# THE LOCAL LOGARITHMIC BRUNN-MINKOWSKI INEQUALITY FOR ZONOIDS

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Abstract. The aim of this note is to show that the local form of the logarithmic Brunn-Minkowski conjecture holds for zonoids. The proof uses a variant of the Bochner method due to Shenfeld and the author.

## 1. Introduction

1.1. The classical Brunn-Minkowski inequality states that

$$\mathrm{Vol}((1-t)K+tL)^{1/n} \geqslant (1-t)\,\mathrm{Vol}(K)^{1/n} + t\,\mathrm{Vol}(L)^{1/n} \tag{1.1}$$

for all $t \in [0,1]$ and convex bodies $K, L$ in $\mathbb{R}^n$, where

$$aK + bL := \{ax + by : x \in K, y \in L\}$$

denotes Minkowski addition. Its importance, both to convexity and to other areas of mathematics, can hardly be overstated; cf. [10]. As is well known, (1.1) is equivalent to the apparently weaker inequality

$$\mathrm{Vol}((1-t)K+tL) \geqslant \mathrm{Vol}(K)^{1-t}\,\mathrm{Vol}(L)^{t} \tag{1.2}$$

where the arithmetic mean on the right-hand side has been replaced by the geometric mean. Clearly (1.1) implies (1.2), as the geometric mean is smaller than the arithmetic mean; the converse implication follows by rescaling $K, L$ [10, §4].

As part of their study of the Minkowski problem for cone volume measures, Böröczky, Lutwak, Yang and Zhang [4] asked whether one could replace also the "arithmetic mean" $(1-t)K + tL$ on the left-hand side of the Brunn-Minkowski inequality by a certain kind of "geometric mean": that is, whether

$$\mathrm{Vol}(K^{1-t}L^{t}) \overset{?}{\geqslant} \mathrm{Vol}(K)^{1-t}\,\mathrm{Vol}(L)^{t}, \tag{1.3}$$

where the meaning of $K^{1-t}L^t$ must be carefully defined (see (1.7) below). As the geometric mean is smaller than the arithmetic mean, this would yield an improvement of the classical Brunn-Minkowski inequality. While such an improved inequality turns out to be false for general convex bodies, it was conjectured in [4] that such an improved inequality holds whenever $K, L$ are symmetric convex bodies (that is, $K = -K$ and $L = -L$), which they proved to be true in dimension 2. In higher dimensions, this *logarithmic Brunn-Minkowski conjecture* remains open.

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1.2. It is readily seen that the Brunn-Minkowski inequality (1.1) and the logarithmic Brunn-Minkowski conjecture (1.3) are equivalent to concavity of the functions

$$\varphi : t \mapsto \mathrm{Vol}((1-t)K + tL)^{1/n} \qquad \text{and} \qquad \psi : t \mapsto \log \mathrm{Vol}(K^{1-t}L^t)$$

for all convex bodies $K, L$ and symmetric convex bodies $K, L$ in $\mathbb{R}^n$, respectively. We can therefore obtain equivalent formulations of (1.1) and (1.3) by considering the first- and second-order conditions for concavity of $\varphi$ and $\psi$.

In order to formulate the resulting inequalities, we must first recall some additional notions (we refer to [21] for a detailed treatment). It was shown by Minkowski that the volume of convex bodies is a polynomial in the sense that for any convex bodies $K_1, \ldots, K_m$ in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m > 0$, we have

$$\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \mathsf{V}(K_{i_1}, \ldots, K_{i_n})\, \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $\mathsf{V}(K_1, \ldots, K_n)$, called *mixed volumes*, are nonnegative, symmetric in their arguments, and homogeneous and additive in each argument under Minkowski addition. Moreover, mixed volumes admit the integral representation

$$\mathsf{V}(K_1, \ldots, K_n) = \frac{1}{n} \int h_{K_1} dS_{K_2,\ldots,K_n}, \tag{1.4}$$

where the *mixed area measure* $S_{K_2,\ldots,K_n}$ is a finite measure on $S^{n-1}$ and $h_K(x) := \sup_{z\in K} \langle z, x\rangle$ denotes the support function of a convex body $K$.

In view of the above definitions, it is now straightforward to obtain equivalent formulations of the Brunn-Minkowski inequality in terms of mixed volumes; see, e.g., [21, pp. 381–382 and 406]. In the sequel, we denote by $\mathcal{K}^n$ ($\mathcal{K}^n_s$) the family of all (symmetric) convex bodies in $\mathbb{R}^n$ with nonempty interior.

**Lemma 1.1** (Minkowski)**.** *The following are equivalent:*

1. *For all $K, L \in \mathcal{K}^n$ and $t \in [0, 1]$, the Brunn-Minkowski inequality* (1.1) *holds.*
2. *For all $K \in \mathcal{K}^n$, we have*
$$\mathsf{V}(L, K, \ldots, K) \geqslant \mathrm{Vol}(L)^{1/n}\, \mathrm{Vol}(K)^{1-1/n} \quad \forall\, L \in \mathcal{K}^n. \tag{1.5}$$
3. *For all $K \in \mathcal{K}^n$, we have*
$$\mathsf{V}(L, K, \ldots, K)^2 \geqslant \mathsf{V}(L, L, K, \ldots, K)\, \mathrm{Vol}(K) \quad \forall\, L \in \mathcal{K}^n. \tag{1.6}$$

*Proof.* If we apply (1.5)–(1.6) with $K \leftarrow (1-s)K + sL$ and $L \leftarrow (1-r)K + rL$, then a simple computation shows that Minkowski's first inequality (1.5) is nothing other than the first-order concavity condition $\varphi(r) \leqslant \varphi(s) + \varphi'(s)(r-s)$, while Minkowski's second inequality (1.6) is the second-order condition $\varphi''(s) \leqslant 0$. □

Before we state an analogous reformulation of (1.3), we must first give a precise definition of $K^{1-t}L^t$. To motivate this definition, recall that the arithmetic mean of convex bodies is characterized by its support function $h_{(1-t)K+tL} = (1-t)h_K + th_L$. We may therefore attempt to define $K^{1-t}L^t$ as the convex body whose support function is the geometric mean $h_K^{1-t}h_L^t$. However, the latter need not be the support function of any convex body. We therefore define $K^{1-t}L^t$ in general as the largest convex body whose support function is dominated by $h_K^{1-t}h_L^t$, that is,

$$K^{1-t}L^t := \{z \in \mathbb{R}^n : \langle z, x\rangle \leqslant h_K(x)^{1-t} h_L(x)^t \text{ for all } x \in \mathbb{R}^n\}. \tag{1.7}$$

We can now formulate the following analogue of Lemma 1.1.

