ON MINKOWSKI'S MONOTONICITY PROBLEM

RAMON VAN HANDEL AND SHOUDA WANG

Abstract. We address an old open question in convex geometry that dates back to the work of Minkowski: what are the equality cases of the monotonicity of mixed volumes? The problem is equivalent to that of providing a geometric characterization of the support of mixed area measures. A conjectural characterization was put forward by Schneider (1985), but has been verified to date only for special classes of convex bodies. In this paper we resolve one direction of Schneider's conjecture for arbitrary convex bodies in $\mathbb{R}^n$, and resolve the full conjecture in $\mathbb{R}^3$. Among the implications of these results is a mixed counterpart of the classical fact, due to Monge, Hartman–Nirenberg, and Pogorelov, that a surface with vanishing Gaussian curvature is a ruled surface.

1. Introduction and main results

The foundation for the modern theory of convex geometry was laid by Minkowski in a seminal 1903 paper [11] and in a longer manuscript from around the same time that was published posthumously [12]. A number of basic questions that were raised in these works remain open to this day. The aim of the present paper is to significantly advance what is known about one of these problems: the characterization of the equality cases of the monotonicity of mixed volumes. We also discuss implications to the related notion of mixed Hessian measures, and to mixed analogues of the solution of homogeneous Monge-Ampère equations.

1.1. Mixed volumes and the monotonicity problem.

For any convex bodies $K_1, \ldots, K_m$ in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$, the volume

$$\mathrm{Vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \mathsf{V}_n(K_{i_1}, \ldots, K_{i_n})\, \lambda_{i_1} \cdots \lambda_{i_n}$$

is a homogeneous polynomial. Its coefficients $\mathsf{V}_n(C_1, \ldots, C_n)$, called *mixed volumes*, are important geometric parameters that capture many familiar quantities (volume, surface area, mean width, projection volumes, ...) as special cases [2, 21].

One of the simplest properties of mixed volumes is their monotonicity: if $K \subseteq L$ and $C_1, \ldots, C_{n-1}$ are convex bodies in $\mathbb{R}^n$, then

$$\mathsf{V}_n(K, C_1, \ldots, C_{n-1}) \leq \mathsf{V}_n(L, C_1, \ldots, C_{n-1}). \tag{1.1}$$

This paper is concerned with the following open problem of Minkowski [12, §28].

Problem 1.1. When does equality hold in (1.1)?

2020 *Mathematics Subject Classification.* 52A39; 52A40; 35J96.

Key words and phrases. Mixed volumes; mixed areas measures; mixed Hessian measures; extremum problems; convex geometry; homogeneous Monge-Ampère equations.

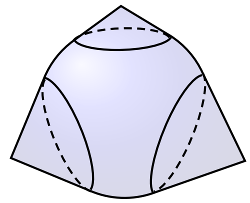

FIGURE 1.1. A cap body of B: i.e., the convex hull of B with a finite or countable number of points so that the cones emanating from the points are disjoint.

The analogous question for volume is trivial: $K \subseteq L$ and $\mathrm{Vol}(K) = \mathrm{Vol}(L) > 0$ imply $K = L$. In contrast, (1.1) has a rich family of equality cases that gives rise to surprising phenomena. Let us illustrate this with an example in $\mathbb{R}^3$. Thoughout this paper, B always denotes the Euclidean unit ball.

Example 1.2. For any convex body K in $\mathbb{R}^3$, the surface area and mean width can be expressed as $\mathsf{V}_3(K, K, B) = \frac{1}{3}\mathrm{S}(K)$ and $\mathsf{V}_3(K, B, B) = \frac{2\pi}{3}\mathrm{W}(K)$, respectively. Thus (1.1) gives rise to the geometric inequalities

$$2\pi \,\mathrm{inr}(K)\, \mathrm{W}(K) \le \mathrm{S}(K) \le 2\pi \,\mathrm{outr}(K)\, \mathrm{W}(K),$$

where $\mathrm{inr}(K)$ and $\mathrm{outr}(K)$ denote the inradius and outradius of K. The equality cases in these inequalities correspond to solutions of variational problems: among all convex bodies in $\mathbb{R}^3$ with unit outradius (inradius), which maximize (minimize) the ratio of surface area to mean width? Even though the two inequalities arise in a completely symmetric manner, their extremals are very different.

- A convex body in $\mathbb{R}^3$ with unit outradius maximizes the ratio of surface area to mean width if and only if it is a translate of the Euclidean unit ball B.
- A convex body in $\mathbb{R}^3$ with unit inradius minimizes the ratio of surface area to mean width if and only if it is a translate of a cap body of B (Figure 1.1).

This special case of Problem 1.1 is due to Favard [4], see [21, Theorem 7.6.17].

The following reformulation of Problem 1.1 will be the main focus of this paper. Recall that the support function h_K of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(u) = \sup_{x \in K} \langle u, x \rangle.$$

Geometrically, $h_K(u)$ is the (signed) distance to the origin of the supporting hyperplane of K with outer normal direction $u \in S^{n-1}$. Mixed volumes can be expressed in terms of the support function of one of the bodies as

$$\mathsf{V}_n(K, C_1, \ldots, C_{n-1}) = \frac{1}{n} \int h_K \, d\mathrm{S}_{C_1,\ldots,C_{n-1}},$$

where $\mathrm{S}_{C_1,\ldots,C_{n-1}}$ is the *mixed area measure* on S^{n-1}. Now note that if $K \subseteq L$, then $h_L - h_K \ge 0$ pointwise, and thus the following are equivalent:

a. Equality holds in (1.1).

b. K and L have the same supporting hyperplanes with outer normal direction in the support of the measure $\mathrm{S}_{C_1,\ldots,C_{n-1}}$.

We can therefore reformulate Problem 1.1 as follows.

Problem 1.3. Provide a geometric characterization of $\operatorname{supp} \mathrm{S}_{C_1,\ldots,C_{n-1}}$.

Beside its fundamental interest in convex geometry, Problem 1.3 is closely connected with the long-standing problem of providing a complete characterization of the equality cases of the Alexandrov–Fenchel inequality for general convex bodies [19, 23], and has further implications that will be discussed in §1.4.

While Problem 1.3 has to date been resolved only for special classes of convex bodies, a precise conjectural picture that is consistent with all known cases was put forward by Schneider in 1985 [19]. We presently discuss Schneider's conjecture and recall the previously known results in this direction.

1.2. **Schneider's conjecture.** To motivate the conjectural picture, it is instructive to first consider the special case that $C_1, \ldots, C_{n-1}$ are all polytopes. In this setting, the mixed area measure $\mathrm{S}_{C_1,\ldots,C_{n-1}}$ is atomic with [21, (5.22)]

$$\mathrm{S}_{C_1,\ldots,C_{n-1}}(\{u\}) = \mathsf{V}_{n-1}(F(C_1,u),\ldots,F(C_{n-1},u)) \tag{1.2}$$

for all $u \in S^{n-1}$, where $F(C,u)$ denotes the exposed face of C with normal direction u (see §2.2). Thus to understand the support of the mixed area measure of polytopes, it suffices to understand when mixed volumes are positive. The latter is classical and admits an elementary proof, see [21, Theorem 5.1.8].

Fact 1.4. *The following are equivalent for convex bodies $C_1, \ldots, C_n$ in $\mathbb{R}^n$.*

a. $\mathsf{V}_n(C_1,\ldots,C_n) > 0$.

b. There exist segments $I_i \subseteq C_i$, $i \in [n]$ with linearly independent directions.

c. $\dim\left(\sum_{i\in I} C_i\right) \geq |I|$ *for all* $I \subseteq [n]$.

The above ingredients suffice to provide a satisfactory answer to Problem 1.3 for polytopes: combining (1.2) and Fact 1.4 yields

$$\operatorname{supp} \mathrm{S}_{C_1,\ldots,C_{n-1}} = \left\{u \in S^{n-1} : \dim\left(F(C_I,u)\right) \geq |I| \text{ for all } I \subseteq [n-1]\right\}$$

whenever $C_1, \ldots, C_{n-1}$ are convex polytopes in $\mathbb{R}^n$, where we define

$$C_I = \textstyle\sum_{i\in I} C_i.$$

However, this conclusion fails to extend to general convex bodies: for example, if $C_1, \ldots, C_{n-1}$ are strictly convex, then $\dim(F(C_I,u)) = 0$ for all u, I.

It was realized by Schneider [19] that the characterization remains meaningful for general convex bodies if $F(C,u)$ is replaced by the "tangent space" $T(C,u)^\perp$ of C with normal direction u, where the touching cone $T(C,u)$ is defined as the unique face of a normal cone of C so that $T(C,u)$ contains u in its relative interior. Equivalent formulations of the following definition are given in §2.4.

Definition 1.5. Let $C_1, \ldots, C_{n-1}$ be convex bodies in $\mathbb{R}^n$. Then $u \in S^{n-1}$ is called $(C_1,\ldots,C_{n-1})$-*extreme* if $\dim\left(T(C_I,u)^\perp\right) \geq |I|$ for all $I \subseteq [n-1]$.

Conjecture 1.6 (Schneider). *Let $C_1, \ldots, C_{n-1}$ be convex bodies in $\mathbb{R}^n$. Then*

$$\operatorname{supp} \mathrm{S}_{C_1,\ldots,C_{n-1}} = \operatorname{cl}\left\{u \in S^{n-1} : u \textit{ is } (C_1,\ldots,C_{n-1})\textit{-extreme}\right\}.$$

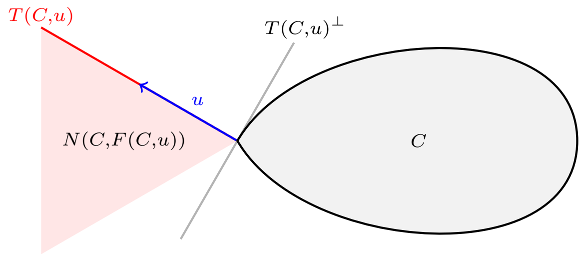

FIGURE 1.2. Illustration of a normal direction u of a convex body C in $\mathbb{R}^2$ that is C-extreme but not C-exposed. In this case, the "tangent space" $T(C,u)^\perp$ is only tangent to the boundary of C in one direction.

To date, Conjecture 1.6 has been verified only for special classes of convex bodies. In particular, the conjecture is known to hold in the following cases:

- $C_1,\ldots,C_{n-1}$ are convex polytopes, as explained above [19];
- $C_1=\cdots=C_k$ and $C_{k+1},\ldots,C_{n-1}$ are smooth and strictly convex [18, 19];[1]
- $C_1,\ldots,C_{n-1}$ are zonoids [20, 23, 8] or "polyoids" [8];
- various combinations of the above cases [23, 8].

The proofs of all these results rely crucially on the special structure of the bodies involved. While all known cases are consistent with Conjecture 1.6, none of the known results to date sheds any light on its validity for general convex bodies.

1.3. Main results.

1.3.1. *Upper bound.* Our first main result proves one half of Conjecture 1.6 in full generality: $\operatorname{supp} \mathrm{S}_{C_1,\ldots,C_{n-1}}$ is always included in the closure of the set of $(C_1,\ldots,C_{n-1})$-extreme directions. We will in fact prove a significantly stronger result of which this is a consequence, as we now explain.

The notion of a "tangent space" of C with normal direction u is somewhat ambiguous in cases where u does not lie in the relative interior of a normal cone of C. This situation is illustrated in Figure 1.2: in this case $T(C,u)^\perp$ is only tangent to the body in one direction. This situation can be avoided by replacing the touching cone $T(C,u)$ by the normal cone $N(C,F(C,u))$ of C at $F(C,u)$ (that is, the smallest normal cone of C that contains u) in Definition 1.5.

Definition 1.7. Let $C_1,\ldots,C_{n-1}$ be convex bodies in $\mathbb{R}^n$. Then $u\in S^{n-1}$ is called $(C_1,\ldots,C_{n-1})$-*exposed* if $\dim\big(N(C_I,F(C_I,u))^\perp\big)\ge |I|$ for all $I\subseteq[n-1]$.

The following is the first main result of this paper.

Theorem 1.8. *For any convex bodies $C_1,\ldots,C_{n-1}$ in $\mathbb{R}^n$, we have*

$$\mathrm{S}_{C_1,\ldots,C_{n-1}}\big(\big\{u\in S^{n-1}: u \textit{ is not } (C_1,\ldots,C_{n-1})\textit{-exposed}\big\}\big)=0.$$

This result immediately implies the following.

[1] The strict convexity assumption is not needed and can be removed along the lines of [23, §14].

Corollary 1.9. *For any convex bodies $C_1, \dots, C_{n-1}$ in $\mathbb{R}^n$, we have*

$$\begin{aligned}\operatorname{supp} \mathrm{S}_{C_1,\dots,C_{n-1}} &\subseteq \operatorname{cl}\big\{u \in S^{n-1} : u \text{ is } (C_1,\dots,C_{n-1})\text{-exposed}\big\}\\ &\subseteq \operatorname{cl}\big\{u \in S^{n-1} : u \text{ is } (C_1,\dots,C_{n-1})\text{-extreme}\big\}.\end{aligned}$$

Proof. By the definition of the support of a measure, the first inclusion is equivalent to the statement that the *interior* of the set of non-$(C_1, \dots, C_{n-1})$-exposed directions has $\mathrm{S}_{C_1,\dots,C_{n-1}}$-measure zero. This follows trivially from Theorem 1.8. For the second inclusion, it suffices to note that as $T(C, u)$ is a subset of $N(C, F(C, u))$, any $(C_1, \dots, C_{n-1})$-exposed direction is *a fortiori* $(C_1, \dots, C_{n-1})$-extreme. □

1.3.2. *Lower bound.* Our second main result fully resolves Conjecture 1.6 in $\mathbb{R}^3$. This will again follow as a consequence of a somewhat more general result.

Theorem 1.10. *For any convex bodies K, L in $\mathbb{R}^n$, we have*

$$\operatorname{supp} \mathrm{S}_{K,\dots,K,L} = \operatorname{cl}\big\{u \in S^{n-1} : u \text{ is } (K,\dots,K,L)\text{-extreme}\big\}.$$

In particular, this fully resolves Conjecture 1.6 in dimension $n = 3$.

The works of Minkowski [11, 12] were set exclusively in $\mathbb{R}^3$, and thus Theorem 1.10 resolves Minkowski's monotonicity problem in its original setting. The obstacle to fully resolving Conjecture 1.6 in higher dimensions will be explained in §1.5.

Remark 1.11. Using the methods of [19, 20, 23, 8], one can readily generalize Theorem 1.10 by combining it with previously known cases; e.g., one may consider $\mathrm{S}_{K,L,C_1,\dots,C_{n-3}}$ where $C_1, \dots, C_{n-3}$ are smooth. We do not spell out such variations on Theorem 1.10 as they do not shed any new light on Conjecture 1.6.

Remark 1.12. While Theorem 1.10 considers a special case of mixed area measures in $\mathbb{R}^n$, we emphasize the result holds for *arbitrary* convex bodies K, L, in contrast to previous results that were restricted to special classes of bodies.

On the other hand, Theorem 1.10 is new even in the special case that the support functions of K, L are smooth. At first sight, this setting would appear to be far easier than the general case, as the mixed area measure has a simple explicit formula: if $C_1, \dots, C_{n-1}$ have smooth support functions, then

$$d\mathrm{S}_{C_1,\dots,C_{n-1}} = \mathsf{D}_{n-1}(D^2 h_{C_1}, \dots, D^2 h_{C_{n-1}})\, d\omega,$$

where ω denotes the Lebesgue measure on S^{n-1}, $D^2 h_C(u)$ denotes the restriction of the Hessian $\nabla^2 h_C(u)$ in $\mathbb{R}^n$ to $u^\perp$, and the mixed discriminant $\mathsf{D}_n(M_1, \dots, M_n)$ of n-dimensional matrices M_i is defined analogously to mixed volumes as

$$\det(\lambda_1 M_1 + \cdots + \lambda_n M_n) = \sum_{i_1,\dots,i_n} \mathsf{D}_n(M_{i_1}, \dots, M_{i_n})\, \lambda_{i_1} \cdots \lambda_{i_n};$$

see, e.g., [22, §2]. Thus using the analogue of Fact 1.4 for mixed discriminants due to Panov [15, Theorem 1] yields the characterization

$$\operatorname{supp} \mathrm{S}_{C_1,\dots,C_{n-1}} = \operatorname{cl}\big\{u \in S^{n-1} : \operatorname{rank}\big(D^2 h_{C_I}(u)\big) \geq |I| \text{ for all } I \subseteq [n-1]\big\}$$

for any convex bodies $C_1, \dots, C_{n-1}$ in $\mathbb{R}^n$ with smooth support functions. However, this is not a satisfactory characterization since it is analytic rather than geometric in nature. The problem of deducing Conjecture 1.6 from this analytic description is closely related to classical problems on the solution of homogeneous Monge-Ampère equations, which will be discussed in §1.4.3 below. At present, Theorem 1.10 is the only general result in this direction even in the case of smooth support functions.

