MIXED VOLUMES AND THE BOCHNER METHOD

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Abstract. At the heart of convex geometry lies the observation that the volume of convex bodies behaves as a polynomial. Many geometric inequalities may be expressed in terms of the coefficients of this polynomial, called mixed volumes. Among the deepest results of this theory is the Alexandrov-Fenchel inequality, which subsumes many known inequalities as special cases. The aim of this note is to give new proofs of the Alexandrov-Fenchel inequality and of its matrix counterpart, Alexandrov's inequality for mixed discriminants, that appear conceptually and technically simpler than earlier proofs and clarify the underlying structure. Our main observation is that these inequalities can be reduced by the spectral theorem to certain trivial "Bochner formulas".

1. Introduction and main ideas

Much of the foundation for the modern theory of convex geometry was put forward by H. Minkowski around the turn of the 20th century. One of the central notions in Minkowski's theory arises from the fundamental fact that the volume of convex bodies in $\mathbb{R}^n$ behaves as a homogeneous polynomial of degree n: that is, for any convex bodies $K_1, \ldots, K_m \subset \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m > 0$, we have

$$\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \mathsf{V}(K_{i_1}, \ldots, K_{i_n})\, \lambda_{i_1} \cdots \lambda_{i_n}. \tag{1.1}$$

The coefficients $\mathsf{V}(K_{i_1}, \ldots, K_{i_n})$ of this polynomial are called *mixed volumes*. Given this observation, it seems natural to expect that many geometric properties of convex bodies may be expressed in terms of relations between mixed volumes. This viewpoint plays a major role in Minkowski's work on convex geometry [15], and lies at the heart of what is now called the Brunn-Minkowski theory [6, 16]. Among the deepest results of this theory is the Alexandrov-Fenchel inequality, which subsumes many geometric inequalities as special cases.

Theorem 1.1 (Alexandrov-Fenchel inequality)**.** *We have*

$$\mathsf{V}(K, L, C_1, \ldots, C_{n-2})^2 \geq \mathsf{V}(K, K, C_1, \ldots, C_{n-2})\, \mathsf{V}(L, L, C_1, \ldots, C_{n-2})$$

for any convex bodies $K, L, C_1, \ldots, C_{n-2}$ *in* $\mathbb{R}^n$.

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The cases $n = 2, 3$ are special in that they can be derived from the Brunn-Minkowski inequality, as was already shown by Minkowski himself [15, p. 261]. However, this approach only yields special cases of Theorem 1.1 in higher dimension. A (questionable) proof of Theorem 1.1 was announced, but never published, by W. Fenchel [9]. Finally, two different but closely related proofs were obtained by A. D. Alexandrov [1, 2] using a homotopy method due to Hilbert [13]. It was realized much later that Theorem 1.1 has connections with algebraic geometry through the Hodge index theorem, which led to the development of algebraic and complex geometric proofs [8, 12, 17]. Despite these diverse viewpoints, the inequality and its proofs are generally considered to be conceptually deep. We refer to [16, 4] for further remarks on the history and significance of Theorem 1.1.

The aim of this note is to give a new proof of the Alexandrov-Fenchel inequality that appears to be conceptually and technically simpler than previous proofs. The basic ingredients of our proof were already introduced by Minkowski, Hilbert, and Alexandrov. However, by means of a very simple but apparently overlooked device, we will replace the main part of Alexandrov's proof by a one-line computation. We believe the resulting approach is particularly intuitive and sheds new light on why the inequality holds. In the remainder of the introduction we describe the basic elements of our proof; the details are filled in in subsequent sections.

1.1. **Mixed volumes and mixed discriminants.** Mixed volumes are defined by considering the volume of the sum $K+L := \{x+y : x \in K, y \in L\}$ of convex bodies. We would like to think of volume as a polynomial on the space of convex bodies. However, this is somewhat awkward, as convex bodies do not form a vector space. To address this issue, we identify each convex body K with its *support function*

$$h_K(x) := \sup_{y\in K} \langle y, x\rangle.$$

Geometrically, $h_K(x)$ is the distance to the origin of the supporting hyperplane of K whose normal direction is $x \in S^{n-1}$. As K can be recovered by intersecting all its supporting halfspaces, h_K and K uniquely determine each other.

The advantage of working with support functions is that they map set addition into scalar addition: $h_{aK+bL} = ah_K + bh_L$. To understand the behavior of volume under addition, it is therefore natural to express $\mathrm{Vol}(K)$ in terms of h_K: we have

$$\mathrm{Vol}(K) = \frac{1}{n}\int_{S^{n-1}} h_K \det(D^2 h_K)\, d\omega, \tag{1.2}$$

where ω denotes the surface measure on S^{n-1} and $D^2h_K(x)$ denotes the restriction of the Hessian of $h_K : \mathbb{R}^n \to \mathbb{R}$ to the tangent space of S^{n-1} at the point x (this classical computation is recalled in section 2.2). With this representation in hand, it is immediately clear that volume is a polynomial in the sense of (1.1): the integrand in (1.2) is a polynomial of degree n in h_K in the usual sense (as D^2h_K is an $(n-1)$-dimensional matrix), and the conclusion follows directly.

Remark 1.2. As written, the representation (1.2) only makes sense for smooth convex bodies, that is, when h_K is a C^2 function on S^{n-1}. However, any convex body can be approximated by smooth bodies, and mixed volumes are continuous with respect to this approximation [6, §27–§29]. We therefore can and will assume in the sequel that all convex bodies are sufficiently smooth.

We can similarly represent mixed volumes in terms of support functions. As mixed volumes are defined as the coefficients of the polynomial (1.1), we must first define the analogous coefficients of the determinant: that is, for any $(n-1)$-dimensional matrices $M_1, \ldots, M_m$ and $\lambda_1, \ldots, \lambda_m > 0$, we define

$$\det(\lambda_1 M_1 + \cdots + \lambda_m M_m) = \sum_{i_1,\ldots,i_{n-1}=1}^{m} \mathsf{D}(M_{i_1}, \ldots, M_{i_{n-1}})\, \lambda_{i_1} \cdots \lambda_{i_{n-1}}. \tag{1.3}$$

The coefficients $\mathsf{D}(M_{i_1}, \ldots, M_{i_{n-1}})$ are called *mixed discriminants.* Following a similar argument to the proof of (1.2), we obtain the following representation:

$$\mathsf{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_1} \mathsf{D}(D^2 h_{K_2}, \ldots, D^2 h_{K_n})\, d\omega. \tag{1.4}$$

It is important to note that mixed volumes are, by definition, symmetric in their arguments, even though this is not obvious from the representation (1.4). For this reason (1.4) does not follow trivially from (1.2). However, one can prove (1.4) by a small modification of the proof of (1.2), as we will recall in section 2.2 below.

Now that we obtained a natural representation of mixed volumes, how might one go about proving Theorem 1.1? In view of (1.4), one may ask first whether there is an analogue of Theorem 1.1 for mixed discriminants. This is indeed the case.

Theorem 1.3 (Alexandrov's mixed discriminant inequality)**.** *Let A be any $(n-1)$-dimensional symmetric matrix, and let $B, M_1, \ldots, M_{n-3}$ be $(n-1)$-dimensional positive semidefinite matrices. Then we have*

$$\mathsf{D}(A, B, M_1, \ldots, M_{n-3})^2 \geq \mathsf{D}(A, A, M_1, \ldots, M_{n-3})\, \mathsf{D}(B, B, M_1, \ldots, M_{n-3}).$$

Theorem 1.3 is a matrix inequality and does not necessarily belong to convex geometry. Given this inequality, it might seem that the Alexandrov-Fenchel inequality should be a simple consequence of Theorem 1.3 and the representation (1.4). This is far from clear, however. Had the inequality signs in Theorems 1.1 and 1.3 been reversed, then the former would follow directly from the latter by the Cauchy-Schwarz inequality. However, the inequalities being such as they are, Cauchy-Schwarz goes in the wrong direction and there is no reason to expect, *a priori*, that Theorem 1.3 should imply Theorem 1.1.