**Theorem 1.2** ([4, 8, 14, 7, 17, 15])**.** *The following are equivalent:*

1. *For all $K, L \in \mathcal{K}_s^n$ and $t \in [0,1]$, the log-Brunn-Minkowski inequality (1.3) holds.*
2. *For all $K \in \mathcal{K}_s^n$, we have*
$$\int h_K \log\left(\frac{h_L}{h_K}\right) dS_{K,\ldots,K} \geqslant \mathrm{Vol}(K)\log\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right) \quad \forall\, L \in \mathcal{K}_s^n. \tag{1.8}$$
3. *For all $K \in \mathcal{K}_s^n$, we have*
$$\frac{\mathsf{V}(L,K,\ldots,K)^2}{\mathrm{Vol}(K)} \geqslant \frac{n-1}{n}\,\mathsf{V}(L,L,K,\ldots,K) + \frac{1}{n^2}\int \frac{h_L^2}{h_K}\, dS_{K,\ldots,K} \quad \forall\, L \in \mathcal{K}_s^n. \tag{1.9}$$

The difficulty in the proof of Theorem 1.2 is that the map $t \mapsto K^{1-t}L^t$ can be nonsmooth: if it were the case that $h_{K^{1-t}L^t} = h_K^{1-t}h_L^t$ for all $t \in [0,1]$, the result would follow easily from the first- and second-order conditions for concavity of $\psi$. That the conclusion remains valid using the correct definition (1.7) is a nontrivial fact that has been established through the combined efforts of several groups.

*Remark* 1.3. The notation (1.7) is nonstandard: $K^{1-t}L^t$ is often denoted in the literature as $(1-t)K +_0 tL$, as it coincides with the $q \to 0$ limit of $L^q$-Minkowski addition. As the latter notation is somewhat confusing (the geometric mean is not defined by the rescaled bodies $(1-t)K$ and $tL$), and as only geometric means are used in this paper, we have chosen a nonstandard but more suggestive notation.

1.3. It was shown in [4] that the logarithmic Brunn-Minkowski conjecture holds in dimension $n = 2$. In dimensions $n \geqslant 3$, however, the conjecture has been proved to date only under special symmetry assumptions: when $K, L$ are complex [19] or unconditional [20] bodies (see also [3] for a generalization). In both cases the conjecture is established by replacing the geometric mean (1.7) by a smaller set whose construction requires the special symmetries, which yields strictly stronger inequalities than are conjectured for general bodies.

Even for a fixed reference body $K$, the validity of the inequalities (1.8) and (1.9) for *all* $L \in \mathcal{K}_s^n$ (i.e., in the absence of additional symmetries) appears to be unknown except in one very special family of examples: it follows from [14, 15] that (1.8) and (1.9) hold when $K$ is the $\ell_p^n$-ball with $2 \leqslant p < \infty$ and sufficiently large $n$, as well as for affine images and sufficiently small perturbations of these bodies. Note, however, that the analysis of these examples shows that they satisfy even stronger inequalities that cannot hold for general bodies (local $L^q$-Brunn-Minkowski inequalities with $q = -\frac{1}{4}$ [14, Theorem 10.4]), so that they do not approach the extreme cases of the logarithmic Brunn-Minkowski conjecture.[1]

The aim of this note is to contribute some further evidence toward the validity of the logarithmic Brunn-Minkowski conjecture. Recall that a convex body $K \in \mathcal{K}_s^n$ is called a *zonoid* if it is the limit of Minkowski sums of segments. The first main result of this note is the following theorem.

**Theorem 1.4.** *Let $K \in \mathcal{K}_s^n$ be a zonoid. Then the local logarithmic Brunn-Minkowski inequality (1.9) holds for all $L \in \mathcal{K}_s^n$.*

Our second main result settles the equality cases of Theorem 1.4.

[1] For one extreme case, the $\ell_\infty^n$-ball, the validity of (1.9) may be verified by an explicit computation, see, e.g., [14, Theorem 10.2]. This does not follow as a limiting case of the general result [14, Theorem 10.4] on $\ell_p^n$-balls, however, as the latter only holds for $n \geqslant n_0(p) \to \infty$ as $p \to \infty$.

**Definition.** A vector $u \in S^{n-1}$ is called an *$r$-extreme normal vector* of a convex body $K$ if there do not exist linearly independent normal vectors $u_1, \ldots, u_{r+2}$ at a boundary point of $K$ such that $u = u_1 + \cdots + u_{r+2}$.

**Theorem 1.5.** *Let $K \in \mathcal{K}_s^n$ be a zoonoid. Then equality holds in (1.9) if and only if*

1. *$K = C_1 + \cdots + C_m$ for some $1 \leqslant m \leqslant n$ and zonoids $C_1, \ldots, C_m$ such that $\dim(C_1) + \cdots + \dim(C_m) = n$; and*
2. *there exist $a_1, \ldots, a_m \geqslant 0$ such that $L$ and $a_1C_1 + \cdots + a_mC_m$ have the same supporting hyperplanes in all 1-extreme normal directions of $K$.*

Theorem 1.4 does not suffice to conclude that the logarithmic Brunn-Minkowski inequality (1.3) holds when $K, L$ are zonoids, as $K^{1-t}L^t$ is generally not a zonoid. Nonetheless, by combining Theorems 1.4–1.5 with [15, Theorem 2.1] we can deduce validity of the logarithmic Minkowski inequality (1.8), albeit without its equality cases. Some further implications will be given in section 5.

