SHEPHARD’S INEQUALITIES, HODGE-RIEMANN RELATIONS, AND A CONJECTURE OF FEDOTOV

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Abstract. A well-known family of determinantal inequalities for mixed volumes of convex bodies were derived by Shephard from the Alexandrov-Fenchel inequality. The classic monograph Geometric Inequalities by Burago and Zalgaller states a conjecture on the validity of higher-order analogues of Shephard’s inequalities, which is attributed to Fedotov. In this note we disprove Fedotov’s conjecture by showing that it contradicts the Hodge-Riemann relations for simple convex polytopes. Along the way, we make some expository remarks on the linear algebraic and geometric aspects of these inequalities.

1. Introduction

1.1. Let $K_1, \ldots, K_m$ be convex bodies in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m > 0$. One of the most basic facts of convex geometry, due to H. Minkowski, is that the volume of convex bodies is a homogeneous polynomial in the sense that

$$\text{Vol}(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \ldots, i_n = 1}^m V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$ 

The coefficients $V(K_1, \ldots, K_n)$, called mixed volumes, define a large family of natural geometric parameters of convex bodies, and play a central role in convex geometry [5, 12]. Mixed volumes are always nonnegative, are symmetric in their arguments, and are additive and homogeneous in each argument.

The fundamental inequality in the theory of mixed volumes is the following.

**Theorem 1.1** (Alexandrov-Fenchel). For convex bodies $K, L, C_1, \ldots, C_{n-2}$ in $\mathbb{R}^n$

$$V(K, L, C_1, \ldots, C_{n-2})^2 \geq V(K, K, C_1, \ldots, C_{n-2}) V(L, L, C_1, \ldots, C_{n-2}).$$

Numerous inequalities in convex geometry may be derived from the Alexandrov-Fenchel inequality, cf. [5, §20] and [12, §7.4]. The starting point for this note is a well-known family of determinantal inequalities, due to Shephard [16], that extend the Alexandrov-Fenchel inequality to more than $n$ bodies.

**Theorem 1.2** (Shephard). Given convex bodies $K_1, \ldots, K_m, C_1, \ldots, C_{n-2}$ in $\mathbb{R}^n$, define the $m \times m$ symmetric matrix $M$ by setting

$$M_{ij} := V(K_i, K_j, C_1, \ldots, C_{n-2}).$$

Then

$$(-1)^m \det M \leq 0.$$
The special case \( m = 2 \) of Theorem 1.2 is just a reformulation of the Alexandrov-Fenchel inequality, and Shephard’s inequalities may thus be viewed as a considerable generalization of the Alexandrov-Fenchel inequality. However, as is shown by Shephard (and as we will explain later in this note), the general inequalities may in fact be deduced from the \( m = 2 \) case by a simple linear algebraic argument. In the case \( m = 3 \), this result dates back already to Minkowski [11, p. 478].

1.2. The classic monograph Geometric Inequalities by Burago and Zalgaller states a conjecture on the validity of a higher-order generalization of Theorem 1.2, which is attributed to Fedotov [5, §20.6]. Let us recall the statement of this conjecture. In the sequel, we will frequently employ the notation

\[ V(K_1[m_1], K_2[m_2], \ldots, K_r[m_r]) := V(K_1, \ldots, K_1, K_2, \ldots, K_2, \ldots, K_r, \ldots, K_r) \]

when convex bodies are repeated multiple times in the arguments of a mixed volume.

**Conjecture 1.3** (Fedotov). Let \( k \leq n/2 \), and let \( K_1, \ldots, K_m, C_1, \ldots, C_{n-2k} \) be convex bodies in \( \mathbb{R}^n \). Define the \( m \times m \) symmetric matrix \( M \) by setting

\[ M_{ij} := V(K_i[k], K_j[k], C_1, \ldots, C_{n-2k}) \]

Then

\[ (-1)^m \det M \leq 0. \]

If true, this conjecture would entail a considerable generalization of Shephard’s inequalities. The conjecture is rather appealing, as it is easily verified to be true in two extreme cases that have a different flavor.

**Lemma 1.4.** Conjecture 1.3 is valid in the following two cases:

a. When \( k = 1 \) and \( m \) is arbitrary.

b. When \( m = 2 \) and \( k \) is arbitrary.

