Abstract. In a seminal paper “Volumen und Oberfläche” (1903), Minkowski introduced the basic notion of mixed volumes and the corresponding inequalities that lie at the heart of convex geometry. The fundamental importance of characterizing the extremals in these inequalities was already emphasized by Minkowski himself, but has to date only been resolved in special cases. In this paper, we completely settle the extremals in Minkowski’s quadratic inequality, confirming a conjecture of R. Schneider. Our proof is based on the representation of mixed volumes of arbitrary convex bodies as Dirichlet forms associated to certain highly degenerate elliptic operators. A key ingredient of the proof is a quantitative rigidity property associated to these operators.

1. Introduction

1.1. History of the problem. The systematic study of the geometry of convex bodies dates back to the work of Brunn and Steiner in the 1880s. It is however arguably the work of Minkowski that laid the foundation for the modern theory of convex geometry. In his seminal paper “Volumen und Oberfläche” (1903) [24], and in an unfinished manuscript “Theorie der konvexen Körper” that was published posthumously [25], Minkowski introduces the basic notion of mixed volumes and the corresponding inequalities that play a central role in the modern theory [7, 33]. The aim of this paper is to settle a fundamental question arising from Minkowski’s original paper that has hitherto remained open.

As was customary at that time, Minkowski restricted attention to 3-dimensional bodies. While our main results are formulated in any dimension, let us first explain the problem investigated here in its original context. Let $K_1, K_2, K_3$ be convex bodies in $\mathbb{R}^3$, and let $\lambda_1, \lambda_2, \lambda_3 > 0$. The starting point for Minkowski’s theory is the fact that the volume of convex bodies is a homogeneous polynomial: that is,

$$\mathrm{Vol}(\lambda_1K_1 + \lambda_2K_2 + \lambda_3K_3) = \sum_{i_1, i_2, i_3=1}^{3} V(K_{i_1}, K_{i_2}, K_{i_3}) \lambda_{i_1} \lambda_{i_2} \lambda_{i_3},$$

where we denote $\lambda K + \mu L := \{\lambda x + \mu y : x \in K, y \in L\}$. The coefficients $V(K, L, M)$ in this polynomial are called mixed volumes. They are nonnegative, symmetric in their arguments, and linear in each argument. Mixed volumes admit various natural geometric interpretations, and give rise to many familiar notions as special cases. For example, if $K$ is any convex body and $B$ denotes the (Euclidean) unit ball in...
\[ \text{Vol}(K) = V(K, K, K), \quad \text{Surf}(K) = 3V(K, K, B), \quad W(K) = \frac{2V(K, B, B)}{\text{Vol}(B)}, \]

respectively. We refer to [33] for a detailed exposition.

Once the central role of mixed volumes has been realized, it is natural to expect that many geometric properties of convex bodies may be expressed in terms of relations between mixed volumes. This viewpoint lies at the heart of Minkowski’s theory. In particular, Minkowski established [24, p. 479] the following fundamental inequality for three convex bodies \( K, L, M \) in \( \mathbb{R}^3 \):

\[ V(K, L, M)^2 \geq V(K, K, M)V(L, L, M). \quad (1.1) \]

We refer to (1.1) as (the 3-dimensional case of) Minkowski’s quadratic inequality. This inequality unifies and extends many geometric inequalities for 3-dimensional convex bodies, including the isoperimetric inequality, the Brunn-Minkowski inequality, Urysohn’s inequality, etc. [33]. Our interest here is in the following:

**Question.** For which \( K, L, M \) is equality attained in (1.1)?

This question arises in Minkowski’s 1903 paper in the following context. Minkowski viewed (1.1) as a far-reaching generalization of the isoperimetric inequality

\[ \text{Surf}(K) \geq 3\text{Vol}(K)^{2/3}\text{Vol}(B)^{1/3} \]

for convex bodies \( K \) in \( \mathbb{R}^3 \), which may be written in terms of mixed volumes as 
\[ V(K, K, B)^3 \geq V(K, K, K)^2V(B, B, B). \]

The isoperimetric inequality follows from (1.1) by combining the special cases \( M = K, L = B \) and \( M = L = B \), which may be viewed as generalized isoperimetric inequalities. To interpret such inequalities as genuine isoperimetric statements, however, one must understand their equality cases or extremals: these are precisely the bodies \( K \) that maximize the right-hand side when the left-hand side is fixed. For example, the only equality cases of the isoperimetric inequality are Euclidean balls; thus among all bodies \( K \) with fixed surface area, volume is maximized if and only if \( K \) is a ball. Similarly,

\[ V(K, K, B)^2 \geq V(K, B, B)V(K, K) \]

has the following isoperimetric interpretation: *among all \( K \) with fixed surface area, the product of volume and mean width is maximized if and only if \( K \) attains equality in (1.2).* Remarkably, it turns out that such generalized isoperimetric problems can possess many unusual extremals, in sharp contrast to the classical isoperimetric theorem. For example, equality holds in (1.2) when \( K \) is any cap body of the ball, i.e., the convex hull of \( B \) with a finite or countable number of points so that the cones emanating from the points are disjoint (cf. Figure 1.1).
These striking observations motivate the attention paid by Minkowski to the extremals in his inequalities. In particular, in [24, p. 477], he asserts that the only extremals in (1.2) are cap bodies of the ball. No proof of this statement appears, however, in [24] or in any of Minkowski’s other works [25], and it seems unlikely that Minkowski had a correct proof of this fact. Nonetheless, the statement is correct, as was shown 40 years later by Bol [6]. On the other hand, Minkowski does not formulate any general conjecture on the extremals in (1.1). The characterization of these extremals is a long-standing problem in convex geometry (cf. [14, p. 248], [1, p. 80], [7, §52 and §55], [31], [33, section 7.6]) that has so far remained open.

From now on, we will consider the general case of $n$-dimensional convex bodies. If $K_1, \ldots, K_m$ are convex bodies in $\mathbb{R}^n$, then we have

$$\text{Vol}(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \ldots, i_n = 1}^m \text{Vol}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}. \quad (1.3)$$

Repeating the proof of (1.1) in higher dimension yields the following [7, p. 99].

**Theorem 1.1** (Minkowski’s quadratic inequality). We have

$$\text{Vol}(K, L, M, \ldots, M)^2 \geq \text{Vol}(K, K, M, \ldots, M) \text{Vol}(L, L, M, \ldots, M) \quad (1.4)$$

for any convex bodies $K, L, M$ in $\mathbb{R}^n$.

The main result of this paper is a complete characterization of the extremals in (1.4), settling in particular the problem left open by Minkowski.

Minkowski’s inequality is a special case of the Alexandrov-Fenchel inequality

$$\text{Vol}(K, L, C_1, \ldots, C_{n-2})^2 \geq \text{Vol}(K, K, C_1, \ldots, C_{n-2}) \text{Vol}(L, L, C_1, \ldots, C_{n-2}),$$

which has numerous applications and connections with various areas of mathematics and is a deep result in its own right, cf. [1, 33, 8, 2, 34]. The characterization of its equality cases is a well-known open problem. A conjecture on the extremals of the Alexandrov-Fenchel inequality was formulated by Schneider [31], but only limited cases have been settled [33, section 7.6]. Our result confirms Schneider’s conjecture in the setting of Minkowski’s quadratic inequality. In particular, we fully settle the extremals of the Alexandrov-Fenchel inequality in the 3-dimensional case that was already known to Minkowski. Along the way, we develop a number of new ideas in convex geometry that are of independent interest (cf. sections 2 and 4).

Beside their direct significance to the foundations of convex geometry, the extremals in Minkowski’s quadratic inequality and in the Alexandrov-Fenchel inequality are closely connected to several other problems. They arise, for example, in the study of infinitesimal rigidity of convex surfaces [15, 5], uniqueness in the mixed analogue of Minkowski’s problem [33, Theorem 7.4.2], and in graph theory [20]. Moreover, remarkable connections with algebraic geometry [8, section 27] relate extremals of the Alexandrov-Fenchel inequality to those of the Hodge inequality, which are not well understood. These connections are orthogonal to the main contribution of this paper, so we will not develop them further here.

### 1.2. Main results

Before we can formulate our main results, we must recall some basic notions of convex geometry. Here and throughout the paper, our standard reference will be the encyclopedic monograph [33].

Throughout the remainder of this paper we fix the dimension $n \geq 3$. A convex body is a nonempty compact convex subset of $\mathbb{R}^n$. Given a convex body $K$ and
a vector $u \in \mathbb{R}^n \setminus \{0\}$, we denote by $F(K, u)$ the unique face of $K$ with outer normal vector $u$. The supporting hyperplane of $K$ in the normal direction $u$ is the hyperplane $F(K, u) + u^\perp$. A key role in the present context will be played by normal directions that are extreme in the following sense [33, p. 85].

**Definition 1.2.** A vector $u \in \mathbb{R}^n \setminus \{0\}$ is called an $r$-extreme normal vector of a convex body $K$ if there do not exist linearly independent normal vectors $u_1, \ldots, u_{r+2}$ at one and the same boundary point of $K$ such that $u = u_1 + \cdots + u_{r+2}$.

For example, if $K$ is a polytope, then $u$ is an $r$-extreme normal vector if and only if it is an outer normal of a face of $K$ of dimension $\dim(K) = n - 1 - r$.

We are now ready to describe the extremals in Minkowski's quadratic inequality (1.4). We must distinguish several cases, depending on the dimension of $M$. Let us observe at the outset that (1.4) is invariant under translation and scaling of each body, so that the extremals must be invariant under homothety as well.

We begin by considering the main case where $M$ has nonempty interior. The following result confirms a conjecture of Schneider [31], cf. [33, section 7.6].

**Theorem 1.3 (Extremals: full-dimensional case).** Let $M \subset \mathbb{R}^n$ be a convex body with nonempty interior, and let $K, L \subset \mathbb{R}^n$ be arbitrary convex bodies such that $V(K, L, M, \ldots, M) > 0$. Then we have

$$V(K, L, M, \ldots, M)^2 = V(K, K, M, \ldots, M) V(L, L, M, \ldots, M)$$

if and only if there exist $a \geq 0$ and $v \in \mathbb{R}^n$ such that $K$ and $aL + v$ have the same supporting hyperplanes in all $1$-extreme normal directions of $M$.

When $M$ is lower-dimensional, the conclusion of Theorem 1.3 is no longer valid and additional extremals appear. A suitable modification of Schneider’s conjecture in this case was proposed by Ewald and Tondorf [13], see also [32, section 4.2]. The following result confirms this conjecture in the present setting.

**Theorem 1.4 (Extremals: lower-dimensional case).** Let $M \subset \mathbb{R}^n$ be a convex body with empty interior, so that $M - M \subset w^\perp$ for some $w \in S^{n-1}$. Let $K, L \subset \mathbb{R}^n$ be arbitrary convex bodies such that $V(K, L, M, \ldots, M) > 0$. Then we have

$$V(K, L, M, \ldots, M)^2 = V(K, K, M, \ldots, M) V(L, L, M, \ldots, M)$$

if and only if $\hat{L} := \frac{V(K, L, M, \ldots, M)}{V(L, L, M, \ldots, M)} L$ satisfies that $K + F(\hat{L}, w)$ and $\hat{L} + F(K, w)$ have the same supporting hyperplanes in all $1$-extreme normal directions of $M$.

The only case that remains to be considered is $V(K, L, M, \ldots, M) = 0$, in which case equality in (1.4) can only arise for the trivial reason $V(K, L, M, \ldots, M) = 0$. The characterization of positivity of mixed volumes is well-known and is unrelated to the questions in this paper. For completeness, we spell out the resulting characterization without further comment [33, Theorem 5.1.8].

**Theorem 1.5 (Extremals: trivial case).** Let $K, L, M \subset \mathbb{R}^n$ be convex bodies such that $V(K, L, M, \ldots, M) = 0$. Then we have

$$V(K, L, M, \ldots, M)^2 = V(K, K, M, \ldots, M) V(L, L, M, \ldots, M)$$

if and only if one of the following holds: $\dim(K) = 0$; $\dim(L) = 0$; $\dim(K + L) < 2$; $\dim(M) < n - 2$; $\dim(K + M) < n - 1$; $\dim(L + M) < n - 1$; $\dim(K + L + M) < n$.

We also note for completeness that $V(L, L, M, \ldots, M) = 0$ holds if and only if $\dim(L) = 2$, $\dim(M) < n - 2$, or $\dim(L + M) < n$. Thus the different cases covered by Theorems 1.3–1.5 are completely determined by the dimensions of $L, M, L + M$. 
1.3. Prior work. Let us briefly discuss previously known cases of our results.

Bol [6] proved Theorem 1.3 in the special case \( L = M \) (cf. [33, Theorem 7.6.19]). In this case, the characterization states that up to homothety, \( L \) is an \((n - 2)\)-tangential body of \( K \), i.e., \( K \) and \( L \) have the same supporting hyperplanes in 1-extreme normal directions of \( L \). In the 3-dimensional case, all normal directions are 1-extreme except those in the interior of the normal cone of a vertex; from this, it is not difficult to deduce that for bodies \( K, L \) in \( \mathbb{R}^3 \), \( L \) is a 1-tangential body of \( K \) if and only if \( L \) is a cap body of \( K \) as described in section 1.1 [7, section 12].

While Bol's result is an important case of Theorem 1.3, somewhat surprisingly the method used in his proof sheds essentially no light on the general problem. To explain why, let us first note that if equality is attained in Minkowski’s quadratic inequality, we may always assume (by suitably rescaling \( L \)) that

\[
V(K, K, M, \ldots, M) = V(K, L, M, \ldots, M) = V(L, L, M, \ldots, M) \tag{1.5}
\]

as \( (1.4) \) is invariant under scaling. By means of a delicate geometric argument using the method of inner parallel bodies, Bol was able to show that if \( (1.5) \) holds with \( L = M \), then it must be the case that \( K \subseteq L \) (up to translation of \( L \)). However, a basic property of mixed volumes is their mononicity: one always has

\[
V(K, K, M, \ldots, M) \leq V(K, L, M, \ldots, M) \leq V(L, L, M, \ldots, M)
\]

when \( K \subseteq L \). Thus Bol’s argument shows that when \( L = M \), the extremals of Minkowski’s quadratic inequality are reduced to equality cases of the monotonicity of mixed volumes. The latter are much simpler to analyze, as was already done for \( L = M \) by Minkowski himself [25, p. 227] (cf. [33, Theorem 7.6.17]). Unfortunately, the key ingredient of this argument is simply false for general \( K, L, M \).

Example 1.6. Let \( M = \text{conv}\{B, x, -x\} \), where \( B \) is a ball in \( \mathbb{R}^3 \) and \( x \notin B \). Let \( L \) be any body that has the same supporting hyperplanes as \( B \) in all normal directions except those in the interior of the normal cone of \( M \) at \( x \), and let \( K = -L \) (cf. Figure 1.2). Then the supporting hyperplanes of \( K \) and \( L \) coincide in all 1-extreme normal directions of \( M \), so by Theorem 1.3 we have equality in \( (1.4) \). Moreover, by symmetry \( V(K, K, M, \ldots, M) = V(L, L, M, \ldots, M) \), so \( (1.5) \) holds. But clearly \( K + v \not\subseteq L \) and \( L \not\subseteq K + v \) for any \( v \). Thus \( (1.5) \) need not imply any monotonicity.

It therefore appears that Bol’s theorem arises essentially by coincidence as a very special feature of the case \( L = M \). In particular, it cannot explain the mechanism that gives rise to the extremals in the general setting of Theorem 1.3.

Prior to the present work, Bol’s theorem was the only known result for general convex bodies. In addition, some cases where \( L \neq M \) have been proved for special types of bodies, such as when \( M \) is a ball or a zonoid; we refer to [33, section 7.6] for a review of such results, whose proofs rely on specific features of these special
bodies. Most importantly for our purposes, the validity of Theorems 1.3 and 1.4 was previously established by Schneider in the case that $M$ is a simple polytope, cf. [33, Theorem 7.6.21] and [32, Theorem 4.2] (in the lower-dimensional case, the result holds for any polytope). As will be explained below, our approach to Theorems 1.3 and 1.4 was partially inspired by Schneider’s results for polytopes. The key contribution of this paper, however, is the introduction of new tools that open the door to the investigation of arbitrary convex bodies.

1.4. Organization of this paper. The rest of this paper is organized as follows. Section 2 is devoted to a high-level overview of the main ingredients of our proofs, and of how they fit together. Section 3 reviews, mostly without proofs, some basic results of convex geometry and functional analysis that will be used throughout the paper. Section 4 is devoted to the construction and basic properties of certain highly degenerate elliptic operators that lie at the core of our analysis of Minkowski’s quadratic inequality. Sections 5 and 6 develop the two main ingredients of our proof of Theorem 1.3: a weak stability theorem and a quantitative rigidity theorem for Minkowski’s inequality. In particular, the proof of Theorem 1.3 will be completed in section 6. Finally, section 7 is devoted to the proof of Theorem 1.4.

2. Overview

The formulation of our main results is of a purely geometric nature. Nonetheless, it will shortly become clear that the core difficulty in our proofs does not arise from geometry, but rather from analytic questions: at the heart of our results lies an analysis of the behavior of certain highly degenerate elliptic operators. The aim of this section is to give a high-level overview of the main ingredients of our proofs and how they fit together, in order to help the reader navigate the rest of the paper. We have deliberately kept this overview as concise as possible, as further discussion is best postponed until after formal definitions have been given.

2.1. The Hilbert method. The original proofs derive Minkowski’s quadratic inequality as a limiting case of simpler inequalities, such as the Brunn-Minkowski inequality [7, sections 49 and 52]. These proofs do not lend themselves to the study of extremals, however, as extremals are not preserved by taking limits. Instead, our starting point will be the direct proof of Minkowski’s inequality due to Hilbert [18, Chapter XIX] (cf. [7, section 52]) which also forms the foundation for the Alexandrov-Fenchel inequality. Let us briefly explain its basic premise.

It will be convenient to identify a convex body $K$ with its support function

$$h_K(u) := \sup_{y \in K} \langle y, u \rangle.$$ 

Geometrically, if $u \in S^{n-1}$, then $h_K(u)$ is the distance to the origin of the supporting hyperplane of $K$ in the direction $u$; as any convex body is the intersection of its supporting halfspaces, $h_K : S^{n-1} \to \mathbb{R}$ uniquely determines $K$. Support functions have the basic property $h_{\lambda K + \mu L} = \lambda h_K + \mu h_L$, i.e., they map linear combinations of sets to linear combinations of functions. In particular, we may view

$$(h_K, h_L) \mapsto V(K, L, M, \ldots, M)$$

as a symmetric quadratic form of the support functions. By linearity of mixed volumes, this quadratic form can be uniquely extended to the linear space of differences
of support functions (which contains $C^2(S^{n-1})$, cf. Lemma 3.5 below). Minkowski’s inequality (1.4) is nothing other than the statement, that this quadratic form satisfies a reverse form of the Cauchy-Schwarz inequality.

We are therefore led to ask which quadratic forms satisfy reverse Cauchy-Schwarz inequalities. A particularly useful characterization arises if we consider closed symmetric quadratic forms on a Hilbert space, i.e., forms $E(f,g) = \langle f, A g \rangle$ associated to a self-adjoint operator $A$. The reason this setting is powerful is that we can bring spectral theory to bear on the problem: $E$ satisfies a reverse Cauchy-Schwarz inequality if and only if $A$ has a one-dimensional positive eigenspace. Let us formulate for future reference the following general statement for (possibly unbounded) self-adjoint operators on a Hilbert space, whose proof may be found in section 3.3.

**Lemma 2.1 (Hyperbolic quadratic forms).** Let $A$ be a self-adjoint operator on a Hilbert space $H$ with $0 < \sup \text{spec } A < \infty$, and let $E(f,g)$ be the associated closed quadratic form. Then the following are equivalent:

1. $E(f,g)^2 \geq E(f,f)E(g,g)$ for all $f,g \in \text{Dom } E$ such that $E(g,g) > 0$.
2. $\text{rank } 1_{(0,\infty)}(A) = 1$.

Moreover, if either (hence both) of these conditions is satisfied, the following are equivalent for given $f,g \in \text{Dom } E$ such that $E(g,g) > 0$:

1’. $E(f,g)^2 = E(f,f)E(g,g)$.
2’. $f - ag \in \ker A \subset \text{Dom } A$ for some $a \in \mathbb{R}$.

It is not clear, a priori, that mixed volumes fit into the setting of Lemma 2.1. However, Hilbert realized that for sufficiently smooth bodies, a classical representation formula of Minkowski (see, e.g., Lemma 3.7) can be used to write

$$V(K,L,M,\ldots,M) = \langle h_K, A h_L \rangle_{L^2(\omega)},$$

where $\omega$ denotes the surface measure on $S^{n-1}$ and $A$ is a certain elliptic second order differential operator. Elliptic regularity theory shows that $A$ is essentially self-adjoint on $L^2(\omega)$ and has compact resolvent. Hilbert exploited this regularity, using a clever homotopy argument, to establish condition 2 of Lemma 2.1. This proves Minkowski’s inequality for smooth bodies $M$, and the general case follows by approximation. As a key step in his proof, Hilbert shows that

$$\ker A = \{ x \mapsto \langle v, x \rangle : v \in \mathbb{R}^n \}$$

consists precisely of the linear functions restricted to $S^{n-1}$. Thus for smooth bodies $M$, Lemma 2.1 implies that equality holds in Minkowski’s inequality if and only if $h_K - ah_L = \langle v, \cdot \rangle$ for some $a,v$, i.e., when $K, L$ are homothetic $K = aL + v$.