**Corollary 1.6.** *Let $K \in \mathcal{K}_s^n$ be a zonoid. Then the logarithmic Minkowski inequality (1.8) holds for all $L \in \mathcal{K}_s^n$.*

It appears somewhat unlikely that our results make major progress in themselves toward the full resolution of the logarithmic Brunn-Minkowski conjecture; as is the case for other well-known conjectures in convex geometry (see, e.g., [11]), zonoids form a very special class of convex bodies that provide only modest insight into the behavior of general convex bodies. Nonetheless, let us highlight several interesting features of the main results of this note:

- Theorem 1.4 possesses many nontrivial equality cases; therefore, in contrast to the setting of previous results in dimensions $n \geqslant 3$ for general $L \in \mathcal{K}_s^n$, the class of zonoids includes many extreme cases of the logarithmic Brunn-Minkowski conjecture. (Theorem 1.5 supports the conjectured equality cases in [3].)
- Unlike in dimensions $n \geqslant 3$, every planar symmetric convex body is a zonoid. The $n = 2$ case of the logarithmic Brunn-Minkowski conjecture that was settled in [4] may therefore be viewed in a new light as a special case of our results (modulo the nontrivial Theorem 1.2). In fact, the proof of Theorem 1.4 will work in a completely analogous manner for $n = 2$ and $n \geqslant 3$.
- The $\ell_p^n$-ball is a zonoid for every $n$ and $2 \leqslant p \leqslant \infty$ [2, Theorem 6.6]. Our results therefore capture as special cases all explicit examples of convex bodies $K$ for which (1.8) and (1.9) were previously known to hold.[2]

Before we proceed, let us briefly sketch some key ideas behind the proofs.

1.4. It was a fundamental insight of Hilbert [12, Chapter XIX] that mixed volumes of sufficiently smooth convex bodies admit a spectral interpretation. To this end, given any sufficiently smooth convex body $K \in \mathcal{K}^n$, Hilbert constructs an elliptic differential operator $\mathscr{A}_K$ (see section 2.2 for a precise definition) and a measure $d\mu_K := \frac{1}{nh_K} dS_{K,\ldots,K}$ on $S^{n-1}$ with the following properties:

[2]However, the methods of [14, 15] provide complementary information that does not follow from our results. For example, the estimates of [14] imply that for any $2 < p < \infty$ and $n \geqslant n_0(p)$, all $K \in \mathcal{K}_s^n$ that are sufficiently close to the $\ell_p^n$-ball in a quantitative sense satisfy the $L^q$-Minkowski inequality with $q = -\frac{1}{4}$. More generally, it is shown in [15] that for any $K \in \mathcal{K}_s^n$, there exists $K' \in \mathcal{K}_s^n$ with $K \subseteq K' \subseteq 8K$ so that $K'$ satisfies the $L^q$-Minkowski inequality with $q = -\frac{1}{4}$.

- $\mathscr{A}_K$ defines a self-adjoint operator on $L^2(\mu_K)$ with discrete spectrum.
- $\mathscr{A}_K h_K = h_K$, that is, $h_K$ is an eigenfunction with eigenvalue 1.
- $\mathsf{V}(L, M, K, \ldots, K) = \langle h_L, \mathscr{A}_K h_M\rangle$ for all $L, M \in \mathcal{K}^n$.

Using these properties, it is readily verified that (1.6) is equivalent to

$$\langle f, \mathscr{A}_K f\rangle \leqslant 0 \quad \text{for} \quad f = h_L - \frac{\langle h_L, h_K\rangle}{\|h_K\|^2} h_K, \tag{1.10}$$

where we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm of $L^2(\mu_K)$. As $f$ in (1.10) is the projection of $h_L$ on $\{h_K\}^\perp$, we obtain:

**Lemma 1.7** (Hilbert)**.** *Items 1–3 of Lemma 1.1 are equivalent to:*

4. *For every sufficiently smooth convex body $K \in \mathcal{K}^n$, any eigenfunction $\mathscr{A}_K f = \lambda f$ with $\langle f, h_K\rangle = 0$ has eigenvalue $\lambda \leqslant 0$.*

The condition of Lemma 1.7 is optimal, as $\mathscr{A}_K$ always has eigenfunctions with eigenvalue 0: any linear function $\ell(x) = h_{\{v\}}(x) = \langle v, x\rangle$ satisfies $\mathscr{A}_K \ell = 0$. If we restrict attention to *symmetric* convex bodies $K, L \in \mathcal{K}^n_s$, however, only *even* functions $f(x) = f(-x)$ arise in (1.10), and it is certainly possible that all even eigenfunctions of $\mathscr{A}_K$ have strictly negative eigenvalues. It was observed by Kolesnikov and Milman [14] that the logarithmic Brunn-Minkowski conjecture may be viewed as a quantitative form of this phenomenon: as (1.9) is equivalent to

$$\langle f, \mathscr{A}_K f\rangle \leqslant -\frac{1}{n-1}\|f\|^2 \quad \text{for} \quad f = h_L - \frac{\langle h_L, h_K\rangle}{\|h_K\|^2} h_K,$$

the following conclusion follows readily.

**Lemma 1.8** (Kolesnikov-Milman)**.** *Items 1–3 of Theorem 1.2 are equivalent to:*

4. *For every sufficiently smooth symmetric convex body $K \in \mathcal{K}^n_s$, any* even *eigenfunction $\mathscr{A}_K f = \lambda f$ with $\langle f, h_K\rangle = 0$ has eigenvalue $\lambda \leqslant -\frac{1}{n-1}$.*

It is verified in [14, Theorem 10.4] that when $K$ is the $\ell^n_p$-ball for $2 \leqslant p < \infty$, any even eigenfunction of $\mathscr{A}_K$ orthogonal to $h_K$ has eigenvalue $\lambda$ with $n\lambda \to -\infty$ as $n \to \infty$. This shows that such $K$ satisfy the condition of Lemma 1.8 for large $n$, but does not explain the significance of the threshold $-\frac{1}{n-1}$.

A new approach to the study of the spectral properties of $\mathscr{A}_K$ was discovered by Shenfeld and the author in [22]. This approach, called the *Bochner method* in view of its analogy to the classical Bochner method in differential geometry, has found several surprising applications both inside and outside convex geometry. The Bochner method was already used in [22] to provide new proofs of the Alexandrov-Fenchel inequality, a much deeper result of which (1.6) is a special case, and of the Alexandrov mixed discriminant inequality. Subsequent applications outside convexity include the proof of certain properties of Lorentzian polynomials in [9, 5] and the striking results of [6], where the method is used to prove numerous combinatorial inequalities. The paper [15] contains another application to the study of isomorphic variants of the $L^q$-Minkowski problem.

