

# THE EXTREMALS OF MINKOWSKI'S QUADRATIC INEQUALITY

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ABSTRACT. In a seminal paper “Volumen und Oberfläche” (1903), Minkowski introduced the basic notion of mixed volumes and the corresponding inequalities that lie at the heart of convex geometry. The fundamental importance of characterizing the extremals of these inequalities was already emphasized by Minkowski himself, but has to date only been resolved in special cases. In this paper, we completely settle the extremals of Minkowski’s quadratic inequality, confirming a conjecture of R. Schneider. Our proof is based on the representation of mixed volumes of arbitrary convex bodies as Dirichlet forms associated to certain highly degenerate elliptic operators. A key ingredient of the proof is a quantitative rigidity property associated to these operators.

## 1. INTRODUCTION

**1.1. History of the problem.** The systematic study of the geometry of convex bodies dates back to the work of Brunn and Steiner in the 1880s. It is however arguably the work of Minkowski that laid the foundation for the modern theory of convex geometry. In his seminal paper “Volumen und Oberfläche” (1903) [29], and in an unfinished manuscript “Theorie der konvexen Körper” that was published posthumously [30], Minkowski introduced the basic notion of mixed volumes and the corresponding inequalities that play a central role in the modern theory [8, 40]. The aim of this paper is to settle a fundamental question arising from Minkowski’s original paper that has hitherto remained open.

As was customary at that time, Minkowski restricted attention to 3-dimensional bodies. While our main results are formulated in any dimension, let us first explain the problem investigated here in its original context. Let  $K_1, K_2, K_3$  be convex bodies in  $\mathbb{R}^3$  (throughout this paper, a *convex body* is a nonempty compact convex set). The starting point for Minkowski’s theory is the fact that the volume of convex bodies is a homogeneous polynomial: for any  $\lambda_1, \lambda_2, \lambda_3 \geq 0$ , we have

$$\text{Vol}(\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3) = \sum_{i_1, i_2, i_3=1}^3 \mathsf{V}(K_{i_1}, K_{i_2}, K_{i_3}) \lambda_{i_1} \lambda_{i_2} \lambda_{i_3}, \quad (1.1)$$

where we denote  $\lambda K + \mu L := \{\lambda x + \mu y : x \in K, y \in L\}$ . The coefficients  $\mathsf{V}(K, L, M)$  in this polynomial are called *mixed volumes*. They are nonnegative, symmetric in their arguments, and linear in each argument. Mixed volumes admit various natural geometric interpretations, and give rise to many familiar notions as special cases.

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For example, if  $K$  is any convex body and  $B$  denotes the (Euclidean) unit ball, then the volume, surface area, and mean width of  $K$  may be expressed as

$$\text{Vol}(K) = \mathbb{V}(K, K, K), \quad \text{S}(K) = 3\mathbb{V}(B, K, K), \quad \text{W}(K) = \frac{3}{2\pi}\mathbb{V}(B, B, K),$$

respectively. We refer to [40, 9, 8] for a detailed exposition.

Once the central role of mixed volumes has been realized, it is natural to expect that many geometric properties of convex bodies may be expressed in terms of relations between mixed volumes. This perspective lies at the heart of Minkowski's theory. In particular, Minkowski established [29, p. 479] the following fundamental inequality for three convex bodies  $K, L, M$  in  $\mathbb{R}^3$ :

$$\mathbb{V}(K, L, M)^2 \geq \mathbb{V}(K, K, M)\mathbb{V}(L, L, M). \quad (1.2)$$

We refer to (1.2) as (the 3-dimensional case of) *Minkowski's quadratic inequality*. This inequality unifies many classical geometric inequalities for 3-dimensional convex bodies, and gives rise to numerous new ones.

**Example 1.1.** The special cases of (1.2) that involve only  $\text{Vol}$ ,  $\text{S}$ , and  $\text{W}$  are

$$\text{S}(K)^2 \geq 6\pi \text{W}(K) \text{Vol}(K), \quad \pi \text{W}(K)^2 \geq \text{S}(K).$$

When combined, these recover the classical isoperimetric and Urysohn inequalities

$$\text{S}(K)^3 \geq 36\pi \text{Vol}(K)^2, \quad \text{W}(K)^3 \geq \frac{6}{\pi} \text{Vol}(K).$$

**Example 1.2.** Define  $C_t := (1-t)K + tL$  for  $t \in [0, 1]$ . Then  $\frac{d^2}{dt^2} \text{Vol}(C_t)^{1/3} = -\frac{2}{(1-t)^2} \text{Vol}(C_t)^{-5/3} \{ \mathbb{V}(L, C_t, C_t)^2 - \mathbb{V}(L, L, C_t)\mathbb{V}(C_t, C_t, C_t) \} \leq 0$  by (1.2). Thus

$$\text{Vol}((1-t)K + tL)^{1/3} \geq (1-t)\text{Vol}(K)^{1/3} + t\text{Vol}(L)^{1/3},$$

that is, we recover the Brunn-Minkowski inequality as another consequence of (1.2).

These are only some basic examples of the breadth of implications of (1.2), cf. [9, 40]; many others are obtained from more general choices of convex bodies or from higher-dimensional analogues that will be discussed below.

Minkowski viewed (1.2) as a far-reaching generalization of the classical isoperimetric inequality. To deduce a genuine isoperimetric statement, however, an inequality in itself does not suffice: one must also understand the associated extremal problem. For example, the classical isoperimetric theorem states that among all bodies  $K$  with fixed volume, surface area is minimized if and only if  $K$  is a ball. This extremal principle follows from the isoperimetric inequality  $\text{S}(K)^3 \geq 36\pi \text{Vol}(K)^2$  once it is understood that balls are the unique *equality cases* or *extremals* of this inequality: these are precisely the bodies for which the left-hand side is minimized when the right-hand side is fixed. It is far from obvious what is the analogous statement for the general inequality (1.2). These considerations motivate the following:

**Question.** For which  $K, L, M$  is equality attained in (1.2)?

Minkowski realized that striking new phenomena arise from this question already in the most basic example. Consider the inequality

$$\mathbb{V}(B, K, K)^2 \geq \mathbb{V}(B, B, K)\mathbb{V}(K, K, K), \quad (1.3)$$

which is equivalent to  $\text{S}(K)^2 \geq 6\pi \text{W}(K) \text{Vol}(K)$ . This inequality has the following isoperimetric interpretation: *among all bodies  $K$  with fixed volume and mean width,*

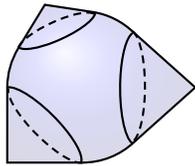


FIGURE 1.1. A cap body of the ball.

the surface area is minimized if and only if  $K$  attains equality in (1.3). Remarkably, it turns out that this generalized isoperimetric problem possesses many unusual extremals, in sharp contrast to the classical isoperimetric theorem. For example, equality holds in (1.3) whenever  $K$  is any cap body of the ball, that is, the convex hull of  $B$  with a finite or countable number of points so that the “caps” emanating from these points are disjoint (see Figure 1.1).

These observations motivate the attention paid by Minkowski to the extremals of his inequalities. In particular, in [29, p. 477], he asserts that the only extremals of (1.3) are cap bodies of the ball. No proof of this statement appears, however, in [29, 30], and it seems unlikely that Minkowski had a correct proof of this fact. Nonetheless, the statement is correct, as was shown 40 years later by Bol [7] using methods that are specific to this special case. On the other hand, Minkowski does not formulate any general conjecture on the extremals of (1.2). The characterization of these extremals is a long-standing open problem in convex geometry; see, for example, [15, p. 248], [1, p. 80], [8, §52 and §55], [36], [9, §20.5], [40, §7.6]. The main results of this paper will provide a complete solution to this problem.

**1.2. Setting and results.** The results of this paper will in fact be developed in a more general setting than the 3-dimensional case considered by Minkowski. We will presently describe the general setting considered in this paper, as well as how the results of this paper fit in a broader context.

The theory of mixed volumes is by no means restricted to three dimensions: a full theory was developed by many authors, following key contributions by Bonnesen, Fenchel, Favard, and Alexandrov. If  $K_1, \dots, K_m$  are convex bodies in  $\mathbb{R}^n$ , then

$$\text{Vol}(\lambda_1 K_1 + \dots + \lambda_m K_m) = \sum_{i_1, \dots, i_n=1}^m \mathbf{V}(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n} \quad (1.4)$$

for  $\lambda_1, \dots, \lambda_m \geq 0$  in direct analogy with the 3-dimensional case (1.1). This identity serves as the definition of mixed volumes in dimension  $n$ .

With this general definition in hand, one can directly adapt the proof of (1.2) to higher dimension [8, p. 99], which yields the inequality

$$\mathbf{V}(K, L, M, \dots, M)^2 \geq \mathbf{V}(K, K, M, \dots, M) \mathbf{V}(L, L, M, \dots, M) \quad (1.5)$$

for any convex bodies  $K, L, M$  in  $\mathbb{R}^n$ . We will henceforth refer to (1.5) as (the  $n$ -dimensional case of) *Minkowski's quadratic inequality*. The main result of this paper is a *complete characterization of the extremals of (1.5)*. In particular, the 3-dimensional case of our main result fully settles the extremal problem that arises in Minkowski's 1903 paper. As the structure of the family of extremals is somewhat intricate, we postpone a detailed description of our main results to section 2.

The characterization of the extremals provided by our main results was conjectured by Schneider [36] (where credit for a special case is given to Loritz) in a more general setting. We now briefly describe the broader context of Schneider’s conjectures, and how it relates to the setting of this paper.

Minkowski’s quadratic inequality (1.5) involves mixed volumes of three convex bodies  $K, L, M$ . In the 3-dimensional case, all mixed volumes are of this form. However, in the  $n$ -dimensional case, mixed volumes may in principle involve  $n$  distinct convex bodies. It was conjectured by Fenchel that Minkowski’s quadratic inequality can be generalized to arbitrary  $n$ -dimensional mixed volumes; such a general inequality, however, does not follow from Minkowski’s methods. Its existence was finally proved by Alexandrov in 1937 [1], who showed that

$$V(K, L, C_1, \dots, C_{n-2})^2 \geq V(K, K, C_1, \dots, C_{n-2})V(L, L, C_1, \dots, C_{n-2}) \quad (1.6)$$

for any convex bodies  $K, L, C_1, \dots, C_{n-2}$  in  $\mathbb{R}^n$ . The fundamental inequality (1.6) is known as the *Alexandrov-Fenchel inequality*.

The Alexandrov-Fenchel inequality has numerous applications and connections with various areas of mathematics and is a deep result in its own right, cf. [1, 40, 9, 2, 41]. The characterization of its equality cases is a well-known open problem that dates back to Alexandrov’s 1937 paper.<sup>1</sup> Not even a plausible conjecture was available on the nature of these equality cases, until detailed conjectures were proposed by Schneider in 1985 [36]. To date, only very limited cases of Schneider’s conjectures have been settled, cf. [40, section 7.6] and the references therein. Our main results confirm Schneider’s conjectures in the setting of Minkowski’s quadratic inequality (which already fully captures the 3-dimensional case). While a part of our proofs is specific to this setting (cf. Remark 6.3), many techniques that are introduced in this paper apply to general mixed volumes and may be of independent interest. Some key elements of our approach are described in section 1.3.

Beside their direct significance to the foundations of convex geometry, the extremals of Minkowski’s quadratic inequality and of the Alexandrov-Fenchel inequality are closely connected to several other problems. They arise, for example, in the study of infinitesimal rigidity of convex surfaces [17, 5], in the mixed analogue of Minkowski’s uniqueness problem [40, Theorem 7.4.2], and in graph theory [24]. Moreover, remarkable connections with algebraic geometry [9, section 27] relate extremals of the Alexandrov-Fenchel inequality to those of the Hodge index theorem, which are not well understood. These connections are orthogonal to the main contribution of this paper, so we will not develop them further here.

**1.3. Some key ideas.** In first instance, it may be expected that the characterization of the extremals of the Minkowski and Alexandrov-Fenchel inequalities should follow from a careful analysis of the proofs of these inequalities. It turns out, however, that none of the classical proofs provides information on the cases of equality: the proofs rely on strong regularity assumptions (such as smooth bodies or polytopes with restricted face directions) under which only trivial equality cases arise, and deduce the general result by approximation. The study of the nontrivial extremals requires one to work directly with general convex bodies, whose analysis gives rise to basic open questions in the foundation of convex geometry.

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<sup>1</sup>Alexandrov writes [1, p. 80]: “Serious difficulties occur in determining the conditions for equality to hold in the general inequalities just derived. This question has not been solved even for the Minkowski quadratic inequality in three dimensional space.”

From their definition, it is clear that mixed volumes are well defined for arbitrary convex bodies. The behavior of mixed volumes  $V(K_1, \dots, K_n)$  as a function of the bodies  $K_1, \dots, K_n$  is far from clear, however. In the early literature on convex geometry [29, 8], explicit formulae for mixed volumes were only available for smooth bodies or for polytopes, which do not provide a basis for the direct study of general convex bodies. An important advance in convex geometry, due independently to Fenchel and Jessen [16] and Alexandrov [1] in the late 1930s, was the explicit formulation of mixed volumes as a functional of a *single* body  $K \mapsto V(K, C_1, \dots, C_{n-1})$ . These authors discovered that for arbitrary convex bodies  $C_1, \dots, C_{n-1}$  in  $\mathbb{R}^n$ , there exists a finite measure  $S_{C_1, \dots, C_{n-1}}$  on  $S^{n-1}$  so that

$$V(K, C_1, \dots, C_{n-1}) = \frac{1}{n} \int h_K dS_{C_1, \dots, C_{n-1}},$$

where  $h_K : S^{n-1} \rightarrow \mathbb{R}$  denotes the support function  $h_K(x) := \sup_{y \in K} \langle y, x \rangle$ . The *mixed area measure*  $S_{C_1, \dots, C_{n-1}}$  plays a fundamental role in convex geometry: its introduction made it possible to study classical problems that were previously restricted to smooth bodies or polytopes (such as the Minkowski existence and uniqueness theorems [40, chapter 8]) in the setting of general convex bodies.

The Minkowski and Alexandrov-Fenchel inequalities, however, capture a more subtle phenomenon: (1.6) expresses the behavior of mixed volumes as a functional of *two* bodies  $(K, L) \mapsto V(K, L, C_1, \dots, C_{n-2})$ . The structure of this “quadratic form”, and in particular its spectral theory, is central both to proofs of the inequality and to the study of its extremals (cf. Lemma 3.1). To date, however, such a spectral theory is known to exist only for very smooth bodies or polytopes with identical face directions, by virtue of the special representations that are available in these cases. The latter suffices to prove the Minkowski and Alexandrov-Fenchel inequalities by approximation, but provides no information on their nontrivial extremals.

One of the most basic ideas contained in the present paper is the introduction of an explicit formulation of mixed volumes as a function of two bodies, which gives rise to a spectral theory for mixed volumes of arbitrary convex bodies. What we show (Theorem 5.1) is that for any collection of convex bodies  $C_1, \dots, C_{n-2}$  in  $\mathbb{R}^n$ , there is a self-adjoint operator  $\mathcal{A}_{C_1, \dots, C_{n-2}}$  on the Hilbert space  $L^2(S^{n-1}, S_{B, C_1, \dots, C_{n-2}})$  so that mixed volumes can be expressed as the quadratic form

$$V(K, L, C_1, \dots, C_{n-2}) = \langle h_K, \mathcal{A}_{C_1, \dots, C_{n-2}} h_L \rangle_{L^2(S^{n-1}, S_{B, C_1, \dots, C_{n-2}})}$$

for any convex bodies  $K, L$  (in particular,  $h_K$  is guaranteed to lie in the domain of the quadratic form for every convex body  $K$ ; see section 4.3 for a brief review of the relevant notions of functional analysis). The operator  $\mathcal{A}_{C_1, \dots, C_{n-2}}$  should be viewed as the natural bilinear counterpart of the mixed area measure  $S_{C_1, \dots, C_{n-1}}$ , providing a fundamental new tool for the study of mixed volumes of general convex bodies in the spirit of the Alexandrov-Fenchel-Jessen theory.

It should be emphasized that the introduction of mixed area measures and operators does not in itself solve any problem in convexity. For general convex bodies, these objects are quite abstract: even the structure of mixed area measures is still not fully understood more than 80 years after their introduction. However, these objects provide an essential foundation for the formulation and investigation of concrete problems involving mixed volumes of general convex bodies. In the proof of our main results, we will exploit the fact that tractable expressions for the operator

$\mathcal{A}_{C_1, \dots, C_{n-2}}$  exist for smooth bodies and for polytopes to develop quantitative rigidity estimates that enable us to pass to the limit of general convex bodies (while the existence of such expressions in the smooth case dates back to Hilbert [21], they appear to be new even for polytopes where they give rise to “quantum graphs” [4]). The structure of our proof, and in particular the quantitative estimates that lie at its core, will be described in further detail in section 3.

**1.4. Organization of this paper.** The rest of this paper is organized as follows. The main results of this paper are formulated in section 2. Section 3 is devoted to a high-level overview of the main ingredients of our proofs, and of how they fit together. Section 4 reviews, mostly without proofs, some basic results of convex geometry and functional analysis that will be used throughout the paper. Section 5 is devoted to the construction and basic properties of the self-adjoint operators that form the foundation for our analysis. Sections 6 and 7 develop the two main ingredients of the proof of our main result: a weak stability theorem and a quantitative rigidity theorem for Minkowski’s inequality. In particular, the proof of our main results will be completed in section 7 in the case that  $M$  in (1.5) is full-dimensional. Finally, section 8 is devoted to the case that  $M$  in (1.5) is lower-dimensional.

## 2. THE EXTREMALS OF MINKOWSKI’S QUADRATIC INEQUALITY

**2.1. Main results.** Before we can formulate our main results, we must recall some basic notions of convex geometry. Here and throughout the paper, our standard reference on convexity will be the monograph [40].

Throughout the remainder of this paper we fix the dimension  $n \geq 3$ . A convex body is a nonempty compact convex subset of  $\mathbb{R}^n$ . Given a convex body  $K$  and a vector  $u \in \mathbb{R}^n \setminus \{0\}$ , we denote by  $F(K, u)$  the unique exposed face of  $K$  with outer normal vector  $u$ . The supporting hyperplane of  $K$  in the normal direction  $u$  is the hyperplane  $F(K, u) + u^\perp$ . A key role in the present context will be played by normal directions that are extreme in the following sense [40, p. 85].

**Definition 2.1.** A vector  $u \in \mathbb{R}^n \setminus \{0\}$  is called an  $r$ -extreme normal vector of a convex body  $K$  if there do not exist linearly independent normal vectors  $u_1, \dots, u_{r+2}$  at a boundary point of  $K$  such that  $u = u_1 + \dots + u_{r+2}$ .

For example, if  $K$  is a polytope, then  $u$  is an  $r$ -extreme normal vector if and only if it is an outer normal of a face of  $K$  of dimension  $\dim F(K, u) \geq n - 1 - r$ .

We are now ready to describe the extremals of Minkowski’s quadratic inequality (1.5). We must distinguish several cases, depending on the dimension of  $M$ . Let us observe at the outset that (1.5) is invariant under translation and scaling of each body, so that the extremals must be invariant under homothety as well.

We begin by considering the main case where  $M$  has nonempty interior. The following result confirms a conjecture of Schneider [36], cf. [40, section 7.6].

**Theorem 2.2 (Extremals: full-dimensional case).** *Let  $M \subset \mathbb{R}^n$  be a convex body with nonempty interior, and let  $K, L \subset \mathbb{R}^n$  be arbitrary convex bodies such that  $\mathbb{V}(L, L, M, \dots, M) > 0$ . Then we have*

$$\mathbb{V}(K, L, M, \dots, M)^2 = \mathbb{V}(K, K, M, \dots, M) \mathbb{V}(L, L, M, \dots, M)$$

*if and only if there exist  $a \geq 0$  and  $v \in \mathbb{R}^n$  such that  $K$  and  $aL + v$  have the same supporting hyperplanes in all 1-extreme normal directions of  $M$ .*

When  $M$  is lower-dimensional, the conclusion of Theorem 2.2 is no longer valid and additional extremals appear. A suitable modification of Schneider's conjecture in this case was proposed by Ewald and Tondorf [14], see also [39, section 4.2]. The following result confirms this conjecture in the present setting.

**Theorem 2.3 (Extremals: lower-dimensional case).** *Let  $M \subset \mathbb{R}^n$  be a convex body with empty interior, so that  $M - M \subset w^\perp$  for some  $w \in S^{n-1}$ . Let  $K, L \subset \mathbb{R}^n$  be arbitrary convex bodies such that  $\mathcal{V}(L, L, M, \dots, M) > 0$ . Then we have*

$$\mathcal{V}(K, L, M, \dots, M)^2 = \mathcal{V}(K, K, M, \dots, M) \mathcal{V}(L, L, M, \dots, M)$$

*if and only if  $\tilde{L} := \frac{\mathcal{V}(K, L, M, \dots, M)}{\mathcal{V}(L, L, M, \dots, M)}L$  satisfies that  $K + F(\tilde{L}, w)$  and  $\tilde{L} + F(K, w)$  have the same supporting hyperplanes in all 1-extreme normal directions of  $M$ .*

The only case that remains to be considered is  $\mathcal{V}(L, L, M, \dots, M) = 0$ , in which case equality in (1.5) can only arise for the trivial reason  $\mathcal{V}(K, L, M, \dots, M) = 0$ . The characterization of positivity of mixed volumes is well-known and is unrelated to the questions in this paper. For completeness, we spell out the resulting characterization without further comment [40, Theorem 5.1.8].

**Theorem 2.4 (Extremals: trivial case).** *Let  $K, L, M \subset \mathbb{R}^n$  be convex bodies such that  $\mathcal{V}(L, L, M, \dots, M) = 0$ . Then we have*

$$\mathcal{V}(K, L, M, \dots, M)^2 = \mathcal{V}(K, K, M, \dots, M) \mathcal{V}(L, L, M, \dots, M)$$

*if and only if one of the following holds:  $\dim K = 0$ ;  $\dim L = 0$ ;  $\dim(K + L) \leq 1$ ;  $\dim M \leq n - 3$ ;  $\dim(K + M) \leq n - 2$ ;  $\dim(L + M) \leq n - 2$ ;  $\dim(K + L + M) \leq n - 1$ .*

We also note for completeness that  $\mathcal{V}(L, L, M, \dots, M) = 0$  holds if and only if  $\dim L \leq 1$ ,  $\dim M \leq n - 3$ , or  $\dim(L + M) \leq n - 1$ . Thus the different cases covered by Theorems 2.2–2.4 are completely determined by the dimensions of  $L, M, L + M$ .

**2.2. Examples and prior work.** In order to understand the statement of Theorem 2.2, it is instructive to revisit the case of cap bodies. It is readily seen by inspection of Figure 1.1 that the normal directions of a 3-dimensional cap body  $K$  that are *not* 1-extreme are exactly those that lie in the interior of the normal cone of the vertex of one of its caps. Therefore, any supporting hyperplane of  $K$  whose normal direction is 1-extreme also supports the ball  $B$  from which the cap body was formed. Thus Theorem 2.2 explains precisely why cap bodies arise as the extremals of the special case (1.3) of Minkowski's quadratic inequality.

The extremals of (1.3), and more generally of the case  $L = M$  of Theorem 2.2, were obtained by Bol in 1943 [7]. Prior to the present paper, this was the only case of (1.5) whose extremals have been characterized for general convex bodies. Despite its importance, Bol's proof turns out to be fundamentally based on the following very special feature of the case  $L = M$ . Note that if equality is attained in (1.5), we may always assume (by rescaling  $L$ ) that

$$\mathcal{V}(K, K, M, \dots, M) = \mathcal{V}(K, L, M, \dots, M) = \mathcal{V}(L, L, M, \dots, M) \quad (2.1)$$

as (1.5) is invariant under scaling. By means of a delicate geometric argument using the method of inner parallel bodies, Bol was able to show that if (2.1) holds with  $L = M$ , then it must be the case that  $K \subseteq L$  (up to translation of  $L$ ). This reduces the extremal problem of Minkowski's quadratic inequality to equality cases of the monotonicity of mixed volumes, which were already characterized in the case

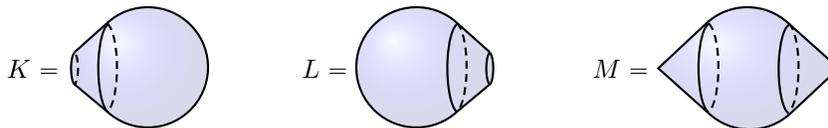


FIGURE 2.1. A non-monotone equality case.

$L = M$  by Minkowski himself [30, p. 227] (cf. [40, Theorem 7.6.17]). Unfortunately, the entire premise of this argument fails for general  $K, L, M$ .

**Example 2.5.** Let  $M = \text{conv}\{B, x, -x\}$ , where  $B$  is a ball in  $\mathbb{R}^3$  and  $x \notin B$ . Let  $L$  be any body that has the same supporting hyperplanes as  $B$  in all normal directions except those in the interior of the normal cone of  $M$  at  $x$ , and let  $K = -L$  (cf. Figure 2.1). Then the supporting hyperplanes of  $K$  and  $L$  coincide in all 1-extreme normal directions of  $M$ , so by Theorem 2.2 we have equality in (1.5). Moreover, by symmetry  $V(K, K, M, \dots, M) = V(L, L, M, \dots, M)$ , so (2.1) holds. But clearly  $K + v \not\subseteq L$  and  $L \not\subseteq K + v$  for any  $v$ . Thus (2.1) need not imply any monotonicity.

While the method of inner parallel bodies that is introduced by Bol has found many other applications in convex geometry, the above example illustrates why Bol's methods cannot explain the more general extremals of (1.5).

Other than Bol's theorem, special cases of the extremal problem have been proved under various regularity assumptions on the bodies involved. If  $M$  is a smooth convex body (that is, each boundary point has a unique normal), then all normal directions are 1-extreme, so that equality can arise only in the trivial case that  $K$  and  $L$  are homothetic. The lack of nontrivial extremals in this case was proved by Hilbert [21] when  $M$  has a sufficiently smooth boundary (see section 3.1), and by Schneider [38] for general smooth bodies. Similarly, no nontrivial equality cases can arise when  $K, L, M$  are all strongly isomorphic simple polytopes; this case follows from Alexandrov's proof of the Alexandrov-Fenchel inequality [1]. Another special case, where  $M$  is a zonoid and  $K, L$  are symmetric bodies, was proved by Schneider in [37]. The case that  $M$  is lower-dimensional was investigated by Ewald and Tondorf [14] when  $K, L, M$  are polytopes; a simple example that illustrates Theorem 2.3 may be found in their paper (see also [40, p. 429]).

Most importantly for our purposes, the validity of Theorems 2.2 and 2.3 was previously established by Schneider in the case that  $M$  is a simple polytope and  $K, L$  are arbitrary convex bodies; cf. [40, Theorem 7.6.21] and [39, Theorem 4.2] (in the lower-dimensional case,  $M$  need not be simple). As will be explained below, our approach to Theorems 2.2 and 2.3 was partially inspired by Schneider's results for polytopes. The key contribution of this paper, however, is the introduction of new tools that open the door to the investigation of arbitrary convex bodies.

### 3. OVERVIEW

The formulation of our main results is of a purely geometric nature. Nonetheless, it will shortly become clear that the core difficulty in our proofs does not arise from geometry, but rather from analytic questions: at the heart of our results lies an analysis of the behavior of certain highly degenerate elliptic operators. The aim of this section is to give a high-level overview of the main ingredients of our proofs and how they fit together, in order to help the reader navigate the rest of the paper.

