Optimal bounds on the Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky equation, i. e.

$$\partial_t u + \partial_x(\tfrac{1}{2}u^2) + \partial_x^2 u + \partial_x^4 u \;=\; 0$$

is a "normal form" for many processes which lead to complex dynamics in space and time (one example is the roughening of the crystal surface in epitaxial growth). Numerical simulations show that after an initial layer, the statistical properties of the solution are independent of the initial data and the system size L (defined by the period $u(t,x+L) = u(t,x)$). More precisely, the energy $\int u^2\,dx$ is equally distributed over all wave numbers $|k| \ll 1$.

Unfortunately, PDE theory is far from a rigorous understanding of this phenomenon. Over the past 20 years, bounds on the space-time average $\langle(|\partial_x|^\alpha u)^2\rangle^{1/2}$ of (fractional) derivatives $|\partial_x|^\alpha u$ of u in terms of L have been established and improved. The best available result states that $\langle(|\partial_x|^\alpha u)^2\rangle^{1/2}$ $= o(L)$ for all $0 \le \alpha \le 2$.

In this talk, I shall present the new bound

$$\langle(|\partial_x|^\alpha u)^2\rangle^{1/2} \;=\; O(\ln^{5/3} L)$$

for $1/3 < \alpha \le 2$. This seems the first result in favor of an extensive behavior — albeit only up to a logarithm and for a restricted range of fractional derivatives.

The proof estentially relies on an extension of Oleinik's principle to the inhomogeneous inviscid Burgers' equation $\partial_t u + \partial_x(\frac{1}{2}u^2) \;=\; f$. ¿From this extension we learn that the quadratic term $\partial_x(\frac{1}{2}u^2)$, which is conservative, *acts* like a coercive term in the sense that we obtain a priori estimates as if $\int \partial_x(\frac{1}{2}u^2)\,u\,dx \;\sim\; \int ||\partial_x|^{1/3}u|^3\,dx$.

If time permits, we will show how another aspect of the proof yields a bound on the energy spectrum of the forced Navier-Stokes equation. Part of the presented work is joint with Dorian Goldman and with Fabio Ramos.