Theorem 1.3 was in fact used by Alexandrov in one part of his study of the Alexandrov-Fenchel inequality. However, in this proof Theorem 1.3 is used very indirectly, and the relationship between Theorems 1.1 and 1.3 has remained somewhat mysterious. Indeed, many other inequalities are known for mixed discriminants, but most such inequalities are simply false in the context of mixed volumes (e.g., [3]).

The new observation of this note is that when viewed in the right way, the Alexandrov-Fenchel inequality will prove to be a *direct* consequence of Alexandrov's inequality for mixed discriminants. This not only yields a simpler proof, but also demystifies the relationship between Theorems 1.1 and 1.3. We believe this conceptual simplification significantly clarifies the structure of these inequalities. Once the basic idea has been understood, we will find that the same idea can be used to give a simple new proof of Theorem 1.3.

1.2. **Hyperbolic inequalities.** Before we can explain the main idea of this note, we must recall the basic structure behind the Alexandrov-Fenchel inequalities. By definition, mixed volumes and mixed discriminants are symmetric multilinear functions of their arguments. Therefore, Theorems 1.1 and 1.3 may be viewed as statements about certain *quadratic forms*: Theorem 1.1 is concerned with the quadratic form $(h_K, h_L) \mapsto \mathsf{V}(K, L, C_1, \ldots, C_{n-2})$, while Theorem 1.3 is concerned with the quadratic form $(A, B) \mapsto \mathsf{D}(A, B, M_1, \ldots, M_{n-3})$. From this perspective, both Theorems 1.1 and 1.3 can be interpreted as stating that the relevant quadratic form satisfies a *reverse* form of the Cauchy-Schwarz inequality.

It is instructive to recall more generally when quadratic forms satisfy Cauchy-Schwarz inequalities. For example, it is a basic fact of linear algebra that a symmetric quadratic form $\langle x, Ax\rangle$ on $\mathbb{R}^d$ satisfies the Cauchy-Schwarz inequality $\langle x, Ay\rangle^2 \le \langle x, Ax\rangle\langle y, Ay\rangle$ if and only if the matrix A is positive or negative semidefinite. The validity of the reverse Cauchy-Schwarz inequality can be characterized in an entirely analogous manner, see section 2.4 for a short proof.

Lemma 1.4 (Hyperbolic quadratic forms)**.** *Let A be a symmetric matrix. Then the following conditions are equivalent:*

1. *$\langle x, Ay\rangle^2 \ge \langle x, Ax\rangle\langle y, Ay\rangle$ for all x, y such that $\langle y, Ay\rangle \ge 0$.*
2. *The positive eigenspace of A has dimension at most one.*

The conclusion remains valid if A is a self-adjoint operator on a Hilbert space with a discrete spectrum, provided the vectors x, y are chosen in the domain of A.

To apply Lemma 1.4 to the Alexandrov-Fenchel inequality, we may reason as follows. Fix bodies $C_1, \ldots, C_{n-2}$, and define

$$\tilde{\mathscr{A}} f := \frac{1}{n}\mathsf{D}(D^2 f, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}}). \tag{1.5}$$

Then the representation (1.4) can be expressed as

$$\mathsf{V}(K, L, C_1, \ldots, C_{n-2}) = \langle h_K, \tilde{\mathscr{A}} h_L\rangle_{L^2(\omega)}.$$

Note that $\tilde{\mathscr{A}}$ is a second-order differential operator on S^{n-1}. It will follow from basic properties of mixed discriminants and mixed volumes that $\tilde{\mathscr{A}}$ is elliptic and symmetric on $L^2(\omega)$. Thus standard elliptic regularity theory shows that $\tilde{\mathscr{A}}$ is self-adjoint and that it has a discrete spectrum and a simple top eigenvalue (cf. section 3). Therefore, by Lemma 1.4, the Alexandrov-Fenchel inequality is *equivalent* to the statement that $\tilde{\mathscr{A}}$ has exactly one positive eigenvalue.

1.3. **The Bochner method.**

Up to this point we have not formally made any progress towards proving the Alexandrov-Fenchel inequality: we have merely reformulated the statement of Theorem 1.1 as an equivalent spectral problem. The key question in the proof of Theorem 1.1 is why the relevant spectral property actually holds. What is new in this note is the realization that this follows almost immediately from Theorem 1.3 by a one-line computation.

Let us sketch the relevant argument. It is convenient to normalize the operator $\tilde{\mathscr{A}}$ such that its top eigenvalue is 1. Let us call the normalized operator $\mathscr{A}$. As $\mathscr{A} f$ is defined by a mixed discriminant (1.5), what can be deduced from Theorem 1.3 is an inequality for $(\mathscr{A} f)^2$: indeed, when we choose the appropriate normalization, integrating both sides of Theorem 1.3 will immediately yield the inequality

$$\langle \mathscr{A} f, \mathscr{A} f\rangle \ge \langle f, \mathscr{A} f\rangle, \tag{1.6}$$

where the inner product is the one associated to the normalized operator (cf. section 3). By plugging in for f any eigenfunction of $\mathscr{A}$, it follows that any eigenvalue λ

of $\mathscr{A}$ must satisfy $\lambda^2 \geq \lambda$. But as the normalization was chosen such that $\lambda_{\max} = 1$, this can evidently only happen if either $\lambda = 1$ or $\lambda \leq 0$, concluding the proof.

This very simple device sheds light on the reason why an inequality for mixed volumes can be deduced from an inequality for mixed discriminants: as our inequalities are spectral in nature, the spectral theorem reduces the problem of bounding the square of the quadratic form of an operator to that of bounding the square of the operator itself. Once this idea has been understood, it becomes apparent that it explains also other aspects of the Alexandrov-Fenchel theory. For example, the same principle will give a new proof of Theorem 1.3.

While our approach has apparently been overlooked in the literature on the Alexandrov-Fenchel inequality,[1] the underlying idea is classical in Riemannian geometry: it was used by Lichnerowicz [14] to lower bound the spectral gap of the Laplacian on a Riemannian manifold with positive Ricci curvature. In this setting, the analogue of (1.6) is established by means of a technique known as the Bochner method. This analogy is not a coincidence: for example, in the case $C_1 = \cdots = C_{n-2} = B_2$ (the Euclidean unit ball), it turns out that (1.6) reduces exactly to a Bochner formula for the Laplacian on S^{n-1}, see section 6.3 below. We emphasize, however, that no Riemannian geometry will be used in our proofs.

1.4. **Organization of this paper.** The rest of this note is organized as follows. Section 2 recalls basic facts about mixed volumes and mixed discriminants. In section 3, we prove Theorem 1.1 assuming validity of Theorem 1.3. In section 4, our method is adapted to prove Theorem 1.3 itself. In section 5 we sketch an alternative proof of Theorem 1.1 that uses polytopes instead of smooth bodies; while we find this approach less illuminating, it has the advantage of using only matrices and avoiding the use of elliptic operators. Finally, section 6 contains some concluding remarks that places our approach in context.

2. Basic facts

The aim of this section is to recall the basic properties of mixed volumes and mixed discriminants that will be needed in the sequel. The material in this section is standard, see, e.g., [6, 16]. We have nonetheless chosen to include (almost) full proofs, both in order to make our exposition accessible to non-experts and to emphasize that the facts recalled in this section are indeed elementary. Readers who are familiar with basic properties of mixed volumes and mixed discriminants are encouraged to skip ahead directly to section 3.

2.1. **Convex bodies and support functions.** A *convex body* is a nonempty compact convex subset of $\mathbb{R}^n$. We will mostly work with bodies that are sufficiently smooth so that the representation formulas stated in section 1 are valid. Let us make this requirement more precise.

As support functions are 1-homogeneous functions on $\mathbb{R}^n$, let us first consider such functions more generally. First of all, a 1-homogeneous function $f : \mathbb{R}^n \to \mathbb{R}$, i.e., $f(x) = \|x\| f(x/\|x\|)$, is clearly uniquely determined by its values on S^{n-1}. Conversely, the latter identity uniquely extends any function $f : S^{n-1} \to \mathbb{R}$ to

[1]However, a recent paper of Wang [17] uses various algebraic identities in Kähler geometry, including a Bochner-type formula, to give a complex-geometric proof of the Alexandrov-Fenchel inequality. While the connection with our elementary methods is unclear to us, [17] provided the initial inspiration to pursue the ideas in this paper.

a 1-homogeneous function on $\mathbb{R}^n$. Now note that if f is 1-homogeneous and C^2, then ∇f is 0-homogeneous, so that $\nabla^2 f(x)x = 0$. The Hessian of f is therefore completely determined by the restriction of the linear map $\nabla^2 f(x) : \mathbb{R}^n \to \mathbb{R}^n$ to the tangent space $x^\perp$ of the sphere. We denote this restriction as $D^2 f(x) : x^\perp \to x^\perp$.[2] If we begin instead with a C^2 function f on S^{n-1}, then we denote by $D^2 f(x)$ for $x \in S^{n-1}$ the restricted Hessian of its 1-homogeneous extension.