The above discussion shows that, while innocent for the purpose of proving Minkowski’s inequality, the smoothness assumption on $M$ completely eliminates its nontrivial extremals. The key difficulty we face in the analysis of extremals is to make sense of the above ideas for arbitrary non-smooth bodies $M$.

### 2.2. Mixed volumes and Dirichlet forms

It is far from clear that it is possible, even in principle, to define self-adjoint representations of mixed volumes for non-smooth bodies $M$. For example, one may verify that when $M$ is a polytope, the quadratic form $(h_K, h_L) \mapsto V(K,L,M,\ldots,M)$ is not closable on $L^2(\omega)$ and therefore does not admit any self-adjoint representation on this space. For this reason, self-adjoint representations have only appeared in the literature in special
cases: for smooth bodies (due to Hilbert [18]), and for certain special families of polytopes with common face normals (due to Alexandrov [1]).

The first step towards our analysis of extremals is the realization that mixed volumes do in fact admit a self-adjoint representation for any body $M$, provided one chooses the Hilbert space appropriately. In the sequel, $S_{K_1,\ldots,K_{n-1}}$ denotes the mixed area measure on $S^{n-1}$, whose definition is postponed to section 3.

**Theorem 2.2.** For any convex body $M$, there is a self-adjoint operator $\mathcal{A}$ on $L^2(S_{B,M,\ldots,M})$ such that the associated closed quadratic form $\mathcal{E}(f,g)$ satisfies

$$h_K, h_L \in \text{Dom } \mathcal{E} \quad \text{and} \quad V(K,L,M,\ldots,M) = \mathcal{E}(h_K, h_L)$$

for all convex bodies $K, L$.

The operator $\mathcal{A}$ of Theorem 2.2 is a highly degenerate elliptic operator in the sense of the theory of Dirichlet forms [16, 3]. Its existence and basic properties will be investigated in section 4 (in the general setting of arbitrary mixed volumes). While this operator is, in general, a rather abstract object, it provides the foundation for extending Hilbert’s method to arbitrary convex bodies $M$.

Theorem 2.2 and Lemma 2.1 reduce the study of extremals in Minkowski’s inequality to the characterization of $\ker \mathcal{A}$. Translation-invariance of mixed volumes implies that $\ker \mathcal{A}$ always contains the linear functions, so that homothetic bodies $K, L$ trivially yield equality. It should be emphasized at this point that there are two distinct mechanisms for the appearance of nontrivial extremals:

a. It is possible that $\ker \mathcal{A}$ may be strictly larger than the set of linear functions. Its additional (nonlinear) elements can give rise to new extremals.

b. Even if $\ker \mathcal{A}$ contains only linear functions, new extremals can arise as the underlying measure $S_{B,M,\ldots,M}$ need not be supported on the entire sphere. Thus Lemma 2.1 only guarantees that $h_K - ah_L = \langle v, \cdot \rangle S_{B,M,\ldots,M}$-a.e.

The statements of Theorems 1.3 and 1.4 can now be interpreted in a new light. A fundamental result of Schneider (Theorem 3.4 below) states that

$$\text{supp } S_{B,M,\ldots,M} = \text{cl}\{u \in S^{n-1} : u \text{ is a 1-extreme normal vector of } M\}.$$ 

Thus Theorem 1.3 shows that when $M$ has nonempty interior, only mechanism b. arises. In contrast, both a. and b. arise when the body $M$ has empty interior, which explains the appearance of additional extremals in Theorem 1.4.

The proofs of these facts occupy the main part of this paper. The proof of Theorem 1.4 will turn out to be conceptually simpler, as in this case one can compute explicitly the operator $\mathcal{A}$ and its kernel. This will be done in section 7.

The main difficulty lies in the proof of Theorem 1.3, as in this case we do not have an explicit description of the operator $\mathcal{A}$ that is amenable to computation.

2.3. **Weak stability and rigidity.** The most natural starting point for understanding the extremals of a given inequality is to attempt to deduce these from a careful examination of the proof of the inequality. In the case of Minkowski’s inequality, however, there is a surprising and apparently fundamental obstacle to such an approach. In view of Lemma 2.1 and Theorem 2.2, it is clear that $S_{B,M,\ldots,M}$ is

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1 More precisely, our results show that $\ker \mathcal{A} \cap \{h_Q - h_R : Q, R \text{ convex bodies} \}$ consists of linear functions only, which is equivalent to Theorem 1.3 by Lemma 2.1 (see also Remark 7.8).
the relevant measure for the study of the extremals. However, it is a different measure $S_{M,...,M}$ that appears naturally (explicitly or implicitly) in the various known proofs of Minkowski’s inequality [1, 22, 34]. As (cf. Theorem 3.4)

$$ \text{supp} S_{M,...,M} = \text{cl} \{ u \in S^{n-1} : u \text{ is a 0-extreme normal vector of } M \}, $$

the support of $S_{M,...,M}$ is generally much smaller than that of $S_{B,M,...,M}$, and does not suffice to characterize the extremals (see Example 2.4 below). Thus there appears to be a disconnect between the setting of this paper that gives rise to a good analytic theory, and the setting used in the proofs of Minkowski’s inequality that is needed to exploit the algebraic structure of mixed volumes.

The conceptual challenge behind the proof of Theorem 1.3 is to understand how to surmount this obstacle. To this end, we will prove that the operator $A$ satisfies a strong rigidity property: once the extremals in Minkowski’s inequality have been fixed in the 0-extreme directions of $M$, their extension to the 1-extreme directions of $M$ is uniquely determined. This property closes the gap between the information provided by proofs of Minkowski’s inequality and its extremals.

More precisely, we will proceed in two steps. First, in section 5, we obtain weak control on the extremals by refining an approach to Minkowski’s inequality that was recently developed by Kolesnikov and Milman [21, 22].

**Theorem 2.3.** Let $K, L, M$ be convex bodies so that $M$ has nonempty interior and $V(L, L, M, \ldots, M) > 0$. If equality holds in (1.4), then there exist $a \geq 0$, $v \in \mathbb{R}^n$ so that $h_K(x) - ah_L(x) = \langle v, x \rangle$ for all $x \in \text{supp} S_{M,...,M}$.

We emphasize that Theorem 2.3, on its own, provides only very weak information on the extremals, as is illustrated by the following example.

**Example 2.4.** When $M$ is a polytope, the 0-extreme directions are precisely the facet normals. Thus if, for example, $K \subseteq L = M$ is an equality case, Theorem 2.3 only implies that $K$ touches every facet of $M$. While this is nontrivial geometric information, this property is much too weak to characterize the extremals. For example, if $K$ is the unit ball and $L = M$ is the unit cube, then $K$ touches every facet of $M$, but this is not an equality case of Minkowski’s inequality.

In the second step, we amplify the weak control provided by Theorem 2.3 to fully characterize the extremals. The following rigidity property, which will be proved in section 6, is one of the central results of this paper.

**Theorem 2.5.** Let $K, L, M$ be convex bodies so that $M$ has nonempty interior and $V(L, L, M, \ldots, M) > 0$. If equality holds in (1.4) and $h_K(x) = h_L(x)$ for all $x \in \text{supp} S_{M,...,M}$, then necessarily $h_K(x) = h_L(x)$ for all $x \in \text{supp} S_{B,M,...,M}$.

Unfortunately, we cannot prove Theorems 2.3 and 2.5 directly, as we do not have a sufficiently explicit description of $A$ for general convex bodies. Instead, we will prove quantitative versions of both these theorems for special convex bodies. A quantitative form of Theorem 2.3, i.e., a weak stability form of Minkowski’s inequality, will be proved for smooth bodies using Reilly’s formula in Riemannian geometry. A quantitative form of Theorem 2.5, i.e., a quantitative rigidity theorem, will be proved for polytopes: in this case $A$ is a “quantum graph” (cf. section 4.2), and the proof will exploit a stability estimate for the solution of the Dirichlet problem for $A$ with boundary data on $\text{supp} S_{M,...,M}$. In both cases, a key aspect of the proof is to discover the correct quantitative formulation that does not degenerate when we take the appropriate limit to approximate arbitrary $M$. 
Remark 2.6. The formulation of Theorems 2.3 and 2.5 was inspired by the proof of a result of Schneider [33, Theorem 7.6.21] for the case that $M$ is a simple polytope. In this setting, the statement of Theorem 2.3 can be deduced from Alexandrov’s polytope proof of the Alexandrov-Fenchel inequality [1], while the statement of Theorem 2.5 follows from a form of Minkowski’s uniqueness theorem. However, these qualitative statements do not allow one to pass to the limit of general convex bodies; the tools that are needed to do so are developed in this paper.

On the other hand, Schneider’s result goes beyond Minkowski’s inequality to cover some additional cases of the Alexandrov-Fenchel inequality. Similarly, most of the techniques that are developed in this paper are not specific to Minkowski’s inequality, and extend to general mixed volumes. The only ingredient of this paper that is fundamentally restricted to Minkowski’s classical setting is Theorem 2.3, for reasons that will become clear in section 5. While a replacement for this argument in the Alexandrov-Fenchel setting will require new ideas, we expect that the techniques that are introduced in this paper could provide a basis for further developments.

3. Preliminaries

The aim of this section is to recall some basic definitions and facts from convex geometry and functional analysis that will be used in the sequel.

We highlight at the outset the following convention. In Minkowski’s inequalities and in many arguments in this paper, mixed volumes $V(K,L,C_1,\ldots,C_{n-2})$ of convex bodies $K,L,C_1,\ldots,C_{n-2}$ in $\mathbb{R}^n$ are considered fixed $C_1,\ldots,C_{n-2}$, and only $K$ and $L$ are varied. We therefore introduce once and for all the notation $C := (C_1,\ldots,C_{n-2})$, $M := (M,\ldots,M\overbrace{\cdots\cdots}^{n-2})$, and we will write $V(K,L,C) := V(K,L,C_1,\ldots,C_{n-2})$, $S_{B,M} := S_{B,M\overbrace{\cdots\cdots}^{n-2}}$, etc.

Throughout this paper, we always denote by $B$ the Euclidean unit ball in $\mathbb{R}^n$.

3.1. Mixed volumes and mixed area measures. Mixed volumes are defined by (1.3). They have the following basic properties [33, section 5.1].

**Lemma 3.1.** Let $K,K',K_1,\ldots,K_n$ be convex bodies in $\mathbb{R}^n$.

- a. $V(K_1,\ldots,K_n) = Vol(K)$.
- b. $V(K_1,\ldots,K_n)$ is symmetric and multilinear in its arguments.
- c. $V(K_1,\ldots,K_n) \geq 0$.
- d. $V(K_1,\ldots,K_n) \geq V(K',K_2,\ldots,K_n)$ if $K \supseteq K'$.
- e. $V(K_1,\ldots,K_n)$ is invariant under translation $K_i \mapsto K_i + v_i$.

There is a close connection between mixed volumes and mixed area measures. The area measure of a convex body $K$ in $\mathbb{R}^n$ is the measure on $S^{n-1}$ defined by

$$S(K,A) = H^{n-1} \{ \{ x \in \partial K : x \in F(K,u) \text{ for some } u \in A \} \},$$

where $H^k$ is the $k$-Hausdorff measure and $A \subseteq S^{n-1}$. That is, $S(K,A)$ is the surface measure of the part of the boundary of $K$ with outer normal vectors in $A$. Just like volume, the area measure $S(K,\cdot)$ is polynomial in $K$, cf. [33, p. 279]:

$$S(\lambda_1 K_1 + \cdots + \lambda_m K_m, A) = \sum_{i_1,\ldots,i_{n-1}=1}^m S_{K_{i_1},\ldots,K_{i_{n-1}}}(A) \lambda_{i_1} \cdots \lambda_{i_{n-1}}$$
for \( \lambda_1, \ldots, \lambda_m \geq 0 \). The measures \( S_{K_1, \ldots, K_{n-1}} \) are called mixed area measures. They have the following properties [33, section 5.1].

**Lemma 3.2.** Let \( K, K_1, \ldots, K_{n-1} \) be convex bodies in \( \mathbb{R}^n \).

a. \( S_{K, \ldots, K} = S(K, \cdot) \).

b. \( S_{K_1, \ldots, K_{n-1}} \) is symmetric and multilinear in its arguments.

c. \( S_{K_1, \ldots, K_{n-1}} \geq 0 \).

d. \( S_{K_1, \ldots, K_{n-1}} \) is invariant under translation \( K_i \mapsto K_i + v_i \).

e. \( \int (v, x) S_{K_1, \ldots, K_{n-1}}(dx) = 0 \) for all \( v \in \mathbb{R}^n \).

The central relation between mixed volumes and mixed area measures is the following representation formula [33, Theorem 5.1.7]:

\[
V(K_1, \ldots, K_n) = \frac{1}{n} \int h_{K_1} dS_{K_2, \ldots, K_n}.
\]  

(3.1)

Note that while \( K_1 \) and \( K_2, \ldots, K_n \) play different roles in this representation, the expression is nonetheless symmetric under permutation of the \( K_i \) by Lemma 3.1(b).

We now record two important facts. First, mixed volumes and mixed area measures are continuous in the topology of Hausdorff convergence (i.e., \( K^{(s)} \to K \) if and only if \( \|h_{K^{(s)}} - h_K\|_\infty \to 0 \)); see the proof of [33, Theorem 5.1.7].

**Theorem 3.3.** Suppose that \( K_1^{(s)}, \ldots, K_n^{(s)} \) are convex bodies such that \( K_i^{(s)} \to K_i \) as \( s \to \infty \) in the sense of Hausdorff convergence. Then

\[
V(K_1^{(s)}, \ldots, K_n^{(s)}) \to V(K_1, \ldots, K_n), \quad S_{K_1^{(s)}, \ldots, K_{n-1}^{(s)}} \xrightarrow{\text{m}} S_{K_1, \ldots, K_{n-1}}
\]

as \( s \to \infty \), where the limit of measures is in the sense of weak convergence.

Second, we have the following support characterization [33, Theorem 4.5.3].

**Theorem 3.4.** Let \( M \) be a convex body in \( \mathbb{R}^n \) with nonempty interior. Then

\[
\text{supp } S_{M, M} = \text{cl}\{\{u \in S^{n-1} : u \text{ is a } 0\text{-extreme normal vector of } M\},
\]

\[
\text{supp } S_{B, M} = \text{cl}\{\{u \in S^{n-1} : u \text{ is a } 1\text{-extreme normal vector of } M\}.
\]

Let us finally note that as mixed volumes and mixed area measures are linear functionals of the underlying bodies (and hence of their support functions), their definitions extend naturally by linearity to functions that are differences of two support functions [33, section 5.2]. By a slight abuse of notation, we will write

\[
V(f, K_2, \ldots, K_n) := V(K, K_2, \ldots, K_n) - V(K', K_2, \ldots, K_n),
\]

\[
S_{f, K_2, \ldots, K_{n-1}} := S_{K, K_2, \ldots, K_{n-1}} - S_{K', K_2, \ldots, K_{n-1}}
\]

for functions of the form \( f = h_K - h_{K'} \). We may similarly extend further arguments by linearity to write \( V(f, g, K_3, \ldots, K_n), \) etc. In particular, as will be recalled in the following section, any \( C^2 \) function on the sphere can be written as the difference of two support functions, so that mixed volumes and mixed area measures are well defined when their arguments are arbitrary \( C^2 \) functions. Of course, \( V(f, K_2, \ldots, K_n) \) need not be nonnegative and \( S_{f, K_2, \ldots, K_{n-1}} \) may be a signed measure.
3.2. Smooth convex bodies. For sufficiently smooth bodies, mixed volumes and area measures may be expressed in a more explicit form, cf. [34, section 2].

To define the appropriate regularity, we recall that the support function $h_K$ may be viewed either as a function on the sphere $S^{n-1}$ or, equivalently, as a 1-homogeneous function on $\mathbb{R}^n$. Now suppose that $h_K$ is a $C^2$ function. Then $\nabla h_K$ is 0-homogeneous, so the derivative of $\nabla h_K$ in the radial direction vanishes. The Hessian of $h_K$ in $\mathbb{R}^n$ may therefore be viewed as a linear mapping from the tangent space of the sphere to itself. We denote this mapping by $D^2 h_K$. For an arbitrary $C^2$ function $f : S^{n-1} \to \mathbb{R}$, the restricted Hessian $D^2 f$ is defined analogously by applying the above construction to the 1-homogeneous extension of $f$. We may also express this notion intrinsically as $D^2 f = \nabla^2 h_K f + f I$ in terms of the covariant Hessian on the sphere. We have the following basic fact [34, section 2.1]; here and below, $M > 0 (M \geq 0)$ denotes that the matrix $M$ is positive definite (semidefinite).

**Lemma 3.5.** Let $f : S^{n-1} \to \mathbb{R}$ be a $C^2$ function. Then $f = h_K$ for some convex body $K$ if and only if $D^2 f \geq 0$. In particular, any $C^2$ function satisfies $f = h_K - h_L$ for some convex bodies $K, L$ (as $D^2(f + h_{\lambda B}) = D^2 f + \lambda I \geq 0$ for large $\lambda$).

We now formulate the following definition.

**Definition 3.6.** A convex body $K$ is of class $C^k_+$ ($k \geq 2$) if $h_K$ is $C^k$ and $D^2 h_K > 0$.

It can be shown [33, sections 2.5 and 3.4] that $K$ is of class $C^k_+$ if and only if its boundary $\partial K$ is a $C^k$-submanifold of $\mathbb{R}^n$, and the function $n_K : \partial K \to S^{n-1}$ that maps each boundary point to its (unique) outer normal is a $C^{k-1}$-diffeomorphism.

For a $C^k_+$ body $K$, one can obtain an explicit expression for the area measure $S(K, \cdot)$ by using the outer normal map $n_K$ to perform a change of variables in its definition. Using the basic fact $n_K^{-1} = \nabla h_K$ [33, Corollary 1.7.3], this yields

$$S(K, d\omega) = \det(D^2 h_K) d\omega,$$

where $\omega$ denotes the surface measure on $S^{n-1}$. To extend this expression to mixed area measures, note that we can write

$$\det(\lambda_1 M_1 + \cdots + \lambda_m M_m) = \sum_{i_1, \ldots, i_{n-1}=1}^m D(M_{i_1}, \ldots, M_{i_{n-1}}) \lambda_{i_1} \cdots \lambda_{i_{n-1}}$$

for any $(n-1)$-dimensional matrices $M_i$ and $\lambda_i \geq 0$, as $M \mapsto \det M$ is a homogeneous polynomial. The coefficients $D(M_1, \ldots, M_{n-1})$ are called mixed discriminants. The definition of mixed area measures and (3.1) now yield the following.

**Lemma 3.7.** Let $K_1, \ldots, K_n$ be convex bodies in $\mathbb{R}^n$ of class $C^2_+$. Then

$$S_{K_2, \ldots, K_n}(d\omega) = D(D^2 h_{K_2}, \ldots, D^2 h_{K_n}) d\omega,$$

$$V(K_1, \ldots, K_n) = \frac{1}{n!} \int h_{K_1} D(D^2 h_{K_2}, \ldots, D^2 h_{K_n}) d\omega.$$

Let us recall some basic properties of mixed discriminants.

**Lemma 3.8.** Let $M, M', M_1, \ldots, M_{n-1}$ be symmetric $(n-1)$-dimensional matrices.

a. $D(M, \ldots, M) = \det(M)$.

b. $D(M', M, \ldots, M) = \frac{1}{n!} \text{Tr}[\text{cof}(M)M']$.

c. $D(M_1, \ldots, M_{n-1})$ is symmetric and multilinear in its arguments.

d. $D(M, M_2, \ldots, M_{n-1}) \geq D(M', M_2, \ldots, M_{n-1})$ if $M \geq M'$, $M_2, \ldots, M_{n-1} \geq 0$. 

These properties may be found in [34, Lemma 2.6] except part (b), which follows readily by differentiation $\frac{d}{dt} \det(M + tM')|_{t=0} = (n-1)D(M', M, \ldots, M)$.

We finally recall that the directional derivatives of support functions have a geometric meaning for any (non-smooth) convex body [33, Theorem 1.7.2].

**Lemma 3.9.** Let $K$ be any convex body in $\mathbb{R}^n$. Then $\nabla_x h_K(u) = h_{F(K,u)}(x)$ for all $u, x \in S^{n-1}$, where $\nabla_x$ denotes the directional derivative in $\mathbb{R}^n$ in direction $x$.

### 3.3. Self-adjoint operators and quadratic forms

Let $H$ be a (real) Hilbert space and let $\mathcal{E}(f,g)$ be a bilinear map defined on a dense subspace $\text{Dom} \mathcal{E} \subseteq H$. We call $\mathcal{E}$ a nonnegative form if $\mathcal{E}(f,g) = \mathcal{E}(g,f)$ and $\mathcal{E}(f,f) \geq 0$. A nonnegative form is closed if $\text{Dom} \mathcal{E}$ is complete for the norm $\|f\|^2 + \mathcal{E}(f,f))^{1/2}$.

