The Nonlinear Stability of Rotating Black Holes

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The Kerr spacetime, discovered by Kerr in 1963 [Ke] as an example of algebraically special metric, is an asymptotically flat solution of the vacuum Einstein equations representing a rotating black hole in equilibrium. The Kerr spacetime possesses a 2-parameter isometry group such that the two generating Killing fields commute and the integral curves of one of the Killing fields are in a neighborhood of spatial infinity timelike curves, so the associated subgroup corresponds to a group of time translations in this neighborhood, while the integral curves of the other Killing field are space-like circles, so the associated subgroup corresponds to a group of rotations in a plane. The group orbits are then cylinders. Moreover the distribution of planes orthogonal to the group orbits is integrable. The integral surfaces are then mutually isometric. We thus have a quotient manifold $(\mathcal{Q}, h)$ represented by any one of these orthogonal surfaces. The metric g of the spacetime manifold $\mathcal{M}$ at the group orbit corresponding to $q \in \mathcal{Q}$ is given by:

$$g(q) = f(q) + h(q)$$

where $f(q)$ is the metric of the group orbit. In Boyer-Lindquist coordinates we have coordinates (t, ϕ) on the group orbits and (r, θ) on the orthogonal surfaces, and the two Killing fields are:

$$\frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial \phi}$$

The coordinate ϕ is an azimuthal angle, so ϕ is taken modulo 2π. The coordinate θ is a polar angle, so $0 \leq \theta \leq \pi$. The circles which are the integral curves of the rotation field $\partial/\partial\phi$ degenerate to points at $\theta = 0$ and $\theta = \pi$, the two components of the rotation axis. The group orbits degenerate there to lines, the integral curves of $\partial/\partial t$. The metrics f and h are given by:

$$f = -dt^2 + \frac{2Mr}{\rho^2}(a \sin^2 \theta d\phi - dt)^2 + (r^2 + a^2) \sin^2 \theta d\phi^2$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta$$

and:

$$h = \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right)$$

where

$$\Delta = r^2 - 2Mr + a^2$$

The Kerr spacetime is actually a 2-parameter family of spacetimes, the parameters being a and M. The positive parameter M is the mass of the black hole, while its angular momentum is a vector J directed along the axis of rotation, which we can think of as the z axis, and the parameter a is the ratio J^z/M. The spacetime represents a black hole if $|a| \leq M$. The extreme case $|a| = M$ being exceptional, we shall confine attention to the subextremal range $|a| < M$. In the case $a = 0$ the spacetime reduces to the static, spherically symmetric Schwarzschild solution known since 1916 [Sc], which represents a nonrotating black hole in equilibrium.

We have

$$\Delta = (r - r_+)(r - r_-) \quad \text{where} \quad r_\pm = M \pm \sqrt{M^2 - a^2}$$

From the above expression for the metric f of the group orbits we obtain:

$$\det f = -\Delta \sin^2 \theta$$

hence the group orbits are timelike cylinders for $r > r_+$ and $r < r_-$, spacelike cylinders for $r_+ > r > r_-$, and null cylinders for $r = r_+$ and $r = r_-$.

The metric h of the quotient $\mathcal{Q}$ is singular at the two roots $r_\pm$ of Δ. These are only coordinate singularities of the spacetime metric g. To elliminate them we introduce in place of t and ϕ the coordinates v and ξ, where

$$v = t + \int (r^2 + a^2)\Delta^{-1} dr, \quad \xi = \phi + \int a\Delta^{-1} dr$$

The metric g of $\mathcal{M}$ is expressed in terms of the new coordinates as;

$$g = f + 2dvdr - 2a \sin^2 \theta d\xi dr + \rho^2 d\theta^2$$

where f is again the metric of the group orbits:

$$f = f_{vv} dv^2 + 2f_{v\xi} dv d\xi + f_{\xi\xi} d\xi^2$$

with

$$f_{vv} = f_{tt} = -\left(1 - \tfrac{2M}{\rho^2}\right)$$

$$f_{v\xi} = f_{t\phi} = -\tfrac{2Mr}{\rho^2} a \sin^2 \theta$$

$$f_{\xi\xi} = f_{\phi\phi} = \left(r^2 + a^2 + \tfrac{2Mr}{\rho^2} a^2 \sin^2 \theta\right) \sin^2 \theta$$