1.4. **Implications.**

1.4.1. *Extreme and exposed directions.*

An unexpected implication of Theorem 1.8 is that it sheds light on the relation between extreme and exposed directions.

Corollary 1.13. *If Conjecture 1.6 holds for given convex bodies $C_1, \ldots, C_{n-1}$ in $\mathbb{R}^n$, then every $(C_1, \ldots, C_{n-1})$-extreme direction $u \in S^{n-1}$ is a limit $u = \lim_k u_k$ of $(C_1, \ldots, C_{n-1})$-exposed directions $u_k \in S^{n-1}$.*

Proof. Theorem 1.8 and Conjecture 1.6 imply that the closures of the sets of $(C_1, \ldots, C_{n-1})$-extreme and $(C_1, \ldots, C_{n-1})$-exposed directions coincide. □

Thus, for example, Theorem 1.10 implies that

$$\mathrm{cl}\,\big\{u \in S^{n-1} : u \text{ is } (K, L)\text{-extreme}\big\} = \mathrm{cl}\,\big\{u \in S^{n-1} : u \text{ is } (K, L)\text{-exposed}\big\}$$

for any convex bodies K, L in $\mathbb{R}^3$.

Another special case of Corollary 1.13 gives a new proof of the following result. A vector $u \in S^{n-1}$ is called an *r-extreme normal direction* of K if $\dim T(K, u) \leq r+1$, and is called an *r-exposed normal direction* if $\dim N(K, F(K, u)) \leq r+1$. It is clear that these definitions are equivalent to u being $(K[n-r-1], B[r])$-extreme and $(K[n-r-1], B[r])$-exposed, respectively, where $C[k]$ denotes that C is repeated k times. Since Conjecture 1.6 holds in this setting [18], it follows that any r-extreme normal direction of a convex body is the limit of r-exposed normal directions. This fact was previously established by a direct argument in [21, Theorem 2.2.9].

1.4.2. *Mixed Hessian measures.*

We now describe an analogue of mixed area measures for convex functions rather than convex bodies. Such measures are readily obtained [7] by polarizing the Monge-Ampère measure [5, Chapter 1], and have been investigated (in a somewhat more general setting) by Trudinger and Wang [24] as part of their study of nonlinear Dirichlet problems.

In the following, we fix an open convex set $\Omega \subseteq \mathbb{R}^n$, and denote by $\mathrm{Conv}(\Omega)$ the set of all proper convex functions f on $\mathbb{R}^n$ with $\Omega \subseteq \mathrm{dom}\, f$. If $f_1, \ldots, f_n \in \mathrm{Conv}(\Omega)$ are smooth, their *mixed Hessian measure* is the measure on Ω with density

$$\frac{d\mathrm{H}_{f_1,\ldots,f_n}}{dx} = \mathsf{D}_n(\nabla^2 f_1, \ldots, \nabla^2 f_n)$$

with respect to the Lebesgue measure. The definition of $\mathrm{H}_{f_1,\ldots,f_n}$ can be extended to arbitrary $f_1, \ldots, f_n \in \mathrm{Conv}(\Omega)$ by continuity [24, Theorem 2.4].

For any $f \in \mathrm{Conv}(\Omega)$ and $x \in \Omega$, denote by $L(f, x)$ the largest convex subset of Ω on which f is affine and that has x in its relative interior. The set $L(f, x)$ is the analogue for convex functions of the notion of a touching cone for convex bodies (indeed, the support function h_C is affine on every normal cone of C; for the precise relation between mixed Hessian measures and mixed area measures, see [7, Corollary 4.2]). In the following, we let $\bar{L}(f, x) = \mathrm{span}\{L(f, x) - x\}$.

We can now transcribe Conjecture 1.6 to the setting of mixed Hessian measures.

Definition 1.14. Let $f_1, \ldots, f_n \in \mathrm{Conv}(\Omega)$. Then a point $x \in \Omega$ is called $(f_1, \ldots, f_n)$-*extreme* if $\dim\big(\bar{L}(f_I, x)^\perp\big) \geq |I|$ for all $I \subseteq [n]$.

Conjecture 1.15. *Let $f_1, \ldots, f_n \in \mathrm{Conv}(\Omega)$. Then*

$$\operatorname{supp} \mathrm{H}_{f_1,\ldots,f_n} = \mathrm{cl}\,\big\{x \in \Omega : x \textit{ is } (f_1, \ldots, f_n)\textit{-extreme}\big\}.$$

It can be shown using results of Hug–Mussnig–Ulivelli [7] that Conjecture 1.6 and Conjecture 1.15 are in fact equivalent; see §5. By the same technique, we can readily transcribe the main results of this paper to mixed Hessian measures.

Corollary 1.16. *For any $f_1, \ldots, f_n \in \mathrm{Conv}(\Omega)$, we have*

$$\operatorname{supp} \mathrm{H}_{f_1,\ldots,f_n} \subseteq \mathrm{cl}\,\big\{x \in \Omega : x \textit{ is } (f_1, \ldots, f_n)\textit{-extreme}\big\}.$$

Moreover, for any $f, g \in \mathrm{Conv}(\Omega)$, we have

$$\operatorname{supp} \mathrm{H}_{f,\ldots,f,g} = \mathrm{cl}\,\big\{x \in \Omega : x \textit{ is } (f, \ldots, f, g)\textit{-extreme}\big\}.$$

In particular, this fully resolves Conjecture 1.15 in dimension $n = 2$.

1.4.3. *A mixed Hartman–Nirenberg–Pogorelov theorem.* A classical fact that dates back to Monge [9, §23.7], and in more precise form to Hartman and Nirenberg [6] and Pogorelov [16, §IX.4], is that a surface with vanishing Gauss curvature is a ruled surface: it is foliated by straight lines and planar regions.

More concretely, let $f : D \to \mathbb{R}$ be a smooth (not necessarily convex) function on $D \subset \mathbb{R}^2$. The graph of f defines a surface in $\mathbb{R}^3$, whose Gauss curvature vanishes if

$$\det(\nabla^2 f) = 0 \quad \text{on} \quad D, \tag{1.3}$$

that is, if f it a solution of the homogeneous Monge-Ampère equation on D. That $\det(\nabla^2 f(x)) = 0$ means that $\nabla^2 f(x)$ has a nontrivial kernel—that is, the second derivative of f vanishes in some direction—but this local condition does not in itself imply that f must be affine in that direction. It is a nontrivial fact that when $\det(\nabla^2 f)$ vanishes in an open domain D, this local condition can be integrated to yield the global property that D is foliated by regions on which f is affine. The following formulation is due to Hartman and Nirenberg [6, §3].

Theorem 1.17 (Hartman–Nirenberg)**.** *Let $D \subset \mathbb{R}^2$ be an open connected set, let $f : D \to \mathbb{R}$ be of class C^2, and suppose that* (1.3) *holds. Let*

$$R = \{x \in D : \nabla^2 f = 0 \textit{ in a neighborhood of } x\}.$$

Then for every $x \in D \backslash R$, there is an affine line $x \in L \subset \mathbb{R}^2$ so that f is affine on the connected component I of $L \cap D$ that contains x, and $I \cap R = \varnothing$.

It is clear that f is affine on every connected component of the open set R; these are the planar regions of the surface defined by f. Theorem 1.17 states that the non-planar part of the surface is foliated by lines that extend to the boundary. Such a result for non-smooth surfaces is due to Pogorelov [16, §IX.4].

The problems investigated in this paper may be viewed as mixed analogues of these classical results. More concretely, given two smooth functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ on a domain $D \subseteq \mathbb{R}^2$, we aim to characterize when

$$\mathsf{D}_2(\nabla^2 f, \nabla^2 g) = 0 \quad \text{on} \quad D. \tag{1.4}$$

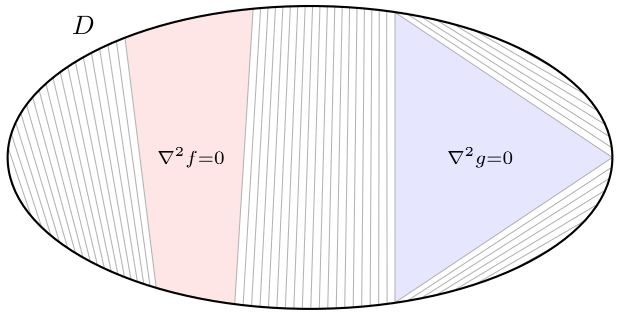

FIGURE 1.3. Illustration of of Corollary 1.18. Outside the planar regions of f and g, the domain D is foliated by lines on which f and g are simultaneously affine.

When $f = g$, this is precisely (1.3). However, the case $f \neq g$ can be of a very different nature, as $\mathsf{D}_2(M_1, M_2) = 0$ need not have any implication for the kernels of M_1, M_2: for example, (1.4) holds when $f(x) = \|x\|^2$ and g is any harmonic function on D, neither of which define ruled surfaces.

When M_1, M_2 are positive semidefinite, however, their kernels do characterize when $\mathsf{D}_2(M_1, M_2) = 0$: this holds if and only if $M_1 = 0$, or $M_2 = 0$, or $\ker M_1 = \ker M_2$ with $\dim\ker M_i = 1$ [15]. Therefore, by analogy with the local-to-global phenomenon captured by Theorem 1.17, it is natural to conjecture that (1.4) has the following solution for *convex* f, g: each $x \in D$ is either contained in a planar region of f or of g, or in an affine line on which f and g are simultaneously affine; see Figure 1.3. That this is the case is a consequence of Theorem 1.10.

Corollary 1.18. *Let $\Omega \subseteq \mathbb{R}^2$ be an open convex set and $D \subseteq \Omega$ be an open connected set. Let $f, g \in \mathrm{Conv}(\Omega)$ be of class C^2, and suppose that (1.4) holds. Let*

$$R = \{x \in D : \nabla^2 f = 0 \text{ in a neighborhood of } x\} \cup \{x \in D : \nabla^2 g = 0 \text{ in a neighborhood of } x\}.$$

Then for every $x \in D \backslash R$, there is an affine line $x \in L \subset \mathbb{R}^2$ so that f and g are both affine on the connected component I of $L \cap D$ that contains x, and $I \cap R = \varnothing$.

We have formulated Corollary 1.18 for smooth convex functions to emphasize the analogy with Theorem 1.17. However, the result is an immediate consequence of the following result in the nonsmooth case.

Corollary 1.19. *Let $\Omega \subseteq \mathbb{R}^2$ be an open convex set and $D \subseteq \Omega$ be an open connected set. Let $f, g \in \mathrm{Conv}(\Omega)$ satisfy $\mathrm{H}_{f,g}(D) = 0$, and define*

$$R = \{x \in D : \dim L(f, x) = 2 \text{ or } \dim L(g, x) = 2\}.$$

Then for every $x \in D \backslash R$, there is an affine line $x \in L \subset \mathbb{R}^2$ so that f and g are both affine on the connected component I of $L \cap D$ that contains x, and $I \cap R = \varnothing$.

1.5. **On higher dimensions.** The proof of Theorem 1.8 is based on an argument with a geometric measure theory flavor that will be explained in §3. This argument works in the same manner in any dimension. In contrast, Theorem 1.10 is restricted to special mixed area measures that fully capture only the three-dimensional case.

The reason for this is not merely technical in nature: there is a fundamental obstacle that arises in higher dimensions, as we presently explain.

The proof of Theorem 1.10 is based on the following simple observation that will be proved in §2.4 below. Here P_E denotes the orthogonal projection onto E.

Lemma 1.20. *Let $K, C_1, \ldots, C_{n-2}$ be convex bodies in $\mathbb{R}^n$, and let $u \in S^{n-1}$ be a $(K, C_1, \ldots, C_{n-2})$-extreme direction. Then there exists $v \in T(K,u)^\perp$ so that u is a $(\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{n-2})$-extreme direction.*

Lemma 1.20 states that the property of being an extreme direction is preserved under projection. Therefore, if it were true that

$$u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{n-2}} \quad \overset{?}{\Longrightarrow} \quad u \in \operatorname{supp} \mathrm{S}_{K, C_1, \ldots, C_{n-2}}, \tag{1.5}$$

then the lower bound in Conjecture 1.6 would follow by induction on the dimension (and thus the full conjecture would follow by Theorem 1.8). Unfortunately, it turns out that (1.5) does not always hold. In fact, we will provide an essentially complete understanding of the behavior of the support under projection.

Theorem 1.21. *Let $K, C_1, \ldots, C_{n-2}$ be convex bodies in $\mathbb{R}^n$.*

a. If $\dim T(K,u) = 1$ or $\dim T(K,u) = n-1$, the following holds: if there exists $v \in T(K,u)^\perp$ so that $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{n-2}}$, then $u \in \operatorname{supp} \mathrm{S}_{K, C_1, \ldots, C_{n-2}}$.

b. If $1 < \dim T(K,u) < n-1$, it can be the case that $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{n-2}}$ for every $v \in T(K,u)^\perp$, but $u \notin \operatorname{supp} \mathrm{S}_{K, C_1, \ldots, C_{n-2}}$.

Only the case $\dim T(K,u) = 1$ will be needed for the proof of Theorem 1.10; this case will be proved in §4. The case $\dim T(K,u) = n-1$ and part *b.* are included to clarify the basic obstruction to extending the lower bound to higher dimensions; their proofs are postponed until §6.

Theorem 1.21 shows that performing induction on the dimension by projection must fail in dimensions $n \geq 4$. Thus further progress on the lower bound in Conjecture 1.6 will likely require a fundamentally new ingredient.

1.6. **Organization of this paper.** The remainder of this paper is organized as follows. In §2, we recall some basic notions of convex geometry that appear throughout the paper, and we establish some elementary facts that will be used in the sequel. The main results of this paper, Theorems 1.8 and 1.10, are proved in §3 and §4, respectively. The corresponding results for mixed Hessian measures and the mixed Hartman-Nirenberg-Pogorelov problem are developed in §5. Finally, the behavior of the support of mixed area measures in higher dimensions is discussed in §6.

2. Preliminaries

The aim of this section is to recall a number of basic notions of convex geometry and to establish some elementary facts that will be used throughout this paper. Our standard reference on convexity is the excellent monograph [21].

2.1. **Basic notions.** By a *convex body* we mean a nonempty compact convex set. We denote by $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$, by $\mathcal{K}^n_n$ the set of all convex bodies in $\mathbb{R}^n$ with nonempty interior, and $\mathcal{K}^n_{(o)}$ the set of all convex bodies in $\mathbb{R}^n$ that contain the origin in their interior.

For $K \in \mathcal{K}^n$, we denote by $\operatorname{relint} K$ the relative interior and by $\operatorname{relbd} K$ the relative boundary of K, that is, the interior (boundary) of K viewed as a convex body in the affine hull of K. The polar dual body of $K \in \mathcal{K}^n_{(o)}$ is defined by

$$K^\circ = \{u \in \mathbb{R}^n : \langle u, x\rangle \le 1 \text{ for all } x \in K\}.$$

The polar is an involution $K^{\circ\circ} = K$. Moreover,

$$h_{K^\circ}(x) = \|x\|_K \quad \text{for all } x \in \mathbb{R}^n,$$

where the Minkowski functional of K is defined by $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$.