We have deliberately kept this overview as concise as possible, as further discussion is best postponed until after formal definitions have been given.

**3.1. The Hilbert method.** The original proofs derive Minkowski's quadratic inequality as a limiting case of other inequalities, such as the Brunn-Minkowski inequality [8, sections 49 and 52]. These proofs do not lend themselves to the study of extremals, however, as extremals are not preserved by taking limits. Instead, our starting point will be the direct approach to Minkowski's inequality due to Hilbert [21, Chapter XIX] (cf. [8, section 52]), which also forms the foundation for the Alexandrov-Fenchel inequality. Let us briefly explain its basic premise.

It will be convenient to identify a convex body  $K$  with its support function

$$h_K(u) := \sup_{y \in K} \langle y, u \rangle.$$

Geometrically, if  $u \in S^{n-1}$ , then  $h_K(u)$  is the signed distance to the origin of the supporting hyperplane of  $K$  in the direction  $u$ ; as any convex body is the intersection of its supporting halfspaces,  $h_K : S^{n-1} \rightarrow \mathbb{R}$  uniquely determines  $K$ . Support functions satisfy  $h_{\lambda K + \mu L} = \lambda h_K + \mu h_L$  for  $\lambda, \mu \geq 0$ , i.e., they map linear operations on sets to linear operations on functions. In particular, we may view

$$(h_K, h_L) \mapsto \mathcal{V}(K, L, M, \dots, M)$$

as a symmetric quadratic form of the support functions. By linearity of mixed volumes, this quadratic form can be uniquely extended to the linear space of differences of support functions (which contains  $C^2(S^{n-1})$ , cf. Lemma 4.5 below). Minkowski's inequality (1.5) is nothing other than the statement that this quadratic form satisfies a reverse form of the Cauchy-Schwarz inequality.

We are therefore led to ask which quadratic forms satisfy reverse Cauchy-Schwarz inequalities. A particularly useful characterization arises if we consider closed symmetric quadratic forms on a Hilbert space, i.e., forms  $\mathcal{E}(f, g) = \langle f, \mathcal{A}g \rangle$  associated to a self-adjoint operator  $\mathcal{A}$ . The reason this setting is powerful is that we can bring spectral theory to bear on the problem:  $\mathcal{E}$  satisfies a reverse Cauchy-Schwarz inequality if and only if  $\mathcal{A}$  has a one-dimensional positive eigenspace. Let us formulate for future reference a general statement for (possibly unbounded) self-adjoint operators on a Hilbert space, whose proof is given in section 4.3.<sup>2</sup>

**Lemma 3.1 (Hyperbolic quadratic forms).** *Let  $\mathcal{A}$  be a self-adjoint operator on a Hilbert space  $H$  with  $0 < \text{sup spec } \mathcal{A} < \infty$ , and let  $\mathcal{E}(f, g)$  be the associated closed quadratic form. Then the following are equivalent:*

1.  $\mathcal{E}(f, g)^2 \geq \mathcal{E}(f, f) \mathcal{E}(g, g)$  for all  $f, g \in \text{Dom } \mathcal{E}$  such that  $\mathcal{E}(g, g) > 0$ .
2.  $\text{rank } 1_{(0, \infty)}(\mathcal{A}) = 1$ .

*Moreover, if either (hence both) of these conditions is satisfied, the following are equivalent for given  $f, g \in \text{Dom } \mathcal{E}$  such that  $\mathcal{E}(g, g) > 0$ :*

- 1'.  $\mathcal{E}(f, g)^2 = \mathcal{E}(f, f) \mathcal{E}(g, g)$ .
- 2'.  $f - ag \in \ker \mathcal{A} \subset \text{Dom } \mathcal{A}$  for some  $a \in \mathbb{R}$ .

<sup>2</sup>Here and in the sequel, we use the following notation for a self-adjoint operator  $\mathcal{A}$  and the associated closed quadratic form  $\mathcal{E}$ :  $\text{spec } \mathcal{A}$  denotes the spectrum,  $\ker \mathcal{A}$  the kernel, and  $\text{rank } \mathcal{A}$  the rank (i.e., the dimension of the range) of  $\mathcal{A}$ ; and  $\text{Dom } \mathcal{A}$  and  $\text{Dom } \mathcal{E}$  denote the domains of  $\mathcal{A}$  and  $\mathcal{E}$ . A brief review of the relevant notions of functional analysis may be found in section 4.3.

It is not clear, a priori, that mixed volumes fit into the setting of Lemma 3.1. However, Hilbert realized that *for sufficiently smooth bodies*, a classical representation formula of Minkowski (see, e.g., Lemma 4.7) can be used to write

$$V(K, L, M, \dots, M) = \langle h_K, \mathcal{A}h_L \rangle_{L^2(\omega)},$$

where  $\omega$  denotes the surface measure on  $S^{n-1}$  and  $\mathcal{A}$  is a certain elliptic second order differential operator. Elliptic regularity theory (cf. section 4.4) shows that  $\mathcal{A}$  has a self-adjoint extension on  $L^2(\omega)$  with compact resolvent. Hilbert exploited this regularity, using a clever homotopy argument, to prove condition 2 of Lemma 3.1. This yields Minkowski's inequality for smooth bodies  $M$ , and the general case follows by approximation. As a key step in his proof, Hilbert shows that

$$\ker \mathcal{A} = \{x \mapsto \langle v, x \rangle : v \in \mathbb{R}^n\}$$

consists precisely of the linear functions restricted to  $S^{n-1}$ . Thus for *smooth* bodies  $M$ , Lemma 3.1 implies that equality holds in Minkowski's inequality if and only if  $h_K - ah_L = \langle v, \cdot \rangle$  for some  $a, v$ , i.e., when  $K, L$  are homothetic  $K = aL + v$ .

The above discussion shows that, while innocent for the purpose of proving Minkowski's inequality, the smoothness assumption on  $M$  completely eliminates its nontrivial extremals. The key difficulty we face in the analysis of extremals is to make sense of the above ideas for arbitrary non-smooth bodies  $M$ .

**3.2. Mixed volumes and Dirichlet forms.** It is far from clear that it is possible, even in principle, to define self-adjoint representations of mixed volumes for non-smooth bodies  $M$ . While we have argued above that  $(h_K, h_L) \mapsto V(K, L, M, \dots, M)$  defines a symmetric quadratic form on  $C^2(S^{n-1})$ , it is a basic fact of functional analysis that not every such form is associated to a self-adjoint operator. A self-adjoint representation is crucial, however, for the applicability of spectral theory in the infinite-dimensional setting. The basic criterion for a quadratic form to admit a self-adjoint representation is that it is *closable*, see section 4.3 below. This property may not be taken for granted. For example, one may verify that when  $M$  is a polytope, the quadratic form  $(h_K, h_L) \mapsto V(K, L, M, \dots, M)$  is not closable in  $L^2(\omega)$  and therefore does not admit any self-adjoint representation on this space. For this reason, self-adjoint representations have only appeared in the literature in special cases: for smooth bodies (due to Hilbert [21]), and for certain special families of polytopes with common face normals (due to Alexandrov [1]).

The first step towards our analysis of extremals is the realization that mixed volumes do in fact admit a self-adjoint representation for *any* body  $M$ , provided one chooses the Hilbert space appropriately. Let us recall that the mixed area measure  $S_{K_1, \dots, K_{n-1}}$  on  $S^{n-1}$  was introduced in section 1.3.

**Theorem 3.2.** *For any convex body  $M$ , there is a self-adjoint operator  $\mathcal{A}$  on  $L^2(S_{B, M, \dots, M})$  such that the associated closed quadratic form  $\mathcal{E}(f, g)$  satisfies*

$$h_K, h_L \in \text{Dom } \mathcal{E} \quad \text{and} \quad V(K, L, M, \dots, M) = \mathcal{E}(h_K, h_L)$$

for all convex bodies  $K, L$ .

The operator  $\mathcal{A}$  of Theorem 3.2 is a highly degenerate elliptic operator in the sense of the theory of Dirichlet forms [18, 3]. Its existence and basic properties will be investigated in section 5 (in the general setting of arbitrary mixed volumes). While this operator is, in general, a rather abstract object, it provides the foundation for extending Hilbert's method to arbitrary convex bodies  $M$ .

Theorem 3.2 and Lemma 3.1 reduce the study of extremals of Minkowski's inequality to the characterization of  $\ker \mathcal{A}$ . Translation-invariance of mixed volumes implies that  $\ker \mathcal{A}$  always contains the linear functions, so that homothetic bodies  $K, L$  trivially yield equality. It should be emphasized at this point that there are two distinct mechanisms for the appearance of nontrivial extremals:

- a. It is possible that  $\ker \mathcal{A}$  may be strictly larger than the set of linear functions. Its additional (nonlinear) elements can give rise to new extremals.
- b. Even if  $\ker \mathcal{A}$  contains only linear functions, new extremals can arise as the underlying measure  $S_{B, M, \dots, M}$  need not be supported on the entire sphere. Thus Lemma 3.1 only guarantees that  $h_K - ah_L = \langle v, \cdot \rangle S_{B, M, \dots, M}$ -a.e.

The statements of Theorems 2.2 and 2.3 can now be interpreted in a new light. A fundamental result of Schneider (Theorem 4.4 below) states that

$$\text{supp } S_{B, M, \dots, M} = \text{cl}\{u \in S^{n-1} : u \text{ is a 1-extreme normal vector of } M\}.$$

Thus Theorem 2.2 shows that when  $M$  has nonempty interior, only mechanism b. arises.<sup>3</sup> In contrast, both a. and b. arise when the body  $M$  has empty interior, which explains the appearance of additional extremals in Theorem 2.3.

The proofs of these facts occupy the main part of this paper. The proof of Theorem 2.3 will turn out to be conceptually simpler, as in this case one can compute explicitly the operator  $\mathcal{A}$  and its kernel. This will be done in section 8. The main difficulty lies in the proof of Theorem 2.2, as in this case we do not have an explicit description of the operator  $\mathcal{A}$  that is amenable to computation.

**Remark 3.3.** That the extremals of (1.5) are determined by the support of the measure  $S_{B, M, \dots, M}$  was conjectured by Schneider [36] (in the general setting of the Alexandrov-Fenchel inequality). Even before we explain how the extremal characterization is proved, however, we observe that the abstract statement of Theorem 3.2 already demystifies why this particular measure appears: this is precisely the measure that must be chosen on the sphere so that the quadratic form  $(h_K, h_L) \mapsto \mathbb{V}(K, L, M, \dots, M)$  is closable. Thus the existence of a self-adjoint representation is far from being merely a technical device: it explains the fundamental structure of the extremals of Minkowski's quadratic inequality.

**3.3. Weak stability and rigidity.** The most natural starting point for understanding the extremals of a given inequality is to attempt to deduce these from a careful examination of the proof of the inequality. In the case of Minkowski's inequality, however, there is a surprising and apparently fundamental obstacle to such an approach. In view of Lemma 3.1 and Theorem 3.2, it is clear that  $S_{B, M, \dots, M}$  is the relevant measure for the study of the *extremals*. However, it is a different measure  $S_{M, \dots, M}$  that appears naturally (explicitly or implicitly) in the various known proofs of Minkowski's *inequality* [1, 26, 41].<sup>4</sup> As (cf. Theorem 4.4)

$$\text{supp } S_{M, \dots, M} = \text{cl}\{u \in S^{n-1} : u \text{ is a 0-extreme normal vector of } M\},$$

the support of  $S_{M, \dots, M}$  is generally much smaller than that of  $S_{B, M, \dots, M}$ , and does not suffice to characterize the extremals (see Example 3.5 below). Thus there

<sup>3</sup>More precisely, our results show that  $\ker \mathcal{A} \cap \{h_Q - h_R : Q, R \text{ convex bodies}\}$  consists of linear functions only, which is equivalent to Theorem 2.2 by Lemma 3.1 (see also Remark 8.8).

<sup>4</sup>We emphasize that the quadratic form  $(h_K, h_L) \mapsto \mathbb{V}(K, L, M, \dots, M)$  is not closable on  $L^2(S_{M, \dots, M})$  unless one assumes strong regularity assumptions on  $M$  (such as smoothness), which explains why regularity assumptions play a fundamental role in proofs of the inequality.

appears to be a disconnect between the setting of this paper that gives rise to a good analytic theory, and the setting used in the proofs of Minkowski's inequality that is needed to exploit the algebraic structure of mixed volumes.

The conceptual challenge behind the proof of Theorem 2.2 is to understand how to surmount this obstacle. To this end, we will prove that the operator  $\mathcal{A}$  satisfies a strong rigidity property: once the extremals of Minkowski's inequality have been fixed in the 0-extreme directions of  $M$ , their extension to the 1-extreme directions of  $M$  is uniquely determined. This property closes the gap between the information provided by proofs of Minkowski's inequality and its extremals.

More precisely, we will proceed in two steps. First, in section 6, we obtain weak control on the extremals by refining an approach to Minkowski's inequality that was recently developed by Kolesnikov and Milman [25, 26].

**Theorem 3.4.** *Let  $K, L, M$  be convex bodies so that  $M$  has nonempty interior and  $V(L, L, M, \dots, M) > 0$ . If equality holds in (1.5), then there exist  $a \geq 0$ ,  $v \in \mathbb{R}^n$  so that  $h_K(x) - ah_L(x) = \langle v, x \rangle$  for all  $x \in \text{supp } S_{M, \dots, M}$ .*

We emphasize that Theorem 3.4, on its own, provides only very weak information on the extremals, as is illustrated by the following example.

**Example 3.5.** When  $M$  is a polytope, the 0-extreme directions are precisely the facet normals. Thus, for example, if equality holds in (1.5) for convex bodies  $K$  and  $L = M$  with nonempty interior, then Theorem 3.4 only implies that  $K$  touches every facet of a homothet of  $M$ . While this is nontrivial geometric information, this property is much too weak to characterize the extremals. For example, if  $K$  is the unit ball and  $L = M$  is the unit cube, then  $K$  touches every facet of  $M$ , but this is not an equality case of Minkowski's inequality.

In the second step, we amplify the weak control provided by Theorem 3.4 to fully characterize the extremals. The following rigidity property, which will be proved in section 7, is one of the central results of this paper.

**Theorem 3.6.** *Let  $K, L, M$  be convex bodies so that  $M$  has nonempty interior and  $V(L, L, M, \dots, M) > 0$ . If equality holds in (1.5) and  $h_K(x) = h_L(x)$  for all  $x \in \text{supp } S_{M, \dots, M}$ , then necessarily  $h_K(x) = h_L(x)$  for all  $x \in \text{supp } S_{B, M, \dots, M}$ .*

Unfortunately, we cannot prove Theorems 3.4 and 3.6 directly, as we do not have a sufficiently explicit description of  $\mathcal{A}$  for general convex bodies. Instead, we will prove quantitative versions of both these theorems for special convex bodies. A quantitative form of Theorem 3.4, i.e., a weak stability form of Minkowski's inequality, will be proved for smooth bodies using Reilly's formula in Riemannian geometry. A quantitative form of Theorem 3.6, i.e., a quantitative rigidity theorem, will be proved for polytopes: in this case  $\mathcal{A}$  is a "quantum graph" (cf. section 5.2), and the proof will exploit a stability estimate for the solution of the Dirichlet problem for  $\mathcal{A}$  with boundary data on  $\text{supp } S_{M, \dots, M}$ . In both cases, a key aspect of the proof is to discover the correct quantitative formulation that does not degenerate when we take the appropriate limit to approximate arbitrary  $M$ .

**Remark 3.7.** The formulation of Theorems 3.4 and 3.6 was inspired by the proof of a result of Schneider [40, Theorem 7.6.21] for the case that  $M$  is a simple polytope. In this setting, the statement of Theorem 3.4 can be deduced from Alexandrov's polytope proof of the Alexandrov-Fenchel inequality [1], while the statement of

Theorem 3.6 follows from a form of Minkowski's uniqueness theorem. However, these qualitative statements do not allow one to pass to the limit of general convex bodies; the tools that are needed to do so are developed in this paper.

On the other hand, Schneider's result goes beyond Minkowski's inequality to cover some additional cases of the Alexandrov-Fenchel inequality. Similarly, most of the techniques that are developed in this paper are not specific to Minkowski's inequality, and extend to general mixed volumes. The only ingredient of this paper that is fundamentally restricted to Minkowski's classical setting is Theorem 3.4, for reasons that will become clear in section 6. While a replacement for this argument in the Alexandrov-Fenchel setting will require new ideas, we expect that the techniques that are introduced in this paper could provide a basis for further developments.

#### 4. PRELIMINARIES

The aim of this section is to recall some basic definitions and facts from convex geometry and functional analysis that will be used in the sequel.

We highlight at the outset the following convention. In Minkowski's inequalities and in many arguments in this paper, mixed volumes  $V(K, L, C_1, \dots, C_{n-2})$  of convex bodies  $K, L, C_1, \dots, C_{n-2}$  in  $\mathbb{R}^n$  are considered for fixed  $C_1, \dots, C_{n-2}$ , and only  $K$  and  $L$  are varied. We therefore introduce once and for all the notation

$$\mathcal{C} := (C_1, \dots, C_{n-2}), \quad \mathcal{M} := \underbrace{(M, \dots, M)}_{n-2},$$

and we will write  $V(K, L, \mathcal{C}) := V(K, L, C_1, \dots, C_{n-2})$ ,  $S_{B, \mathcal{M}} := S_{B, M, \dots, M}$ , etc. Throughout this paper, we always denote by  $B$  the Euclidean unit ball in  $\mathbb{R}^n$ .

By convention, we will consider linear operations  $\lambda_1 K_1 + \dots + \lambda_m K_m$  on convex bodies to always have positive coefficients  $\lambda_1, \dots, \lambda_m \geq 0$ . The linearity of mixed volumes  $K \mapsto V(K, C_1, \dots, C_{n-1})$  should be interpreted in this sense. In contrast, differences of support functions form a genuine linear space (over the reals).

**4.1. Mixed volumes and mixed area measures.** Mixed volumes are defined by (1.4). They have the following basic properties [40, section 5.1].

**Lemma 4.1.** *Let  $K, K', K_1, \dots, K_n$  be convex bodies in  $\mathbb{R}^n$ .*

- a.  $V(K, \dots, K) = \text{Vol}(K)$ .
- b.  $V(K_1, \dots, K_n)$  is a symmetric and multilinear functional of  $K_1, \dots, K_n$ .
- c.  $V(K_1, \dots, K_n) \geq 0$ .
- d.  $V(K, K_2, \dots, K_n) \geq V(K', K_2, \dots, K_n)$  if  $K \supseteq K'$ .
- e.  $V(K_1, \dots, K_n)$  is invariant under translation  $K_i \mapsto K_i + v_i$ .

There is a close connection between mixed volumes and mixed area measures that we describe presently. The *surface area measure* of a convex body  $K$  in  $\mathbb{R}^n$  is the measure on  $S^{n-1}$  such that for any measurable set  $A \subseteq S^{n-1}$

$$S(K, A) := \mathcal{H}^{n-1}(\{x \in \partial K : x \in F(K, u) \text{ for some } u \in A\}),$$

where  $\mathcal{H}^k$  denotes the  $k$ -dimensional Hausdorff measure. That is,  $S(K, A)$  is the surface area of the part of the boundary of  $K$  with outer normal vectors in  $A$ . Just like volume, the surface area measure  $S(K, \cdot)$  is polynomial in  $K$ , cf. [40, p. 279]:

$$S(\lambda_1 K_1 + \dots + \lambda_m K_m, A) = \sum_{i_1, \dots, i_{n-1}=1}^m S_{K_{i_1}, \dots, K_{i_{n-1}}}(A) \lambda_{i_1} \cdots \lambda_{i_{n-1}}$$

for  $\lambda_1, \dots, \lambda_m \geq 0$ . The measures  $S_{K_1, \dots, K_{n-1}}$  are called *mixed area measures*. They have the following properties [40, section 5.1].

**Lemma 4.2.** *Let  $K, K_1, \dots, K_{n-1}$  be convex bodies in  $\mathbb{R}^n$ .*

- a.  $S_{K, \dots, K} = S(K, \cdot)$ .
- b.  $S_{K_1, \dots, K_{n-1}}$  is a symmetric and multilinear functional of  $K_1, \dots, K_{n-1}$ .
- c.  $S_{K_1, \dots, K_{n-1}} \geq 0$ .
- d.  $S_{K_1, \dots, K_{n-1}}$  is invariant under translation  $K_i \mapsto K_i + v_i$ .
- e.  $\int \langle v, x \rangle S_{K_1, \dots, K_{n-1}}(dx) = 0$  for all  $v \in \mathbb{R}^n$ .

The key relation between mixed volumes and mixed area measures is the representation formula [40, Theorem 5.1.7] that we already introduced in section 1.3:

$$\mathbf{V}(K_1, \dots, K_n) = \frac{1}{n} \int h_{K_1} dS_{K_2, \dots, K_n}. \quad (4.1)$$

Note that while  $K_1$  and  $K_2, \dots, K_n$  play different roles in this representation, the expression is nonetheless symmetric under permutation of the  $K_i$  by Lemma 4.1(b).

We now record two important facts. First, mixed volumes and mixed area measures are continuous in the topology of Hausdorff convergence (i.e.,  $K^{(s)} \rightarrow K$  if and only if  $\|h_{K^{(s)}} - h_K\|_\infty \rightarrow 0$ ); see the proof of [40, Theorem 5.1.7].

**Theorem 4.3.** *Suppose that  $K_1^{(s)}, \dots, K_n^{(s)}$  are convex bodies such that  $K_i^{(s)} \rightarrow K_i$  as  $s \rightarrow \infty$  in the sense of Hausdorff convergence. Then*

$$\mathbf{V}(K_1^{(s)}, \dots, K_n^{(s)}) \rightarrow \mathbf{V}(K_1, \dots, K_n), \quad S_{K_1^{(s)}, \dots, K_{n-1}^{(s)}} \xrightarrow{w} S_{K_1, \dots, K_{n-1}}$$

as  $s \rightarrow \infty$ , where the limit of measures is in the sense of weak convergence.

Second, we have the following support characterization [40, Theorem 4.5.3].

**Theorem 4.4.** *Let  $M$  be a convex body in  $\mathbb{R}^n$  with nonempty interior. Then*

$$\begin{aligned} \text{supp } S_{M, \mathcal{M}} &= \text{cl}\{u \in S^{n-1} : u \text{ is a 0-extreme normal vector of } M\}, \\ \text{supp } S_{B, \mathcal{M}} &= \text{cl}\{u \in S^{n-1} : u \text{ is a 1-extreme normal vector of } M\}. \end{aligned}$$

Let us finally note that as mixed volumes and mixed area measures are linear functionals of the underlying bodies (and hence of their support functions), their definitions extend naturally by linearity to functions that are differences of two support functions [40, section 5.2]. By a slight abuse of notation, we will write

$$\begin{aligned} \mathbf{V}(f, K_2, \dots, K_n) &:= \mathbf{V}(K, K_2, \dots, K_n) - \mathbf{V}(K', K_2, \dots, K_n), \\ S_{f, K_2, \dots, K_{n-1}} &:= S_{K, K_2, \dots, K_{n-1}} - S_{K', K_2, \dots, K_{n-1}} \end{aligned}$$

for functions of the form  $f = h_K - h_{K'}$ . We may similarly extend further arguments by linearity to write  $\mathbf{V}(f, g, K_3, \dots, K_n)$ , etc. In particular, as will be recalled in the following section, any  $C^2$  function on the sphere can be written as the difference of two support functions, so that mixed volumes and mixed area measures are well defined when their arguments are arbitrary  $C^2$  functions. Of course,  $\mathbf{V}(f, K_2, \dots, K_n)$  need not be nonnegative and  $S_{f, K_2, \dots, K_{n-1}}$  may be a signed measure.

**4.2. Smooth convex bodies.** For sufficiently smooth bodies, mixed volumes and area measures may be expressed in a more explicit form, cf. [41, section 2].

To define the appropriate regularity, we recall that the support function  $h_K$  may be viewed either as a function on the sphere  $S^{n-1}$  or, equivalently, as a 1-homogeneous function on  $\mathbb{R}^n$ . Now suppose that  $h_K$  is a  $C^2$  function on  $\mathbb{R}^n \setminus \{0\}$ . Then  $\nabla h_K$  is 0-homogeneous, so the derivative of  $\nabla h_K$  in the radial direction vanishes. The Hessian  $\nabla^2 h_K(x)$  in  $\mathbb{R}^n$  may therefore be viewed as a linear mapping from  $x^\perp$  to itself. We denote this mapping by  $D^2 h_K$ . For an arbitrary  $C^2$  function  $f : S^{n-1} \rightarrow \mathbb{R}$ , the restricted Hessian  $D^2 f$  is defined analogously by applying the above construction to the 1-homogeneous extension of  $f$ . We may also express this notion intrinsically as  $D^2 f = \nabla_{S^{n-1}}^2 f + fI$  in terms of the covariant Hessian on the sphere. We now have the following classical fact [41, section 2.1]; here and below,  $A > 0$  ( $A \geq 0$ ) denotes that the matrix  $A$  is positive definite (semidefinite).

**Lemma 4.5.** *Let  $f : S^{n-1} \rightarrow \mathbb{R}$  be a  $C^2$  function. Then  $f = h_K$  for some convex body  $K$  if and only if  $D^2 f \geq 0$ . In particular, any  $C^2$  function satisfies  $f = h_K - h_L$  for some convex bodies  $K, L$  (as  $D^2(f + h_{\lambda B}) = D^2 f + \lambda I \geq 0$  for large  $\lambda$ ).*

We now formulate the following definition.

**Definition 4.6.** A convex body  $K$  is of class  $C_+^k$  ( $k \geq 2$ ) if  $h_K$  is  $C^k$  and  $D^2 h_K > 0$ .

It can be shown [40, sections 2.5 and 3.4] that  $K$  is of class  $C_+^k$  if and only if its boundary  $\partial K$  is a  $C^k$ -submanifold of  $\mathbb{R}^n$ , and the function  $n_K : \partial K \rightarrow S^{n-1}$  that maps each boundary point to its (unique) outer normal is a  $C^{k-1}$ -diffeomorphism.

For a  $C_+^2$  body  $K$ , one can obtain an explicit expression for the surface area measure  $S(K, \cdot)$  by using the outer normal map  $n_K$  to perform a change of variables in its definition. Using the basic fact  $n_K^{-1} = \nabla h_K$  [40, Corollary 1.7.3], this yields

$$S(K, d\omega) = \det(D^2 h_K) d\omega,$$

where  $\omega$  denotes the surface measure on  $S^{n-1}$ . Geometrically, this expression states that the density of the surface area measure of a  $C_+^2$  body  $K$  at the point  $u \in S^{n-1}$  is the reciprocal Gauss curvature of its boundary  $\partial K$  at the point  $n_K^{-1}(u)$ .

To extend this expression to mixed area measures, note that we can write

$$\det(\lambda_1 A_1 + \cdots + \lambda_m A_m) = \sum_{i_1, \dots, i_{m-1}=1}^m D(A_{i_1}, \dots, A_{i_{m-1}}) \lambda_{i_1} \cdots \lambda_{i_{m-1}}$$

for any  $(n-1)$ -dimensional matrices  $A_i$  and  $\lambda_i \geq 0$ , as  $A \mapsto \det A$  is a homogeneous polynomial. The coefficients  $D(A_1, \dots, A_{n-1})$  are called *mixed discriminants*. The definition of mixed area measures and (4.1) now yield the following.

