

SHEPHARD'S INEQUALITIES, HODGE-RIEMANN RELATIONS, AND A CONJECTURE OF FEDOTOV

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ABSTRACT. A well-known family of determinantal inequalities for mixed volumes of convex bodies were derived by Shephard from the Alexandrov-Fenchel inequality. The classic monograph *Geometric Inequalities* by Burago and Zalgaller states a conjecture on the validity of higher-order analogues of Shephard's inequalities, which is attributed to Fedotov. In this note we disprove Fedotov's conjecture by showing that it contradicts the Hodge-Riemann relations for simple convex polytopes. Along the way, we make some expository remarks on the linear algebraic and geometric aspects of these inequalities.

1. INTRODUCTION

1.1. Let K_1, \dots, K_m be convex bodies in \mathbb{R}^n and $\lambda_1, \dots, \lambda_m > 0$. One of the most basic facts of convex geometry, due to H. Minkowski, is that the volume of convex bodies is a homogeneous polynomial in the sense that

$$\text{Vol}(\lambda_1 K_1 + \dots + \lambda_m K_m) = \sum_{i_1, \dots, i_m=1}^m \mathbf{V}(K_{i_1}, \dots, K_{i_m}) \lambda_{i_1} \cdots \lambda_{i_m}.$$

The coefficients $\mathbf{V}(K_1, \dots, K_n)$, called mixed volumes, define a large family of natural geometric parameters of convex bodies, and play a central role in convex geometry [5, 14]. Mixed volumes are always nonnegative, are symmetric in their arguments, and are additive and homogeneous in each argument.

The fundamental inequality in the theory of mixed volumes is the following.

Theorem 1.1 (Alexandrov-Fenchel). *For convex bodies $K, L, C_1, \dots, C_{n-2}$ in \mathbb{R}^n*

$$\mathbf{V}(K, L, C_1, \dots, C_{n-2})^2 \geq \mathbf{V}(K, K, C_1, \dots, C_{n-2}) \mathbf{V}(L, L, C_1, \dots, C_{n-2}).$$

Numerous inequalities in convex geometry may be derived from the Alexandrov-Fenchel inequality, cf. [5, §20] and [14, §7.4]. The starting point for this note is a well-known family of determinantal inequalities, due to Shephard [18], that extend the Alexandrov-Fenchel inequality to more than n bodies.

Theorem 1.2 (Shephard). *Given convex bodies $K_1, \dots, K_m, C_1, \dots, C_{n-2}$ in \mathbb{R}^n , define the $m \times m$ symmetric matrix M by setting*

$$M_{ij} := \mathbf{V}(K_i, K_j, C_1, \dots, C_{n-2}).$$

Then

$$(-1)^m \det M \leq 0.$$

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The special case $m = 2$ of Theorem 1.2 is just a reformulation of the Alexandrov-Fenchel inequality, and Shephard's inequalities may thus be viewed as a considerable generalization of the Alexandrov-Fenchel inequality. However, as is shown by Shephard (and as we will explain later in this note), the general inequalities may in fact be deduced from the $m = 2$ case by a simple linear algebraic argument. In the case $m = 3$, this result dates back already to Minkowski [12, p. 478].

1.2. The classic monograph *Geometric Inequalities* by Burago and Zalgaller states a conjecture on the validity of a higher-order generalization of Theorem 1.2, which is attributed to Fedotov [5, §20.6]. Let us recall the statement of this conjecture. In the sequel, we will frequently employ the notation

$$\mathbb{V}(K_1[m_1], K_2[m_2], \dots, K_r[m_r]) := \mathbb{V}(\underbrace{K_1, \dots, K_1}_{m_1}, \underbrace{K_2, \dots, K_2}_{m_2}, \dots, \underbrace{K_r, \dots, K_r}_{m_r})$$

when convex bodies are repeated multiple times in the arguments of a mixed volume.

Conjecture 1.3 (Fedotov). *Let $k \leq n/2$, and let $K_1, \dots, K_m, C_1, \dots, C_{n-2k}$ be convex bodies in \mathbb{R}^n . Define the $m \times m$ symmetric matrix M by setting*

$$M_{ij} := \mathbb{V}(K_i[k], K_j[k], C_1, \dots, C_{n-2k}).$$

Then

$$(-1)^m \det M \leq 0.$$

If true, this conjecture would entail a considerable generalization of Shephard's inequalities. The conjecture is rather appealing, as it is easily verified to be true in two extreme cases that have a different flavor.

Lemma 1.4. *Conjecture 1.3 is valid in the following two cases:*

- a. *When $k = 1$ and m is arbitrary.*
- b. *When $m = 2$ and k is arbitrary.*

Proof. Case a is nothing other than Theorem 1.2. To prove b, it suffices to note that iterating the Alexandrov-Fenchel inequality yields [14, (7.63)]

$$\begin{aligned} \mathbb{V}(K_1[k], K_2[l], C_1, \dots, C_{n-k-l})^{k+l} \geq \\ \mathbb{V}(K_1[k+l], C_1, \dots, C_{n-k-l})^k \mathbb{V}(K_2[k+l], C_1, \dots, C_{n-k-l})^l \end{aligned}$$

for any $k, l \geq 1$, $k + l \leq n$. The case $k = l$ is readily seen to be equivalent to b. \square

The main purpose of this note is to explain that Conjecture 1.3 fails when one goes beyond the special cases of Lemma 1.4. More precisely, we will prove:

Theorem 1.5. *For every $k > 1$, Conjecture 1.3 is false for some $m > 2$.*

1.3. In order to explain how we will disprove Conjecture 1.3, it is useful to first briefly recall some of its history.

Despite the fundamental nature of the Alexandrov-Fenchel inequality, no really elementary proof of it is known. Alexandrov gave two different (but closely related) proofs in the 1930s: a combinatorial proof using strongly isomorphic polytopes [2], and an analytic proof using elliptic operators [3]. Further remarks on its history and on more modern proofs may be found in [14, 15].