The proofs of Theorems 1.4–1.5 provide yet another illustration of the utility of the Bochner method. By using a variation on the method of [22], we obtain a "Bochner identity" which relates the spectral condition of Lemma 1.8 in dimension $n$ to the inequality (1.9) in dimension $n-1$. The conclusion then follows by induction on the dimension. One interesting feature of this proof is that it provides an explanation for the appearance of the mysterious value $-\frac{1}{n-1}$ in Lemma 1.8.[3] While the specific formulas derived in this note rely on the zonoid assumption, our approach may provide some hope that other variations on the Bochner method could lead to further progress toward the logarithmic Brunn-Minkowski conjecture.

1.5. The rest of this note is organized as follows. In section 2, we briefly recall some background from convex geometry that will be needed in the proofs, and we recall the basic idea behind the Bochner method as developed in [22]. Theorem 1.4 is proved in section 3, and Theorem 1.5 is proved in section 4. Finally, section 5 spells out some implications of Theorem 1.4, including the proof of Corollary 1.6.

## 2. Preliminaries

Throughout this note, we will use without comment the standard properties of mixed volumes and mixed area measures: that they are nonnegative, symmetric and multilinear in their arguments, and continuous under Hausdorff convergence. We refer to the monograph [21] for a detailed treatment, or to [22, §2], [23, §4] for a brief review of such basic properties. The aim of this section is to recall some further notions that will play a central role in the sequel: the behavior of mixed volumes under projections, the construction of the Hilbert operator $\mathscr{A}_K$ for sufficiently smooth convex bodies, and the Bochner method of [22].

The following notation will often be used: if $f = h_K - h_L$ is a difference of support functions of convex bodies, then we define [21, §5.2]

$$\mathsf{V}(f, C_1, \ldots, C_{n-1}) := \mathsf{V}(K, C_1, \ldots, C_{n-1}) - \mathsf{V}(L, C_1, \ldots, C_{n-1}),$$
$$S_{f,C_1,\ldots,C_{n-2}} := S_{K,C_1,\ldots,C_{n-2}} - S_{L,C_1,\ldots,C_{n-2}}.$$

We similarly define $\mathsf{V}(f, g, C_1, \ldots, C_{n-2})$ by linearity when $f, g$ are differences of support functions, etc. Mixed volumes and area measures of differences of support functions are still symmetric and multilinear, but need not be nonnegative.

2.1. **Projections and zonoids.** Let $E \subseteq \mathbb{R}^n$ be a linear subspace of dimension $k$, and let $C_1, \ldots, C_k$ be convex bodies in $E$. Then we denote by $\mathsf{V}(C_1, \ldots, C_k)$ and $S_{C_1,\ldots,C_{k-1}}$ the mixed volume and mixed area measure computed in $E \simeq \mathbb{R}^k$. We will often view $S_{C_1,\ldots,C_{k-1}}$ as a measure on $\mathbb{R}^n$ that is supported in $E$. The projection of a convex body $C$ in $\mathbb{R}^n$ onto $E$ will be denoted as $\mathsf{P}_E C$.

The following basic formulas relate mixed volumes and mixed area measures of convex bodies to those of their projections.

**Lemma 2.1.** *For any $u \in S^{n-1}$ and $C_1, \ldots, C_{n-1} \in \mathcal{K}^n$, we have*

$$\frac{n}{2}\mathsf{V}([-u, u], C_1, \ldots, C_{n-1}) = \mathsf{V}(\mathsf{P}_{u^\perp} C_1, \ldots, \mathsf{P}_{u^\perp} C_{n-1}),$$
$$\frac{n-1}{2} S_{[-u,u],C_1,\ldots,C_{n-2}} = S_{\mathsf{P}_{u^\perp} C_1,\ldots,\mathsf{P}_{u^\perp} C_{n-2}}.$$

[3]As was pointed out in [14], the eigenvalue $-\frac{1}{n-1}$ is attained when $K$ is the cube, so that Lemma 1.8 may be interpreted as stating that the second eigenvalue of $\mathscr{A}_K$ is maximized by the cube. This interpretation does not explain, however, *why* this should be the case. In any case, there are many maximizers other than cubes, as is already illustrated by Theorem 1.5.

*Proof.* The first identity is [21, (5.77)]. To prove the second identity, note that the first identity may be rewritten using (1.4) as

$$\frac{1}{2}\int f\,dS_{[-u,u],C_1,\dots,C_{n-2}} = \frac{1}{n-1}\int f\,dS_{\mathsf{P}_{u^\perp}C_1,\dots,\mathsf{P}_{u^\perp}C_{n-2}}$$

for $f = h_{C_{n-1}}$, where we used that $h_{\mathsf{P}_E C}(u) = h_C(u)$ for $u \in E$. The identity extends by linearity to any difference of support functions $f = h_K - h_L$. But as any $f \in C^2(S^{n-1})$ is of this form (cf. Lemma 2.4 below), the conclusion follows. □

A body $K \in \mathcal{K}^n_s$ is called a *zonoid* if it is the limit of Minkowski sums of segments $[-u,u]$. The following equivalent definition [21, Theorem 3.5.3] is well known.

**Definition 2.2.** $K \in \mathcal{K}^n_s$ is called a *zonoid* if

$$h_K(x) = \int h_{[-u,u]}(x)\,\eta(du)$$

for some finite even measure $\eta$ on $S^{n-1}$, called the *generating measure* of $K$.

The significance of zonoids for our purposes is that mixed volumes of zonoids can be expressed in terms of mixed volumes of projections by Lemma 2.1 and linearity. One simple illustration of this is the following fact.