**Lemma 4.7.** *Let  $K_1, \dots, K_n$  be convex bodies in  $\mathbb{R}^n$  of class  $C_+^2$ . Then*

$$\begin{aligned} S_{K_2, \dots, K_n}(d\omega) &= D(D^2 h_{K_2}, \dots, D^2 h_{K_n}) d\omega, \\ \mathcal{V}(K_1, \dots, K_n) &= \frac{1}{n} \int h_{K_1} D(D^2 h_{K_2}, \dots, D^2 h_{K_n}) d\omega. \end{aligned}$$

Let us recall some basic properties of mixed discriminants.

**Lemma 4.8.** *Let  $A, B, A_1, \dots, A_{n-1}$  be symmetric  $(n-1)$ -dimensional matrices.*

- a.  $D(A, \dots, A) = \det(A)$ .
- b.  $D(B, A, \dots, A) = \frac{1}{n-1} \text{Tr}[\text{cof}(A)B]$ .

- c.  $D(A_1, \dots, A_{n-1})$  is symmetric and multilinear in its arguments.  
d.  $D(A, A_2, \dots, A_{n-1}) \geq D(B, A_2, \dots, A_{n-1})$  if  $A \geq B$ ,  $A_2, \dots, A_{n-1} \geq 0$ .

These properties may be found in [41, Lemma 2.6] except part (b), which follows readily by differentiation  $\frac{d}{dt} \det(A + tB)|_{t=0} = (n-1)D(B, A, \dots, A)$ .

We finally recall that the directional derivatives of support functions have a geometric meaning for any (non-smooth) convex body [40, Theorem 1.7.2].

**Lemma 4.9.** *Let  $K$  be any convex body in  $\mathbb{R}^n$ . Then  $\nabla_x h_K(u) = h_{F(K,u)}(x)$  for all  $u, x \in S^{n-1}$ , where  $\nabla_x$  denotes the directional derivative in  $\mathbb{R}^n$  in direction  $x$ .*

**4.3. Self-adjoint operators and quadratic forms.** It is an elementary fact of linear algebra that every symmetric quadratic form  $Q$  on  $\mathbb{R}^n$  can be represented in terms of a symmetric matrix  $A$  as  $Q(x, y) = \langle x, Ay \rangle$ . Moreover, the spectral theorem states that  $A$  can be diagonalized by a proper choice of basis. Analogous notions exist in infinite dimension, but the definitions and properties of the relevant objects are more subtle. The aim of this section is to give a brief review of the requisite notions; the reader is referred to [33, 13] for textbook treatments.

**Self-adjointness and the spectral theorem.** Let  $H$  be a real<sup>5</sup> separable Hilbert space. A *linear operator* on  $H$  is a linear map  $\mathcal{L} : \text{Dom } \mathcal{L} \rightarrow H$  whose domain is a dense subspace  $\text{Dom } \mathcal{L} \subseteq H$ . The specification of the domain of  $\mathcal{L}$  is part of its definition. For any linear operator  $\mathcal{L}$ , the associated *adjoint operator*  $\mathcal{L}^*$  is defined by  $\langle \mathcal{L}^*x, y \rangle = \langle x, \mathcal{L}y \rangle$  for all  $y \in \text{Dom } \mathcal{L}$  and  $x \in \text{Dom } \mathcal{L}^*$ , where

$$\text{Dom } \mathcal{L}^* := \{x \in H : \text{there is } z \in H \text{ so that } \langle x, \mathcal{L}y \rangle = \langle z, y \rangle \text{ for all } y \in \text{Dom } \mathcal{L}\}.$$

A linear operator  $\mathcal{L}$  is *self-adjoint* if  $\mathcal{L} = \mathcal{L}^*$  (in particular,  $\text{Dom } \mathcal{L} = \text{Dom } \mathcal{L}^*$ ). Note that the condition  $\langle \mathcal{L}x, y \rangle = \langle x, \mathcal{L}y \rangle$  for  $x, y \in \text{Dom } \mathcal{L}$  is *not* sufficient to ensure self-adjointness (such  $\mathcal{L}$  is said to be *symmetric*).

The spectral theorem for self-adjoint operators [33, Theorem VIII.4] states that any self-adjoint operator can be diagonalized.

**Theorem 4.10** (Spectral theorem). *For any self-adjoint operator  $\mathcal{L}$ , there exists a measure space  $(\Omega, \mu)$  with a finite measure  $\mu$ , an invertible linear isometry  $U : H \rightarrow L^2(\Omega, \mu)$ , and a measurable function  $\eta : \Omega \rightarrow \mathbb{R}$  such that*

- a.  $\text{Dom } \mathcal{L} = \{U^{-1}\psi : \psi \in L^2(\Omega, \mu) \text{ such that } \eta\psi \in L^2(\Omega, \mu)\}$ ; and  
b.  $U\mathcal{L}U^{-1}\psi = \eta\psi$  for every  $\psi \in L^2(\Omega, \mu)$ .

The spectral theorem gives rise to a spectral calculus in direct analogy to the finite-dimensional case. Let  $\mathcal{L}, \Omega, \mu, U, \eta$  be as in Theorem 4.10. Then for any measurable function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , we can define a self-adjoint operator  $\varphi(\mathcal{L})$  with

$$\text{Dom } \varphi(\mathcal{L}) := \{U^{-1}\psi : \psi \in L^2(\Omega, \mu) \text{ such that } (\varphi \circ \eta)\psi \in L^2(\Omega, \mu)\}$$

and  $\varphi(\mathcal{L})x := U^{-1}(\varphi \circ \eta)Ux$  for  $x \in \text{Dom } \varphi(\mathcal{L})$ .

The *spectrum*  $\text{spec } \mathcal{L}$  of a self-adjoint operator is the essential range of the function  $\eta$  in the spectral theorem.  $\mathcal{L}$  is *nonnegative* if  $\text{spec } \mathcal{L} \subseteq [0, \infty)$ .

<sup>5</sup>While most references on spectral theory assume complex  $H$ , the relevant notions apply readily to the real case [28, Remark 20.18]; we work with real  $H$  primarily for notational convenience.

**Nonnegative forms.** We now consider the correspondence between self-adjoint operators and symmetric quadratic forms. Let  $\mathcal{E} : \text{Dom } \mathcal{E} \times \text{Dom } \mathcal{E} \rightarrow \mathbb{R}$  be a bilinear map that is defined on a dense subspace  $\text{Dom } \mathcal{E} \subseteq H$ . We call  $\mathcal{E}$  a *nonnegative form* if  $\mathcal{E}(f, g) = \mathcal{E}(g, f)$  and  $\mathcal{E}(f, f) \geq 0$ . A nonnegative form is *closed* if  $\text{Dom } \mathcal{E}$  is complete for the norm  $\|f\| := [\|f\|^2 + \mathcal{E}(f, f)]^{1/2}$ .

It is a basic fact that closed nonnegative forms are in one-to-one correspondence with nonnegative self-adjoint operators on  $H$ : for any nonnegative self-adjoint operator  $\mathcal{L}$ , the form  $\mathcal{E}(f, g) := \langle \mathcal{L}^{1/2} f, \mathcal{L}^{1/2} g \rangle$  with  $\text{Dom } \mathcal{E} = \text{Dom } \mathcal{L}^{1/2}$  is closed; while for every closed nonnegative form  $\mathcal{E}$ , there is a nonnegative self-adjoint operator  $\mathcal{L}$  such that the above representation holds [13, Theorem 4.4.2].

Note that when  $g \in \text{Dom } \mathcal{L}$ , we may also write  $\mathcal{E}(f, g) = \langle f, \mathcal{L}g \rangle$ ; in general, however,  $\text{Dom } \mathcal{E}$  is considerably larger than  $\text{Dom } \mathcal{L}$ . On the other hand, any minimizer  $f$  of  $\mathcal{E}(f, f)$  must exhibit additional regularity  $f \in \text{Dom } \mathcal{L}$ .

**Lemma 4.11.** *Let  $\mathcal{E}$  be a closed nonnegative form and  $\mathcal{L}$  the associated self-adjoint operator. Then  $f \in \text{Dom } \mathcal{E}$ ,  $\mathcal{E}(f, f) = 0$  if and only if  $f \in \text{Dom } \mathcal{L}$ ,  $\mathcal{L}f = 0$ .*

*Proof.* As  $\text{Dom } \mathcal{L} = \{f \in \text{Dom } \mathcal{L}^{1/2} : \mathcal{L}^{1/2} f \in \text{Dom } \mathcal{L}^{1/2}\}$  by definition of  $\mathcal{L}^{1/2}$ , the conclusion follows from  $\mathcal{E}(f, f) = \|\mathcal{L}^{1/2} f\|^2 = \langle f, \mathcal{L}f \rangle$  for  $f \in \text{Dom } \mathcal{L}$ .  $\square$

**Semibounded forms.** In the context of Lemma 3.1, the operator  $\mathcal{A}$  is not nonnegative but rather bounded from above. The above notions extend directly to this setting modulo a change in notation. A densely defined symmetric bilinear map  $\mathcal{E}(f, g)$  is called a *c-semibounded form* if  $\mathcal{E}(f, f) \leq c\|f\|^2$ . Then  $\mathcal{E}'(f, g) := c\langle f, g \rangle - \mathcal{E}(f, g)$  defines a nonnegative form, and  $\mathcal{E}$  is said to be closed if  $\mathcal{E}'$  is closed, that is, if  $\text{Dom } \mathcal{E}$  is complete for the norm  $\|f\| := [(1 + c)\|f\|^2 - \mathcal{E}(f, f)]^{1/2}$  (here  $1 + c$  may be replaced by any  $C \geq 1 + c$ , as this defines an equivalent norm).

By applying the correspondence between nonnegative self-adjoint operators and closed nonnegative forms to  $\mathcal{E}'$ , we obtain the analogous statements for the semibounded form  $\mathcal{E}$ . In particular, for any closed  $c$ -semibounded form  $\mathcal{E}$ , there is a self-adjoint operator  $\mathcal{A} \leq cI$  so that  $\mathcal{E}(f, g) = \langle f, \mathcal{A}g \rangle$  for  $g \in \text{Dom } \mathcal{A}$ . Conversely, any self-adjoint operator  $\mathcal{A} \leq cI$  defines such a closed  $c$ -semibounded form  $\mathcal{E}$ .

**Friedrichs extension.** It is often not a trivial matter to define self-adjoint operators. In practice, an operator is often naturally specified on a smaller domain, and one must understand how to extend its definition to a larger domain on which it is self-adjoint. Semibounded forms provide a powerful tool for this purpose.

A semibounded form  $\mathcal{E}$  is *closable* if it can be extended to a larger domain on which it is closed. The smallest closed extension is called the *closure*  $\bar{\mathcal{E}}$ , and  $\text{Dom } \bar{\mathcal{E}}$  is the completion of  $\text{Dom } \mathcal{E}$  with respect to the norm  $\|\cdot\|$  defined above.

The following procedure is known as Friedrichs extension [13, Theorem 4.4.5].

**Lemma 4.12.** *Let  $\mathcal{A}$  be a densely defined operator on  $H$  such that the bilinear map  $\mathcal{E}(f, g) = \langle f, \mathcal{A}g \rangle$  is symmetric and semibounded on the domain of  $\mathcal{A}$ . Then  $\mathcal{E}$  is closable and its closure defines a self-adjoint extension of  $\mathcal{A}$ .*

**Hyperbolic forms.** We conclude this section by proving Lemma 3.1. By the spectral theorem, we may assume without loss of generality in the proof that  $H = L^2(\Omega, \mu)$ ,  $\mathcal{A}\psi = \eta\psi$  with  $0 < \text{ess sup } \eta < \infty$ , and  $\mathcal{E}(f, g) = \int fg\eta d\mu$  with  $\text{Dom } \mathcal{E} = \{f \in L^2(\Omega, \mu) : |\int f^2\eta d\mu| < \infty\}$ , for which the various arguments become explicit. Note that in this setting, the condition  $\text{rank } 1_{(0, \infty)}(\mathcal{A}) = 1$  holds if and only if  $\mu$  has an atom at some point  $\alpha \in \Omega$  with  $\eta(\alpha) > 0$  and  $\eta(\omega) \leq 0$  for  $\mu$ -a.e.  $\omega \neq \alpha$ .

*Proof of Lemma 3.1.* We prove each implication separately.

**2**  $\Rightarrow$  **1**. Suppose condition 2 holds. Then the spectral theorem implies that there exists  $v \in \text{Dom } \mathcal{A}$ ,  $\lambda > 0$  such that  $\mathcal{A}v = \lambda v$ , and such that the following holds:

$$\mathcal{E}(h, h) \leq 0 \quad \text{for } h \in \text{Dom } \mathcal{E}, \quad h \perp v. \quad (4.2)$$

Let  $f, g \in \text{Dom } \mathcal{E}$  such that  $\mathcal{E}(g, g) > 0$ . Then  $\langle g, v \rangle \neq 0$  by (4.2). Define  $z = f - ag$  with  $a = \langle f, v \rangle / \langle g, v \rangle$ . Then  $z \perp v$ , so applying (4.2) again yields

$$0 \geq \mathcal{E}(z, z) = \mathcal{E}(f, f) - 2a\mathcal{E}(f, g) + a^2\mathcal{E}(g, g) \geq \mathcal{E}(f, f) - \frac{\mathcal{E}(f, g)^2}{\mathcal{E}(g, g)},$$

where the last inequality follows by minimizing over  $a$ . Condition 1 follows readily.

**1**  $\Rightarrow$  **2**. Suppose condition 2 is violated. As  $0 < \text{sup spec } \mathcal{A} < \infty$ , the spectral theorem implies that  $H_+ := 1_{(0, \infty)}(\mathcal{A})H$  satisfies  $H_+ \subset \text{Dom } \mathcal{A}$ ,  $\mathcal{A}H_+ \subseteq H_+$ ,  $\dim H_+ \geq 2$ , and  $\mathcal{E}(g, g) > 0$  for  $g \in H_+ \setminus \{0\}$ . That is,  $\mathcal{A}$  is a bounded positive definite operator on its positive eigenspace, which has dimension at least two.

Now choose any nonzero  $g \in H_+$  and  $f \in H_+$  such that  $f \perp \mathcal{A}g$ . Then  $\mathcal{E}(f, f) > 0$ ,  $\mathcal{E}(g, g) > 0$ , and  $\mathcal{E}(f, g) = \langle f, \mathcal{A}g \rangle = 0$ . Thus condition 1 is violated.

**1'**  $\Rightarrow$  **2'**. Suppose condition 1' holds. Then each inequality in the proof of **2**  $\Rightarrow$  **1** must be equality. In particular,  $\mathcal{E}(z, z) = 0$ . Now note that by the spectral theorem,  $z \in H_- := 1_{(-\infty, 0]}(\mathcal{A})H$  and  $\mathcal{A}$  is a nonpositive operator on  $H_-$ . By Lemma 4.11, we obtain  $z \in \text{Dom } \mathcal{A}$  and  $\mathcal{A}z = 0$ . Thus we have established condition 2'.

**2'**  $\Rightarrow$  **1'**. Suppose that condition 2' holds. As  $z = f - ag \in \ker \mathcal{A}$ , we have  $0 = \mathcal{E}(z, z) = \mathcal{E}(f, f) - a\mathcal{E}(g, g)$ , so  $a = \mathcal{E}(f, g) / \mathcal{E}(g, g)$ . But then

$$0 = \mathcal{E}(z, z) = \mathcal{E}(f, f) - \frac{\mathcal{E}(f, g)^2}{\mathcal{E}(g, g)},$$

where we used again  $z \in \ker \mathcal{A}$ . Condition 1' follows.  $\square$

**4.4. Elliptic operators.** In finite dimension, the spectrum of a symmetric matrix always coincides with its set of eigenvalues. This is not the case for general self-adjoint operators, however, unless additional regularity assumptions are imposed.

The basic criterion for a semibounded self-adjoint operator  $\mathcal{A} \leq cI$  to possess a nice eigendecomposition is that it has a *compact resolvent*. For our purposes, the term need not be defined precisely, as we may take it to be synonymous with the following equivalent statement [34, Theorem XIII.64]: there is an orthonormal basis  $\{\psi_n\}_{n \geq 1} \subset \text{Dom } \mathcal{A}$  that spans  $H$ , and  $c \geq \lambda_1 \geq \lambda_2 \geq \dots$  with  $\lambda_n \rightarrow -\infty$ , so that  $\mathcal{A}\psi_n = \lambda_n\psi_n$  for each  $n$  (and thus  $\text{spec } \mathcal{A} = \{\lambda_n\}_{n \geq 1}$ ). Just as for matrices, the eigenvalues  $\lambda_n$  may be computed by the min-max principle [34, section XIII.1].

**Lemma 4.13.** *Let  $\mathcal{A} \leq cI$  be a semibounded self-adjoint operator with compact resolvent, with eigenvalues  $\{\lambda_n\}_{n \geq 1}$  and eigenvectors  $\{\psi_n\}_{n \geq 1}$ . Then*

$$\lambda_n = \inf_{L \subset H: \dim L = n-1} \sup_{f \perp L} \frac{\langle f, \mathcal{A}f \rangle}{\|f\|^2} = \sup_{f \perp \{\psi_1, \dots, \psi_{n-1}\}} \frac{\langle f, \mathcal{A}f \rangle}{\|f\|^2}.$$

We now recall a very useful example of self-adjoint operators with compact resolvent: elliptic second-order differential operators on compact manifolds. For concreteness, we limit the discussion to the special case of such operators that will be needed in this paper; see [19, 20, 13] for a comprehensive treatment.

Let  $M$  be a compact, connected Riemannian manifold (without boundary). Define the measure  $d\mu := \rho d\text{Vol}_M$ , where  $\rho > 0$  is a positive smooth function on  $M$ . Moreover, let  $V$  be a smooth function on  $M$ , and let  $A$  be a smooth symmetric  $(1, 1)$ -tensor field on  $M$ . We now define for any  $f \in C^2$

$$\mathcal{A}f(x) := \text{Tr}[A(x)\nabla^2 f(x)] + V(x)f(x).$$

The operator  $\mathcal{A}$  is called *elliptic* if  $A(x)$  is positive definite for every  $x \in M$ , and is called *symmetric* if  $\langle f, \mathcal{A}g \rangle_{L^2(M, \mu)} = \langle \mathcal{A}f, g \rangle_{L^2(M, \mu)}$  for all  $f, g \in C^2$ .

Now note that by integration by parts, we have for  $f, g \in C^2$

$$\langle f, \mathcal{A}g \rangle_{L^2(M, \mu)} = - \int f \langle \rho^{-1} \text{div}(\rho A), \nabla g \rangle d\mu - \int \langle \nabla f, A \nabla g \rangle d\mu + \int V f g d\mu.$$

If  $\mathcal{A}$  is symmetric, it follows readily that  $\text{div}(\rho A) = 0$ . Therefore, if  $\mathcal{A}$  is a symmetric elliptic operator, then  $\langle f, \mathcal{A}f \rangle_{L^2(M, \mu)} \leq (\sup V) \|f\|_{L^2(M, \mu)}^2$  for all  $f \in C^2$ . We may then apply Lemma 4.12 to extend  $\mathcal{A}$  to a self-adjoint operator on  $L^2(M, \mu)$ . As no confusion can arise, we will denote the Friedrichs extension by the same symbol  $\mathcal{A}$ , and refer to it as a self-adjoint elliptic operator.

The following result summarizes some basic facts of elliptic regularity theory; we refer to [19, section 8.12] and [20, chapter 10] for a detailed treatment.

**Proposition 4.14.** *Let  $\mathcal{A}$  be a self-adjoint elliptic operator on  $(M, \mu)$ . Then  $\mathcal{A}$  has compact resolvent. Moreover, its top eigenvalue is simple  $\lambda_1 > \lambda_2$ , and its eigenfunction  $\psi_k$  has the same sign everywhere on  $M$  if and only if  $k = 1$ .*

## 5. MIXED VOLUMES AND DIRICHLET FORMS

The aim of this section is to show that mixed volumes of arbitrary convex bodies can be represented as closed quadratic forms associated to self-adjoint operators. This was stated in the setting of Minkowski's quadratic inequality as Theorem 3.2 above, which is a special case of the following general theorem.

**Theorem 5.1.** *Let  $\mathcal{C} = (C_1, \dots, C_{n-2})$  be convex bodies in  $\mathbb{R}^n$  with  $S_{B, \mathcal{C}} \neq 0$ . Then there is a self-adjoint operator  $\mathcal{A}$  on  $L^2(S_{B, \mathcal{C}})$  with  $C^2(S^{n-1}) \subset \text{Dom } \mathcal{A}$  such that:*

- $\text{spec } \mathcal{A} \subseteq (-\infty, 0] \cup \{\frac{1}{n}\}$ .
- $\text{rank } 1_{(0, \infty)}(\mathcal{A}) = 1$  and  $\mathcal{A}1 = \frac{1}{n}1$ .
- $\mathcal{A}\ell = 0$  for any linear function  $\ell : x \mapsto \langle v, x \rangle$  on  $S^{n-1}$  ( $v \in \mathbb{R}^n$ ).

*Moreover, the closed quadratic form  $\mathcal{E}$  associated to  $\mathcal{A}$  satisfies the following:*

- $h_K \in \text{Dom } \mathcal{E}$  for every convex body  $K$  in  $\mathbb{R}^n$ .
- $\mathcal{E}(h_K, h_L) = V(K, L, \mathcal{C})$  for any convex bodies  $K, L$  in  $\mathbb{R}^n$ .

The above theorem states, in particular, that linear functions are always in the kernel of  $\mathcal{A}$ . For the reasons explained in section 3, the central question in the study of the extremals of the Alexandrov-Fenchel inequality is whether these are the *only* elements of  $\ker \mathcal{A}$  that are differences of support functions. The main part of this paper (sections 6–8) will be devoted to settling this question in the setting  $C_1 = \dots = C_{n-2} = M$  of Minkowski's quadratic inequality.

The operator  $\mathcal{A}$  of Theorem 5.1 is defined somewhat abstractly. To actually work with such operators, one would like to have a more explicit formulation. Explicit constructions can be obtained in various special cases that will play an important role in the remainder of this paper. When  $C_1, \dots, C_{n-2}$  are  $C_+^\infty$  bodies,  $\mathcal{A}$  is a classical elliptic operator on  $S^{n-1}$  in the sense of section 4.4. This case

is already used in the proof of Theorem 5.1, which will be given in section 5.1. When  $C_1, \dots, C_{n-2}$  are polytopes, it turns out that  $\mathcal{A}$  is a quantum graph [4]; this setting will be developed in detail in section 5.2. A third setting in which  $\mathcal{A}$  can be explicitly described will be encountered in section 8 below.

In complete generality, we do not know how to give an expression for  $\mathcal{A}$  that is amenable to explicit computations. Nonetheless, we may always view  $\mathcal{A}$  as a highly degenerate elliptic second-order operator in the sense of the theory of Dirichlet forms [18, 3]. While the latter theory will not be needed in the remainder of this paper, we briefly develop this viewpoint in section 5.3 in order to highlight the general structure that lies behind the representation of Theorem 5.1.

**Remark 5.2.** It should be emphasized at the outset that while  $\mathcal{A}$  may be thought of quite generally as a kind of elliptic operator on the sphere, it does not necessarily possess some of the nice regularity properties of classical elliptic operators on compact manifolds. In particular, it turns out that  $\mathcal{A}$  need not have a compact resolvent, as we will see in section 8 below (cf. Remark 8.7).

**Remark 5.3.** The assumption  $S_{B,C} \neq 0$  in Theorem 5.1 is innocuous. Indeed, if  $S_{B,C} \equiv 0$ , then it follows (for example, using Lemma 5.4 below) that  $V(K, L, C) = 0$  for all convex bodies  $K, L$ , so that the representation of mixed volumes is trivial.

**5.1. Proof of Theorem 5.1.** Throughout this section, the assumptions and notation of Theorem 5.1 are in force. The proof is based on the following observation.

**Lemma 5.4.** *For any  $C^2$  function  $f$  on  $S^{n-1}$ , we have  $S_{f,C} \ll S_{B,C}$  and*

$$\left\| \frac{dS_{f,C}}{dS_{B,C}} \right\|_{\infty} \leq \|D^2 f\|_{\infty},$$

where we use the notation  $\|D^2 f\|_{\infty} := \sup_{x \in S^{n-1}} \|D^2 f(x)\|$ .

*Proof.* Let  $C_i^{(s)}$  be convex bodies of class  $C_+^{\infty}$  such that  $C_i^{(s)} \rightarrow C_i$  as  $s \rightarrow \infty$  in the sense of Hausdorff convergence for every  $i = 1, \dots, n-2$ . The existence of such smooth approximations is classical, cf. [40, section 3.4].

For  $\mathcal{C}^{(s)} = (C_1^{(s)}, \dots, C_{n-2}^{(s)})$ , Lemmas 4.7 and 4.5 imply

$$S_{f,\mathcal{C}^{(s)}}(d\omega) = D(D^2 f, D^2 h_{C_1^{(s)}}, \dots, D^2 h_{C_{n-2}^{(s)}}) d\omega$$

for any  $C^2$  function  $f$ . In particular, as  $D^2 h_B = I$ , we have

$$\frac{dS_{f,\mathcal{C}^{(s)}}}{dS_{B,\mathcal{C}^{(s)}}} = \frac{D(D^2 f, D^2 h_{C_1^{(s)}}, \dots, D^2 h_{C_{n-2}^{(s)}})}{D(I, D^2 h_{C_1^{(s)}}, \dots, D^2 h_{C_{n-2}^{(s)}})}$$

(note that the denominator is strictly positive by Lemma 4.8(d), so the expression is well-defined). As  $\lambda_{\min}(D^2 f)I \leq D^2 f \leq \lambda_{\max}(D^2 f)I$ , Lemma 4.8(d) shows that

$$\left| \frac{dS_{f,\mathcal{C}^{(s)}}}{dS_{B,\mathcal{C}^{(s)}}} \right| \leq \|D^2 f\|$$

pointwise. Therefore, Theorem 4.3 implies that

$$\begin{aligned} \left| \int g dS_{f,\mathcal{C}} \right| &= \lim_{s \rightarrow \infty} \left| \int g dS_{f,\mathcal{C}^{(s)}} \right| \\ &\leq \limsup_{s \rightarrow \infty} \int |g| \|D^2 f\| dS_{B,\mathcal{C}^{(s)}} \\ &\leq \|D^2 f\|_\infty \int |g| dS_{B,\mathcal{C}} \end{aligned} \quad (5.1)$$

for every continuous function  $g$  on  $S^{n-1}$ . In particular,

$$g \mapsto \int g dS_{f,\mathcal{C}}$$

is a bounded linear map from  $C^0(S^{n-1}) \subset L^1(S_{B,\mathcal{C}})$  to  $\mathbb{R}$ , and therefore extends uniquely to a bounded linear functional on  $L^1(S_{B,\mathcal{C}})$ . As  $L^1(S_{B,\mathcal{C}})^* = L^\infty(S_{B,\mathcal{C}})$ ,

$$\int g dS_{f,\mathcal{C}} = \int g \varrho dS_{B,\mathcal{C}}$$

for some  $\varrho \in L^\infty(S_{B,\mathcal{C}})$ . This proves absolute continuity  $S_{f,\mathcal{C}} \ll S_{B,\mathcal{C}}$ , and the bound on  $\varrho$  follows by taking the supremum in (5.1) over  $g$  with  $\int |g| dS_{B,\mathcal{C}} \leq 1$ .  $\square$

We will also need the following simple consequence.