$\partial/\partial v$, $\partial/\partial\xi$ being the two Killing fields expressed in terms of the new coordinates. The metric g expressed in the new coordinates is manifestly regular at the roots of Δ. Here we shall only be concerned with the region $r > r_+ - \varepsilon$ for some small $\varepsilon > 0$, which contains only the larger root r_+.

As is evident from the above form of the metric g, in view of the preceding expression for $\det f$, the hypersurface $r = r_+$, which corresponds to $t = +\infty$, is an outgoing null hypersurface. This is the *future event horizon* $\mathcal{H}^+$, the future boundary of the *domain of outer communications*, which is is general defined as the causal past of future null infinity, and corresponds here to the region where $r > r_+$. On the group orbits which are the null cylinders corresponding to $r = r_+$ the null generator is

$$\frac{\partial}{\partial t} + \omega_+ \frac{\partial}{\partial \phi} \quad \text{or} \quad \frac{\partial}{\partial v} + \omega_+ \frac{\partial}{\partial \xi}$$

in the old and new coordinates respectively. This Killing field is the null generator of $\mathcal{H}^+$. Here, the constant:

$$\omega_+ = \frac{a}{r_+^2 + a^2}$$

is the *angular velocity* of $\mathcal{H}^+$. Any particle falling into $\mathcal{H}^+$ is seen from spatial infinity to approach $\mathcal{H}^+$ asymptotically, but never to actually cross it, acquiring in the limit $t \to \infty$ the angular velocity ω_+.

On hypersurface in the closure of the domain of outer communications where

$$\rho^2 - 2Mr = 0$$

f_{tt} and f_{vv} vanish. This is the hypersurface $r = r_0(\theta)$, where

$$r_0(\theta) = M + \sqrt{M^2 - a^2 \cos^2\theta}$$

It is called the *stationary limit surface* because the Killing field $\partial/\partial t$ is timelike outside this hypersurface but null on this hypersurface, thus it is the limit of the region where a particle world line can be an integral curve of the stationary Killing field. The stationary limit surface lies outside $\mathcal{H}^+$, thus in the interior of the domain of outer communications, everywhere except at the poles $\theta = 0, \pi$ where the two hypersurfaces touch. The region between the stationary limit surface and $\mathcal{H}^+$ is called the *ergosphere*. In this region a particle is compelled to rotate in azimuth in the same sense as the black hole, its angular velocity becoming equal to that of the black hole at $\mathcal{H}^+$. In the ergosphere a particle can have negative energy, meaning that, with p being a particle momentum, that is a covector the evaluation of which on any future-directed timelike vector is positive, the particle energy, which is the quantity $p(\partial/\partial t)$, can take negative values. This is because the stationary Killing field $\partial/\partial t$ is spacelike in the ergosphere.

At the time the Kerr solution was discovered, and during the time 1968-1971 when I was a graduate student in physics at Princeton in the group of John Wheeler and our professor had encouraged us to study the fascinating properties of the Kerr solution, the colleagues in astrophysics considered all this as pure fantasy having nothing to do with reality. But the situation changed dramatically in the intervening years, and now black holes, in fact rotating black holes, is a central theme in astrophysics. In recent decades a compact object at the center of our Milky Way galaxy named Sagittarius A*, with a mass of 4 million solar masses, was identified as a supermassive black hole. The observations, by tracking of orbiting stars, led to the award to Rienhard Genzel and Andrea Ghez of the 2020 Nobel Prize in physics, shared with Roger Penrose for his theoretical contributions to general relativity. The identification of Saggitarius A* as a supermassive black hole was confirmed in 2022 by an image produced by the Event Horizon Telescope. Even more impressive is the supermassive black hole with a mass of 6 billion solar masses at the center of the giant spherical galaxy M87 at the core of the Virgo cluster of galaxies, 55 million light years from Earth. The image of this black hole, captured in 2019, was in fact the first to be produced by the Event Horizon Telescope. Associated with the M87 black hole are ultra-relativistic plasma jets along each of the two components of the rotation axis. Supermassive black holes are now thought to be the engines driving all Active Galactic Nuclei and Quasars.