The restricted Hessian $D^2 f$ appears naturally when performing calculus with support functions. For example, we have the following basic result.[3]

Lemma 2.1. *Let $f : S^{n-1} \to \mathbb{R}$ be a C^2 function. Then $f = h_K$ for some convex body K if and only if $D^2 f(x) \geq 0$ for all $x \in S^{n-1}$.*

Proof. As support functions are convex, clearly $D^2 h_K \geq 0$. Conversely, suppose that $D^2 f \geq 0$. Then the 1-homogeneous extension of f is convex, so it can be written as the supremum of affine functions $f(x) = \sup_{y \in A}\{\langle y, x\rangle - f^*(y)\}$. It is readily verified that 1-homogeneity implies $f^* = 0$, and that A is bounded as f is finite. Thus $f(x) = \sup_{y\in A}\langle y, x\rangle = h_{\overline{\mathrm{conv}}(A)}(x)$. $\square$

An key corollary is that any C^2 function is a difference of support functions.

Corollary 2.2. *Let $f : S^{n-1} \to \mathbb{R}$ be a C^2 function and L be a convex body such that $D^2 h_L > 0$. Then there is a convex body K and $a > 0$ such that $f = a(h_K - h_L)$. In particular, any C^2 function on S^{n-1} is the difference of two support functions.*

Proof. As S^{n-1} is compact and f, h_L are C^2 functions, we have $D^2 f \geq -\alpha I$ and $D^2 h_L \geq \beta I$ for some $\alpha, \beta > 0$. Thus $g := f + (\alpha/\beta)h_L$ satisfies $D^2 g \geq 0$, so $f = (\alpha/\beta)(h_K - h_L)$ for some convex body K by Lemma 2.1. We may always choose $L = B_2$ to be the Euclidean ball (as $D^2 h_{B_2} = I$). $\square$

A convex body K is of class C^k_+ ($k \geq 2$) if its support function h_K is C^k and satisfies $D^2 h_K > 0$. Such bodies will allow us to perform all the calculus we need; see [16, section 2.5] for a detailed study of the regularity of such bodies. For our purposes, working with C^∞_+ bodies entails no loss of generality, cf. Remark 1.2. As the approximation argument is unrelated to the topic of this paper, we omit further discussion and refer instead to [16, sections 3.4 and 5.1].

2.2. **Representation of volumes and mixed volumes.** We now prove (1.2) and (1.4). To prove (1.2), we first use the divergence theorem to write $\mathrm{Vol}(K)$ as an integral over ∂K; then we change variables using the outer unit normal vector $n_K : \partial K \to S^{n-1}$ to map the integral to S^{n-1}. The term $\det(D^2 h_K)$ that appears in (1.2) is precisely the Jacobian of this transformation.

Lemma 2.3. *Let K be a C^2_+ convex body. Then*

$$\mathrm{Vol}(K) = \frac{1}{n}\int_{S^{n-1}} h_K \det(D^2 h_K)\, d\omega.$$

Proof. By the divergence theorem,

$$\mathrm{Vol}(K) = \frac{1}{n}\int_K \mathrm{div}(x)\, dx = \int_{\partial K} \langle x, n_K(x)\rangle\, d\omega_K(x),$$

[2] By choosing a basis of $x^\perp$, one may express $D^2 f(x)$ as an $(n-1)$-dimensional matrix. However, we only use determinants and mixed discriminants of such matrices which are basis-independent.

[3] The notation $M > 0$ ($M \geq 0$) denotes that M is positive definite (positive semidefinite).

where ω_K is the surface measure on ∂K and n_K is the outer unit normal. Now note that ∇h_K (the gradient is in $\mathbb{R}^n$) maps $u \in S^{n-1}$ to $\nabla h_K(u) = \arg\max_{y\in K}\langle y, u\rangle \in \partial K$. As $D^2 h_K > 0$, the map $\nabla h_K : S^{n-1} \to \partial K$ is a diffeomorphism. Thus

$$\mathrm{Vol}(K) = \frac{1}{n}\int_{S^{n-1}} \langle \nabla h_K, n_K(\nabla h_K)\rangle \det(D^2 h_K)\, d\omega$$

by the change of variables formula. It remains to note that $\nabla h_K = n_K^{-1}$: indeed, as $\langle y - x, n_K(x)\rangle \le 0$ for $x \in \partial K$ and $y \in K$ by convexity, we have $\nabla h_K(n_K(x)) = \arg\max_{y\in K}\langle y, n_K(x)\rangle = x$. As clearly $\langle \nabla h_K(u), u\rangle = \max_{y\in K}\langle y, u\rangle = h_K(u)$, it follows that $\langle \nabla h_K, n_K(\nabla h_K)\rangle = h_K$, and the proof is complete. □

Lemma 2.3 shows that volume is a polynomial in the sense of (1.1), but this does not immediately yield (1.4): choosing $K = \lambda_1 K_1 + \cdots + \lambda_n K_n$ in Lemma 2.3 and using (1.3) would give (1.4) averaged over all permutations of $K_1, \ldots, K_n$. To prove a non-symmetric representation, it is convenient to first prove a special case.

Lemma 2.4. *Let K, L be C^2_+ convex bodies. Then*

$$\mathsf{V}(K, L, \ldots, L) = \frac{1}{n}\int_{S^{n-1}} h_K \det(D^2 h_L)\, d\omega.$$

Proof. The idea is to repeat the proof of Lemma 2.3, but replacing $\mathrm{div}(x)$ by $\mathrm{div}(Y)$ for some suitably chosen vector field Y. More precisely, let Y be a bounded Lipschitz vector field. Then $I - t\nabla Y$ is nonsingular for sufficiently small t. Therefore

$$\lim_{t\to 0}\frac{1}{t}\bigg\{\int_{\mathbb{R}^n} 1_L(x - tY(x))\, dx - \mathrm{Vol}(L)\bigg\} =$$
$$\lim_{t\to 0}\int_{\mathbb{R}^n} 1_L(x - tY(x))\,\frac{1 - \det(I - t\nabla Y(x))}{t}\, dx = \int_L \mathrm{div}(Y)\, dx = \int_{\partial L}\langle Y, n_L\rangle\, d\omega_L,$$

where we used the change of variables formula in the first step, and the divergence theorem in the last step. Now take the supremum on both sides over Lipschitz vector fields Y taking values in K. As $1_L(x - tY(x)) \le 1_{L+tK}(x)$ for any such Y,

$$\begin{aligned} n\mathsf{V}(K, L, \ldots, L) &= \lim_{t\to 0}\frac{\mathrm{Vol}(L + tK) - \mathrm{Vol}(L)}{t} \\ &\ge \int_{\partial L} h_K(n_L)\, d\omega_L = \int_{S^{n-1}} h_K \det(D^2 h_L)\, d\omega, \end{aligned}$$

where we changed variables in the last step using ∇h_L as in Lemma 2.3.

To obtain the reverse inequality, note that by Corollary 2.2, there is a C^2_+ body C and $a > 0$ such that $-h_K = a(h_C - h_L)$. As mixed volumes are linear in each argument (this follows from (1.1)), $\mathsf{V}(K, L, \ldots, L) = a(\mathrm{Vol}(L) - \mathsf{V}(C, L, \ldots, L))$. Applying the above inequality to $\mathsf{V}(C, L, \ldots, L)$ and Lemma 2.3, we readily obtain the reversed inequality for $\mathsf{V}(K, L, \ldots, L)$. □

Choosing $K = K_1$, $L = \lambda_2 K_2 + \cdots + \lambda_n K_n$ in Lemma 2.4, and applying the definitions (1.1) and (1.3) of mixed volumes and discriminants, directly yields (1.4).