**Lemma 5.5.** *Let  $f, g \in C^2(S^{n-1})$  satisfy  $f = g$   $S_{B,\mathcal{C}}$ -a.e. Then  $S_{f,\mathcal{C}} = S_{g,\mathcal{C}}$ .*

*Proof.* Let  $h \in C^2(S^{n-1})$ . Then

$$\int h dS_{f,\mathcal{C}} - \int h dS_{g,\mathcal{C}} = n \mathcal{V}(h, f - g, \mathcal{C}) = \int (f - g) dS_{h,\mathcal{C}} = 0,$$

where we have used (4.1), the symmetry of mixed volumes, and Lemma 5.4. As this holds for any  $h \in C^2(S^{n-1})$ , the conclusion follows.  $\square$

The idea behind Theorem 5.1 is now simple. By Lemma 5.4, we can define

$$\mathcal{A}f := \frac{1}{n} \frac{dS_{f,\mathcal{C}}}{dS_{B,\mathcal{C}}} \quad \text{for } f \in C^2(S^{n-1}), \quad (5.2)$$

and Lemma 5.5 ensures that  $\mathcal{A}$  is well-defined as a linear operator on  $C^2(S^{n-1}) \subset L^2(S_{B,\mathcal{C}})$  (here we implicitly identified those functions in  $C^2$  that agree up to  $S_{B,\mathcal{C}}$ -null sets). We aim to extend  $\mathcal{A}$  to a bona fide self-adjoint operator by Friedrichs extension. To this end, we must show that  $\mathcal{A}$  is semibounded.

**Lemma 5.6.** *For every  $f \in C^2(S^{n-1})$ , we have*

$$n \int f \mathcal{A}f dS_{B,\mathcal{C}} = \int f dS_{f,\mathcal{C}} \leq \int f^2 dS_{B,\mathcal{C}}.$$

*Proof.* Let  $C_i^{(s)}$  be convex bodies of class  $C_+^\infty$  such that  $C_i^{(s)} \rightarrow C_i$  as  $s \rightarrow \infty$ . Then

$$\int f dS_{f,\mathcal{C}^{(s)}} \rightarrow \int f dS_{f,\mathcal{C}}, \quad \int f^2 dS_{B,\mathcal{C}^{(s)}} \rightarrow \int f^2 dS_{B,\mathcal{C}}$$

as  $s \rightarrow \infty$  by Theorem 4.3. It therefore suffices to assume in the remainder of the proof that  $C_1, \dots, C_{n-2}$  are convex bodies of class  $C_+^\infty$ .

For  $C_+^\infty$  bodies, we may write as in the proof of Lemma 5.4

$$\mathcal{A}f = \frac{1}{n} \frac{dS_{f,\mathcal{C}}}{dS_{B,\mathcal{C}}} = \frac{1}{n} \frac{D(D^2 f, D^2 h_{C_1}, \dots, D^2 h_{C_{n-2}})}{D(I, D^2 h_{C_1}, \dots, D^2 h_{C_{n-2}})}.$$

Using  $D^2f = \nabla_{S^{n-1}}^2 f + fI$  and Lemma 4.8(d), we see that  $\mathcal{A}$  is an elliptic operator in the sense of section 4.4. Moreover, by (4.1) and the symmetry of mixed volumes,  $\langle f, \mathcal{A}g \rangle_{L^2(S_{B,C})} = \mathbf{V}(f, g, \mathcal{C})$  is symmetric in  $f, g$ . It follows from section 4.4 that  $\mathcal{A}$  has a self-adjoint extension for which  $\lambda := \sup \text{spec } \mathcal{A}$  is the unique eigenvalue associated to a positive eigenfunction. As  $\mathcal{A}1 = \frac{1}{n}1$ , we have  $\lambda = \frac{1}{n}$  and thus

$$\int f dS_{f,C} = n \langle f, \mathcal{A}f \rangle_{L^2(S_{B,C})} \leq \|f\|_{L^2(S_{B,C})}^2,$$

concluding the proof.  $\square$

We are now ready to prove Theorem 5.1.

*Proof of Theorem 5.1.* For  $f, g \in C^2(S^{n-1})$ , define  $\mathcal{A}f$  as in (5.2) and let

$$\mathcal{E}(f, g) := \langle f, \mathcal{A}g \rangle_{L^2(S_{B,C})} = \mathbf{V}(f, g, \mathcal{C}).$$

By the symmetry of mixed volumes and Lemma 5.6,  $\mathcal{E}$  is a densely defined and  $\frac{1}{n}$ -semibounded symmetric quadratic form on  $L^2(S_{B,C})$ . It follows from Lemma 4.12 that  $\mathcal{E}$  is closable and that its closure defines a self-adjoint extension of  $\mathcal{A}$ . This concludes the construction of the objects  $\mathcal{A}$  and  $\mathcal{E}$  that appear in the statement of Theorem 5.1. We now proceed to verify each of the claimed properties.

**Proof of d and e.** By construction,  $\mathcal{E}(h_K, h_L) = \mathbf{V}(K, L, \mathcal{C})$  whenever  $K, L$  are  $C_+^2$  bodies. Now let  $K$  be an arbitrary convex body, and let  $K^{(s)}$  be convex bodies of class  $C_+^2$  such that  $K^{(s)} \rightarrow K$  as  $s \rightarrow \infty$ . That

$$\|h_{K^{(s)}} - h_K\|_{L^2(S_{B,C})}^2 \leq n \mathbf{V}(B, B, \mathcal{C}) \|h_{K^{(s)}} - h_K\|_{\infty}^2 \rightarrow 0$$

as  $s \rightarrow \infty$  follows immediately from Hausdorff convergence. Moreover,

$$\begin{aligned} \mathcal{E}(h_{K^{(s)}} - h_{K^{(t)}}, h_{K^{(s)}} - h_{K^{(t)}}) &= \\ \mathbf{V}(K^{(s)}, K^{(s)}, \mathcal{C}) - 2\mathbf{V}(K^{(s)}, K^{(t)}, \mathcal{C}) + \mathbf{V}(K^{(t)}, K^{(t)}, \mathcal{C}) &\rightarrow 0 \end{aligned}$$

as  $s, t \rightarrow \infty$  by Theorem 4.3. Thus  $h_K$  is in the completion of  $C^2(S^{n-1})$  for the norm  $[2\|f\|_{L^2(S_{B,C})}^2 - \mathcal{E}(f, f)]^{1/2}$ . Therefore, by construction,  $h_K \in \text{Dom } \mathcal{E}$  and

$$\mathcal{E}(h_K, h_K) = \lim_{s \rightarrow \infty} \mathcal{E}(h_{K^{(s)}}, h_{K^{(s)}}) = \lim_{s \rightarrow \infty} \mathbf{V}(K^{(s)}, K^{(s)}, \mathcal{C}) = \mathbf{V}(K, K, \mathcal{C}).$$

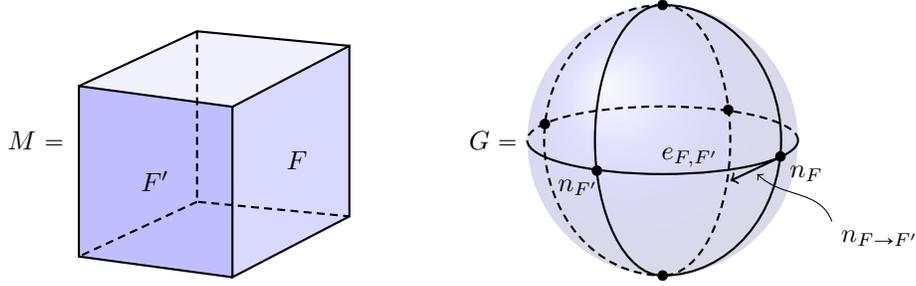
The analogous conclusion for  $\mathcal{E}(h_K, h_L)$  follows by polarization.

**Proof of a and b.** Note first that as  $h_B = 1$ , it follows immediately from the definition (5.2) that  $\mathcal{A}1 = \frac{1}{n}1$ . Now suppose  $\text{rank } 1_{(0,\infty)}(\mathcal{A}) > 1$ , and choose any nonzero  $f \in 1_{(0,\infty)}(\mathcal{A})L^2(S_{B,C})$  so that  $\langle f, 1 \rangle_{L^2(S_{B,C})} = 0$ . Choose  $f_s \in C^2(S^{n-1})$  so that  $\|f_s - f\|_{L^2(S_{B,C})} \rightarrow 0$  and  $\mathcal{E}(f_s, f_s) \rightarrow \mathcal{E}(f, f)$  as  $s \rightarrow \infty$  (the existence of such a sequence is guaranteed by construction as  $f \in \text{Dom } \mathcal{E}$ ). By Lemma 4.5, for all  $s$  and for all sufficiently large  $t > 0$  (depending on  $s$ ), there is a  $C_+^2$  convex body  $K_t$  such that  $h_{K_t^{(s)}} = t + f_s$ . Therefore, by the Alexandrov-Fenchel inequality,

$$\begin{aligned} \langle t + f_s, \mathcal{A}1 \rangle_{L^2(S_{B,C})}^2 &= \mathbf{V}(K_t^{(s)}, B, \mathcal{C})^2 \geq \mathbf{V}(K_t^{(s)}, K_t^{(s)}, \mathcal{C}) \mathbf{V}(B, B, \mathcal{C}) \\ &= \langle t + f_s, \mathcal{A}(t + f_s) \rangle_{L^2(S_{B,C})} \langle 1, \mathcal{A}1 \rangle_{L^2(S_{B,C})}. \end{aligned}$$

Expanding the squares and using  $\mathcal{A}1 = \frac{1}{n}1$  yields

$$\langle f_s, 1 \rangle_{L^2(S_{B,C})}^2 \geq n^2 \mathcal{E}(f_s, f_s) \mathbf{V}(B, B, \mathcal{C}).$$

FIGURE 5.1. Metric graph  $G$  associated to a polytope  $M$ .

In particular, letting  $s \rightarrow \infty$  yields  $\mathcal{E}(f, f) \leq 0$ , where we used that  $\mathbf{V}(B, B, \mathcal{C}) > 0$  by the assumption  $S_{B, \mathcal{C}} \neq 0$ . But this contradicts the assumption that  $f \in 1_{(0, \infty)}(\mathcal{A})L^2(S_{B, \mathcal{C}})$ . Thus we have shown that  $\text{rank } 1_{(0, \infty)}(\mathcal{A}) = 1$ , proving part *b*. Part *a* now follows as an immediate consequence.

**Proof of c.** Let  $\ell : x \mapsto \langle v, x \rangle$  be a linear function. Then  $\ell = h_{K+v} - h_K$ , so

$$\langle f, \mathcal{A}\ell \rangle_{L^2(S_{B, \mathcal{C}})} = \mathbf{V}(f, K + v, \mathcal{C}) - \mathbf{V}(f, K, \mathcal{C}) = 0$$

for every  $f \in C^2(S^{n-1})$  by Lemma 4.1(e). Thus  $\mathcal{A}\ell = 0$ .  $\square$

**5.2. Polytopes and quantum graphs.** The aim of this section is to provide an explicit description of the objects that appear in Theorem 5.1 in the case where  $C_1, \dots, C_{n-2}$  are polytopes. In this case, it turns out that  $\mathcal{A}$  is a “quantum graph”: the one-dimensional Laplacian on the edges of a certain metric graph, with appropriately chosen boundary conditions at the vertices [4]. For simplicity, we will develop this result in detail in the Minkowski case  $C_1 = \dots = C_{n-2} = M$  that is of primary interest in this paper, and explain at the end of the section how the representation for general polytopes  $C_1, \dots, C_{n-2}$  may be obtained.

Throughout this section, let  $M$  be a fixed polytope in  $\mathbb{R}^n$  with nonempty interior. We denote by  $\mathcal{F}$  the set of facets of  $M$ , and by  $n_F$  the outer unit normal vector of  $F \in \mathcal{F}$ .<sup>6</sup> We say that  $F, F' \in \mathcal{F}$  are neighbors  $F \sim F'$  if  $\dim(F \cap F') = n - 2$ .

We now associate to  $M$  a metric graph  $G = (V, E)$  inscribed in  $S^{n-1}$  that will play a basic role in the sequel. The vertices of  $G$  are the facet normals

$$V = \{n_F : F \in \mathcal{F}\} \subset S^{n-1}.$$

Moreover, for every pair of neighboring facets  $F \sim F'$ , we connect the corresponding vertices by an edge  $e_{F, F'} \subset S^{n-1}$  that is the (shortest) geodesic segment in  $S^{n-1}$  connecting  $n_F$  and  $n_{F'}$ . Thus the undirected edges of  $G$  are

$$E = \{e_{F, F'} : F \sim F'\}.$$

The length of the edge  $e_{F, F'}$  will be denoted as  $l_{F, F'} := \mathcal{H}^1(e_{F, F'})$ .

We will parametrize each edge by arclength. To this end, it is convenient to fix an arbitrary orientation of the edges by introducing a total ordering  $\leq$  on  $\mathcal{F}$ . When  $F \leq F'$ , the edge  $e_{F, F'}$  will be parametrized as  $\theta \in [0, l_{F, F'}]$ , where  $\theta = 0$  corresponds to  $n_F$  and  $\theta = l_{F, F'}$  corresponds to  $n_{F'}$ . For a function  $f : e_{F, F'} \rightarrow \mathbb{R}$ ,

<sup>6</sup>For a smooth body  $K$ , we denote by  $n_K : \partial K \rightarrow S^{n-1}$  the outer unit normal map on its boundary (cf. section 4.2). For a polytope  $K$ , the outer unit normal map is constant in the interior of each facet  $F$ ; we then denote the facet normal as  $n_F \in S^{n-1}$  by a slight abuse of notation.

we will write  $f' = \frac{df}{d\theta}$  with respect to this parametrization. We denote by  $H^k$  the usual Sobolev space of functions on an edge with  $k$  weak derivatives in  $L^2$ .

A function defined on all the edges of  $G$  will be denoted  $f : G \rightarrow \mathbb{R}$ , and its (weak) derivative  $f'$  is defined as above on each edge. In particular, any function  $f : S^{n-1} \rightarrow \mathbb{R}$  may be viewed as a function on  $G$  by restricting it to the edges, and then its derivative  $f'$  is defined as above. We will write  $f \in C^0(G)$  if  $f : G \rightarrow \mathbb{R}$  is continuous on the edges as well as at each vertex.

Finally, we will denote by  $n_{F \rightarrow F'}$  the unit tangent vector to the geodesic segment  $e_{F, F'}$  at  $n_F$  in the direction of  $n_{F'}$ . The directional derivative of a function  $f : S^{n-1} \rightarrow \mathbb{R}$  at  $n_F$  in the direction  $n_{F \rightarrow F'}$  will be denoted  $\nabla_{n_{F \rightarrow F'}} f(n_F)$ . In terms of the arclength parametrization, we may evidently write

$$\nabla_{n_{F \rightarrow F'}} f(n_F) = f'|_{e_{F, F'}}(0) \mathbf{1}_{F' \geq F} - f'|_{e_{F, F'}}(l_{F, F'}) \mathbf{1}_{F' \leq F}.$$

Note, in particular, that  $\nabla_{n_{F \rightarrow F'}} f(n_F)$  is well defined whenever  $f|_{e_{F, F'}} \in H^2$  as  $H^2(e_{F, F'}) \subset C^1(e_{F, F'})$  by the Sobolev embedding theorem [19, Theorem 7.26]. The various objects that we have defined above are illustrated in Figure 5.1.

We are now ready to state the main result of this section.

**Theorem 5.7.** *Let  $M$  be a polytope in  $\mathbb{R}^n$  with nonempty interior, and define its metric graph  $G$  as above. Then for any bounded measurable  $f : S^{n-1} \rightarrow \mathbb{R}$ , we have*

$$\begin{aligned} \int f dS_{M, \mathcal{M}} &= \sum_{F \in \mathcal{F}} \mathcal{H}^{n-1}(F) f(n_F), \\ \int f dS_{B, \mathcal{M}} &= \frac{1}{n-1} \sum_{e_{F, F'} \in E} \mathcal{H}^{n-2}(F \cap F') \int_{e_{F, F'}} f d\mathcal{H}^1. \end{aligned}$$

Moreover, the self-adjoint operator  $\mathcal{A}$  and closed quadratic form  $\mathcal{E}$  on  $L^2(S_{B, \mathcal{M}})$  defined in Theorem 5.1 can be expressed in this case as follows:

$$\mathcal{A}f = \frac{1}{n} \{f'' + f\}$$

where

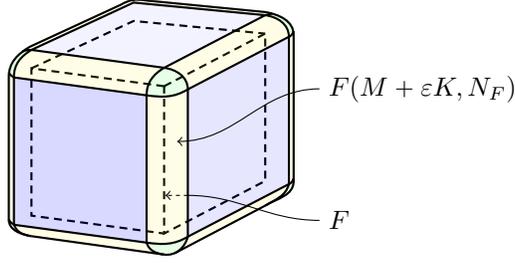
$$\begin{aligned} \text{Dom } \mathcal{A} &= \left\{ f \in C^0(G) : f|_{e_{F, F'}} \in H^2 \text{ for all } e_{F, F'} \in E, \right. \\ &\quad \left. \sum_{F' : F' \sim F} \mathcal{H}^{n-2}(F \cap F') \nabla_{n_{F \rightarrow F'}} f(n_F) = 0 \text{ for all } F \in \mathcal{F} \right\}, \end{aligned}$$

and

$$\mathcal{E}(f, g) = \frac{1}{n(n-1)} \sum_{e_{F, F'} \in E} \mathcal{H}^{n-2}(F \cap F') \int_{e_{F, F'}} \{fg - f'g'\} d\mathcal{H}^1$$

where  $\text{Dom } \mathcal{E} = \{f \in C^0(G) : f|_{e_{F, F'}} \in H^1 \text{ for all } e_{F, F'} \in E\}$ .

In the remainder of this section, we fix the setting and notation of Theorem 5.7. The starting point for the proof is an integral formula for the mixed area measures  $S_{K, \mathcal{M}}$ . The cases  $K = M$  and  $K = B$  that appear in the first part of the statement of Theorem 5.7 are classical, cf. [40, eq. (4.24)]. The analogous representation for general convex bodies  $K$  is also well known to experts, cf. [40, eq. (7.175)] (where it is stated without proof) or [23, Corollary 2.13] for a more general setting. For completeness, we include a full proof for the case that will be needed here.

FIGURE 5.2. Decomposition of the boundary of  $M + \varepsilon K$ .

**Proposition 5.8.** *Let  $K$  be a convex body of class  $C_+^2$ . Then*

$$\int f dS_{K,\mathcal{M}} = \frac{1}{n-1} \sum_{e_{F,F'} \in E} \mathcal{H}^{n-2}(F \cap F') \int_{e_{F,F'}} (h_K'' + h_K) f d\mathcal{H}^1.$$

*Proof.* Denote by  $\mathcal{F}_i$  the collection of  $i$ -dimensional faces of  $M$ , and define

$$N_F := \{u \in S^{n-1} : F(M, u) = F\}.$$

Then  $\{N_F : F \in \bigcup_i \mathcal{F}_i\}$  is a partition of  $S^{n-1}$ . Note that if  $F, F' \in \mathcal{F} = \mathcal{F}_{n-1}$  are neighboring facets  $F \sim F'$ , we have  $F \cap F' \in \mathcal{F}_{n-2}$  and  $N_{F \cap F'} = e_{F,F'}$ .

To compute  $S_{K,\mathcal{M}}$ , we will compute the area measure  $S(M + \varepsilon K, \cdot)$ , and then deduce the mixed area measure from its definition given in section 4.1. To compute the area measure, we partition the boundary of the set  $M + \varepsilon K$  into disjoint sets  $F(M + \varepsilon K, N_F) := \bigcup_{u \in N_F} F(M + \varepsilon K, u)$  corresponding to boundary points with normals in  $N_F$ , and apply the change of variables formula to each of these sets. This partition is illustrated in Figure 5.2 (in the figure  $M$  is a cube and  $K = B$ ).

**Step 1.** Fix  $F \in \mathcal{F}_i$ . We first conveniently parametrize  $F(M + \varepsilon K, N_F)$ .

Recall that as  $K$  is a  $C_+^2$ -body, the outer normal map  $n_K : \partial K \rightarrow S^{n-1}$  is a  $C^1$ -diffeomorphism and  $n_K^{-1} = \nabla h_K$ , cf. section 4.2. Therefore

$$F(M + \varepsilon K, u) = F(M, u) + \varepsilon F(K, u) = F + \varepsilon \nabla h_K(u)$$

for any  $u \in N_F$  by Lemma 4.9. It follows that  $F(M + \varepsilon K, N_F)$  is the image of the map  $\iota : F \times N_F \rightarrow F(M + \varepsilon K, N_F)$  defined by  $\iota(x, u) := x + \varepsilon \nabla h_K(u)$ .

**Step 2.** We now show that  $\iota$  is a  $C^1$ -diffeomorphism.

Let  $L_F := \text{span}(F - F)$  be the tangent space of  $F$ , and note that  $N_F \subset S^{n-1} \cap L_F^\perp$ . Assume for simplicity that  $0 \in F$ , so that  $F \subset L_F$  (when considering a single face, we may always reduce to this setting by translation). Then we may express

$$\iota(x, u) = (x + \varepsilon P_{L_F} \nabla h_K(u), \varepsilon P_{L_F^\perp} \nabla h_K(u)) \in L_F \oplus L_F^\perp,$$

where  $P_L$  denotes orthogonal projection onto  $L$ .

Now note that as  $h_{P_{L_F^\perp} K}(u) = h_K(P_{L_F^\perp} u)$ , differentiating yields that

$$P_{L_F^\perp} \nabla h_K(u) = \nabla h_{P_{L_F^\perp} K}(u) \quad \text{for } u \in S^{n-1} \cap L_F^\perp.$$

As  $P_{L_F^\perp} K$  is a  $C_+^2$  body in  $L_F^\perp$ ,  $n_{P_{L_F^\perp} K} : \partial P_{L_F^\perp} K \rightarrow S^{n-1} \cap L_F^\perp$  is a  $C^1$ -diffeomorphism and  $n_{P_{L_F^\perp} K}^{-1} = P_{L_F^\perp} \nabla h_K$ . It now follows that  $\iota$  is a diffeomorphism, as  $\iota$  is  $C^1$  and

$$\iota^{-1}(z, v) = (z - \varepsilon P_{L_F} \nabla h_K(n_{P_{L_F^\perp} K}(\varepsilon^{-1} v)), n_{P_{L_F^\perp} K}(\varepsilon^{-1} v))$$

for  $(z, v) \in F(M + \varepsilon K, N_F) \subset L_F \oplus L_F^\perp$  is also  $C^1$ .

It is readily seen that the differential  $dt$  has a block-triangular form with respect to  $L_F \oplus L_F^\perp$ , and that its determinant may be written as  $\det(\varepsilon D^2 h_{P_{L_F^\perp} K})$  where  $D^2 h_{P_{L_F^\perp} K}$  is computed in  $L_F^\perp$  (equivalently,  $D^2 h_{P_{L_F^\perp} K}$  is the projection of the Hessian  $\nabla^2 h_K$  in  $\mathbb{R}^n$  on the tangent space of  $S^{n-1} \cap L_F^\perp$ ).

**Step 3.** Now recall that  $\{N_F\}$  partitions  $S^{n-1}$ . We can therefore write

$$\begin{aligned} S(M + \varepsilon K, A) &= \sum_{i=0}^{n-1} \sum_{F \in \mathcal{F}_i} \mathcal{H}^{n-1}(\{x \in F(M + \varepsilon K, u) \text{ for some } u \in A \cap N_F\}) \\ &= \sum_{i=0}^{n-1} \varepsilon^{n-1-i} \sum_{F \in \mathcal{F}_i} \mathcal{H}^i(F) \int_{N_F} 1_A \det(D^2 h_{P_{L_F^\perp} K}) d\mathcal{H}^{n-1-i}, \end{aligned}$$

where we used the map  $\iota$  to perform a change of variables in the second line. By the definition of mixed area measures in section 4.1, we have

$$\begin{aligned} (n-1) S_{K, \mathcal{M}}(A) &= \left. \frac{d}{d\varepsilon} S(M + \varepsilon K, A) \right|_{\varepsilon=0^+} \\ &= \sum_{F \in \mathcal{F}_{n-2}} \mathcal{H}^{n-2}(F) \int_{N_F} 1_A \det(D^2 h_{P_{L_F^\perp} K}) d\mathcal{H}^1. \end{aligned}$$

But note that for every  $(n-2)$ -face  $F$ , the set  $N_F$  is an edge of the quantum graph. In this case  $D^2 h_{P_{L_F^\perp} K}$  is a scalar function (as  $S^{n-1} \cap L_F^\perp$  is one-dimensional) and coincides precisely with  $h_K'' + h_K$  on each edge. Thus the proof is complete.  $\square$

We also record a simple but essential observation. It immediately implies, for example, that  $C^2(S^{n-1}) \subset \text{Dom } \mathcal{A}$  in Theorem 5.7.

**Lemma 5.9.** *For every  $F \in \mathcal{F}$ , we have*

$$\sum_{F': F' \sim F} \mathcal{H}^{n-2}(F \cap F') n_{F \rightarrow F'} = 0.$$

*Proof.* Consider  $F$  as an  $(n-1)$ -dimensional convex body in  $\text{aff } F$ . The facets of  $F$  are precisely the sets  $F \cap F'$  for  $F' \sim F$ , and the corresponding facet normals are  $n_{F \rightarrow F'}$ . Thus the conclusion follows from Lemma 4.2(e).  $\square$

We now proceed to the proof of Theorem 5.7.

*Proof of Theorem 5.7.* The expressions for  $S_{M, \mathcal{M}}$  and  $S_{B, \mathcal{M}}$  follow immediately from the definition of mixed area measures (cf. section 4.1) and Proposition 5.8.

In the remainder of the proof, we define the operator  $\mathcal{A}$  as in the statement of Theorem 5.7. Our aim is to show that the operator thus defined coincides with the general construction of Theorem 5.1 in the present setting. To this end, consider first  $f \in C^2(S^{n-1})$ . By Lemma 5.9, we have  $f \in \text{Dom } \mathcal{A}$ . Moreover,

$$\mathcal{A}f := \frac{1}{n} \{f'' + f\} = \frac{1}{n} \frac{dS_{f, \mathcal{M}}}{dS_{B, \mathcal{M}}}$$

by Proposition 5.8. Thus the restriction of  $\mathcal{A}$  to  $C^2(S^{n-1}) \subset \text{Dom } \mathcal{A}$  agrees with the operator defined in (5.2). It remains to show that  $\mathcal{A}$  as defined in Theorem 5.7 is self-adjoint, that  $\mathcal{E}$  in Theorem 5.7 is the associated closed quadratic form, and that  $\mathcal{A}$  agrees with the Friedrichs extension of its restriction to  $C^2(S^{n-1})$ .

Consider any  $f, g \in \text{Dom } \mathcal{A}$ . Then

$$\langle g, \mathcal{A}f \rangle_{L^2(S_{B, \mathcal{M}})} = \frac{1}{n(n-1)} \sum_{e_{F, F'} \in E} \mathcal{H}^{n-2}(F \cap F') \int_{e_{F, F'}} (f'' + f) g d\mathcal{H}^1$$

by definition. Integrating by parts yields

$$\begin{aligned} \int_{e_{F, F'}} (f'' + f) g d\mathcal{H}^1 = \\ \int_{e_{F, F'}} \{fg - f'g'\} d\mathcal{H}^1 - \nabla_{n_{F \rightarrow F'}} f(n_F)g(n_F) - \nabla_{n_{F' \rightarrow F}} f(n_{F'})g(n_{F'}). \end{aligned}$$

Multiplying this expression by  $\mathcal{H}^{n-2}(F \cap F')$  and summing over  $e_{F, F'} \in E$  yields

$$\langle g, \mathcal{A}f \rangle_{L^2(S_{B, \mathcal{M}})} = \frac{1}{n(n-1)} \sum_{e_{F, F'} \in E} \mathcal{H}^{n-2}(F \cap F') \int_{e_{F, F'}} \{fg - f'g'\} d\mathcal{H}^1,$$

where the boundary terms vanish by the definition of  $\text{Dom } \mathcal{A}$ . We have therefore shown that  $\mathcal{E}(f, g) = \langle g, \mathcal{A}f \rangle_{L^2(S_{B, \mathcal{M}})}$  for all  $f, g \in \text{Dom } \mathcal{A}$ . It is now a standard exercise to prove that  $\mathcal{A}$  is self-adjoint and that  $\mathcal{E}$  is the associated closed quadratic form with the given domains; see, e.g., [4, Theorems 1.4.4 and 1.4.11].

