+2)} &= \mathcal{P}_1^{(n+1)}\delta\Psi_9^{(n+2)} + (\mathcal{P}_1^{(n+1)} - \mathcal{P}_1^{(n)})\Psi_9^{(n+1)}, \\
&\Delta^{(n+1)}(\hat{a}^{(n+1)}\Psi_{10}^{(n+2)}) - \Delta^{(n)}(\hat{a}^{(n)}\Psi_{10}^{(n+1)}) \\
&= \Delta^{(n+1)}\delta(\hat{a}^{(n+1)}\Psi_{10}^{(n+2)}) + (\Delta^{(n+1)} - \Delta^{(n)})(\hat{a}^{(n)}\Psi_{10}^{(n+1)}), \\
&\quad \overline{\hat{a}^{(n+1)}\Psi_{10}^{(n+2)}}^{(n+1)} - \overline{\hat{a}^{(n)}\Psi_{10}^{(n+1)}}^{(n)} \\
&= \overline{\delta(\hat{a}^{(n+1)}\Psi_{10}^{(n+2)})} + \overline{\hat{a}^{(n)}\Psi_{10}^{(n+1)}}^{(n+1)} - \overline{\hat{a}^{(n)}\Psi_{10}^{(n+1)}}^{(n)}.
\end{aligned}$$

Moreover, we have, in view of (4.19),

$$\begin{aligned}
\delta\widehat{\mathcal{K}}_{\ell \leq 1}^{(n+2)} &= \mathcal{P}_1 \left(\mathcal{P}_1^{(n+1)}\mathcal{P}_2^{(n+1)}(\hat{a}^{(n+1)}\Psi_8^{(n+1)}) - \mathcal{P}_1^{(n)}\mathcal{P}_2^{(n)}(\hat{a}^{(n)}\Psi_8^{(n)}) \right)_{\ell \leq 1} \\
&\quad - \frac{1}{2}(\hat{a}^{(n+1)}\Delta^{(n+1)}\Psi_7^{(n+2)} - \hat{a}^{(n)}\Delta^{(n)}\Psi_7^{(n+1)})_{\ell=1} \\
&\quad - \frac{1}{2}(\hat{a}^{(n+1)}\Delta^{(n+1)}\Psi_7^{(n+1)} - \hat{a}^{(n)}\Delta^{(n)}\Psi_7^{(n)})_{\ell=0} + (\Gamma_1^{(n+1)} \cdot \Gamma_2^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1} \\
&= -\frac{1}{2}(\hat{a}^{(n+1)}\Delta^{(n+1)}\delta\Psi_7^{(n+2)})_{\ell=1} + \delta\mathcal{R}_{lower}^{(n+1)},
\end{aligned}$$

where

$$\begin{aligned}\delta\mathcal{R}_{lower}^{(n+1)} &:= \mathcal{P}_1 \left(\mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} (\hat{a}^{(n+1)} \Psi_8^{(n+1)}) - \mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} (\hat{a}^{(n)} \Psi_8^{(n)}) \right)_{\ell \leq 1} \\ &\quad - \frac{1}{2} (\hat{a}^{(n+1)} \Delta^{(n+1)} \Psi_7^{(n+1)} - \hat{a}^{(n)} \Delta^{(n)} \Psi_7^{(n+1)})_{\ell=1} \\ &\quad - \frac{1}{2} (\hat{a}^{(n+1)} \Delta^{(n+1)} \Psi_7^{(n+1)} - \hat{a}^{(n)} \Delta^{(n)} \Psi_7^{(n)})_{\ell=0} + (\Gamma_1^{(n+1)} \cdot \Gamma_2^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_2^{(n)})_{\ell \leq 1}.\end{aligned}$$