Corollary 2.5. *Let $K_1, \ldots, K_n$ be C^2_+ convex bodies. Then*

$$\mathsf{V}(K_1, \ldots, K_n) = \frac{1}{n}\int_{S^{n-1}} h_{K_1}\mathsf{D}(D^2 h_{K_2}, \ldots, D^2 h_{K_n})\, d\omega.$$

2.3. **Basic properties of mixed volumes and mixed discriminants.** We now proceed to recall the basic properties of mixed volumes and mixed discriminants.

Lemma 2.6 (Properties of mixed discriminants)**.** *Let $M, M_1, \ldots, M_{n-1}$ be symmetric $(n-1)$-dimensional matrices and U be an $(n-1)$-dimensional matrix.*

(a) $\mathsf{D}(M, \ldots, M) = \det(M)$.
(b) $\mathsf{D}(M_1, \ldots, M_{n-1})$ *is symmetric and multilinear in its arguments.*
(c) $\mathsf{D}(UM_1U^*, \ldots, UM_{n-1}U^*) = \det(UU^*)\mathsf{D}(M_1, \ldots, M_{n-1})$.
(d) $\mathsf{D}(M_1, \ldots, M_{n-1}) \geq 0$ *if* $M_1, M_2, \ldots, M_{n-1} \geq 0$.
(e) $\mathsf{D}(M_1, \ldots, M_{n-1}) > 0$ *if* $M_2, \ldots, M_{n-1} > 0$ *and* $M_1 \geq 0$, $M_1 \neq 0$.
(f) $\mathsf{D}(e_ie_i^*, M_2, \ldots, M_{n-1}) = \frac{1}{n-1}\mathsf{D}(M_2^{\langle i\rangle}, \ldots, M_{n-1}^{\langle i\rangle})$, *where* $\{e_i\}$ *is the standard basis in* $\mathbb{R}^{n-1}$ *and* $M^{\langle i\rangle}$ *is obtained from* M *by removing its* i*-th row and column.*

Remark 2.7. Note that, by definition, the mixed discriminant of k-dimensional matrices has k arguments. Therefore, as no confusion can arise, we denote mixed discriminants in every dimension by the same symbol D (e.g., as in Lemma 2.6(f)).

Proof. Parts (a) and (b) follow directly from the definition (1.3). Part (c) also follows from (1.3) using $\det(UMU^*) = \det(UU^*)\det(M)$. For the remaining parts, it is useful to compute the mixed discriminant of rank one matrices. Let $v_1, \ldots, v_{n-1} \in \mathbb{R}^{n-1}$ be the columns of a matrix V. Then $\det\big(\sum_{i=1}^{n-1} v_iv_i^*\big) = \det(VV^*) = \det(V)^2$. By scaling v_i we obtain $\det\big(\sum_{i=1}^{n-1} \lambda_iv_iv_i^*\big) = \lambda_1 \cdots \lambda_{n-1}\det(V)^2$, so (1.3) implies

$$\mathsf{D}(v_1v_1^*, \ldots, v_{n-1}v_{n-1}^*) = \frac{\det(V)^2}{(n-1)!} \geq 0. \tag{2.1}$$

Part (d) now follows from linearity of mixed discriminants, as any $M \geq 0$ can be written as the sum of rank one matrices of the form vv^*. If $M_1 \geq 0$, $M_1 \neq 0$ and $M_i > 0$ for $i \geq 2$, we can write $M_i = M_i' + v_iv_i^*$ for each i where $M_i' \geq 0$ and $v_1, \ldots, v_{n-1}$ are linearly independent. Then part (e) follows by observing that $\mathsf{D}(v_1v_1^*, \ldots, v_{n-1}v_{n-1}^*) > 0$ by (2.1). Finally, part (f) follows for $M_i = v_iv_i^*$ directly from (2.1), and extends to general M_i by linearity. □

Lemma 2.8 (Properties of mixed volumes)**.** *Let $K, K_1, \ldots, K_n$ be convex bodies.*

(a) $\mathsf{V}(K, \ldots, K) = \mathrm{Vol}(K)$.
(b) $\mathsf{V}(K_1, \ldots, K_n)$ *is symmetric and multilinear in its arguments.*
(c) $\mathsf{V}(K_1, \ldots, K_n)$ *is invariant under translation* $K_i \mapsto K_i + z_i$.
(d) $\mathsf{V}(K_1, \ldots, K_n) \geq 0$.

Proof. Parts (a) and (b) follow directly from the definition (1.1). Part (c) also follows from (1.1) using $\mathrm{Vol}(K) = \mathrm{Vol}(K + z)$. To prove part ($d$), we may assume without loss of generality that $0 \in K_1$ by translation-invariance, which implies $h_{K_1} \geq 0$. Then part (d) follows for C_+^2 bodies from Corollary 2.5 and Lemma 2.6(d), and for general bodies by approximation (cf. Remark 1.2). □

2.4. **Hyperbolic quadratic forms.** We conclude with a proof of Lemma 1.4; we in fact add an equivalent condition that will be useful in the proof of Theorem 1.3.

Lemma 2.9 (Hyperbolic quadratic forms)**.** *Let A be a symmetric matrix. Then the following conditions are equivalent:*

1. $\langle x, Ay\rangle^2 \geq \langle x, Ax\rangle\langle y, Ay\rangle$ *for all* x, y *such that* $\langle y, Ay\rangle \geq 0$.
2. *There exists a vector* w *such that* $\langle x, Ax\rangle \leq 0$ *for all* x *such that* $\langle x, Aw\rangle = 0$.

3. The positive eigenspace of A has dimension at most one.

The conclusion remains valid if A is a self-adjoint operator on a Hilbert space with a discrete spectrum, provided the vectors x, y, w are chosen in the domain of A.

Proof. If A is negative semidefinite, the conclusion is trivial. Let us therefore assume that A has an eigenvector v with positive eigenvalue $\lambda > 0$.

$3 \Rightarrow 2$: by assumption, the second-largest eigenvalue λ_2 of A is nonpositive, so

$$0 \geq \lambda_2 = \max\{\langle x, Ax\rangle : \|x\| = 1,\ \langle x, v\rangle = 0\}.$$

As $\lambda\langle x, v\rangle = \langle x, Av\rangle$, we may choose $w = v$.

$2 \Rightarrow 1$: assume $\langle y, Ay\rangle > 0$ (else the conclusion is trivial). Then $\langle y, Aw\rangle \neq 0$, so we may define $z = x - ay$ with $a = \langle x, Aw\rangle/\langle y, Aw\rangle$. As $\langle z, Aw\rangle = 0$, we obtain

$$0 \geq \langle z, Az\rangle = \langle x, Ax\rangle - 2a\langle x, Ay\rangle + a^2\langle y, Ay\rangle \geq \langle x, Ax\rangle - \frac{\langle x, Ay\rangle^2}{\langle y, Ay\rangle},$$

where the last inequality is obtained by minimizing over a.

$1 \Rightarrow 3$: let $u \perp v$ be an eigenvector of A with eigenvalue μ. Then we obtain $0 = \langle v, Au\rangle^2 \geq \lambda\mu\|v\|^2\|u\|^2$. As $\lambda > 0$, we must have $\mu \leq 0$. □

Remark 2.10. The assumption that A has a discrete spectrum ensures that the proof extends *verbatim* to the infinite-dimensional setting (for the variational characterization of eigenvalues used in the proof of $3 \Rightarrow 2$, see, e.g., [11, eq. (8.94)]). This assumption is not really necessary, see [8, p. 184] for a more general formulation. However, the present simple formulation suffices for our purposes.

3. The Alexandrov-Fenchel inequality

In this section we will prove the Alexandrov-Fenchel inequality assuming the validity of Alexandrov's inequality for mixed discriminants. The idea of the proof was already explained in section 1.3, and it remains to spell out the details.

Throughout this section, we fix C_+^∞ convex bodies $C_1, \ldots, C_{n-2}$. For reasons that will become clear shortly, we will also assume that $0 \in \operatorname{int} C_1$. The latter entails no loss of generality: C_+^∞ bodies have nonempty interior, and thus we may assume $0 \in \operatorname{int} C_1$ by translation-invariance of mixed volumes (Lemma 2.8(c)).