It is a basic fact that closed nonnegative forms are in one-to-one correspondence with nonnegative self-adjoint operators on $H$. Indeed, for any nonnegative self-adjoint operator $\mathcal{L}$, the form $\mathcal{E}(f,g) := \langle \mathcal{L}^{1/2}f, \mathcal{L}^{1/2}g \rangle$ with $\text{Dom} \mathcal{E} = \text{Dom} \mathcal{L}^{1/2}$ is closed; while for every closed nonnegative form $\mathcal{E}$, there is a nonnegative self-adjoint operator $\mathcal{L}$ such that the above representation holds [12, Theorem 4.4.2].

The following classical result provides a powerful method to construct self-adjoint operators (the Friedrichs extension), cf. [12, Theorem 4.4.5].

**Lemma 3.10.** Let $\mathcal{L}$ be a densely defined operator on $H$ such that the bilinear map $\mathcal{E}(f,g) = \langle f, \mathcal{L}g \rangle$ is symmetric and nonnegative on the domain of $\mathcal{L}$. Then $\mathcal{E}$ is closable and its closure defines a self-adjoint extension of $\mathcal{L}$.

Let us also note the following basic fact. The key point here is that if $f \in \text{Dom} \mathcal{E}$ and $\mathcal{E}(f,f) = 0$, then we obtain additional regularity $f \in \text{Dom} \mathcal{L}$.

**Lemma 3.11.** Let $\mathcal{E}(f,g)$ be a closed nonnegative form and $\mathcal{L}$ the associated self-adjoint operator. Then $f \in \text{Dom} \mathcal{E}$ if and only if $f \in \text{Dom} \mathcal{L}$.

**Proof.** $\text{Dom} \mathcal{L} = \{ f \in \text{Dom} \mathcal{L}^{1/2} : \langle f, \mathcal{L}^{1/2}f \rangle \in \text{Dom} \mathcal{L}^{1/2} \}$ by [12, Theorem 4.3.4]. The conclusion follows from $\mathcal{E}(f,f) = \| \mathcal{L}^{1/2}f \|^2 = \langle f, \mathcal{L}f \rangle$ for $f \in \text{Dom} \mathcal{L}$. $\square$

In the context of Lemma 2.1, the relevant operator is not nonnegative but rather bounded from above. The above notions extend readily to this setting. To this end, a densely defined symmetric bilinear map $\mathcal{E}(f,g)$ is called a $c$-seminbounded form if $\mathcal{E}(f,f) \leq c\|f\|^2$. Then $\mathcal{E}'(f,g) := c(f,g) - \mathcal{E}(f,g)$ is a nonnegative form, and $\mathcal{E}$ is said to be closed if $\mathcal{E}'$ is closed. Using the analogous property for $\mathcal{E}'$, we find that for any closed $c$-seminbounded form $\mathcal{E}$, there is a self-adjoint operator $\mathcal{A} \leq cI$ such that $\mathcal{E}(f,g) = \langle f, \mathcal{A}g \rangle$ for $g \in \text{Dom} \mathcal{A}$. Conversely, any self-adjoint operator $\mathcal{A} \leq cI$ defines a closed $c$-seminbounded form $\mathcal{E}(f,g) = c(f,g) - \langle (cI - \mathcal{A})^{1/2}f, (cI - \mathcal{A})^{1/2}g \rangle$.

As the situation will generally be clear from context, we will often simply speak of the closed quadratic form associated to a self-adjoint operator (which is either nonnegative or semibounded), as in the statement of Lemma 2.1.

We conclude this section by proving Lemma 2.1 (for a simpler setting, see [34, Lemma 2.8]). In the sequel, self-adjoint operators and their functional calculus will be used without further comment; we refer to [28, 12] as basic references.

**Proof of Lemma 2.1.** We prove each implication separately.

2 $\Rightarrow$ 1. Suppose condition 2 holds. Then the spectral theorem implies that there exists $v \in \text{Dom} \mathcal{A}$, $\lambda > 0$ such that $\mathcal{A}v = \lambda v$, and such that the following holds:

$$\mathcal{E}(h, h) \leq 0 \quad \text{for } h \in \text{Dom} \mathcal{E}, \ h \perp v. \quad (3.2)$$
Let \( f, g \in \text{Dom} \mathcal{E} \) such that \( \mathcal{E}(g, g) > 0 \). Then \( \langle g, v \rangle \neq 0 \) by (3.2). Define \( z = f - ag \) with \( a = \langle f, v \rangle / \langle g, v \rangle \). Then \( z \perp v \), so applying (3.2) again yields
\[
0 \geq \mathcal{E}(z, z) = \mathcal{E}(f, f) - 2a \mathcal{E}(f, g) + a^2 \mathcal{E}(g, g) \geq \mathcal{E}(f, f) - \frac{\mathcal{E}(f, g)^2}{\mathcal{E}(g, g)},
\]
where the last inequality follows by minimizing over \( a \). Condition 1 follows readily.

1 \( \Rightarrow \) 2. Suppose condition 2 is violated. As \( 0 < \sup \text{spec} \mathcal{A} < \infty \), the spectral theorem implies that \( H_+ := 1_{(0, \infty)}(\mathcal{A})H \) satisfies \( H_+ \subset \text{Dom} \mathcal{A}, \mathcal{A}H_+ \subset H_+ \), \( \dim H_+ \geq 2 \), and \( \mathcal{E}(g, g) > 0 \) for \( g \in H_+ \setminus \{0\} \). That is, \( \mathcal{A} \) is a bounded positive definite operator on its positive eigenspace, which has dimension at least two.

Now choose any nonzero \( g \in H_+ \) and \( f \in H_+ \) such that \( f \perp \mathcal{A}g \). Then \( \mathcal{E}(f, f) > 0, \mathcal{E}(g, g) > 0 \), and \( \mathcal{E}(f, g) = \langle f, \mathcal{A}g \rangle = 0 \). Thus condition 1 is violated.

1’ \( \Rightarrow \) 2’. Suppose condition 1’ holds. Then each inequality in the proof of 2 \( \Rightarrow \) 1 must be equality. In particular, \( \mathcal{E}(z, z) = 0 \). Now note that by the spectral theorem, \( z \in H_- := 1_{(-\infty, 0]}(\mathcal{A})H \) and \( \mathcal{A} \) is a nonpositive operator on \( H_- \). Thus by Lemma 3.11, we have \( z \in \text{Dom} \mathcal{A} \) and \( \mathcal{A}z = 0 \). Thus we have established condition 2’.

2’ \( \Rightarrow \) 1’. Suppose that condition 2’ holds. As \( z = f - ag \in \ker \mathcal{A} \), we have \( 0 = \mathcal{E}(z, g) = \mathcal{E}(f, g) - a \mathcal{E}(g, g) \), so \( a = \mathcal{E}(f, g) / \mathcal{E}(g, g) \). But then
\[
0 = \mathcal{E}(z, z) = \mathcal{E}(f, f) - \frac{\mathcal{E}(f, g)^2}{\mathcal{E}(g, g)},
\]
where we used again \( z \in \ker \mathcal{A} \). Condition 1’ follows.

4. Mixed volumes and Dirichlet forms

The aim of this section is to show that mixed volumes of arbitrary convex bodies can be represented as closed quadratic forms associated to self-adjoint operators. This was stated in the setting of Minkowski’s quadratic inequality as Theorem 2.2 above, which is a special case of the following general theorem.

**Theorem 4.1.** Let \( \mathcal{C} = (C_1, \ldots, C_{n-2}) \) be convex bodies in \( \mathbb{R}^n \) with \( S_{B, \mathcal{C}} \neq 0 \). Then there is a self-adjoint operator \( \mathcal{A} \) on \( L^2(S_{B, \mathcal{C}}) \) with \( C^2(S_{n-1}) \subset \text{Dom} \mathcal{A} \) such that:

a. \( \text{spec} \mathcal{A} \subset (-\infty, 0] \cup \{\frac{1}{n}\} \).

b. \( \text{rank} 1_{(0, \infty)}(\mathcal{A}) = 1 \) and \( \mathcal{A}1 = \frac{1}{n}1 \).

c. \( \mathcal{A} \ell = 0 \) for any linear function \( \ell : x \mapsto \langle v, x \rangle \) on \( S_{n-1} \) (\( v \in \mathbb{R}^n \)).

Moreover, the closed quadratic form \( \mathcal{E} \) associated to \( \mathcal{A} \) satisfies the following:

d. \( h_K \in \text{Dom} \mathcal{E} \) for every convex body \( K \) in \( \mathbb{R}^n \).

e. \( \mathcal{E}(h_K, h_L) = V(K, L, \mathcal{C}) \) for any convex bodies \( K, L \) in \( \mathbb{R}^n \).

The above theorem states, in particular, that linear functions are always in the kernel of \( \mathcal{A} \). For the reasons explained in section 2, the central question in the study of the extremals of the Alexandrov-Fenchel inequality is whether these are the only elements of \( \ker \mathcal{A} \) that are differences of support functions. The main part of this paper (sections 5–7) will be devoted to settling this question in the setting \( C_1 = \cdots = C_{n-2} = M \) of Minkowski’s quadratic inequality.

The operator \( \mathcal{A} \) of Theorem 4.1 is defined somewhat abstractly. To actually work with such operators, one would like to have a more explicit formulation. Explicit constructions can be obtained in various special cases that will play an important role in the remainder of this paper. When \( C_1, \ldots, C_{n-2} \) are \( C^2_+ \) bodies,
\(\mathcal{A}\) is a classical elliptic second-order differential operator on \(S^{n-1}\). This case is already used in the proof of Theorem 4.1, which will be given in section 4.1. When \(C_1, \ldots, C_{n-2}\) are polytopes, it turns out that \(\mathcal{A}\) is a quantum graph [4]; this setting will be developed in detail in section 4.2. A third setting in which \(\mathcal{A}\) can be explicitly described will be encountered in section 7 below.

In complete generality, we do not know how to give an expression for \(\mathcal{A}\) that is amenable to explicit computations. Nonetheless, we may always view \(\mathcal{A}\) as a highly degenerate elliptic second-order operator in the sense of the theory of Dirichlet forms [16, 3]. While the latter theory will not be needed in the remainder of this paper, we briefly develop this viewpoint in section 4.3 in order to highlight the general structure that lies behind the representation of Theorem 4.1.

**Remark 4.2.** It should be emphasized at the outset that while \(\mathcal{A}\) may be thought of quite generally as an elliptic second order operator on the sphere, it does not necessarily possess some of the nice regularity properties of classical elliptic operators on compact manifolds. In particular, \(\mathcal{A}\) need not have compact resolvent (i.e., its essential spectrum may be nonempty), as we will see in section 7 below.

**Remark 4.3.** The assumption \(S_{B,C} \neq 0\) in Theorem 4.1 is innocuous. Indeed, if \(S_{B,C} \equiv 0\), then it follows (for example, using Lemma 4.4 below) that \(V(K, L, C) = 0\) for all convex bodies \(K, L\), so that the representation of mixed volumes is trivial.

### 4.1. Proof of Theorem 4.1

Throughout this section, the assumptions and notation of Theorem 4.1 are in force. The proof is based on the following observation.

**Lemma 4.4.** For any \(C^2\) function \(f\) on \(S^{n-1}\), we have \(S_{f,C} \ll S_{B,C}\) and

\[
\left\| \frac{dS_{f,C}}{dS_{B,C}} \right\|_{\infty} \leq \|D^2 f\|_{\infty},
\]

where we use the notation \(\|D^2 f\|_{\infty} := \sup_{x \in S^{n-1}} \|D^2 f(x)\|\).

**Proof.** Let \(C^{(s)}_i\) be convex bodies of class \(C^2\) such that \(C^{(s)}_i \rightarrow C_i\) as \(s \rightarrow \infty\) in the sense of Hausdorff convergence for every \(i = 1, \ldots, n-2\). The existence of such smooth approximations is classical, cf. [33, section 3.4].

For \(C^{(s)} = (C^{(s)}_1, \ldots, C^{(s)}_{n-2})\), Lemmas 3.7 and 3.5 imply

\[
S_{f,C^{(s)}}(d\omega) = D(D^2 f, D^2 h_{C_1^{(s)}}, \ldots, D^2 h_{C_{n-2}^{(s)}}) d\omega
\]

for any \(C^2\) function \(f\). In particular, as \(D^2 h_B = I\), we have

\[
\frac{dS_{f,C^{(s)}}}{dS_{B,C^{(s)}}} = \frac{D(D^2 f, D^2 h_{C_1^{(s)}}, \ldots, D^2 h_{C_{n-2}^{(s)}})}{D(I, D^2 h_{C_1^{(s)}}, \ldots, D^2 h_{C_{n-2}^{(s)}})}
\]

(note that the denominator is strictly positive by Lemma 3.8(d), so the expression is well-defined). As \(\lambda_{\min}(D^2 f) I \leq D^2 f \leq \lambda_{\max}(D^2 f) I\), Lemma 3.8(d) shows that

\[
\left\| \frac{dS_{f,C^{(s)}}}{dS_{B,C^{(s)}}} \right\| \leq \|D^2 f\|
\]
and Lemma 4.5 ensures that $A$ is a bounded linear map from $L^1$ to $L^\infty$. Therefore, Theorem 3.3 implies that

$$\left| \int g \, dS_{f,C} \right| = \lim_{s \to \infty} \left| \int g \, dS_{f,C(s)} \right| \leq \limsup_{s \to \infty} \int |g| \|D^2 f\| \, dS_{B,C(s)} \leq \|D^2 f\|_\infty \int |g| \, dS_{B,C}$$

(4.1)

for every continuous function $g$ on $S^{n-1}$. In particular,

$$g \mapsto \int g \, dS_{f,C}$$

is a bounded linear map from $C^0(S^{n-1}) \subset L^1(S_{B,C})$ to $\mathbb{R}$, and therefore extends uniquely to a bounded linear functional on $L^1(S_{B,C})$. As $L^1(S_{B,C})^* = L^\infty(S_{B,C})$,

$$\int g \, dS_{f,C} = \int g \, dS_{B,C}$$

for some $g \in L^\infty(S_{B,C})$. This proves absolute continuity $S_{f,C} \ll S_{B,C}$, and the bound on $g$ follows by taking the supremum in (4.1) over $g$ with $\int |g| \, dS_{B,C} \leq 1$. □

We will also need the following simple consequence.

**Lemma 4.5.** Let $f, g \in C^2(S^{n-1})$ satisfy $f = g$ $S_{B,C}$-a.e. Then $S_{f,C} = S_{g,C}$.

**Proof.** Let $h \in C^2(S^{n-1})$. Then

$$\int h \, dS_{f,C} - \int h \, dS_{g,C} = n \mathcal{V}(h, f - g, C) = \int (f - g) \, dS_{h,C} = 0,$$

where we have used (3.1), the symmetry of mixed volumes, and Lemma 4.4. As this holds for any $h \in C^2(S^{n-1})$, the conclusion follows. □

The idea behind Theorem 4.1 is now simple. By Lemma 4.4, we can define

$$\mathcal{A}f := \frac{1}{n} \frac{dS_{f,C}}{dS_{B,C}}$$

for $f \in C^2(S^{n-1})$,

(4.2)

and Lemma 4.5 ensures that $\mathcal{A}$ is well-defined as a linear operator on $C^2(S^{n-1}) \subset L^2(S_{B,C})$ (here we implicitly identified those functions in $C^2$ that agree up to $S_{B,C}$-null sets). We aim to extend $\mathcal{A}$ to a bona fide self-adjoint operator by Friedrichs extension. To this end, we must show that $\mathcal{A}$ is semibounded.

**Lemma 4.6.** For every $f \in C^2(S^{n-1})$, we have

$$n \int f \, \mathcal{A}f \, dS_{B,C} = \int f \, dS_{f,C} \leq \int f^2 \, dS_{B,C}.$$

**Proof.** Let $C_i^{(s)}$ be convex bodies of class $C^2_+$ such that $C_i^{(s)} \to C_i$ as $s \to \infty$. Then

$$\int f \, dS_{f,C(s)} \to \int f \, dS_{f,C}, \quad \int f^2 \, dS_{B,C(s)} \to \int f^2 \, dS_{B,C}$$

as $s \to \infty$ by Theorem 3.3. It therefore suffices to assume in the remainder of the proof that $C_1, \ldots, C_{n-2}$ are convex bodies of class $C^2_+$.

For $C^2_+$ bodies, we may write as in the proof of Lemma 4.4

$$\mathcal{A}f = \frac{1}{n} \frac{dS_{f,C}}{dS_{B,C}} = \frac{1}{n} \frac{D(D^2 f, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})}{D(I, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})}.$$
Using $D^2 f = \nabla^2 f - f$ and Lemma 3.8(d), we see that in this case $\mathcal{A}$ is a uniformly elliptic second-order differential operator. Moreover, by (3.1) and the symmetry of mixed volumes, $(f, \mathcal{A}g)_{L^2(S_{B_C})} = V(f,g,C)$ is a symmetric quadratic form. It follows from standard elliptic regularity theory [17, section 8.12] that $\mathcal{A}$ has a self-adjoint extension for which $A_{\lambda}$ concludes the construction of $A_{\epsilon}$.

**Proof of Theorem 4.1.**

4.1. We now proceed to verify each of the claimed properties.

By the symmetry of mixed volumes and Lemma 4.6, concluding the proof. □

**Proof of Theorem 4.1.** For $f, g \in C^2(S^{n-1})$, define $\mathcal{A} f$ as in (4.2) and let

$$E(f, g) := \langle f, \mathcal{A} g \rangle_{L^2(S_{B_C})} = V(f, g, C).$$

By the symmetry of mixed volumes and Lemma 4.6, $E$ is a densely defined and $\frac{1}{n}$-semibounded symmetric quadratic form on $L^2(S_{B_C})$. It follows from Lemma 3.10 that $E$ is closable and that its closure defines a self-adjoint extension of $\mathcal{A}$. This concludes the construction of $\mathcal{A}'$ and $E$ as announced in the statement of Theorem 4.1. We now proceed to verify each of the claimed properties.

**Proof of d and e.** By construction, $E(h_K, h_L) = V(K, L, C)$ whenever $K, L$ are $C^2_+$ bodies. Now let $K$ be an arbitrary convex body, and let $K^{(s)}$ be convex bodies of class $C^2_+$ such that $K^{(s)} \to K$ as $s \to \infty$. That

$$\|h_{K^{(s)}} - h_K\|_{L^2(S_{B_C})}^2 \leq n \sup_{B, B_C} \|h_{K^{(s)}} - h_K\|_2^2 \to 0$$

as $s \to \infty$ follows immediately from Hausdorff convergence. Moreover,

$$E(h_{K^{(s)}}, h_{K^{(s)}}, h_{K^{(s)}}) = V(K^{(s)}, K^{(s)}, C) + 2V(K^{(s)}, K^{(t)}, C) + V(K^{(t)}, K^{(t)}, C) \to 0$$

as $s, t \to \infty$ by Theorem 3.3. Thus $h_K$ is in the completion of $C^2(S^{n-1})$ for the norm $\|2\|_{L^2(S_{B_C})} - E(f, f)^{1/2}$. Therefore, by construction, $h_K \in \text{Dom} E$ and

$$E(h_K, h_K) = \lim_{s \to \infty} E(h_{K^{(s)}}, h_{K^{(s)}}) = \lim_{s \to \infty} V(K^{(s)}, K^{(s)}, C) = V(K, K, C).$$

The analogous conclusion for $E(h_K, h_L)$ follows by polarization.

**Proof of a and b.** Note first that as $h_B = 1$, it follows immediately from the definition (4.2) that $\mathcal{A} 1 = \frac{1}{n} 1$. Now suppose rank $1_{(0, \infty)}(\mathcal{A}) > 1$, and choose any nonzero $f \in 1_{(0, \infty)}(\mathcal{A}) L^2(S_{B_C})$ so that $\langle f, 1 \rangle_{L^2(S_{B_C})} = 0$. Choose $f_s \in C^2(S^{n-1})$ so that $\|f_s - f\|_{L^2(S_{B_C})} \to 0$ and $E(f_s, f_s) \to E(f, f)$ as $s \to \infty$ (the existence of such a sequence is guaranteed by construction as $f \in \text{Dom} E$). By Lemma 3.5, for all $s$ and for all sufficiently large $t > 0$ (depending on $s$), there is a $C^2_+$ convex body $K_t$ such that $h_{K_t^{(s)}} = t + f_s$. Therefore, by the Alexandrov-Fenchel inequality,

$$\langle t + f_s, \mathcal{A} 1 \rangle_{L^2(S_{B_C})}^2 = V(K_t^{(s)}, B, C)^2 \geq V(K_t^{(s)}, K_t^{(s)}, C) V(B, B, C)$$

$$= \langle t + f_s, \mathcal{A}(t + f_s) \rangle_{L^2(S_{B_C})} : \langle 1, \mathcal{A} 1 \rangle_{L^2(S_{B_C})}. $$

Expanding the squares and using $\mathcal{A} 1 = \frac{1}{n} 1$ yields

$$\langle f_s, 1 \rangle_{L^2(S_{B_C})}^2 \geq n^2 E(f_s, f_s) V(B, B, C).$$
In particular, letting $s \to \infty$ yields $E(f, f) \leq 0$, where we used that $V(B, B, C) > 0$ by the assumption $S_{B, C} \neq 0$. But this contradicts the assumption that $f \in L^2(S_{B, C})$. Thus we have shown that $\text{rank } 1_{(0, \infty)}(\mathcal{A}) = 1$, proving part $b$. Part $a$ now follows as an immediate consequence.