In the 1970s, unexpected connections were discovered between the theory of mixed volumes and algebraic geometry. In particular, a remarkable identity due to

Bernstein and Kushnirenko [5, Theorem 27.1.2] shows that the number of solutions $z \in (\mathbb{C} \setminus \{0\})^n$ of a generic system of polynomial equations $p_1(z) = 0, \dots, p_n(z) = 0$ with given monomials coincides with the mixed volume of an associated family of lattice polytopes in \mathbb{R}^n (i.e., polytopes with vertices in \mathbb{Z}^n).

Motivated by these developments, Fedotov [7] proposed a simple proof of the Alexandrov-Fenchel inequality using only basic properties of polynomials. Fedotov further notes that his method even yields the more general Conjecture 1.3, which is stated in [7] as a theorem. These results were included in the Russian edition of the monograph of Burago and Zalgaller. Unfortunately, Fedotov's elementary approach turns out to contain a serious flaw, which renders his method of proof invalid. A correct algebraic proof of the Alexandrov-Fenchel inequality was given by Teissier and Khovanskii using nontrivial machinery, namely a reduction to the Hodge index theorem of algebraic geometry. The latter proof is included in the English translation of Burago-Zalgaller [5, §27], but does not settle the validity of Fedotov's higher-order analogue of Shephard's inequalities [5, §20.6].

On the other hand, the algebraic connection yields other higher-order inequalities. The Alexandrov-Fenchel inequality is analogous to a Hodge-Riemann relation of degree 1 in the cohomology ring of a smooth projective variety [6, 8]. Hodge-Riemann relations of higher degree give rise to new inequalities in convex geometry. Such inequalities were first stated by McMullen [11] for strongly isomorphic simple polytopes as a byproduct of his work on the g -conjecture. Their geometric significance was greatly clarified by Timorin [20], whose formulation is readily interpreted in terms of explicit inequalities for mixed volumes. Very recently, some special cases were extended also to smooth convex bodies in [9, 1, 10].

The proof of Theorem 1.5 may now be explained as follows. Using the properties of hyperbolic quadratic forms, we will first reformulate Conjecture 1.3 as a higher-order Alexandrov-Fenchel inequality. In this equivalent formulation, it will be evident that this inequality contradicts the Hodge-Riemann relation of degree 2. Thus the results of McMullen and Timorin imply that Conjecture 1.3 is false. Beside disproving the conjecture, a more expository aim of this note is to draw attention to some basic linear algebraic and geometric aspects of the above inequalities (none of which are really new here) in the context of classical convexity.

Remark. It should be noted that Fedotov's conjecture as stated in [5, §20.6] is somewhat more general than Conjecture 1.3: the matrix M considered there is

$$M_{ij} := V(K_i[k], K_j[l], C_1, \dots, C_{n-k-l})$$

for any $k, l \geq 1$ such that $k + l \leq n$. Lemma 1.4 extends to this setting: the case $k = l = 1$ and general m reduces to Shephard's inequalities, while the case $m = 2$ and general k, l is obtained by multiplying the inequality [14, (7.63)] used in the proof of Lemma 1.4 by the same inequality with the roles of k, l reversed. When $k \neq l$, however, the matrix M is not symmetric, and the spectral interpretation of the conjecture becomes unclear. Given that we show the conjecture fails for general m already in the symmetric case $k = l$, it seems implausible that the nonsymmetric case $k \neq l$ has any merit, and we do not consider it further in this note.

1.4. The remainder of this note is organized as follows. In section 2, we recall some basic properties of hyperbolic quadratic forms that will be used in the sequel. We also briefly discuss Shephard's inequalities and clarify their equality cases. In section 3 we formulate the Hodge-Riemann relations for strongly isomorphic simple

polytopes, due to McMullen and Timorin, entirely in the language of classical convexity. Finally, section 4 completes the proof of Theorem 1.5.

While the proof of Theorem 1.5 explains clearly *why* Fedotov's conjecture must fail, the construction is rather indirect. Once the proof has been understood, however, it is not difficult to engineer an explicit counterexample, which will be done in section 5. Beside further illustrating the basic construction, this example will show that we may in fact choose $m = 3$ in Theorem 1.5.

We conclude this note by highlighting a puzzling aspect of the Hodge-Riemann relations: even though their statement makes sense in principle for arbitrary convex bodies, the Hodge-Riemann relations have only been proved for special classes of bodies (e.g., strongly isomorphic simple polytopes). In section 6, we will illustrate by means of a simple example that the Hodge-Riemann relations may fail for general convex bodies. This highlights the rather unusual nature of the Hodge-Riemann relations as compared to other inequalities in convex geometry.

2. LINEAR ALGEBRA

The aim of this section is to explain that the connection between the Alexandrov-Fenchel and Shephard inequalities has nothing to do with convexity, but is rather a simple linear-algebraic fact. The results of this section are known in various forms, see, e.g., [4, Theorem 4.4.6], [15, Lemma 2.9], or [16, Lemma 3.1], but we provide simple self-contained proofs for the variants needed here.

2.1. Hyperbolic matrices. We begin by giving a spectral interpretation of the Alexandrov-Fenchel inequality. In the sequel, a matrix M will be called positive if $M_{ij} > 0$ for all i, j . For $y \in \mathbb{R}^m$, we write $y \geq 0$ ($y > 0$) if $y_i \geq 0$ ($y_i > 0$) for all i . The linear span of all eigenvectors of a symmetric matrix M with positive eigenvalues will be called the positive eigenspace of M .