Finally, we claim that  $\mathcal{A}$  is the Friedrichs extension of its restriction to  $C^2(S^{n-1})$ . Indeed, by applying Lemma 5.10 below to every edge, we can write any  $f \in \text{Dom } \mathcal{E}$  as the limit with respect to the norm  $[2\|f\|_{L^2(S_{B, \mathcal{M}})}^2 - \mathcal{E}(f, f)]^{1/2}$  of functions  $f_k \in C^0(G)$  that are  $C^\infty$  on each edge and constant in a neighborhood of each vertex. By a classical extension argument [27, Lemma 2.26], any such function may be extended to a  $C^\infty$  function on the entire sphere  $S^{n-1}$ . We have therefore shown that  $\mathcal{E}$  is the closure of its restriction to  $C^2(S^{n-1})$ , completing the proof.  $\square$

Above we used the following simple approximation argument. We recall that  $H^1([0, l]) \subset C^0([0, l])$  by the Sobolev embedding theorem [19, Theorem 7.26], so that the value of a function  $h \in H^1$  is well defined at every point.

**Lemma 5.10.** *Let  $h : [0, l] \rightarrow \mathbb{R}$  satisfy  $h \in H^1$ . Then there exists a sequence of functions  $h_k : [0, l] \rightarrow \mathbb{R}$  such that for all sufficiently large  $k$  the following hold:*

- $h_k \in C^\infty$ .
- $h_k(x) = h(0)$  on  $x \in [0, a_k]$  and  $h_k(x) = h(l)$  on  $x \in [l - a_k, l]$  for some  $a_k > 0$ .
- $h_k \rightarrow h$  and  $h'_k \rightarrow h'$  in  $L^2$  as  $k \rightarrow \infty$ .

*Proof.* The argument is classical; see, e.g., [13, Theorem 7.2.1]. By convolving with a smooth mollifier, we first construct  $C^\infty$  functions  $u_k$  such that  $u_k \rightarrow h$  and  $u'_k \rightarrow h'$  in  $L^2$  [19, section 7.6]. By the Sobolev embedding theorem,  $u_k \rightarrow h$  uniformly. We may therefore assume without loss of generality that  $u_k(0) = h(0)$  and  $u_k(l) = h(l)$  for every  $k$ ; otherwise this may be accomplished by adding a linear function to  $u_k$  without changing the above approximation properties.

Now define for every  $m \geq 4/l$  a function  $\iota_m : [0, l] \rightarrow [0, l]$  with the following properties:  $\iota_m$  is  $C^\infty$  and  $0 \leq \iota'_m \leq 3$ ;  $\iota_m(x) = x$  for  $x \in [\frac{2}{m}, l - \frac{2}{m}]$ ;  $\iota_m(x) = 0$  for  $x \in [0, \frac{1}{m}]$ ; and  $\iota_m(x) = l$  for  $x \in [l - \frac{1}{m}, l]$ . Then we have  $u_k \circ \iota_m \rightarrow u_k$  and  $(u_k \circ \iota_m)' = (u'_k \circ \iota_m)\iota'_m \rightarrow u'_k$  in  $L^2$  as  $m \rightarrow \infty$  for any  $k$ . Moreover,  $u_k \circ \iota_m$  clearly satisfies properties a. and b. of the statement of the lemma. We may therefore construct  $\{h_k\}$  by extracting a diagonal subsequence of  $\{u_k \circ \iota_m\}_{k, m \geq 1}$ .  $\square$

We conclude this section with two remarks.

**Remark 5.11** (General mixed volumes). For simplicity, we have restricted attention in this section to the special case of Theorem 5.1 where  $C_1 = \dots = C_{n-2} = M$ . This is the only setting that will be needed in the sequel. However, Theorem 5.7 is readily extended to the setting where  $C_1, \dots, C_{n-2}$  are arbitrary polytopes. Let us briefly sketch the relevant constructions, leaving the details to the reader.

Let  $C_1, \dots, C_{n-2}$  be polytopes. Then the polytopes  $M_\lambda := \lambda_1 C_1 + \dots + \lambda_{n-2} C_{n-2}$  are strongly isomorphic for all choices of  $\lambda_1, \dots, \lambda_{n-2} > 0$  [40, Corollary 2.4.12]; in particular, all  $M_\lambda$  induce the same metric graph  $G = (V, E)$ , whose vertices are the facet normals of  $M_1 := C_1 + \dots + C_{n-2}$  and whose edges are indexed by the  $(n-2)$ -faces of  $M_1$ . Now let  $e \in E$ , and let  $\tilde{F}^e$  be the associated  $(n-2)$ -face of  $M_1$ . By Lemma 4.9, there exists for each  $i$  a face  $\tilde{F}_i^e$  of  $C_i$  such that  $\tilde{F}^e = \tilde{F}_1^e + \dots + \tilde{F}_{n-2}^e$ . It follows from Proposition 5.8 and the definitions in section 4.1 that

$$\int f dS_{K,c} = \frac{1}{n-1} \sum_{e \in E} \mathcal{V}(\tilde{F}_1^e, \dots, \tilde{F}_{n-2}^e) \int_e (h_K'' + h_K) f d\mathcal{H}^1$$

for every convex body  $K$  of class  $C_+^2$ , where the mixed volume that appears here is computed in the  $(n-2)$ -dimensional subspace spanned by  $\tilde{F}^e$  (modulo translation of the faces  $\tilde{F}_i^e$ ). Using this expression we may readily adapt Theorem 5.7 to the present setting with the same proof. The underlying graph is now the metric graph associated to  $M_1$ , and the weights  $\mathcal{H}^{n-2}(F \cap F')$  are replaced everywhere by the mixed volumes  $\mathcal{V}(\tilde{F}_1^e, \dots, \tilde{F}_{n-2}^e)$ . Let us note that some of these weights may vanish in the general case, unlike in the more restricted setting of Theorem 5.7.

**Remark 5.12** (The kernel of a quantum graph). Lemma 3.1 states that the study of equality cases in the Minkowski or Alexandrov-Fenchel inequalities is equivalent to understanding  $\ker \mathcal{A}$ . In the setting of Theorem 5.7, we can attempt to compute this kernel explicitly. Indeed, if  $\mathcal{A}f = 0$ , then we must evidently have  $f|_e(\theta) = a_e \cos(\theta) + b_e \sin(\theta)$  on every edge  $e \in E$ . Thus we only need to compute the vector of coefficients  $(a_e, b_e)_{e \in E}$ . By using the vertex boundary conditions of  $\text{Dom } \mathcal{A}$ , it can be shown that  $f \in \ker \mathcal{A}$  if and only if the vector of coefficients is in the kernel of a certain matrix, cf. [4, section 3.6]. In the present setting, this matrix turns out to be precisely what appears in Alexandrov's polytope proof of the Alexandrov-Fenchel inequality [1]. This is essentially the idea behind Schneider's proof of the equality cases of the Alexandrov-Fenchel inequality in the case that  $C_1, \dots, C_{n-2}$  are strongly isomorphic simple polytopes [40, Theorem 7.6.21].

The polytope setting is rather special, however, and such direct computations cannot be performed for general bodies. One may view the methods that will be developed in subsequent sections as a kind of quantitative replacement for this argument, that enables us to pass to the limiting case of arbitrary convex bodies.

**5.3. Dirichlet forms.** Our aim in this section is to provide some additional insight into the structure of the operator defined in a somewhat abstract manner by Theorem 5.1. The results of this section will not be used elsewhere in this paper, but are included in order to clarify our general constructions.

In two special cases, when  $C_1, \dots, C_{n-2}$  are polytopes and when they are bodies of class  $C_+^\infty$ , we have seen that  $\mathcal{A}$  can be expressed explicitly as a second-order differential operator. Another explicit setting will be encountered in section 8 below. In the general setting, however, it is not so clear what such an operator may look like: it is perfectly possible in general for a limit of second-order differential operators to exhibit quite different behavior [31]. The study of such operators is

enabled in a very general setting by the theory of Dirichlet forms [18, 3]. In the present setting, however, we can give a more concrete representation that avoids the abstract theory. The main result of this section is the following.

**Theorem 5.13.** *Let  $C_1, \dots, C_{n-2}$  be arbitrary convex bodies in  $\mathbb{R}^n$  with  $S_{B,C} \neq 0$ , and let  $\mathcal{A}$  and  $\mathcal{E}$  be the operator and quadratic form on  $L^2(S_{B,C})$  of Theorem 5.1. Then there exists a measurable function  $A : S^{n-1} \rightarrow \text{Sym}_n(\mathbb{R})$  such that<sup>7</sup>  $A(x)x = 0$ ,  $A(x) \geq 0$ , and  $\text{Tr}[A(x)] = 1$  for all  $x \in S^{n-1}$ , and such that*

$$\mathcal{A}f = \frac{1}{n} \text{Tr}[A D^2 f], \quad \mathcal{E}(f, g) = \frac{1}{n} \int \{fg - \langle \nabla_{S^{n-1}} f, A \nabla_{S^{n-1}} g \rangle\} dS_{B,C}$$

for every  $f, g \in C^2(S^{n-1})$ , where  $\nabla_{S^{n-1}}$  is the covariant derivative on  $S^{n-1}$ .

From a practical viewpoint, the problem with Theorem 5.13 is that we do not have a useful explicit expression for the tensor field  $A$  when  $C_1, \dots, C_{n-2}$  are arbitrary bodies. For this reason, this result will not be used in the rest of the paper. Nonetheless, Theorem 5.13 shows that the objects in Theorem 5.1 may be viewed rather concretely as highly degenerate elliptic second-order differential operators.

**Remark 5.14.** As we are working on  $L^2(S_{B,C})$ , the tensor field  $A$  need only be defined  $S_{B,C}$ -a.e. It may be viewed as a kind of Riemannian metric on  $\text{supp } S_{B,C}$ , cf. [22]. For example, for polytopes it follows from Remark 5.11 that  $\text{rank } A = 1$  a.e., which reflects the fact that the underlying metric graph is one-dimensional.

The main observation behind the proof of Theorem 5.13 is the following.

**Lemma 5.15.** *Define the functions  $\ell_i : S^{n-1} \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$  as  $\ell_i(x) := x_i$ . Then the following are equivalent.*

1. *There exists a measurable function  $A : S^{n-1} \rightarrow \text{Sym}_n(\mathbb{R})$  such that  $A(x)x = 0$  and  $\text{Tr}[A(x)] = 1$  for all  $x \in S^{n-1}$ , and*

$$\mathcal{A}f = \frac{1}{n} \text{Tr}[A D^2 f] \quad \text{for all } f \in C^2(S^{n-1}).$$

2. *We have*

$$dS_{f,C} = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \bar{f}}{\partial x_i \partial x_j} dS_{\ell_i \ell_j, C} \quad \text{for all } f \in C^2(S^{n-1}),$$

where  $\bar{f}$  is the 1-homogeneous extension of  $f$  to  $\mathbb{R}^n$ .

*Proof.* To prove  $2 \Rightarrow 1$ , recall that as  $\bar{f}$  is 1-homogeneous, its Hessian in  $\mathbb{R}^n$  satisfies  $\nabla^2 \bar{f}(x)x = 0$  (cf. section 4.2). If we therefore define  $\bar{A}(x)$ ,  $A(x) \in \text{Sym}_n(\mathbb{R})$  as

$$\bar{A}_{ij} := \frac{1}{2} \frac{dS_{\ell_i \ell_j, C}}{dS_{B,C}}, \quad A(x) := P_{x^\perp} \bar{A}(x) P_{x^\perp}$$

using Lemma 5.4, where  $P_{x^\perp}$  denotes the orthogonal projection on  $x^\perp$  in  $\mathbb{R}^n$ , then

$$\mathcal{A}f = \frac{1}{n} \frac{dS_{f,C}}{dS_{B,C}} = \frac{1}{n} \text{Tr}[A D^2 f]$$

by (5.2). That  $\text{Tr}[A(x)] = 1$  follows by choosing  $f = 1$  and using that  $\mathcal{A}1 = \frac{1}{n}1$  by Theorem 5.1. Thus the claim is established.

<sup>7</sup>As in the definition of  $D^2 f$  in section 4.2, we view  $A(x)$  as a matrix acting on the tangent space  $x^\perp$  at each  $x \in S^{n-1}$  (that is,  $A$  is a measurable  $(1, 1)$ -tensor field on  $S^{n-1}$ ).

To prove the converse implication  $1 \Rightarrow 2$ , note first that  $f = \bar{f}(\ell_1, \dots, \ell_n)$ . Substituting this into the expression for  $\mathcal{A}f$  and using the chain rule gives

$$n\mathcal{A}f = f + \sum_{i=1}^n \frac{\partial \bar{f}}{\partial x_i} \operatorname{Tr}[A \nabla_{S^{n-1}}^2 \ell_i] + \sum_{i,j=1}^n \frac{\partial^2 \bar{f}}{\partial x_i \partial x_j} \langle \nabla_{S^{n-1}} \ell_i, A \nabla_{S^{n-1}} \ell_j \rangle, \quad (5.3)$$

where we used  $D^2 f = \nabla_{S^{n-1}}^2 f + fI$  and  $\operatorname{Tr}[A(x)] = 1$ . As  $\ell_i$  is a linear function, we obtain  $\operatorname{Tr}[A \nabla_{S^{n-1}}^2 \ell_i] = n\mathcal{A}\ell_i - \ell_i = -\ell_i$  by Theorem 5.1. But note that

$$\sum_{i=1}^n \ell_i(x) \frac{\partial \bar{f}}{\partial x_i}(x) = \langle x, \nabla \bar{f}(x) \rangle = \bar{f}(x)$$

by 1-homogeneity. Thus the first two terms on the right-hand side of (5.3) cancel. To simplify the last term, we use again the chain rule to write

$$\begin{aligned} 2\langle \nabla_{S^{n-1}} \ell_i, A \nabla_{S^{n-1}} \ell_j \rangle &= \operatorname{Tr}[A \nabla_{S^{n-1}}^2 (\ell_i \ell_j)] - \ell_i \operatorname{Tr}[A \nabla_{S^{n-1}}^2 \ell_j] - \ell_j \operatorname{Tr}[A \nabla_{S^{n-1}}^2 \ell_i] \\ &= n\mathcal{A}(\ell_i \ell_j) + \ell_i \ell_j. \end{aligned}$$

But as

$$\sum_{i,j=1}^n \ell_i(x) \ell_j(x) \frac{\partial^2 \bar{f}}{\partial x_i \partial x_j}(x) = \langle x, \nabla^2 \bar{f}(x) x \rangle = 0$$

by 1-homogeneity, we have shown that

$$\mathcal{A}f = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \bar{f}}{\partial x_i \partial x_j} \mathcal{A}(\ell_i \ell_j).$$

The proof is concluded by invoking again (5.2).  $\square$

*Proof of Theorem 5.13.* Suppose first that  $C_1, \dots, C_{n-2}$  are convex bodies of class  $C_+^\infty$ . Then we obtain as in the proof of Lemma 5.4

$$\mathcal{A}f = \frac{1}{n} \frac{\operatorname{D}(D^2 f, D^2 h_{C_1}, \dots, D^2 h_{C_{n-2}})}{\operatorname{D}(I, D^2 h_{C_1}, \dots, D^2 h_{C_{n-2}})} \quad \text{for } f \in C^2(S^{n-1}).$$

By linearity of mixed discriminants, this expression may be written as

$$\mathcal{A}f = \frac{1}{n} \operatorname{Tr}[A D^2 f]$$

for a continuous function  $A : S^{n-1} \rightarrow \operatorname{Sym}_n(\mathbb{R})$ . As  $D^2 f$  only acts on  $x^\perp$ , we may choose  $A(x)x = 0$  without loss of generality. Moreover, that  $A(x) \geq 0$  follows from Lemma 4.8(d). Finally, that  $\operatorname{Tr}[A(x)] = 1$  follows by choosing  $f = 1$  (so that  $D^2 f = I$ ) in the above expressions for  $\mathcal{A}$ .

Now let  $C_1, \dots, C_{n-2}$  be arbitrary convex bodies. Note that by Theorem 4.3, the second condition of Lemma 5.15 is preserved by taking Hausdorff limits. As we have established above the first condition of Lemma 5.15 for bodies of class  $C_+^\infty$ , we may extend this conclusion to arbitrary bodies by approximation as in the proof of Lemma 5.6. We have therefore shown that

$$\mathcal{A}f = \frac{1}{n} \operatorname{Tr}[A D^2 f]$$

for some  $A : S^{n-1} \rightarrow \operatorname{Sym}_n(\mathbb{R})$  such that  $A(x)x = 0$  and  $\operatorname{Tr}[A(x)] = 1$ .

We now show that  $A(x) \geq 0$ , that is, that  $\mathcal{A}$  is (semi)elliptic. To this end, define the quadratic function  $q_v(x) := \langle v, x \rangle^2$  for  $v \in \mathbb{R}^n$ . Then

$$\nabla_{S^{n-1}}^2 q_v(x) = 2P_{x^\perp} v (P_{x^\perp} v)^* - 2q_v(x)I,$$

where  $P_{x^\perp}$  is the orthogonal projection on  $x^\perp$  and  $z^*$  denotes the transpose of the column vector  $z$ . It follows that

$$\mathcal{A}q_v + \frac{1}{n}q_v = \frac{1}{n}\operatorname{Tr}[A D^2 q_v] + \frac{1}{n}q_v = \frac{2}{n}\langle v, Av \rangle,$$

where we used  $A(x)x = 0$  and  $\operatorname{Tr}[A(x)] = 1$ . Thus the condition  $A \geq 0$  is equivalent to the statement that  $\mathcal{A}q_v + \frac{1}{n}q_v \geq 0$  pointwise for every  $v \in \mathbb{R}^n$ , or, equivalently by (5.2), that  $d\mu_v := dS_{q_v, \mathcal{C}} + q_v dS_{B, \mathcal{C}}$  is a (nonnegative) measure for every  $v \in \mathbb{R}^n$ . But we have already established ellipticity in the case that  $C_1, \dots, C_{n-2}$  are of class  $C_+^\infty$ , and the nonnegativity of  $\mu_v$  is preserved by Hausdorff convergence due to Theorem 4.3. We can therefore conclude that  $A \geq 0$  for arbitrary convex bodies by approximation as in the proof of Lemma 5.6.

It remains to compute the quadratic form associated to  $\mathcal{A}$ . Note that

$$\mathcal{E}(f, g) = \langle f, \mathcal{A}g \rangle_{L^2(S_{B, \mathcal{C}})} = \frac{1}{2} \int \left\{ f \mathcal{A}g + g \mathcal{A}f - \mathcal{A}(fg) + \frac{1}{n}fg \right\} dS_{B, \mathcal{C}}$$

for  $f, g \in C^2(S^{n-1})$ , where we have used that  $\mathcal{A}$  is self-adjoint and  $\mathcal{A}1 = \frac{1}{n}1$ . But

$$\mathcal{A}(fg) = \frac{1}{n} \{ \operatorname{Tr}[A \nabla_{S^{n-1}}^2(fg)] + fg \} = \frac{2}{n} \langle \nabla_{S^{n-1}} f, A \nabla_{S^{n-1}} g \rangle + f \mathcal{A}g + g \mathcal{A}f - \frac{1}{n}fg$$

by the product rule. We have therefore shown that

$$\mathcal{E}(f, g) = \frac{1}{n} \int \{ fg - \langle \nabla_{S^{n-1}} f, A \nabla_{S^{n-1}} g \rangle \} dS_{B, \mathcal{C}}$$

for  $f, g \in C^2(S^{n-1})$ . The proof is complete.  $\square$

## 6. A WEAK STABILITY THEOREM

The aim of this section is to prove a weak stability result that will be used in section 7 below as input to the main part of the proof of Theorem 2.2. The main result of this section is the following quantitative form of Theorem 3.4; the latter follows immediately by combining the following theorem with Theorem 4.4.

**Theorem 6.1.** *Let  $M$  be a convex body in  $\mathbb{R}^n$  with  $0 \in \operatorname{int} M$ . Then there is a constant  $C_M > 0$ , depending only on  $M$  and on the dimension  $n$ , so that*

$$\begin{aligned} \mathbf{V}(K, L, \mathcal{M})^2 &\geq \mathbf{V}(K, K, \mathcal{M}) \mathbf{V}(L, L, \mathcal{M}) + \\ &C_M \mathbf{V}(L, L, \mathcal{M}) \inf_{v \in \mathbb{R}^n, a \geq 0} \int (h_K - ah_L - \langle v, \cdot \rangle)^2 \frac{dS_{M, \mathcal{M}}}{h_M} \end{aligned}$$

for all convex bodies  $K, L$  in  $\mathbb{R}^n$ .

Theorem 6.1 is different in spirit than most of the theory developed in this paper, in that it is  $S_{M, \mathcal{M}}$  rather than  $S_{B, \mathcal{M}}$  that appears here as the reference measure. As  $\operatorname{supp} S_{M, \mathcal{M}}$  is much smaller than  $\operatorname{supp} S_{B, \mathcal{M}}$ , only weak information on the extremals may be extracted from this result (cf. Example 3.5). Nonetheless, this weak information provides crucial input to the quantitative rigidity analysis of the following section, and we therefore develop it first.

The idea behind the proof of Theorem 6.1 is as follows. Lemma 3.1 shows that Minkowski's quadratic inequality follows if one can show that an associated self-adjoint operator has a one-dimensional positive eigenspace. One may readily modify the proof of this fact to show that if, in addition, the zero eigenvalue of the operator is separated from the rest of the spectrum by a positive gap, then one obtains a quantitative improvement along the lines of Theorem 6.1.

It was recently observed by Kolesnikov and Milman [26] in their study of local  $L^p$ -Brunn-Minkowski inequalities that, in the case where all bodies involved are smooth and symmetric, such a spectral separation may be established by means of a differential-geometric technique (albeit when the operator is normalized differently than in Theorem 5.1, which is responsible for the presence of  $S_{M,\mathcal{M}}$  rather than  $S_{B,\mathcal{M}}$  in Theorem 6.1). The approach of [26] may also be used to obtain a quantitative result for non-symmetric bodies, as we will show in section 6.1. In this case, however, the resulting bound no longer has a direct spectral interpretation. Nonetheless, we will show in section 6.2 that the requisite spectral property can be recovered, for a suitably normalized operator, using the min-max principle. Theorem 6.1 then follows for smooth bodies by reasoning as in the proof Lemma 3.1, and the proof is concluded by smooth approximation. The background from Riemannian geometry that is needed in this section may be found, e.g., in [10].

**Remark 6.2.** By exploiting the linear equivariance of the quantities that appear in Theorem 6.1, it may be shown that the constant  $C_M$  can in fact be chosen to depend on the dimension  $n$  only. This property, which is important in the theory of [26], is not relevant in our setting. We therefore do not repeat the arguments leading to this observation, and refer the interested reader to [26].

**6.1. An extrinsic formulation.** Behind the proof of Theorem 6.1 lies a different perspective on mixed volumes than we have encountered so far. By definition we have  $V(K, K, \mathcal{M}) = \frac{1}{n(n-1)} \frac{d^2}{dt^2} \text{Vol}(M+tK)|_{t=0+}$ , so that mixed volumes of this kind may be viewed as arising from the second variation of the volume of the convex body  $M$ . First and second variation formulae play a classical role in Riemannian geometry, for example, in the theory of minimal surfaces [10, 11]. In this setting, however, the variation formulae are not expressed on the sphere as in Lemma 4.7, but rather in terms of the extrinsic geometry of  $\partial M$  viewed as a hypersurface in  $\mathbb{R}^n$ . These two viewpoints are related by using the outer normal map  $n_M$  as a change of variables, as we did in section 4.2. The change of perspective is useful, however, as it enables us to exploit classical techniques from Riemannian geometry.

**Remark 6.3.** This is the main point in this paper where we rely specifically on the restricted setting of Minkowski's quadratic inequality, as opposed to the general Alexandrov-Fenchel inequality: the mixed volumes that appear in (1.5) are precisely those that arise as second variations of the volume of  $M$ . For general mixed volumes, the body  $M$  no longer plays any distinguished role, and it seems unlikely that the methods of this section could be useful in this context. In contrast, the techniques of sections 5, 7, and 8 do not appear to be fundamentally tied to the special setting of (1.5), and could potentially be adapted to a much more general context.

Throughout this section, we will work with convex bodies  $K, L, M$  in  $\mathbb{R}^n$  of class  $C_+^\infty$ . We will also assume that  $0 \in \text{int } M$ , so that  $h_M > 0$ . We denote by  $\Pi := \nabla n_M$  the second fundamental form of  $\partial M$  (viewed, as usual, as a symmetric linear map  $\Pi(x) : T_x \partial M \rightarrow T_x \partial M$ ). As in previous sections, the symbols  $\nabla$ ,  $\nabla_{S^{n-1}}$ , and  $\nabla_{\partial M}$  denote covariant differentiation in  $\mathbb{R}^n$ ,  $S^{n-1}$ , and  $\partial M$ , respectively.

We begin by making explicit the second variation formula alluded to above. Following [12], we will derive the formula by a change of variables.

**Lemma 6.4.** *Let  $K, L, M$  be convex bodies in  $\mathbb{R}^n$  of class  $C_+^\infty$ . Then*

$$\begin{aligned} n(n-1)V(K, L, \mathcal{M}) &= \int_{\partial M} (h_K \circ n_M)(h_L \circ n_M) \operatorname{Tr}[\Pi] dx \\ &\quad - \int_{\partial M} \langle \nabla_{\partial M}(h_K \circ n_M), \Pi^{-1} \nabla_{\partial M}(h_L \circ n_M) \rangle dx. \end{aligned}$$

*Proof.* First note that by Lemma 4.7 and Lemma 4.8(b), we have

$$V(K, L, \mathcal{M}) = \frac{1}{n(n-1)} \int_{S^{n-1}} h_K \operatorname{Tr}[\operatorname{cof}(D^2 h_M) D^2 h_L] d\omega.$$

Let  $\mathcal{L}f := \operatorname{Tr}[\operatorname{cof}(D^2 h_M) D^2 f]$ . It follows from the symmetry of mixed volumes that  $\mathcal{L}$  is a symmetric operator on  $C^2(S^{n-1}) \subset L^2(\omega)$ . We may therefore write

$$\begin{aligned} n(n-1)V(K, L, \mathcal{M}) &= \frac{1}{2} \int_{S^{n-1}} \{h_K \mathcal{L}h_L + h_L \mathcal{L}h_K - \mathcal{L}(h_K h_L) + h_K h_L \mathcal{L}1\} d\omega \\ &= \int_{S^{n-1}} \{h_K h_L \operatorname{Tr}[\operatorname{cof}(D^2 h_M)] - \langle \nabla_{S^{n-1}} h_K, \operatorname{cof}(D^2 h_M) \nabla_{S^{n-1}} h_L \rangle\} d\omega, \end{aligned}$$

where we used  $D^2 f = \nabla_{S^{n-1}}^2 f + fI$  and the product rule.