**Proof.** Case \( a \) is nothing other than Theorem 1.2. To prove \( b \), it suffices to note that iterating the Alexandrov-Fenchel yields the inequality [12, (7.63)]

\[ V(K_1[k], K_2[l], C_1, \ldots, C_{n-k-l})^{k+l} \geq V(K_1[k+l], C_1, \ldots, C_{n-k-l})^k V(K_2[k+l], C_1, \ldots, C_{n-k-l})^l \]

for any \( k, l \geq 1, k + l \leq n \). The case \( k = l \) is readily seen to be equivalent to \( b \). \( \square \)

The main purpose of this note is to explain that Conjecture 1.3 fails when one goes beyond the special cases of Lemma 1.4. More precisely, we will prove:

**Theorem 1.5.** For every \( k > 1 \), Conjecture 1.3 is false for some \( m > 2 \).
Bernstein and Kushnirenko [5, Theorem 27.1.2] shows that the number of solutions 
\( z \in (\mathbb{C}\setminus\{0\})^n \) of a generic system of polynomial equations 
\( p_1(z) = 0, \ldots, p_n(z) = 0 \) with given monomials coincides with the mixed volume of an associated family of 
lattice polytopes in \( \mathbb{R}^n \) (i.e., polytopes with vertices in \( \mathbb{Z}^n \)).

Motivated by these developments, Fedotov [7] proposed a simple proof of the
Alexandrov-Fenchel inequality using only basic properties of polynomials. Fedotov
further notes that his method even yields the more general Conjecture 1.3, which
is stated in [7] as a theorem. These results were included in the Russian edition
of the monograph of Burago and Zalgaller. Unfortunately, Fedotov’s elementary
approach turns out to contain a serious flaw, which renders his method of proof
invalid. A correct algebraic proof of the Alexandrov-Fenchel inequality was given by
Teissier and Khovanskii using nontrivial machinery, viz. a reduction to the Hodge
index theorem of algebraic geometry. The latter proof is included in the English
translation of Burago-Zalgaller [5, §27], but does not settle the validity of Fedotov’s
higher-order analogue of Shephard’s inequalities [5, §20.6].

On the other hand, the algebraic connection yields other higher-order inequalities. The Alexandrov-Fenchel inequality is analogous to a Hodge-Riemann relation
of degree 1 in the cohomology ring of a smooth projective variety [6, 8]. Hodge-
Riemann relations of higher degree give rise to new inequalities in convex geometry.
Such inequalities were first stated by McMullen [10] for strongly isomorphic simple
polytopes as a byproduct of his work on the \( g \)-conjecture. Their geometric signifi-
cance was greatly clarified by Timorin [17], whose formulation is readily interpreted
in terms of explicit inequalities for mixed volumes. Very recently, some special cases
were extended also to smooth convex bodies in [9, 1].

The proof of Theorem 1.5 may now be explained as follows. Using the prop-
erties of hyperbolic quadratic forms, we will first reformulate Conjecture 1.3 as a
higher-order Alexandrov-Fenchel inequality. In this equivalent formulation, it will
be evident that this inequality contradicts the Hodge-Riemann relation of degree 2.
Thus the results of McMullen and Timorin imply that Conjecture 1.3 is false. Beside
disproving the conjecture, a more expository aim of this note is to draw attention
to some basic linear algebraic and geometric aspects of the above inequalities (none
of which are really new here) in the context of classical convexity.

Remark. It should be noted that Fedotov’s conjecture as stated in [5, §20.6] is
somewhat more general than Conjecture 1.3: the matrix \( M \) considered there is
\[
M_{ij} := \mathcal{V}(K_i[k], K_j[l], C_1, \ldots, C_{n-k-l})
\]
for any \( k, l \geq 1 \) such that \( k + l \leq n \). Lemma 1.4 extends to this setting: the case
\( k = l = 1 \) and general \( m \) reduces to Shephard’s inequalities, while the case \( m = 2 \)
and general \( k, l \) is obtained by multiplying the inequality [12, (7.63)] used in the
proof of Lemma 1.4 by the same inequality with the roles of \( k, l \) reversed. When
\( k \neq l \), however, the matrix \( M \) is not symmetric, and the spectral interpretation of
the conjecture becomes unclear. Given that we show the conjecture fails for general
\( m \) already in the symmetric case \( k = l \), it seems implausible that the nonsymmetric
case \( k \neq l \) has any merit, and we do not consider it further in this note.

1.4. The remainder of this note is organized as follows. In section 2, we recall
some basic properties of hyperbolic quadratic forms that will be used in the sequel.
We also briefly discuss Shephard’s inequalities and clarify their equality cases. In
section 3 we formulate the Hodge-Riemann relations for strongly isomorphic simple
polytopes, due to McMullen and Timorin, entirely in the language of classical convexity. Finally, section 4 completes the proof of Theorem 1.5.