Proof of $c$. Let $\ell : x \mapsto \langle v, x \rangle$ be a linear function. Then $\ell = h_{K + v} - h_K$, so

$$\langle f, \mathcal{A} \ell \rangle_{L^2(S_{B, C})} = V(f, K + v, C) - V(f, K, C) = 0$$

for every $f \in C^2(S^{n-1})$ by Lemma 3.1(e). Thus $\mathcal{A} \ell = 0$. $\square$

4.2. Polytopes and quantum graphs. The aim of this section is to provide an explicit description of the objects that appear in Theorem 4.1 in the case where $C_1, \ldots, C_{n-2}$ are polytopes. In this case, it turns out that $\mathcal{A}$ is a “quantum graph”: the one-dimensional Laplacian on the edges of a certain metric graph, with appropriately chosen boundary conditions at the vertices (cf. [4]). For simplicity, we will develop this result in detail in the Minkowski case $C_1 = \cdots = C_{n-2} = M$ that is of primary interest in this paper, and sketch at the end of the section how the representation for general polytopes $C_1, \ldots, C_{n-2}$ may be obtained.

Throughout this section, let $M$ be a fixed polytope in $\mathbb{R}^n$ with nonempty interior. We denote by $\mathcal{F}$ the set of facets of $M$, and by $n_F$ the outer unit normal vector of $F \in \mathcal{F}$.\footnote{Recall that for a smooth body $K$, we denote by $n_K : \partial K \to S^{n-1}$ the outer unit normal map on its boundary (cf. section 3.2). As a facet $F$ of a polytope is associated to a unique outer unit normal vector, however, we simply denote this vector as $n_F \in S^{n-1}$ by a slight abuse of notation.} We say that $F, F' \in \mathcal{F}$ are neighbors if $F \sim F'$, where $\sim$ stands for the (shortest) geodesic segment in $S^{n-1}$ connecting $n_F$ and $n_{F'}$. Thus the undirected edges of $G$ are

$$E = \{e_{F, F'} : F \sim F'\}.$$ 

The length of the edge $e_{F, F'}$ will be denoted as $l_{F, F'} := \mathcal{H}^1(e_{F, F'})$.

We will parametrize each edge by arclength. To this end, it is convenient to fix an arbitrary orientation of the edges by introducing a total ordering $\preceq$ on $\mathcal{F}$. When $F \preceq F'$, the edge $e_{F, F'}$ will be parametrized as $\theta \in [0, l_{F, F'}]$, where $\theta = 0$ corresponds to $n_F$ and $\theta = l_{F, F'}$ corresponds to $n_{F'}$. For a function $f : e_{F, F'} \to \mathbb{R}$,

---

**Figure 4.1.** Metric graph $G$ associated to a polytope $M$.
we will write \( f' = \frac{df}{dt} \) with respect to this parametrization. We denote by \( H^k \) the usual Sobolev space of functions on an edge with \( k \) weak derivatives in \( L^2 \).

A function defined on all the edges of \( G \) will be denoted \( f : G \to \mathbb{R} \), and its (weak) derivative \( f' \) is defined as above on each edge. In particular, any function \( f : S^{n-1} \to \mathbb{R} \) may be viewed as a function on \( G \) by restricting it to the edges, and then its derivative \( f' \) is defined as above. We will write \( f \in C^0(G) \) if \( f : G \to \mathbb{R} \) is continuous on the edges as well as at each vertex.

Finally, we will denote by \( n_{F,F'} \) the unit tangent vector to the geodesic segment \( e_{F,F'} \) at \( n_F \) in the direction of \( n_{F'} \). The directional derivative of a function \( f : S^{n-1} \to \mathbb{R} \) at \( n_F \) in the direction \( n_{F,F'} \) will be denoted \( \nabla_{n_{F,F'}} f(n_F) \). In terms of the arclength parametrization, we may evidently write

\[
\nabla_{n_{F,F'}} f(n_F) = f'|_{e_{F,F'}}(0) 1_{F' \geq F} - f'|_{e_{F,F'}}(l_{F,F'}) 1_{F' \leq F}.
\]

The various objects that we have defined are illustrated in Figure 4.1.

We are now ready to state the main result of this section.

**Theorem 4.7.** Let \( M \) be a polytope in \( \mathbb{R}^n \) with nonempty interior, and define the associated metric graph \( G \) as above. Then for any \( f : S^{n-1} \to \mathbb{R} \), we have

\[
\int f \, ds_{M,M} = \sum_{F \in F} H^{n-1}(F) f(n_F),
\]

\[
\int f \, ds_{B,M} = \frac{1}{n-1} \sum_{e_{F,F'} \in E} H^{n-2}(F \cap F') \int_{e_{F,F'}} f \, dH^1.
\]

Moreover, the self-adjoint operator \( \mathcal{A} \) and closed quadratic form \( \mathcal{E} \) on \( L^2(S_{B,M}) \) defined in Theorem 4.1 can be expressed in this case as follows:

\[
\mathcal{A} f = \frac{1}{n} \{ f'' + f \}
\]

where

\[
\text{Dom} \mathcal{A} = \left\{ f \in C^0(G) : f|_{e_{F,F'}} \in H^2 \text{ for all } e_{F,F'} \in E, \quad \sum_{F' : F' \cap F} H^{n-2}(F \cap F') \nabla_{n_{F,F'}} f(n_F) = 0 \text{ for all } F \in F \right\},
\]

and

\[
\mathcal{E}(f,g) = \frac{1}{n(n-1)} \sum_{e_{F,F'} \in E} H^{n-2}(F \cap F') \int_{e_{F,F'}} \{ f g - f' g' \} \, dH^1
\]

where \( \text{Dom} \mathcal{E} = \{ f \in C^0(G) : f|_{e_{F,F'}} \in H^1 \text{ for all } e_{F,F'} \in E \} \).

In the rest of this section, we fix the setting and notation of Theorem 4.7. The starting point for the proof is the following representation of \( S_{K,M} \) for smooth convex bodies \( K \). Such a statement was given in [33, eq. (7.175)] without proof (and with missing normalization); for completeness, we include a full proof here.

**Proposition 4.8.** Let \( K \) be a convex body of class \( C^2_+ \). Then

\[
\int f \, ds_{K,M} = \frac{1}{n-1} \sum_{e_{F,F'} \in E} H^{n-2}(F \cap F') \int_{e_{F,F'}} (h''_K + h_K) f \, dH^1.
\]
Proof. Denote by \( F_i \) the collection of \( i \)-dimensional faces of \( M \), and define
\[
N_F := \{ u \in S^{n-1} : F(M, u) = F \}.
\]
Then \( \{ N_F : F \in \bigcup_i F_i \} \) is a partition of \( S^{n-1} \). Note that if \( F, F' \in F = F_{n-2} \) are neighboring facets \( F \sim F' \), we have \( F \cap F' \in F_{n-2} \) and \( N_{F \cap F'} = e_{F,F'} \).

To compute \( S_{K,M} \), we will compute the area measure \( S(M + \varepsilon K, \cdot) \), and then deduce the mixed area measure from its definition given in section 3.1. To compute the area measure, we partition the boundary of the set \( M + \varepsilon K \) into disjoint sets \( F(M + \varepsilon K, N_F) := \bigcup_{u \in N_F} F(M + \varepsilon K, u) \) corresponding to boundary points with normals in \( N_F \), and apply the change of variables formula to each of these sets. This partition is illustrated in Figure 4.2 (in the figure \( M \) is a cube and \( K = B \)).

**Step 1.** Fix \( F \in F_i \). We first conveniently parametrize \( F(M + \varepsilon K, N_F) \).

Recall that as \( K \) is a \( C^2 \)-body, the outer normal map \( n_K : \partial K \to S^{n-1} \) is a \( C^1 \)-diffeomorphism and \( n_K^{-1} = \nabla h_K \), cf. section 3.2. Therefore
\[
F(M + \varepsilon K, u) = F(M, u) + \varepsilon F(K, u) = F + \varepsilon \nabla h_K(u)
\]
for any \( u \in N_F \) by Lemma 3.9. It follows that \( F(M + \varepsilon K, N_F) \) is the image of the map \( \iota : F \times N_F \to F(M + \varepsilon K, N_F) \) defined by \( \iota(x, u) := x + \varepsilon \nabla h_K(u) \).

**Step 2.** We now show that \( \iota \) is a \( C^1 \)-diffeomorphism.

Let \( L_F := \text{span}(F-F) \) be the tangent space of \( F \), and note that \( N_F \subset S^{n-1} \cap L_F^\perp \).

Assume for simplicity that \( 0 \in F \), so that \( F \subset L_F \) (when considering a single face, we may always reduce to this setting by translation). Then we may express
\[
\iota(x, u) = (x + \varepsilon P_{L_F} \nabla h_K(u), \varepsilon P_{L_F^\perp} \nabla h_K(u)) \in L_F \oplus L_F^\perp,
\]
where \( P_L \) denotes orthogonal projection onto \( L \).

Now note that as \( h_{P_{L_F}^\perp K}(u) = h_K(P_{L_F}^\perp P_{L_F}^\perp u) \), differentiating yields that
\[
P_{L_F^\perp} \nabla h_K(u) = \nabla h_{P_{L_F}^\perp K}(u) \quad \text{for } u \in S^{n-1} \cap L_F^\perp.
\]
As \( P_{L_F^\perp} K \) is a \( C^2 \) body in \( L_F^\perp \), \( n_{P_{L_F^\perp} K} : \partial P_{L_F^\perp} K \to S^{n-1} \cap L_F^\perp \) is a \( C^1 \)-diffeomorphism and \( n_{P_{L_F}^\perp K}^{-1} = P_{L_F^\perp} \nabla h_K \). It now follows that \( \iota \) is a diffeomorphism, as \( \iota \) is \( C^1 \) and
\[
\iota^{-1}(z, v) = (z - \varepsilon P_{L_F} \nabla h_K(n_{P_{L_F}^\perp K}(\varepsilon^{-1}v)), n_{P_{L_F}^\perp K}(\varepsilon^{-1}v))
\]
for \( (z, v) \in F(M + \varepsilon K, N_F) \subset L_F \oplus L_F^\perp \) is also \( C^1 \).

It is readily seen that the differential \( d\iota \) has a block-triangular form with respect to \( L_F \oplus L_F^\perp \), and that its determinant may be written as \( \det(\varepsilon D^2 h_{P_{L_F}^\perp K}) \).
where $D^2 h_{P_{L^p} K}$ is computed in $L^p_\mathbb{R}$ (equivalently, $D^2 h_{P_{L^p} K}$ is the projection of the Hessian $\nabla^2 h_K$ in $\mathbb{R}^n$ on the tangent space of $S^{n-1} \cap L^p_\mathbb{R}$).

**Step 3.** Now recall that \{\{N_F\}\} partitions $S^{n-1}$. We can therefore write

$$S(M + \varepsilon K, A) = \sum_{i=0}^{n-1} \sum_{F \in \mathcal{F}_i} \mathcal{H}^{n-1}(\{x \in F(M + \varepsilon K, u) \text{ for some } u \in A \cap N_F\})$$

$$= \sum_{i=0}^{n-1} \varepsilon^{n-1-i} \sum_{F \in \mathcal{F}_i} \mathcal{H}^i(F) \int_{N_F} 1_A \det(D^2 h_{P_{L^p} K}) d\mathcal{H}^{n-1-i},$$

where we used the map $\iota$ to perform a change of variables in the second line. By the definition of mixed area measures in section 3.1, we have

$$(n-1) S_{K,M}(A) = \left. \frac{d}{d\varepsilon} S(M + \varepsilon K, A) \right|_{\varepsilon=0}$$

$$= \sum_{F \in \mathcal{F}_{n-2}} \mathcal{H}^{n-2}(F) \int_{N_F} 1_A \det(D^2 h_{P_{L^p} K}) d\mathcal{H}^1.$$ But note that for every $(n-2)$-face $F$, the set $N_F$ is an edge of the quantum graph. In this case $D^2 h_{P_{L^p} K}$ is a scalar function (as $S^{n-1} \cap L^p_\mathbb{R}$ is one-dimensional) and coincides precisely with $h_K'' + h_K$ on each edge. Thus the proof is complete. \(\square\)

We also record a simple but essential observation. It immediately implies, for example, that $C^2(S^{n-1}) \subset \text{Dom} \mathcal{A}$ in Theorem 4.7.

**Lemma 4.9.** For every $F \in \mathcal{F}$, we have

$$\sum_{F': F' \sim F} \mathcal{H}^{n-2}(F \cap F') n_{F \rightarrow F'} = 0.$$  

**Proof.** Consider $F$ as an $(n-1)$-dimensional convex body in $\text{aff} F$. The facets of $F$ are precisely the sets $F \cap F'$ for $F' \sim F$, and the corresponding facet normals are $n_{F \rightarrow F'}$. Thus the conclusion follows from Lemma 3.2(e). \(\square\)

We now proceed to the proof of Theorem 4.7.

**Proof of Theorem 4.7.** The expressions for $S_{M,M}$ and $S_{B,M}$ follow immediately from the definition of mixed area measures (cf. section 3.1) and Proposition 4.8.

Next, let $f \in C^2(S^{n-1})$. By Lemma 4.9, we have $f \in \text{Dom} \mathcal{A}$. Moreover,

$$\mathcal{A} f = \frac{1}{n} \frac{dS_{I,M}}{dS_{B,M}}$$

by Proposition 4.8. Thus the restriction of $\mathcal{A}$ to $C^2(S^{n-1}) \subset \text{Dom} \mathcal{A}$ agrees with the operator defined in (4.2). It remains to show that $\mathcal{A}$ as defined in Theorem 4.7 is self-adjoint, that $\mathcal{E}$ in Theorem 4.7 is the associated closed quadratic form, and that $\mathcal{A}$ agrees with the Friedrichs extension of its restriction to $C^2(S^{n-1})$.

To this end, consider first $f, g \in \text{Dom} \mathcal{A}$. Then

$$\langle g, \mathcal{A} f \rangle_{L^2(S_{B,M})} = \frac{1}{n(n-1)} \sum_{e_{F', F} \in E} \mathcal{H}^{n-2}(F \cap F') \int_{e_{F', F}} (f'' + f) g d\mathcal{H}^1.$$
by definition. Integrating by parts yields
\[ \int_{E_p,p'} (f'' + f) g \, d\mathcal{H}^1 = \int_{E_p,p'} \{fg - f'g'\} \, d\mathcal{H}^1 - \nabla_{n_{p,p'}} f(n_F)g(n_F) - \nabla_{n_{p',p}} f(n_{F'})g(n_{F'}). \]
The vertex boundary conditions of \( \text{Dom} \, \mathcal{A} \) ensure that the boundary terms vanish when we sum over all edges in the expression for \( (g, \mathcal{A} f)_{L^2(S_{B,M})} \). We have therefore shown that \( \mathcal{E}(f,g) = (g, \mathcal{A} f)_{L^2(S_{B,M})} \) for all \( f, g \in \text{Dom} \, \mathcal{A} \). It is now a standard exercise to prove that \( \mathcal{A} \) is self-adjoint and that \( \mathcal{E} \) is the associated closed quadratic form with the given domains; see, e.g., [4, Theorems 1.4.4 and 1.4.11].

Finally, we claim that \( \mathcal{A} \) is the Friedrichs extension of its restriction to \( C^2(S^{n-1}) \). Indeed, by a standard argument (as in the proof of [12, Theorem 7.2.1]), any \( f \in \text{Dom} \, \mathcal{E} \) is the limit with respect to the norm \( \| f \|_{L^2(S_{B,M})} \) of \( C^0(G) \) that are \( C^\infty \) on each edge and constant in a neighborhood of each vertex. But by a classical extension argument [23, Lemma 2.26] any such function may be extended to a \( C^\infty \) function on the entire sphere \( S^{n-1} \). We have therefore shown that \( \mathcal{E} \) is the closure of its restriction to \( C^2(S^{n-1}) \), completing the proof. \( \square \)

We conclude this section with two remarks.

**Remark 4.10** (General mixed volumes). For simplicity, we have restricted attention in this section to the special case of Theorem 4.1 where \( C_1 = \cdots = C_{n-2} = M \). This is the only setting that will be needed in the sequel. However, Theorem 4.7 is readily extended to the setting where \( C_1, \ldots, C_{n-2} \) are arbitrary polytopes. Let us briefly sketch the relevant constructions, leaving the details to the reader.

Let \( C_1, \ldots, C_{n-2} \) be polytopes. Then the polytopes \( M_\lambda := \lambda_1 C_1 + \cdots + \lambda_{n-2} C_{n-2} \) are strongly isomorphic for all choices of \( \lambda_1, \ldots, \lambda_{n-2} > 0 \) [33, Corollary 2.4.12]; in particular, all \( M_\lambda \) induce the same metric graph \( G = (V, E) \), whose vertices are the facet normals of \( M_\lambda = C_1 + \cdots + C_{n-2} \) and whose edges are indexed by the \((n-2)\)-faces of \( M_\lambda \). Now let \( e \in E \), and let \( \tilde{F}_e \) be the associated \((n-2)\)-face of \( M_\lambda \). By Lemma 3.9, there exists for each \( i \) a face \( \tilde{F}_i \) of \( C_i \) such that \( \tilde{F}_e = \tilde{F}_1 + \cdots + \tilde{F}_{n-2} \). It follows from Proposition 4.8 and the definitions in section 3.1 that
\[ \int f \, dS_{K,E} = \frac{1}{n-1} \sum_{e \in E} V(\tilde{F}_1, \ldots, \tilde{F}_{n-2}) \int (h_K - h_{F}) \, f \, d\mathcal{H}^1 \]
for every convex body \( K \) of class \( C^2_+ \), where the mixed volume that appears here is computed in the \((n-2)\)-dimensional subspace spanned by \( \tilde{F}_e \) (modulo translation of the faces \( F_i \)). Using this expression we may readily adapt Theorem 4.7 to the present setting with the same proof. The underlying graph is now the metric graph associated to \( M_1 \), and the weights \( \mathcal{H}^{n-2}(F \cap F') \) are replaced everywhere by the mixed volumes \( V(F_1, \ldots, F_{n-2}) \). Let us note that some of these weights may vanish in the general case, unlike in the more restricted setting of Theorem 4.7.

**Remark 4.11** (The kernel of a quantum graph). Lemma 2.1 states that the study of equality cases in the Minkowski or Alexandrov-Fenchel inequalities is equivalent to understanding \( \text{ker} \, \mathcal{A} \). In the setting of Theorem 4.7, we can attempt to compute this kernel explicitly. Indeed, if \( \mathcal{A} f = 0 \), then we must evidently have \( f|_e(\theta) = a_e \cos(\theta) + b_e \sin(\theta) \) on every edge \( e \in E \). Thus we only need to compute the vector
of coefficients \((a_e, b_e)_{e \in E}\). By using the vertex boundary conditions of \(\text{Dom} \mathcal{A}\), it can be shown that \(f \in \ker \mathcal{A}\) if and only if the vector of coefficients is in the kernel of a certain matrix, cf. [4, section 3.6]. In the present setting, this matrix turns out to be precisely what appears in Alexandrov’s polytope proof of the Alexandrov-Fenchel inequality [1]. This is essentially the idea behind Schneider’s proof of the equality cases of the Alexandrov-Fenchel inequality in the case that \(C_1, \ldots, C_{n-2}\) are strongly isomorphic simple polytopes [33, Theorem 7.6.21].

The polytope setting is rather special, however, and such direct computations cannot be performed for general bodies. One may view the methods that will be developed in subsequent sections as a kind of quantitative replacement for this argument, that enables us to pass to the limiting case of arbitrary convex bodies.

4.3. Dirichlet forms. Our aim in this section is to provide some additional insight into the structure of the operator defined in a somewhat abstract manner by Theorem 4.1. The results of this section will not be used elsewhere in this paper, but are included in order to clarify our general constructions.

In two special cases, when \(C_1, \ldots, C_{n-2}\) are polytopes and when they are bodies of class \(C^2\), we have seen that \(\mathcal{A}\) can be expressed explicitly as a second-order differential operator. Another explicit setting will be encountered in section 7 below. In the general setting, however, it is not so clear what such an operator may look like: it is perfectly possible in general for a limit of second-order differential operators to exhibit quite different behavior [26]. The study of such operators is enabled in a very general setting by the theory of Dirichlet forms [16, 3]. In the present setting, however, we can give a more concrete representation that avoids the abstract theory. The main result of this section is the following.

**Theorem 4.12.** Let \(C_1, \ldots, C_{n-2}\) be arbitrary convex bodies in \(\mathbb{R}^n\) with \(S_{B,C} \neq 0\), and let \(\mathcal{A}\) and \(\mathcal{E}\) be the operator and quadratic form on \(L^2(S_{B,C})\) of Theorem 4.1. Then there exists a measurable function \(A : S^{n-1} \to \text{Sym}_n(\mathbb{R})\) such that\(^3\)

\[
\mathcal{A}f = \frac{1}{n} \text{Tr}[A D^2 f], \quad \mathcal{E}(f, g) = \frac{1}{n} \int \{fg - \langle \nabla_{S^{n-1}} f, A \nabla_{S^{n-1}} g \rangle\} \, dS_{B,C}
\]

for every \(f, g \in C^2(S^{n-1})\), where \(\nabla_{S^{n-1}}\) is the covariant derivative on \(S^{n-1}\).

From a practical viewpoint, the problem with Theorem 4.12 is that we do not have a useful explicit expression for the tensor field \(A\) when \(C_1, \ldots, C_{n-2}\) are arbitrary bodies. For this reason, this result will not be used in the rest of the paper. Nonetheless, Theorem 4.12 shows that the objects in Theorem 4.1 may be viewed rather concretely as highly degenerate elliptic second-order differential operators.