For any $u \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we define

$$\begin{aligned} H^+_{u,t} &= \{x \in \mathbb{R}^n : \langle u, x\rangle \ge t\},\\ H^-_{u,t} &= \{x \in \mathbb{R}^n : \langle u, x\rangle \le t\},\\ H_{u,t} &= \{x \in \mathbb{R}^n : \langle u, x\rangle = t\}. \end{aligned}$$

For $u \in S^{n-1}$ and $\varepsilon > 0$, $B(u, \varepsilon)$ denotes the open ε-ball in S^{n-1} with center u.

2.2. **Facial structure.**

2.2.1. *Faces and exposed faces.* A *face* F of a convex body $K \in \mathcal{K}^n$ is a convex subset of K such that $x, y \in K$ and $\frac{x+y}{2} \in F$ implies $x, y \in F$. For $u \in S^{n-1}$,

$$F(K, u) = \{x \in K : \langle u, x\rangle = h_K(u)\} = K \cap H_{u,h_K(u)}$$

is called the *exposed face* of K with normal direction u. Exposed faces are additive under Minkowski addition, that is,

$$F(K + L, u) = F(K, u) + F(L, u) \tag{2.1}$$

for all $K, L \in \mathcal{K}^n$ and $u \in S^{n-1}$; see [21, Theorem 1.7.5].

2.2.2. *Normal cones.* For any $x \in K$, the *normal cone* of K at x is defined by

$$N(K, x) = \{u \in \mathbb{R}^n : \langle u, x\rangle = h_K(u)\}.$$

This is precisely the dual notion to an exposed face. Dual to (2.1), we have

$$N(K + L, x + y) = N(K, x) \cap N(L, y) \tag{2.2}$$

for all $K, L \in \mathcal{K}^n$ and $x \in K$, $y \in L$; see [21, Theorem 2.2.1].

The *metric projection map* $p_K : \mathbb{R}^n \to K$ sends each $x \in \mathbb{R}^n$ to the point in K that is closest to x. The metric projection is closely related to the normal cone:

$$p_K^{-1}(x) = x + N(K, x)$$

for every $x \in K$; see [21, p. 81].

Let F be a face of K. Then every point $x \in \operatorname{relint} F$ yields the same normal cone $N(K, x)$, which will also be denoted as $N(K, F)$; see [21, p. 83]. Since relative interior distributes over Minkowski addition [17, Corollary 6.6.2], the property (2.2) extends to $N(K+L, F(K+L,u)) = N(K, F(K,u)) \cap N(L, F(L,u))$.

2.2.3. *Touching cones.* For any $u \in S^{n-1}$, the *touching cone* $T(K,u)$ is defined as the unique face of $N(K, F(K,u))$ that contains u in its relative interior. The distinction between touching and normal cones is illustrated in Figure 1.2.

Remark 2.1. Since $N(K, F(K,u))$ is itself a face of every normal cone $N(K,x)$ that contains u, the touching cone $T(K,u)$ can be equivalently defined as the unique face of any normal cone of K so that $T(K,u)$ contains u in its relative interior.

Just as normal cones are dual to exposed faces, touching cones are dual to faces. This duality can be made precise as follows [25, §1.2.3]. For any $K \in \mathcal{K}^n_{(o)}$, the map $F \mapsto \mathbb{R}_+ F = \{\lambda x : \lambda \geq 0, x \in F\}$ defines a bijection

$$\{\text{faces of } K^\circ\} \xrightarrow{\sim} \{\text{touching cones of } K\}.$$

The restriction of this map to the set of exposed faces of K° defines a bijection

$$\{\text{exposed faces of } K^\circ\} \xrightarrow{\sim} \{\text{normal cones of } K\}.$$

The inverse map is given by $T \mapsto T \cap \operatorname{bd} K^\circ$ for any touching cone T of K.

2.3. **Properties of touching cones.** A number of elementary properties of touching cones will be needed throughout this paper. We record their proofs here. We begin with the following simple observation.

Lemma 2.2. *Let $v \in T(K,u)$. Then $T(K,v)$ is the unique face of $T(K,u)$ that contains v in its relative interior.*

Proof. By definition, $T(K,u)$ is a face of $N(K,x)$ for some $x \in K$. Let F be the unique face of $T(K,u)$ that contains v in its relative interior. Then F is also a face of $N(K,x)$ [21, Theorem 2.1.1], and thus $F = T(K,v)$ by Remark 2.1. □

Our next observation is the analogue of (2.2) for touching cones.

Lemma 2.3. *Let $K, L \in \mathcal{K}^n$ and $u \in S^{n-1}$. Then*

$$T(K+L,u) = T(K,u) \cap T(L,u).$$

Proof. We first note that u is in the relative interior of $T(K,u) \cap T(L,u)$, as the relative interior distributes over finite intersections [17, Theorem 6.5]. On the other hand, let $T(K,u)$ be a face of $N(K,x)$, and let $T(L,u)$ be a face of $N(L,y)$. Then clearly $a, b \in N(K,x) \cap N(L,y)$ with $\frac{a+b}{2} \in T(K,u) \cap T(L,u)$ implies that $a, b \in T(K,u) \cap T(L,u)$. Thus we have shown that $T(K,u) \cap T(L,u)$ is a face of $N(K+L, x+y) = N(K,x) \cap N(L,y)$, which concludes the proof by Remark 2.1. □

Next, we clarify the behavior of touching cones under projection.

Lemma 2.4. *Let $K \in \mathcal{K}^n$ and $u, v \in S^{n-1}$ with $u \in v^\perp$. Then*

$$T(\mathrm{P}_{v^\perp} K, u) = T(K,u) \cap v^\perp,$$

where we view $\mathrm{P}_{v^\perp} K$ as a convex body in $v^\perp$.

Proof. Let $T(K,u)$ be a face of $N(K,x)$, and note that

$$N(\mathrm{P}_{v^\perp}K, \mathrm{P}_{v^\perp}x) = \{w \in v^\perp : \langle w, x\rangle = h_K(w)\} = N(K,x) \cap v^\perp.$$

It follows by the same argument as in the proof of Lemma 2.3 that u is in the relative interior of $T(K,u) \cap v^\perp$ and that $T(K,u) \cap v^\perp$ is a face of $N(K,x) \cap v^\perp$. This concludes the proof by Remark 2.1. □

We finally record an implication between normal and touching cone inclusion.

Lemma 2.5. *Let $K, L \in \mathcal{K}^n$ and $x \in \mathrm{bd}\, K$, $y \in \mathrm{bd}\, L$ with $N(K,x) \subseteq N(L,y)$. Then $T(K,u) \subseteq T(L,u)$ for every $u \in N(K,x)$.*

Proof. It follows by the argument in the proof of Lemma 2.3 and by Remark 2.1 that u is in the relative interior of $T(K,u) \cap T(L,u)$, and that $T(K,u) \cap T(L,u)$ is a face of $N(K,x) \cap N(L,y) = N(K,x)$. Thus $T(K,u) \cap T(L,u) = T(K,u)$. □

2.4. **Extreme directions.** The notion of a $(C_1, \ldots, C_{n-1})$-extreme direction was defined in Definition 1.5. This notion has a number of equivalent formulations that are analogous to the equivalent conditions of Fact 1.4.

Lemma 2.6. *Let $C_1, \ldots, C_{n-1} \in \mathcal{K}^n$ and $u \in S^{n-1}$. The following are equivalent:*
a. u is $(C_1, \ldots, C_{n-1})$-extreme.
b. $\dim\big(\sum_{i \in I} T(C_i,u)^\perp\big) \geq |I|$ for all $I \subseteq [n-1]$.
c. There exist lines $L_i \subseteq T(C_i,u)^\perp$, $i \in [n-1]$ with linearly independent directions.

Proof. By Lemma 2.3, *a.* states that

$$\dim\big(\textstyle\bigcap_{i\in I} T(C_i,u)\big) = \dim\big(T(C_I,u)\big) \leq n - |I|$$

for all I, while *b.* states that

$$\dim\big(\textstyle\bigcap_{i\in I} \mathrm{span}\, T(C_i,u)\big) \leq n - |I|$$

for all I. The equivalence of these two conditions follows from the fact that u is in the relative interior of each $T(C_i,u)$. The equivalence of *b.* and *c.* is a general fact of linear algebra that is proved, e.g., in [20, Lemma 2.3]. □

We emphasize that the counterpart of Lemma 2.6 for $(C_1, \ldots, C_{n-1})$-exposed directions does not hold: it may happen that the normal cones $N(C_i, F(C_i,u))$ do not have any common point in their relative interiors, and thus their intersection may have strictly smaller dimension than the intersection of their linear spans. That Definition 1.7 provides the natural notion of a $(C_1, \ldots, C_{n-1})$-exposed direction will become evident from the proof of Theorem 1.8.

On the other hand, Lemma 2.6 extends readily to its counterpart for convex functions that was defined in Definition 1.14.

Lemma 2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be an open convex set, and let $f_1, \ldots, f_n \in \mathrm{Conv}(\Omega)$ and $x \in \Omega$. The following are equivalent:*
a. x is $(f_1, \ldots, f_n)$-extreme.
b. $\dim\big(\sum_{i\in I} \bar{L}(f_i,x)^\perp\big) \geq |I|$ for all $I \subseteq [n]$.
c. There exist lines $L_i \subseteq \bar{L}(f_i,x)^\perp$, $i \in [n]$ with linearly independent directions.

Proof. The proof is identical to that of Lemma 2.6 using that

$$L(f_I, x) = \textstyle\bigcap_{i\in I} L(f_i, x).$$

To verify this is the case, note that as

$$f_I\big(\tfrac{y+z}{2}\big) - \tfrac{f_I(y)+f_I(z)}{2} = \sum_{i\in I}\big(f_i\big(\tfrac{y+z}{2}\big) - \tfrac{f_i(y)+f_i(z)}{2}\big)$$

and each term in the sum is nonnegative, f_I is affine on a set A if and only if f_i is affine on A for every $i \in I$. The claim therefore follows from the fact that the relative interior distributes over finite intersections [17, Theorem 6.5]. □

We can now prove Lemma 1.20 in the introduction.

Proof of Lemma 1.20. Let u be $(K, C_1, \ldots, C_{n-2})$-extreme. By Lemma 2.6, there exist lines $L_0 \subseteq T(K, u)^\perp$ and $L_i \subseteq T(C_i, u)^\perp$, $i = 1, \ldots, n-2$ with linearly independent directions. Let $L_0 = \mathbb{R}v$. Then the lines $\mathrm{P}_{v^\perp} L_1, \ldots, \mathrm{P}_{v^\perp} L_{n-2}$ have linearly independent directions. Moreover, since each element of $\mathrm{P}_{v^\perp} L_i$ is a linear combination of an element of L_i and v, we have

$$\mathrm{P}_{v^\perp} L_i \subseteq T(C_i, u)^\perp + \mathbb{R}v = \big(T(C_i, u) \cap v^\perp\big)^\perp = T(\mathrm{P}_{v^\perp} C_i, u)^\perp$$

by Lemma 2.4. Thus u is $(\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{n-2})$-extreme by Lemma 2.6. □

2.5. **Mixed volumes and mixed area measures.** The most basic properties of mixed volumes $\mathsf{V}_n(C_1, \ldots, C_n)$ and mixed area measures $\mathrm{S}_{C_1,\ldots,C_{n-1}}$ are that they are additive and 1-homogeneous in each argument C_i, symmetric in their arguments C_i, and that they are nonnegative and translation invariant. For the theory of mixed volumes and mixed area measures, we refer to the monograph [21].

Let $f = h_K - h_L$ be a difference of support functions of convex bodies $K, L \in \mathcal{K}^n$. By multilinearity, we can uniquely extend the definitions of mixed volumes and mixed area measures to such functions by defining [21, §5.2]

$$\mathsf{V}_n(f, C_1, \ldots, C_{n-1}) = \mathsf{V}_n(K, C_1, \ldots, C_{n-1}) - \mathsf{V}_n(L, C_1, \ldots, C_{n-1}),$$
$$\mathrm{S}_{f,C_1,\ldots,C_{n-2}} = \mathrm{S}_{K,C_1,\ldots,C_{n-2}} - \mathrm{S}_{L,C_1,\ldots,C_{n-2}}.$$

If $f_1, \ldots, f_n$ are differences of support functions, we can iterate this construction to define $\mathsf{V}_n(f_1, \ldots, f_n)$ and $\mathrm{S}_{f_1,\ldots,f_{n-1}}$. Note that these functional extensions of mixed volumes and mixed area measures are no longer necessarily nonnegative. Any $f \in C^2(S^{n-1})$ is a difference of support functions [21, Lemma 1.7.8].

We now recall the behavior of mixed volumes under projection onto a hyperplane [21, Theorem 5.3.1]: for any $v \in S^{n-1}$, we have

$$n\,\mathsf{V}_n([0, v], C_1, \ldots, C_{n-1}) = \mathsf{V}_{n-1}(\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{n-1}).$$

Since $h_{\mathrm{P}_E C} = h_C|_E$, this implies that

$$n\,\mathsf{V}_n([0, v], f_1, \ldots, f_{n-1}) = \mathsf{V}_{n-1}(f_1|_{v^\perp}, \ldots, f_{n-1}|_{v^\perp})$$

for any differences of support functions $f_1, \ldots, f_{n-1}$.

In the opposite direction, we will require the following.

Lemma 2.8. *Let $C_1,\dots,C_{n-2}\in\mathcal{K}^n$, and let $u,v\in S^{n-1}$ with $u\in v^\perp$. Then $u\in\operatorname{supp}\mathrm{S}_{\mathrm{P}_{v^\perp}C_1,\dots,\mathrm{P}_{v^\perp}C_{n-2}}$ implies that $u\in\mathrm{S}_{B,C_1,\dots,C_{n-2}}$.*

Proof. The projection formula for mixed volumes implies that [23, Remark 8.6]

$$(n-1)\,\mathrm{S}_{[0,v],C_1,\dots,C_{n-2}}=\mathrm{S}_{\mathrm{P}_{v^\perp}C_1,\dots,\mathrm{P}_{v^\perp}C_{n-2}} \tag{2.3}$$

if we view the right-hand side as a measure on $\mathbb{R}^n$ that is supported in $v^\perp$. The result follows as $\operatorname{supp}\mathrm{S}_{[0,v],C_1,\dots,C_{n-2}}\subseteq\operatorname{supp}\mathrm{S}_{B,C_1,\dots,C_{n-2}}$ by [21, Lemma 7.6.15]. □

Finally, for any $C\in\mathcal{K}^n$, the *area measure* $\mathrm{S}_{C[n-1]}$ has the representation

$$\mathrm{S}_{C[n-1]}(A)=\mathcal{H}^{n-1}\big(n_C^{-1}(A)\big), \tag{2.4}$$

for $A\subseteq S^{n-1}$ [21, §5.1], where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-Hausdorff measure and

$$n_C^{-1}(A)=\bigcup_{u\in A}F(C,u)\subseteq\operatorname{bd}C$$

is the set of points in $\operatorname{bd}C$ that have a normal direction in A.

3. Proof of the upper bound

The main aim of this section is to prove the following.

Theorem 3.1. *For any $K,L\in\mathcal{K}^n$ and $k=0,\dots,n-2$, we have*

$$\mathrm{S}_{K[k+1],L[n-k-2]}\big(\{u\in S^{n-1}:\dim N(K,F(K,u))\geq n-k\}\big)=0.$$

Let us first explain why this suffices to conclude the proof of Theorem 1.8.

Proof of Theorem 1.8. By definition,

$$\{u\in S^{n-1}: u\text{ is not }(C_1,\dots,C_{n-1})\text{-exposed}\}=\bigcup_{k=0}^{n-2}\bigcup_{|I|=k+1}\{u\in S^{n-1}:\dim N(C_I,F(C_I,u))\geq n-k\}.$$

Thus it suffices to show that

$$\mathrm{S}_{C_1,\dots,C_{n-1}}\big(\{u\in S^{n-1}:\dim N(C_I,F(C_I,u))\geq n-k\}\big)=0$$

for all k and $I\subseteq[n-1]$ with $|I|=k+1$. But as $\mathrm{S}_{C_1,\dots,C_{n-1}}\leq\mathrm{S}_{C_I[|I|],C_{I^c}[n-1-|I|]}$ by the additivity of mixed area measures, this follows from Theorem 3.1. □

The remainder of this section is devoted to the proof of Theorem 3.1. We begin by sketching the main idea behind the proof.