**Lemma 2.3.** *In dimension 2, any symmetric convex body $K \in \mathcal{K}^2_s$ is a zonoid with*

$$h_K(x) = \frac{1}{4}\int h_{[-u^\dagger,u^\dagger]}(x)\,S_K(du),$$

*where for $u \in S^1$ we denote by $u^\dagger \in S^1$ the clockwise rotation of $u$ by the angle $\frac{\pi}{2}$.*

*Proof.* Note first that $h_{[-u^\dagger,u^\dagger]}(x) = |\langle u^\dagger, x\rangle| = |\langle u, x^\dagger\rangle| = h_{[-x^\dagger,x^\dagger]}(u)$. Thus by Lemma 2.1, we can write for any $K \in \mathcal{K}^2$

$$\frac{1}{2}\int h_{[-u^\dagger,u^\dagger]}(x)\,S_K(du) = \mathsf{V}([-x^\dagger,x^\dagger],K) = \mathrm{Vol}(\mathsf{P}_{\mathrm{span}\{x\}}K) = h_K(x) + h_K(-x).$$

As $K \in \mathcal{K}^2_s$ is symmetric, we have $h_K(x) + h_K(-x) = 2h_K(x)$. □

2.2. **Smooth bodies and the Hilbert operator.** A support function $h_K$ may be viewed either as a function on $S^{n-1}$, or as a 1-homogeneous function on $\mathbb{R}^n$. In particular, if $h_K$ is a $C^2$ function on $S^{n-1}$, then its gradient $\nabla h_K$ in $\mathbb{R}^n$ is 0-homogeneous, and thus its Hessian $\nabla^2 h_K(x)$ in $\mathbb{R}^n$ is a linear map from $x^\perp$ to itself. We denote the restriction of $\nabla^2 h_K(x)$ to $x^\perp$ as $D^2 h_K(x)$. For a general function $f \in C^2(S^{n-1})$, the restricted Hessian $D^2 f$ is defined analogously by applying the above construction to the 1-homogeneous extension of $f$.

We now recall the following basic facts [22, §2.1]. Here we write $A \geqslant 0$ ($A > 0$) to indicate that a symmetric matrix $A$ is positive semidefinite (positive definite).

**Lemma 2.4.** *Let $f \in C^2(S^{n-1})$. Then the following hold:*

a. *$f = h_K$ for some convex body $K$ if and only if $D^2 f \geqslant 0$.*
b. *For any convex body $L$ such that $h_L \in C^2(S^{n-1})$ and $D^2 h_L > 0$, there is a convex body $K$ and $a > 0$ so that $f = a(h_K - h_L)$.*

A particularly useful class of bodies is the following.

**Definition 2.5.** $K \in \mathcal{K}^n$ is of class $C^k_+$ ($k \geqslant 2$) if $h_K \in C^k(S^{n-1})$ and $D^2 h_K > 0$.

For our purposes, the importance of $C^k_+$ bodies is that they admit certain explicit representations of mixed volumes and mixed area measures. To define these, let us first recall that the mixed discriminant $\mathsf{D}(A_1,\ldots,A_{n-1})$ of $(n-1)$-dimensional matrices $A_1,\ldots,A_{n-1}$ is defined by the formula

$$\det(\lambda_1 A_1 + \cdots + \lambda_m A_m) = \sum_{i_1,\ldots,i_{n-1}=1}^{m} \mathsf{D}(A_{i_1},\ldots,A_{i_{n-1}})\,\lambda_{i_1}\cdots\lambda_{i_{n-1}}$$

in analogy with the definition of mixed volumes. Mixed discriminants are symmetric and multilinear in their arguments, and $\mathsf{D}(A_1,\ldots,A_{n-1}) > 0$ for $A_1,\ldots,A_{n-1} > 0$. Moreover, we have the Alexandrov mixed discriminant inequality

$$\mathsf{D}(A,B,M_1,\ldots,M_{n-3})^2 \geqslant \mathsf{D}(A,A,M_1,\ldots,M_{n-3})\,\mathsf{D}(B,B,M_1,\ldots,M_{n-3}) \tag{2.1}$$

whenever $B,M_1,\ldots,M_{n-3} \geqslant 0$ and $A$ is a symmetric matrix. For these and other facts about mixed discriminants, see [22, §2.3 and §4].

With these definitions in place, we have the following [23, Lemma 4.7].

**Lemma 2.6.** *Let $C_1,\ldots,C_{n-1} \in \mathcal{K}^n$ be of class $C^2_+$. Then*

$$dS_{C_1,\ldots,C_{n-1}} = \mathsf{D}(D^2 h_{C_1},\ldots,D^2 h_{C_{n-1}})\,d\omega,$$

$$\mathsf{V}(K,C_1,\ldots,C_{n-1}) = \frac{1}{n}\int h_K\,\mathsf{D}(D^2 h_{C_1},\ldots,D^2 h_{C_{n-1}})\,d\omega$$

*for any convex body $K$, where $\omega$ denotes the surface measure on $S^{n-1}$.*

We now introduce the spectral interpretation of mixed volumes due to Hilbert. For simplicity, we will only consider the special case that is needed in this note; the same construction applies to general mixed volumes (cf. [22]).

Fix a body $K \in \mathcal{K}^n$ of class $C^2_+$ with the origin in its interior (so that $h_K > 0$). Then we define a measure $\mu_K$ on $S^{n-1}$ as

$$d\mu_K := \frac{1}{nh_K} dS_{K,\ldots,K} = \frac{1}{nh_K}\mathsf{D}(D^2 h_K,\ldots,D^2 h_K)\,d\omega,$$

and define the second-order differential operator $\mathscr{A}_K$ on $S^{n-1}$ as

$$\mathscr{A}_K f := h_K \frac{\mathsf{D}(D^2 f, D^2 h_K,\ldots,D^2 h_K)}{\mathsf{D}(D^2 h_K,\ldots,D^2 h_K)}$$

for $f \in C^2(S^{n-1})$. The positivity of mixed discriminants of positive definite matrices implies that $\mathscr{A}_K$ is elliptic. Standard facts of elliptic regularity theory therefore imply the following, cf. [22, §3] or [14, Theorem 5.3]:

- $\mathscr{A}_K$ extends to a self-adjoint operator on $L^2(\mu_K)$ with $\mathrm{Dom}(\mathscr{A}_K) = H^2(S^{n-1})$.
- $\mathscr{A}_K$ has a discrete spectrum, that is, it has a countable sequence of eigenvalues $\lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \cdots$ of tending to $-\infty$, and its eigenfunctions span $L^2(\mu_K)$.
- $\lambda_1 = 1$ is a simple eigenvalue, whose eigenspace is spanned by $h_K$.

These facts will be invoked in the sequel without further comment.