Lemma 2.1. *Let M be a symmetric positive matrix. The following are equivalent:*

1. *The positive eigenspace of M is one-dimensional.*
2. *$\langle x, My \rangle^2 \geq \langle x, Mx \rangle \langle y, My \rangle$ for all $x \geq 0$ and $y \geq 0$.*
3. *$\langle x, My \rangle = 0$ implies $\langle x, Mx \rangle \leq 0$ for all x and $y \geq 0$, $y \neq 0$.*

Proof. As M is a positive matrix, the Perron-Frobenius theorem implies that it has at least one eigenvector $v > 0$ with positive eigenvalue.

$3 \Rightarrow 1$: Let $x \perp v$ be any other eigenvector of M . Then $\langle x, Mv \rangle = 0$, so 3 implies $\langle x, Mx \rangle \leq 0$. Thus the eigenvalue associated to x must be nonpositive.

$1 \Rightarrow 2$: It follows from 1 that M is negative semidefinite on v^\perp . Fix $x, y \geq 0$; we may assume $y \neq 0$ (else the inequality is trivial), so that $\langle y, v \rangle > 0$ and $\langle y, My \rangle > 0$. If we define $z = x - ay$ with $a = \langle x, v \rangle / \langle y, v \rangle$, then $z \in v^\perp$, so

$$0 \geq \langle z, Mz \rangle = \langle x, Mx \rangle - 2a \langle x, My \rangle + a^2 \langle y, My \rangle \geq \langle x, Mx \rangle - \frac{\langle x, My \rangle^2}{\langle y, My \rangle}.$$

$2 \Rightarrow 3$: We first show that 2 remains valid for any x (not just $x \geq 0$). Suppose first that $y > 0$. Then $x + by \geq 0$ when b is chosen sufficiently large, so 2 implies

$$\langle x + by, My \rangle^2 \geq \langle x + by, M(x + by) \rangle \langle y, My \rangle.$$

Expanding both sides of this inequality shows that all terms involving b cancel, so $\langle x, My \rangle^2 \geq \langle x, Mx \rangle \langle y, My \rangle$ for any x and $y > 0$. This conclusion remains valid for

any $y \geq 0$ by applying the above argument with $y \leftarrow y + \varepsilon v$ and letting $\varepsilon \rightarrow 0$. Now $\mathfrak{3}$ follows immediately once we note that $y \geq 0$, $y \neq 0$ implies $\langle y, My \rangle > 0$. \square

In the sequel, a symmetric (but not necessarily positive) matrix that has a one-dimensional positive eigenspace will be called *hyperbolic*.

2.2. Shephard's inequalities. An $m \times m$ hyperbolic matrix M has 1 positive and $m-1$ nonpositive eigenvalues. It is therefore immediately obvious that such a matrix satisfies $(-1)^m \det M \leq 0$ (as the determinant is the product of the eigenvalues). Shephard's inequalities follow directly from this observation.

Proof of Theorem 1.2. We may assume without loss of generality that all the convex bodies have nonempty interior, so that M is a positive matrix (otherwise we may replace $K_i \leftarrow K_i + \varepsilon B$, $C_i \leftarrow C_i + \varepsilon B$ for any body B with nonempty interior, and take $\varepsilon \rightarrow 0$ in the final inequality.) Condition 2 of Lemma 2.1 is immediate from the Alexandrov-Fenchel inequality (Theorem 1.1 with $K = \sum_i x_i K_i$ and $L = \sum_i y_i K_i$). Thus M is hyperbolic by Lemma 2.1, which implies $(-1)^m \det M \leq 0$. \square

While this is only tangentially related to the rest of this note, let us take the opportunity to clarify the cases of equality in Shephard's inequalities.

Proposition 2.2. *In the setting and notations of Theorem 1.2, we have $\det M = 0$ if and only if there are linearly independent vectors $x, y > 0$ such that $K = \sum_i x_i K_i$, $L = \sum_i y_i K_i$ yield equality in the Alexandrov-Fenchel inequality of Theorem 1.1.*

Proof. We must show $\det M = 0$ if and only if $\langle x, My \rangle^2 = \langle x, Mx \rangle \langle y, My \rangle$ for some linearly independent $x, y > 0$. We may assume $M \neq 0$ (else the result is trivial).

Suppose first that $\det M = 0$. Then there exists $z \in \ker M$, $z \neq 0$. Choose any $y > 0$ that is linearly independent of z . Evidently $\langle z, My \rangle^2 = \langle z, Mz \rangle \langle y, My \rangle$. But as this identity is invariant under the replacement $z \leftarrow z + by$ (as in the proof of $\mathfrak{2} \Rightarrow \mathfrak{3}$ of Lemma 2.1), we may choose $x = z + by > 0$ for b sufficiently large.

Now suppose $\langle x, My \rangle^2 = \langle x, Mx \rangle \langle y, My \rangle$ for linearly independent $x, y > 0$. Then

$$q(v) := \langle x + v, My \rangle^2 - \langle x + v, M(x + v) \rangle \langle y, My \rangle$$

satisfies $q(0) = 0$, and $q(v) \geq 0$ for all v in a neighborhood of 0 by the Alexandrov-Fenchel inequality. Thus $\nabla q(0) = 0$, which yields $z = \langle y, My \rangle x - \langle x, My \rangle y \in \ker M$. Moreover, $z \neq 0$ as x, y are linearly independent. Thus $\det M = 0$. \square

Proposition 2.2 reduces the equality cases of Shephard's inequalities to those of the Alexandrov-Fenchel inequality. The characterization of the latter is a long-standing open problem [14, §7.6], which was recently settled in several important cases in [17, 16]. This problem remains open in full generality.

2.3. A Sylvester criterion. While any hyperbolic $m \times m$ matrix M trivially satisfies $(-1)^m \det M \leq 0$, the converse implication clearly does not hold: the sign of the determinant does not determine the number of positive eigenvalues. However, the implication can be reversed if the determinant condition holds for all principal submatrices of M . This hyperbolic analogue of the classical Sylvester criterion may be proved in essentially the same manner.¹

In the following, we denote for any $m \times m$ symmetric matrix M and subset $I \subseteq [m]$ by $M_I := (M_{ij})_{i,j \in I}$ the associated principal submatrix.