3.1. **Outline of the proof.** We begin with an elementary observation: the conclusion of Theorem 3.1 is equivalent to the statement that for all $\ell>k$

$$\mathrm{S}_{K[\ell],L[n-\ell-1]}\big(\{u\in S^{n-1}:\dim N(K,F(K,u))\geq n-k\}\big)=0,$$

since the set inside the measure increases if we replace k by $\ell-1$. Thus the conclusion of Theorem 3.1 is also equivalent to

$$\mathrm{S}_{(K+tL)[n-1]}\big(\{u\in S^{n-1}:\dim N(K,F(K,u))\geq n-k\}\big)=O(t^{n-k-1})$$

as $t \downarrow 0$, since $\mathrm{S}_{(K+tL)[n-1]} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} t^{n-\ell-1} \mathrm{S}_{K[\ell],L[n-\ell-1]}$. We aim to establish such a property by exploiting the representation (2.4) of $\mathrm{S}_{(K+tL)[n-1]}(A)$ as the area of the set of boundary points of $K+tL$ with a normal direction in A.

The basic intuition behind the proof is that the set of points in $\operatorname{bd} K$ that have a normal direction u with $\dim N(K, F(K,u)) \geq n-k$ is expected to behave as a k-dimensional set. Thus the corresponding boundary points of $K+tL$ are contained in an L-shaped "tube" with radius t around a k-dimensional subset of $\operatorname{bd} K$, which therefore has area $O(t^{n-k-1})$ regardless of the choice of L.

That the set of k-singular boundary points of K—that is, boundary points with normal cone of dimension at least $n-k$—has dimension k is made precise by classical results of Anderson and Klee [1], see also [21, Theorem 2.2.5]: they show that this set can be covered by countably many compact sets of finite k-Hausdorff measure. This does not suffice for our purposes, however, since it is unclear whether this covering of a subset of $\operatorname{bd} K$ can be obtained by pulling back a covering of the corresponding normal directions in S^{n-1}. To make the argument work, we must first prove the existence of such a cover that behaves nicely under the Gauss map. We can subsequently make precise the idea that an L-shaped "tube" around any element of this cover has area of order $O(t^{n-k-1})$, concluding the proof.

3.2. **A Lipschitz cover of k-singular points.** The aim of this section is to prove the following result, which refines the results of [1] and [21, Theorem 2.2.5].

Proposition 3.2. *Fix $K \in \mathcal{K}^n$ and $k \in \{0,\ldots,n-1\}$. Then there exists a family of Lipschitz maps $u_m : [0,1]^k \to \operatorname{bd} K$ such that every face F of K with $\dim N(K,F) \geq n-k$ is contained in $\operatorname{Im} u_m$ for some $m \in \mathbb{N}$.*

We begin with some basic notations. The affine Grassmannian $\mathrm{Graff}(n,k)$ is defined as the set of all k-dimensional affine subspaces of $\mathbb{R}^n$. Denote by $\mathrm{M}_{n-k,n}$ the set of all $(n-k)\times n$ matrices with rank $n-k$. Then the map

$$H : \mathrm{M}_{n-k,n} \times \mathbb{R}^n \to \mathrm{Graff}(n,k)$$

defined by

$$H(M,y) = \{x \in \mathbb{R}^n : M(x-y) = 0\}$$

is surjective. The following trivial fact will be used below.

Fact 3.3. *Consider $M \in \mathrm{M}_{n-k,n}$ so that $M = [\,M_1 \,|\, M_2\,]$ with $M_1 \in \mathbb{R}^{(n-k)\times(n-k)}$ invertible and $M_2 \in \mathbb{R}^{(n-k)\times k}$. Define $g : \mathbb{R}^k \times \mathrm{M}_{n-k,n} \times \mathbb{R}^n \to \mathbb{R}^{n-k}$ as*

$$g(z; M, y) = M_1^{-1}(My - M_2 z).$$

Then

$$H(M,y) = \big\{\big(g(z;M,y), z\big) : z \in \mathbb{R}^k\big\}.$$

In the following, we denote by 0_p the vector in $\mathbb{R}^p$ and by $0_{p\times q}$ the matrix in $\mathbb{R}^{p\times q}$ all of whose entries are zero, and by $\mathbf{I}_p$ the identity matrix in $\mathbb{R}^{p\times p}$. We also recall that p_K denotes the metric projection map of K, see §2.2.2.

The following is the key step in the proof of Proposition 3.2.

Lemma 3.4. *Fix $K \in \mathcal{K}^n$ and $k \in \{0,\ldots,n-1\}$, and let S be a closed ball with $K \subset \operatorname{int} S$. For every face F of K with $\dim N(K,F) = n-k$, there is a nonempty open set $O \subseteq \mathrm{M}_{n-k,n} \times \mathbb{R}^n$ so that $F \subseteq p_K(H(M,y) \cap S)$ for all $(M,y) \in O$.*

Proof. Since the statement is invariant under rotation and translation of K, we can assume without loss of generality that

$$N(K,F) \subseteq \mathbb{R}^{n-k} \times \{0_k\}, \qquad F = \{0_{n-k}\} \times F' \subset \{0_{n-k}\} \times \mathbb{R}^k.$$

We will first find a single point (M_*, y_*) that satisfies the desired conclusion, and then show it is stable under perturbation.

To construct (M_*, y_*), we choose any $y_* \in \operatorname{relint} N(K,F)$ of sufficiently small norm so that $F + y_* \subset \operatorname{int} S$, and let $M_* = [\,\mathbf{I}_{n-k}\,|\,0_{(n-k)\times k}\,] \in \mathrm{M}_{n-k,n}$. Then

$$F + y_* \subset H(M_*, y_*) \cap S,$$

which implies that $F \subseteq p_K(H(M_*, y_*) \cap S)$ (see §2.2.2).

By construction, $y_* = (y'_*, 0_k)$ for some $y'_* \in \mathbb{R}^{n-k}$. Fix an open neighborhood U' of y'_* such that $U = U' \times \{0_k\} \subset N(K,F)$ and $F + U \subset S$. In the following, we adopt the notation defined in Fact 3.3. Note first that

$$g(z; M_*, y_*) = y'_*$$

for all $z \in \mathbb{R}^k$. By continuity of the map g and as $\mathbf{I}_{n-k}$ is invertible, there is an open neighborhood $\tilde{O} \subset \mathbb{R}^k \times \mathrm{M}_{n-k,n} \times \mathbb{R}^n$ of $F' \times \{M_*\} \times \{y_*\}$ so that

$$g(z; M, y) \in U' \quad \text{and} \quad M_1 \text{ is invertible} \quad \text{for all} \quad (z, M, y) \in \tilde{O}.$$

By the generalized tube lemma [14, §26, Ex. 9], there exists an open neighborhood $O \subset \mathrm{M}_{n-k,n} \times \mathbb{R}^n$ of (M_*, y_*) so that $F' \times O \subset \tilde{O}$.

Denote by $n(x; M, y) = (g(x'; M, y), 0_k)$ for any $x = (0_{n-k}, x') \in F$. The choice of U', the construction of $\tilde{O}$, and Fact 3.3 ensure that

$$x + n(x; M, y) \in H(M, y) \cap S, \qquad n(x; M, y) \in N(K, F)$$

for all $(M, y) \in O$ and $x \in F$. Thus $F \subseteq p_K(H(M, y) \cap S)$ for all $(M, y) \in O$. □

We can now complete the proof of Proposition 3.2.

Proof of Proposition 3.2. Fix a closed ball S that contains K in its interior. For every $\ell \in \{0, \dots, k\}$, $M \in \mathrm{M}_{n-\ell,n}$, and $y \in \mathbb{R}^n$, the set $H(M, y) \cap S$ is either empty or a closed ball of dimension at most k. In the latter case, we can certainly find a Lipschitz map $f : [0,1]^k \to H(M, y)$ so that $H(M, y) \cap S \subseteq \operatorname{Im} f$.

Let $(f_m)_{m\in\mathbb{N}}$ be a family of such maps obtained by applying this construction to all $\ell \in \{0, \dots, k\}$ and all (M, y) in a countable dense subset of $\mathrm{M}_{n-\ell,n} \times \mathbb{R}^n$. Lemma 3.4 ensures that every face F of K with $\dim N(K,F) = n - \ell \geq n - k$ is contained in $\operatorname{Im} u_m$ for some m, where we define $u_m = p_K \circ f_m$. It remains to note that u_m is Lipschitz as p_K is Lipschitz [21, Theorem 1.2.1]. □

3.3. **Proof of Theorem 3.1.** Throughout the proof of Theorem 3.1, fix $K, L \in \mathcal{K}^n$, $k \in \{0, \dots, n-2\}$, and a family $(u_m)_{m\in\mathbb{N}}$ of Lipschitz maps $u_m : [0,1]^k \to \operatorname{bd} K$ as in Proposition 3.2. Define the sets $A_m \subseteq S^{n-1}$ as

$$A_m = \big\{v \in S^{n-1} : F(K, v) \subseteq \operatorname{Im} u_m\big\}.$$

Then Proposition 3.2 ensures that

$$\{v \in S^{n-1} : \dim N(K, F(K,v)) \geq n-k\} \subseteq \bigcup_{m\in\mathbb{N}} A_m.$$

By the argument in §3.1, it suffices to show that

$$\mathrm{S}_{(K+tL)[n-1]}(A_m) = O(t^{n-k-1}) \quad \text{as} \quad t \downarrow 0$$

for every $m \in \mathbb{N}$.

Proof of Theorem 3.1. The representation (2.4) yields

$$\mathrm{S}_{(K+tL)[n-1]}(A_m) = \mathcal{H}^{n-1}\Bigg(\bigcup_{v\in A_m} F(K+tL, v)\Bigg).$$

By the definition of A_m and (2.1), we have

$$\bigcup_{v\in A_m} F(K+tL, v) \subseteq (\operatorname{Im} u_m + tL) \cap \operatorname{bd}(K+tL).$$

Let $[0,1]^k = \bigcup_{i=1}^{N_t} B_i$ be a covering of $[0,1]^k$ by $N_t = O(t^{-k})$ balls B_i of radius t. Then $\operatorname{Im} u_m \subseteq \bigcup_{i=1}^{N_t} u_m(B_i)$, and we can estimate

$$\mathrm{S}_{(K+tL)[n-1]}(A_m) \leq \sum_{i=1}^{N_t} \mathcal{H}^{n-1}\big((u_m(B_i) + tL) \cap \operatorname{bd}(K+tL)\big).$$

Note that $u_m(B_i) + tL$ is contained in a ball $\tilde{B}_i$ of radius $(\operatorname{Lip}(u_m) + \operatorname{diam}(L))t$. As $K_1 \cap \operatorname{bd} K_2 \subseteq \operatorname{bd}(K_1 \cap K_2)$ for any convex bodies K_1, K_2, we obtain

$$\mathcal{H}^{n-1}\big((u_m(B_i) + tL) \cap \operatorname{bd}(K+tL)\big) \leq \mathcal{H}^{n-1}\big(\operatorname{bd}\big(\tilde{B}_i \cap (K+tL)\big)\big).$$

Now recall that $\mathcal{H}^{n-1}(\operatorname{bd} C) = n\mathsf{V}_n(B, C, \ldots, C)$ for every $C \in \mathcal{K}^n$ [21, (5.53)], so $\mathcal{H}^{n-1}(\operatorname{bd} K_1) \leq \mathcal{H}^{n-1}(\operatorname{bd} K_2)$ for any convex bodies $K_1 \subseteq K_2$. Thus

$$\mathcal{H}^{n-1}\big(\operatorname{bd}\big(\tilde{B}_i \cap (K+tL)\big)\big) \leq \mathcal{H}^{n-1}\big(\operatorname{bd} \tilde{B}_i\big) = ct^{n-1}$$

with $c = (\operatorname{Lip}(u_m) + \operatorname{diam}(L))^{n-1}\mathcal{H}^{n-1}(S^{n-1})$. We therefore obtain

$$\mathrm{S}_{(K+tL)[n-1]}(A_m) \leq ct^{n-1}N_t = O(t^{n-k-1}),$$

concluding the proof. □

4. Proof of the lower bound

The aim of this section is to prove Theorem 1.10. We again begin by sketching the main idea behind the proof before proceeding to the details.

4.1. **Outline of the proof.** We aim to characterize $\operatorname{supp} \mathrm{S}_{K[n-2],L}$ for $K, L \in \mathcal{K}^n$ by induction on the dimension n. The basic principle for doing so was explained in §1.5, but the procedure cannot be implemented directly as (1.5) does not always hold. Instead, we will perform the induction in an indirect manner that requires only the case $\dim T(K,u) = 1$ of (1.5), which holds by Theorem 1.21.

Remark 4.1. In the following, we will take for granted that Conjecture 1.6 is known to hold for the area measure $\mathrm{S}_{K[n-1]}$, that is, that

$$\operatorname{supp} \mathrm{S}_{K[n-1]} = \operatorname{cl}\left\{u \in S^{n-1} : \dim T(K,u) = 1\right\}.$$

This is a well known result, see, e.g., [21, Lemma 4.5.2]; we include a short proof in Corollary 4.5 below that follows trivially from the methods used in the proof of Theorem 1.21. Note that this implies, in particular, that $\operatorname{supp} \mathrm{S}_{K[n-2],L}$ is characterized for $n = 2$, which serves as the base case of our induction on n.

Let us now sketch the idea behind the induction. By Corollary 1.9 and as the support of a measure is a closed set, it suffices to show that any $(K[n-2], L)$-extreme direction $u \in S^{n-1}$ is contained in $\operatorname{supp} \mathrm{S}_{K[n-2],L}$. We fix such a direction u in the following. Let us consider two nearly complementary cases.

1. If $\dim T(K,u) = 1$, then the approach described in §1.5 can be applied directly: Lemma 1.20 yields $v \in u^\perp$ so that u is $(\mathrm{P}_{v^\perp} K[n-3], \mathrm{P}_{v^\perp} L)$-extreme, and thus the induction hypothesis implies that $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} K[n-3], \mathrm{P}_{v^\perp} L}$. We now conclude that $u \in \operatorname{supp} \mathrm{S}_{K[n-2],L}$ by the case $\dim T(K,u) = 1$ of Theorem 1.21.
2. Now suppose that $\dim T(K,u') > 1$ for all u' in a neighborhood of u. Then we have $u \notin \operatorname{supp} \mathrm{S}_{K[n-1]}$. It therefore suffices to show that
$$u \in \operatorname{supp}\left(\mathrm{S}_{K[n-2],L} + \mathrm{S}_{K[n-1]}\right) = \operatorname{supp} \mathrm{S}_{K[n-2],K+L}.$$
To this end, note that it is easily checked using Definition 1.5 that the fact that u is $(K[n-2], L)$-extreme implies that it is also $(K[n-2], K+L)$-extreme, as well as that $\dim T(K+L,u) = 1$. We can therefore once again proceed as in §1.5 to achieve the desired conclusion using Theorem 1.21.

These two cases almost, but not quite, suffice to prove Theorem 1.10: the above arguments leave the case that $\dim T(K,u) > 1$, but u lies on the boundary of the set of u' with $\dim T(K,u') = 1$, unresolved.

To handle this boundary case, we will use to our advantage the fact that the measure $\mathrm{S}_{C_1,\ldots,C_{n-1}}$ does *not* uniquely determine the convex bodies $C_1, \ldots, C_{n-1}$. We will show in §4.3 how one can modify the body K so that it satisfies the condition of the second case above, without changing the mixed area measure $\mathrm{S}_{K[n-2],L}$. Then the second case suffices to complete the proof of Theorem 1.10 (and, somewhat surprisingly, the first case no longer needs to be considered separately).

4.2. **The case** $\dim T(K,u) = 1$ **of Theorem 1.21.** Before we proceed to the proof of Theorem 1.10, we must prove the case $\dim T(K,u) = 1$ of Theorem 1.21 that is needed here. The remaining parts of Theorem 1.21 will be proved in §6.