For this view to rest on a solid theoretical foundation however, it is necessary that the dynamical stability of the stationary black hole solutions to general small, but finite, perturbations be mathematically established.

Recently the papers [KS] by Klainerman and Szeftel and [GKS] by Giorgi, Klainerman and Szeftel were published, which taken together solve this stability problem in the case that the angular momentum to squared mass ratio of the black hole is suitably small. This marks a milestone in the mathematical development of general relativity theory. In reaching this milestone several mathematicians have contributed over the preceding two decades, and dramatic developments have taken place resulting in the solution of problems which seemed out of reach only a short while ago.

The papers in question were preceded in particular by the monograph [KSa] by Klainerman and Szeftel which was the first to address a problem in nonlinear theory in the same context albeit under a special symmetry assumption, and by the paper [DHRT] by Dafermos, Holzegel, Rodniaski and Taylor which constructed the solutions corresponding to a codimension 3 subset of the space of initial data.

A subfamily of the Kerr family is the Schwarzschild family of non-rotating static black holes. However an attempt to address a simpler problem of the nonlinear stability of the Schwarzschild family would be misguided, because a generic finite perturbation of the initial data for Schwarzschild will result in the black hole acquiring some spin. Unless a symmetry condition is imposed, such as polarized axisymmetry as in [KSa], or else the initial data are selected out of a codimension 3 subset chosen so that the final black hole spin vanishes as in [DHRT], in the general nonlinear theory only the stability of the Kerr family is a sensible problem, the only possible simplification being a smallness restriction on the angular momentum to squared mass ratio.

Nevertheless, a proof of nonlinear stability necessarily rests on understanding the decay properties of the solutions of the linearized Einstein equations which govern infinitesimal perturbations, and the linearization can be about the Schwarzschild family as well as about the more general Kerr family.

The work of mathematicians was preceded by the work of physicists in the period 1957-1989, which formulated the relevant equations to which the linearized Einstein equations reduce in ways which played a basic role in the later work of mathematicians, first in the context of the linearized theory and then, when suitably extended, also in the nonlinear theory. The first work was that of Regge and Wheeler [RW] who formulated the linearized about Schwarzschild equations from a metric perturbation point of view in the form of wave-type equations, the *Regge-Wheeler equations*. Another point of view, based on the Newmann-Penrose formalism which uses a null frame field (see [NP]) was introduced by Teukolsky [Teu] and is well suited to the Kerr solutions because of the presence in those solutions of distinguished null frame fields, the principal null frames. Teukolsky discovered that α and $\underline{\alpha}$, the extreme curvature components relative to such a principal null frame field satisfy decoupled wave equations, the *Teukolsky equations*, which separate into modes. Chandrasekhar then found a transformation, the *Chandrasekhar transformation*, expounded in his book [Ch], which relates the null frame field approach to the metric perturbation approach. transforming, for each mode, the Teukolsky equations to Regge-Wheeler-type equations.

While the work of physicists in formulating suitable equations for the perturbations did play a fundamental role in the subsequent work of mathematicians, as far as stability results are concerned, these were derived through the separation of variables and the analysis of the resulting modes and where essentially limited to showing that there are no exponentially growing modes. The goal was reached by Whiting [W] who showed that the solutions of the linearized equations about any member of the Kerr family do not contain any exponentially growing modes.