Now note that as  $n_M^{-1} = \nabla h_M$ , we have

$$D^2 h_M \circ n_M = \nabla n_M^{-1} \circ n_M = (\nabla n_M)^{-1} = \Pi^{-1}.$$

Similarly, we can compute by the chain rule

$$\nabla_{S^{n-1}} f \circ n_M = \Pi^{-1} \nabla_{\partial M}(f \circ n_M).$$

Finally, as  $D^2 h_M > 0$ , we may write

$$\operatorname{cof}(D^2 h_M) = (D^2 h_M)^{-1} \det(D^2 h_M).$$

The proof is completed by changing variables according to  $n_M^{-1}$  in the expression for  $V(K, L, \mathcal{M})$  and using the above identities.  $\square$

The quantities that appear in Lemma 6.4 are strongly reminiscent of the following classical formula of Reilly, obtained by integrating the Bochner formula on a manifold with boundary; for the proof, we refer to [10, Lemma A.17].

**Lemma 6.5.** *Let  $M \subset \mathbb{R}^n$  be a compact set with  $C^\infty$  boundary. Then*

$$\begin{aligned} \int_M (\Delta u)^2 dx &= \int_M \operatorname{Tr}[(\nabla^2 u)^2] dx \\ &\quad + \int_{\partial M} \{\operatorname{Tr}[\Pi] u_n^2 + \langle \nabla_{\partial M} u, \Pi \nabla_{\partial M} u \rangle - 2 \langle \nabla_{\partial M} u_n, \nabla_{\partial M} u \rangle\} dx \end{aligned}$$

for any  $u \in C^\infty(M)$ , where we defined the normal derivative  $u_n := \langle n_M, \nabla u \rangle$ .

That Minkowski's inequality may be deduced from Reilly's formula was observed in a special case by Reilly himself [35], and more generally by Kolesnikov and Milman [25]. In particular, it was noticed in [26] that the latter proof admits a quantitative improvement when the bodies are symmetric. The following result is a straightforward adaptation of [26, section 6] to the non-symmetric case.

**Proposition 6.6.** *Let  $M$  be a convex body in  $\mathbb{R}^n$  of class  $C_+^\infty$  with  $0 \in M$ . Then*

$$\int_{\partial M} \langle \nabla_{\partial M} g, \Pi^{-1} \nabla_{\partial M} g \rangle dx - \int_{\partial M} g^2 \operatorname{Tr}[\Pi] dx \geq \frac{r^2}{nR^2} \int_{\partial M} \frac{g^2}{h_M \circ n_M} dx$$

for any  $C^\infty$  function  $g : \partial M \rightarrow \mathbb{R}$  such that  $\int_{\partial M} g(x) dx = 0$  and  $\int_{\partial M} xg(x) dx = 0$ , where  $r, R > 0$  are chosen such that  $rB \subseteq M \subseteq RB$ .

*Proof.* As  $\int_{\partial M} g dx = 0$ , there exists a  $C^\infty$  solution  $u$  to the Neumann problem

$$\begin{aligned} \Delta u &= 0 && \text{on } M, \\ u_n &= g && \text{on } \partial M, \end{aligned}$$

see, e.g., [42, section 5.7]. Applying Lemma 6.5 yields

$$\begin{aligned} 0 &= \int_M \operatorname{Tr}[(\nabla^2 u)^2] dx + \int_{\partial M} \{g^2 \operatorname{Tr}[\Pi] + \langle \nabla_{\partial M} u, \Pi \nabla_{\partial M} u \rangle - 2\langle \nabla_{\partial M} g, \nabla_{\partial M} u \rangle\} dx \\ &\geq \int_M \operatorname{Tr}[(\nabla^2 u)^2] dx + \int_{\partial M} \{g^2 \operatorname{Tr}[\Pi] - \langle \nabla_{\partial M} g, \Pi^{-1} \nabla_{\partial M} g \rangle\} dx, \end{aligned}$$

where we used that  $\Pi > 0$  as  $M$  is a body of class  $C_+^\infty$ . To complete the proof, it remains to lower bound the first term on the right-hand side.

To this end, note that  $\operatorname{div}(\langle v, x \rangle \nabla u) = \langle v, \nabla u \rangle$  for any  $v \in \mathbb{R}^n$ , as  $\Delta u = 0$ . Thus

$$\int_M \nabla u(x) dx = \int_{\partial M} x u_n(x) dx = \int_{\partial M} x g(x) dx = 0.$$

By a classical Poincaré inequality of Payne and Weinberger [32], it follows that

$$\int_M \operatorname{Tr}[(\nabla^2 u)^2] dx \geq \frac{\pi^2}{4R^2} \int_M \|\nabla u\|^2 dx.$$

As  $r \leq h_M \leq R$  and  $h_M(n_M(x)) = \langle x, n_M(x) \rangle$ , we can now estimate

$$\begin{aligned} r^2 \int_{\partial M} \frac{g^2}{h_M \circ n_M} dx &\leq \int_{\partial M} u_n^2 \langle x, n_M \rangle dx \leq \int_{\partial M} \|\nabla u\|^2 \langle x, n_M \rangle dx \\ &= \int_M \operatorname{div}(x \|\nabla u\|^2) dx \\ &\leq (n+1) \int_M \|\nabla u\|^2 dx + R^2 \int_M \operatorname{Tr}[(\nabla^2 u)^2] dx \\ &\leq \left(1 + \frac{4(n+1)}{\pi^2}\right) R^2 \int_M \operatorname{Tr}[(\nabla^2 u)^2] dx, \end{aligned}$$

where we used the divergence theorem and

$$\operatorname{div}(x \|\nabla u\|^2) = n \|\nabla u\|^2 + 2 \langle \nabla u, \nabla^2 u x \rangle \leq (n+1) \|\nabla u\|^2 + \|\nabla^2 u x\|^2.$$

The proof follows readily by combining the above estimates (for aesthetic reasons, we have estimated  $1 + \frac{4(n+1)}{\pi^2} \leq n$  for  $n \geq 3$  in the statement).  $\square$

Lemma 6.4 shows that the left-hand side of the expression in Proposition 6.6 is nothing other than a mixed volume  $-n(n-1)\mathcal{V}(\tilde{g}, \tilde{g}, \mathcal{M})$  for  $\tilde{g} := g \circ n_M^{-1}$ , but the significance of the inequality may not be immediately obvious. We will presently see that Proposition 6.6 controls a gap in the spectrum of a certain self-adjoint operator associated to mixed volumes (different from the one in Theorem 5.1). This observation has a number of interesting consequences. For symmetric bodies, it implies a local form of the  $L^p$ -Brunn-Minkowski inequality with  $p < 1$ , which

was the problem investigated in [26]. In the present setting, the relevant spectral property will form the basis for the proof of Theorem 6.1.

**6.2. Proof of Theorem 6.1.** Let us fix, for the time being, a convex body  $M$  in  $\mathbb{R}^n$  of class  $C_+^\infty$  with  $0 \in \text{int } M$ . Define an operator and a measure on  $S^{n-1}$  by

$$\tilde{\mathcal{A}}f := \frac{1}{n(n-1)} h_M \text{Tr}[(D^2 h_M)^{-1} D^2 f], \quad d\tilde{\nu} := \frac{\det(D^2 h_M)}{h_M} d\omega$$

for  $f \in C^\infty(S^{n-1})$ . By Lemma 4.7 and Lemma 4.8(b), we have

$$\mathbb{V}(f, g, \mathcal{M}) = \langle f, \tilde{\mathcal{A}}g \rangle_{L^2(\tilde{\nu})} \quad \text{for } f, g \in C^\infty(S^{n-1}).$$

Consequently, we observe the following basic facts:

- $\tilde{\mathcal{A}}$  is an elliptic operator (as  $D^2 h_M > 0$ ).
- $\tilde{\mathcal{A}}$  defines a symmetric quadratic form  $\langle f, \tilde{\mathcal{A}}g \rangle_{L^2(\tilde{\nu})}$  for  $f, g \in C^\infty(S^{n-1})$ .
- $\tilde{\mathcal{A}}$  has a self-adjoint extension with compact resolvent. Moreover, as  $\tilde{\mathcal{A}}h_M = \frac{1}{n}h_M$ , its largest eigenvalue is  $\frac{1}{n}$  and this eigenvalue is simple (cf. section 4.4).
- $\tilde{\mathcal{A}}\ell = 0$  for any linear function  $\ell(x) = \langle v, x \rangle$ ,  $v \in \mathbb{R}^n$ .

It should be emphasized that the normalization chosen in the definition of  $\tilde{\mathcal{A}}$  is very different than the one employed in section 5; in particular, the present operator makes sense only for smooth bodies  $M$ , and does not give rise to a well-behaved limiting operator for arbitrary (non-smooth) bodies. Nonetheless, the present normalization is the appropriate one for exploiting Proposition 6.6.

**Lemma 6.7.** *Let  $M$  be a convex body in  $\mathbb{R}^n$  of class  $C_+^\infty$  with  $rB \subseteq M \subseteq RB$ . Whenever  $f \in C^\infty(S^{n-1})$  satisfies  $f \perp \text{span}\{h_M, \ell : \ell \text{ is linear}\}$  in  $L^2(\tilde{\nu})$ , we have*

$$\langle f, \tilde{\mathcal{A}}f \rangle_{L^2(\tilde{\nu})} \leq -\frac{1}{n^2(n-1)} \frac{r^2}{R^2} \|f\|_{L^2(\tilde{\nu})}^2.$$

*Proof.* The statement is spectral in nature. As  $\tilde{\mathcal{A}}$  has a compact resolvent, it has a discrete spectrum and a complete set of eigenfunctions (section 4.4). As stated above, the largest eigenvalue of  $\tilde{\mathcal{A}}$  is  $\frac{1}{n}$  and its one-dimensional eigenspace is spanned by  $h_M$ . Moreover, Minkowski's inequality implies as in the proof of Theorem 5.1 that  $\text{spec } \tilde{\mathcal{A}} \subseteq (-\infty, 0] \cup \{\frac{1}{n}\}$ , and all linear functions are eigenfunctions with eigenvalue 0. Therefore, by Lemma 4.13,

$$\sup_{f \perp \text{span}\{h_M, \ell : \ell \text{ is linear}\}} \frac{\langle f, \tilde{\mathcal{A}}f \rangle_{L^2(\tilde{\nu})}}{\|f\|_{L^2(\tilde{\nu})}^2} \leq \sup_{f \perp L} \frac{\langle f, \tilde{\mathcal{A}}f \rangle_{L^2(\tilde{\nu})}}{\|f\|_{L^2(\tilde{\nu})}^2}$$

for any linear space  $L \subset C^\infty(S^{n-1})$  with  $\dim L \leq n+1$ .

Now let  $f : S^{n-1} \rightarrow \mathbb{R}$  be  $C^\infty$ . Choosing  $g := f \circ n_M$  in Proposition 6.6 and changing variables as in Lemma 6.4, we find that

$$n(n-1) \langle f, \tilde{\mathcal{A}}f \rangle_{L^2(\tilde{\nu})} = n(n-1) \mathbb{V}(f, f, \mathcal{M}) \leq -\frac{r^2}{nR^2} \|f\|_{L^2(\tilde{\nu})}^2$$

whenever

$$\int f h_M d\tilde{\nu} = 0, \quad \int f h_M \nabla h_M d\tilde{\nu} = 0.$$

If we therefore choose  $L = \text{span}\{h_M, h_M \langle v, \nabla h_M \rangle : v \in \mathbb{R}^n\}$ , then we have shown

$$\sup_{f \perp L} \frac{\langle f, \tilde{\mathcal{A}}f \rangle_{L^2(\tilde{\nu})}}{\|f\|_{L^2(\tilde{\nu})}^2} \leq -\frac{1}{n^2(n-1)} \frac{r^2}{R^2}.$$

As clearly  $\dim L \leq n + 1$ , the proof is complete.  $\square$

Before we proceed to the proof of Theorem 6.1, we state a simple lemma.

**Lemma 6.8.** *For any convex body  $M$  with  $0 \in \text{int } M$ , we have*

$$G_M := \int xx^* \frac{S_{M, \mathcal{M}}(dx)}{h_M(x)} > 0.$$

*Proof.* If the conclusion were false, then there must exist  $w \in \mathbb{R}^n \setminus \{0\}$  such that  $\langle w, G_M w \rangle = 0$ . This would imply that  $\langle w, x \rangle = 0$  for all 0-extreme normal vectors  $x$  of  $M$  by Theorem 4.4. But that is impossible, as a convex body with nonempty interior is the intersection of its regular supporting halfspaces [40, Theorem 2.2.6]; if the normals of all these halfspaces were orthogonal to  $w$ , then  $M$  would be noncompact, contradicting the definition of a convex body.  $\square$

We are now ready to complete the proof of Theorem 6.1.

*Proof of Theorem 6.1.* We may clearly assume that  $\mathbf{V}(L, L, \mathcal{M}) > 0$ , as otherwise the statement reduces to Minkowski's quadratic inequality. Then  $\mathbf{V}(L, M, \mathcal{M}) > 0$  as well, as by Minkowski's inequality  $\mathbf{V}(L, M, \mathcal{M})^2 \geq \mathbf{V}(L, L, \mathcal{M}) \text{Vol}(M) > 0$ .

Let us first consider the case that  $M$  is of class  $C_+^\infty$  with  $rB \subseteq M \subseteq RB$  for  $r, R > 0$ , and that  $K, L$  are of class  $C_+^\infty$ . Using  $\langle h_L, h_M \rangle_{L^2(\tilde{\nu})} = n\mathbf{V}(L, M, \mathcal{M}) > 0$  and Lemma 6.8, we can define  $a \geq 0$  and  $v \in \mathbb{R}^n$  as

$$a := \frac{\langle h_K, h_M \rangle_{L^2(\tilde{\nu})}}{\langle h_L, h_M \rangle_{L^2(\tilde{\nu})}} = \frac{\mathbf{V}(K, M, \mathcal{M})}{\mathbf{V}(L, M, \mathcal{M})}, \quad v := \int (h_K(x) - ah_L(x)) G_M^{-1} x \frac{S_{M, \mathcal{M}}(dx)}{h_M(x)}.$$

Then it is readily verified (using  $d\tilde{\nu} = dS_{M, \mathcal{M}}/h_M$ ) that

$$\delta := h_K - ah_L - \langle v, \cdot \rangle \perp \text{span}\{h_M, \ell : \ell \text{ is linear}\} \quad \text{in } L^2(\tilde{\nu}).$$

Applying Lemma 6.7 and translation-invariance of mixed volumes yields

$$\begin{aligned} -\frac{1}{n^2(n-1)} \frac{r^2}{R^2} \|\delta\|_{L^2(\tilde{\nu})}^2 &\geq \mathbf{V}(\delta, \delta, \mathcal{M}) \\ &= \mathbf{V}(K, K, \mathcal{M}) - 2a\mathbf{V}(K, L, \mathcal{M}) + a^2\mathbf{V}(L, L, \mathcal{M}) \\ &\geq \mathbf{V}(K, K, \mathcal{M}) - \frac{\mathbf{V}(K, L, \mathcal{M})^2}{\mathbf{V}(L, L, \mathcal{M})}, \end{aligned}$$

where we minimized over  $a$  in the last inequality. The conclusion follows readily.

We now consider the general case where  $K, L, M$  are arbitrary convex bodies in  $\mathbb{R}^n$  and  $0 \in \text{int } M$ . It is classical [40, section 3.4] that we may choose convex bodies  $K^{(s)}, L^{(s)}, M^{(s)}$  of class  $C_+^\infty$  such that  $K^{(s)} \rightarrow K, L^{(s)} \rightarrow L, M^{(s)} \rightarrow M$  as  $s \rightarrow \infty$  in the sense of Hausdorff convergence. Note that as  $0 \in \text{int } M$ , there exist  $r, R > 0$  such that  $rB \subseteq M^{(s)} \subseteq RB$  for all  $s$  sufficiently large. Thus we have shown that

$$\begin{aligned} \mathbf{V}(K^{(s)}, L^{(s)}, \mathcal{M}^{(s)})^2 &\geq \mathbf{V}(K^{(s)}, K^{(s)}, \mathcal{M}^{(s)}) \mathbf{V}(L^{(s)}, L^{(s)}, \mathcal{M}^{(s)}) \\ &\quad + C_M \mathbf{V}(L^{(s)}, L^{(s)}, \mathcal{M}^{(s)}) \int (h_{K^{(s)}} - a^{(s)} h_{L^{(s)}} - \langle v^{(s)}, \cdot \rangle)^2 \frac{dS_{M^{(s)}, \mathcal{M}^{(s)}}}{h_{M^{(s)}}} \end{aligned}$$

for all  $s$  sufficiently large, where  $a^{(s)}, v^{(s)}$  are chosen as in the first part of the proof and we defined  $C_M := r^2/(R^2 n^2(n-1))$ . It remains to take  $s \rightarrow \infty$  in this inequality. Convergence of the mixed volumes follows directly from Theorem 4.3. Moreover, that  $a^{(s)} \rightarrow a$  and  $v^{(s)} \rightarrow v$  for some  $a \geq 0, v \in \mathbb{R}^n$  follows readily from Theorem 4.3 and the explicit expressions for  $a, v$  given in the first part of the proof. Thus the

integrand in the above inequality converges uniformly to  $(h_K - ah_L - \langle v, \cdot \rangle)^2/h_M$ , which implies convergence of the integral. This concludes the proof.  $\square$

## 7. A QUANTITATIVE RIGIDITY THEOREM

The weak stability result of the previous section implies that if equality holds in (1.5) (and  $M$  has nonempty interior and  $V(L, L, \mathcal{M}) > 0$ ) then, up to homothety,  $K$  and  $L$  have the same supporting hyperplanes in the 0-extreme normal directions of  $M$ . This is however far from characterizing the extremals of Minkowski's quadratic inequality, as was illustrated in Example 3.5. Nonetheless, we will show that this weak information can be amplified to recover the full equality cases, because the extremals of Minkowski's inequality turn out to be very rigid: once they are fixed in the 0-extreme directions of  $M$ , their extension to the 1-extreme directions of  $M$  is uniquely determined. This rigidity property, formulated above as Theorem 3.6, lies at the heart of our proof of Theorem 2.2.

Theorem 3.6 is an immediate consequence of the following quantitative result.

**Theorem 7.1.** *Let  $M$  be a convex body in  $\mathbb{R}^n$  with nonempty interior. There exist  $C_M > 0$  and a measure  $\mu_M$ , depending only on  $M$ , so that  $\text{supp } \mu_M \subseteq \text{supp } S_{M, \mathcal{M}}$  and*

$$V(K, L, \mathcal{M})^2 \geq V(K, K, \mathcal{M})V(L, L, \mathcal{M}) + C_M V(L, L, \mathcal{M}) \{ \|h_K - h_L\|_{L^2(S_{B, \mathcal{M}})}^2 - \|h_K - h_L\|_{L^2(\mu_M)}^2 \}$$

for all convex bodies  $K, L$  in  $\mathbb{R}^n$ .

The formulation of Theorem 7.1 is subtle due to the measure  $\mu_M$  appearing here. As will be explained in the proof, this measure does not appear to have a canonical geometric interpretation. Ideally, one would have liked to prove a version of Theorem 7.1 where  $\mu_M$  is replaced by the measure  $S_{M, \mathcal{M}}/h_M$  that appears in Theorem 6.1. If this were possible, then one would even obtain a sharp quantitative analogue of Theorem 2.2 (that is, a stability form of Minkowski's quadratic inequality). It is far from clear, however, how such a result might be proved: we do not know how to directly relate the measures  $S_{M, \mathcal{M}}$  and  $S_{B, \mathcal{M}}$ . Fortunately, to characterize the extremals it suffices to work with the measure  $\mu_M$  in Theorem 7.1, which may be viewed as a projection of  $S_{B, \mathcal{M}}$  on the 0-extreme normal vectors of  $M$ .

**Remark 7.2.** Spectrally, a stability form of Minkowski's quadratic inequality

$$V(K, L, \mathcal{M})^2 \stackrel{?}{\geq} V(K, K, \mathcal{M})V(L, L, \mathcal{M}) + C_M V(L, L, \mathcal{M}) \inf_{v \in \mathbb{R}^n, a \geq 0} \|h_K - ah_L - \langle v, \cdot \rangle\|_{L^2(S_{B, \mathcal{M}})}^2$$

may be shown as in the proof of Lemma 3.1 to be equivalent to the following: (i) the kernel of the operator  $\mathcal{A}$  in Theorem 5.1 consists only of linear functions (which characterizes the extremals); and (ii) the remainder of the spectrum is separated from zero by a positive constant (which quantifies the deficit). If  $\mathcal{A}$  were to have compact resolvent, then (ii) would follow directly from (i) by discreteness of the spectrum. Unfortunately, as we will see in section 8, it is not true in general that  $\mathcal{A}$  has compact resolvent. For this reason, it is far from clear whether we might expect even in principle to replace  $\mu_M$  by  $S_{M, \mathcal{M}}/h_M$  in Theorem 7.1. Understanding the answer to this question would be of considerable interest.

The rest of this section is organized as follows. In section 7.1, we complete the proof of Theorem 2.2 using Theorems 6.1 and 7.1. Sections 7.2 and 7.3 are devoted to the proof of Theorem 7.1. In section 7.2, we consider the special case where  $M$  is a polytope. We then extend the conclusion to general bodies  $M$  in section 7.3.

**7.1. Proof of Theorem 2.2.** Before we proceed to the proof of Theorem 7.1, let us show how Theorems 6.1 and 7.1 combine to complete the proof of Theorem 2.2. Note that the *if* direction of Theorem 2.2 was already proved by Schneider [36] by a slightly different method; it is the *only if* direction that is new. For completeness, we include a proof here of both directions using the methods of this paper.

*Proof of Theorem 2.2.* In the following, we assume that  $K, L, M$  are convex bodies in  $\mathbb{R}^n$  such that  $M$  has nonempty interior and  $V(L, L, \mathcal{M}) > 0$ .

Suppose first that there exist  $a \geq 0$  and  $v \in \mathbb{R}^n$  so that  $K$  and  $aL + v$  have the same supporting hyperplanes in all 1-extreme normal directions of  $M$ . Then

$$h_K - ah_L = \langle v, \cdot \rangle \quad S_{B, \mathcal{M}\text{-a.e.}}$$

by Theorem 4.4. Therefore, denoting by  $\mathcal{A}$  and  $\mathcal{E}$  the operator and quadratic form of Theorem 5.1, we have  $h_K - ah_L \in \ker \mathcal{A}$  and  $\mathcal{E}(h_L, h_L) = V(L, L, \mathcal{M}) > 0$ . Thus equality in (1.5) follows from Lemma 3.1.

Conversely, suppose that we have equality in (1.5). By translation-invariance of mixed volumes, we may assume without loss of generality that  $0 \in \text{int } M$ . Then Theorem 6.1 implies that there exist  $a \geq 0$  and  $v \in \mathbb{R}^n$  such that

$$\delta := h_K - ah_L - \langle v, \cdot \rangle = 0 \quad S_{M, \mathcal{M}\text{-a.e.}}$$

(note that the infimum in Theorem 6.1 is clearly attained, as it is the minimum of a nonnegative quadratic function).

By continuity, it follows that  $\delta$  vanishes on  $\text{supp } \mu_M \subseteq \text{supp } S_{M, \mathcal{M}}$ , where  $\mu_M$  is as in Theorem 7.1. Consequently, applying Theorem 7.1 with  $L \mapsto aL + v$  yields

$$\delta = h_K - ah_L - \langle v, \cdot \rangle = 0 \quad S_{B, \mathcal{M}\text{-a.e.}},$$

where we have used the invariance of Minkowski's quadratic inequality under translation and scaling of  $L$ . By continuity, it follows that  $\delta$  vanishes on  $\text{supp } S_{B, \mathcal{M}}$ . Thus Theorem 4.4 implies that  $K$  and  $aL + v$  have the same supporting hyperplanes in all 1-extreme normal directions of  $M$ , completing the proof.  $\square$

**7.2. Proof of Theorem 7.1: polytopes.** In this section we consider the case that  $M$  is a polytope with nonempty interior. At a qualitative level, the rigidity property of the extremals of Minkowski's inequality admits in this case a very intuitive interpretation. Suppose we have equality in (1.5), so that  $f := h_K - ah_L \in \ker \mathcal{A}$  for some  $a \geq 0$ . Suppose in addition that we have fixed the values of  $f$  in the 0-extreme normal directions of  $M$ , which are in this case the vertices of the metric graph associated to  $M$ . Then it follows from Theorem 5.7 that  $f$  solves the Dirichlet problem  $f'' + f = 0$  on each edge of the metric graph with boundary data on the vertices. It is readily verified by explicit computation that this one-dimensional Dirichlet problem has a unique solution as long as the lengths of all the edges are less than  $\pi$ , which must be the case as  $M$  has nonempty interior. Thus the value of  $f$  is uniquely determined on the 1-extreme normal directions of  $M$  once we have fixed its values on the 0-extreme normal directions.

This intuitive argument appears to be rather special to the case of polytopes: for a general body  $M$ , the structure of the sets of 0- and 1-extreme normal vectors can

be highly irregular, and it is far from clear even how to make sense of the Dirichlet problem in this setting. Instead, we will proceed by developing a quantitative formulation of the above intuition for polytopes. The key point is to find the “right” formulation that does not degenerate when we approximate an arbitrary convex body  $M$  by polytopes. Once such a formulation has been found, we will be able to extend its conclusion to the general setting by taking limits.

We now proceed to make these ideas precise. In the rest of this subsection,  $M$  will be a polytope in  $\mathbb{R}^n$  with nonempty interior, and we adopt without further comment the definitions and notation of section 5.2. Our starting point is the following Poincaré-type inequality on a single edge of the metric graph.

**Lemma 7.3.** *Let  $M$  be a polytope in  $\mathbb{R}^n$  with nonempty interior, and let  $F \sim F'$  be neighboring facets. Then for any function  $f \in H^1(e_{F,F'})$  and  $0 < \varepsilon < 1$ , we have*

$$l_{F,F'}^2 \int_{e_{F,F'}} (f')^2 d\mathcal{H}^1 \geq (1 - \varepsilon)^2 \pi^2 \int_{e_{F,F'}} f^2 d\mathcal{H}^1 - \frac{2}{\varepsilon} l_{F,F'} \{f(n_F)^2 + f(n_{F'})^2\}.$$

*Proof.* Assume without loss of generality that  $F \leq F'$ , and recall that we parametrize functions  $f : e_{F,F'} \rightarrow \mathbb{R}$  as  $f(\theta)$  for  $\theta \in [0, l_{F,F'}]$ , where  $\theta = 0$  corresponds to vertex  $n_F$  and  $\theta = l_{F,F'}$  corresponds to vertex  $n_{F'}$ . Define the function

$$y(\theta) := \cos \left( \frac{(1 - \varepsilon)\pi}{l_{F,F'}} \left( \theta - \frac{l_{F,F'}}{2} \right) \right),$$

and note that  $y(\theta) > 0$  for  $\theta \in [0, l_{F,F'}]$ . Defining  $g := f/y$ , we compute

$$\begin{aligned} \int_0^{l_{F,F'}} (f')^2 d\theta &= \int_0^{l_{F,F'}} \{(g')^2 y^2 + g^2 (y')^2 + (g^2)' y y'\} d\theta \\ &= \int_0^{l_{F,F'}} \{(g')^2 y^2 - g^2 y y''\} d\theta + g^2 y y' \Big|_0^{l_{F,F'}}, \end{aligned}$$

where we integrated the last term by parts. But note that

$$y y'' = -\frac{(1 - \varepsilon)^2 \pi^2}{l_{F,F'}^2} y^2, \quad \frac{y'}{y}(0) = -\frac{y'}{y}(l_{F,F'}) = \frac{(1 - \varepsilon)\pi}{l_{F,F'}} \tan \left( \frac{(1 - \varepsilon)\pi}{2} \right) \leq \frac{2}{l_{F,F'} \varepsilon}.$$

It follows that

$$\int_0^{l_{F,F'}} (f')^2 d\theta \geq \frac{(1 - \varepsilon)^2 \pi^2}{l_{F,F'}^2} \int_0^{l_{F,F'}} f^2 d\theta - \frac{2}{l_{F,F'} \varepsilon} \{f(l_{F,F'})^2 + f(0)^2\}.$$

Rearranging this expression yields the conclusion.  $\square$

**Remark 7.4.** Let us note that Lemma 7.3 may indeed be viewed as a quantitative formulation of uniqueness of the Dirichlet problem on an edge. Indeed, suppose  $f_1, f_2$  both satisfy  $f_i'' + f_i = 0$  on  $e_{F,F'}$ , and that  $f_1, f_2$  agree on the vertices  $n_F, n_{F'}$ . Then applying Lemma 7.3 to  $f = f_1 - f_2$  and letting  $\varepsilon \rightarrow 0$  yields

$$0 = -l_{F,F'}^2 \int_{e_{F,F'}} f(f'' + f) d\mathcal{H}^1 = l_{F,F'}^2 \int_{e_{F,F'}} \{(f')^2 - f^2\} d\mathcal{H}^1 \geq C \int_{e_{F,F'}} f^2 d\mathcal{H}^1$$

with  $C = \pi^2 - l_{F,F'}^2$ , where we integrated by parts in the second equality. Thus provided  $l_{F,F'} < \pi$ , the two solutions must coincide  $f_1 = f_2$ .