The point of this construction is that, by Lemmas 2.4 and 2.6, we evidently have

$$\langle f, \mathscr{A}_K g\rangle := \int f\,\mathscr{A}_K g\,d\mu_K = \mathsf{V}(f,g,K,\ldots,K)$$

for any $f,g \in C^2(S^{n-1})$. Mixed volumes of this type may therefore be viewed as quadratic forms of the operator $\mathscr{A}_K$, which furnishes various geometric inequalities with a spectral interpretation as explained in section 1.4.

When $K \in \mathcal{K}_s^n$ is symmetric, it is readily verified from the definitions that $\mu_K$ is an even measure, and that $\mathscr{A}_K$ leaves the spaces $L^2(\mu_K)_{\mathrm{even}}$ and $L^2(\mu_K)_{\mathrm{odd}}$ of even and odd functions on $S^{n-1}$ invariant. As $L^2(\mu_K) = L^2(\mu_K)_{\mathrm{even}} \oplus L^2(\mu_K)_{\mathrm{odd}}$, it follows that when $K \in \mathcal{K}_s^n$ any $f \in L^2(\mu_K)_{\mathrm{even}}$ can be expressed as a linear combination of the *even* eigenfunctions of $\mathscr{A}_K$; cf. [14, §5.1].

2.3. **The Bochner method.** As was explained in section 1.4, Minkowski's second inequality (1.6), and thus the Brunn-Minkowski inequality, is equivalent to the statement that the second largest eigenvalue of $\mathscr{A}_K$ satisfies $\lambda_2 \leqslant 0$ (cf. Lemma 1.7). This idea was exploited by Hilbert to give a spectral proof of the Brunn-Minkowski inequality by means of an eigenvalue continuity argument.

A new proof of the above spectral condition was discovered by Shenfeld and the author in [22]. This proof is based on the elementary fact that the condition $\lambda_2 \leqslant 0$ would follow directly from the Lichnerowicz condition

$$\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle \geqslant \langle f, \mathscr{A}_K f\rangle \quad \text{for all } f. \tag{2.2}$$

Indeed, if $\mathscr{A}_K f = \lambda f$, then (2.2) yields $\lambda^2 \geqslant \lambda$, i.e., $\lambda \geqslant 1$ or $\lambda \leqslant 0$. As $1 = \lambda_1 > \lambda_2$ by elliptic regularity theory, the conclusion $\lambda_2 \leqslant 0$ follows. While this is merely a reformulation of the problem, the beauty of (2.2) is that it is an immediate consequence of the following identity that admits a one-line proof.

**Lemma 2.7** (Bochner identity)**.** *Let $K \in \mathcal{K}^n$ be of class $C^2_+$ and let $f \in C^2$. Then*

$$\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle - \langle f, \mathscr{A}_K f\rangle = \int \frac{h_K}{n}\left\{\frac{\mathsf{D}(D^2 f, D^2 h_K, \ldots, D^2 h_K)^2}{\mathsf{D}(D^2 h_K, \ldots, D^2 h_K)} - \mathsf{D}(D^2 f, D^2 f, D^2 h_K, \ldots, D^2 h_K)\right\} d\omega. \tag{2.3}$$

*Proof.* The identity is immediate from the definitions of $\mathscr{A}_K$ and $\mu_K$, and as

$$\langle f, \mathscr{A}_K f\rangle = \mathsf{V}(K, f, f, K, \ldots, K) = \frac{1}{n}\int h_K\, \mathsf{D}(D^2 f, D^2 f, D^2 h_K, \ldots, D^2 h_K)\, d\omega$$

by Lemma 2.6 and as mixed volumes are symmetric in their arguments. □

To deduce (2.2), it remains to recall that the integrand in (2.3) is nonnegative by the following special case of the mixed discriminant inequality (2.1):

$$\mathsf{D}(A, B, \ldots, B)^2 \geqslant \mathsf{D}(A, A, B, \ldots, B)\, \mathsf{D}(B, \ldots, B). \tag{2.4}$$

In other words, the Bochner method reduces Minkowski's second inequality (1.6) to its linear-algebraic counterpart (2.4). This interpretation of the Bochner method will form the starting point for the main results of this note.

*Remark* 2.8. Lemma 2.7 is a trivial reformulation of the proof of [22, Lemma 3.1], as is explained in [22, §6.3]. Moreover, it is observed there that in the special case that $K$ is the Euclidean ball, (2.3) is precisely the classical (integrated) Bochner formula on $S^{n-1}$. One may therefore naturally view the above approach as an analogue of the Bochner method of differential geometry.

The identity (2.3) was recently rediscovered by Milman [15]. A new insight of [15] is that (2.3) may in fact be viewed as a true Bochner formula in the sense of differential geometry for any body $K$ of class $C^2_+$, by introducing a special centro-affine connection on $\partial K$. This interpretation does not appear to extend, however, to more general situations: for example, neither the more general identity that was used in [22] to prove the Alexandrov-Fenchel inequality, nor the "Bochner identities" of this note, are true Bochner formulas in the strictly formal sense, but should rather be viewed as a loose analogues of such a formula. The merits of taking a more liberal view on the Bochner method are illustrated by its diverse applications not only in convexity, but also in algebra and combinatorics [22, 9, 5, 6].

*Remark* 2.9. The Bochner method should not be confused with a different method to prove Brunn-Minkowski inequalities that was developed by Reilly [18] and considerably refined by Kolesnikov and Milman in [13, 14]. The basis for Reilly's method is an integrated form of the classical Bochner formula on $\mathbb{R}^n$ (or on a manifold), combined with the solution of a certain Neumann problem. This method appears to be unrelated to the Bochner method for the operator $\mathscr{A}_K$.

## 3. Proof of Theorem 1.4

The main step in the proof of Theorem 1.4 is the following analogue of (2.2).

**Theorem 3.1.** *Let $K \in \mathcal{K}^n_s$ be a zonoid of class $C^2_+$ and $f \in C^2(S^{n-1})_{\mathrm{even}}$. Then*

$$\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle \geqslant \frac{n-2}{n-1}\langle f, \mathscr{A}_K f\rangle + \frac{1}{n-1}\langle f, f\rangle. \tag{3.1}$$

Before we proceed, let us complete the proof of Theorem 1.4.