**Remark 4.13.** As we are working on \(L^2(S_{B,C})\), the tensor field \(A\) need only be defined \(S_{B,C}\)-a.e. It may be viewed as a kind of Riemannian metric on \(\text{supp} \ S_{B,C}\), cf. [19]. For example, for polytopes it follows from Remark 4.10 that \(\text{rank} \ A = 1\) a.e., which reflects the fact that the underlying metric graph is one-dimensional.

The main observation behind the proof of Theorem 4.12 is the following.

**Lemma 4.14.** Let \(\ell_i(x) := x_i, i = 1, \ldots, n\). Then following are equivalent.

\(^3\) As in the definition of \(D^2 f\) in section 3.2, we view \(A(x)\) as a matrix acting on the tangent space \(x^+\) at each \(x \in S^{n-1}\) (that is, \(A\) is a measurable \((1, 1)\)-tensor field on \(S^{n-1}\)).
1. There is a map \( A : S^{n-1} \rightarrow \text{Sym}_n(\mathbb{R}) \) such that \( A(x)x = 0 \), \( \text{Tr}[A(x)] = 1 \), and
\[
\mathcal{A} f = \frac{1}{n} \text{Tr}[AD^2 f] \quad \text{for all } f \in C^2(S^{n-1}).
\]

2. We have
\[
dS_{f,c} = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j} \, dS_{\ell_i,\ell_j,c} \quad \text{for all } f \in C^2(S^{n-1}),
\]
where \( \tilde{f} \) is the 1-homogeneous extension of \( f \) to \( \mathbb{R}^n \).

Proof. To prove 2 \( \Rightarrow \) 1, recall that as \( \tilde{f} \) is 1-homogeneous, its Hessian in \( \mathbb{R}^n \) satisfies \( \nabla^2 \tilde{f}(x)x = 0 \) (cf. section 3.2). If we therefore define using Lemma 4.4
\[
\tilde{A}_{ij} := \frac{1}{2} \frac{dS_{\ell_i,\ell_j,c}}{dS_{B,c}}, \quad A(x) := P_{x^\perp} \tilde{A}(x) P_{x^\perp}
\]
where \( P_{x^\perp} \) denotes the orthogonal projection on \( x^\perp \), then clearly
\[
\mathcal{A} f = \frac{1}{n} \frac{dS_{f,c}}{dS_{B,c}} = \frac{1}{n} \text{Tr}[AD^2 f]
\]
by (4.2). That \( \text{Tr}[A(x)] = 1 \) follows by choosing \( f = 1 \) and using that \( \mathcal{A} 1 = \frac{1}{n} 1 \) by Theorem 4.1. Thus the claim is established.

To prove the converse implication 1 \( \Rightarrow \) 2, note first that \( f = \tilde{f}(\ell_1, \ldots, \ell_n) \).
Substituting this into the expression for \( \mathcal{A} f \) and using the chain rule gives
\[
n \mathcal{A} f = f + \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial x_i} \text{Tr}[A \nabla^2_{S^{n-1}} \ell_i] + \sum_{i,j=1}^{n} \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j} \langle \nabla^2_{S^{n-1}} \ell_i, A \nabla^2_{S^{n-1}} \ell_j \rangle,
\]
where we used that \( D^2 f = \nabla^2_{S^{n-1}} f + f I \) and \( \text{Tr}[A(x)] = 1 \). As \( \ell_i \) is a linear function, we obtain \( \text{Tr}[A \nabla^2_{S^{n-1}} \ell_i] = n \mathcal{A} \ell_i - \ell_i = -\ell_i \) by Theorem 4.1. But note that
\[
\sum_{i=1}^{n} \ell_i(x) \frac{\partial \tilde{f}}{\partial x_i}(x) = \langle x, \nabla \tilde{f}(x) \rangle = \tilde{f}(x)
\]
by 1-homogeneity. Thus the first two terms on the right-hand side of (4.3) cancel.
To simplify the last term, we use again the chain rule to write
\[
2 \langle \nabla_{S^{n-1}} \ell_i, A \nabla_{S^{n-1}} \ell_j \rangle = \text{Tr}[A \nabla^2_{S^{n-1}} (\ell_i, \ell_j)] - \ell_i \text{Tr}[A \nabla^2_{S^{n-1}} \ell_j] - \ell_j \text{Tr}[A \nabla^2_{S^{n-1}} \ell_i]
\]
\[
= n \mathcal{A} (\ell_i, \ell_j) + \ell_i \ell_j.
\]
But as
\[
\sum_{i,j=1}^{n} \ell_i(x) \ell_j(x) \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(x) = \langle x, \nabla^2 \tilde{f}(x)x \rangle = 0
\]
by 1-homogeneity, we have shown that
\[
\mathcal{A} f = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j} \mathcal{A} (\ell_i, \ell_j).
\]

The proof is concluded by invoking again (4.2). \( \square \)
Proof of Theorem 4.12. Suppose first that \( C_1, \ldots, C_{n-2} \) are convex bodies of class \( C^2_+ \). Then we obtain as in the proof of Lemma 4.4

\[
\mathcal{A} f = \frac{1}{n} D(Df, D^2h_{C_1}, \ldots, D^2h_{C_{n-2}}) \text{ for } f \in C^2(S^{n-1}).
\]

By linearity of mixed discriminants, this expression may be written as

\[
\mathcal{A} f = \frac{1}{n} \text{Tr}[A D^2f]
\]

for some \( A : S^{n-1} \to \text{Sym}_n(\mathbb{R}) \). As \( D^2f \) only acts on \( x^\perp \), we may choose \( A(x)x = 0 \) without loss of generality. Moreover, \( A(x) \geq 0 \) follows from Lemma 3.8(d). That \( \text{Tr}[A(x)] = 1 \) follows by substituting \( D^2f \mapsto I \) in the above expressions for \( \mathcal{A} \).

Now let \( C_1, \ldots, C_{n-2} \) are arbitrary convex bodies. Note that by Theorem 3.3, the second condition of Lemma 4.14 is preserved by taking Hausdorff limits. As we have established above the first condition of Lemma 4.14 for bodies of class \( C^2_+ \), we may extend this conclusion to arbitrary bodies by approximation as in the proof of Lemma 4.6. We have therefore shown that

\[
\mathcal{A} f = \frac{1}{n} \text{Tr}[A D^2f]
\]

for some \( A : S^{n-1} \to \text{Sym}_n(\mathbb{R}) \) such that \( A(x)x = 0 \) and \( \text{Tr}[A(x)] = 1 \).

We now show that \( A(x) \geq 0 \), that is, that \( \mathcal{A} \) is elliptic. To this end, define the quadratic function \( q_v(x) := \langle v, x \rangle^2 \) for \( v \in \mathbb{R}^n \). Then

\[
\nabla^2 q_v(x) = 2P_{x^\perp} v (P_{x^\perp} v)^* - 2q_v(x) I,
\]

where \( P_{x^\perp} \) is the orthogonal projection on \( x^\perp \). It follows that

\[
\mathcal{A} q_v + \frac{1}{n} q_v = \frac{1}{n} \text{Tr}[A D^2 q_v] + \frac{1}{n} q_v = \frac{2}{n} \langle v, Av \rangle,
\]

where we used \( A(x)x = 0 \) and \( \text{Tr}[A(x)] = 1 \). Thus the ellipticity condition \( A \geq 0 \) is equivalent to the statement that \( \mathcal{A} q_v + \frac{1}{n} q_v \geq 0 \) pointwise for every \( v \in \mathbb{R}^n \), or, equivalently by (4.2), that \( d\mu_v := dS_{q_v, C} + q_v dS_{B, C} \) is a (nonnegative) measure for every \( v \in \mathbb{R}^n \). But we have already shown ellipticity in the case that \( C_1, \ldots, C_{n-2} \) are of class \( C^2_+ \), and the nonnegativity of \( \mu_v \) is preserved by Hausdorff convergence due to Theorem 3.3. We can therefore establish ellipticity for arbitrary bodies by approximation as in the proof of Lemma 4.6.

It remains to compute the quadratic form associated to \( \mathcal{A} \). Note that

\[
\mathcal{E}(f, g) = (f, \mathcal{A} g)_{L^2(S_B, C)} = \frac{1}{2} \left\{ f \mathcal{A} g + g \mathcal{A} f - \mathcal{A}(fg) + \frac{1}{n} fg \right\} dS_{B, C}
\]

for \( f, g \in C^2(S^{n-1}) \), where we have used that \( \mathcal{A} \) is self-adjoint and \( \mathcal{A} 1 = \frac{1}{n} 1 \). But

\[
\mathcal{A}(fg) = \frac{1}{n} \{ \text{Tr}[A \nabla^2_{S^{n-1}}(fg)] + fg \} = \frac{2}{n} \langle \nabla_{S^{n-1}} f, A \nabla_{S^{n-1}} g \rangle + f \mathcal{A} g + g \mathcal{A} f - \frac{1}{n} fg
\]

by the product rule. We have therefore shown that

\[
\mathcal{E}(f, g) = \frac{1}{n} \int \{ fg - \langle \nabla_{S^{n-1}} f, A \nabla_{S^{n-1}} g \rangle \} dS_{B, C}
\]

for \( f, g \in C^2(S^{n-1}) \). The proof is complete. \( \square \)
5. A weak stability theorem

The aim of this section is to prove a weak stability result that will be used in section 6 below as input to the main part of the proof of Theorem 1.3. The main result of this section is the following quantitative form of Theorem 2.3; the latter follows immediately by combining the following theorem with Theorem 3.4.

**Theorem 5.1.** Let $M$ be a convex body in $\mathbb{R}^n$ with $0 \in \text{int} M$. Then there is a constant $C_M > 0$, depending only on $M$ and on the dimension $n$, so that

$$V(K, L, M)^2 \geq V(K, K, M) V(L, L, M) + C_M V(L, L, M) \inf_{v \in \mathbb{R}^n, a \geq 0} \int (h_K - ah_L - \langle v, \cdot \rangle)^2 \frac{dS_{M,M}}{h_M}$$

for all convex bodies $K, L$ in $\mathbb{R}^n$.

Theorem 5.1 is different in spirit than most of the theory developed in this paper, in that it is $S_{M,M}$ rather than $S_{B,M}$ that appears here as the reference measure. As $\text{supp} S_{M,M}$ is much smaller than $\text{supp} S_{B,M}$, only weak information on the extremals may be extracted from this result (cf. Example 2.4). Nonetheless, this weak information provides crucial input to the quantitative rigidity analysis of the following section, and we therefore develop it first.

The idea behind the proof of Theorem 5.1 is as follows. Lemma 2.1 shows that Minkowski's quadratic inequality follows if one can show that an associated self-adjoint operator has a one-dimensional positive eigenspace. One may readily modify the proof of this fact to show that if, in addition, the zero eigenvalue of the operator is separated from the rest of the spectrum by a positive gap, then one obtains a quantitative improvement along the lines of Theorem 5.1.

It was recently observed by Kolesnikov and Milman [22] in their study of local $L^p$-Brunn-Minkowski inequalities that, in the case where all bodies involved are smooth and symmetric, such a spectral separation may be established by means of a differential-geometric technique (albeit when the operator is normalized differently than in Theorem 4.1, which is responsible for the presence of $S_{M,M}$ rather than $S_{B,M}$ in Theorem 5.1). The approach of [22] may also be used to obtain a quantitative result for non-symmetric bodies, as we will show in section 5.1. In this case, however, the resulting bound no longer has a direct spectral interpretation. Nonetheless, we will show in section 5.2 that the requisite spectral property can be recovered, for a suitably normalized operator, using the min-max principle. Theorem 5.1 then follows for smooth bodies by reasoning as in the proof Lemma 2.1, and the proof is concluded by smooth approximation. The background from Riemannian geometry that is needed in this section may be found, e.g., in [9].

**Remark 5.2.** By exploiting the linear equivariance of the quantities that appear in Theorem 5.1, it may be shown that the constant $C_M$ can in fact be chosen to depend on the dimension $n$ only. This property, which is important in the theory of [22], is not relevant in our setting. We therefore do not repeat the arguments leading to this observation, and refer the interested reader to [22].

5.1. An extrinsic formulation. Behind the proof of Theorem 5.1 lies a different perspective on mixed volumes than we have encountered so far. By definition we have $V(K, K, M) = \frac{1}{n(n-1)} \frac{d}{dt} \text{Vol}(M + tK)|_{t=0}$, so that mixed volumes of this kind may be viewed as arising from the second variation of the volume of the convex...
body $M$. First and second variation formulae play a classical role in Riemannian geometry, for example, in the theory of minimal surfaces [9, 10]. In this setting, however, the variation formulae are not expressed on the sphere as in Lemma 3.7, but rather in terms of the extrinsic geometry of $\partial M$ viewed as a hypersurface in $\mathbb{R}^n$. These two viewpoints are related by using the outer normal map $n_M$ as a change of variables, as we did in section 3.2. The change of perspective is useful, however, as it enables us to exploit classical techniques from Riemannian geometry.

**Remark 5.3.** This is the main point in this paper where we rely specifically on the restricted setting of Minkowski’s second inequality, as opposed to the general Alexandrov-Fenchel inequality: the mixed volumes that appear in (1.4) are precisely those that arise as second variations of the volume of $M$. For general mixed volumes, the body $M$ no longer plays any distinguished role, and it seems unlikely that the methods of this section could be useful in this context. In contrast, the techniques of sections 4, 6, and 7 do not appear to be fundamentally tied to the special setting of (1.4), and could potentially be adapted to a much more general context.

Throughout this section, we will work with convex bodies $K, L, M$ in $\mathbb{R}^n$ of class $C^\infty_+$. We will also assume that $0 \in \text{int} \ M$, so that $h_M > 0$. We denote by $\Pi := \nabla n_M$ the second fundamental form of $\partial M$ (viewed, as usual, as a symmetric linear map $\Pi(x) : T_x \partial M \to T_x \partial M$). As in previous sections, the symbols $\nabla$, $\nabla_{S^{n-1}}$, and $\nabla_{\partial M}$ denote covariant differentiation in $\mathbb{R}^n$, $S^{n-1}$, and $\partial M$, respectively.

We begin by making explicit the second variation formula alluded to above. Following [11], we will derive the formula by a change of variables.

**Lemma 5.4.** Let $K, L, M$ be convex bodies in $\mathbb{R}^n$ of class $C^\infty_+$. Then

$$n(n - 1) V(K, L, M) = \int_{\partial M} \left( h_K \circ n_M \right) (h_L \circ n_M) \text{Tr}[\Pi] \, dx$$

$$- \int_{\partial M} \langle \nabla_{\partial M} (h_K \circ n_M), \Pi^{-1} \nabla_{\partial M} (h_L \circ n_M) \rangle \, dx.$$ 

**Proof.** First note that by Lemma 3.7 and Lemma 3.8(b), we have

$$V(K, L, M) = \frac{1}{n(n-1)} \int_{S^{n-1}} h_K \text{Tr}[\text{cof}(D^2 h_M) D^2 h_L] \, d\omega.$$ 

Let $\mathcal{L} f := \text{Tr}[\text{cof}(D^2 h_M) D^2 f]$. It follows from the symmetry of mixed volumes that $\mathcal{L}$ is a symmetric operator on $C^2(S^{n-1}) \subset L^2(\omega)$. We may therefore write

$$n(n - 1) V(K, L, M)$$

$$= \frac{1}{2} \int_{S^{n-1}} \{ h_K \mathcal{L} h_L + h_L \mathcal{L} h_K - \mathcal{L} \langle h_K h_L \rangle + h_K h_L \} \, d\omega$$

$$= \int_{S^{n-1}} \{ h_K h_L \text{Tr}[\text{cof}(D^2 h_M)] - \langle \nabla_{S^{n-1}} h_K, \text{cof}(D^2 h_M) \nabla_{S^{n-1}} h_L \rangle \} \, d\omega,$$

where we used $D^2 f = \nabla^2_{S^{n-1}} f + f I$ and the product rule.

Now note that as $n_M^{-1} = \nabla h_M$, we have

$$D^2 h_M \circ n_M = \nabla n_M^{-1} \circ n_M = (\nabla n_M)^{-1} = \Pi^{-1}.$$

Similarly, we can compute by the chain rule

$$\nabla_{S^{n-1}} f \circ n_M = \Pi^{-1} \nabla_{\partial M} (f \circ n_M).$$
Finally, as $D^2 h_M > 0$, we may write

$$\text{cof}(D^2 h_M) = (D^2 h_M)^{-1} \det(D^2 h_M).$$

The proof is completed by changing variables according to $n_M^{-1}$ in the expression for $V(K, L, M)$ and using the above identities. \(\square\)

The quantities that appear in Lemma 5.4 are strongly reminiscent of the following classical formula of Reilly, obtained by integrating the Bochner formula on a manifold with boundary; for the proof, we refer to [9, Lemma A.17].

**Lemma 5.5.** Let $M \subset \mathbb{R}^n$ be a compact set with $C^\infty$ boundary. Then

$$\int_M (\Delta u)^2 \, dx = \int_M \text{Tr}[(\nabla^2 u)^2] \, dx + \int_{\partial M} \{2\text{Tr}[\Pi] u_n^2 + \langle \nabla_{\partial M} u, \Pi \nabla_{\partial M} u \rangle - 2 \langle \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \} \, dx$$

for any $u \in C^\infty(M)$, where we defined the normal derivative $u_n := \langle n_M, \nabla u \rangle$.

That Minkowski’s inequality may be deduced from Reilly’s formula was observed in a special case by Reilly himself [30], and more generally by Kolesnikov and Milman [21]. In particular, it was noticed in [22] that the latter proof admits a quantitative improvement when the bodies are symmetric. The following result is a straightforward adaptation of [22, section 6] to the non-symmetric case.

**Proposition 5.6.** Let $M$ be a convex body in $\mathbb{R}^n$ of class $C^\infty$ with $0 \in M$. Then

$$\int_{\partial M} \langle \nabla_{\partial M} g, \Pi^{-1} \nabla_{\partial M} g \rangle \, dx \geq \frac{1}{n+2} \frac{r^2}{R^2} \int_{\partial M} \frac{g^2}{h_M \circ n_M} \, dx$$

for any $C^\infty$ function $g : \partial M \to \mathbb{R}$ such that $\int_{\partial M} g(x) \, dx = 0$ and $\int_{\partial M} xg(x) \, dx = 0$, where $r, R > 0$ are chosen such that $rB \subseteq M \subseteq RB$.

**Proof.** As $\int_{\partial M} g \, dx = 0$, there exists a $C^\infty$ solution $u$ to the Neumann problem

$$\Delta u = 0 \quad \text{on } M,$$

$$u_n = g \quad \text{on } \partial M,$$

see, e.g., [35, section 5.7]. Applying Lemma 5.5 and minimizing over $\nabla_{\partial M} u$ yields

$$0 = \int_M \text{Tr}[(\nabla^2 u)^2] \, dx + \int_{\partial M} \{g^2 \text{Tr}[\Pi] + \langle \nabla_{\partial M} u, \Pi \nabla_{\partial M} u \rangle - 2 \langle \nabla_{\partial M} u, \nabla_{\partial M} u \rangle \} \, dx \geq \int_M \text{Tr}[(\nabla^2 u)^2] \, dx + \int_{\partial M} \{g^2 \text{Tr}[\Pi] - \langle \nabla_{\partial M} g, \Pi^{-1} \nabla_{\partial M} g \rangle \} \, dx,$$

where we used that $\Pi > 0$ as $M$ is a body of class $C^\infty$. To complete the proof, it remains to lower bound the first term on the right-hand side.

To this end, note that $\text{div}(\langle v, x \rangle \nabla u) = \langle v, \nabla u \rangle$ for any $v \in \mathbb{R}^n$, as $\Delta u = 0$. Thus

$$\int_M \nabla u(x) \, dx = \int_{\partial M} xu_n(x) \, dx = \int_{\partial M} xg(x) \, dx = 0.$$

By a classical Poincaré inequality of Payne and Weinberger [27], it follows that

$$\int_M \text{Tr}[(\nabla^2 u)^2] \, dx \geq \frac{\pi^2}{4R^2} \int_M \|\nabla u\|^2 \, dx.$$
As \( r \leq h_M \leq R \) and \( h_M(n_M(x)) = \langle x, n_M(x) \rangle \), we can now estimate
\[
R^2 \int_{\partial M} g^2 h_M^\circ n_M \, dx \leq \int_{\partial M} u^2 \langle x, n_M \rangle \, dx \leq \int_{\partial M} \|\nabla u\|^2 \langle x, n_M \rangle \, dx
\]
\[
= \int_M \text{div}(x\|\nabla u\|^2) \, dx
\]
\[
\leq (n + 1) \int_M \|\nabla u\|^2 \, dx + R^2 \int_M \text{Tr}[(\nabla^2 u)^2] \, dx
\]
\[
\leq \left(1 + \frac{4(n+1)}{\pi^2}\right) R^2 \int_M \text{Tr}[(\nabla^2 u)^2] \, dx,
\]
where we used the divergence theorem and
\[
\text{div}(x\|\nabla u\|^2) = n\|\nabla u\|^2 + 2\langle \nabla u, \nabla^2 u \rangle \leq (n + 1)\|\nabla u\|^2 + \|\nabla^2 u\|^2.
\]
The proof follows readily by combining the above estimates (for aesthetic reasons, we have estimated \(1 + \frac{4(n+1)}{\pi^2} \leq n + 2\) in the statement). \(\square\)

Lemma 5.4 shows that the left-hand side of the expression in Proposition 5.6 is nothing other than a mixed volume, but the significance of the inequality may not be immediately obvious. We will presently see that Proposition 5.6 controls a gap in the spectrum of a certain self-adjoint operator associated to mixed volumes (different from the one in Theorem 4.1). This has a number of interesting consequences. For symmetric bodies, it implies a local form of the \( L^p \)-Brunn-Minkowski inequality with \( p < 1 \), which is the problem investigated in [22]. In the present setting, the relevant spectral property forms the basis for the proof of Theorem 5.1.