¹The author learned the elementary approach used here from lecture notes of M. Hladik.

Lemma 2.3. *For a symmetric positive $m \times m$ matrix, the following are equivalent:*

1. *The positive eigenspace of M is one-dimensional.*
2. *$(-1)^{|I|} \det M_I \leq 0$ for all $I \subseteq [m]$.*

Proof. To prove $1 \Rightarrow 2$, note first that condition 2 of Lemma 2.1 is inherited by all its principal submatrices M_I (as one may restrict to x, y supported on I). The conclusion therefore follows immediately from Lemma 2.1.

To prove $2 \Rightarrow 1$, we argue by induction on m . For $m = 2$, it suffices to note that as M has at least one positive eigenvalue by the Perron-Frobenius theorem, $\det M \leq 0$ implies that its other eigenvalue must be nonpositive.

Now let $m > 2$ and assume the result has been proved in dimensions up to $m - 1$. Then 2 implies that M_I is hyperbolic for all $I \subsetneq [m]$. By the Perron-Frobenius theorem, M has an eigenvector v with positive eigenvalue. Now suppose 1 fails, that is, M is not hyperbolic. Then there must be another eigenvector $w \perp v$ with positive eigenvalue. As $(-1)^m \det M \leq 0$, there must then be a third eigenvector $u \perp \{v, w\}$ with nonnegative eigenvalue. Choose any $i \in [m]$ such that $u_i \neq 0$ and let $I = [m] \setminus \{i\}$. Choose $a, b \in \mathbb{R}$ so that $x := v - au$ and $y := w - bu$ satisfy $x_i = y_i = 0$. By construction, x, y are linearly independent and $\langle z, Mz \rangle > 0$ for all $z \in \text{span}\{x, y\}$, $z \neq 0$. As x, y are supported on I , this implies M_I has a positive eigenspace of dimension at least two, contradicting the induction hypothesis. \square

It follows immediately from Lemma 2.3 that Conjecture 1.3 is equivalent to the statement that the matrix M is hyperbolic. This observation will form the basis for the proof of Theorem 1.5 in section 4: we will show that hyperbolicity of M contradicts the Hodge-Riemann relations for simple convex polytopes.

3. HODGE-RIEMANN RELATIONS

The Hodge-Riemann relations in algebraic geometry give rise to higher order analogues of the Alexandrov-Fenchel inequality [11, 20]. While these inequalities are not usually stated in this form in the literature, they may be equivalently formulated as explicit inequalities between mixed volumes. The aim of this section is to draw attention to this elementary formulation of the Hodge-Riemann relations in terms of familiar objects from classical convex geometry.

Recall that a convex polytope in \mathbb{R}^n is called *simple* if it has nonempty interior and each vertex is contained in exactly n facets. In the following, let us fix an arbitrary simple polytope Λ in \mathbb{R}^n , and denote by $\mathcal{P}(\Lambda)$ the collection of polytopes that are *strongly isomorphic* to Λ : that is, $P \in \mathcal{P}(\Lambda)$ if and only if

$$\dim F(P, u) = \dim F(\Lambda, u) \quad \text{for all } u \in S^{n-1},$$

where $F(P, u)$ denotes the face of P with normal direction u . For the basic properties of simple and strongly isomorphic polytopes, the reader is referred to [14, §2.4]. For the purposes of this note, the only significance of these definitions is that they are needed for the validity of the following theorem (see section 6).

Theorem 3.1 (McMullen-Timorin). *Fix $n \geq 2$ and a simple polytope $\Lambda \in \mathbb{R}^n$, and let $m \geq 1$, $k \leq n/2$, $K_1, \dots, K_m, L, C_1, \dots, C_{n-2k} \in \mathcal{P}(\Lambda)$, and $x \in \mathbb{R}^m$. If*

$$\sum_i x_i \mathcal{V}(K_i[k], M[k-1], L, C_1, \dots, C_{n-2k}) = 0 \tag{3.1}$$

holds for every $M \in \mathcal{P}(\Lambda)$, then

$$(-1)^k \sum_{i,j} x_i x_j \mathbf{V}(K_i[k], K_j[k], C_1, \dots, C_{n-2k}) \geq 0. \quad (3.2)$$

Moreover, the statement is nontrivial in the sense that for any $n \geq 2$ and $k \leq n/2$, there is a simple polytope $\Lambda = L = C_1 = \dots = C_{n-2k}$ in \mathbb{R}^n , $m \geq 1$, $K_1, \dots, K_m \in \mathcal{P}(\Lambda)$, and $x \in \mathbb{R}^m$ so that (3.1) holds and the inequality in (3.2) is strict.

The case $k = 1$ of Theorem 3.1 is nothing other than the Alexandrov-Fenchel inequality. To see why this is so, assume without loss of generality that $L = \sum_i y_i K_i$ for some $y \geq 0$, $y \neq 0$ (otherwise let $m \leftarrow m + 1$ and $K_{m+1} \leftarrow L$), and define

$$M_{ij} = \mathbf{V}(K_i, K_j, C_1, \dots, C_{n-2}).$$

Then the statement of Theorem 3.1 for $k = 1$ may be formulated as

$$\langle x, My \rangle = 0 \quad \text{implies} \quad \langle x, Mx \rangle \leq 0$$

for any x and $y \geq 0$, $y \neq 0$. Thus by Lemma 2.1, the inequality of Theorem 3.1 in the case $k = 1$ is equivalent to the Alexandrov-Fenchel inequality for convex bodies in $\mathcal{P}(\Lambda)$. As any collection of convex bodies can be approximated by simple strongly isomorphic polytopes [14, Theorem 2.4.15], the general case of the Alexandrov-Fenchel inequality is further equivalent to this special case.