The proof relies on a characterization of directions u with $\dim T(K,u) = 1$, which is dual to the characterization of extreme boundary points of a convex body in [21, Lemma 1.4.6]. Only the implication $a \Rightarrow b$ will be used in the sequel.

Lemma 4.2. *For any $K \in \mathcal{K}^n$ and $u \in S^{n-1}$, the following are equivalent.*

a. $\dim T(K,u) = 1$.

b. For every $\varepsilon > 0$, there exists $K' \in \mathcal{K}^n_n$ so that $K \subset K'$, $h_{K'}(u) > h_K(u)$, and $h_{K'}(v) = h_K(v)$ for all $v \in S^{n-1} \backslash B(u,\varepsilon)$.

Proof. We first prove $a \Rightarrow b$. As the statement is invariant under translation, we assume without loss of generality that $\operatorname{relint} K$ contains the origin.

If $\dim K \le n-2$, then $\dim T(K,u) > 1$ for all u and there is nothing to prove. If $\dim K = n-1$, then $\dim T(K,u) = 1$ implies that $u \in K^\perp$. Let $K' = \operatorname{conv}\{K, tu\}$ with $t > 0$. Then $h_{K'}(u) = t > 0 = h_K(u)$. Moreover, for every $\varepsilon > 0$, we can choose $t > 0$ so that $h_{K'} = h_K$ on $S^{n-1}\backslash B(u,\varepsilon)$, since as $t \downarrow 0$ the normal direction of each supporting hyperplane of K' that does not touch K converges to u.

It remains to consider the case that $\dim K = n$, so that $K \in \mathcal{K}^n_{(o)}$. By the duality explained in §2.2.3, $\dim T(K,u) = 1$ implies that

$$T(K,u) \cap \operatorname{bd} K^\circ = \{x\} \quad \text{with} \quad x = \frac{u}{\|u\|_{K^\circ}}$$

is a 0-dimensional face of K°. By [21, Lemma 1.4.6], there exists for every $\varepsilon > 0$ a choice of $v \in S^{n-1}$ and $t \in \mathbb{R}_+$ so that $x \in \operatorname{int} H^+_{v,t}$ and

$$K^\circ \cap H^+_{v,t} \subset \mathbb{R}_+ B(u,\varepsilon).$$

Now define $K' = (K^\circ \cap H^-_{v,t})^\circ \supset K$. Then

$$h_{K'}(v) = \|v\|_{K^\circ \cap H^-_{v,t}} = \|v\|_{K^\circ} = h_K(v)$$

for all $v \in S^{n-1}\backslash B(u,\varepsilon)$. Moreover, as $x \notin K^\circ \cap H^-_{v,t}$, we have

$$h_{K'}(u) = \|u\|_{K^\circ \cap H^-_{v,t}} > \|u\|_{K^\circ} = h_K(u),$$

which concludes the proof of $a \Rightarrow b$.

To prove the converse implication $b \Rightarrow a$, suppose that $\dim T(K,u) > 1$. Since $u \in \operatorname{relint} T(K,u)$, we can find distinct $v_1, v_2 \in T(K,u)$ so that $u = \frac{v_1+v_2}{2}$. Now assume for sake of contradiction that the conclusion of part b holds, and choose $\varepsilon > 0$ sufficiently small that $v_1, v_2 \notin B(u,\varepsilon)$. Then

$$h_K(u) = \frac{h_K(v_1) + h_K(v_2)}{2} = \frac{h_{K'}(v_1) + h_{K'}(v_2)}{2} \ge h_{K'}(u),$$

where we used that h_K is affine on $T(K,u)$ in the first equality, and that $h_{K'}$ is convex in the inequality. This entails a contradiction, concluding the proof. ☐

Recall the following classical result [21, Theorems 7.3.1 and 7.4.2].

Theorem 4.3 (Alexandrov–Fenchel inequality)**.** *For any $K, L, C_1, \ldots, C_{n-2} \in \mathcal{K}^n$*

$$\mathsf{V}_n(K,L,C_1,\ldots,C_{n-2})^2 \ge \mathsf{V}_n(K,K,C_1,\ldots,C_{n-2})\,\mathsf{V}_n(L,L,C_1,\ldots,C_{n-2}).$$

If in addition $\mathsf{V}_n(K,L,C_1,\ldots,C_{n-2}) > 0$, then equality holds if and only if

$$\mathrm{S}_{K,C_1,\ldots,C_{n-2}} = \frac{\mathsf{V}_n(K,K,C_1,\ldots,C_{n-2})}{\mathsf{V}_n(K,L,C_1,\ldots,C_{n-2})}\,\mathrm{S}_{L,C_1,\ldots,C_{n-2}}.$$

The following is the main observation of this section.

Lemma 4.4. *Let $K, C_1, \ldots, C_{n-2} \in \mathcal{K}^n$ and $u \in S^{n-1}$ with $\dim T(K,u) = 1$. If $u \notin \operatorname{supp} \mathrm{S}_{K,C_1,\ldots,C_{n-2}}$, then $u \notin \operatorname{supp} \mathrm{S}_{L,C_1,\ldots,C_{n-2}}$ for every $L \in \mathcal{K}^n$.*

Proof. As we assume that $u \notin \operatorname{supp} \mathrm{S}_{K,C_1,\ldots,C_{n-2}}$, there exists $\varepsilon > 0$ so that $\mathrm{S}_{K,C_1,\ldots,C_{n-2}}(B(u,\varepsilon)) = 0$. Let $K' \supset K$ be the convex body that is provided by part b of Lemma 4.2 for this choice of ε.

Suppose that $\mathsf{V}_n(K', K, C_1, \ldots, C_{n-2}) = 0$. Recall that $\dim K' = n$ by construction, while $\dim K \geq n-1$ as $\dim T(K,u) = 1$. Thus by Fact 1.4, there exists $I \subseteq [n-2]$ so that $\dim C_I < |I|$. Then Fact 1.4 readily yields that $\mathrm{S}_{L,C_1,\ldots,C_{n-2}} = 0$ for every $L \in \mathcal{K}^n$, concluding the proof in this case.

We may therefore assume that $\mathsf{V}_n(K', K, C_1, \ldots, C_{n-2}) > 0$. Note that by construction $h_{K'} = h_K$ on $S^{n-1} \backslash B(u,\varepsilon) \supseteq \operatorname{supp} \mathrm{S}_{K,C_1,\ldots,C_{n-2}}$, so that

$$\mathsf{V}_n(K', K, C_1, \ldots, C_{n-2}) = \mathsf{V}_n(K, K, C_1, \ldots, C_{n-2}).$$

Therefore

$$\begin{aligned}\mathsf{V}_n(K', K, C_1, \ldots, C_{n-2})^2 &= \mathsf{V}_n(K', K, C_1, \ldots, C_{n-2})\,\mathsf{V}_n(K, K, C_1, \ldots, C_{n-2}) \\ &\leq \mathsf{V}_n(K', K', C_1, \ldots, C_{n-2})\,\mathsf{V}_n(K, K, C_1, \ldots, C_{n-2}),\end{aligned}$$

where we used that $K \subseteq K'$. Thus Theorem 4.3 yields $\mathrm{S}_{h_{K'}-h_K,C_1,\ldots,C_{n-2}} = 0$. In particular, integrating the function h_L with respect to this measure yields

$$0 = \mathsf{V}_n(L, h_{K'} - h_K, C_1, \ldots, C_{n-2}) = \frac{1}{n}\int (h_{K'} - h_K)\, d\mathrm{S}_{L,C_1,\ldots,C_{n-2}}$$

for every $L \in \mathcal{K}^n$. Since the function $h_{K'} - h_K$ is nonnegative on S^{n-1} and is strictly positive in a neighborhood of u, we conclude that $u \notin \operatorname{supp} \mathrm{S}_{L,C_1,\ldots,C_{n-2}}$. ◻

The following follows directly.

Proof of the case $\dim T(K,u) = 1$ *of Theorem 1.21.* The conclusion is immediate by choosing $L = [0, v]$ in Lemma 4.4 and applying the projection formula in §2.5. ◻

As a further simple illustration of the utility of Lemma 4.4, let us give another proof of the following well known result (see, e.g., [21, §4.5]).

Corollary 4.5. *For any* $K \in \mathcal{K}^n$*, we have*

$$\operatorname{supp} \mathrm{S}_{K[n-1]} = \operatorname{cl}\big\{u \in S^{n-1} : \dim T(K,u) = 1\big\}.$$

Proof. As the upper bound holds by Corollary 1.9, it suffices to show that every $u \in S^{n-1}$ with $\dim T(K,u) = 1$ satisfies $u \in \operatorname{supp} \mathrm{S}_{K[n-1]}$. Suppose the latter does not hold. Then we may repeatedly apply Lemma 4.4 with $L = B$ to conclude that $u \notin \mathrm{S}_{B[n-1]}$. But this entails a contradiction, as $\mathrm{S}_{B[n-1]}$ is the Lebesgue measure on S^{n-1} by (2.4) and thus $\operatorname{supp} \mathrm{S}_{B[n-1]} = S^{n-1}$. ◻

4.3. **Proof of Theorem 1.10.** As was explained in §4.1, the main difficulty in the proof is to modify K so that it has the desired properties without changing the mixed area measure. We begin with a useful observation.

Lemma 4.6. *Let* $K \in \mathcal{K}^n$*,* $E \subseteq S^{n-1}$*, and* $f : E \to \mathbb{R}$ *such that*

$$K = \bigcap_{v \in E} H^-_{v,f(v)}.$$

Then every $u \in S^{n-1}$ *with* $\dim T(K,u) = 1$ *is in the closure of* E.

Proof. Let $u \in S^{n-1}$ with $\dim T(K,u) = 1$, and suppose that $u \notin \operatorname{cl} E$. Choose $\varepsilon > 0$ so that $B(u,\varepsilon) \cap E = \varnothing$, and let $K' \supset K$ be the resulting convex body that is provided by part *b* of Lemma 4.2. Then $h_{K'}(v) = h_K(v)$ for all $v \in E$ and thus

$$K' \subseteq \bigcap_{v\in E} H^-_{v,h_{K'}(v)} = \bigcap_{v\in E} H^-_{v,h_K(v)} \subseteq \bigcap_{v\in E} H^-_{v,f(v)} = K,$$

where we used the obvious fact that $h_K(v) \le f(v)$ for every $v \in S^{n-1}$. This entails a contradiction, since $h_{K'}(u) > h_K(u)$. □

We can now construct the desired modification of K.

Lemma 4.7. *Let $K, C_1, \ldots, C_{n-2} \in \mathcal{K}^n$, $u \in S^{n-1}$, and $\varepsilon > 0$ so that $B(u,\varepsilon)$ lies strictly inside a hemisphere. Suppose that $\mathrm{S}_{K,C_1,\ldots,C_{n-2}}(B(u,\varepsilon)) = 0$. Then*

$$\hat{K} = \bigcap_{v\in S^{n-1}\setminus B(u,\varepsilon)} H^-_{v,h_K(v)} \supseteq K$$

is a convex body that satisfies the following properties.

a. $\mathrm{S}_{\hat{K},C_1,\ldots,C_{n-2}} = \mathrm{S}_{K,C_1,\ldots,C_{n-2}}$.

b. $\dim T(\hat{K},v) > 1$ *for every* $v \in B(u,\varepsilon)$.

c. If in addition $u \in \operatorname{supp} \mathrm{S}_{B,C_1,\ldots,C_{n-2}}$*, then* $T(\hat{K},u) \subseteq T(K,u)$.

Proof. We first note that as $B(u,\varepsilon)$ lies strictly inside a hemisphere, it follows that $\hat{K} \subseteq \bigcap_{v\in S^{n-1}\setminus B(u,\varepsilon)} H^-_{v,\|h_K\|_\infty}$ is compact and thus $\hat{K} \in \mathcal{K}^n$.

We now prove part *a*. Suppose first that $\dim K = n$. Then we can proceed as in the proof of Lemma 4.4. Indeed, if $\mathsf{V}_n(\hat{K}, K, C_1, \ldots, C_{n-2}) = 0$, then Fact 1.4 yields $I \subseteq [n-2]$ with $\dim C_I < |I|$, and thus $\mathrm{S}_{\hat{K},C_1,\ldots,C_{n-2}} = \mathrm{S}_{K,C_1,\ldots,C_{n-2}} = 0$. If $\mathsf{V}_n(\hat{K}, K, C_1, \ldots, C_{n-2}) > 0$, then the Alexandrov–Fenchel argument of Lemma 4.4 extends *verbatim* to the present setting as $h_{\hat{K}}(v) = h_K(v)$ for all $v \in S^{n-1}\setminus B(u,\varepsilon)$. This concludes the proof of part *a* for $\dim K = n$.

If $\dim K < n$, define $K_t = K + t\hat{B}$, where $\hat{B} = \operatorname{conv}\{B, su\}$ and $s > 1$ is chosen so that $N(\hat{B}, su) = \mathbb{R}_+ \operatorname{cl} B(u,\varepsilon)$. Then $\mathrm{S}_{\hat{B},C_1,\ldots,C_{n-2}}(B(u,\varepsilon)) = 0$ by Theorem 1.8, so we can apply part *a* to K_t to conclude that $\mathrm{S}_{\hat{K}_t,C_1,\ldots,C_{n-2}} = \mathrm{S}_{K_t,C_1,\ldots,C_{n-2}}$. The conclusion of part *a* follows by the continuity of mixed area measures [21, p. 281], as $K_t \to K$ and $\hat{K}_t \to \hat{K}$ as $t \downarrow 0$ [21, Lemma 7.5.2].

Part *b* follows directly from the definition of $\hat{K}$ by Lemma 4.6 and as $S^{n-1}\setminus B(u,\varepsilon)$ is closed (as $B(u,\varepsilon)$ is an open ball by definition).

To prove part *c*, note that integrating both sides of part *a* with respect to h_B, and using that mixed volumes are symmetric in their arguments, yields

$$\int (h_{\hat{K}} - h_K)\, d\mathrm{S}_{B,C_1,\ldots,C_{n-2}} = \int h_B \, d\mathrm{S}_{h_{\hat{K}}-h_K,C_1,\ldots,C_{n-2}} = 0.$$

Since $h_{\hat{K}} - h_K \ge 0$ and $u \in \operatorname{supp} \mathrm{S}_{B,C_1,\ldots,C_{n-2}}$, this implies that $h_{\hat{K}}(u) = h_K(u)$. As $K \subseteq \hat{K}$, it follows that $F(K,u) \subseteq F(\hat{K},u)$. Now fix any $x \in F(K,u)$. Since $x \in \operatorname{bd} K \cap \operatorname{bd} \hat{K}$ and $K \subseteq \hat{K}$, any supporting hyperplane of $\hat{K}$ that contains x must also be a supporting hyperplane of K, so we must have $u \in N(\hat{K},x) \subseteq N(K,x)$. The conclusion of part *c* now follows from Lemma 2.5. □

We can now conclude the proof of Theorem 1.10.

Proof of Theorem 1.10. Let $K, L \in \mathcal{K}^n$. As the upper bound is provided by Corollary 1.9, it suffices to show that every $(K[n-2], L)$-extreme direction $u \in S^{n-1}$ is contained in $\operatorname{supp} \mathrm{S}_{K[n-2],L}$. We will do so by induction on the dimension n. The base case $n = 2$ is provided by Corollary 4.5; we assume in the remainder of the proof that the result has been proved up to dimension $n - 1$.

Fix a $(K[n-2], L)$-extreme direction $u \in S^{n-1}$, and assume for sake of contradiction that $u \notin \operatorname{supp} \mathrm{S}_{K[n-2],L}$. Then there exists $\varepsilon > 0$ sufficiently small so that $\mathrm{S}_{K[n-2],L}(B(u, \varepsilon)) = 0$. Let $\hat{K}$ be the modified body of Lemma 4.7. Then applying part *a* of Lemma 4.7 repeatedly yields $\mathrm{S}_{\hat{K}[n-2],L} = \mathrm{S}_{K[n-2],L}$. In particular,

$$u \notin \operatorname{supp} \mathrm{S}_{\hat{K}[n-2],L}.$$

We must now establish the following.