5.2. Proof of Theorem 5.1. Let us fix, for the time being, a convex body \( M \) in \( \mathbb{R}^n \) of class \( C_+^\infty \) with \( 0 \in \text{int} M \). Define an operator and a measure on \( S^{n-1} \) by
\[
\tilde{A} f := \frac{1}{n(n-1)} h_M \text{Tr}[(D^2 h_M)^{-1} D^2 f], \quad \tilde{\nu} := \frac{\text{det}(D^2 h_M)}{h_M} \, d\omega
\]
for \( f \in C^\infty(S^{n-1}) \). By Lemma 3.7 and Lemma 3.8(b), we have
\[
V(f, g, M) = \langle f, \tilde{A} g \rangle_{L^2(\tilde{\nu})} \text{ for } f, g \in C^\infty(S^{n-1}).
\]
Consequently, we observe the following basic facts:
- \( \tilde{A} \) is a uniformly elliptic operator (as \( D^2 h_M > 0 \)).
- \( \tilde{A} \) defines a symmetric quadratic form \( \langle f, \tilde{A} g \rangle_{L^2(\tilde{\nu})} \) for \( f, g \in C^\infty(S^{n-1}) \).
- \( \tilde{A} \) is essentially self-adjoint and has compact resolvent. Moreover, as \( \tilde{A} h_M = \frac{1}{n} h_M \), its largest eigenvalue is \( \frac{1}{n} \) and this eigenvalue is simple. (This follows from standard elliptic regularity theory [17, section 8.12] and [12, Lemma 1.2.2].)
- \( \tilde{A} \ell = 0 \) for any linear function \( \ell(x) = \langle v, x \rangle \), \( v \in \mathbb{R}^n \).

It should be emphasized that the normalization chosen in the definition of \( \tilde{A} \) is very different than the one employed in section 4; in particular, the present operator makes sense only for smooth bodies \( M \), and does not give rise to a well-behaved limiting operator for arbitrary (non-smooth) bodies. Nonetheless, the present normalization is the appropriate one for exploiting Proposition 5.6.
Lemma 5.7. Let $M$ be a convex body in $\mathbb{R}^n$ of class $C^\infty_+$ with $rB \subseteq M \subseteq RB$. Whenever $f \in C^\infty(S^{n-1})$ satisfies $f \perp \text{span}\{h_M, \ell : \ell \text{ is linear}\}$ in $L^2(\nu)$, we have

$$\langle f, \tilde{A} f \rangle_{L^2(\nu)} \leq - \frac{1}{n(n-1)(n+2)} \frac{r^2}{R^2} \|f\|_{L^2(\nu)}^2.$$ 

Proof. The statement is spectral in nature. As $\tilde{A}$ has a compact resolvent, it has a discrete spectrum and a complete set of eigenfunctions [17, section 8.12]. As stated above, the largest eigenvalue of $\tilde{A}$ is $\frac{1}{n}$ and its one-dimensional eigenspace is spanned by $h_M$. Moreover, Minkowski’s inequality implies as in the proof of Theorem 4.1 that $\text{spec } \tilde{A} \subseteq (-\infty, 0] \cup \{\frac{1}{n}\}$, and all linear functions are eigenfunctions with eigenvalue 0. Therefore, by the min-max theorem [29, section XIII.1],

$$\sup_{f \perp \text{span}\{h_M, \ell : \ell \text{ is linear}\}} \frac{\langle f, \tilde{A} f \rangle_{L^2(\nu)}}{\|f\|_{L^2(\nu)}^2} \leq \sup_{f \perp L} \frac{\langle f, \tilde{A} f \rangle_{L^2(\nu)}}{\|f\|_{L^2(\nu)}^2}$$

for any linear space $L \subset C^\infty(S^{n-1})$ with $\dim L \leq n + 1$.

Now let $f : S^{n-1} \rightarrow \mathbb{R}$ be $C^\infty$. Choosing $g := f \circ n_M$ in Proposition 5.6 and changing variables as in Lemma 5.4, we find that

$$n(n-1) \langle f, \tilde{A} f \rangle_{L^2(\nu)} = n(n-1) \mathcal{V}(f, f, \mathcal{M}) \leq - \frac{1}{n(n-1)(n+2)} \frac{r^2}{R^2} \int_0^1 f h_M \, d\nu = 0.$$ 

Whenever

$$\int_0^1 f h_M \, d\nu = 0, \quad \int_0^1 f h_M \, d\nu = 0.$$ 

If we therefore choose $L = \text{span}\{h_M, h_M \langle v, \nabla h_M \rangle : v \in \mathbb{R}^n\}$, then we have shown

$$\sup_{f \perp L} \frac{\langle f, \tilde{A} f \rangle_{L^2(\nu)}}{\|f\|_{L^2(\nu)}^2} \leq - \frac{1}{n(n-1)(n+2)} \frac{r^2}{R^2}.$$ 

As clearly $\dim L \leq n + 1$, the proof is complete. \hfill \square

We are now ready to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. We may clearly assume that $\mathcal{V}(L, L, \mathcal{M}) > 0$, as otherwise the statement reduces to Minkowski’s quadratic inequality. Then $\mathcal{V}(L, M, \mathcal{M}) > 0$ as well, by Minkowski’s inequality $\mathcal{V}(L, M, \mathcal{M})^2 \geq \mathcal{V}(L, L, \mathcal{M}) \text{Vol}(M) > 0$.

Let us first consider the case that $M$ is of class $C^\infty_+$ with $rB \subseteq M \subseteq RB$ for $r, R > 0$, and that $K, L$ are of class $C^\infty_\infty$. As $\langle h_L, h_M \rangle_{L^2(\nu)} = n\mathcal{V}(L, M, \mathcal{M}) > 0$, we may clearly choose $a \geq 0$ and $v \in \mathbb{R}^n$ such that

$$\delta := h_K - ah_L - \langle v, \cdot \rangle \perp \text{span}\{h_M, \ell : \ell \text{ is linear}\} \text{ in } L^2(\nu).$$

Applying Lemma 5.7 and translation-invariance of mixed volumes yields

$$- \frac{1}{n(n-1)(n+2)} \frac{r^2}{R^2} \|\delta\|_{L^2(\nu)}^2 \geq \mathcal{V}(\delta, \delta, \mathcal{M})$$

$$= \mathcal{V}(K, K, \mathcal{M}) - 2a\mathcal{V}(K, L, \mathcal{M}) + a^2\mathcal{V}(L, L, \mathcal{M})$$

$$\geq \mathcal{V}(K, K, \mathcal{M}) - \frac{\mathcal{V}(K, L, \mathcal{M})^2}{\mathcal{V}(L, L, \mathcal{M})},$$

where we minimized over $a$ in the last inequality. The conclusion follows readily.

We now consider the general case where $K, L, M$ are arbitrary convex bodies in $\mathbb{R}^n$ and $0 \in \text{int } M$. It is classical [33, section 3.4] that we may choose convex bodies $K^{(s)}, L^{(s)}, M^{(s)}$ of class $C^\infty_+$ such that $K^{(s)} \rightarrow K$, $L^{(s)} \rightarrow L$, $M^{(s)} \rightarrow M$ as $s \rightarrow \infty$. This completes the proof of Theorem 5.1.
in the sense of Hausdorff convergence. Note that as $0 \in \text{int } M$, there exist $r, R > 0$ such that $rB \subseteq M(s) \subseteq RB$ for all $s$ sufficiently large. Thus we have shown that

$$V(K(s), L(s), M(s))^2 \geq V(K(s), R(s), M(s)) V(L(s), L(s), M(s)) + C_M V(L(s), L(s), M(s)) \int (h_K(s) - a(s) h_L(s) - \langle h(s), \cdot \rangle)^2 dS_{M(s), M(s)}$$

for all $s$ sufficiently large, where $a(s), v(s)$ are chosen as in the first part of the proof and we defined $C_M := r^2/(R^2 n(n - 1)(n + 2))$.

We would like to take $s \to \infty$ in the above inequality to conclude the proof of Theorem 5.1. Convergence of the mixed volumes follows directly from Theorem 3.3. In addition, convergence of the integral would follow from Theorem 3.3 if we could show that $a(s) \to a$ and $v(s) \to v$ for some $a \geq 0, v \in \mathbb{R}^n$, as then the integrand would converge uniformly to $(h_K - ah_L - \langle v, \cdot \rangle)^2/h_M$.

To establish the requisite convergence, let us examine the explicit expressions for $a(s)$ and $v(s)$. Setting $d\tilde{b}(s) := dS_{M(s), M(s)}/\tilde{h}_M(s)$, we clearly have

$$a(s) = \frac{(h_K(s), h_M(s))_{L^2(\rho^{(s)})}}{(h_L(s), h_M(s))_{L^2(\rho^{(s)})}} = \frac{V(K(s), M(s), M(s))}{V(L(s), M(s), M(s))},$$

which converges as $s \to \infty$ by Theorem 3.3 and $V(L, M, M) > 0$.

To proceed, consider first for any body $M$ with $0 \in \text{int } M$ the matrix

$$G_M := \int xx^* \frac{S_{M,M}(dx)}{h_M(x)}.$$

We claim that $G_M > 0$. Indeed, if not, then by Theorem 3.4 there exists $w \in \mathbb{R}^n \setminus \{0\}$ such that $\langle w, x \rangle = 0$ for all 0-extreme normal vectors $x$ of $M$. But this is impossible, as a convex body with nonempty interior is the intersection of its regular supporting halfspaces [33, Theorem 2.2.6]; if the normals of all these halfspaces were orthogonal to $w$, then $M$ would be noncompact, contradicting the definition of a convex body. With this observation in hand, it follows readily that

$$v(s) = \int (h_K(s)(x) - a(s) h_L(s)(x)) G^{-1}_{M(s)} x \frac{S_{M(s), M(s)}(dx)}{h_M(s)(x)},$$

which converges as $s \to \infty$ by Theorem 3.3.

\[ \square \]

6. A QUANTITATIVE RIGIDITY THEOREM

The weak stability result of the previous section implies that if equality holds in (1.4) (and $M$ has nonempty interior and $V(L, L, M) > 0$) then, up to homothety, $K$ and $L$ have the same supporting hyperplanes in the 0-extreme normal directions of $M$. This is however far from characterizing the extremals of Minkowski’s quadratic inequality, as was illustrated in Example 2.4. Nonetheless, we will show that this weak information can be amplified to recover the full equality cases, because the extremals in Minkowski’s inequality turn out to be very rigid: once they are fixed in the 0-extreme directions of $M$, their extension to the 1-extreme directions of $M$ is uniquely determined. This rigidity property, formulated above as Theorem 2.5, lies at the heart of our proof of Theorem 1.3.

Theorem 2.5 is an immediate consequence of the following quantitative result.
Theorem 6.1. Let $M$ be a convex body in $\mathbb{R}^n$ with nonempty interior. There exist $C_M > 0$ and a measure $\mu_M$, depending only on $M$, so that $\text{supp} \mu_M \subseteq \text{supp} S_{M,M}$ and

$$V(K, L, M)^2 \geq V(K, K, M)V(L, L, M) + C_M V(L, L, M)\{\|h_K - h_L\|_{L^2(S_{B,M})}^2 - \|h_K - h_L\|_{L^2(\mu_M)}^2\}$$

for all convex bodies $K, L$ in $\mathbb{R}^n$.

The formulation of Theorem 6.1 is subtle due to the measure $\mu_M$ appearing here. As will be explained in the proof, this measure does not appear to have a canonical geometric interpretation. Ideally, one would have liked to prove a version of Theorem 6.1 where $\mu_M$ is replaced by the measure $S_{M,M}/h_M$ that appears in Theorem 5.1. If this were possible, then one would even obtain a sharp quantitative analogue of Theorem 1.3 (that is, a stability form of Minkowski’s quadratic inequality). It is far from clear, however, how such a result might be proved: we do not know how to directly relate the measures $S_{M,M}$ and $S_{B,M}$. Fortunately, to characterize the extremals it suffices to work with the measure $\mu_M$ in Theorem 6.1, which may be viewed as a projection of $S_{B,M}$ on the 0-extreme normal vectors of $M$.

Remark 6.2. Spectrally, a stability form of Minkowski’s quadratic inequality

$$V(K, L, M)^2 \geq V(K, K, M)V(L, L, M) + C_M V(L, L, M)\inf_{v \in \mathbb{R}^n, a \geq 0} \|h_K - ah_L - \langle v, \cdot \rangle\|_{L^2(S_{B,M})}^2$$

may be shown as in the proof of Lemma 2.1 to be equivalent to the following: (i) the kernel of the operator $\mathcal{A}$ in Theorem 4.1 consists only of linear functions (which characterizes the extremals); and (ii) the remainder of the spectrum is separated from zero by a positive constant (which quantifies the deficit). If $\mathcal{A}$ were to have compact resolvent, then (ii) would follow directly from (i) by discreteness of the spectrum. Unfortunately, as we will see in section 7, it is not true in general that $\mathcal{A}$ has compact resolvent. For this reason, it is far from clear whether we might expect even in principle to replace $\mu_M$ by $S_{M,M}/h_M$ in Theorem 6.1. Understanding the answer to this question would be of considerable interest.

The rest of this section is organized as follows. In section 6.1, we complete the proof of Theorem 1.3 using Theorems 5.1 and 6.1. Sections 6.2 and 6.3 are devoted to the proof of Theorem 6.1. In section 6.2, we consider the special case where $M$ is a polytope. We then extend the conclusion to general bodies $M$ in section 6.3.

6.1. Proof of Theorem 1.3. Before we proceed to the proof of Theorem 6.1, let us show how Theorems 5.1 and 6.1 combine to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. In the following, we assume that $K, L, M$ are convex bodies in $\mathbb{R}^n$ such that $M$ has nonempty interior and $V(L, L, M) > 0$.

Suppose first that there exist $a \geq 0$ and $v \in \mathbb{R}^n$ so that $K$ and $aL + v$ have the same supporting hyperplanes in all 1-extreme normal directions of $M$. Then

$$h_K - ah_L = \langle v, \cdot \rangle \quad S_{B,M}\text{-a.e.}$$

by Theorem 3.4. Therefore, denoting by $\mathcal{A}$ and $\mathcal{E}$ the operator and quadratic form of Theorem 4.1, we have $h_K - ah_L \in \ker \mathcal{A}$ and $\mathcal{E}(h_L, h_L) = V(L, L, M) > 0$. Thus equality in (1.4) follows from Lemma 2.1.
Conversely, suppose that we have equality in (1.4). By translation-invariance of mixed volumes, we may assume without loss of generality that $0 \in \text{int} \, M$. Then Theorem 5.1 implies that there exist $a \geq 0$ and $v \in \mathbb{R}^n$ such that
\[ \delta := h_K - ah_L - \langle v, \cdot \rangle = 0 \quad S_{M,A} \text{-a.e.} \]
By continuity, it follows that $\delta$ vanishes on $\text{supp} \, \mu_M \subseteq \text{supp} \, S_{M,A}$, where $\mu_M$ is as in Theorem 6.1. Consequently, applying Theorem 6.1 with $L \mapsto aL + v$ yields
\[ \delta = h_K - ah_L - \langle v, \cdot \rangle = 0 \quad S_{B,M} \text{-a.e.}, \]
where we have used the invariance of Minkowski’s quadratic inequality under translation and scaling of $L$. By continuity, it follows that $\delta$ vanishes on $\text{supp} \, S_{B,M}$. Thus Theorem 3.4 implies that $K$ and $aL + v$ have the same supporting hyperplanes in all 1-extreme normal directions of $M$, completing the proof. \hfill $\square$

6.2. Proof of Theorem 6.1: polytopes. In this section we consider the case that $M$ is a polytope with nonempty interior. At a qualitative level, the rigidity property of the extremals in Minkowski’s inequality admits in this case a very intuitive interpretation. Suppose we have equality in (1.4), so that $f := h_K - ah_L \in \ker a^\dagger$ for some $a \geq 0$. Suppose in addition that we have fixed the values of $f$ in the 0-extreme normal directions of $M$, which are in this case the vertices of the metric graph associated to $M$. Then it follows from Theorem 4.7 that $f$ solves the Dirichlet problem $f'' + f = 0$ on each edge of the metric graph with boundary data on the vertices. It is readily verified by explicit computation that this one-dimensional Dirichlet problem has a unique solution as long as the lengths of all the edges are less than $\pi$, which must be the case as $M$ has nonempty interior. Thus the value of $f$ is uniquely determined on the 1-extreme normal directions of $M$ once we have fixed its values on the 0-extreme normal directions.

This intuitive argument appears to be rather special to the case of polytopes: for general bodies $M$, the structure of the sets of 0- and 1-extreme normal vectors can be highly irregular, and it is far from clear even how to make sense of the Dirichlet problem in this setting. Instead, we will proceed by developing a quantitative formulation of the above intuition for polytopes. The key point is to find the “right” formulation that does not degenerate when we approximate an arbitrary convex body $M$ by polytopes. Once such a formulation has been found, we will be able to extend its conclusion to the general setting by taking limits.

We now proceed to make these ideas precise. In the rest of this subsection, $M$ will be a polytope in $\mathbb{R}^n$ with nonempty interior, and we adopt without further comment the definitions and notation of section 4.2. Our starting point is the following Poincaré-type inequality on a single edge of the metric graph.

**Lemma 6.3.** Let $M$ be a polytope in $\mathbb{R}^n$ with nonempty interior, and let $F \sim F'$ be neighboring facets. Then for any function $f \in H^1(e_{F,F'})$ and $0 < \varepsilon < 1$, we have
\[ l_{F,F'}^2 \int_{e_{F,F'}} |f|^2 \, d\mathcal{H}^1 \geq (1 - \varepsilon)^2 \pi^2 \int_{e_{F,F'}} f^2 \, d\mathcal{H}^1 - \frac{2}{\varepsilon} l_{F,F'} \{ f(n_F)^2 + f(n_{F'})^2 \}. \]

**Proof.** Assume without loss of generality that $F \leq F'$, and recall that we parameterize functions $f : e_{F,F'} \to \mathbb{R}$ as $f(\theta)$ for $\theta \in [0, l_{F,F'}]$, where $\theta = 0$ corresponds to vertex $n_F$ and $\theta = l_{F,F'}$ corresponds to vertex $n_{F'}$. Define the function
\[ y(\theta) := \cos \left( \frac{(1 - \varepsilon) \pi}{l_{F,F'}} \left( \frac{\theta - l_{F,F'}}{2} \right) \right), \]
and note that \( y(\theta) > 0 \) for \( \theta \in [0, l_{F,F'}] \). Defining \( g := f/y \), we compute

\[
\int_0^{l_{F,F'}} (f')^2 \, d\theta = \int_0^{l_{F,F'}} \{(g')^2 y^2 + g^2 (y')^2 + (g') y y' \} \, d\theta
= \int_0^{l_{F,F'}} \{(g')^2 y^2 - g^2 y y'' \} \, d\theta + g^2 y y' \bigg|_0^{l_{F,F'}} ,
\]

where we integrated the last term by parts. But note that

\[
y y'' = -\frac{(1 - \varepsilon)^2 \pi^2}{l_{F,F'}^2} y^2 , \quad \frac{y'}{y}(0) = -\frac{y'}{y}(l_{F,F'}) = \frac{(1 - \varepsilon)\pi}{l_{F,F'}} \tan \left( \frac{(1 - \varepsilon)\pi}{2} \right) \leq \frac{2}{l_{F,F'}\varepsilon} .
\]

It follows that

\[
\int_0^{l_{F,F'}} (f')^2 \, d\theta \geq \frac{(1 - \varepsilon)^2 \pi^2}{l_{F,F'}^2} \int_0^{l_{F,F'}} f^2 \, d\theta - \frac{2}{l_{F,F'}\varepsilon} \{f(l_{F,F'})^2 + f(0)^2\} .
\]

Rearranging this expression yields the conclusion. \( \square \)

**Remark 6.4.** Let us note that Lemma 6.3 may indeed be viewed as a quantitative formulation of uniqueness of the Dirichlet problem on an edge. Indeed, suppose \( f_1, f_2 \) both satisfy \( f'' + f = 0 \) on \( e_{F,F'} \), and that \( f_1, f_2 \) agree on the vertices \( n_F, n_{F'} \). Then applying Lemma 6.3 to \( f = f_1 - f_2 \) and letting \( \varepsilon \to 0 \) yields

\[
0 = -l_{F,F'}^2 \int_{e_{F,F'}} f(f'' + f) \, d\mathcal{H}^1 \geq l_{F,F'}^2 \int_{e_{F,F'}} \{(f')^2 - f^2\} \, d\mathcal{H}^1 \geq C \int_{e_{F,F'}} f^2 \, d\mathcal{H}^1
\]

with \( C = \pi^2 - l_{F,F'}^2 \), where we integrated by parts in the second equality. Thus provided \( l_{F,F'} < \pi \), the two solutions must coincide \( f_1 = f_2 \).