2. Linear algebra

The aim of this section is to explain that the connection between the Alexandrov-Fenchel and Shephard inequalities has nothing to do with convexity, but is rather a simple linear-algebraic fact. The results of this section are known in various forms, see, e.g., [4, Theorem 4.4.6], [13, Lemma 2.9], or [15, Lemma 3.1], but we provide simple self-contained proofs for the variants needed here.

2.1. Hyperbolic matrices. A matrix $M$ is called positive if $M_{ij} > 0$ for all $i, j$. For $y \in \mathbb{R}^m$, we write $y \succeq 0 (y > 0)$ if $y_i \geq 0 (y_i > 0)$ for all $i$. We begin by giving a spectral interpretation of the Alexandrov-Fenchel inequality.

**Lemma 2.1.** Let $M$ be a symmetric positive matrix. The following are equivalent:

1. The positive eigenspace of $M$ is one-dimensional.
2. $\langle x, My \rangle^2 \geq \langle x, Mx \rangle \langle y, My \rangle$ for all $x \succeq 0$ and $y \succeq 0$.
3. $\langle x, My \rangle = 0$ implies $\langle x, Mx \rangle \leq 0$ for all $x$ and $y \succeq 0$, $y \neq 0$.

**Proof.** As $M$ is a positive matrix, the Perron-Frobenius theorem implies that it has at least one eigenvector $v > 0$ with positive eigenvalue.

$3 \Rightarrow 1$: Let $x \perp v$ be any other eigenvector of $M$. Then $\langle x, Mv \rangle = 0$, so $3$ implies $\langle x, Mx \rangle \leq 0$. Thus the eigenvalue associated to $x$ must nonpositive.

$1 \Rightarrow 2$: It follows from 1 that $M$ is negative semidefinite on $v^+$. Fix $x, y \succeq 0$; we may assume $y \neq 0$ (else the inequality is trivial), so that $\langle y, v \rangle > 0$ and $\langle y, Mv \rangle > 0$.

If we define $z = x - ay$ with $a = \langle x, v \rangle/\langle y, v \rangle$, then $z \in v^+$, so

$$0 \geq \langle z, Mz \rangle = \langle x, Mx \rangle - 2a\langle x, My \rangle + a^2\langle y, My \rangle \geq \langle x, Mx \rangle - \frac{\langle x, My \rangle^2}{\langle y, My \rangle}.$$

$2 \Rightarrow 3$: We first show that $2$ remains valid for any $x$ (not just $x \succeq 0$). Suppose first that $y > 0$. Then $x + by \succeq 0$ when $b$ is chosen sufficiently large, so $2$ implies

$$\langle x + by, My \rangle^2 \geq \langle x + by, M(x + by) \rangle \langle y, My \rangle.$$

Expanding both sides of this inequality shows that all terms involving $b$ cancel, so $\langle x, My \rangle^2 \geq \langle x, Mx \rangle \langle y, My \rangle$ for any $x$ and $y > 0$. This conclusion remains valid for any $y \succeq 0$ by applying the above argument with $y \leftarrow y + \varepsilon v$ and letting $\varepsilon \to 0$. Now $3$ follows immediately once we note that $y \geq 0$, $y \neq 0$ implies $\langle y, My \rangle > 0$. \hfill \Box

In the sequel, a symmetric (but not necessarily positive) matrix that has a one-dimensional positive eigenspace will be called *hyperbolic*. 

2.2. Shephard’s inequalities. An $m \times m$ hyperbolic matrix $M$ has $1$ positive and $m-1$ nonpositive eigenvalues. It is therefore immediately obvious that such a matrix satisfies $(-1)^m \det M \leq 0$ (as the determinant is the product of the eigenvalues). Shephard’s inequalities follow directly from this observation.

**Proof of Theorem 1.2.** We may assume without loss of generality that all the convex bodies have nonempty interior, so that $M$ is a positive matrix (otherwise we may replace $K_i \leftarrow K_i + \varepsilon B$, $C_i \leftarrow C_i + \varepsilon B$ for any body $B$ with nonempty interior, and take $\varepsilon \to 0$ in the final inequality.) Condition 2 of Lemma 2.1 is immediate from the
Alexandrov-Fenchel inequality (Theorem 1.1 with $K = \sum_i x_i K_i$ and $L = \sum_i y_i K_i$). Thus M is hyperbolic by Lemma 2.1, which implies $(-1)^m \det M \leq 0$. 