For $k > 1$, the statement of Theorem 3.1 may be viewed as an analogue of the Alexandrov-Fenchel inequality for $M_{ij} = \mathbf{V}(K_i[k], K_j[k], C_1, \dots, C_{n-2k})$. Thus the Hodge-Riemann relations are reminiscent of Conjecture 1.3, but their formulation is considerably more subtle. In section 4, we will show that the Hodge-Riemann relations in fact contradict Conjecture 1.3, disproving the latter.

The aim of the rest of this section is to convince the reader that the statement of Theorem 3.1 given here in terms of mixed volumes is equivalent to the statement of the Hodge-Riemann relations as given in [20]. The reader who is primarily interested in Theorem 1.5 may safely jump ahead to section 4.

To explain the formulation of [20], we must first introduce some additional notation. Let $u_1, \dots, u_N \in S^{n-1}$ be the normal directions of the facets of Λ . For any $P \in \mathcal{P}(\Lambda)$, we denote by $h_P \in \mathbb{R}^N$ its support vector

$$(h_P)_i := \sup_{y \in P} \langle y, u_i \rangle.$$

Then there is a homogenous polynomial $V : \mathbb{R}^N \rightarrow \mathbb{R}$ of degree n , called the *volume polynomial*, so that $\text{Vol}(P) = V(h_P)$ for every $P \in \mathcal{P}(\Lambda)$ [14, §5.2]. Moreover, as $\mathcal{P}(\Lambda)$ is closed under addition [14, §2.4], it follows immediately from the definition of mixed volumes that we have for any $P_1, \dots, P_n \in \mathcal{P}(\Lambda)$

$$\mathbf{V}(P_1, \dots, P_n) = \frac{1}{n!} D_{h_{P_1}} \cdots D_{h_{P_n}} V,$$

where D_h denotes the directional derivative in direction h . In this notation, the Hodge-Riemann relations are formulated in [20, p. 385] as follows:

Theorem 3.2. *Let $k \leq n/2$, $L, C_1, \dots, C_{n-2k} \in \mathcal{P}(\Lambda)$, and let $\alpha = \sum_{|I|=k} \alpha_I D^I$ be a homogeneous differential operator of order k with constant coefficients. If*

$$\alpha D_{h_L} D_{h_{C_1}} \cdots D_{h_{C_{n-2k}}} V = 0, \quad (3.3)$$

then

$$(-1)^k \alpha^2 D_{h_{C_1}} \cdots D_{h_{C_{n-2k}}} V \geq 0. \quad (3.4)$$

Moreover, equality is attained if and only if $\alpha V = 0$.

To write Theorem 3.2 in terms of mixed volumes, we need the following.

Lemma 3.3. *For any homogeneous differential operator $\alpha = \sum_{|I|=k} \alpha_I D^I$, there exist $m \geq 1$, $K_1, \dots, K_m \in \mathcal{P}(\Lambda)$, and $x \in \mathbb{R}^m$ so that $\alpha = \sum_i x_i (D_{h_{K_i}})^k$.*

Proof. We first recall that for any $z \in \mathbb{R}^N$, $h_\Lambda + \varepsilon z$ is the support vector of some polytope $K \in \mathcal{P}(\Lambda)$ for sufficiently small ε (as Λ is simple, cf. [14, Lemma 2.4.13]). We may therefore write $z = h_L - h_{L'}$ where $L = \varepsilon^{-1}K$ and $L' = \varepsilon^{-1}\Lambda$.

Now denote by e_1, \dots, e_N the standard coordinate basis of \mathbb{R}^N . By the above observation, we may write $e_i = h_{L_i} - h_{L'_i}$ for $L_i, L'_i \in \mathcal{P}(\Lambda)$. We can therefore write

$$\alpha = \sum_{i_1 \leq \dots \leq i_k} \alpha_{i_1, \dots, i_k} (D_{h_{L_{i_1}}} - D_{h_{L'_{i_1}}}) \cdots (D_{h_{L_{i_k}}} - D_{h_{L'_{i_k}}}).$$

By expanding the product, we may evidently express α as a linear combination of differential operators of the form $D_{h_{R_{i_1}}} \cdots D_{h_{R_{i_k}}}$ with $R_i \in \mathcal{P}(\Lambda)$. But as

$$D_{h_{R_{i_1}}} \cdots D_{h_{R_{i_k}}} = \frac{1}{k!} \sum_{\delta \in \{0,1\}^k} (-1)^{k+\delta_1+\dots+\delta_k} (D_{h_{\delta_1 R_{i_1} + \dots + \delta_k R_{i_k}}})^k$$

by the polarization formula [5, p. 137], the proof is readily concluded. \square

We are now ready to show that the Hodge-Riemann relations expressed by Theorems 3.1 and 3.2 are equivalent. First, note that $\alpha D_{h_L} D_{h_{C_1}} \cdots D_{h_{C_{n-2k}}} V$ in (3.3) is a homogeneous polynomial of degree $k-1$. Thus (3.3) is equivalent to the statement that $\beta \alpha D_{h_L} D_{h_{C_1}} \cdots D_{h_{C_{n-2k}}} V = 0$ for every homogeneous differential operator β of order $k-1$. By Lemma 3.3, the statement of Theorem 3.2 (without the equality case) may be equivalently formulated as follows: if

$$\alpha (D_{h_M})^{k-1} D_{h_L} D_{h_{C_1}} \cdots D_{h_{C_{n-2k}}} V = 0$$

for all $M \in \mathcal{P}(\Lambda)$, then

$$(-1)^k \alpha^2 D_{h_{C_1}} \cdots D_{h_{C_{n-2k}}} V \geq 0.$$

That (3.3)–(3.4) imply (3.1)–(3.2) follows immediately by choosing the differential operator $\alpha = \sum_i x_i (D_{h_{K_i}})^k$. Conversely, that (3.1)–(3.2) imply (3.3)–(3.4) follows as any α can be expressed as $\alpha = \sum_i x_i (D_{h_{K_i}})^k$ by Lemma 3.3.