However mode stability was insufficient to demonstrate, even in the case of the scalar wave equation on a Schwarzschild background, the decay of solutions with compactly supported initial data. This is because of the presence of *trapped null geodesics* which neither fall into the black hole nor escape to infinity, along which solutions can concentrate for arbitrarily long times. In the case of the Schwarzschild spacetime the trapped null geodesics are contained in a timelike hypersurface, a timelike cylinder, in the exterior of the black hole. In the case of the wave equation on a Kerr background, even the boundedness of the solutions could not be established. This is because of the presence, as we have seen above, of a region surrounding a Kerr black hole, the ergosphere, where the Killing vectorfield generating time translations at spatial infinity is spacelike and as a consequence particles can have negative energy. Waves of negative energy falling into the black hole are accompanied by waves of increased positive energy escaping to infinity, in a phenomenon called "superradiace". Moreover, in the case of Kerr spacetime the trapped null geodesics are not confined to a timelike hypersurface but extend to a spacetime region in the vicinity of the black hole. The only simplification which occurs for Kerr spacetimes corresponding to small angular momentum to squared mass ratio is that in this case the region where trapped null geodesics occur does not intersect the ergosphere.

Mathematicians began studying these problems in 2003, the first work being that of Blue and Soffer [BSo] which established local energy decay for the solutions of the wave equation on Schwarzschild spacetime. This was established using a multiplier vectorfield with a coercive deformation tensor which degenerates on the timelike cylinder containing the trapped null geodesics. The method originates, in the case of the wave equation on Minkowski spacetime, in the paper of Morawetz [Mo], and for this reason estimates of this type are called *Morawetz estimates*. The result of [BSo] was soon improved to a uniform local energy decay result by Blue and Sterbenz [BSt].

A new insight was provided by Dafermos and Rodniaski in [DR1] with the understanding of the redshift effect present in a neighborhood of the event horizon, boundary of the black hole. This insight was employed to construct the *redshift vectorfield* with the help of which they achieved control of the solutions in the vicinity of the event horizon. Another important method, that of r^p weighted estimates, was introduced by Dafermos and Rodnianski in [DR2] and used to derive the uniform decay properties of the solutions.

In the case of the wave equation on Kerr spacetime, by reason of the presence of the ergosphere and the associate phenomenon of superradiance, as remarked above, even the boundedness of solutions proved difficult to establish. This was first done by Dafermos and Rodnianski in [DR3], in the case of small angular momentum to squared mass ratio of the black hole, using a decomposition of the wave function into a higher and lower frequency parts, with the higher frequency part containing the frequencies displaying the phenomenon of trapping and the lower frequency part containing the frequencies displaying the phenomenon of superradiace, the crucial observation that in the case of small angular momentum to squared mass ratio of the black hole *the superradiant frequencies are not trapped*. See also the exposition in [DR4]. This result was soon followed by the proof in this case of local energy decay in [TaTo]. The derivation of a decay estimate for the wave equation on Kerr in the full subextremal range $|J|/M^2 < 1$ of the magnitude $|J| = |a|M$ of the angular momentum J to squared mass M of the black hole was estabished by Dafermos, Rodnianski and Shlapentokh-Rothman in [DRS], using a subtle continuity argument.

Another development in the same area was the work [AB] by Andersson and Blue. This was inspired by the work of Carter in [Ca] who discovered that Kerr possesses a *Killing tensor* $K^{\mu\nu}$ and used it to study the motion of test particles on Kerr, the quantity $K^{\mu\nu}p_\mu p_\nu$, where p_μ is the particle momentum, being a constant of the motion. Making the quantum correspondence according to which to p_μ there corresponds the operator $\partial/\partial x^\mu$, they defined a 2nd order differential operator acting on the wave function, and used it to prove decay for the wave equation on Kerr in the case of small $|J|/M^2$, purely in physical space, avoiding entirely frequency decompositions. This work laid the foundation for the wave equations estimates in [GKS], as it is robust with respect to perturbations of the underlying metric.