While this is only tangentially related to the rest of this note, let us take the opportunity to clarify the cases of equality in Shephard’s inequalities.

**Proposition 2.2.** In the setting and notations of Theorem 1.2, we have $\det M = 0$ if and only if there are linearly independent vectors $x, y > 0$ such that $K = \sum_i x_i K_i$, $L = \sum_i y_i K_i$ yield equality in the Alexandrov-Fenchel inequality of Theorem 1.1.

**Proof.** We must show $\det M = 0$ if and only if $\langle x, My \rangle^2 = \langle x, Mx \rangle \langle y, My \rangle$ for some linearly independent $x, y > 0$. We may assume $M \neq 0$ (else the result is trivial).

Suppose first that $\det M = 0$. Then there exists $z \in \ker M$, $z \neq 0$. Choose any $y > 0$ that is linearly independent of $z$. Evidently $\langle z, My \rangle^2 = \langle z, Mz \rangle \langle y, My \rangle$. But as this identity is invariant under the replacement $z \leftarrow z + by$ (as in the proof of 2$\Rightarrow$3 of Lemma 2.1), we may choose $x = z + by > 0$ for $b$ sufficiently large.

Now suppose $\langle x, My \rangle^2 = \langle x, Mx \rangle \langle y, My \rangle$ for linearly independent $x, y > 0$. Then

$$q(v) := \langle x + v, My \rangle^2 - \langle x + v, M(x + v) \rangle \langle y, My \rangle$$

satisfies $q(0) = 0$, and $q(v) \geq 0$ for all $v$ in a neighborhood of 0 by the Alexandrov-Fenchel inequality. Thus $\nabla q(0) = 0$, which yields $z = \langle y, My \rangle x - \langle xMy \rangle y \in \ker M$. Moreover, $z \neq 0$ as $x, y$ are linearly independent. Thus $\det M = 0$. 

Proposition 2.2 reduces the equality cases of Shephard’s inequalities to those of the Alexandrov-Fenchel inequality. The characterization of the latter is a long-standing open problem [12, §7.6], which was recently settled in several important cases in [14, 15]. This problem remains open in full generality.

### 2.3. A Sylvester criterion.

While any hyperbolic $m \times m$ matrix $M$ trivially satisfies $(-1)^m \det M \leq 0$, the converse implication clearly does not hold: the sign of the determinant does not determine the number of positive eigenvalues. However, the implication can be reversed if the determinant condition holds for all principal submatrices of $M$. This hyperbolic analogue of the classical Sylvester criterion may be proved in essentially the same manner.\(^1\)

In the following, we denote for any $m \times m$ symmetric matrix $M$ and subset $I \subseteq [m]$ by $M_I := (M_{ij})_{i,j \in I}$ the associated principal submatrix.

**Lemma 2.3.** For a symmetric positive $m \times m$ matrix, the following are equivalent:

1. The positive eigenspace of $M$ is one-dimensional.
2. $(-1)^{|I|} \det M_I \leq 0$ for all $I \subseteq [m]$.

**Proof.** To prove 1$\Rightarrow$2, note first that condition 2 of Lemma 2.1 is inherited by all its principal submatrices $M_I$ (as one may restrict to $x, y$ supported on $I$). The conclusion therefore follows immediately from Lemma 2.1.

To prove 2$\Rightarrow$1, we argue by induction on $m$. For $m = 2$, it suffices to note that as $M$ has at least one positive eigenvalue by the Perron-Frobenius theorem, $\det M \leq 0$ implies that its other eigenvalue must be nonpositive.

Now let $m > 2$ and assume the result has been proved in dimensions up to $m - 1$. Then 2 implies that $M_I$ is hyperbolic for all $I \subseteq [m]$. By the Perron-Frobenius theorem, $M$ has an eigenvector $v$ with positive eigenvalue. Now suppose 1 fails, that is, $M$ is not hyperbolic. Then there must be another eigenvector $w \perp v$ with