We begin by expressing mixed volume as the quadratic form of a suitably chosen operator. While the most obvious choice is (1.5), we do not know much *a priori* about where its eigenvalues are located. Instead, we will choose a different normalization that fixes the top eigenvalue. To this end, let us define

$$\mathscr{A} f := \frac{h_{C_1}\mathsf{D}(D^2 f, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})}{\mathsf{D}(D^2 h_{C_1}, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})}$$

for any C^2 function f. That is, $\mathscr{A} f$ is obtained by rescaling the operator of (1.5) by some positive function. Correspondingly, if we define a measure on S^{n-1} by

$$d\mu := \frac{1}{n}\frac{\mathsf{D}(D^2 h_{C_1}, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})}{h_{C_1}}\,d\omega,$$

then (1.4) can clearly be written as

$$\mathsf{V}(K, L, C_1, \ldots, C_{n-2}) = \langle h_K, \mathscr{A} h_L\rangle_{L^2(\mu)} := \int h_K\,\mathscr{A} h_L\,d\mu.$$

Note that all the above objects are well defined, as $h_{C_1} > 0$ because we assumed $0 \in \operatorname{int} C_1$, and as $\mathsf{D}(D^2 h_{C_1}, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}}) > 0$ by Lemma 2.6(e).

The point of scaling the operator in this manner is that now, by definition, $\mathscr{A} h_{C_1} = h_{C_1}$. Thus $\mathscr{A}$ has eigenvalue 1, and an associated eigenvector h_{C_1} that is strictly positive. Let us collect a few basic facts about the operator $\mathscr{A}$.

- $\mathscr{A}$ is a uniformly elliptic operator (it is increasing as a function of $D^2 f$ in the positive semidefinite order); this follows from Lemma 2.6(e).
- $\mathscr{A}$ defines a symmetric quadratic form $\langle f, \mathscr{A} g\rangle_{L^2(\mu)} = \langle g, \mathscr{A} f\rangle_{L^2(\mu)}$ for $f, g \in C^2$; this follows from Lemma 2.8(b) and Corollary 2.2.
- $\mathscr{A}$ extends to a self-adjoint operator with a discrete spectrum; its largest eigenvalue is 1 and the corresponding eigenspace is spanned by h_{C_1}; and all its eigenfunctions are C^∞. This follows from standard elliptic regularity theory [11, §8.12].

These facts may be viewed in essence as an infinite-dimensional analogue of the Perron-Frobenius theorem [5]: a uniformly elliptic operator on a compact manifold behaves much like a positive matrix, in particular, it has a unique positive eigenvector and the associated eigenvalue is maximal. The use of elliptic operators is convenient but not essential; an alternative approach is sketched in section 5.

We now arrive at the key observation of this paper.

Lemma 3.1. *For any function $f \in C^2$, we have*
$$\langle \mathscr{A} f, \mathscr{A} f\rangle_{L^2(\mu)} \geq \langle f, \mathscr{A} f\rangle_{L^2(\mu)}.$$

Proof. In the present notation, the statement of Theorem 1.3 can be written as
$$(\mathscr{A} f)^2 \geq h_{C_1}^2 \frac{\mathsf{D}(D^2 f, D^2 f, D^2 h_{C_2}, \ldots, D^2 h_{C_{n-2}})}{\mathsf{D}(D^2 h_{C_1}, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})}.$$
Integrating both sides with respect to μ yields
$$\begin{aligned}\int (\mathscr{A} f)^2\, d\mu &\geq \frac{1}{n}\int h_{C_1} \mathsf{D}(D^2 f, D^2 f, D^2 h_{C_2}, \ldots, D^2 h_{C_{n-2}})\, d\omega \\ &= \frac{1}{n}\int f\, \mathsf{D}(D^2 f, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})\, d\omega = \langle f, \mathscr{A} f\rangle_{L^2(\mu)},\end{aligned}$$
where we used the symmetry of mixed volumes to exchange the role of h_{C_1} and f (using Corollary 2.5, Lemma 2.8(b), and Corollary 2.2). ☐

The proof of the Alexandrov-Fenchel inequality is now almost immediate.

Proof of Theorem 1.1. Let f be an eigenfunction of $\mathscr{A}$ with eigenvalue λ. Then Lemma 3.1 yields $\lambda^2 \geq \lambda$, so $\lambda \geq 1$ or $\lambda \leq 0$. Thus the positive eigenspace of $\mathscr{A}$ is spanned by h_{C_1}, and we conclude by invoking Lemma 1.4. ☐

Remark 3.2. The proof of Theorem 1.1 shows that $\mathscr{A}$ has a one-dimensional positive eigenspace, so the Alexandrov-Fenchel inequality follows from Lemma 1.4. While we did not use this in the proof, we stated in the introduction that the Alexandrov-Fenchel inequality is in fact *equivalent* to this spectral statement. This may not be entirely obvious, however, as the Alexandrov-Fenchel inequality only yields condition 1 of Lemma 1.4 when x, y are support functions.

For completeness, let us show that the spectral property of $\mathscr{A}$ is in fact also a consequence of the Alexandrov-Fenchel inequality. Let f be any C^2 function. By Corollary 2.2, $f + ah_{C_1}$ is a support function for a sufficiently large, so that
$$\langle f + ah_{C_1}, \mathscr{A} h_{C_1}\rangle_{L^2(\mu)}^2 \geq \langle f + ah_{C_1}, \mathscr{A}(f + ah_{C_1})\rangle_{L^2(\mu)} \langle h_{C_1}, \mathscr{A} h_{C_1}\rangle_{L^2(\mu)}$$

by the Alexandrov-Fenchel inequality. Expanding both sides yields

$$\langle f, \mathscr{A} h_{C_1}\rangle^2_{L^2(\mu)} \geq \langle f, \mathscr{A} f\rangle_{L^2(\mu)} \langle h_{C_1}, \mathscr{A} h_{C_1}\rangle_{L^2(\mu)}.$$

If we now choose $f \perp h_{C_1}$ to be any eigenfunction of $\mathscr{A}$ with eigenvalue μ, this inequality shows that $\mu \leq 0$, establishing the claim.

4. Alexandrov's mixed discriminant inequality

In this section we will prove Theorem 1.3 using the same method as in section 3. The main new difficulty is that the mixed discriminant inequality is an inequality for matrices rather than for vectors: as matrix multiplication is noncommutative, it is not clear how to define the normalized operator as in the previous section. It turns out that a second application of Lemma 2.9 allows us to reduce the problem to a special case where the relevant matrices are diagonal; the latter can be handled by repeating almost verbatim the argument of section 3.

In the present setting, the proof proceeds by induction on the dimension. Let us first dispose of the base of the induction, which follows from a trivial computation.

Lemma 4.1. *Let A, B be 2×2 matrices. Then $\mathsf{D}(A,B)^2 \geq \mathsf{D}(A,A)\,\mathsf{D}(B,B)$.*

Proof. The general case is reduced to the case $B = I$ by applying Lemma 2.6(c) with $U = B^{-1/2}$ to both sides of the inequality. Moreover, by an appropriate choice of basis, we may assume without loss of generality that A is diagonal. Then we have $\det(A + tI) = (a_{11} + t)(a_{22} + t)$, so $\mathsf{D}(A, I) = \frac{1}{2}(a_{11} + a_{22})$ and $\mathsf{D}(A, A) = a_{11}a_{22}$. Thus the desired inequality $(a_{11} + a_{22})^2 \geq 4a_{11}a_{22}$ is elementary. □

We now proceed with the induction argument: in the remainder of this section we assume that Theorem 1.3 is valid for $(n-1)$-dimensional matrices (for $n \geq 3$), and we will show that it must also be valid for n-dimensional matrices.

We begin by proving a "commutative" special case: note that the quadratic form in the following proof acts on vectors rather than matrices.