Next, we note that when \( M \) has nonempty interior, then the lengths \( l_{F,F'} \) of all edges must be bounded away from \( \pi \). The following lemma quantifies this idea.

**Lemma 6.5.** Let \( M \) be a polytope in \( \mathbb{R}^n \) such that \( rB \subseteq M \subseteq RB \). Then

\[
\tan \left( \frac{l_{F,F'}}{2} \right) \leq \frac{R}{r} \quad \text{for all } e_{F,F'} \in E .
\]

In particular, we can estimate

\[
l_{F,F'}^2 \leq \pi^2 \left( 1 - \frac{r^2}{R^2 + r^2} \right) \quad \text{for all } e_{F,F'} \in E .
\]

**Proof.** Define for every facet \( F \in \mathcal{F} \) the supporting hyperplane

\[
H_F := \{ x \in \mathbb{R}^n : \langle n_F, x \rangle = h_M(n_F) \} .
\]

Let \( F \sim F' \) be neighboring facets. We make the following claims.

a. We claim that \( h_M(n_F) \geq r \) and \( h_M(n_{F'}) \geq r \). Indeed, note that as \( rB \subseteq M \), we have \( h_M(n_F) \geq h_B(n_F) = r \), and similarly for \( F' \).

b. We claim that \( H_F \cap H_{F'} \cap RB \neq \emptyset \). Indeed, this follows readily by noting that \( F \cap F' \subset H_F \cap H_{F'} \) and \( \emptyset \neq F \cap F' \subset M \subseteq RB \).

Now note that we can write for any \( x \in \mathbb{R}^n \)

\[
\|x\|^2 = \langle x, n_F \rangle^2 + \frac{(\langle x, n_F \rangle - \cos(l_{F,F'}) \langle x, n_F \rangle)^2}{\sin(l_{F,F'})^2} + \|P_{\langle n_F, n_F \rangle^\perp} x\|^2 ,
\]
where we used \( \langle n_F, n_{F'} \rangle = \cos(l_{F,F'}) \). Thus
\[
R^2 \geq \inf_{x \in H_F \cap H_{F'}} \| x \|^2 = h_M(n_F)^2 + \frac{(h_M(n_{F'}) - \cos(l_{F,F'})h_M(n_F))^2}{\sin(l_{F,F'})^2}
\]
by claim b above. Applying claim a yields
\[
R^2 \geq r^2 \left( 1 + \frac{(1 - \cos(l_{F,F'}))^2}{\sin(l_{F,F'})^2} \right)
\]
provided \( \frac{\pi}{2} \leq l_{F,F'} < \pi \). It follows that
\[
\tan \left( \frac{l_{F,F'}}{2} \right) = \frac{1 - \cos(l_{F,F'})}{\sin(l_{F,F'})} \leq \frac{R}{r}.
\]
Indeed, for \( \frac{\pi}{2} \leq l_{F,F'} < \pi \) this is immediate from the previous expression, while for \( l_{F,F'} < \frac{\pi}{2} \) this follows as \( \tan(\frac{\pi}{2}) = 1 \leq \frac{R}{r} \). To deduce the second part of the statement, it remains to note that \( 4 \arctan(x) \leq \pi x^2/(1 + x^2) \).

\[\square\]

Combining Lemmas 6.3 and 6.5 yields the following.

**Corollary 6.6.** Let \( M \) be a polytope in \( \mathbb{R}^n \) such that \( rB \subseteq M \subseteq RB \). Then for any neighboring facets \( F \sim F' \) of \( M \) and any function \( f \in H^1(\epsilon_{F,F'}) \), we have
\[
\int_{\epsilon_{F,F'}} \{(f')^2 - f^2\} \, d\mathcal{H}^1 \geq \frac{r^2}{2R^2} \int_{\epsilon_{F,F'}} f^2 \, d\mathcal{H}^1 - \frac{4R^2}{r^2} l_{F,F'} \{f(n_F)^2 + f(n_{F'})^2\}.
\]

**Proof.** Applying Lemma 6.5 to the left-hand side of Lemma 6.3 and rearranging the resulting expression yields the following inequality:
\[
\int_{\epsilon_{F,F'}} \{(f')^2 - f^2\} \, d\mathcal{H}^1 \geq \frac{R^2 + r^2}{R^2} (1 - \epsilon)^2 - 1 \int_{\epsilon_{F,F'}} f^2 \, d\mathcal{H}^1 - \frac{R^2 + r^2}{R^2} \frac{2}{\epsilon \pi^2} l_{F,F'} \{f(n_F)^2 + f(n_{F'})^2\}.
\]
Now choose \( \epsilon = \frac{r^2}{4(R^2 + r^2)} \). Then
\[
\frac{R^2 + r^2}{R^2} (1 - \epsilon)^2 - 1 \geq \frac{R^2 + r^2}{R^2} (1 - 2\epsilon) - 1 = \frac{r^2}{2R^2},
\]
while
\[
\frac{R^2 + r^2}{\epsilon \pi^2} = \frac{8}{\pi^2} \frac{(R^2 + r^2)^2}{\epsilon \pi^2} \leq \frac{32 R^2}{\pi^2 r^2}.
\]
To conclude, we estimate \( \frac{32}{\pi^2} \leq 4 \) for aesthetic appeal. \[\square\]

We are now ready to prove a form of Theorem 6.1 for polytopes.

**Proposition 6.7.** Let \( M \) be a polytope in \( \mathbb{R}^n \) such that \( rB \subseteq M \subseteq RB \). Define a measure \( \mu_M \) on the vertices of the associated metric graph by setting
\[
\mu_M(\{n_F\}) := \frac{1}{n - 1} \sum_{F': F' \sim F} \mathcal{H}^{n-2}(F \cap F') l_{F,F'}
\]
for all facets \( F \) of \( M \). Then we have
\[
\mathcal{V}(K,L,M)^2 \geq \mathcal{V}(K,M) \mathcal{V}(L,M) + \mathcal{V}(L,M) \left( \int_{2nR^2} (h_K - h_L)^2 \, dS_{B,M} - \frac{4R^2}{nr^2} \int (h_K - h_L)^2 \, d\mu_M \right)
\]
for all convex bodies \( K,L \) in \( \mathbb{R}^n \).
Proof. Let $\mathcal{E}$ be the quadratic form of Theorem 4.7 and $f \in \text{Dom} \mathcal{E}$. Multiplying the inequality of Corollary 6.6 by $\mathcal{H}^{n-2}(F \cap F')$ and summing over all edges yields

$$0 \geq \mathcal{E}(f, f) + \frac{r^2}{2nR^2} \int f^2 dS_{B, M} - \frac{4R^2}{nr^2} \int f^2 d\mu_M.$$

Now let $f = h_K - h_L$. Then we obtain

$$0 \geq \mathcal{V}(K, K, M) - 2\mathcal{V}(K, L, M) + \mathcal{V}(L, L, M) + \frac{r^2}{2nR^2} \int (h_K - h_L)^2 dS_{B, M} - \frac{4R^2}{nr^2} \int (h_K - h_L)^2 d\mu_M.$$

It remains to note that

$$\mathcal{V}(K, K, M) - 2\mathcal{V}(K, L, M) + \mathcal{V}(L, L, M) \geq \mathcal{V}(K, K, M) - \frac{\mathcal{V}(K, L, M)^2}{\mathcal{V}(L, L, M)}$$

when $\mathcal{V}(L, L, M) > 0$, so the conclusion follows readily in this case. On the other hand, when $\mathcal{V}(L, L, M) = 0$ the conclusion is trivial. \hfill \Box

6.3. Proof of Theorem 6.1: general case. In order to prove Theorem 6.1 for an arbitrary convex body $M$, we will approximate it by polytopes and take limits in Proposition 6.7. The main issue that we will encounter is to understand the behavior of the measure $\mu_M$ under taking limits.

At first sight, one might hope that $\mu_M$ is a natural geometric object that remains meaningful for arbitrary convex bodies, just like $S_{B, M}$ or $S_{M, M}$. This does not appear to be the case, however. It is important to note that even within the class of polytopes, the measure $\mu_M$ is not continuous with respect to Hausdorff convergence, as is illustrated by the following example.

Example 6.8. Consider a cube $M_\varepsilon$ with one of its edges sliced off at width $\varepsilon$; this construction is illustrated in Figure 6.1. Then $M_\varepsilon$ has, for all $\varepsilon > 0$, an additional facet $F$ as compared to $M_0$. It is readily seen that $\inf_{\varepsilon > 0} \mu_M_\varepsilon(\{n_F\}) > 0$, while $\mu_{M_0}(\{n_F\}) = 0$. Thus $M_\varepsilon \to M_0$ but $\mu_{M_\varepsilon} \not\to \mu_{M_0}$ as $\varepsilon \to 0$.

For a polytope $M$, Theorem 4.7 shows that the mass assigned by $\mu_M$ to a vertex of the metric graph is precisely the $S_{B, M}$-measure of its incident edges. We may therefore view $\mu_M$ as a kind of projection of $S_{B, M}$ onto the 0-extreme normal vectors of $M$. It is not clear, however, what this might mean for a general convex body $M$, and the above example illustrates that one cannot hope to canonically define such projections by approximation of general bodies by polytopes. Nonetheless, as

$$\mu_M(S^{n-1}) = 2 S_{B, M}(S^{n-1}) = 2n \mathcal{V}(B, B, M)$$
by Theorem 4.7, the total mass of $\mu_M$ is uniformly bounded for any convergent sequence of polytopes, and we may therefore extract a weakly convergence subsequence of these measures. While the limiting measure is not uniquely defined by the limiting body, we can nonetheless guarantee it satisfies our desired properties by working with specially chosen polytope approximations.

**Lemma 6.9.** Let $M$ be any convex body in $\mathbb{R}^n$ with $0 \in \text{int } M$. Then there exists a sequence of polytopes $M_k$ in $\mathbb{R}^n$ with the following properties:

a. $M_k \to M$ in Hausdorff metric.
b. There exist $r, R > 0$ so that $rB \subseteq M_k \subseteq RB$ for all $k$.
c. $\text{supp } S_{M_k, M_k} \subseteq \text{supp } S_{M, M}$ for all $k$.
d. $\mu_{M_k}$ converges weakly to a limiting measure $\mu_M$ with $\text{supp } \mu_M \subseteq \text{supp } S_{M, M}$.

**Proof.** Recall that a regular boundary point of $M$ is a point in $\partial M$ that has a unique outer normal vector; in particular, the normal vector at a regular boundary point is 0-extreme [33, section 2.2]. Choose a countable dense subset of the regular boundary points of $M$, and let $\{n_i\}_{i \geq 1}$ be the corresponding normal directions. Then $\{n_i\} \subseteq \text{supp } S_{M, M}$ by Theorem 3.4. Moreover, as a convex body with nonempty interior is the intersection of its regular supporting halfspaces [33, Theorem 2.2.6], we have

$$M = \bigcap_{i \geq 1} \{x \in \mathbb{R}^n : \langle x, n_i \rangle \leq h_M(n_i) \}.$$  

Now define

$$M_k' := \bigcap_{1 \leq i \leq k} \{x \in \mathbb{R}^n : \langle x, n_i \rangle \leq h_M(n_i) \}.$$  

Then we have the following properties.

i. $M_k'$ is a polytope for all sufficiently large $k$.
ii. $M_k' \to M$ as $k \to \infty$ in Hausdorff metric by [33, Lemma 1.8.2].
iii. $rB \subseteq M \subseteq M_k'$ for all $k$ with $r > 0$, as $0 \in \text{int } M$.
iv. $M_k' \subseteq RB$ for all sufficiently large $k$ with $R = \text{diam } M$ by property ii.
v. $\text{supp } S_{M_k', M_k'} \subseteq \{n_i\}_{1 \leq i \leq k} \subseteq \text{supp } S_{M, M}$ for all $k$ by Theorem 3.4.

Now note that when $M_k' \subseteq RB$, we can estimate

$$\mu_{M_k'}(S^{n-1}) = 2n \text{Vol}(B, B, M_k') \leq 2n R^{n-2} \text{Vol}(B).$$  

By property iv, the mass of $\mu_{M_k'}$ is uniformly bounded for all sufficiently large $k$.

We may therefore extract a subsequence $\{M_k\}$ of $\{M_k'\}$ such that $\mu_{M_k}$ converges weakly to a limiting measure $\mu_M$ (by weak compactness of bounded sets of measures on $S^{n-1}$), and such that properties a–c in the statement of the Lemma hold. It remains to show that $\text{supp } \mu_M \subseteq \text{supp } S_{M, M}$; this follows immediately, however, from the fact that $\text{supp } \mu_{M_k} = \text{supp } S_{M_k, M_k} \subseteq \text{supp } S_{M, M}$ for all $k$.  

We can now complete the proof of Theorem 6.1.

**Proof of Theorem 6.1.** By translation-invariance of mixed volumes and mixed area measures, we may assume without loss of generality that $0 \in \text{int } M$. Define the sequence of polytopes $M_k$ and the measure $\mu_M$ as in Lemma 6.9. Applying Proposition 6.7 to $M_k$ and taking the limit as $k \to \infty$, the conclusion follows readily from Theorem 3.3. (For aesthetic reasons, we have rescaled the definition of the measure $\mu_M$ in the statement of Theorem 6.1 so that only a single constant $C_M$ appears; this makes no difference, of course, to the statement of the result.)  

□
7. The lower-dimensional case

In the setting of Theorem 1.3, we have seen that the extremals of Minkowski’s inequality have a simple spectral interpretation: the kernel of the operator $\mathcal{A}$ of Theorem 4.1 always contains the linear functions, and Theorem 1.3 shows that when $M$ has nonempty interior, these are the only elements of the kernel that are differences of support functions of convex bodies.

When $M$ is a lower-dimensional body, however, Theorem 1.4 states that new equality cases appear. Thus, unlike in the full-dimensional case, nonlinear differences of support functions can appear in the kernel of $\mathcal{A}$. This may suggest that the lower-dimensional situation is more complicated, as we must understand the new elements of the kernel. In fact, somewhat surprisingly, the lower-dimensional situation turns out to be considerably simpler: when $M$ has empty interior, the operator $\mathcal{A}$ can be described explicitly in complete generality (i.e., not just in special cases such as smooth bodies or polytopes). Once the operator has been constructed, we will be able to compute its kernel directly, and the proof of Theorem 1.4 will follow. These ideas will be developed in the remainder of this section.

To gain some insight into the lower-dimensional situation, it is instructive to consider first the case of a lower-dimensional polytope $M \subset w^\perp$ for some $w \in S^{n-1}$. The following discussion is illustrated in Figure 7.1. To understand the operator associated to $M$, we first approximate it by the “cylinder” $M_\varepsilon := M + \varepsilon [0, w]$ which has nonempty interior. The body $M_\varepsilon$ has two types of facets:

1. Two facets with normals $\pm w$ are translates of $M$.
2. The remaining facet normals are the normals of the $(n-2)$-faces of $M$ in $w^\perp$.

The body $M_\varepsilon$ defines a quantum graph according to Theorem 4.7. We now formally let $\varepsilon \to 0$ and investigate what happens to the quantum graph in the limit. For each pair of facets $F \sim F'$ of $M_\varepsilon$ of type 2, we evidently have $\mathcal{H}^{n-2}(F \cap F') = O(\varepsilon)$. Thus all edges in the quantum graph associated to $M_\varepsilon$ that lie in $w^\perp$ vanish as $\varepsilon \to 0$. Consequently, the limiting graph has an extremely simple structure: it has exactly two vertices at the antipodal points $\pm w$; and its edges are the geodesic arcs between $\pm w$ in the directions of the $(n-2)$-faces of $M$ in $w^\perp$.

This structure also explains why additional equality cases appear in the lower-dimensional setting: as all edges of the graph associated to $M$ have length $\pi$, the
solution to the Dirichlet problem \( \mathcal{A} f = \frac{1}{n} \{ f'' + f \} = 0 \) on each edge is no longer unique. Indeed, if \( f \) is such a solution on a given edge, then \( \theta \mapsto f(\theta) + a \sin(\theta) \) is also a solution with the same boundary data on the vertices for any \( a \in \mathbb{R} \). This resonance phenomenon results in many new elements of the kernel of \( \mathcal{A} \) in the setting of lower-dimensional polytopes. Because of the simple structure of the graph, however, one can compute all elements of \( \ker \mathcal{A} \) explicitly, and we encounter none of the challenges that arose in the full-dimensional setting.

It is not difficult to work out the details of the above argument for polytopes. In this case, Theorem 1.4 was proved in [32, Theorem 4.2] (see also [13]) from a somewhat different perspective. However, we will show below that essentially the same construction remains valid when \( M \) is any lower-dimensional convex body. In this case, the operator \( \mathcal{A} \) turns out to be very similar to the quantum graph of a lower-dimensional polytope, except there may now be an infinite (even uncountable) number of edges in the graph. Some care must be taken, therefore, to construct this operator properly and to compute its domain, which will be done in section 7.1. Once this has been accomplished, however, the proof of Theorem 1.4 will follow readily in section 7.2 from an explicit computation of \( \ker \mathcal{A} \).

### 7.1. Construction of the operator

Throughout this section, we fix \( w \in S^{n-1} \) and a convex body \( M \subset w^\perp \). We define the measure \( S_M \) on \( S^{n-1} \cap w^\perp \) to be the area measure of \( M \) when viewed as a convex body in \( w^\perp \), that is,

\[
S_M(A) := \mathcal{H}^{n-2}(\{ x \in w^\perp : x \in F(M, u) \text{ for some } u \in A \})
\]

for \( A \subseteq S^{n-1} \cap w^\perp \). We will assume that \( \dim M \geq n-2 \), so that \( S_M \neq 0 \).

In view of the structure illustrated in Figure 7.1, it will be convenient to parametrize \( S^{n-1} \) in polar coordinates \( (\theta,z) \in [0,\pi] \times (S^{n-1} \cap w^\perp) \) as

\[
\iota : [0,\pi] \times (S^{n-1} \cap w^\perp) \to S^{n-1}, \quad \iota(\theta,z) := w \cos \theta + z \sin \theta.
\]

Note that the parametrization is unique except at \( \theta \in \{0,\pi\} \), where \( \iota(0,z) = w \) and \( \iota(\pi,z) = -w \) for every \( z \). Therefore, a continuous function \( f \in C^0(S^{n-1}) \) is given in this parametrization by a function \( f(\theta,z) \) such that \( f(0,\cdot) \) and \( f(\pi,\cdot) \) are constant functions. Note, however, that the directional derivatives \( \frac{\partial f}{\partial \theta}(\theta,\cdot) \) are generally not constant functions at \( \theta \in \{0,\pi\} \) even when \( f \in C^1(S^{n-1}) \).

The main result of this section is the following.

**Theorem 7.1.** Let \( w \in S^{n-1} \) and let \( M \subset w^\perp \) be a convex body with \( \dim M \geq n-2 \). Then for any \( f : S^{n-1} \to \mathbb{R} \), we have

\[
\int f\,dS_{B,M} = \frac{1}{n-1} \int_0^{\pi} \int_{S^{n-1} \cap w^\perp} f(\theta,z) S_M(dz)\,d\theta.
\]

Moreover, the operator \( \mathcal{A} \) defined by

\[
\mathcal{A} f(\theta,z) = \frac{1}{n} \left\{ \frac{\partial^2 f}{\partial \theta^2}(\theta,z) + f(\theta,z) \right\}
\]
Thus \( HF_2 \).

1. \( F \) convex body \( M \) normal of \( b \) by \( F \) is self-adjoint on \( L^2(S_{B,M}) \) and satisfies all the properties of Theorem 4.1.

**Remark 7.2.** With some additional work, one can show that the operator \( \mathcal{A} \) of Theorem 7.1 is in fact the one constructed in the proof of Theorem 4.1, that is, it is the Friedrichs extension of \((4.2)\) in the present setting. This is not needed, however, for the applications of this theorem, and the particularly simple structure of the present setting enables us to short-circuit some technical arguments.

The proof of Theorem 7.1 is similar to that of Theorem 4.7. We begin by making precise the procedure illustrated in Figure 7.1.

**Lemma 7.3.** Let \( w \in S^{n-1} \), let \( M \subset w^\perp \) be a convex body with \( \dim M \geq n - 2 \), and let \( K \) be a convex body of class \( C^2 \). Then

\[
\int f \, dS_{K,M} = \frac{1}{n - 1} \int_0^\pi \int_{S^{n-1} \cap w^\perp} \left\{ \frac{\partial^2 h_K(\theta,z)}{\partial \theta^2} + h_K(\theta,z) \right\} f(\theta,z) \, S_M(dz) \, d\theta.
\]

**Proof.** Suppose first that \( M \) is a polytope of dimension \( \dim M = n - 1 \). Denote by \( \mathcal{F}_M \) the set of its \((n-2)\)-dimensional faces, and by \( n_F \in S^{n-1} \cap w^\perp \) the outer normal of \( F \in \mathcal{F}_M \) when viewed as a convex body in \( w^\perp \). Now define for \( \varepsilon > 0 \) the convex body \( M_\varepsilon := M + \varepsilon [0,w] \) in \( \mathbb{R}^n \). Then \( M_\varepsilon \) has the following facets:

1. \( F(M_\varepsilon,w) = M + \varepsilon w \) and \( F(M_\varepsilon,-w) = M \).
2. \( F(M_\varepsilon,n_F) = F + \varepsilon [0,w] \) for \( F \in \mathcal{F}_M \).