All the above results concern the wave equation on Schwarzschild, or, more generally, on Kerr spacetime. As for the actual linearized Einstein equations, these were first addressed, in the case of the Schwarzschild background, by Dafermos, Holzegel and Rodnianski in [DHR]. There is a fundamental difference of this problem from the problem of wave equation on Schwarzschild. This is the fact there are infinitesimal perturbations of the initial data for Schwarzschild which lead to solutions of the linearized equations *which do not decay in time*. These are of two kinds. First, we have the perturbations which correspond to initial data for nearby Kerr solutions, including that to a Schwarzschild solution of slightly different mass. And second, we have the perturbations which correspond to a different description of the same Schwarzschild solution, in particular the description from the point of view of a slightly moving frame. These must be considered separately, and modded out, in order to obtain solutions which decay in time. In the case of the second kind of perturbations this is done in [DHR] by adding what the authors call "a pure gauge solution".

The paper [DHR] makes use of a physical space version of of the Chandrasekhar transformation to prove boundedness and decay of the solutions of the Teukolsky equations. This approach, of estimating the extreme curvature components by passing from the Teukolsky equations to Regge-Wheeler type equations to which the methods developed for the treatment of the wave equation can be applied, is basic to all further developments in the subject.

In the case of the linearized Einstein equations about a member of the Kerr family with $|J|/M^2 \ll 1$ the papers [Ma] and [DHRa] were the first to study the Teukolsky equations for the extreme curvature components, proving boundedness and decay. These results were superceeded by the results of Teixeira da Costa and Y. Shlapentokh-Rothman [TdCSR1], [TdCSR2] which estabish boundedness and decay for the Teukolsky equations on Kerr in the full subextremal range $|J|/M^2 < 1$. Also, subsequent to [Ma] and [DHRa], the first linear stability results for the full system of linearized Einstein equations about Kerr with $|J|/M^2 \ll 1$ were established independently by [ABBM] and [HHV].

Passing from linearized theory to the actual nonlinear theory involves not only adapting the techniques developed in the linear framework to the situation where the background is not fixed but is itself evolving, but, more importantly, in handling the aspects of the nonlinear problem which are entirely absent in linearized theory. These aspects are in connection with the fact that a finite perturbation of the initial data results in waves escaping to infinity and carrying off energy, linear and angular momentum. As a consequence of energy and angular momentum being carried off, the mass and spin of the final black hole differ from the corresponding quantities associated to the initial black hole. As a consequence of linear momentum being carried off, the final black hole is not at rest relative to the center of mass frame of the initial black hole, but is moving relative to it with a certain velocity. The corresponding momentum, called *recoil momentum*, is familiar to anyone who has fired a bullet with a rifle. These back reaction effects are entirely absent in linearized theory, being quadratic in the amplitude of the perturbation. Capturing the mass, spin and recoil momentum of the final black hole is crucial in applying a decay argument along the lines of the linearized theory, because it is relative to the final background that the waves decay.

As already mentioned above, the monograph [KSa] by Klainerman and Szeftel was the first to address the problem of stability of black holes in the nonlinear theory, albeit under the special symmetry condition of polarized axisymmetry. Under this condition the angular momentum vanishes throughout the evolution, thus so does the initial as well as the final black hole spin. Nevertheless linear momentum as well as energy is carried off to infinity by the waves, thus there is indeed non-trivial recoil momentum, except for the fact that it is aligned with the axis of symmetry.

To capture the final background Klainerman and Szeftel introduced, first in the polarized axisymmetric context of the monograph [KSa] and then in the general context of perturbations of an initial Kerr black hole with small angular momentum to squared mass ratio in [KS1] and [KS2], a new geometric concept which they call *generally covariant modulated* or GCM surfaces. It is by means of these surfaces, which come with a distinguished pair of null vectorfields defined along them, that they succeed in capturing the final background in the papers [KS] and [GKS]. The geometric construction is "teleological" in so far as it starts from the essential part Σ_* of a spacelike hypersurface at late times (see in this connection [Sh]), in the framework of a bootstrap argument, and proceeds toward the past to a neighborhood of the hypersurface where the initial data are given. We postpone further discussion of the ideas introduced in the papers [KS] and [GKS] until after we have described the content of the paper [DHRT] which preceded them.