Lemma 4.2. *Let $n \geq 3$ and let $M_2, \ldots, M_{n-2}$ be n-dimensional positive definite matrices. Then for any n-dimensional* diagonal *matrix Z, we have*

$$\mathsf{D}(Z, I, I, M_2, \ldots, M_{n-2})^2 \geq \mathsf{D}(Z, Z, I, M_2, \ldots, M_{n-2})\,\mathsf{D}(I, I, I, M_2, \ldots, M_{n-2}).$$

(*When $n = 3$, the statement should be read as $\mathsf{D}(Z, I, I)^2 \geq \mathsf{D}(Z, Z, I)\,\mathsf{D}(I, I, I)$.*)

Proof. Define for $x, y \in \mathbb{R}^n$ the quadratic form

$$\begin{aligned} Q(x,y) &:= \mathsf{D}(\mathrm{diag}(x), \mathrm{diag}(y), I, M_2, \ldots, M_{n-2}) \\ &= \frac{1}{n}\sum_{i=1}^n x_i\,\mathsf{D}(\mathrm{diag}(y)^{\langle i\rangle}, I^{\langle i\rangle}, M_2^{\langle i\rangle}, \ldots, M_{n-2}^{\langle i\rangle}), \end{aligned}$$

where we used Lemma 2.6(f) (recall that $M^{\langle i\rangle}$ is the $(n-1)$-dimensional matrix obtained from the n-dimensional matrix M by removing its ith row and column). This formula will play the role of (1.4) in the present setting.

We now proceed as in section 3. Define the $n \times n$ matrix A and $p \in \mathbb{R}^n$ by

$$\begin{aligned} (Ay)_i &:= \frac{\mathsf{D}(\mathrm{diag}(y)^{\langle i\rangle}, I^{\langle i\rangle}, M_2^{\langle i\rangle}, \ldots, M_{n-2}^{\langle i\rangle})}{\mathsf{D}(I^{\langle i\rangle}, I^{\langle i\rangle}, M_2^{\langle i\rangle}, \ldots, M_{n-2}^{\langle i\rangle})}, \\ p_i &:= \frac{1}{n}\mathsf{D}(I^{\langle i\rangle}, I^{\langle i\rangle}, M_2^{\langle i\rangle}, \ldots, M_{n-2}^{\langle i\rangle}) \end{aligned}$$

for $y \in \mathbb{R}^n$. Then $Q(x,y) = \langle x, Ay\rangle_{\ell^2(p)}$, where $\langle x,y\rangle_{\ell^2(p)} := \sum_i x_i y_i p_i$. As $Q(x,y)$ is symmetric, A is self-adjoint on $\ell^2(p)$. Moreover, clearly $A1 = 1$. Finally, note that A is a positive matrix by Lemma 2.6(e). Therefore, by the Perron-Frobenius theorem [5, Theorem 1.4.4], A has largest eigenvalue 1 and this eigenvalue is simple.

Now recall that we assumed the validity of Theorem 1.3 for $(n-1)$-dimensional matrices. The latter implies, exactly as in the proof of Lemma 3.1, that

$$(Ay)_i^2 p_i \geq \frac{1}{n}\mathsf{D}(\mathrm{diag}(y)^{\langle i\rangle}, \mathrm{diag}(y)^{\langle i\rangle}, M_2^{\langle i\rangle}, \ldots, M_{n-2}^{\langle i\rangle}).$$

Summing both sides over i and applying Lemma 2.6(f) yields

$$\langle Ay, Ay\rangle_{\ell^2(p)} \geq \mathsf{D}(I, \mathrm{diag}(y), \mathrm{diag}(y), M_2, \ldots, M_{n-2}) = \langle y, Ay\rangle_{\ell^2(p)}.$$

By choosing y to be an eigenvector of A, we find that any eigenvalue λ of A satisfies $\lambda^2 \geq \lambda$, so $\lambda \geq 1$ or $\lambda \leq 0$. But as 1 is the maximal eigenvalue and this eigenvalue is simple, we have shown that A has a one-dimensional positive eigenspace. Therefore, Lemma 2.9($3 \Rightarrow 1$) implies the desired conclusion $Q(x,1)^2 \geq Q(x,x)\,Q(1,1)$. ☐

It remains to show that the mixed discriminant inequality for arbitrary n-dimensional matrices can be reduced to the special case of Lemma 4.2.

Corollary 4.3. *Let $n \geq 3$ and let $B, M_1, \ldots, M_{n-2}$ be n-dimensional positive semidefinite matrices. Then for any n-dimensional symmetric matrix A, we have*

$$\mathsf{D}(A, B, M_1, \ldots, M_{n-2})^2 \geq \mathsf{D}(A, A, M_1, \ldots, M_{n-2})\,\mathsf{D}(B, B, M_1, \ldots, M_{n-2}).$$

Proof. We may assume without loss of generality that $M_1, \ldots, M_{n-2}$ are positive definite (otherwise, replace M_i by $M_i + \varepsilon I$ and let $\varepsilon \to 0$ at the end). Moreover, applying Lemma 2.6(c) with $U = M_1^{-1/2}$, we may assume that $M_1 = I$.

We now define the quadratic form $\mathbf{Q}(Z, Z') := \mathsf{D}(Z, Z', I, M_2, \ldots, M_{n-2})$ on the space of n-dimensional symmetric matrices. It follows immediately from Lemma 4.2 and Lemma 2.6(e) that $\mathbf{Q}(Z, I) = 0$ implies $\mathbf{Q}(Z,Z) \leq 0$ for any diagonal matrix Z. The same conclusion follows for any symmetric matrix Z, as we may always reduce to the diagonal case by a change of basis. Thus $\mathbf{Q}(A,B)^2 \geq \mathbf{Q}(A,A)\,\mathbf{Q}(B,B)$ by Lemma 2.9($2 \Rightarrow 1$), which concludes the proof. ☐

5. An alternative approach using polytopes

Two different approaches to the proof of the Alexandrov-Fenchel inequality appear already in Alexandrov's work. One approach [2] focuses attention on smooth bodies, which gives rise to elliptic operators. The other (historically earlier) approach [1] is to focus instead on polytopes. Because polytopes have a finite number of normal directions, the role of elliptic operators is replaced here by finite-dimensional matrices. The latter may be considered more "elementary", in that the proof requires in principle only linear algebra and basic geometry.

The present authors find computations with polytopes somewhat less clean and intuitive than the smooth approach. Nonetheless, the polytope method is of interest in its own right. The aim of this section is to sketch how our methods may be implemented in the polytope setting. The following discussion is not fully self-contained; we refer to [16] for proofs of the basic polytope representations of mixed volumes, and focus on adapting the our methods to this context.

Let $P_1, \ldots, P_n$ be polytopes in $\mathbb{R}^n$. We denote by $F(P, u)$ the face of the polytope P with normal direction $u \in S^{n-1}$. The following expression[4] is the analogue for polytopes of the representation (1.4) of mixed volumes of C^2_+ bodies [16, (5.23)]:

$$\mathsf{V}(P_1, \ldots, P_n) = \frac{1}{n} \sum_{u \in S^{n-1}} h_{P_1}(u)\, \mathsf{V}(F(P_2, u), \ldots, F(P_n, u)). \tag{5.1}$$

Implicit in the notation is that $\mathsf{V}(F(P_2, u), \ldots, F(P_n, u))$ is nonzero only at a finite number of points u on the sphere; it suffices to restrict the sum to the normal directions of the facets ($(n-1)$-dimensional faces) of $P_2 + \cdots + P_n$.

We would like to think of the restriction of h_{P_i} to the relevant normal directions as finite-dimensional vectors, and of mixed volume as a quadratic form of such vectors. The problem with (5.1) is that $\mathsf{V}(F(P_2, u), \ldots, F(P_n, u))$ is not naturally expressed in terms of h_{P_2}, but rather in terms of $h_{F(P_2,u)}$. It is therefore unclear how we may view (5.1) as a quadratic form of the support vectors of the original polytopes. It turns out that this can be done, and that one can recover various properties of mixed volumes that appeared naturally in the smooth setting, if one restricts attention to certain "nice" families of polytopes.

In the following, we will call polytopes $P_1, \ldots, P_n$ *strongly isomorphic* if

$$\dim F(P_1, u) = \dim F(P_2, u) = \cdots = \dim F(P_n, u) \text{ for all } u \in S^{n-1}.$$

In this setting, the sum in (5.1) ranges over the common normal directions Ω of the facets of P_i, and $h_{F(P_i,u)}$ is a linear function (independent of i) of the restriction of h_{P_i} to Ω [16, p. 276]. We also recall that a polytope P in $\mathbb{R}^n$ is called *simple* if it has nonempty interior and each vertex is contained in exactly n facets.