\(^1\)The author learned the elementary approach used here from lecture notes of M. Hladík.
positive eigenvalue. As \((-1)^m \det M \leq 0\), there must then be a third eigenvector 
\(u \perp \{v, w\}\) with nonnegative eigenvalue. Choose any \(i \in [m]\) such that 
\(u_i \neq 0\) and let \(I = [m] \setminus \{i\}\). Choose \(a, b \in \mathbb{R}\) so that
\(x := v - au\) and \(y := w - bu\) satisfy \(x_i = y_i = 0\). By construction, \(x, y\) are linearly independent and 
\(\langle z, Mz \rangle > 0\) for all \(z \in \text{span}\{x, y\}, z \neq 0\). As \(x, y\) are supported on \(I\), this implies 
\(M_I\) has a positive eigenspace of dimension at least two, contradicting the induction hypothesis. \(\square\)

It follows immediately from Lemma 2.3 that Conjecture 1.3 is equivalent to the statement that 
the matrix \(M\) is hyperbolic. This observation will form the basis for the proof of Theorem 1.5 in 
section 4: we will show that hyperbolicity of \(M\) contradicts the Hodge-Riemann relations for 
simple convex polytopes.

3. Hodge-Riemann relations

The Hodge-Riemann relations in algebraic geometry give rise to higher order analogues 
of the Alexandrov-Fenchel inequality [10, 17]. While these inequalities are not usually stated in 
this form in the literature, they may be equivalently formulated as explicit inequalities between mixed volumes. The aim of this section is to draw attention to this elementary formulation of the Hodge-Riemann relations 
in terms of familiar objects from classical convex geometry.

Recall that a convex polytope in \(\mathbb{R}^n\) is called simple if it has nonempty interior and 
each vertex is contained in exactly \(n\) facets. In the following, let us fix an arbitrary simple polytope \(\Delta\) in \(\mathbb{R}^n\), and denote by \(P(\Delta)\) the collection of polytopes that are strongly isomorphic to \(\Delta\): that is, \(P \in P(\Delta)\) if and only if
\[
\dim F(P, u) = \dim F(\Delta, u) \quad \text{for all } u \in S^{n-1},
\]
where \(F(P, u)\) denotes the face of \(P\) with normal direction \(u\). For the basic properties of simple and strongly isomorphic polytopes, the reader is referred to [12, §2.4]. For the purposes of this paper, the only significance of these definitions is that they are needed for the validity of the following theorem.

**Theorem 3.1** (McMullen-Timorin). *Fix \(n \geq 2\) and a simple polytope \(\Delta \in \mathbb{R}^n\), and let \(m \geq 1, k \leq n/2, K_1, \ldots, K_m, L, C_1, \ldots, C_{n-2k} \in P(\Delta)\), and \(x \in \mathbb{R}^m\). If
\[
\sum_i x_i V(K_i[k], M[k-1], L, C_1, \ldots, C_{n-2k}) = 0
\]
holds for every \(M \in P(\Delta)\), then
\[
(-1)^k \sum_{i,j} x_i x_j V(K_i[k], K_j[k], C_1, \ldots, C_{n-2k}) \geq 0.
\]
Moreover, the statement is nontrivial in the sense that for any \(n \geq 2\) and \(k \leq n/2\), 
there is a simple polytope \(\Delta = L = C_1 = \cdots = C_{n-2k} \in \mathbb{R}^n\), \(m \geq 1, K_1, \ldots, K_m \in P(\Delta)\), and \(x \in \mathbb{R}^m\) so that (3.1) holds and the inequality in (3.2) is strict.

The case \(k = 1\) of Theorem 3.1 is nothing other than the Alexandrov-Fenchel inequality. To see why this is so, assume without loss of generality that \(L = \sum_i y_i K_i\) for some \(y \geq 0, y \neq 0\) (otherwise let \(m \leftarrow m + 1\) and \(K_{m+1} \leftarrow L\), and define
\[
M_{ij} = V(K_i, K_j, C_1, \ldots, C_{n-2}).
\]
Then the statement of Theorem 3.1 for \(k = 1\) may be formulated as
\[
\langle x, My \rangle = 0 \quad \text{implies} \quad \langle x, Mx \rangle \leq 0.
\]
for any $x$ and $y \geq 0$, $y \neq 0$. Thus by Lemma 2.1, the inequality of Theorem 3.1 in the case $k = 1$ is equivalent to the Alexandrov-Fenchel inequality for convex bodies in $\mathcal{P}(\Delta)$. As any collection of convex bodies can be approximated by simple strongly isomorphic polytopes [12, Theorem 2.4.15], the general case of the Alexandrov-Fenchel inequality is further equivalent to this special case.