It remains to check that the Hodge-Riemann relations are nontrivial. This is certainly not obvious at first sight: the condition (3.1) is a very strong one (as it must hold for *any* $M \in \mathcal{P}(\Lambda)$), and it is not clear *a priori* that it can be satisfied in any nontrivial situation. To show this is the case, consider the special case where $L = C_1 = \dots = C_{n-2k} = \Lambda$, and define the spaces

$$P_k := \{\alpha : \alpha (D_{h_\Lambda})^{n-2k+1} V = 0\}, \quad I := \{\alpha : \alpha V = 0\}.$$

The remarkable combinatorial theory underlying the Hodge-Riemann relations enables us to compute [20, Corollary 5.3.4]

$$\dim(P_k/I) = h_k - h_{k-1},$$

where (h_1, \dots, h_n) is the so-called h -vector of Λ . To show the Hodge-Riemann relations are nontrivial, it suffices to construct a simple polytope Λ in \mathbb{R}^n whose h -vector satisfies $h_k > h_{k-1}$ for $k \leq n/2$, as by Theorem 3.2 this ensures the existence

of α so that (3.3) holds and the inequality in (3.4) is strict (by Lemma 3.3, this implies the corresponding statement of Theorem 3.1 for some m, K_1, \dots, K_m, x). But such an example is easily identified: e.g., we may choose Λ to be the unit cube in \mathbb{R}^n , whose h -vector is given by $h_k = \binom{n}{k}$ by the computations in [20, p. 387] (note that $\Lambda = [0, 1] \times \dots \times [0, 1]$ and use the product formula for H -polynomials).

4. PROOF OF THEOREM 1.5

We first consider the special case that $k = 2$.

Proof of Theorem 1.5 for $k = 2$. Fix any $n \geq 4$ and let $k = 2$. By the second part of Theorem 3.1, we may choose a simple polytope $\Lambda = L = C_1 = \dots = C_{n-4}$ in \mathbb{R}^n , $m \geq 1$, polytopes $K_1, \dots, K_m \in \mathcal{P}(\Lambda)$, and $x \in \mathbb{R}^m$ so that (3.1) holds and the inequality in (3.2) is strict. In the following, we will denote $K_{m+1} := \Lambda$.

Now define the $(m+1) \times (m+1)$ matrix

$$M_{ij} := \mathcal{V}(K_i[2], K_j[2], \Lambda[n-4]),$$

and let $y = e_{m+1}$. Then (3.1) with $M = \Lambda$ implies

$$\langle x, My \rangle = 0,$$

while the strict inequality in (3.2) may be written as

$$\langle x, Mx \rangle > 0.$$

Note that M is a positive matrix, as all bodies in $\mathcal{P}(\Lambda)$ are full-dimensional. Thus M is not hyperbolic by Lemma 2.1. In particular, by Lemma 2.3, there exists $I \subseteq [m+1]$ so that $(-1)^{|I|} \det M_I > 0$. The latter contradicts Conjecture 1.3. \square

Informally, the above proof works as follows. By Lemma 2.3, Fedotov's Conjecture 1.3 is equivalent to the statement that the matrix M is hyperbolic. However, when $k = 2$, the Hodge-Riemann relation (3.2) yields an inequality in the opposite direction from the one that holds for hyperbolic matrices by Lemma 2.1. Thus the Hodge-Riemann relation contradicts Fedotov's conjecture.

Precisely the same argument works whenever $k \geq 2$ is even. Curiously, however, the argument fails when k is odd, as then (3.2) and hyperbolicity yield inequalities in the same direction. To prove Theorem 1.5 for arbitrary k , we will use a different argument: rather than applying the Hodge-Riemann relation of degree k , we will instead reduce the problem for any $k > 2$ back to the case $k = 2$.

Proof of Theorem 1.5 for general k . Fix any $n \geq 6$ and $2 < k \leq n/2$. Choose Λ , $m, K_1, \dots, K_{m+1}, x, y$, and M as in the proof of the $k = 2$ case. Note first that

$$\begin{aligned} M_{ij} &:= \mathcal{V}(K_i[2], K_j[2], \Lambda[n-4]) \\ &= \mathcal{V}(K_i[2], \Lambda[k-2], K_j[2], \Lambda[k-2], \Lambda[n-2k]) \\ &= \frac{1}{(k!)^2} \sum_{\delta, \varepsilon \in \{0,1\}^k} (-1)^{k+\delta_1+\dots+\delta_k} (-1)^{k+\varepsilon_1+\dots+\varepsilon_k} \mathcal{V}(K_{i\delta}[k], K_{j\varepsilon}[k], \Lambda[n-2k]) \end{aligned}$$

by the polarization formula [5, p. 137], where

$$K_{i\delta} := (\delta_1 + \delta_2)K_i + (\delta_3 + \dots + \delta_k)\Lambda.$$

Define the $(m+1)(2^k-1) \times (m+1)(2^k-1)$ positive matrix

$$\tilde{M}_{i\delta, j\varepsilon} := \mathcal{V}(K_{i\delta}[k], K_{j\varepsilon}[k], \Lambda[n-2k])$$

for $i, j \in [m+1]$, $\delta, \varepsilon \in \{0, 1\}^k \setminus (0, \dots, 0)$, and define $\tilde{x}, \tilde{y} \in \mathbb{R}^{(m+1)(2^k-1)}$ as

$$\tilde{x}_{i\delta} = \frac{(-1)^{k+\delta_1+\dots+\delta_k} x_i}{k!}, \quad \tilde{y}_{i\delta} = 1_{i=m+1} 1_{\delta=(1,0,\dots,0)}.$$