Lemma 5.1. *Let $P_3, \ldots, P_n$ be simple strongly isomorphic polytopes in $\mathbb{R}^n$, and let $\Omega \subset S^{n-1}$ be the common normal directions of facets of P_i. Denote by $\mathrm{h}_{P_i} := (h_{P_i}(u))_{u \in \Omega} \in \mathbb{R}^{|\Omega|}$ the support vector of P_i. Then:*

(a) For every $x \in \mathbb{R}^{|\Omega|}$ and polytope P strongly isomorphic to P_i, there is a polytope Q strongly isomorphic to P_i and $a > 0$ such that $x = a(\mathrm{h}_Q - \mathrm{h}_P)$.

(b) There is a $|\Omega|$-dimensional symmetric matrix $\tilde{A}$ such that

$$(\tilde{A}\mathrm{h}_P)_u = \frac{1}{n}\mathsf{V}(F(P, u), F(P_3, u), \ldots, F(P_n, u))$$

for every $u \in \Omega$ and polytope P strongly isomorphic to P_i.

(c) $\tilde{A} = L + D$ for an irreducible nonnegative matrix L and diagonal matrix D.

Moreover, any family of convex bodies $C_1, \ldots, C_n$ can be approximated arbitrarily well in the Hausdorff metric by simple strongly isomorphic polytopes $P_1, \ldots, P_n$.

Proof. Part (a) follows from [16, Lemma 2.4.13]. Parts (b) and (c) may be read off from the explicit expression given in the proof of [16, Lemma 5.1.5]; in particular, irreducibility follows as the facet graph of a polytope is connected (this standard fact follows by duality from [7, Theorem 15.5]). That arbitrary bodies may be approximated by simple strongly isomorphic polytopes is [16, Theorem 2.4.15]. □

[4] By definition $F(P_i, u)$, $i = 2, \ldots, n$ all lie in the $(n-1)$-dimensional space $u^\perp \subset \mathbb{R}^n$ modulo translation. By a slight abuse of notation, we denote by $\mathsf{V}(F(P_2, u), \ldots, F(P_n, u))$ the $(n-1)$-dimensional mixed volume of the translated faces in $u^\perp$ (cf. Remark 2.7).

In comparison with the smooth setting, part (a) of this lemma is analogous to Corollary 2.2; $\tilde{A}$ is analogous to (1.5); and part (c) corresponds to ellipticity.

It will be convenient to extend mixed volumes linearly as follows: whenever $x = \mathrm{h}_Q - \mathrm{h}_{Q'}$ for polytopes Q, Q' strongly isomorphic to P_i, we define

$$\mathsf{V}(x, P_2, \ldots, P_n) := \mathsf{V}(Q, P_2, \ldots, P_n) - \mathsf{V}(Q', P_2, \ldots, P_n),$$

and for $u \in \Omega$

$$\begin{aligned}&\mathsf{V}(F(x,u), F(P_3,u), \ldots, F(P_n,u)) := \\ &\qquad \mathsf{V}(F(Q,u), F(P_3,u), \ldots, F(P_n,u)) - \mathsf{V}(F(Q',u), F(P_3,u), \ldots, F(P_n,u))\end{aligned}$$

(the latter notation is justified by Lemma 5.1(b)). By Lemma 5.1 and the representation (5.1), we can then write for any $x, y \in \mathbb{R}^{|\Omega|}$

$$\begin{aligned}(\tilde{A}x)_u &= \frac{1}{n}\mathsf{V}(F(x,u), F(P_3,u), \ldots, F(P_n,u)), \\ \langle x, \tilde{A}y\rangle &= \mathsf{V}(x, y, P_3, \ldots, P_n).\end{aligned}$$

We are now ready to prove the Alexandrov-Fenchel inequality for polytopes.

Theorem 5.2. *Let $P, P_3, \ldots, P_n$ be simple strongly isomorphic polytopes in $\mathbb{R}^n$ with common facet directions $\Omega \subset S^{n-1}$. Then for every $x \in \mathbb{R}^{|\Omega|}$*

$$\mathsf{V}(x, P, P_3, \ldots, P_n)^2 \geq \mathsf{V}(x, x, P_3, \ldots, P_n)\mathsf{V}(P, P, P_3, \ldots, P_n).$$

In particular, by the last part of Lemma 5.1, this implies Theorem 1.1.

Proof. The proof will proceed by induction on the dimension n.

For $n = 2$, the Alexandrov-Fenchel inequality $\mathsf{V}(K,L)^2 \geq \mathsf{V}(K,K)\,\mathsf{V}(L,L)$ follows easily from the Brunn-Minkowski theorem [16, Theorem 7.2.1]. This implies the result when $x = \mathrm{h}_Q$ is the support vector of a polytope strongly isomorphic to P. The general case $x \in \mathbb{R}^{|\Omega|}$ now follows from Lemma 5.1(a) as in Remark 3.2.

We now proceed to the induction step; that is, we will assume the theorem is valid for polytopes in $\mathbb{R}^{n-1}$ with $n \geq 3$, and aim to conclude it is also valid for polytopes in $\mathbb{R}^n$. To this end, define the $|\Omega|$-dimensional matrix A and $p \in \mathbb{R}^{|\Omega|}$ as

$$\begin{aligned}(Ax)_u &:= \frac{h_{P_3}(u)\mathsf{V}(F(x,u), F(P_3,u), \ldots, F(P_n,u))}{\mathsf{V}(F(P_3,u), F(P_3,u), \ldots, F(P_n,u))}, \\ p_u &:= \frac{1}{n}\frac{\mathsf{V}(F(P_3,u), F(P_3,u), \ldots, F(P_n,u))}{h_{P_3}(u)}\end{aligned}$$

(as in section 3, we assume without loss of generality that $h_{P_3} > 0$). By definition, $\mathsf{V}(x, y, P_3, \ldots, P_n) = \langle x, Ay\rangle_{\ell^2(p)}$. Thus, as mixed volumes are symmetric, A is self-adjoint on $\ell^2(p)$. Moreover, A was defined so that $A\mathrm{h}_{P_3} = \mathrm{h}_{P_3}$. By Lemma 5.1$(c)$, the Perron-Frobenius theorem [5, Theorem 1.4.4] (applied to $A + cI$ for c sufficiently large) implies A has largest eigenvalue 1 and that this is a simple eigenvalue.

Now note that the facets of simple strongly isomorphic polytopes with a given normal direction are simple (cf. [7, Theorem 12.15] for this basic fact) and strongly isomorphic (by definition). Thus the induction hypothesis implies

$$\begin{aligned}(Ax)_u^2 p_u &= \frac{h_{P_3}(u)}{n}\frac{\mathsf{V}(F(x,u), F(P_3,u), \ldots, F(P_n,u))^2}{\mathsf{V}(F(P_3,u), F(P_3,u), \ldots, F(P_n,u))} \\ &\geq \frac{h_{P_3}(u)}{n}\mathsf{V}(F(x,u), F(x,u), F(P_4,u), \ldots, F(P_n,u)).\end{aligned}$$

Summing over u and using (5.1) and symmetry of mixed volumes yields

$$\langle Ax, Ax\rangle_{\ell^2(p)} \geq \mathsf{V}(P_3, x, x, P_4, \ldots, P_n) = \langle x, Ax\rangle_{\ell^2(p)}.$$

Choosing x to be an eigenvector of A, we find that any eigenvalue λ of A satisfies $\lambda^2 \geq \lambda$, so $\lambda \geq 1$ or $\lambda \leq 0$. But as 1 is the maximal eigenvalue of A and as it is a simple eigenvalue, the conclusion follows immediately from Lemma 1.4. □

6. Concluding remarks

6.1. **Alexandrov's proof.** Alexandrov's proof of the Alexandrov-Fenchel inequality [2] is very different in spirit than the method used in section 3. For sake of comparison, let us briefly sketch his approach.