For $k > 1$, the statement of Theorem 3.1 may be viewed as an analogue of the Alexandrov-Fenchel inequality for $M_{ij} = V(K_i[k], K_j[k], C_1, \ldots, C_{n-2k})$. Thus the Hodge-Riemann relations are reminiscent of Conjecture 1.3, but their formulation is considerably more subtle. In section 4, we will show that the Hodge-Riemann relations in fact contradict Conjecture 1.3, disproving the latter.

Remark 3.2. In the special case $k = 1$, we have argued by approximation that Theorem 3.1 remains valid for arbitrary convex bodies (not just for simple strongly isomorphic polytopes). However, it is far from clear whether this is still the case when $k \geq 2$; it is possible that some restriction on the class of bodies is required for the validity of Hodge-Riemann relations of degree $k \geq 2$.

The case $k = 1$ is special because the statement may be reformulated in this case as a single inequality by Lemma 2.1. As inequalities are preserved by taking limits, the result is then amenable to approximation. However, when $k \geq 2$, no meaningful formulation of this kind is known for general convex bodies. It is therefore unclear how to apply an approximation argument, as the equality (3.1) need not be stable under approximation (that is, if (3.1) holds for a given collection of convex bodies, it is unclear whether they can be approximated by simple strongly isomorphic polytopes in such a way that (3.1) remains valid for the approximations).

On the other hand, it is expected that the validity of Hodge-Riemann relations should extend to “ample” families of convex bodies other than simple strongly isomorphic polytopes. In particular, one may conjecture that the statement of Theorem 3.1 remains valid if the class $\mathcal{P}(\Delta)$ is replaced by the class $C_\mathcal{P}^e$ of convex bodies whose boundaries are smooth and have strictly positive curvature. Some initial progress in this direction may be found in the recent papers [9, 1].

The aim of the rest of this section is to convince the reader that the statement of Theorem 3.1 given here in terms of mixed volumes is equivalent to the statement of the Hodge-Riemann relations as given in [17]. The reader who is primarily interested in Theorem 1.5 may safely jump ahead to section 4.

To explain the formulation of [17], we must first introduce some additional notation. Let $u_1, \ldots, u_N \in S^{n-1}$ be the normal directions of the facets of $\Delta$. For any $P \in \mathcal{P}(\Delta)$, we denote by $h_P \in \mathbb{R}^N$ its support vector $(h_P)_i := \sup_{y \in P} \langle y, u_i \rangle$.

Then there is a homogeneous polynomial $V : \mathbb{R}^N \to \mathbb{R}$ of degree $n$, called the volume polynomial, so that $\text{Vol}(P) = V(h_P)$ for every $P \in \mathcal{P}(\Delta)$ [12, §5.2]. Moreover, as $\mathcal{P}(\Delta)$ is closed under addition [12, §2.4], it follows immediately from the definition of mixed volumes that we have for any $P_1, \ldots, P_n \in \mathcal{P}(\Delta)$

$$V(P_1, \ldots, P_n) = \frac{1}{n!} D_{h_{P_1}} \cdots D_{h_{P_n}} V,$$

where $D_h$ denotes the directional derivative in direction $h$. In this notation, the Hodge-Riemann relations are formulated in [17, p. 385] as follows:
Lemma 3.4. For any homogeneous differential operator \( \alpha = \sum_{|I|=k} \alpha_I D^I \) be a homogeneous differential operator of order \( k \) with constant coefficients. If

\[
\alpha D_{hL} D_{hC_1} \cdots D_{hC_{n-2k}} V = 0,
\]

then

\[
(-1)^k \alpha^2 D_{hC_1} \cdots D_{hC_{n-2k}} V \geq 0.
\]

Moreover, equality is attained if and only if \( \alpha V = 0 \).

To write Theorem 3.3 in terms of mixed volumes, we need the following.

Lemma 3.4. For any homogeneous differential operator \( \alpha = \sum_{|I|=k} \alpha_I D^I \), there exist \( m \geq 1 \), \( K_1, \ldots, K_m \in \mathcal{P}(\Delta) \), and \( x \in \mathbb{R}^m \) so that \( \alpha = \sum_i x_i (D_{hK_i})^k \).

Proof. We first recall that for any \( z \in \mathbb{R}^N \), \( h_\Delta + \varepsilon z \) is the support vector of some polytope \( K \in \mathcal{P}(\Delta) \) for sufficiently small \( \varepsilon \) (as \( \Delta \) is simple, cf. [12, Lemma 2.4.13]). We may therefore write \( z = h_L - h_{L'} \) where \( L = \varepsilon^{-1} K \) and \( L' = \varepsilon^{-1} \Delta \).