Claim. The vector u is $(\hat{K}[n-2], L)$-extreme.

Proof. As u is $(K[n-2], L)$-extreme, Lemma 1.20 yields $v \in T(K, u)^{\perp}$ so that u is $(\mathrm{P}_{v^\perp} K[n-3], \mathrm{P}_{v^\perp} L)$-extreme. Thus $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} K[n-3], \mathrm{P}_{v^\perp} L}$ by the induction hypothesis, which implies $u \in \operatorname{supp} \mathrm{S}_{B,K[n-3],L}$ by Lemma 2.8. Part *c* of Lemma 4.7 therefore yields $T(\hat{K}, u) \subseteq T(K, u)$, which readily implies the claim. □

We now proceed as was explained in §4.1. We have

$$u \notin \operatorname{supp} \mathrm{S}_{\hat{K}[n-1]}$$

by Corollary 4.5 and part *b* of Lemma 4.7. Therefore

$$u \notin \operatorname{supp} \left(\mathrm{S}_{\hat{K}[n-2],L} + \mathrm{S}_{\hat{K}[n-1]} \right) = \operatorname{supp} \mathrm{S}_{\hat{K}[n-2],\hat{K}+L}.$$

On the other hand, as u is $(\hat{K}[n-2], L)$-extreme, u is also $(\hat{K}[n-2], \hat{K} + L)$-extreme and $\dim T(\hat{K} + L, u) = 1$. By Lemma 1.20, there exists $v \in u^{\perp}$ so that u is $(\mathrm{P}_{v^\perp} \hat{K}[n-2])$-extreme, and thus $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} \hat{K}[n-2]}$ by Corollary 4.5. The case $\dim T(K, u) = 1$ of Theorem 1.21 now yields $u \in \operatorname{supp} \mathrm{S}_{\hat{K}[n-2],\hat{K}+L}$. This entails a contradiction, concluding the proof. □

5. Mixed Hessian measures

The aim of this section is to derive the implications of our main results for mixed Hessian measures. In particular, we will prove Corollaries 1.16 and 1.19.

It should be emphasized that mixed area and Hessian measures are not really distinct notions: mixed areas measures of convex bodies in $\mathbb{R}^n$ are a special case of mixed Hessian measures of convex functions on $\mathbb{R}^n$ [7, Corollary 4.2], while mixed Hessian measures of convex functions on $\mathbb{R}^n$ are a special case of mixed area measures of convex bodies in $\mathbb{R}^{n+1}$ [7, Corollary 4.9]. Such connections were used in [3, 7] to translate some previously known special cases of Conjecture 1.6 to the setting of mixed Hessian measures. Some care is needed, however, in extending these connections to the general setting considered here.

5.1. **From mixed area to mixed Hessian measures.** The aim of this section is to recall how mixed Hessian measures can be obtained as a special case of mixed area measures. The construction in this section is taken from [7].

Denote by $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ the class of lower-semicontinuous proper convex functions $g : \mathbb{R}^n \to (-\infty, \infty]$ such that $\mathrm{dom}\, g$ is compact, and let

$$\mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n) = \{g^* : g \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)\}.$$

Here g^* denotes the convex conjugate

$$g^*(x) = \sup_{y\in\mathbb{R}^n} \{\langle x, y\rangle - g(y)\}.$$

Let $f \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$. Then $f^* \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, as any lower-semicontinuous convex function g satisfies $g^{**} = g$ [21, Theorem 1.6.13]. As f^* is lower-semicontinuous,

$$\mathrm{epi}(f^*) = \{(x, t) \in \mathbb{R}^{n+1} : f^*(x) \le t\}$$

is a closed convex set, where we write $(x, t) \in \mathbb{R}^{n+1}$ with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Definition 5.1. For every $f \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$, define $K_f \in \mathcal{K}^{n+1}$ as

$$K_f = \mathrm{epi}(f^*) \cap \{(x, t) \in \mathbb{R}^{n+1} : t \le r_f\}$$

where $r_f = \max_{x\in\mathrm{dom}\, f^*} f^*(x) < \infty$.

In the following, we denote by S^n_- the negative hemisphere in S^n:

$$S^n_- = \{(y, t) \in S^n : t < 0\}.$$

Then the map $\phi : \mathbb{R}^n \to S^n_-$ defined by

$$\phi(x) = \left(\frac{x}{\sqrt{1 + \|x\|^2}}, -\frac{1}{\sqrt{1 + \|x\|^2}}\right)$$

is a homeomorphism with inverse $\phi^{-1}(y, t) = -\frac{y}{t}$. The following result, which is a restatement of [7, Corollary 4.9], relates the pushforward $\phi_*\mathrm{H}_{f_1,\dots,f_n}$ of the mixed Hessian measure to the mixed area measure $\mathrm{S}_{K_{f_1},\dots,K_{f_n}}$ restricted to S^n_-.

Lemma 5.2. *Let $f_1, \dots, f_n \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$. Then the measures $\phi_*\mathrm{H}_{f_1,\dots,f_n}$ and $\mathrm{S}_{K_{f_1},\dots,K_{f_n}}|_{S^n_-}$ are mutually absolutely continuous with*

$$\frac{d\phi_*\mathrm{H}_{f_1,\dots,f_n}}{d\mathrm{S}_{K_{f_1},\dots,K_{f_n}}}(\phi(x)) = \frac{1}{\sqrt{1 + \|x\|^2}}.$$

In particular, $\mathrm{supp}\, \mathrm{H}_{f_1,\dots,f_n} = \phi^{-1}\big(S^n_- \cap \mathrm{supp}\, \mathrm{S}_{K_{f_1},\dots,K_{f_n}}\big)$.

5.2. **Extreme directions and localization.** In order to translate our main results to the setting of mixed Hessian measures, we must clarify the connection between $L(f, x)$ and $T(K_f, u)$. We begin with the following simple lemma.

Lemma 5.3. *For any $f \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have $f(x) = h_{K_f}(x, -1)$.*

Proof. We readily compute

$$h_{K_f}(x, -1) = \sup_{(y,t): f^*(y)\le t\le r_f} \{\langle y, x\rangle - t\} = \sup_y \{\langle y, x\rangle - f^*(y)\} = f(x)$$

by the definition of K_f. □

Consequently, we have the following.

Lemma 5.4. *For any $f \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have*

$$L(f,x) = \phi^{-1}\big(S^n_- \cap T(K_f, \phi(x))\big).$$

Proof. Any convex set $B \subseteq \mathbb{R}^n$ defines a convex cone $A = \mathbb{R}_+\phi(B)$ in $\mathbb{R}_+S^n_-$. We claim that f is affine on B if and only if h_{K_f} is linear on A: indeed,

$$f\left(\frac{t}{t+t'}x + \frac{t'}{t+t'}x'\right) = \frac{t}{t+t'}f(x) + \frac{t'}{t+t'}f(x') \quad \Longleftrightarrow$$
$$h_{K_f}(tx + t'x', -t-t') = h_{K_f}(tx, -t) + h_{K_f}(t'x', -t')$$

for any $x, x' \in B$ and $t, t' > 0$ by Lemma 5.3.

Now note that for any convex cone A in $\mathbb{R}_+S^n_-$, the convex set $B \subseteq \mathbb{R}^n$ defined by $B = \phi^{-1}(A \cap S^n_-)$ satisfies $A = \mathbb{R}_+\phi(B)$. Thus the relation between A and B defines a bijection between convex sets in $\mathbb{R}^n$ and convex cones in $\mathbb{R}_+S^n_-$. The definition of $L(f,x)$ therefore implies that $\mathbb{R}_+\phi(L(f,x))$ is the largest convex cone in $\mathbb{R}_+S^n_-$ that contains $\phi(x)$ in its relative interior on which h_{K_f} is linear, that is,

$$\mathbb{R}_+\phi(L(f,x)) = T(K_f, \phi(x)) \cap \mathbb{R}_+S^n_-.$$

The conclusion follows by inverting this identity. □

Lemma 5.4 immediately yields the following corollary.

Corollary 5.5. *Let $f_1, \ldots, f_n \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$. Then $x \in \mathbb{R}^n$ is $(f_1, \ldots, f_n)$-extreme if and only if $\phi(x) \in S^n$ is $(K_{f_1}, \ldots, K_{f_n})$-extreme.*

We are now nearly ready to prove our main results on mixed Hessian measures. There is, however, one additional technical point that must be dispensed with first. The reduction from mixed Hessian measures to mixed area measures requires us to work with functions in $\mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$, while our results are formulated in the more general setting of functions in $\mathrm{Conv}(\Omega)$ for some open convex set $\Omega \subseteq \mathbb{R}^n$. The following localization lemma reduces the the latter setting to the former one.

Lemma 5.6. *Let $\Omega \subseteq \mathbb{R}^n$ be an open convex set and $\Omega' \subset \Omega$ be convex and compact. Then for any $f \in \mathrm{Conv}(\Omega)$, there exists $g \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$ so that $f = g$ on Ω'.*

Proof. Recall that the indicator I_A of a convex set A is defined by

$$\mathrm{I}_A(x) = \begin{cases} 0 & \text{for } x \in A, \\ +\infty & \text{for } x \notin A. \end{cases}$$

By [21, Theorem 1.5.3], f is Lipschitz on Ω' with Lipschitz constant ℓ. Define

$$g = ((f + \mathrm{I}_{\Omega'})^* + \mathrm{I}_{\ell B})^*.$$

As $\mathrm{dom}(f + \mathrm{I}_{\Omega'}) = \Omega'$ is compact, $(f + \mathrm{I}_{\Omega'})^*$ is a continuous convex function on $\mathbb{R}^n$. Thus $(f + \mathrm{I}_{\Omega'})^* + \mathrm{I}_{\ell B} \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ and therefore $g \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$.

Now note that $f + \mathrm{I}_{\Omega'}$ is lower-semicontinuous, and thus

$$f(x) = \sup_{y \in \mathbb{R}^n} \big\{\langle y, x\rangle - (f + \mathrm{I}_{\Omega'})^*(y)\big\}$$

for $x \in \Omega'$. By [17, Theorem 23.5], the supremum is attained by $y \in \partial f(x) \subseteq \partial(f + \mathrm{I}_{\Omega'})(x)$, where ∂f is the subdifferential. As $\|y\| \leq \ell$ by [17, Theorem 24.7],

$$f(x) = \sup_{\|y\| \leq \ell} \left\{ \langle y, x \rangle - (f + \mathrm{I}_{\Omega'})^*(y) \right\} = g(x)$$

for every $x \in \Omega'$, concluding the proof. ☐

Fix $f_1, \ldots, f_n \in \mathrm{Conv}(\Omega)$ and $x \in \Omega$, and let $\Omega' = \mathrm{cl}\, B(x, \varepsilon)$ for some $\varepsilon > 0$. Construct the functions $g_1, \ldots, g_n \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$ as in Lemma 5.6. Then clearly

$$\mathrm{H}_{f_1,\ldots,f_n}|_{B(x,\varepsilon)} = \mathrm{H}_{g_1,\ldots,g_n}|_{B(x,\varepsilon)}, \qquad T(f_i, x) \cap B(x, \varepsilon) = T(g_i, x) \cap B(x, \varepsilon).$$

In particular, the properties that $u \in \mathrm{supp}\, \mathrm{H}_{f_1,\ldots,f_n}$ or that u is $(f_1, \ldots, f_n)$-extreme are unchanged if we replace $f_1, \ldots, f_n$ by $g_1, \ldots, g_n$.

Corollary 5.7. *Conjecture 1.6 implies Conjecture 1.15.*

Proof. Since the statements of both conjectures are local in nature, it suffices by the remark following Lemma 5.6 to assume that $f_1, \ldots, f_n \in \mathrm{Conv}^*_{\mathrm{cd}}(\mathbb{R}^n)$. The conclusion now follows directly from Lemma 5.2 and Corollary 5.5. ☐

Remark 5.8. Conjectures 1.6 and 1.15 are in fact equivalent; the converse implication to Corollary 5.7 can be readily read off from [7, Corollary 4.2].

The proof of Corollary 1.16 is completely analogous.

Proof of Corollary 1.16. The result follows from Theorems 1.8 and 1.10 by exactly the same argument as in the proof of Corollary 5.7. ☐

We conclude by proving Corollary 1.19. In the proof, we will use without comment that Lemma 2.2 extends directly to the convex function setting.

Proof of Corollary 1.19. As $\mathrm{H}_{f,g}(D) = 0$, Corollary 1.19 shows that none of the points in D are (f, g)-extreme. Thus for any $x \in D \backslash R$, it must be the case that $L(f, x)$ and $L(g, x)$ are both one-dimensional with the same direction.

Let L be the affine line in $\mathbb{R}^2$ that contains both $L(f, x)$ and $L(g, x)$, and let I be the connected component of $L \cap D$ that contains x. We claim that I must be contained in both $L(f, x)$ and $L(g, x)$. Indeed, suppose this is not the case. Then there must exist $y \in I \cap \mathrm{relbd}(L(f, x) \cap L(g, x))$. Therefore $\{y\}$ is a face of either $L(f, x)$ or $L(g, x)$. By Lemma 2.2, this implies that

$$\min\{\dim L(f, y), \dim L(g, y)\} = 0, \qquad \max\{\dim L(f, y), \dim L(g, y)\} \leq 1.$$

But then y is (f, g)-extreme, which entails a contradiction.

As I is contained in both $L(f, x)$ and $L(g, x)$, it follows from Lemma 2.2 that $L(f, y) = L(f, x)$ and $L(g, y) = L(g, x)$ for all $y \in I$. Thus $I \cap R = \varnothing$. ☐

Note that Corollary 1.18 is merely a specialization of Corollary 1.19 to the case that f, g are smooth, and therefore does not require a separate proof.

6. Support and projection

The main aim of this final part of the paper is to clarify the behavior of the supports of mixed area measures under projection. As was explained in §1.5, this presents a basic obstacle to extending the proof of Theorem 1.10 to more general situations. In the first two sections, we will prove the remaining cases of Theorem 1.21. In the last section, we present a simple example further illuminates some aspects of the structure of $(C_1, \ldots, C_{n-1})$-extreme directions.

6.1. **The case $\dim T(K, u) = n-1$ of Theorem 1.21.** Let $K \in \mathcal{K}^n$, $u \in S^{n-1}$ with $\dim T(K, u) = n - 1$. What is special about this situation is that, for sufficiently small $\varepsilon > 0$, the touching cone $T(K, u)$ bisects $B(u, \varepsilon)$. Thus for all $v \in B(u, \varepsilon)$, it must be the case that $T(K, v) \cap B(u, \varepsilon)$ lies entirely on one side of the hyperplane that contains $T(K, u)$. This property makes it possible to flexibly deform K in the direction $T(K, u)^\perp$, as we will presently explain.

Recall that the Minkowski difference of $K, L \in \mathcal{K}^n$ is defined as

$$K \div L = \{x \in \mathbb{R}^n : x + L \subseteq K\}.$$

The proof will be based on the following result of Schneider.

Lemma 6.1. *Let $K \in \mathcal{K}^n_n$ and $L \in \mathcal{K}^n$, and define*

$$K_t = \begin{cases} K + tL & \text{for } t \geq 0, \\ K \div (-t)L & \text{for } t < 0. \end{cases}$$

Suppose that for every $x \in \operatorname{bd} K$, there is $y \in \operatorname{bd} L$ with $N(K, x) \subseteq N(L, y)$. Then

$$\frac{d}{dt} h_{K_t}(u)\bigg|_{t=0} = h_L(u)$$

for every $u \in S^{n-1}$. Moreover, there exists $\delta > 0$ so that

$$\|h_{K_t} - h_K\|_\infty \leq \delta |t|$$

for all sufficiently small $|t|$.

Proof. The first statement is given in [21, Lemma 7.5.4] for $K, L \in \mathcal{K}^n_n$; however, its proof only uses that $K \in \mathcal{K}^n_n$ and extends *verbatim* to any $L \in \mathcal{K}^n$. The second statement is trivial for $t \geq 0$, and is proved in [21, p. 425] for $t < 0$. □

The assumption of Lemma 6.1, which is formulated in terms of normal cones, is automatically implied by the corresponding property of touching cones.