*Proof of Theorem 1.4.* Let $K \in \mathcal{K}^n_s$ be a zonoid of class $C^2_+$. As $\mathscr{A}_K$ is essentially self-adjoint on $C^2(S^{n-1})$, (3.1) extends directly to any even function $f \in \mathrm{Dom}(\mathscr{A}_K)$. Thus if $f$ is any even eigenfunction of $\mathscr{A}_K$ with eigenvalue $\lambda$, Theorem 3.1 yields

$$\lambda^2 \geqslant \frac{n-2}{n-1}\,\lambda + \frac{1}{n-1},$$

i.e., $\lambda \geqslant 1$ or $\lambda \leqslant -\frac{1}{n-1}$. But recall that the largest eigenvalue of $\mathscr{A}_K$ is $\lambda_1 = 1$ and its eigenspace is spanned by $h_K$. Thus any even eigenfunction $f$ of $\mathscr{A}_K$ that is orthogonal to $h_K$ must have eigenvalue $\lambda \leqslant -\frac{1}{n-1}$. In particular, as any $f \in L^2(\mu_K)_{\mathrm{even}}$ is in the linear span of the even eigenfunctions of $\mathscr{A}_K$, we obtain

$$\langle f, \mathscr{A}_K f\rangle \leqslant -\frac{1}{n-1}\langle f, f\rangle \quad \text{whenever } f \in C^2(S^{n-1})_{\mathrm{even}},\ \langle f, h_K\rangle = 0.$$

For any $L \in \mathcal{K}^n_s$ of class $C^2_+$, we may now choose $f = h_L - \frac{\langle h_L, h_K\rangle}{\langle h_K, h_K\rangle} h_K$ and use

$$\langle h_L, \mathscr{A}_K h_L\rangle = \mathsf{V}(L, L, K, \ldots, K), \qquad \langle h_L, h_L\rangle = \frac{1}{n}\int \frac{h_L^2}{h_K}\, dS_{K,\ldots,K},$$
$$\langle h_L, \mathscr{A}_K h_K\rangle = \langle h_L, h_K\rangle = \mathsf{V}(L, K, \ldots, K)$$

to conclude the validity of (1.9) when $K, L$ are of class $C^2_+$.

To conclude the proof, it suffices to show that for any $K, L \in \mathcal{K}^n_s$ such that $K$ is a zonoid, there exist $K_n, L_n \in \mathcal{K}^n_s$ of class $C^2_+$ such that $K_n$ is a zonoid and $K_n \to K$, $L_n \to L$ in the Hausdorff metric; the validity of (1.9) then follows by the continuity of mixed volumes and area measures. Both statements are classical; an approximation of $L$ by $C^2_+$ bodies is given in [21, §3.4], while the approximation of $K$ may be performed, for example, by choosing $h_{K_n} = \int h_{E_{n,u}}\,\eta(du)$ where $\eta$ is the generating measure of $K$ and $E_{n,u}$ are ellipsoids such that $E_{n,u} \to [-u, u]$. □

The remainder of this section is devoted to the proof of Theorem 3.1. In essence, the inequality (3.1) will follow from a "Bochner identity" in the spirit of (2.3). However, rather than reducing the validity of (1.9) to a linear algebraic analogue as was done in section 2.3, the Bochner method will be used here to reduce (1.9) in dimension $n$ to its validity in dimension $n-1$. The conclusion then follows by induction. As will be explained below, the structure of the induction also provides an explanation for the appearance of the mysterious value $-\frac{1}{n-1}$ in Lemma 1.8.

### 3.1. **The induction step.** 

We begin with the following observation.

**Lemma 3.2.** *Let $n \geqslant 3$, $K \in \mathcal{K}_s^n$ be a zonoid, and $f = h_M - h_{M'}$ for $M, M' \in \mathcal{K}_s^n$. Assume that Theorem 1.4 has been proved in dimension $n-1$. Then*

$$\frac{\mathsf{V}([-u,u],f,K,\ldots,K)^2}{\mathsf{V}([-u,u],K,\ldots,K)} \geqslant \frac{n-2}{n-1}\,\mathsf{V}([-u,u],f,f,K,\ldots,K) + \frac{1}{n(n-1)}\int \frac{f^2}{h_K}\,dS_{[-u,u],K,\ldots,K}$$

*for every $u \in S^{n-1}$.*

*Proof.* Assume first that $K$ is a zonoid of class $C_+^2$ and that $f \in C^2(S^{n-1})_{\mathrm{even}}$. Then by Lemma 2.4, there exists a convex body $L \in \mathcal{K}_s^n$ of class $C_+^2$ and $a > 0$ such that $f = a(h_L - h_K)$. By expanding the squares, the inequality in the statement is readily seen to be equivalent to the inequality

$$\frac{\mathsf{V}([-u,u],L,K,\ldots,K)^2}{\mathsf{V}([-u,u],K,\ldots,K)} \geqslant \frac{n-2}{n-1}\,\mathsf{V}([-u,u],L,L,K,\ldots,K) + \frac{1}{n(n-1)}\int \frac{h_L^2}{h_K}\,dS_{[-u,u],K,\ldots,K}.$$

By Lemma 2.1, this is further equivalent to

$$\frac{\mathsf{V}(\mathbf{P}_{u^\perp}L,\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K)^2}{\mathsf{V}(\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K)} \geqslant \frac{n-2}{n-1}\,\mathsf{V}(\mathbf{P}_{u^\perp}L,\mathbf{P}_{u^\perp}L,\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K) + \frac{1}{(n-1)^2}\int \frac{h_{\mathbf{P}_{u^\perp}L}^2}{h_{\mathbf{P}_{u^\perp}K}}\,dS_{\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K},$$

where we used that $h_{\mathbf{P}_{u^\perp}L}(x) = h_L(x)$ for $x \in u^\perp$. But as $\mathbf{P}_{u^\perp}K$ is a zonoid, the latter inequality follows immediately from Theorem 1.4 in dimension $n-1$. It remains to extend the conclusion to general $K$ and $f = h_M - h_{M'}$ by approximating $K, M, M'$ by $C_+^2$ bodies as in the proof of Theorem 1.4. $\square$

We are now ready to perform the induction step in the proof of Theorem 3.1.