Then

$$\langle \tilde{x}, \tilde{M}\tilde{y} \rangle = \langle x, My \rangle = 0, \quad \langle \tilde{x}, \tilde{M}\tilde{x} \rangle = \langle x, Mx \rangle > 0,$$

so \tilde{M} cannot be hyperbolic. The latter contradicts Conjecture 1.3 for the given value of k as in the proof of the case $k = 2$. \square

5. AN EXPLICIT EXAMPLE

The proof of Theorem 1.5 shows that counterexamples to Fedotov's conjecture are prevalent: any simple polytope Λ whose Hodge-Riemann relation of degree 2 is nontrivial (that is, whose h -vector satisfies $h_2 > h_1$, cf. section 3) gives rise to a counterexample to Conjecture 1.3 with $C_1 = \dots = C_{n-2k} = \Lambda$ and some K_1, \dots, K_m strongly isomorphic to Λ . However, the construction itself is rather indirect. The aim of this section is to illustrate the construction by means of a simple explicit example in the case that Λ is the unit cube.

Let $\Lambda = [0, e_1] + \dots + [0, e_n]$ be the unit cube in \mathbb{R}^n . Then any $M \in \mathcal{P}(\Lambda)$ is a parallelepiped of the form $M = M_a + v$ for some $a_1, \dots, a_n > 0$ and $v \in \mathbb{R}^n$, where

$$M_a := a_1[0, e_1] + \dots + a_n[0, e_n].$$

By translation-invariance of mixed volumes, it suffices to consider $v = 0$. We can compute mixed volumes of parallelepipeds using that

$$n! \mathbf{V}([0, e_{i_1}], \dots, [0, e_{i_n}]) = 1_{i_1 \neq \dots \neq i_n}$$

by [14, (5.77)], so that by additivity of mixed volumes

$$n! \mathbf{V}(M_{a^{(1)}}, \dots, M_{a^{(n)}}) = \sum_{i_1 \neq \dots \neq i_n} a_{i_1}^{(1)} \dots a_{i_n}^{(n)}.$$

Using this simple expression, it is not difficult to generate explicit examples.

For example, for the case $n = 4$, $k = 2$, let us define

$$\begin{aligned} K_1 &:= [0, e_1] + [0, e_2], \\ K_2 &:= [0, e_3] + [0, e_4], \\ K_3 &:= [0, e_1] + \dots + [0, e_4] = \Lambda. \end{aligned}$$

Then it is readily verified by means of the above formula that

$$3\mathbf{V}(K_1[2], M, \Lambda) + 3\mathbf{V}(K_2[2], M, \Lambda) - \mathbf{V}(K_3[2], M, \Lambda) = 0$$

for all $M \in \mathcal{P}(\Lambda)$, that is, (3.1) holds with $x_1 = x_2 = 3$ and $x_3 = -1$. (This is most easily seen by using $\Lambda = K_1 + K_2$ and $\mathbf{V}(K_1[3], M) = \mathbf{V}(K_2[3], M) = 0$ for all M .) On the other hand, we compute

$$\begin{aligned} \sum_{i,j} x_i x_j \mathbf{V}(K_i[2], K_j[2]) &= 18\mathbf{V}(K_1[2], K_2[2]) - 6\mathbf{V}(K_1[2], K_3[2]) \\ &\quad - 6\mathbf{V}(K_2[2], K_3[2]) + \mathbf{V}(K_3[2], K_3[2]) = 2, \end{aligned}$$

so that (3.2) holds with strict inequality. It therefore follows from the argument in the proof of Theorem 1.5 that Conjecture 1.3 must fail for $n = 4$, $k = 2$, $m = 3$ when K_1, K_2, K_3 are chosen as above. The author is indebted to the anonymous referee of this note for suggesting this example.

Remark 5.1. Technically speaking the above example does not verify the assumptions of Theorem 3.1, as K_1, K_2 have empty interior and are therefore not strongly isomorphic to Λ . However, the example remains valid if we replace K_1, K_2, x_3 by $K'_1 = K_1 + \varepsilon K_2$, $K'_2 = K_2 + \varepsilon K_1$, and $x'_3 = -1 - 4\varepsilon - \varepsilon^2$ for any $\varepsilon > 0$.

Of course, given any explicit example, one can readily verify directly that Conjecture 1.3 fails without any reference to the Hodge-Riemann relations. However, this obscures the fundamental reason for the failure of Fedotov's conjecture which was essential for the discovery of such counterexamples. On the other hand, the above explicit example provides additional information beyond our main result as stated in Theorem 1.5: it shows that Fedotov's conjecture fails already when $k = 2$ and $m = 3$, that is, in the smallest case that is not covered by Lemma 1.4. The example is readily modified to extend this conclusion to any k .

Lemma 5.2. *For every $k \geq 2$ and $n \geq 2k$, Conjecture 1.3 fails for $m = 3$.*

Proof. Define the following bodies:

$$\begin{aligned} K_1 &= [0, e_1] + \cdots + [0, e_k], \\ K_2 &= [0, e_{k+1}] + \cdots + [0, e_{2k}], \\ K_3 &= [0, e_1] + \cdots + [0, e_{2k}], \\ C_1, \dots, C_{n-2k} &= [0, e_{2k+1}] + \cdots + [0, e_n]. \end{aligned}$$

Then we can compute $M_{ij} := \mathbb{V}(K_i[k], K_j[k], C_1, \dots, C_{n-2k})$ explicitly as

$$M = \begin{bmatrix} 0 & a & a \\ a & 0 & a \\ a & a & b \end{bmatrix}, \quad a = \frac{(k!)^2(n-2k)!}{n!}, \quad b = \frac{(2k)!(n-2k)!}{n!}.$$

Therefore

$$\det M = a^2(2a - b) = \frac{(k!)^4((n-2k)!)^3}{(n!)^3} (2(k!)^2 - (2k)!) < 0$$

whenever $k \geq 2$, contradicting Conjecture 1.3. \square

Remark 5.3. The explicit expression for $n! \mathbb{V}(M_{a^{(1)}}, \dots, M_{a^{(n)}})$ given above is nothing other than the permanent of the matrix whose columns are $a^{(1)}, \dots, a^{(n)}$. It is well known [5, §25.4] that the permanent of a matrix is not only a special case of mixed volumes, but also of mixed discriminants (the linear-algebraic analogue of mixed volumes). The above example therefore shows that the analogue of Fedotov's conjecture for mixed discriminants is also invalid. This should not come as a surprise, as mixed discriminants also satisfy Hodge-Riemann relations [19] and thus the arguments behind Theorem 1.5 extend to this situation.