Lemma 6.2. *Let $K, L \in \mathcal{K}^n$, and suppose that $T(K, v) \subseteq T(L, v)$ for all $v \in S^{n-1}$. Then for every $x \in \operatorname{bd} K$, there is $y \in \operatorname{bd} L$ with $N(K, x) \subseteq N(L, y)$.*

Proof. Given any $x \in \operatorname{bd} K$, we can choose $v \in \operatorname{relint} N(K, x)$ and $y \in \operatorname{relint} F(L, v)$. Then $N(K, x) = T(K, v) \subseteq T(L, v) \subseteq N(L, y)$. □

Now let $K \in \mathcal{K}^n$, $u \in S^{n-1}$ with $\dim T(K, u) = n-1$, and $w \in T(K, u)^\perp \cap S^{n-1}$. Choose $\varepsilon > 0$ sufficiently small that $T(K, u)$ bisects $B(u, \varepsilon)$. Then for every $v \in B(u, \varepsilon)$, we have that $T(K, v) \cap B(u, \varepsilon)$ is contained either in $H^+_{w,0}$, $H_{w,0}$, or $H^-_{w,0}$.

The latter sets are precisely the touching cones of the interval $[0,w]$. This suggests that we aim to apply Lemma 6.1 with $L=[0,w]$.

There are two problems with this idea. First, only the intersection of the touching cones of K with $B(u,\varepsilon)$, rather than the touching cones themselves, are guaranteed to be contained in a touching cone of $[0,w]$. Second, Lemma 6.1 requires that K has nonempty interior, which we have not assumed. The following lemma shows that we can modify K in such a way that the conditions of Lemma 6.1 are satisfied, without changing the part of its boundary with normal directions in $B(u,\varepsilon)$.

Lemma 6.3. *Fix $K\in\mathcal{K}^n$ and $u\in S^{n-1}$ with $\dim T(K,u)=n-1$, and let $w\in T(K,u)^\perp\cap S^{n-1}$. Choose $\varepsilon>0$ sufficiently small that $T(K,u)$ bisects $B(u,\varepsilon)$. Then there exists $\tilde K\in\mathcal{K}^n_n$ with the following properties.*

a. $F(\tilde K,v)=F(K,v)$ for all $v\in B(u,\varepsilon)$.
b. $T(\tilde K,v)\subseteq T([0,w],v)$ for all $v\in S^{n-1}$.

Proof. Let $\hat B=\mathrm{conv}\{B,su\}$, where $s>1$ is chosen so that $N(\hat B,su)=\mathbb{R}_+\,\mathrm{cl}\,B(u,\varepsilon)$. We define $\tilde K=K+\hat B-su$. Then clearly

$$F(\tilde K,v)=F(K,v)+F(\hat B,v)-su=F(K,v)$$

for all $v\in B(u,\varepsilon)$, which verifies part *a*. Now note that

$$\begin{aligned}T(\tilde K,v)&=T(K,v)\cap T(\hat B,v)\\&=\begin{cases}T(K,v)\cap\mathbb{R}_+\,\mathrm{cl}\,B(u,\varepsilon) & \text{for } v\in B(u,\varepsilon),\\ \mathbb{R}_+v & \text{for } v\in S^{n-1}\backslash B(u,\varepsilon).\end{cases}\end{aligned}$$

As $T(K,u)\cap\mathbb{R}_+\,\mathrm{cl}\,B(u,\varepsilon)=w^\perp\cap\mathbb{R}_+\,\mathrm{cl}\,B(u,\varepsilon)$ bisects $B(u,\varepsilon)$ and as any two touching cones of K are either equal or disjoint, part *b* follows readily. ☐

We finally recall the following well known fact, see, e.g., [8, Lemma 2.12].

Lemma 6.4. *Let $K,L,C_1,\dots,C_{n-2}\in\mathcal{K}^n$ and $A\subseteq S^{n-1}$. If $F(K,v)=F(L,v)$ for all $v\in A$, then $\mathrm{S}_{K,C_1,\dots,C_{n-2}}|_A=\mathrm{S}_{L,C_1,\dots,C_{n-2}}|_A$.*

We can now conclude the proof.

Proof of the case $\dim T(K,u)=n-1$ of Theorem 1.21. Fix $K,C_1,\dots,C_{n-2}\in\mathcal{K}^n$ and $u\in S^{n-1}$ so that $\dim T(K,u)=n-1$, and let $w\in T(K,u)^\perp\cap S^{n-1}$. Suppose that $u\notin\mathrm{supp}\,\mathrm{S}_{K,C_1,\dots,C_{n-2}}$. Then there exists $\varepsilon>0$ so that $T(K,u)$ bisects $B(u,\varepsilon)$ and $\mathrm{S}_{K,C_1,\dots,C_{n-2}}(B(u,\varepsilon))=0$. Construct $\tilde K$ as in Lemma 6.3.

By Lemma 6.4 and part *a* of Lemma 6.3, we have $\mathrm{S}_{\tilde K,C_1,\dots,C_{n-2}}(B(u,\varepsilon))=0$. In particular, if $f\in C^2(S^{n-1})$ is chosen such that $f(v)>0$ for $v\in B(u,\varepsilon)$ and $f(v)=0$ for $v\in S^{n-1}\backslash B(u,\varepsilon)$, then we can write

$$\int h_{\tilde K}\,d\mathrm{S}_{f,C_1,\dots,C_{n-2}}=\int f\,d\mathrm{S}_{\tilde K,C_1,\dots,C_{n-2}}=0.$$

Now define $\tilde K_t$ as in Lemma 6.1 with $K\leftarrow\tilde K$ and $L\leftarrow[0,w]$. Then

$$\int h_{\tilde K_t}\,d\mathrm{S}_{f,C_1,\dots,C_{n-2}}=\int f\,d\mathrm{S}_{\tilde K_t,C_1,\dots,C_{n-2}}\geq 0$$

for all t sufficiently small. In particular, this integral is minimized at $t = 0$. By Lemma 6.2 and part b of Lemma 6.3, the assumption of Lemma 6.1 is satisfied. Thus a routine application of dominated convergence yields

$$\int h_{[0,w]}\, d\mathrm{S}_{f,C_1,\ldots,C_{n-2}} = \frac{d}{dt}\int h_{\tilde{K}_t}\, d\mathrm{S}_{f,C_1,\ldots,C_{n-2}}\bigg|_{t=0} = 0.$$

The projection formula (see §2.5) now yields

$$\int f\, d\mathrm{S}_{\mathrm{P}_{w^\perp}C_1,\ldots,\mathrm{P}_{w^\perp}C_{n-2}} = 0,$$

so that $u \notin \operatorname{supp} \mathrm{S}_{\mathrm{P}_{w^\perp}C_1,\ldots,\mathrm{P}_{w^\perp}C_{n-2}}$. We have therefore proved the contrapositive of the desired statement, concluding the proof. $\square$

6.2. **The case** $1 < \dim T(K,u) < n-1$**.** We first prove part b of Theorem 1.21 in the special case that $\dim T(K,u) = 2$. The construction is then readily adapted to any $1 < \dim T(K,u) < n-1$ at the end of the proof.

The idea of the proof is to construct $K \in \mathcal{K}^n$ with the following properties:

a. $\dim T(K,u') = 2$ for all u' in a neighborhood of u.

b. For every $v \in T(K,u)^\perp$, the union of the touching cones $T(K,u')$ that are contained in $v^\perp$ has Hausdorff dimension at most $n-2$. Thus $\dim T(\mathrm{P}_{v^\perp}K,u') = \dim(T(K,u') \cap v^\perp) = 1$ on a dense set of u' in a neighborhood of u.

Therefore, by Corollary 4.5, $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp}K[n-2]}$ for all $v \in T(K,u)^\perp$ but $u \notin \operatorname{supp} \mathrm{S}_{K[n-1]}$, which proves part b of Theorem 1.21 when $\dim T(K,u) = 2$.

It is somewhat more convenient to construct the polar body $L = K^\circ$. The main part of the argument is contained in the following lemma. Throughout this section, we denote by $e_1, \ldots, e_n$ the coordinate basis of $\mathbb{R}^n$.

Lemma 6.5. *For every $n \geq 4$, there exists $L \in \mathcal{K}^n_{(o)}$ with the following properties.*

a. $[e_1 - e_n, e_1 + e_n]$ is a 1-face of L.

b. There is a neighborhood $O \subset \operatorname{bd} L$ of e_1 so that every $y \in O$ lies in the relative interior of a 1-face F_y of L.

c. For every $(n-1)$-dimensional subspace E of $\mathbb{R}^n$ that contains $[e_1 - e_n, e_1 + e_n]$, the union of all the faces $F_y \subset E$ has Hausdorff dimension at most $n-3$.

Proof. Let U be the Euclidean unit ball in $\mathbb{R}^{n-1}$ and define

$$M = \big\{x \in \mathbb{R}^{n-1} : x_1^2 + \cdots + x_{n-2}^2 + (x_{n-1} - tf(x_1,\ldots,x_{n-2}))^2 \leq 1\big\},$$

where f is a smooth function to be chosen below such that $f(1,0,\ldots,0) = 0$. We choose $t > 0$ sufficiently small so that M is strictly convex.

In the following, we will identify U and M with their natural embedding into $e_n^\perp \subset \mathbb{R}^n$. We now define the convex body $L \in \mathcal{K}^n_{(o)}$ as

$$L = \operatorname{conv}\big(M - e_n, U + e_n\big).$$

We must check that this body satisfies all the desired properties.

Part a. It is readily seen that $F(U,e_1) = F(M,e_1) = \{e_1\}$. Therefore $H_{e_1,1}$ is a supporting hyperplane of L and $F(L,e_1) = L \cap H_{e_1,1} = [e_1 - e_n, e_1 + e_n]$.

Part b. Choose $O = B(e_1, \varepsilon) \cap \operatorname{bd} L$ with $\varepsilon < 1$. By construction, every extreme point of L is contained either in $\operatorname{bd} M - e_n$ or $\operatorname{bd} U + e_n$. Thus every $y \in O$ is a non-extreme boundary point of L, and is therefore in the relative interior of a face F_y of L with $\dim F_y \geq 1$. But $F_y \cap (\operatorname{bd} U + e_n)$ and $F_y \cap (\operatorname{bd} M - e_n)$ must be singletons, as U and M are strictly convex. Thus $\dim F_y = 1$ for all $y \in O$.

Part c. Let $y \in O$ and $w \in N(L, y) \cap S^{n-1}$. By part b, the exist points $a \in \operatorname{bd} M$ and $b \in \operatorname{bd} U$ so that $F_y = [a - e_n, b + e_n] \subseteq F(L, w)$. Therefore

$$F(M, u) = \{a\} \quad \text{and} \quad F(U, u) = \{b\} \quad \text{with} \quad u = \frac{\mathrm{P}_{e_n^\perp} w}{\|\mathrm{P}_{e_n^\perp} w\|},$$

where we used that U and M are strictly convex. By the definition of M, that u is a normal vector to M at the boundary point a implies that

$$u = \frac{\Gamma(a)}{\|\Gamma(a)\|} \quad \text{with} \quad \Gamma(a) = \begin{bmatrix} a_1 - (a_{n-1} - tf(a))\, t\partial_1 f(a) \\ \vdots \\ a_{n-2} - (a_{n-1} - tf(a))\, t\partial_{n-2} f(a) \\ a_{n-1} - tf(a) \end{bmatrix}.$$

On the other hand, clearly $b = u$.

Let E be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that contains $[e_1 - e_n, e_1 + e_n]$. Then there exists a vector $(v_2, \dots, v_{n-1}) \neq 0$ so that

$$E = \{x \in \mathbb{R}^n : v_2 x_2 + \cdots + v_{n-1} x_{n-1} = 0\}.$$

That $F_y \subset E$ requires that $a, b \in E$, which yields the system of equations

$$a_1^2 + \cdots + a_{n-2}^2 + (a_{n-1} - tf(a))^2 = 1, \tag{6.1}$$

$$v_2 a_2 + \cdots + v_{n-1} a_{n-1} = 0, \tag{6.2}$$

$$(v_2 \partial_2 f(a) + \cdots + v_{n-2} \partial_{n-2} f(a))(a_{n-1} - tf(a)) + v_{n-1} f(a) = 0. \tag{6.3}$$

It is natural to expect that the set S of solutions $a \in \mathbb{R}^{n-1}$ of this system of 3 equations has Hausdorff dimension at most $n - 4$. We choose the function f in the definition of L so that this is indeed the case for *every* choice of the vector v. An explicit construction of such a function may be found in Appendix A, but its precise form is irrelevant for what follows.

To complete the proof, note that we have shown that every F_y has the form $\left[a - e_n, \frac{\Gamma(a)}{\|\Gamma(a)\|} + e_n\right]$ for $a \in S$. As the map $a \mapsto \frac{\Gamma(a)}{\|\Gamma(a)\|}$ is locally Lipschitz,

$$\bigcup_{y \in O} F_y = \left\{ \lambda(a - e_n) + (1 - \lambda)\left(\frac{\Gamma(a)}{\|\Gamma(a)\|} + e_n\right) : a \in S,\ \lambda \in [0, 1] \right\}$$

is the image by a locally Lipschitz function of a set $S \times [0, 1]$ of Hausdorff dimension at most $n-3$, and thus has dimension at most $n-3$ itself by [10, Theorem 7.5]. □

We can now prove part b of Theorem 1.21 in the case $\dim T(K, u) = 2$.

Corollary 6.6. *There exists $K \in \mathcal{K}^n_{(o)}$ and $u \in S^{n-1}$ with $\dim T(K, u) = 2$ so that $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} K[n-2]}$ for every $v \in T(K, u)^\perp$, but $u \notin \operatorname{supp} \mathrm{S}_{K[n-1]}$.*

Proof. Let $K = L^\circ$, where L is the convex body provided by Lemma 6.5 and let $u = e_1$. Then part *a* of Lemma 6.5 and the duality between faces and touching cones (see §2.2.3) yields $T(K,u) = \mathbb{R}_+[e_1 - e_n, e_1 + e_n]$, and thus $\dim T(K,u) = 2$. Moreover, by the same argument, part *b* of Lemma 6.5 shows that there exists $\varepsilon > 0$ sufficiently small so that $\dim T(K,u') = 2$ for all $u' \in B(u,\varepsilon)$. Therefore, $u \notin \operatorname{supp} \mathrm{S}_{K[n-1]}$ follows directly from Corollary 4.5.

Finally, by the same duality argument, part *c* of Lemma 6.5 shows that for every $v \in T(K,u)^\perp$, the union of all the touching cones $T(K,u') \subset v^\perp$ with $u' \in B(u,\varepsilon)$ has Hausdorff dimension at most $n-2$. Therefore,

$$\dim T(\mathrm{P}_{v^\perp} K, u') = \dim(T(K,u') \cap v^\perp) = 1$$

for a dense subset of $u' \in B(u,\varepsilon) \cap v^\perp$, where we used Lemma 2.4. In particular,

$$u \in \operatorname{cl}\{u' \in S^{n-1} \cap v^\perp : \dim T(\mathrm{P}_{v^\perp} K, u') = 1\},$$

and thus $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} K[n-2]}$ by Corollary 4.5. □

We now finally extend the argument to the general case.

Proof of part b of Theorem 1.21. Let $N \geq 4$ and $1 < k < N-1$. We aim to construct bodies $K, C_1, \ldots, C_{N-2} \in \mathcal{K}^N$ and $u \in S^{N-1}$ with $\dim T(K,u) = k$ so that $u \in \operatorname{supp} \mathrm{S}_{\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{N-2}}$ for every $v \in T(K,u)^\perp$, but $u \notin \operatorname{supp} \mathrm{S}_{K, C_1, \ldots, C_{N-2}}$.