Next, we note that when  $M$  has nonempty interior, then the lengths  $l_{F,F'}$  of all edges must be bounded away from  $\pi$ . The following lemma quantifies this idea.

**Lemma 7.5.** *Let  $M$  be a polytope in  $\mathbb{R}^n$  such that  $rB \subseteq M \subseteq RB$ . Then*

$$\tan\left(\frac{l_{F,F'}}{2}\right) \leq \frac{R}{r} \quad \text{for all } e_{F,F'} \in E.$$

*In particular, we can estimate*

$$l_{F,F'}^2 \leq \pi^2 \left(1 - \frac{r^2}{R^2 + r^2}\right) \quad \text{for all } e_{F,F'} \in E.$$

*Proof.* Define for every facet  $F \in \mathcal{F}$  the supporting hyperplane

$$H_F := \{x \in \mathbb{R}^n : \langle n_F, x \rangle = h_M(n_F)\}.$$

Let  $F \sim F'$  be neighboring facets. We make the following claims.

- We claim that  $h_M(n_F) \geq r$  and  $h_M(n_{F'}) \geq r$ . Indeed, note that as  $rB \subseteq M$ , we have  $h_M(n_F) \geq h_{rB}(n_F) = r$ , and similarly for  $F'$ .
- We claim that  $H_F \cap H_{F'} \cap RB \neq \emptyset$ . Indeed, this follows readily by noting that  $F \cap F' \subset H_F \cap H_{F'}$  and  $\emptyset \neq F \cap F' \subset M \subseteq RB$ .

Now note that we can write for any  $x \in \mathbb{R}^n$

$$\|x\|^2 = \langle x, n_F \rangle^2 + \frac{(\langle x, n_{F'} \rangle - \cos(l_{F,F'}) \langle x, n_F \rangle)^2}{\sin(l_{F,F'})^2} + \|P_{\{n_F, n_{F'}\}^\perp} x\|^2,$$

where we used  $\langle n_F, n_{F'} \rangle = \cos(l_{F,F'})$ . Thus

$$R^2 \geq \inf_{x \in H_F \cap H_{F'}} \|x\|^2 = h_M(n_F)^2 + \frac{(h_M(n_{F'}) - \cos(l_{F,F'}) h_M(n_F))^2}{\sin(l_{F,F'})^2}$$

by claim b above. Applying claim a yields

$$R^2 \geq r^2 \left(1 + \frac{(1 - \cos(l_{F,F'}))^2}{\sin(l_{F,F'})^2}\right)$$

provided  $\frac{\pi}{2} \leq l_{F,F'} < \pi$ . It follows that

$$\tan\left(\frac{l_{F,F'}}{2}\right) = \frac{1 - \cos(l_{F,F'})}{\sin(l_{F,F'})} \leq \frac{R}{r}.$$

Indeed, for  $\frac{\pi}{2} \leq l_{F,F'} < \pi$  this is immediate from the previous expression, while for  $l_{F,F'} < \frac{\pi}{2}$  this follows as  $\tan(\frac{\pi}{4}) = 1 \leq \frac{R}{r}$ . To deduce the second part of the statement, it remains to note that  $4 \arctan(x)^2 \leq \pi^2 x^2 / (1 + x^2)$ .  $\square$

Combining Lemmas 7.3 and 7.5 yields the following.

**Corollary 7.6.** *Let  $M$  be a polytope in  $\mathbb{R}^n$  such that  $rB \subseteq M \subseteq RB$ . Then for any neighboring facets  $F \sim F'$  of  $M$  and any function  $f \in H^1(e_{F,F'})$ , we have*

$$\int_{e_{F,F'}} \{(f')^2 - f^2\} d\mathcal{H}^1 \geq \frac{r^2}{2R^2} \int_{e_{F,F'}} f^2 d\mathcal{H}^1 - \frac{4R^2}{r^2} l_{F,F'} \{f(n_F)^2 + f(n_{F'})^2\}.$$

*Proof.* Applying Lemma 7.5 to the left-hand side of Lemma 7.3 and rearranging the resulting expression yields the following inequality:

$$\begin{aligned} \int_{e_{F,F'}} \{(f')^2 - f^2\} d\mathcal{H}^1 &\geq \left(\frac{R^2 + r^2}{R^2} (1 - \varepsilon)^2 - 1\right) \int_{e_{F,F'}} f^2 d\mathcal{H}^1 \\ &\quad - \frac{R^2 + r^2}{R^2} \frac{2}{\varepsilon \pi^2} l_{F,F'} \{f(n_F)^2 + f(n_{F'})^2\}. \end{aligned}$$

Now choose  $\varepsilon = \frac{r^2}{4(R^2+r^2)}$ . Then

$$\frac{R^2 + r^2}{R^2}(1 - \varepsilon)^2 - 1 \geq \frac{R^2 + r^2}{R^2}(1 - 2\varepsilon) - 1 = \frac{r^2}{2R^2},$$

while

$$\frac{R^2 + r^2}{R^2} \frac{2}{\varepsilon\pi^2} = \frac{8}{\pi^2} \frac{(R^2 + r^2)^2}{r^2 R^2} \leq \frac{32}{\pi^2} \frac{R^2}{r^2}.$$

To conclude, we estimate  $\frac{32}{\pi^2} \leq 4$  for aesthetic appeal.  $\square$

We are now ready to prove a form of Theorem 7.1 for polytopes.

**Proposition 7.7.** *Let  $M$  be a polytope in  $\mathbb{R}^n$  such that  $rB \subseteq M \subseteq RB$ . Define a measure  $\mu_M$  on the vertices of the associated metric graph by setting*

$$\mu_M(\{n_F\}) := \frac{1}{n-1} \sum_{F': F' \sim F} \mathcal{H}^{n-2}(F \cap F') \iota_{F, F'}$$

for all facets  $F$  of  $M$ . Then we have

$$\begin{aligned} \mathbb{V}(K, L, \mathcal{M})^2 &\geq \mathbb{V}(K, K, \mathcal{M}) \mathbb{V}(L, L, \mathcal{M}) \\ &+ \mathbb{V}(L, L, \mathcal{M}) \left( \frac{r^2}{2nR^2} \int (h_K - h_L)^2 dS_{B, \mathcal{M}} - \frac{4R^2}{nr^2} \int (h_K - h_L)^2 d\mu_M \right) \end{aligned}$$

for all convex bodies  $K, L$  in  $\mathbb{R}^n$ .

*Proof.* Let  $\mathcal{E}$  be the quadratic form of Theorem 5.7 and  $f \in \text{Dom } \mathcal{E}$ . Multiplying the inequality of Corollary 7.6 by  $\mathcal{H}^{n-2}(F \cap F')$  and summing over all edges yields

$$0 \geq \mathcal{E}(f, f) + \frac{r^2}{2nR^2} \int f^2 dS_{B, \mathcal{M}} - \frac{4R^2}{nr^2} \int f^2 d\mu_M.$$

Now let  $f = h_K - h_L$ . Then we obtain

$$\begin{aligned} 0 &\geq \mathbb{V}(K, K, \mathcal{M}) - 2\mathbb{V}(K, L, \mathcal{M}) + \mathbb{V}(L, L, \mathcal{M}) \\ &+ \frac{r^2}{2nR^2} \int (h_K - h_L)^2 dS_{B, \mathcal{M}} - \frac{4R^2}{nr^2} \int (h_K - h_L)^2 d\mu_M. \end{aligned}$$

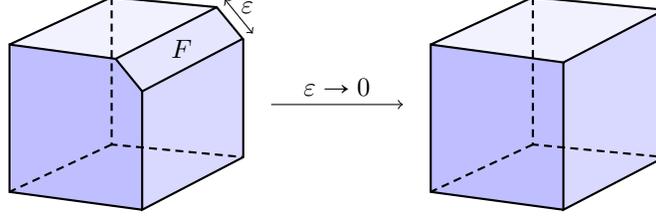
It remains to note that

$$\mathbb{V}(K, K, \mathcal{M}) - 2\mathbb{V}(K, L, \mathcal{M}) + \mathbb{V}(L, L, \mathcal{M}) \geq \mathbb{V}(K, K, \mathcal{M}) - \frac{\mathbb{V}(K, L, \mathcal{M})^2}{\mathbb{V}(L, L, \mathcal{M})}$$

when  $\mathbb{V}(L, L, \mathcal{M}) > 0$ , so the conclusion follows readily in this case. On the other hand, when  $\mathbb{V}(L, L, \mathcal{M}) = 0$  the conclusion is trivial.  $\square$

**7.3. Proof of Theorem 7.1: general case.** In order to prove Theorem 7.1 for an arbitrary convex body  $M$ , we will approximate it by polytopes and take limits in Proposition 7.7. The main issue that we will encounter is to understand the behavior of the measure  $\mu_M$  under taking limits.

At first sight, one might hope that  $\mu_M$  is a natural geometric object that remains meaningful for arbitrary convex bodies, just like  $S_{B, \mathcal{M}}$  or  $S_{M, \mathcal{M}}$ . This does not appear to be the case, however. It is important to note that even within the class of polytopes, the measure  $\mu_M$  is not continuous with respect to Hausdorff convergence, as is illustrated by the following example.

FIGURE 7.1. Example of discontinuity of the measure  $\mu_M$ .

**Example 7.8.** Consider a cube  $M_\varepsilon$  with one of its edges sliced off at width  $\varepsilon$ ; this construction is illustrated in Figure 7.1. Then  $M_\varepsilon$  has, for all  $\varepsilon > 0$ , an additional facet  $F$  as compared to  $M_0$ . It is readily seen that  $\inf_{\varepsilon > 0} \mu_{M_\varepsilon}(\{n_F\}) > 0$ , while  $\mu_{M_0}(\{n_F\}) = 0$ . Thus  $M_\varepsilon \rightarrow M_0$  but  $\mu_{M_\varepsilon} \not\rightarrow \mu_{M_0}$  as  $\varepsilon \rightarrow 0$ .

For a polytope  $M$ , Theorem 5.7 shows that the mass assigned by  $\mu_M$  to a vertex of the metric graph is precisely the  $S_{B,\mathcal{M}}$ -measure of its incident edges. We may therefore view  $\mu_M$  as a kind of projection of  $S_{B,\mathcal{M}}$  onto the 0-extreme normal vectors of  $M$ . It is not clear, however, what this might mean for a general convex body  $M$ , and the above example illustrates that one cannot hope to canonically define such projections by approximation of general bodies by polytopes. Nonetheless, as

$$\mu_M(S^{n-1}) = 2S_{B,\mathcal{M}}(S^{n-1}) = 2nV(B, B, \mathcal{M})$$

by Theorem 5.7, the total mass of  $\mu_M$  is uniformly bounded for any convergent sequence of polytopes, and we may therefore extract a weakly convergent subsequence of these measures (using the classical fact that any bounded family of measures on a compact metric space is relatively sequentially compact for the topology of weak convergence, cf. [6, Theorem 8.6.2]). While the limiting measure is not uniquely defined by the limiting body, we can nonetheless guarantee it satisfies our desired properties by working with specially chosen polytope approximations.

**Lemma 7.9.** *Let  $M$  be any convex body in  $\mathbb{R}^n$  with  $0 \in \text{int } M$ . Then there exists a sequence of polytopes  $M_k$  in  $\mathbb{R}^n$  with the following properties:*

- $M_k \rightarrow M$  in Hausdorff metric.
- There exist  $r, R > 0$  so that  $rB \subseteq M_k \subseteq RB$  for all  $k$ .
- $\text{supp } S_{M_k, \mathcal{M}_k} \subseteq \text{supp } S_{M, \mathcal{M}}$  for all  $k$ .
- $\mu_{M_k}$  converges weakly to a limiting measure  $\mu_M$  with  $\text{supp } \mu_M \subseteq \text{supp } S_{M, \mathcal{M}}$ .

*Proof.* Recall that a regular boundary point of  $M$  is a point in  $\partial M$  that has a unique outer normal vector; in particular, the normal vector at a regular boundary point is 0-extreme [40, section 2.2]. Choose a countable dense subset of the regular boundary points of  $M$ , and let  $\{n_i\}_{i \geq 1}$  be the corresponding normal directions. Then  $\{n_i\} \subseteq \text{supp } S_{M, \mathcal{M}}$  by Theorem 4.4. Moreover, as a convex body with nonempty interior is the intersection of its regular supporting halfspaces [40, Theorem 2.2.6], we have

$$M = \bigcap_{i \geq 1} \{x \in \mathbb{R}^n : \langle x, n_i \rangle \leq h_M(n_i)\}.$$

Now define

$$M'_k := \bigcap_{1 \leq i \leq k} \{x \in \mathbb{R}^n : \langle x, n_i \rangle \leq h_M(n_i)\}.$$

Then we have the following properties.

- i.  $M'_k$  is a polytope for all sufficiently large  $k$ .
- ii.  $M'_k \rightarrow M$  as  $k \rightarrow \infty$  in Hausdorff metric by [40, Lemma 1.8.2].
- iii.  $rB \subseteq M \subseteq M'_k$  for all  $k$  with  $r > 0$ , as  $0 \in \text{int } M$ .
- iv.  $M'_k \subseteq RB$  for all sufficiently large  $k$  with  $R = \text{diam } M$  by property ii.
- v.  $\text{supp } S_{M'_k, \mathcal{M}'_k} \subseteq \{n_i\}_{1 \leq i \leq k} \subseteq \text{supp } S_{M, \mathcal{M}}$  for all  $k$  by Theorem 4.4.

Now note that when  $M'_k \subseteq RB$ , we can estimate

$$\mu_{M'_k}(S^{n-1}) = 2n \mathbf{V}(B, B, \mathcal{M}'_k) \leq 2nR^{n-2} \text{Vol}(B).$$

By property iv, the mass of  $\mu_{M'_k}$  is uniformly bounded for all sufficiently large  $k$ . We may therefore extract a subsequence  $\{M_k\}$  of  $\{M'_k\}$  such that  $\mu_{M_k}$  converges weakly to a limiting measure  $\mu_M$ , and such that properties a–c in the statement of the Lemma hold. It remains to show that  $\text{supp } \mu_M \subseteq \text{supp } S_{M, \mathcal{M}}$ ; this follows immediately, however, as  $\text{supp } \mu_{M_k} = \text{supp } S_{M_k, \mathcal{M}_k} \subseteq \text{supp } S_{M, \mathcal{M}}$  for all  $k$ .  $\square$

We can now complete the proof of Theorem 7.1.

*Proof of Theorem 7.1.* By translation-invariance of mixed volumes and mixed area measures, we may assume without loss of generality that  $0 \in \text{int } M$ . Define the sequence of polytopes  $M_k$  and the measure  $\mu_M$  as in Lemma 7.9. Applying Proposition 7.7 to  $M_k$  and taking the limit as  $k \rightarrow \infty$ , the conclusion follows readily from Theorem 4.3. (For aesthetic reasons, we have rescaled the definition of the measure  $\mu_M$  in the statement of Theorem 7.1 so that only a single constant  $C_M$  appears; this makes no difference, of course, to the statement of the result.)  $\square$

## 8. THE LOWER-DIMENSIONAL CASE

In the setting of Theorem 2.2, we have seen that the extremals of Minkowski's inequality have a simple spectral interpretation: the kernel of the operator  $\mathcal{A}$  of Theorem 5.1 always contains the linear functions, and Theorem 2.2 shows that when  $M$  has nonempty interior, these are the *only* elements of the kernel that are differences of support functions of convex bodies.

When  $M$  is a lower-dimensional body, however, Theorem 2.3 states that new equality cases appear. Thus, unlike in the full-dimensional case, nonlinear differences of support functions can appear in the kernel of  $\mathcal{A}$ . This may suggest that the lower-dimensional situation is more complicated, as we must understand the new elements of the kernel. In fact, somewhat surprisingly, the lower-dimensional situation turns out to be considerably simpler: when  $M$  has empty interior, the operator  $\mathcal{A}$  can be described explicitly in complete generality (i.e., not just in special cases such as smooth bodies or polytopes). Once the operator has been constructed, we will be able to compute its kernel directly, and the proof of Theorem 2.3 will follow. These ideas will be developed in the remainder of this section.

To gain some insight into the lower-dimensional situation, it is instructive to consider first the case of a lower-dimensional polytope  $M \subset w^\perp$  for some  $w \in S^{n-1}$ . The following discussion is illustrated in Figure 8.1. To understand the operator associated to  $M$ , we first approximate it by the ‘‘cylinder’’  $M_\varepsilon := M + \varepsilon[0, w]$  which has nonempty interior. The body  $M_\varepsilon$  has two types of facets:

1. Two facets with normals  $\pm w$  are translates of  $M$ .
2. The remaining facet normals are the normals of the  $(n-2)$ -faces of  $M$  in  $w^\perp$ .

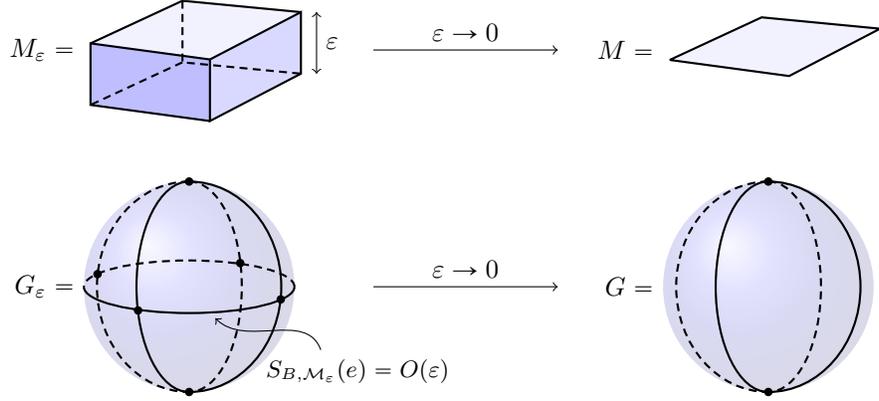


FIGURE 8.1. Metric graph associated to a polytope with empty interior.

The body  $M_\varepsilon$  defines a quantum graph according to Theorem 5.7. We now formally let  $\varepsilon \rightarrow 0$  and investigate what happens to the quantum graph in the limit. For each pair of facets  $F \sim F'$  of  $M_\varepsilon$  of type 2, we evidently have  $\mathcal{H}^{n-2}(F \cap F') = O(\varepsilon)$ . Thus all edges in the quantum graph associated to  $M_\varepsilon$  that lie in  $w^\perp$  vanish as  $\varepsilon \rightarrow 0$ . Consequently, the limiting graph has an extremely simple structure: it has exactly two vertices at the antipodal points  $\pm w$ ; and its edges are the geodesic arcs between  $\pm w$  in the directions of the  $(n-2)$ -faces of  $M$  in  $w^\perp$ .

This structure also explains why additional equality cases appear in the lower-dimensional setting: as all edges of the graph associated to  $M$  have length  $\pi$ , the solution to the Dirichlet problem  $\mathcal{A}f = \frac{1}{n}\{f'' + f\} = 0$  on each edge is no longer unique. Indeed, if  $f$  is such a solution on a given edge, then  $\theta \mapsto f(\theta) + a \sin(\theta)$  is also a solution with the same boundary data on the vertices for any  $a \in \mathbb{R}$ . This resonance phenomenon results in many new elements of the kernel of  $\mathcal{A}$  in the setting of lower-dimensional polytopes. Because of the simple structure of the graph, however, one can compute all elements of  $\ker \mathcal{A}$  explicitly, and we encounter none of the challenges that arose in the full-dimensional setting.

It is not difficult to work out the details of the above argument for polytopes. In this case, Theorem 2.3 was proved in [39, Theorem 4.2] (see also [14]) from a somewhat different perspective. However, we will show below that essentially the same construction remains valid when  $M$  is any lower-dimensional convex body. In this case, the operator  $\mathcal{A}$  turns out to be very similar to the quantum graph of a lower-dimensional polytope, except there may now be an infinite (even uncountable) number of edges in the graph. Some care must be taken, therefore, to construct this operator properly and to compute its domain, which will be done in section 8.1. Once this has been accomplished, however, the proof of Theorem 2.3 will follow readily in section 8.2 from an explicit computation of  $\ker \mathcal{A}$ .

**8.1. Construction of the operator.** Throughout this section, we fix  $w \in S^{n-1}$  and a convex body  $M \subset w^\perp$ . We define the measure  $S_{\mathcal{M}}$  on  $S^{n-1} \cap w^\perp$  to be the area measure of  $M$  when viewed as a convex body in  $w^\perp$ , that is,

$$S_{\mathcal{M}}(A) := \mathcal{H}^{n-2}(\{x \in w^\perp : x \in F(M, u) \text{ for some } u \in A\})$$

for  $A \subseteq S^{n-1} \cap w^\perp$ . We will assume that  $\dim M \geq n-2$ , so that  $S_{\mathcal{M}} \neq 0$ .

In view of the structure illustrated in Figure 8.1, it will be convenient to parametrize  $S^{n-1}$  in polar coordinates  $(\theta, z) \in [0, \pi] \times (S^{n-1} \cap w^\perp)$  as

$$\iota : [0, \pi] \times (S^{n-1} \cap w^\perp) \rightarrow S^{n-1}, \quad \iota(\theta, z) := w \cos \theta + z \sin \theta.$$

Note that the parametrization is unique except at  $\theta \in \{0, \pi\}$ , where  $\iota(0, z) = w$  and  $\iota(\pi, z) = -w$  for every  $z$ . Therefore, a continuous function  $f \in C^0(S^{n-1})$  is given in this parametrization by a function  $f(\theta, z)$  such that  $f(0, \cdot)$  and  $f(\pi, \cdot)$  are constant functions. Note, however, that the directional derivatives  $\frac{\partial f}{\partial \theta}(\theta, \cdot)$  are generally not constant functions at  $\theta \in \{0, \pi\}$  even when  $f \in C^1(S^{n-1})$ .

The main result of this section is the following.

**Theorem 8.1.** *Let  $w \in S^{n-1}$  and let  $M \subset w^\perp$  be a convex body with  $\dim M \geq n-2$ . Then for any  $f : S^{n-1} \rightarrow \mathbb{R}$ , we have*

$$\int f dS_{B, \mathcal{M}} = \frac{1}{n-1} \int_0^\pi \int_{S^{n-1} \cap w^\perp} f(\theta, z) S_{\mathcal{M}}(dz) d\theta.$$

Moreover, the operator  $\mathcal{A}$  defined by

$$\mathcal{A}f(\theta, z) = \frac{1}{n} \left\{ \frac{\partial^2 f}{\partial \theta^2}(\theta, z) + f(\theta, z) \right\}$$

with

$$\begin{aligned} \text{Dom } \mathcal{A} = & \left\{ f \in L^2(S_{B, \mathcal{M}}) : f(\cdot, z) \in H^2((0, \pi)) \text{ for } S_{\mathcal{M}}\text{-a.e. } z, \right. \\ & \frac{\partial^2 f}{\partial \theta^2} \in L^2(S_{B, \mathcal{M}}), f(0, \cdot) \text{ and } f(\pi, \cdot) \text{ are } S_{\mathcal{M}}\text{-a.e. constant,} \\ & \left. \int_{S^{n-1} \cap w^\perp} \frac{\partial f}{\partial \theta}(\theta, z) S_{\mathcal{M}}(dz) = 0 \text{ for } \theta \in \{0, \pi\} \right\} \end{aligned}$$

is self-adjoint on  $L^2(S_{B, \mathcal{M}})$  and satisfies all the properties of Theorem 5.1.

**Remark 8.2.** With some additional work, one can show that the operator  $\mathcal{A}$  of Theorem 8.1 is in fact the one constructed in the proof of Theorem 5.1, that is, it is the Friedrichs extension of (5.2) in the present setting. This is not needed, however, for the applications of this theorem, and the particularly simple structure of the present setting enables us to short-circuit some technical arguments.

The proof of Theorem 8.1 is similar to that of Theorem 5.7. We begin by making precise the procedure illustrated in Figure 8.1.

**Lemma 8.3.** *Let  $w \in S^{n-1}$ , let  $M \subset w^\perp$  be a convex body with  $\dim M \geq n-2$ , and let  $K$  be a convex body of class  $C_+^2$ . Then*

$$\int f dS_{K, \mathcal{M}} = \frac{1}{n-1} \int_0^\pi \int_{S^{n-1} \cap w^\perp} \left\{ \frac{\partial^2 h_K}{\partial \theta^2}(\theta, z) + h_K(\theta, z) \right\} f(\theta, z) S_{\mathcal{M}}(dz) d\theta.$$

*Proof.* Suppose first that  $M$  is a polytope of dimension  $\dim M = n-1$ . Denote by  $\mathcal{F}_M$  the set of its  $(n-2)$ -dimensional faces, and by  $n_F \in S^{n-1} \cap w^\perp$  the outer normal of  $F \in \mathcal{F}_M$  when viewed as a convex body in  $w^\perp$ . Now define for  $\varepsilon > 0$  the convex body  $M_\varepsilon := M + \varepsilon[0, w]$  in  $\mathbb{R}^n$ . Then  $M_\varepsilon$  has the following facets:

1.  $F(M_\varepsilon, w) = M + \varepsilon w$  and  $F(M_\varepsilon, -w) = M$ .
2.  $F(M_\varepsilon, n_F) = F + \varepsilon[0, w]$  for  $F \in \mathcal{F}_M$ .

Thus  $\mathcal{H}^{n-2}(F(M_\varepsilon, n_F) \cap F(M_\varepsilon, n_{F'})) = O(\varepsilon)$  for any  $F, F' \in \mathcal{F}_M$ , so we obtain

$$\begin{aligned} \int f dS_{K, \mathcal{M}_\varepsilon} &= \\ &= \frac{1}{n-1} \sum_{F \in \mathcal{F}_M} \mathcal{H}^{n-2}(F) \int_0^\pi \left\{ \frac{\partial^2 h_K}{\partial \theta^2}(\theta, n_F) + h_K(\theta, n_F) \right\} f(\theta, n_F) d\theta + O(\varepsilon) \end{aligned}$$

for any continuous function  $f$  by Proposition 5.8. Letting  $\varepsilon \rightarrow 0$  using Theorem 4.3, and noting that  $S_{\mathcal{M}}$  is, by definition, the measure defined by  $S_{\mathcal{M}}(\{n_F\}) = \mathcal{H}^{n-2}(F)$  for  $F \in \mathcal{F}_M$ , concludes the proof when  $M$  is an  $(n-1)$ -dimensional polytope.