Now denote by \( e_1, \ldots, e_N \) the standard coordinate basis of \( \mathbb{R}^N \). By the above observation, we may write \( e_i = h_{L_i} - h_{L'_i} \) for \( L_i, L'_i \in \mathcal{P}(\Delta) \). We can therefore write

\[
\alpha = \sum_{i_1 \leq \cdots \leq i_k} \alpha_{i_1 \cdots i_k} (D_{h_{L_{i_1}}} - D_{h_{L'_{i_1}}}) \cdots (D_{h_{L_{i_k}}} - D_{h_{L'_{i_k}}}).
\]

By expanding the product, we may evidently express \( \alpha \) as a linear combination of differential operators of the form \( D_{h_{R_{i_1}}} \cdots D_{h_{R_{i_k}}} \) with \( R_i \in \mathcal{P}(\Delta) \). But as

\[
D_{h_{R_{i_1}}} \cdots D_{h_{R_{i_k}}} = \frac{1}{k!} \sum_{\delta \in \{0,1\}^k} (-1)^{k+\delta_1 + \cdots + \delta_k} (D_{h_{\delta_{R_{i_1}}} \cdots \delta_{R_{i_k}}})^k
\]

by the polarization formula [5, p. 137], the proof is readily concluded. \( \square \)

We are now ready to show that the Hodge-Riemann relations expressed by Theorems 3.1 and 3.3 are equivalent. First, note that \( \alpha D_{hL} D_{hC_1} \cdots D_{hC_{n-2k}} V \) in (3.3) is a homogeneous polynomial of degree \( k-1 \). Thus (3.3) is equivalent to the statement that \( \beta \alpha D_{hL} D_{hC_1} \cdots D_{hC_{n-2k}} V = 0 \) for every homogeneous differential operator \( \beta \) of order \( k-1 \). By Lemma 3.4, the statement of Theorem 3.3 (without the equality case) may be equivalently formulated as follows: if

\[
\alpha(D_{hM})^{k-1} D_{hL} D_{hC_1} \cdots D_{hC_{n-2k}} V = 0
\]

for all \( M \in \mathcal{P}(\Delta) \), then

\[
(-1)^k \alpha^2 D_{hC_1} \cdots D_{hC_{n-2k}} V \geq 0.
\]

That (3.3)–(3.4) imply (3.1)–(3.2) follows immediately by choosing the differential operator \( \alpha = \sum_i x_i (D_{hK_i})^k \). Conversely, that (3.1)–(3.2) imply (3.3)–(3.4) follows as any \( \alpha \) can be expressed as \( \alpha = \sum_i x_i (D_{hK_i})^k \) by Lemma 3.4.

It remains to check that the Hodge-Riemann relations are nontrivial. This is certainly not obvious at first sight: the condition (3.1) is a very strong one (as it must hold for any \( M \in \mathcal{P}(\Delta) \)), and it is not clear a priori that it can be satisfied in any nontrivial situation. To show this is the case, consider the special case where \( L = C_1 = \cdots = C_{n-2k} = \Delta \), and define the spaces

\[
P_k := \{ \alpha : (D_{h\Delta})^{n-2k+1} V = 0 \}, \quad I := \{ \alpha : \alpha V = 0 \}.
\]
The remarkable combinatorial theory underlying the Hodge-Riemann relations enables us to compute [17, Corollary 5.3.4]

$$\dim(P_k/I) = h_k - h_{k-1},$$

where \((h_1, \ldots, h_n)\) is the so-called \(h\)-vector of \(\Delta\). To show the Hodge-Riemann relations are nontrivial, it suffices to construct a simple polytope \(\Delta\) in \(\mathbb{R}^n\) whose \(h\)-vector satisfies \(h_k > h_{k-1}\) for \(k \leq n/2\), as by Theorem 3.3 this ensures the existence of \(\alpha\) so that (3.3) holds and the inequality in (3.4) is strict (by Lemma 3.4, this implies the corresponding statement of Theorem 3.1 for some \(m, K_1, \ldots, K_m, x\)). But such an example is easily identified: e.g., we may choose \(\Delta\) to be the unit cube in \(\mathbb{R}^n\), whose \(h\)-vector is given by \(h_k = \binom{n}{k}\) by the computations in [17, p. 387].

4. Proof of Theorem 1.5

We first consider the special case that \(k = 2\).