To this end, let $n = N - k + 2$, and construct $\bar{K} \in \mathcal{K}^n$ and $\bar{u} \in S^{n-1}$ as in Corollary 6.6. Let K, u be their natural embedding in $\operatorname{span}\{e_1, \ldots, e_n\} \subseteq \mathbb{R}^N$, and

$$C_1 = \cdots = C_{n-2} = K, \qquad C_i = [0, e_{i+2}] \quad \text{for} \quad i = n-1, \ldots, N-2.$$

Note that, by construction,

$$T(K,u) = T(\bar{K}, \bar{u}) \times \mathbb{R}^{N-n}.$$

Thus $\dim T(K,u) = k$. On the other hand, by the projection formula (2.3), the mixed area measures $\mathrm{S}_{K, C_1, \ldots, C_{N-2}}$ and $\mathrm{S}_{\mathrm{P}_{v^\perp} C_1, \ldots, \mathrm{P}_{v^\perp} C_{N-2}}$ reduce to the n-dimensional case in Corollary 6.6, concluding the proof. □

Remark 6.7. The bodies $K, C_1, \ldots, C_{N-2} \in \mathcal{K}^N$ in the above proof have empty interior. However, the construction can be modified as in the proof of Lemma 6.3 to obtain $K, C_1, \ldots, C_{N-2} \in \mathcal{K}^N_N$ that yield the same conclusion.

6.3. **On the continuity of extreme directions.** The construction of the previous section illustrates that the support of mixed area measures can be poorly behaved under projection. One is therefore led to seek other approaches to Conjecture 1.6 that are based on a direct analysis of convex bodies in $\mathbb{R}^n$. The aim of this section is to record an elementary example that illustrates a basic challenge that arises even in the most well-behaved situations.

A tantalizing setting for understanding Conjecture 1.6 is the special case that $h_{C_1}, \ldots, h_{C_{n-1}}$ are smooth, since in this case there is an analytic description of the support that was explained in Remark 1.12. The problem remains open even in this setting. This is particularly surprising since the methods of Hartman and Nirenberg [6] provide powerful information. In particular, [6, §3, Lemma 2] implies that for every convex body K with $h_K \in C^2(S^{n-1})$, there is a dense open set of $u \in S^{n-1}$

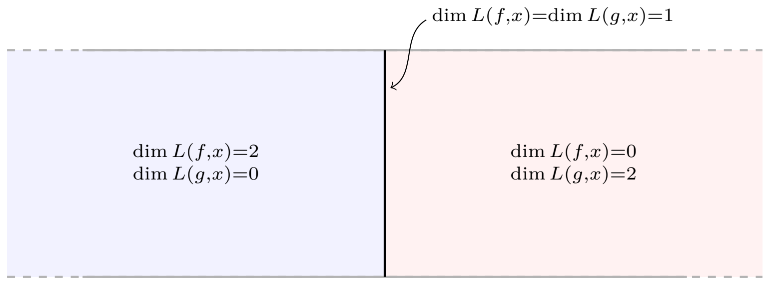

FIGURE 6.1. Illustration of Example 6.8.

for which $\mathrm{rank}(D^2h_K(u)) = \dim(T(K,u)^\perp)$. Thus the analytic characterization of Remark 1.12 agrees with Conjecture 1.6 at almost all points.

In the setting of Hartman and Nirenberg (e.g., for the proof of Theorem 1.17), the above property suffices to obtain the desired conclusion by a continuity argument. A basic problem in the mixed setting, however, is that even when all $v \in B(u,\varepsilon)$ are not $(C_1,\ldots,C_{n-1})$-extreme, the set I that witnesses the failure of extremality (cf. Definition 1.5) can vary with v in a discontinuous fashion.

Example 6.8. Since it leads to simpler expressions, we formulate the example in terms of mixed Hessian measures; the example can be translated to an analogous example of mixed area measures using the correspondence in §5.

Let $\Omega = \mathbb{R}\times(-1,1) \subset \mathbb{R}^2$. The function $h \in \mathrm{Conv}(\Omega)$ defined by

$$h(x_1,x_2) = \frac{x_1^2}{1-x_2^2}$$

is strictly convex whenever $x_1 \neq 0$. We can now define $f,g \in \mathrm{Conv}(\Omega)$ by setting $f(x_1,x_2) = h(x_1,x_2)1_{x_1>0}$ and $g(x_1,x_2) = h(x_1,x_2)1_{x_1<0}$. The structure of the sets $L(f,x)$ and $L(g,x)$ in different regions of Ω is illustrated in Figure 6.1.

We readily see that every $x \in \Omega$ is not (f,g)-extreme, and thus $\mathrm{H}_{f,g} = 0$ by Corollary 1.16. However, the regions $\{x_1 < 0\}$, $\{x_1 = 0\}$, and $\{x_1 > 0\}$ all require a different set I to witness non-extremality (i.e., so that $\dim(\bar{L}(f_I,x)^\perp) < |I|$).

The discontinuous behavior that is illustrated by this example provides a basic obstacle to various approaches for the analysis of mixed area measures.

APPENDIX A. AN EXPLICIT CONSTRUCTION FOR LEMMA 6.5

The aim of this appendix is to construct a function f with the properties required in the proof of Lemma 6.5. Any sufficiently generic choice of f is expected to achieve the same conclusion; we exhibit one explicit example for concreteness.

Lemma A.1. *Define the function*

$$f(x_1,\ldots,x_{n-2}) = (x_1-1)\exp\left(\sum_{j=2}^{n-2}(x_j^3 + jx_j)\right).$$

Then for any $(v_2,\ldots,v_{n-1}) \neq 0$ and $t > 0$, the solution set S of the system of equations (6.1)–(6.3) has Hausdorff dimension at most $n-4$.

Proof. It is clear that the only solution of (6.1) with $a_1 = 1$ is $a = e_1$. We therefore consider in the sequel only solutions with $a_1 \neq 1$. Then (6.3) can be simplified to

$$\left(\sum_{j=2}^{n-2} v_j(3a_j^2 + j)\right)(a_{n-1} - tf(a)) = -v_{n-1}. \tag{A.1}$$

We must now consider two distinct cases.

Case 1. If $v_{n-1} \neq 0$, then (6.2) yields

$$a_{n-1} = -\sum_{j=2}^{n-2} \frac{v_j a_j}{v_{n-1}}.$$

Moreover, as $v_{n-1} \neq 0$, both factors on the left-hand side of (A.1) are nonzero. Substituting the definition of f and the previous equation in (A.1), we obtain

$$a_1 = g(a_2, \ldots, a_{n-2}) = 1 + \frac{1}{t}\left(\frac{v_{n-1}}{\sum_{j=2}^{n-2} v_j(3a_j^2 + j)} - \sum_{j=2}^{n-2} \frac{v_j a_j}{v_{n-1}}\right) \exp\left(-\sum_{j=2}^{n-2}(a_j^3 + ja_j)\right).$$

Finally, combining (6.1), (A.1), and the previous equation yields

$$h(a_2, \ldots, a_{n-2}) = 1 - g(a_2, \ldots, a_{n-2})^2 - a_2^2 - \cdots - a_{n-2}^2 - \frac{v_{n-1}^2}{\left(\sum_{j=2}^{n-2} v_j(3a_j^2 + j)\right)^2} = 0.$$

Thus we have shown that

$$S\backslash\{e_1\} \subseteq \bar{S} = \Bigg\{a \in \mathbb{R}^{n-1} : \sum_{j=2}^{n-2} v_j(3a_j^2 + j) \neq 0,\ h(a_2, \ldots, a_{n-2}) = 0,$$
$$a_1 = g(a_2, \ldots, a_{n-2}),\ a_{n-1} = -\sum_{j=2}^{n-2} \frac{v_j a_j}{v_{n-1}}\Bigg\}.$$

Now note that h is a function of $n-3$ variables that is real analytic on $\operatorname{dom} h = \{(a_2, \ldots, a_{n-2}) : \sum_{j=2}^{n-2} v_j(3a_j^2 + j) \neq 0\}$, and thus the subset of $\operatorname{dom} h$ on which h vanishes has Hausdorff dimension at most $n-4$ [13]. Since a_1 and a_{n-1} are functions of $a_2, \ldots, a_{n-2}$ that are locally Lipschitz on $\operatorname{dom} h$, it follows that the set $\bar{S}$ and thus also S has Hausdorff dimension at most $n-4$ [10, Theorem 7.5].

Case 2. Now suppose that $v_{n-1} = 0$. Then (A.1) yields $S\backslash\{e_1\} = S_1 \cup S_2$ with

$$S_1 = \{a \in S\backslash\{e_1\} : a_{n-1} = tf(a_1, \ldots, a_{n-2})\},$$
$$S_2 = \left\{a \in S\backslash\{e_1\} : \sum_{j=2}^{n-2} v_j(3a_j^2 + j) = 0\right\}.$$

We consider S_1 and $S_2\backslash S_1$ separately.

To control S_1, note that (6.1) and (6.2) imply that

$$S_1 \subseteq \left\{a \in \mathbb{R}^{n-1} : \sum_{j=1}^{n-2} a_j^2 = 1,\ \sum_{j=2}^{n-2} v_j a_j = 0,\ a_{n-1} = tf(a_1, \ldots, a_{n-2})\right\}.$$

In other words, $(a_1,\ldots,a_{n-2})$ lie in the intersection of a sphere with a hyperplane, which has dimension $n-4$. As a_{n-1} is a locally Lipschitz function of $(a_1,\ldots,a_{n-2})$, also S_1 has Hausdorff dimension at most $n-4$ [10, Theorem 7.5].

To control S_2, let $2\le\ell\le n-2$ so that $v_\ell\neq 0$. As $\sum_{j=2}^{n-2} v_j(3a_j^2+j)=0$ by the definition of S_2 and as $\sum_{j=2}^{n-2} v_ja_j=0$ by (6.2), we obtain

$$q(a_2,\ldots,a_{\ell-1},a_{\ell+1},\ldots,a_{n-2}) = \\ 3\sum_{j=2}^{\ell-1} v_ja_j^2+3\sum_{j=\ell+1}^{n-2} v_ja_j^2+3v_\ell\left(\sum_{j=2}^{\ell-1}\frac{v_ja_j}{v_\ell}+\sum_{j=\ell+1}^{n-2}\frac{v_ja_j}{v_\ell}\right)^2+\sum_{j=2}^{n-2} jv_j=0.$$

Note that q is a quadratic function of $n-4$ variables. Thus its zero set can have Hausdorff dimension $n-4$ only if q vanishes identically. In that case we must have $\sum_{j=2}^{n-2} jv_j=0$ and either $v_2=\cdots=v_{\ell-1}=v_{\ell+1}=\cdots=v_{n-2}=0$, or $v_r=-v_\ell$ for some $2\le r\le n-2$, $r\neq\ell$ and $v_j=0$ for $2\le j\le n-2$, $j\neq\ell,r$. Clearly no such v exists, so the zero set of q has Hausdorff dimension at most $n-5$.

Now note that by (6.2), we can write

$$a_\ell=-\sum_{j=2}^{\ell-1}\frac{v_ja_j}{v_\ell}-\sum_{j=\ell+1}^{n-2}\frac{v_ja_j}{v_\ell},$$

while (6.1) implies that

$$a_{n-1}=tf(a_1,\ldots,a_{n-2})\pm\sqrt{1-\sum_{j=1}^{n-2}a_j^2}.$$

Note in particular that the latter equation implies that $\sum_{j=1}^{n-2} a_j^2<1$ on the complement of S_1. Thus $S_2\backslash S_1\subseteq S_+\cup S_-$ with

$$S_\pm=\Bigg\{a\in\mathbb{R}^{n-1}:\sum_{j=1}^{n-2}a_j^2<1,\ q(a_2,\ldots,a_{\ell-1},a_{\ell+1},\ldots,a_{n-2})=0, \\ a_\ell=-\sum_{j=2}^{\ell-1}\frac{v_ja_j}{v_\ell}-\sum_{j=\ell+1}^{n-2}\frac{v_ja_j}{v_\ell},\ a_{n-1}=tf(a_1,\ldots,a_{n-2})\pm\sqrt{1-\sum_{j=1}^{n-2}a_j^2}\Bigg\}.$$

Since the zero set of q has Hausdorff dimension at most $n-5$ and as a_ℓ and a_{n-1} are functions of $a_1,\ldots,a_{\ell-1},a_{\ell+1},\ldots,a_{n-2}$ that are locally Lipschitz for $\sum_{j=1}^{n-2}a_j^2<1$, the sets $S_\pm$ have Hausdorff dimension at most $n-4$ [10, Theorem 7.5]. □

Acknowledgments. The authors were supported in part by NSF grants DMS-2054565 and DMS-2347954. This paper was completed while R.v.H. was a member of the Institute for Advanced Study in Princeton, NJ, which is gratefully acknowledged for providing a fantastic mathematical environment.

References

[1] R. D. Anderson and V. L. Klee, Jr. Convex functions and upper semi-continuous collections. *Duke Math. J.*, 19:349–357, 1952.

[2] T. Bonnesen and W. Fenchel. *Theory of convex bodies.* BCS Associates, Moscow, ID, 1987.

[3] A. Colesanti and D. Hug. Hessian measures of convex functions and applications to area measures. *J. London Math. Soc. (2)*, 71(1):221–235, 2005.
[4] J. Favard. Sur les corps convexes. *J. Math. Pures Appl. (9)*, 12:219–282, 1933.
[5] C. E. Gutiérrez. *The Monge-Ampère equation*, volume 89 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, [Cham], second edition, 2016.
[6] P. Hartman and L. Nirenberg. On spherical image maps whose Jacobians do not change sign. *Amer. J. Math.*, 81:901–920, 1959.
[7] D. Hug, F. Mussnig, and J. Ulivelli. Kubota-type formulas and supports of mixed measures, 2024. Preprint arxiv:2401.16371.
[8] D. Hug and P. A. Reichert. The support of mixed area measures involving a new class of convex bodies. *J. Funct. Anal.*, 287(11):Paper No. 110622, 43, 2024.
[9] M. Kline. *Mathematical thought from ancient to modern times. Vol. 2*. The Clarendon Press, Oxford University Press, New York, second edition, 1990.
[10] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.
[11] H. Minkowski. Volumen und Oberfläche. *Math. Ann.*, 57(4):447–495, 1903.
[12] H. Minkowski. Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs. In D. Hilbert, A. Speiser, and H. Weyl, editors, *Gesammelte Abhandlungen von Hermann Minkowski. Zweiter Band*, pages 131–229. B. G. Teubner, 1911.
[13] B. S. Mityagin. The zero set of a real analytic function. *Mat. Zametki*, 107(3):473–475, 2020.
[14] J. R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, second edition, 2000.
[15] A. A. Panov. Some properties of mixed discriminants. *Mat. Sb. (N.S.)*, 128(170)(3):291–305, 446, 1985.
[16] A. V. Pogorelov. *Extrinsic geometry of convex surfaces*, volume Vol. 35 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1973. Translated from the Russian by Israel Program for Scientific Translations.
[17] R. T. Rockafellar. *Convex analysis*, volume No. 28 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1970.
[18] R. Schneider. Kinematische Berührmaße für konvexe Körper und Integralrelationen für Oberflächenmaße. *Math. Ann.*, 218(3):253–267, 1975.
[19] R. Schneider. On the Aleksandrov-Fenchel inequality. In *Discrete geometry and convexity*, volume 440 of *Ann. New York Acad. Sci.*, pages 132–141. New York Acad. Sci., 1985.
[20] R. Schneider. On the Aleksandrov-Fenchel inequality involving zonoids. *Geom. Dedicata*, 27(1):113–126, 1988.
[21] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*. Cambridge University Press, expanded edition, 2014.
[22] Y. Shenfeld and R. van Handel. Mixed volumes and the Bochner method. *Proc. Amer. Math. Soc.*, 147(12):5385–5402, 2019.
[23] Y. Shenfeld and R. van Handel. The extremals of the Alexandrov-Fenchel inequality for convex polytopes. *Acta Math.*, 231(1):89–204, 2023.
[24] N. S. Trudinger and X.-J. Wang. Hessian measures. III. *J. Funct. Anal.*, 193(1):1–23, 2002.
[25] S. Weis. A note on touching cones and faces. *J. Convex Anal.*, 19(2):323–353, 2012.

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
Email address: rvan@math.princeton.edu

PACM, Princeton University, Princeton, NJ 08544, USA
Email address: shoudawang@princeton.edu