As already mentioned above, the paper [DHRT] involves the selection of the initial data of a codimension 3 subset chosen so that the final black hole spin vanishes, thus the final background is that of a Schwarzschild solution. This allows the authors to make use of the framework and method of [DHR]. As we mentioned above in our discussion of the paper [DHR], there are two kinds of infinitesimal perturbations of the initial data for Schwarzschild, which lead to solutions of the linearized equations *which do not decay in time*, namely the perturbations which correspond to initial data for nearby Kerr solutions, and the perturbations which correspond to a different description of the same Schwarzschild solution, in particular the description from the point of view of a slightly moving frame, and that in [DHR] a pure gauge solution is added to mod out the second kind of perturbations.

What is meant in [DHRT] by gauge is a double null coordinate system based on a pair of optical functions (u, v), the level sets of u and v being outgoing and incoming null hypersurfaces respectively. Two teleologically defined such gauges are employed in [DHRT], one of which is designed to capture a neighborhood of the future event horizon, boundary of the black hole, and the other is designed to give a precise description of future null infinity. In the bootstrap argument involved in the nonlinear setting of [DHRT] there is a final outgoing null hypersurface, taken as far as we wish toward the future in the course of the argument, which anchores the two gauges by being a level set of both of the associated u optical functions. The relation of the two teleologigally defined gauges to the initial data is described in terms of the relation of each to two auxiliary gauges associated to the initial data.

The drawback of this approach from the physical point of view is that the recoil momentum, an intrinsically nonlinear effect, is not manifest, being hidden in the transformation between the teleological gauges and the auxiliary gauges and swamped by the linear effect due to the second kind of perturbation mentioned above, which is dominant for small amplitude perturbations.

We finally come to the papers [KS] and [GKS]. As already mentioned, the main new concept here on which the whole geometric construction depends is that of a GCM surface. In particular this allows a derivation of the recoil momentum : see Remark 8.31 of [KS] and page 2889 of [GKS].

Another notion which plays an important role, in particular in [GKS], is that of *non-integrable horizontal structures.* This notion, first introduced in the paper [GKSa], and the related notion of *principal geodesic structures* introduced in [KS], is natural to perturbations of Kerr because Kerr spacetime itself possesses distinguished pairs of null vectors, the principal null vectors, which span timelike planes, such that the orthogonal distribution of spacelike planes is not integrable. With the aid of this notion generalized Regge-Wheeler equations are derived to which the methods used to treat the wave equation on Kerr can be applied. In this connection a central role is played in [GKS] by the extension of the Andersson-Blue method of [AB] to these generalized Regge-Wheeler equations, as this gives Morawetz-type integrated energy decay estimates from which the required actual decay estimates are derived without recourse to frequency decompositions, provided that $|J|/M^2$ is suitably small.

A simple statement of the Kerr stability theorem proved in [KS] and [GKS] is the following:

Theorem: *The maximal future development of a general, asymptotically flat, initial data set, suitably close to a Kerr initial data set of mass M_0 and angular momentum J_0 and sufficiently small ratio $|J_0|/M_0^2$, possesses a complete future null infinity and converges in its causal past to another nearby Kerr spacetime Kerr of mass M_∞ and angular momentum J_∞ close* *to M_0 and J_0 respectively. The future event horizon is then the boundary of the causal past of future null infinity in the maximal future development.*

Although this result marks a true milestone, it cannot be the end of the line for the black hole stability problem, because of the smallness restriction on the ratio $|J|/M^2$. The general case of the full subextremal range $|J|/M^2 < 1$ remains open. In this regard, the papers [TdCSR1], [TdCSR2] and [Mi] suggest that the general case may not remain open for too long.

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