Thus \( \mathcal{H}^{n-2}(F(M_\varepsilon,n_F) \cap F(M_\varepsilon,n_{F'})) = O(\varepsilon) \) for any \( F, F' \in \mathcal{F}_M \), so we obtain

\[
\int f \, dS_{K,M_\varepsilon} = \frac{1}{n - 1} \sum_{F \in \mathcal{F}_M} \mathcal{H}^{n-2}(F) \int_0^\pi \left\{ \frac{\partial^2 h_K(\theta,n_F)}{\partial \theta^2}(\theta,n_F) + h_K(\theta,n,F) \right\} f(\theta,n_F) \, d\theta + O(\varepsilon)
\]

for any continuous function \( f \) by Proposition 4.8. Letting \( \varepsilon \to 0 \) using Theorem 3.3, and noting that \( S_M \) is, by definition, the measure defined by \( S_M(\{n_F\}) = \mathcal{H}^{n-2}(F) \) for \( F \in \mathcal{F}_M \), concludes the proof when \( M \) is an \((n-1)\)-dimensional polytope.

Now note that any convex body \( M \subset w^\perp \) is the limit in Hausdorff metric of a sequence of \((n-1)\)-dimensional polytopes in \( w^\perp \) [7, p. 39]. Thus the conclusion extends to arbitrary \( M \) by approximation using Theorem 3.3.

The expression for \( S_{B,M} \) in Theorem 7.1 follows immediately from Lemma 7.3.

**Lemma 7.4.** The operator \( \mathcal{A} \) of Theorem 7.1 is self-adjoint.
Integrating by parts as in the beginning of the proof shows that
\[ \langle f, \mathcal{A} \rangle_{L^2(S_{B,M})} = \frac{1}{n(n-1)} \int_0^\pi \int_{S^{n-2}} f(\theta, z) \left\{ \frac{\partial^2 g}{\partial \theta^2}(\theta, z) + g(\theta, z) \right\} S_M(dz) d\theta \]
\[ = \frac{1}{n(n-1)} \int_0^\pi \int_{S^{n-2}} \left\{ f(\theta, z) g(\theta, z) - \frac{\partial f}{\partial \theta}(\theta, z) \frac{\partial g}{\partial \theta}(\theta, z) \right\} S_M(dz) d\theta \]
\[ + \frac{1}{n(n-1)} \int_0^\pi \int_{S^{n-2}} f(\theta, z) \frac{\partial g}{\partial \theta}(\theta, z) S_M(dz) \bigg|_{\theta = \pi} \bigg|_{\theta = 0}, \]
where we integrated by parts. But the definition of Dom \( \mathcal{A} \) ensures that the boundary term vanishes. It follows that \( \mathcal{A} \) is a symmetric operator, and in particular Dom \( \mathcal{A} \subseteq \text{Dom} \mathcal{A}^\ast. \) It therefore remains to prove the converse inclusion.

Fix in the rest of the proof \( f \in \text{Dom} \mathcal{A}^\ast. \) We must show that \( f \) satisfies each of the defining properties of Dom \( \mathcal{A} \).

First, note that for any smooth compactly supported function \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and any \( h \in L^2(S_M) \), the function \( g(\theta, z) := \varphi(\theta)h(z) \) satisfies \( g \in \text{Dom} \mathcal{A}. \) Thus
\[ \frac{1}{n(n-1)} \int_0^\pi \left[ \int_0^\pi f(\theta, z) \{ \varphi''(\theta) + \varphi(\theta) \} d\theta \right] h(z) S_M(dz) = \langle f, \mathcal{A} \varphi \rangle_{L^2(S_{B,M})} \]
\[ = \langle \mathcal{A}^\ast f, \varphi \rangle_{L^2(S_{B,M})} = \frac{1}{n-1} \int_0^\pi \left[ \int_0^\pi \mathcal{A}^\ast f(\theta, z) \varphi(\theta) d\theta \right] h(z) S_M(dz). \]
As \( h \) is arbitrary, we have
\[ \int_0^\pi f(\theta, z) \varphi''(\theta) d\theta = \int_0^\pi (n \mathcal{A}^\ast - I) f(\theta, z) \varphi(\theta) d\theta \quad \text{for } S_M\text{-a.e. } z \]
for any \( \varphi \in C_0^\infty((0, \pi)) \). As \( H^2((0, \pi)) \) is separable [17, section 7.5], this identity remains valid simultaneously for all \( \varphi \in C_0^\infty((0, \pi)) \) (that is, the exceptional set may be chosen independent of \( \varphi \)). As \( (n \mathcal{A}^\ast - I) f(\cdot, z) \in L^2((0, \pi)) \) for \( S_M\text{-a.e. } z \) by Fubini’s theorem, we have shown that \( f(\cdot, z) \in H^2((0, \pi)) \) for \( S_M\text{-a.e. } z \) and that \( \mathcal{A}^\ast f = \frac{1}{n} \{ \partial^2 f \partial \varphi^2 + f \} \) \( S_{B,M}\text{-a.e.} \) (in particular, \( \partial^2 f \partial \varphi^2 \in L^2(S_{B,M}) \)).

It remains only to establish the vertex boundary conditions at \( \theta \in \{0, \pi\} \). To this end, note first that if \( g \in \text{Dom} \mathcal{A} \) is arbitrary, then
\[ \langle f, \mathcal{A} \varphi \rangle_{L^2(S_{B,M})} = \langle \mathcal{A}^\ast f, g \rangle_{L^2(S_{B,M})} = \langle \frac{1}{n} \{ \partial^2 f \partial \varphi^2 + f \}, g \rangle_{L^2(S_{B,M})}. \]
Integrating by parts as in the beginning of the proof shows that
\[ \int_{S^{n-1}} f(\theta, z) g(\theta, z) \frac{\partial g}{\partial \theta}(\theta, z) S_M(dz) \bigg|_{\theta = \pi} \bigg|_{\theta = 0} = 0 \]
for every \( g \in \text{Dom} \mathcal{A}. \) We choose a different test function \( g \) to deduce each boundary condition. First, let \( g(\theta, z) := 1 \pm \cos(\theta) \). Then \( g \in \text{Dom} \mathcal{A} \), so we conclude
\[ \int_{S^{n-1}} \frac{\partial f}{\partial \theta}(\theta, z) S_M(dz) = 0 \quad \text{for } \theta \in \{0, \pi\}. \]
Next, let \( g(\theta, z) := \sin(k\theta) h(z) \) for \( k = 1, 2 \) and \( h \in L^2(S_M) \) with \( \int h dS_M = 0 \). Then \( g \in \text{Dom} \mathcal{A} \), so we conclude that
\[ \int_{S^{n-1}} f(0, z) h(z) S_M(dz) = \int_{S^{n-1}} f(\pi, z) h(z) S_M(dz) = 0. \]
As this holds for all \( h \) of the above form, it must be the case that \( f(0, \cdot) \) and \( f(\pi, \cdot) \) are \( S_M\text{-a.e.} \) constant. The proof is complete. \( \square \)
The reason that the setting of this section is particularly simple is that we can compute the full spectral decomposition of $\mathcal{A}$.

**Lemma 7.5.** The operator $\mathcal{A}$ of Theorem 7.1 satisfies
\[
\text{spec } \mathcal{A} = \{ \lambda_k : k \in \mathbb{Z}_+ \}, \quad \lambda_k := \frac{1 - k^2}{n}.
\]
Moreover, the eigenspace $E_k$ associated to eigenvalue $\lambda_k$ is given by
\[
E_k := \left\{ f : f(\theta, z) = h(z) \sin(k\theta) + a \cos(k\theta), \ a \in \mathbb{R}, \ h \in L^2(S_M), \ \int h \, dS_M = 0 \right\}.
\]

**Proof.** Let $E_k$ be the spaces defined in the statement of the lemma. It is readily verified that $E_k \subset \text{Dom } \mathcal{A}$ and $\mathcal{A} f = \lambda_k f$ for each $k$ and $f \in E_k$.

We now claim that
\[
E := \text{span} \left( \bigcup_{k \geq 0} E_k \right) = L^2(S_{B,M}).
\]
As $S_{B,M}$ is a product measure on $[0, \pi] \times (S^{n-1} \cap \mathbb{S}^1)$, it suffices to show that any function of the form $g(\theta, z) := \varphi(\theta) h(z)$ with $\varphi \in L^2([0, \pi])$ and $h \in L^2(S_M)$ lies in $E$. To this end, note that both $\{ \sin(k\theta) : k \geq 1 \}$ and $\{ \cos(k\theta) : k \geq 0 \}$ are complete orthogonal bases of $L^2([0, \pi])$ (these are the Dirichlet and Neumann eigenfunctions of the Laplacian on $[0, \pi]$, respectively). Thus we may write
\[
\varphi(\theta) = \sum_{k \geq 0} a_k \cos(k\theta) = \sum_{k \geq 1} b_k \sin(k\theta)
\]
in $L^2([0, \pi])$ for some coefficient sequences $a_k, b_k$. Moreover, we can evidently write
\[
h(z) = h_0(z) + c \quad \text{where} \quad h_0 \in L^2(S_M) \quad \text{and} \quad \int h_0 \, dS_M = 0.
\]
Thus
\[
g(\theta, z) = \sum_{k \geq 0} \{ b_k h_0(z) \sin(k\theta) + c a_k \cos(k\theta) \} \in E,
\]
completing the proof of the claim.

Denote by $P_k$ the orthogonal projection in $L^2(S_{B,M})$ onto $E_k$. As the eigenspaces of a self-adjoint operator are orthogonal, it follows that
\[
I = \sum_{k \geq 0} P_k \leq \sum_{k \geq 0} 1_{\{ \lambda_k \}}(\mathcal{A}) = 1_{\{ \lambda_k : k \in \mathbb{Z}_+ \}}(\mathcal{A}) \leq I.
\]
Thus $\text{spec } \mathcal{A} = \{ \lambda_k : k \in \mathbb{Z}_+ \}$ and $P_k = 1_{\{ \lambda_k \}}(\mathcal{A})$ for all $k$. \hfill $\square$

Finally, we make the following simple observation.

**Lemma 7.6.** In the setting of Theorem 7.1, we have $C^2(S^{n-1}) \subset \text{Dom } \mathcal{A}$.

**Proof.** It suffices to note that for $f \in C^2(S^{n-1})$, we have
\[
\int \frac{\partial f}{\partial \theta}(0, z) \, S_M(dz) = \int \langle z, \nabla f(w) \rangle \, S_M(dz) = 0
\]
and
\[
\int \frac{\partial f}{\partial \theta}(\pi, z) \, S_M(dz) = \int \langle z, \nabla f(-w) \rangle \, S_M(dz) = 0
\]
by Lemma 3.2(e). The remaining properties are trivial. \hfill $\square$

We can now complete the proof of Theorem 7.1.
Proof of Theorem 7.1. The expression for $S_{B, M}$ follows from Lemma 7.3, and self-adjointness of $\mathcal{A}$ on $L^2(S_{B, M})$ was proved in Lemma 7.4. Properties $a$–$c$ of Theorem 4.1 can be read off from Lemma 7.5.

Finally, note that by Lemmas 7.3 and 7.6, the operator $\mathcal{A}$ agrees with (4.2) on $C^2(S^{n-1})$. Thus the closed quadratic form associated to $\mathcal{A}$ is a closed extension of its restriction to $C^2(S^{n-1})$. The quadratic form $\mathcal{E}$ of Theorem 4.1 is the smallest such extension (the Friedrichs extension); thus properties $d$–$e$ of Theorem 4.1 remain valid in the present setting (and for any other closed extension of $\mathcal{E}$).

Remark 7.7. Observe that unless $M$ is a polytope, Lemma 7.5 implies that the eigenspaces of $\mathcal{A}$ are infinite-dimensional. In particular, this shows that $\mathcal{A}$ does not have compact resolvent: all eigenvalues $\lambda_k$ for $k \geq 1$ are in the essential spectrum. This provides an explicit example of the issue that was highlighted in Remark 4.2. Let us also note that this phenomenon is not specific to the lower-dimensional setting; for example, it may be verified that a similar situation occurs if we replace $M$ by $M_\varepsilon := M + \varepsilon [0, w]$, which has nonempty interior. We omit the details.

Remark 7.8. Even when $M$ has nonempty interior, we have only established that $\ker \mathcal{A} \cap \{h_Q - h_R : Q, R \text{ convex bodies}\}$ consists of linear functions. A function that is not a difference of support functions does not give rise to extremals in Minkowski’s inequality, so is not relevant for the purposes of this paper. Nonetheless, one may wonder whether it is possible that $\ker \mathcal{A}$ contains such functions.

When $M$ has empty interior, Lemma 7.5 shows that this may in fact happen. Indeed, as the function $h \in L^2(S_M)$ need only be measurable, it is perfectly possible in general to construct eigenfunctions that do not admit a continuous extension to $S^{n-1}$. Such eigenfunctions cannot arise as the difference of support functions, as support functions are always continuous. Thus $\ker \mathcal{A}$ may contain many elements that do not contribute to the characterization of extremals.

It is natural to conjecture that when $M$ has nonempty interior, such examples cannot occur and that $\ker \mathcal{A}$ consists only of linear functions (cf. Remark 6.2). Such a result cannot be achieved, however, by the methods of this paper. The key obstruction is the theory of section 6: that $\text{supp } \mu_M \subseteq \text{supp } S_{M, \mathcal{M}}$ only guarantees that continuous functions that vanish $S_{M, \mathcal{M}}$-a.e. must vanish $\mu_M$-a.e., so we cannot rule out discontinuous elements of $\ker \mathcal{A}$ by this method.

7.2. Proof of Theorem 1.4. By Lemma 2.1, understanding the equality cases of Minkowski’s quadratic inequality reduces to understanding the kernel of $\mathcal{A}$. In the present setting, however, we have already computed the kernel in Lemma 7.5. It therefore remains to furnish its elements with a geometric interpretation.

Before we turn to the proof of Theorem 1.4, however, we must extend the conclusion of Theorem 3.4 to the present setting.

Lemma 7.9. Let $w \in S^{n-1}$ and let $M \subset w^\perp$ be a convex body with $\dim M \geq n - 2$. Then $\text{supp } S_{B, \mathcal{M}} = \text{cl}\{u \in S^{n-1} : u \text{ is a } 1\text{-extreme normal vector of } M\}$.

Proof. As $w$ is normal to every point in $M$, a vector $u \in S^{n-1}$ is normal to a given point in $M$ if and only if its projection $P_{w^\perp} u$ is normal to that point. In particular, it follows readily that $u \neq w$ is a 1-extreme normal vector of $M$ if and only if $P_{w^\perp} u$ is a 0-extreme normal vector of $M$ when viewed as a convex body in $w^\perp$.

Now note that, by the expression for $S_{B, \mathcal{M}}$ given in Theorem 7.1, we have $\text{supp } S_{B, \mathcal{M}} = [0, \pi] \times \text{supp } S_{\mathcal{M}}$ (in polar coordinates). When $\dim M = n - 1$,
we have \( \text{supp} S_M = \partial \{ \text{0-extreme normal vectors of } M \text{ in } w^\perp \} \) by Theorem 3.4, and the conclusion follows. On the other hand, when \( \dim M = n - 2 \), we have \( M \subset \text{span}\{v, w\}^\perp \) for some \( v \perp w \). Thus \( \pm v \) are the only 0-extreme normal directions of \( M \) in \( w^\perp \). On the other hand, it follows readily from the definition that \( \text{supp} S_M = \{ \pm v \} \) in this case, so that the conclusion again follows. \( \square \)

We are now ready to complete the proof of Theorem 1.4.

**Proof of Theorem 1.4.** By translation-invariance of mixed volumes, we may assume without loss of generality that \( M \subset w^\perp \). Moreover, we may assume \( \dim M \geq n - 2 \), as otherwise \( V(L, L, M) = 0 \) for all bodies \( L \) [33, Theorem 5.1.8].

Now let \( L, M \) be any convex bodies in \( \mathbb{R}^n \) with \( V(L, L, M) > 0 \). By Theorem 7.1 and Lemma 2.1, we have equality in Minkowski’s inequality
\[
V(K, L, M)^2 = V(K, K, M)V(L, L, M)
\]
if and only if \( h_K - ah_L \in \ker \mathcal{A} \) for some \( a \in \mathbb{R} \). Thus the proof will be concluded once we establish that the following two statements are equivalent:
1. \( h_K - ah_L \in \ker \mathcal{A} \) for some \( a \in \mathbb{R} \).
2. \( \tilde{L} := \frac{V(K, L, M)}{V(L, L, M)}L \) has the property that \( K + F(\tilde{L}, w) \) and \( L + F(K, w) \) have the same supporting hyperplanes in all 1-extreme normal directions of \( M \).

Let us prove each in turn.

1 \( \Rightarrow \) 2. Let \( h_K - ah_L \in \ker \mathcal{A} \). First, note that
\[
0 = \langle h_L, \mathcal{A}(h_K - ah_L) \rangle_{L^2(S_B, M)} = V(K, L, M) - a V(L, L, M)
\]
by Theorem 7.1. Thus we must have
\[
a = \frac{V(K, L, M)}{V(L, L, M)},
\]
and it follows that \( \tilde{L} = aL \). To proceed, we observe that Lemma 7.5 implies that
\[
h_K(\theta, z) - h_\tilde{L}(\theta, z) = \eta(z) \sin \theta + \alpha \cos \theta \quad S_{B, M}\text{-a.e. } (\theta, z)
\]
for some \( \alpha \in \mathbb{R} \) and \( \eta \in L^2(S_M) \) with \( \int \eta dS_M = 0 \). Evidently
\[
\alpha = h_K(0, z) - h_\tilde{L}(0, z) = h_K(w) - h_\tilde{L}(w),
\]
and
\[
\eta(z) = \frac{\partial h_K}{\partial \theta}(0, z) - \frac{\partial h_\tilde{L}}{\partial \theta}(0, z) = \nabla_z h_K(w) - \nabla_z h_\tilde{L}(w) = h_{F(K, w)}(z) - h_{F(\tilde{L}, w)}(z)
\]
by Lemma 3.9. But note that at the point \( x = w \cos \theta + z \sin \theta \in S^{n-1} \) corresponding to the polar coordinates \( (\theta, z) \), we can write using \( F(K, w) - wh_K(w) \subset w^\perp \)
\[
h_{F(K, w)}(x) = h_{F(K, w)}(z) \sin \theta + h_K(w) \cos \theta.
\]
The analogous formula holds for \( \tilde{L} \), and we conclude that
\[
h_K(x) - h_\tilde{L}(x) = h_{F(K, w)}(x) - h_{F(\tilde{L}, w)}(x) \quad S_{B, M}\text{-a.e. } x.
\]
By continuity of support functions, this identity remains valid for all \( x \in \text{supp} S_{B, M} \), and the implication 1 \( \Rightarrow \) 2 follows by Lemma 7.9.

2 \( \Rightarrow \) 1. By Lemma 7.9 and continuity, we can assume
\[
h_K(x) - h_\tilde{L}(x) = h_{F(K, w)}(x) - h_{F(\tilde{L}, w)}(x) \quad \text{for } x \in \text{supp} S_{B, M}.
\]
We will prove directly that this implies equality in Minkowski’s quadratic inequality (and hence $h_K - ah_L \in \ker \mathcal{A}$ by Lemma 2.1).

We begin by noting that $\text{supp} S_{C,\mathcal{M}} = \text{supp} S_{B,\mathcal{M}}$ for every body $C$ of class $C_2^1$ by Lemma 7.3. Thus Theorem 3.3 implies that $\text{supp} S_{C,\mathcal{M}} \subseteq \text{supp} S_{B,\mathcal{M}}$ for any convex body $C$. Choosing $C = F(K, w)$ and $C = F(\tilde{L}, w)$, respectively, we find

$$V(h_K - h_L, h_F(K, w) - h_F(\tilde{L}, w), \mathcal{M}) = \frac{1}{n} \int (h_K - h_L) dS_{h_F(K, w) - h_F(\tilde{L}, w), \mathcal{M}}$$

$$= \frac{1}{n} \int (h_F(K, w) - h_F(\tilde{L}, w)) dS_{h_F(K, w) - h_F(\tilde{L}, w), \mathcal{M}}$$

$$= V(h_F(K, w) - h_F(\tilde{L}, w), h_F(K, w) - h_F(\tilde{L}, w), \mathcal{M}) = 0,$$

where the last equality holds as $\dim(F(K, w) + F(\tilde{L}, w) + M) < n$. On the other hand, choosing $C = K$ and $C = \tilde{L}$, we obtain similarly

$$V(h_K - h_L, h_K - h_L, \mathcal{M}) = V(h_F(K, w) - h_F(\tilde{L}, w), h_K - h_L, \mathcal{M}) = 0.$$

Thus we have shown that

$$V(K, K, \mathcal{M}) - 2V(K, \tilde{L}, \mathcal{M}) + V(\tilde{L}, \tilde{L}, \mathcal{M}) = 0.$$

The conclusion follows by substituting $\tilde{L} = \frac{V(K, L, \mathcal{M})}{V(L, L, \mathcal{M})}L$ into this expression. \hfill \Box

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**References**


Sherrerd Hall 323, Princeton University, Princeton, NJ 08544, USA

*E-mail address*: yairs@princeton.edu

Fine Hall 207, Princeton University, Princeton, NJ 08544, USA

*E-mail address*: rvan@princeton.edu