Proof of Theorem 1.5 for \(k = 2\). Fix any \(n \geq 4\) and let \(k = 2\). By the second part of Theorem 3.1, we may choose a simple polytope \(\Delta = L = C_1 = \cdots = C_{n-4}\) in \(\mathbb{R}^n\), \(m \geq 1\), polytopes \(K_1, \ldots, K_m \in \mathcal{P}(\Delta)\), and \(x \in \mathbb{R}^m\) so that (3.1) holds and the inequality in (3.2) is strict. In the following, we will denote \(K_{m+1} := \Delta\).

Now define the \((m + 1) \times (m + 1)\) matrix

$$M_{ij} := \mathcal{V}(K_i[2], K_j[2], \Delta[n - 4]),$$

and let \(y = e_{m+1}\). Then (3.1) with \(M = \Delta\) implies

$$\langle x, My \rangle = 0,$$

while the strict inequality in (3.2) may be written as

$$\langle x, Mx \rangle > 0.$$

Note that \(M\) is a positive matrix, as all bodies in \(\mathcal{P}(\Delta)\) are full-dimensional. Thus \(M\) is not hyperbolic by Lemma 2.1. In particular, by Lemma 2.3, there exists \(I \subseteq [m + 1]\) so that \((-1)^{|I|} \det M_I > 0\). The latter contradicts Conjecture 1.3. \(\square\)

Informally, the above proof works as follows. By Lemma 2.3, Fedotov’s Conjecture 1.3 is equivalent to the statement that the matrix \(M\) is hyperbolic. However, when \(k = 2\), the Hodge-Riemann relation (3.2) yields an inequality in the opposite direction from the one that holds for hyperbolic matrices by Lemma 2.1. Thus the Hodge-Riemann relation contradicts Fedotov’s conjecture.

Precisely the same argument works whenever \(k \geq 2\) is even. Curiously, however, the argument fails when \(k\) is odd, as then (3.2) and hyperbolicity yield inequalities in the same direction. To prove Theorem 1.5 for arbitrary \(k\), we will use a different argument: rather than applying the Hodge-Riemann relation of degree \(k\) to \(K_{m+1}\) and \(K_m\), we will instead reduce the problem for any \(k > 2\) back to the case \(k = 2\).

Proof of Theorem 1.5 for general \(k\). Fix any \(n \geq 6\) and \(2 < k \leq n/2\). Choose \(\Delta, m, K_1, \ldots, K_{m+1}, x, y,\) and \(M\) as in the proof of the \(k = 2\) case. Note first that

$$M_{ij} := \mathcal{V}(K_i[2], K_j[2], \Delta[n - 4])$$

$$= \mathcal{V}(K_i[2], \Delta[k - 2], K_j[2], \Delta[k - 2], \Delta[n - 2k])$$

$$= \frac{1}{(k!)^2} \sum_{\delta, \varepsilon \in \{0, 1\}^k} (-1)^{\delta_1 + \cdots + \delta_k} (-1)^{\varepsilon_1 + \cdots + \varepsilon_k} \mathcal{V}(K_{i\delta}[k], K_{j\varepsilon}[k], \Delta[n - 2k])$$
by the polarization formula [5, p. 137], where

\[ K_{i\delta} := (\delta_1 + \delta_2)K_i + (\delta_3 + \cdots + \delta_k)\Delta. \]

Define the \((m+1)(2^k-1) \times (m+1)(2^k-1)\) positive matrix

\[ \tilde{M}_{i\delta,j\varepsilon} := V(K_{i\delta}[k], K_{j\varepsilon}[k], \Delta[m-2k]) \]

for \(i, j \in [m+1], \delta, \varepsilon \in \{0,1\}^k \setminus \{(0, \ldots, 0)\}\), and define \(\tilde{x}, \tilde{y} \in \mathbb{R}^{(m+1)(2^k-1)}\) as

\[ \tilde{x}_{i\delta} = \frac{(-1)^{\delta_1+\cdots+\delta_k}x_i}{k!}, \quad \tilde{y}_{i\delta} = 1_{i=m+1}1_{\delta=(1,0,\ldots,0)}. \]

Then

\[ \langle \tilde{x}, \tilde{M}\tilde{y} \rangle = \langle x, My \rangle = 0, \quad \langle \tilde{x}, \tilde{M}\tilde{x} \rangle = \langle x, Mx \rangle > 0, \]

so \(\tilde{M}\) cannot be hyperbolic. The latter contradicts Conjecture 1.3 for the given value of \(k\) as in the proof of the case \(k = 2\). □

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**References**