Despite the evident similarity between Theorems 1.1 and 1.3, the mixed discriminant inequality is not used in a direct manner in Alexandrov's proof. Rather, it is used to establish an apparently unrelated fact: that the kernel of $\mathscr{A}$ has dimension n (it consists precisely of first-order spherical harmonics). Once this is known, one may establish the requisite spectral property of $\mathscr{A}$ by a homotopy method. For a special choice of bodies (e.g., as in section 6.3 below), an explicit computation shows that the positive eigenspace is one-dimensional. We now interpolate between these special bodies and the given bodies in Theorem 1.1. If the dimension of the positive eigenspace were to increase, then an eigenvalue must cross from below zero to above zero. But then the kernel of the operator must have dimension larger than n at the crossing point, which yields a contradiction.

In contrast, our method appears conceptually and technically simpler, as the mixed discriminant inequality yields the Alexandrov-Fenchel inequality directly by a one-line computation. In particular, we have no need to characterize any other properties of the operator in the proof (such as its kernel). Let us also note that our normalization of $\mathscr{A}$ is slightly different than the one employed by Alexandrov: Alexandrov defined the operator so that h_L, rather than h_{C_1}, is its top eigenvector. With this special choice, the final inequality follows directly without appealing to Lemma 1.4. However, in our approach, the choice h_{C_1} (or, equivalently, h_{C_i} for some i) plays a special role in the proof of Lemma 3.1. By fully exploiting Lemma 1.4 we gain significant flexibility, as is further illustrated in section 4.

6.2. **Equality cases.** It is not hard to deduce from the proof of Lemma 2.9 that equality $\langle x, Ay\rangle^2 = \langle x, Ax\rangle\langle y, Ay\rangle$ holds when $\langle y, Ay\rangle > 0$ if and only if $x - ay \in \ker A$ for some $a \in \mathbb{R}$. Thus Alexandrov's proof (cf. section 6.1), while somewhat circuitous, does provide additional information: it shows that equality holds in Theorem 1.1 for *smooth* bodies if and only if $h_K - ah_L$ is a linear function, i.e., when K and L are homothetic. (This is false for nonsmooth bodies, for which the characterization of equality cases remains open; cf. [16, section 7.6].)

Let us briefly sketch how the equality cases can be deduced from our approach. Let $f \in \ker \mathscr{A}$. Then the inequality in Lemma 3.1 holds with equality, and thus all inequalities in its proof must hold with equality. In particular, one has equality in Theorem 1.3 with $A = D^2 f$, $B = D^2 h_{C_1}$, and $M_i = D^2 h_{C_{i+1}}$. It is known that equality holds in Theorem 1.3 when $B, M_i > 0$ if and only if $A = \lambda B$ for some $\lambda \in \mathbb{R}$. Thus $D^2 f - \lambda D^2 h_{C_1} = 0$ for some $\lambda : S^{n-1} \to \mathbb{R}$. But as $\mathscr{A} f = 0$, we have

$$0 = \frac{\mathsf{D}(D^2 f - \lambda D^2 h_{C_1}, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})}{\mathsf{D}(D^2 h_{C_1}, D^2 h_{C_1}, \ldots, D^2 h_{C_{n-2}})} = -\lambda.$$

Thus we have shown that $D^2 f = 0$, so f must be a linear function.

Using similar reasoning, the abovementioned equality cases of Theorem 1.3 may be deduced from the proof given in section 4. We can similarly recover the equality cases of Theorem 5.2. We omit the details in the interest of space.

6.3. **The Bochner method.** The simple technique of this paper has its origin in the classical bound of Lichnerowicz on the spectral gap of the Laplacian on Riemannian manifolds with positive Ricci curvature [14]. This connection goes beyond an analogy between the proofs, as we will presently explain.

Let us briefly recall Lichnerowicz' argument. Let M be an $(n-1)$-dimensional compact Riemannian manifold. We denote by ∇_M the covariant derivative and by Δ_M the Laplacian. The basic observation of Lichnerowicz is that, by integrating the classical Bochner formula, one obtains the identity (cf. [10, Theorem 4.70])

$$\begin{aligned}\int_M (\Delta_M f)^2 &= \frac{n-1}{n-2}\int_M \mathrm{Ric}_M(\nabla_M f, \nabla_M f)\\ &\quad + \frac{1}{n-2}\int_M \Big\{(n-1)\mathrm{Tr}[(\nabla_M^2 f)^2] - \mathrm{Tr}[\nabla_M^2 f]^2\Big\}. \end{aligned} \tag{6.1}$$

Note that the last term in this expression is always nonnegative by Cauchy-Schwarz. If we specialize to the sphere $M = S^{n-1}$, the Ricci curvature tensor is given by $\mathrm{Ric}_{S^{n-1}}(X,X) = (n-2)\|X\|^2$, and we obtain after integrating by parts

$$\int_{S^{n-1}} (\Delta_{S^{n-1}} f)^2\, d\omega \ge -(n-1)\int_{S^{n-1}} f\Delta_{S^{n-1}} f\, d\omega. \tag{6.2}$$

Thus every eigenvalue λ of $-\Delta_{S^{n-1}}$ (which is positive semidefinite) must satisfy $\lambda^2 \ge (n-1)\lambda$, that is, $\lambda = 0$ or $\lambda \ge n-1$. As noted by Lichnerowicz, this argument applies to any Riemannian manifold M with $\mathrm{Ric}_M(X,X) \ge (n-2)\|X\|^2$.

The idea of Lichnerowicz to use an identity for $(\Delta_M f)^2$ to deduce spectral estimates for Δ_M forms the foundation for our proof of the Alexandrov-Fenchel inequality. However, the proof of (6.2), using the Bochner formula, is very different than the proof of Lemma 3.1. Remarkably, it turns out that not only the inequality (6.2), but even the Bochner identity (6.1) for $M = S^{n-1}$, is implicit in the proof of Lemma 3.1. Thus we may truly think of our method as a "Bochner method".

To recover (6.1) for $M = S^{n-1}$ from the proof of Lemma 3.1, we consider the special case where $C_1 = \cdots = C_{n-2} = B_2$ is the Euclidean ball. Then $h_{B_2} = 1$ and $D^2 h_{B_2} = I$. Differentiating $\det(I + tA)$ with respect to t and using (1.3) yields

$$\begin{aligned}\mathsf{D}(I,\ldots,I) &= \det(I) = 1,\\ \mathsf{D}(A,I,\ldots,I) &= \frac{1}{n-1}\mathrm{Tr}[A],\\ \mathsf{D}(A,A,I,\ldots,I) &= \frac{1}{(n-1)(n-2)}(\mathrm{Tr}[A]^2 - \mathrm{Tr}[A^2]).\end{aligned}$$

Moreover, by differentiating the 1-homogeneous extension $\|x\|f(x/\|x\|)$ of f, we find that $D^2 f = \nabla^2_{S^{n-1}} f + fI$ in terms of the covariant Hessian. In particular, we obtain in this special case $\mathscr{A}f = \frac{1}{n-1}\Delta_{S^{n-1}} f + f$, $d\mu = \frac{1}{n}d\omega$. We now compute

$$\begin{aligned}&\int (\Delta_{S^{n-1}} f)^2\, d\omega + (n-1)\int f\Delta_{S^{n-1}} f\, d\omega\\ &\quad = (n-1)^2\bigg(\int (\mathscr{A}f)^2\, d\omega - \int f\mathscr{A}f\, d\omega\bigg)\end{aligned}$$

$$= (n-1)^2 \int \{\mathsf{D}(D^2 f, I, \ldots, I)^2 - \mathsf{D}(D^2 f, D^2 f, I, \ldots, I)\}\, d\omega$$
$$= \frac{1}{n-2} \int \{(n-1)\mathrm{Tr}[(\nabla^2_{S^{n-1}} f)^2] - \mathrm{Tr}[\nabla^2_{S^{n-1}} f]^2\}\, d\omega.$$

Here the first equality follows by completing the square; the second equality is a reformulation of the proof of Lemma 3.1; and the third equality uses the explicit expressions for mixed volumes and $D^2 f$ given above. Thus we recovered (6.1) for $M = S^{n-1}$ as a special case of the proof of Lemma 3.1.

The connections hinted at here can be developed in far greater generality; however, as the geometric approach is somewhat tangential to the theme of this paper, we omit further discussion. Related ideas, inspired by complex geometry, were also obtained by D. Cordero-Erausquin and B. Klartag (personal communication).

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