Now note that any convex body  $M \subset w^\perp$  is the limit in Hausdorff metric of a sequence of  $(n-1)$ -dimensional polytopes in  $w^\perp$  [8, p. 39]. Thus the conclusion extends to arbitrary  $M$  by approximation using Theorem 4.3.  $\square$

The expression for  $S_{B, \mathcal{M}}$  in Theorem 8.1 follows immediately from Lemma 8.3. We now turn our attention to proving that  $\mathcal{A}$  is self-adjoint on  $L^2(S_{B, \mathcal{M}})$ .

**Lemma 8.4.** *The operator  $\mathcal{A}$  of Theorem 8.1 is self-adjoint.*

*Proof.* Note first that if  $f, g \in \text{Dom } \mathcal{A}$ , then

$$\begin{aligned} \langle f, \mathcal{A}g \rangle_{L^2(S_{B, \mathcal{M}})} &= \frac{1}{n(n-1)} \int_0^\pi \int_{S^{n-1} \cap w^\perp} f(\theta, z) \left\{ \frac{\partial^2 g}{\partial \theta^2}(\theta, z) + g(\theta, z) \right\} S_{\mathcal{M}}(dz) d\theta \\ &= \frac{1}{n(n-1)} \int_0^\pi \int_{S^{n-1} \cap w^\perp} \left\{ f(\theta, z) g(\theta, z) - \frac{\partial f}{\partial \theta}(\theta, z) \frac{\partial g}{\partial \theta}(\theta, z) \right\} S_{\mathcal{M}}(dz) d\theta \\ &\quad + \frac{1}{n(n-1)} \int_{S^{n-1} \cap w^\perp} f(\theta, z) \frac{\partial g}{\partial \theta}(\theta, z) S_{\mathcal{M}}(dz) \Big|_{\theta=0}^{\theta=\pi}, \end{aligned}$$

where we integrated by parts. But the definition of  $\text{Dom } \mathcal{A}$  ensures that the boundary term vanishes. It follows that  $\mathcal{A}$  is a symmetric operator, and in particular  $\text{Dom } \mathcal{A} \subseteq \text{Dom } \mathcal{A}^*$ . It therefore remains to prove the converse inclusion.

Fix in the rest of the proof  $f \in \text{Dom } \mathcal{A}^*$ . We must show that  $f$  satisfies each of the defining properties of  $\text{Dom } \mathcal{A}$ .

First, note that for any smooth compactly supported function  $\varphi \in C_0^\infty((0, \pi))$  and any  $h \in L^2(S_{\mathcal{M}})$ , the function  $g(\theta, z) := \varphi(\theta)h(z)$  satisfies  $g \in \text{Dom } \mathcal{A}$ . Thus

$$\begin{aligned} \frac{1}{n(n-1)} \int \left[ \int_0^\pi f(\theta, z) \{ \varphi''(\theta) + \varphi(\theta) \} d\theta \right] h(z) S_{\mathcal{M}}(dz) &= \langle f, \mathcal{A}g \rangle_{L^2(S_{B, \mathcal{M}})} \\ &= \langle \mathcal{A}^* f, g \rangle_{L^2(S_{B, \mathcal{M}})} = \frac{1}{n-1} \int \left[ \int_0^\pi \mathcal{A}^* f(\theta, z) \varphi(\theta) d\theta \right] h(z) S_{\mathcal{M}}(dz). \end{aligned}$$

As  $h$  is arbitrary, we have

$$\int_0^\pi f(\theta, z) \varphi''(\theta) d\theta = \int_0^\pi (n\mathcal{A}^* - I)f(\theta, z) \varphi(\theta) d\theta \quad \text{for } S_{\mathcal{M}}\text{-a.e. } z$$

for any  $\varphi \in C_0^\infty((0, \pi))$ . As  $H^2((0, \pi))$  is separable [19, section 7.5], this identity remains valid simultaneously for all  $\varphi \in C_0^\infty((0, \pi))$  (that is, the exceptional set may be chosen independent of  $\varphi$ ). As  $(n\mathcal{A}^* - I)f(\cdot, z) \in L^2((0, \pi))$  for  $S_{\mathcal{M}}$ -a.e.  $z$  by Fubini's theorem, we have shown that  $f(\cdot, z) \in H^2((0, \pi))$  for  $S_{\mathcal{M}}$ -a.e.  $z$  and that  $\mathcal{A}^* f = \frac{1}{n} \{ \frac{\partial^2 f}{\partial \theta^2} + f \}$   $S_{B, \mathcal{M}}$ -a.e. (in particular,  $\frac{\partial^2 f}{\partial \theta^2} \in L^2(S_{B, \mathcal{M}})$ ).

It remains only to establish the vertex boundary conditions at  $\theta \in \{0, \pi\}$ . To this end, note first that if  $g \in \text{Dom } \mathcal{A}$  is arbitrary, then

$$\langle f, \mathcal{A}g \rangle_{L^2(S_{B, \mathcal{M}})} = \langle \mathcal{A}^* f, g \rangle_{L^2(S_{B, \mathcal{M}})} = \langle \frac{1}{n} \{ \frac{\partial^2 f}{\partial \theta^2} + f \}, g \rangle_{L^2(S_{B, \mathcal{M}})}.$$

Integrating by parts as in the beginning of the proof shows that

$$\int_{S^{n-1} \cap w^\perp} \left\{ f(\theta, z) \frac{\partial g}{\partial \theta}(\theta, z) - g(\theta, z) \frac{\partial f}{\partial \theta}(\theta, z) \right\} S_{\mathcal{M}}(dz) \Big|_{\theta=0}^{\theta=\pi} = 0$$

for every  $g \in \text{Dom } \mathcal{A}$ . We choose a different test function  $g$  to deduce each boundary condition. First, let  $g(\theta, z) := 1 \pm \cos(\theta)$ . Then  $g \in \text{Dom } \mathcal{A}$ , so we conclude

$$\int_{S^{n-1} \cap w^\perp} \frac{\partial f}{\partial \theta}(\theta, z) S_{\mathcal{M}}(dz) = 0 \quad \text{for } \theta \in \{0, \pi\}.$$

Next, let  $g(\theta, z) := \sin(k\theta)h(z)$  for  $k = 1, 2$  and  $h \in L^2(S_{\mathcal{M}})$  with  $\int h dS_{\mathcal{M}} = 0$ . Then  $g \in \text{Dom } \mathcal{A}$ , so we conclude that

$$\int_{S^{n-1} \cap w^\perp} f(0, z) h(z) S_{\mathcal{M}}(dz) = \int_{S^{n-1} \cap w^\perp} f(\pi, z) h(z) S_{\mathcal{M}}(dz) = 0.$$

As this holds for all  $h$  of the above form, it must be the case that  $f(0, \cdot)$  and  $f(\pi, \cdot)$  are  $S_{\mathcal{M}}$ -a.e. constant. The proof is complete.  $\square$

The reason that the setting of this section is particularly simple is that we can compute the full spectral decomposition of  $\mathcal{A}$ .

**Lemma 8.5.** *The operator  $\mathcal{A}$  of Theorem 8.1 satisfies*

$$\text{spec } \mathcal{A} = \{\lambda_k : k \in \mathbb{Z}_+\}, \quad \lambda_k := \frac{1 - k^2}{n}.$$

Moreover, the eigenspace  $E_k$  associated to eigenvalue  $\lambda_k$  is given by

$$E_k := \left\{ f : f(\theta, z) = h(z) \sin(k\theta) + a \cos(k\theta), \quad a \in \mathbb{R}, \quad h \in L^2(S_{\mathcal{M}}), \quad \int h dS_{\mathcal{M}} = 0 \right\}.$$

*Proof.* Let  $E_k$  be the spaces defined in the statement of the lemma. It is readily verified that  $E_k \subset \text{Dom } \mathcal{A}$  and  $\mathcal{A}f = \lambda_k f$  for each  $k$  and  $f \in E_k$ .

We now claim that

$$E := \overline{\text{span}} \left( \bigcup_{k \geq 0} E_k \right) = L^2(S_{B, \mathcal{M}}).$$

As  $S_{B, \mathcal{M}}$  is a product measure on  $[0, \pi] \times (S^{n-1} \cap w^\perp)$ , it suffices to show that any function of the form  $g(\theta, z) := \varphi(\theta) h(z)$  with  $\varphi \in L^2([0, \pi])$  and  $h \in L^2(S_{\mathcal{M}})$  lies in  $E$ . To this end, note that both  $\{\sin(k\theta) : k \geq 1\}$  and  $\{\cos(k\theta) : k \geq 0\}$  are complete orthogonal bases of  $L^2([0, \pi])$  (these are the Dirichlet and Neumann eigenfunctions of the Laplacian on  $[0, \pi]$ , respectively). Thus we may write

$$\varphi(\theta) = \sum_{k \geq 0} a_k \cos(k\theta) = \sum_{k \geq 1} b_k \sin(k\theta)$$

in  $L^2([0, \pi])$  for some coefficient sequences  $a_k, b_k$ . Moreover, we can evidently write  $h(z) = h_0(z) + c$  where  $h_0 \in L^2(S_{\mathcal{M}})$  and  $\int h_0 dS_{\mathcal{M}} = 0$ . Thus

$$g(\theta, z) = \sum_{k \geq 0} \{ b_k h_0(z) \sin(k\theta) + c a_k \cos(k\theta) \} \in E,$$

completing the proof of the claim.

Denote by  $P_k$  the orthogonal projection in  $L^2(S_{B,\mathcal{M}})$  onto  $E_k$ . As the eigenspaces of a self-adjoint operator are orthogonal, it follows that

$$I = \sum_{k \geq 0} P_k \leq \sum_{k \geq 0} 1_{\{\lambda_k\}}(\mathcal{A}) = 1_{\{\lambda_k : k \in \mathbb{Z}_+\}}(\mathcal{A}) \leq I.$$

Thus  $\text{spec } \mathcal{A} = \{\lambda_k : k \in \mathbb{Z}_+\}$  and  $P_k = 1_{\{\lambda_k\}}(\mathcal{A})$  for all  $k$ .  $\square$

Finally, we make the following simple observation.

**Lemma 8.6.** *In the setting of Theorem 8.1, we have  $C^2(S^{n-1}) \subset \text{Dom } \mathcal{A}$ .*

*Proof.* It suffices to note that for  $f \in C^2(S^{n-1})$ , we have

$$\int \frac{\partial f}{\partial \theta}(0, z) S_{\mathcal{M}}(dz) = \int \langle z, \nabla f(w) \rangle S_{\mathcal{M}}(dz) = 0$$

and

$$\int \frac{\partial f}{\partial \theta}(\pi, z) S_{\mathcal{M}}(dz) = \int \langle z, \nabla f(-w) \rangle S_{\mathcal{M}}(dz) = 0$$

by Lemma 4.2(e). The remaining properties are trivial.  $\square$

We can now complete the proof of Theorem 8.1.

*Proof of Theorem 8.1.* The expression for  $S_{B,\mathcal{M}}$  follows from Lemma 8.3, and self-adjointness of  $\mathcal{A}$  on  $L^2(S_{B,\mathcal{M}})$  was proved in Lemma 8.4. Properties a–c of Theorem 5.1 can be read off from Lemma 8.5.

Finally, note that by Lemmas 8.3 and 8.6, the operator  $\mathcal{A}$  agrees with (5.2) on  $C^2(S^{n-1})$ . Thus the closed quadratic form associated to  $\mathcal{A}$  is a closed extension of its restriction to  $C^2(S^{n-1})$ . The quadratic form  $\mathcal{E}$  of Theorem 5.1 is the smallest such extension (the Friedrichs extension); thus properties d–e of Theorem 5.1 remain valid in the present setting (and for any other closed extension of  $\mathcal{E}$ ).  $\square$

**Remark 8.7.** Unless  $M$  is a polytope, Lemma 8.5 implies that the eigenspaces of  $\mathcal{A}$  are infinite-dimensional. Thus  $\mathcal{A}$  does not have a compact resolvent. This provides an explicit example of the issue that was highlighted in Remark 5.2. Let us also note that this phenomenon is not specific to the lower-dimensional setting; for example, it may be verified that a similar situation occurs if we replace  $M$  by  $M_\varepsilon := M + \varepsilon[0, w]$ , which has nonempty interior. We omit the details.

**Remark 8.8.** Even when  $M$  has nonempty interior, we have only established that  $\ker \mathcal{A} \cap \{h_Q - h_R : Q, R \text{ convex bodies}\}$  consists of linear functions. A function that is not a difference of support functions does not give rise to extremals of Minkowski's inequality, so is not relevant for the purposes of this paper. Nonetheless, one may wonder whether it is possible that  $\ker \mathcal{A}$  contains such functions.

When  $M$  has empty interior, Lemma 8.5 shows that this may in fact happen. Indeed, as the function  $h \in L^2(S_{\mathcal{M}})$  need only be measurable, it is perfectly possible in general to construct eigenfunctions that do not admit a continuous extension to  $S^{n-1}$ . Such eigenfunctions cannot arise as the difference of support functions, as support functions are always continuous. Thus  $\ker \mathcal{A}$  may contain many elements that do not contribute to the characterization of extremals.

It is natural to conjecture that when  $M$  has nonempty interior, such examples cannot occur and that  $\ker \mathcal{A}$  consists *only* of linear functions (cf. Remark 7.2). Such a result cannot be achieved, however, by the methods of this paper. The key obstruction is the theory of section 7: that  $\text{supp } \mu_M \subseteq \text{supp } S_{M,\mathcal{M}}$  only guarantees

that *continuous* functions that vanish  $S_{M,\mathcal{M}}$ -a.e. must vanish  $\mu_M$ -a.e., so we cannot rule out discontinuous elements of  $\ker \mathcal{A}$  by this method.

**8.2. Proof of Theorem 2.3.** By Lemma 3.1, understanding the equality cases of Minkowski's quadratic inequality reduces to understanding the kernel of  $\mathcal{A}$ . In the present setting, however, we have already computed the kernel in Lemma 8.5. It therefore remains to furnish its elements with a geometric interpretation.

Before we turn to the proof of Theorem 2.3, however, we must extend the conclusion of Theorem 4.4 to the present setting.

**Lemma 8.9.** *Let  $w \in S^{n-1}$  and let  $M \subset w^\perp$  be a convex body with  $\dim M \geq n-2$ . Then  $\text{supp } S_{B,\mathcal{M}} = \text{cl}\{u \in S^{n-1} : u \text{ is a 1-extreme normal vector of } M\}$ .*

*Proof.* As  $w$  is normal to every point in  $M$ , a vector  $u \in S^{n-1}$  is normal to a given point in  $M$  if and only if its projection  $P_{w^\perp}u$  is normal to that point. In particular, it follows readily that  $u \neq w$  is a 1-extreme normal vector of  $M$  if and only if  $P_{w^\perp}u$  is a 0-extreme normal vector of  $M$  when viewed as a convex body in  $w^\perp$ .

Now note that, by the expression for  $S_{B,\mathcal{M}}$  given in Theorem 8.1, we have  $\text{supp } S_{B,\mathcal{M}} = [0, \pi] \times \text{supp } S_{\mathcal{M}}$  (in polar coordinates). When  $\dim M = n-1$ , we have  $\text{supp } S_{\mathcal{M}} = \text{cl}\{0\text{-extreme normal vectors of } M \text{ in } w^\perp\}$  by Theorem 4.4, and the conclusion follows. On the other hand, when  $\dim M = n-2$ , we have  $M \subset \text{span}\{v, w\}^\perp$  for some  $v \perp w$ . Thus  $\pm v$  are the only 0-extreme normal directions of  $M$  in  $w^\perp$ . On the other hand, it follows readily from the definition that  $\text{supp } S_{\mathcal{M}} = \{\pm v\}$  in this case, so that the conclusion again follows.  $\square$

We are now ready to complete the proof of Theorem 2.3.

*Proof of Theorem 2.3.* By translation-invariance of mixed volumes, we may assume without loss of generality that  $M \subset w^\perp$ . Moreover, we may assume  $\dim M \geq n-2$ , as otherwise  $V(L, L, \mathcal{M}) = 0$  for all bodies  $L$  [40, Theorem 5.1.8].

Now let  $K, L$  be any convex bodies in  $\mathbb{R}^n$  with  $V(L, L, \mathcal{M}) > 0$ . By Theorem 8.1 and Lemma 3.1, we have equality in Minkowski's inequality

$$V(K, L, \mathcal{M})^2 = V(K, K, \mathcal{M})V(L, L, \mathcal{M})$$

if and only if  $h_K - ah_L \in \ker \mathcal{A}$  for some  $a \in \mathbb{R}$ . Thus the proof will be concluded once we establish that the following two statements are equivalent:

1.  $h_K - ah_L \in \ker \mathcal{A}$  for some  $a \in \mathbb{R}$ .
2.  $\tilde{L} := \frac{V(K, L, \mathcal{M})}{V(L, L, \mathcal{M})}L$  has the property that  $K + F(\tilde{L}, w)$  and  $\tilde{L} + F(K, w)$  have the same supporting hyperplanes in all 1-extreme normal directions of  $M$ .

Let us prove each in turn.

**1  $\Rightarrow$  2.** Let  $h_K - ah_L \in \ker \mathcal{A}$ . First, note that

$$0 = \langle h_L, \mathcal{A}(h_K - ah_L) \rangle_{L^2(S_{B,\mathcal{M}})} = V(K, L, \mathcal{M}) - aV(L, L, \mathcal{M})$$

by Theorem 8.1. Thus we must have

$$a = \frac{V(K, L, \mathcal{M})}{V(L, L, \mathcal{M})},$$

and it follows that  $\tilde{L} = aL$ . To proceed, we observe that Lemma 8.5 implies that

$$h_K(\theta, z) - h_{\tilde{L}}(\theta, z) = \eta(z) \sin \theta + \alpha \cos \theta \quad S_{B,\mathcal{M}}\text{-a.e. } (\theta, z)$$

for some  $\alpha \in \mathbb{R}$  and  $\eta \in L^2(S_{\mathcal{M}})$  with  $\int \eta dS_{\mathcal{M}} = 0$ . Evidently

$$\alpha = h_K(0, z) - h_{\tilde{L}}(0, z) = h_K(w) - h_{\tilde{L}}(w),$$

and

$$\eta(z) = \frac{\partial h_K}{\partial \theta}(0, z) - \frac{\partial h_{\tilde{L}}}{\partial \theta}(0, z) = \nabla_z h_K(w) - \nabla_z h_{\tilde{L}}(w) = h_{F(K,w)}(z) - h_{F(\tilde{L},w)}(z)$$

by Lemma 4.9. But note that at the point  $x = w \cos \theta + z \sin \theta \in S^{n-1}$  corresponding to the polar coordinates  $(\theta, z)$ , we can write using  $F(K, w) - wh_K(w) \subset w^\perp$

$$h_{F(K,w)}(x) = h_{F(K,w)}(z) \sin \theta + h_K(w) \cos \theta.$$

The analogous formula holds for  $\tilde{L}$ , and we conclude that

$$h_K(x) - h_{\tilde{L}}(x) = h_{F(K,w)}(x) - h_{F(\tilde{L},w)}(x) \quad S_{B,\mathcal{M}}\text{-a.e. } x.$$

By continuity of support functions, this identity remains valid for all  $x \in \text{supp } S_{B,\mathcal{M}}$ , and the implication 1  $\Rightarrow$  2 follows by Lemma 8.9.

2  $\Rightarrow$  1. By Lemma 8.9 and continuity, we can assume

$$h_K(x) - h_{\tilde{L}}(x) = h_{F(K,w)}(x) - h_{F(\tilde{L},w)}(x) \quad \text{for } x \in \text{supp } S_{B,\mathcal{M}}.$$

We will prove directly that this implies equality in Minkowski's quadratic inequality (and hence  $h_K - ah_L \in \ker \mathcal{A}$  by Lemma 3.1).

We begin by noting that  $\text{supp } S_{C,\mathcal{M}} = \text{supp } S_{B,\mathcal{M}}$  for every body  $C$  of class  $C_+^2$  by Lemma 8.3. Thus Theorem 4.3 implies that  $\text{supp } S_{C,\mathcal{M}} \subseteq \text{supp } S_{B,\mathcal{M}}$  for any convex body  $C$ . Choosing  $C = F(K, w)$  and  $C = F(\tilde{L}, w)$ , respectively, we find

$$\begin{aligned} & \mathbb{V}(h_K - h_{\tilde{L}}, h_{F(K,w)} - h_{F(\tilde{L},w)}, \mathcal{M}) \\ &= \frac{1}{n} \int (h_K - h_{\tilde{L}}) dS_{h_{F(K,w)} - h_{F(\tilde{L},w)}, \mathcal{M}} \\ &= \frac{1}{n} \int (h_{F(K,w)} - h_{F(\tilde{L},w)}) dS_{h_{F(K,w)} - h_{F(\tilde{L},w)}, \mathcal{M}} \\ &= \mathbb{V}(h_{F(K,w)} - h_{F(\tilde{L},w)}, h_{F(K,w)} - h_{F(\tilde{L},w)}, \mathcal{M}) = 0, \end{aligned}$$

where the last equality holds as  $\dim(F(K, w) + F(\tilde{L}, w) + M) < n$ . On the other hand, choosing  $C = K$  and  $C = \tilde{L}$ , we obtain similarly

$$\mathbb{V}(h_K - h_{\tilde{L}}, h_K - h_{\tilde{L}}, \mathcal{M}) = \mathbb{V}(h_{F(K,w)} - h_{F(\tilde{L},w)}, h_K - h_{\tilde{L}}, \mathcal{M}) = 0.$$

Thus we have shown that

$$\mathbb{V}(K, K, \mathcal{M}) - 2\mathbb{V}(K, \tilde{L}, \mathcal{M}) + \mathbb{V}(\tilde{L}, \tilde{L}, \mathcal{M}) = 0.$$

The conclusion follows by substituting  $\tilde{L} = \frac{\mathbb{V}(K, L, \mathcal{M})}{\mathbb{V}(L, L, \mathcal{M})} L$  into this expression.  $\square$

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## REFERENCES

- [1] A. D. Alexandrov. *Selected works. Part I*. Gordon and Breach Publishers, Amsterdam, 1996.
- [2] S. Artstein-Avidan, A. Giannopoulos, and V. D. Milman. *Asymptotic geometric analysis. Part I*. American Mathematical Society, Providence, RI, 2015.
- [3] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*. Springer, Cham, 2014.
- [4] G. Berkolaiko and P. Kuchment. *Introduction to quantum graphs*. American Mathematical Society, Providence, RI, 2013.
- [5] W. Blaschke. Ein Beweis für die Unverbiegbarkeit geschlossener konvexer Flächen. *Nachr. Ges. Wiss. Göttingen*, pages 607–610, 1912.
- [6] V. I. Bogachev. *Measure theory. Vol. II*. Springer-Verlag, Berlin, 2007.
- [7] G. Bol. Beweis einer Vermutung von H. Minkowski. *Abh. Math. Sem. Hansischen Univ.*, 15:37–56, 1943.
- [8] T. Bonnesen and W. Fenchel. *Theory of convex bodies*. BCS Associates, Moscow, ID, 1987.
- [9] Y. D. Burago and V. A. Zalgaller. *Geometric inequalities*. Springer-Verlag, Berlin, 1988.
- [10] B. Chow, P. Lu, and L. Ni. *Hamilton's Ricci flow*, volume 77 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2006.
- [11] T. H. Colding and W. P. Minicozzi, II. *A course in minimal surfaces*, volume 121 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
- [12] A. Colesanti. From the Brunn-Minkowski inequality to a class of Poincaré-type inequalities. *Commun. Contemp. Math.*, 10(5):765–772, 2008.
- [13] E. B. Davies. *Spectral theory and differential operators*, volume 42 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
- [14] G. Ewald and E. Tondorf. A contribution to equality in Alexandrov-Fenchel's inequality. *Geom. Dedicata*, 50(3):217–233, 1994.
- [15] J. Favard. Sur les corps convexes. *J. Math. Pures Appl. (9)*, 12:219–282, 1933.
- [16] W. Fenchel and B. Jessen. Mengenfunktionen und konvexe Körper. *Danske Vid. Selskab. Mat.-Fys. Medd.*, 16:1–31, 1938.
- [17] P. Filliman. Rigidity and the Alexandrov-Fenchel inequality. *Monatsh. Math.*, 113(1):1–22, 1992.
- [18] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*. Walter de Gruyter & Co., Berlin, extended edition, 2011.
- [19] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [20] A. Grigor'yan. *Heat kernel and analysis on manifolds*, volume 47 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.
- [21] D. Hilbert. *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*. B. G. Teubner, 1912.
- [22] M. Hino. Measurable Riemannian structures associated with strong local Dirichlet forms. *Math. Nachr.*, 286(14-15):1466–1478, 2013.
- [23] D. Hug. *Measures, curvatures and currents in convex geometry*. Habilitation, Universität Freiburg, 2000.
- [24] I. Izestiev. The Colin de Verdière number and graphs of polytopes. *Israel J. Math.*, 178:427–444, 2010.
- [25] A. V. Kolesnikov and E. Milman. Poincaré and Brunn-Minkowski inequalities on the boundary of weighted Riemannian manifolds. *Amer. J. Math.*, 140(5):1147–1185, 2018.
- [26] A. V. Kolesnikov and E. Milman. Local  $L^p$ -Brunn-Minkowski inequalities for  $p < 1$ . *Mem. Amer. Math. Soc.*, 2019. To appear.
- [27] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.
- [28] R. Meise and D. Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press, Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.
- [29] H. Minkowski. Volumen und Oberfläche. *Math. Ann.*, 57(4):447–495, 1903.
- [30] H. Minkowski. *Gesammelte Abhandlungen. Zweiter Band*. B. G. Teubner, 1911.

- [31] U. Mosco. Composite media and asymptotic Dirichlet forms. *J. Funct. Anal.*, 123(2):368–421, 1994.
- [32] L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.*, 5:286–292, 1960.
- [33] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.
- [34] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press, New York-London, 1978.
- [35] R. C. Reilly. Geometric applications of the solvability of Neumann problems on a Riemannian manifold. *Arch. Rational Mech. Anal.*, 75(1):23–29, 1980.
- [36] R. Schneider. On the Aleksandrov-Fenchel inequality. In *Discrete geometry and convexity*, volume 440 of *Ann. New York Acad. Sci.*, pages 132–141. New York Acad. Sci., 1985.
- [37] R. Schneider. On the Aleksandrov-Fenchel inequality involving zonoids. *Geom. Dedicata*, 27(1):113–126, 1988.
- [38] R. Schneider. On the Aleksandrov-Fenchel inequality for convex bodies. I. *Results Math.*, 17(3-4):287–295, 1990.
- [39] R. Schneider. Polytopes and Brunn-Minkowski theory. In *Polytopes: abstract, convex and computational*, volume 440 of *NATO Adv. Sci. Inst. Ser. C*, pages 273–299. Kluwer, 1994.
- [40] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*. Cambridge University Press, expanded edition, 2014.
- [41] Y. Shenfeld and R. van Handel. Mixed volumes and the Bochner method. *Proc. Amer. Math. Soc.*, 147(12):5385–5402, 2019.
- [42] M. E. Taylor. *Partial differential equations I. Basic theory*. Springer, New York, second edition, 2011.

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