6. HODGE-RIEMANN RELATIONS FAIL FOR GENERAL CONVEX BODIES

Beside the disproof of Fedotov's conjecture, an expository aim of this note has been to highlight that the Hodge-Riemann relations of McMullen and Timorin may be interpreted entirely in terms of familiar objects from classical convex geometry: they provide inequalities between mixed volumes that generalize the Alexandrov-Fenchel inequality. From the viewpoint of classical convexity, however, the formulation of Theorem 3.1 exhibits a puzzling aspect. In principle, the statements of the relations (3.1) and (3.2) make sense when K_i, C_i, M, L are arbitrary convex bodies,

but the statement of Theorem 3.1 requires these bodies to be strongly isomorphic simple polytopes. It is not immediately clear why the latter is important: most classical inequalities in convex geometry are either valid for arbitrary convex bodies, or involve geometric quantities that do not make sense in the absence of regularity conditions (such as uniform bounds on the principal curvatures).

We have shown in section 3 that the Hodge-Riemann relation of degree $k = 1$ is equivalent to the Alexandrov-Fenchel inequality for strongly isomorphic polytopes. The inequality then extends readily to arbitrary convex bodies by approximation. This is possible because for $k = 1$ the relations (3.1) and (3.2) can be combined into a single inequality by Lemma 2.1, and this *inequality* is preserved by taking limits. However, a natural analogue of Lemma 2.1 does not hold for $k \geq 2$. It is therefore unclear how to apply an approximation argument, as the *equality* (3.1) need not be stable under approximation (that is, if (3.1) holds for a given collection of convex bodies, they might not be approximated by simple strongly isomorphic polytopes in such a way that (3.1) remains valid for the approximations).

We will presently show by means of a simple example that the Hodge-Riemann relation of degree $k = 2$ can in fact fail for general convex bodies.

Example 6.1. Let B be the Euclidean unit ball in \mathbb{R}^4 , and let $L = \text{conv}\{B, x\}$ for some $x \notin B$, that is, L is a cap body of B . It is a classical fact, which dates back essentially to Minkowski, that [14, Theorem 7.6.17]

$$\mathbb{V}(L, L, B, L) = \mathbb{V}(B, L, B, L) = \mathbb{V}(B, B, B, L) > \mathbb{V}(B, B, B, B).$$

In particular, this gives rise to a nontrivial equality case of the Alexandrov-Fenchel inequality of Theorem 1.1 with $n = 4$, $K = C_1 = B$, $C_2 = L$. The latter implies

$$\mathbb{V}(M, B, B, L) = \mathbb{V}(M, L, B, L)$$

for all convex bodies M , cf. [14, Theorem 7.4.3] or [17, Lemma 3.12].

We will now use these observations to construct a counterexample to the Hodge-Riemann relation of degree $k = 2$ for general convex bodies. Define

$$K_1 := B, \quad K_2 := L, \quad K_3 := B + L.$$

Then

$$\begin{aligned} 3\mathbb{V}(K_1[2], M, L) + \mathbb{V}(K_2[2], M, L) - \mathbb{V}(K_3[2], M, L) = \\ 2\mathbb{V}(M, B, B, L) - 2\mathbb{V}(M, L, B, L) = 0 \end{aligned}$$

for all convex bodies M ; that is, (3.1) is satisfied with $x_1 = 3$, $x_2 = 1$, and $x_3 = -1$. On the other hand, we can compute

$$\sum_{i,j} x_i x_j \mathbb{V}(K_i[2], K_j[2]) = 4\mathbb{V}(B, B, B, B) - 4\mathbb{V}(L, B, B, B) < 0,$$

contradicting the validity of (3.2).

Remark 6.2. There is nothing special about the particular choice of the Euclidean ball in this example: the conclusion remains valid when B is replaced by an arbitrary convex body K and L is a cap body of K as defined in [14, p. 87]. For example, we may take L to be the unit cube in \mathbb{R}^4 and K to be the same cube with one of its corners sliced off. The latter variant of the example shows that the Hodge-Riemann relations can fail for polytopes that are not strongly isomorphic.

The above example suggests that the validity of Hodge–Riemann relations of degree $k \geq 2$ is related to the study of the equality cases of the Alexandrov–Fenchel inequality: indeed, the assumption (3.1) is reminiscent of the equality condition of the Alexandrov–Fenchel inequality (cf. [14, Theorem 7.4.2]), which is precisely what was used to construct the above counterexample. Even though the Alexandrov–Fenchel inequality is stable under approximation, this cannot be used to study its nontrivial equality cases as the latter are destroyed by approximation [13, 16, 17]. The above example shows that for Hodge–Riemann relations of degree $k \geq 2$, this instability is manifested even by the inequality itself.

On the other hand, it is expected that the validity of Hodge–Riemann relations should extend to “ample” families of convex bodies other than simple strongly isomorphic polytopes. In particular, one may conjecture that the statement of Theorem 3.1 remains valid if the class $\mathcal{P}(\Lambda)$ is replaced by the class C_+^∞ of convex bodies whose boundaries are smooth and have strictly positive curvature. Some initial progress in this direction may be found in the recent papers [9, 1, 10].

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