Using this notation, we write

$$\begin{aligned}&\overline{\mathcal{K}_{\ell \leq 1}^{(n+2)}}^{(n+1)} - \overline{\mathcal{K}_{\ell \leq 1}^{(n+1)}}^{(n)} = \overline{\delta \mathcal{K}_{\ell \leq 1}^{(n+2)}}^{(n+1)} + (\overline{\mathcal{K}_{\ell \leq 1}^{(n+1)}}^{(n+1)} - \overline{\mathcal{K}_{\ell \leq 1}^{(n+1)}}^{(n)}) \\ &= -\frac{1}{2} (\hat{a}^{(n+1)} \Delta^{(n+1)} \delta \Psi_7^{(n+2)})_{\ell=1}^{(n+1)} + \overline{\delta \mathcal{R}_{lower}^{(n+1)}}^{(n+1)} + (\overline{\mathcal{K}_{\ell \leq 1}^{(n+1)}}^{(n+1)} - \overline{\mathcal{K}_{\ell \leq 1}^{(n+1)}}^{(n)}).\end{aligned}$$

Then, the system is already of the form (5.62)-(5.66), with

$$\begin{aligned}\mathcal{N}[\delta \Psi_7] &:= \Gamma_1^{(n+1)} \cdot \Gamma_1^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_1^{(n)}, \\ \mathcal{N}[\delta \Psi_8] &:= - \left(\mathcal{P}_1^{(n+1)} \mathcal{P}_2^{(n+1)} - \mathcal{P}_1^{(n)} \mathcal{P}_2^{(n)} \right) (\hat{a}^{(n)} \Psi_8^{(n+1)}) + \frac{1}{2} \left((\hat{a}^{(n+1)} \Delta^{(n+1)} - \hat{a}^{(n)} \Delta^{(n)}) \Psi_7^{(n+1)}, 0 \right) \\ &\quad + \Gamma_1^{(n+1)} \cdot \Gamma_2^{(n+1)} - \Gamma_1^{(n)} \cdot \Gamma_2^{(n)}, \\ \mathcal{N}[\delta \Psi_9] &:= -(\mathcal{P}_1^{(n+1)} - \mathcal{P}_1^{(n)}) \Psi_9^{(n+1)} - \left(0, r^{-4} \int_r^\infty r'^4 (*\mathcal{K}) dr' - r^{-4} \int_r^\infty r'^4 (*\mathcal{K}) dr' \right)^{(n+1)} \\ &\quad + \left(0, r^{-4} \int_r^\infty r'^4 (*\mathcal{K}_{\ell \leq 1}^{(n+1)}) dr' - r^{-4} \int_r^\infty r'^4 (*\mathcal{K}_{\ell \leq 1}^{(n+1)}) dr' \right)^{(n)}, \\ \mathcal{N}[\delta \Psi_{10}] &:= -(\Delta^{(n+1)} - \Delta^{(n)}) (\hat{a}^{(n)} \Psi_{10}^{(n+1)}) + \delta \mathcal{R}_{lower}^{(n+1)} \\ &\quad - (\bar{\mathcal{K}}^{(n+1)} - \bar{\mathcal{K}}^{(n)}) - (\overline{\mathcal{K}_{\ell \leq 1}^{(n+1)}}^{(n+1)} - \overline{\mathcal{K}_{\ell \leq 1}^{(n+1)}}^{(n)}) - \overline{\delta \mathcal{R}_{lower}^{(n+1)}}^{(n)}, \\ \mathcal{N}_{av}[\delta \Psi_{10}] &:= -\overline{\hat{a}^{(n)} \Psi_{10}^{(n+1)}}^{(n+1)} + \hat{a}^{(n)} \Psi_{10}^{(n+1)}.\end{aligned}$$

We then proceed as in Section D.1 to estimate

$$\begin{aligned}r^{-1} \|\mathcal{N}[\delta \Psi_7]\|_{\mathfrak{H}^{s+1}} &\lesssim \varepsilon r^{-4-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1} \|\mathcal{N}[\delta \Psi_8]\|_{\mathfrak{H}^{s-1}} &\lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1} \|\mathcal{N}[\delta \Psi_9]\|_{\mathfrak{H}^s} &\lesssim \varepsilon r^{-4-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1} \|\mathcal{N}[\delta \Psi_{10}]\|_{\mathfrak{H}^{s-1}} &\lesssim \varepsilon r^{-5-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s, \\ r^{-1} \|\mathcal{N}_{av}[\delta \Psi_{10}]\|_{\mathfrak{H}^{s+1}} &\lesssim \varepsilon r^{-3-2\delta} \|\delta(\Psi^{(n+1)}, \gamma^{(n+1)})\|_s.\end{aligned}$$

This concludes the proof of Proposition 5.18.

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