**Proposition 3.3.** *Let $n \geqslant 3$, and assume that Theorem 1.4 has been proved in dimension $n-1$. Then the conclusion of Theorem 3.1 holds in dimension $n$.*

*Proof.* Let $n \geqslant 3$, $f \in C^2(S^{n-1})_{\mathrm{even}}$, and $K \in \mathcal{K}_s^n$ be a zonoid of class $C_+^2$ with generating measure $\eta$. We may write

$$\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle = \frac{1}{n}\int\int h_{[-u,u]}\frac{\mathsf{D}(D^2 f, D^2 h_K,\ldots,D^2 h_K)^2}{\mathsf{D}(D^2 h_K,\ldots,D^2 h_K)}\,d\omega\,\eta(du)$$

by the definitions of $\mathscr{A}_K, \mu_K$ and as $h_K = \int h_{[-u,u]}\,\eta(du)$. Now note that

$$\frac{1}{n}\int h_{[-u,u]}\frac{\mathsf{D}(D^2 f, D^2 h_K,\ldots,D^2 h_K)^2}{\mathsf{D}(D^2 h_K,\ldots,D^2 h_K)}\,d\omega \geqslant \frac{\big(\frac{1}{n}\int h_{[-u,u]}\mathsf{D}(D^2 f, D^2 h_K,\ldots,D^2 h_K)\,d\omega\big)^2}{\frac{1}{n}\int h_{[-u,u]}\mathsf{D}(D^2 h_K,\ldots,D^2 h_K)\,d\omega} = \frac{\mathsf{V}([-u,u],f,K,\ldots,K)^2}{\mathsf{V}([-u,u],K,\ldots,K)}$$

for any $u$ by Cauchy-Schwarz and Lemma 2.6. We therefore obtain

$$\begin{aligned}\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle &\geqslant \int \frac{\mathsf{V}([-u,u],f,K,\ldots,K)^2}{\mathsf{V}([-u,u],K,\ldots,K)}\,\eta(du)\\ &\geqslant \int\left(\frac{n-2}{n-1}\mathsf{V}([-u,u],f,f,K,\ldots,K) + \frac{1}{n(n-1)}\int \frac{f^2}{h_K}\,dS_{[-u,u],K,\ldots,K}\right)\eta(du)\\ &= \frac{n-2}{n-1}\langle f, \mathscr{A}_K f\rangle + \frac{1}{n-1}\langle f,f\rangle\end{aligned}$$

using Lemma 3.2 and $h_K = \int h_{[-u,u]}\,\eta(du)$. □

*Remark* 3.4. While we find it cleaner to formulate the proof of Proposition 3.3 in terms of inequalities, one may in principle interpret this proof as arising from a Bochner *identity* in the spirit of (2.3): indeed, combining the proofs of Lemma 3.2 and Proposition 3.3 yields for $f = a(h_L - h_K)$

$$\begin{aligned}&\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle - \frac{n-2}{n-1}\langle f, \mathscr{A}_K f\rangle - \frac{1}{n-1}\langle f,f\rangle = \\ &\int\!\!\int \frac{h_{[-u,u]}}{h_K}\left(\mathscr{A}_K f - \frac{\mathsf{V}([-u,u],f,K,\ldots,K)}{\mathsf{V}([-u,u],K,\ldots,K)}h_K\right)^2 d\mu_K\,\eta(du) + \\ &\frac{2a^2}{n}\int\Bigg(\frac{\mathsf{V}(\mathbf{P}_{u^\perp}L,\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K)^2}{\mathsf{V}(\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K)} - \frac{n-2}{n-1}\mathsf{V}(\mathbf{P}_{u^\perp}L,\mathbf{P}_{u^\perp}L,\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K)\\ &\qquad - \frac{1}{(n-1)^2}\int \frac{h^2_{\mathbf{P}_{u^\perp}L}}{h_{\mathbf{P}_{u^\perp}K}}\,dS_{\mathbf{P}_{u^\perp}K,\ldots,\mathbf{P}_{u^\perp}K}\Bigg)\eta(du),\end{aligned}$$

where the two terms on the right-hand side are the deficits of the two inequalities used in the proof (the Cauchy-Schwarz inequality and (1.9) in dimension $n-1$, respectively). While it would be difficult to recognize this identity as a Bochner formula in the sense of differential geometry, it plays precisely the same role in the present proof as the Bochner identity (2.3) in section 2.3.

Let us further note that an even eigenfunction $\mathscr{A}_K f = \lambda f$ yields equality in (3.1) if and only if $\lambda = 1$ or $\lambda = -\frac{1}{n-1}$. When this is the case, the right-hand side of the above Bochner identity must vanish. It then follows from the first term on the right that $f$ must be proportional to $h_K$, so that $\lambda = 1$. In other words, when the zonoid $K$ is of class $C^2_+$, any even eigenfunction that is orthogonal to $h_K$ has eigenvalue strictly less than $-\frac{1}{n-1}$, and thus no nontrivial equality cases can arise in (1.9). However, nontrivial equality cases can arise when $K$ is nonsmooth, which will be analyzed in section 4 by a variation on the above argument.

*Remark* 3.5. At first sight, the formulation of the spectral condition of Lemma 1.8 is rather mysterious: what is the significance of the special value $-\frac{1}{n-1}$? The present proof provides one explanation for the appearance of this value: the constants in (3.1) in dimension $n$ are precisely the same as those that appear in (1.9) in dimension $n-1$, so that the preservation of the sharp threshold $\lambda \leqslant -\frac{1}{n-1}$ by induction on the dimension $n$ is explained by the quadratic relation (3.1).

3.2. **The induction base.** By Proposition 3.3 and induction on the dimension, the proof of Theorem 3.1 will be complete in any dimension $n \geqslant 3$ once we establish its validity in dimension $n = 2$. The latter is already known, however, by the results of [4] and Theorem 1.2. On the other hand, as we will presently explain, the $n = 2$ case may also be established directly by exactly the same method as was used in the proof of Proposition 3.3. This shows, in particular, that the Bochner method provides a unified explanation for the validity of Theorem 1.4 in every dimension.

**Lemma 3.6.** *The conclusion of Theorem 3.1 holds in dimension $n = 2$.*

*Proof.* Let $f \in C^2(S^1)_{\mathrm{even}}$, and let $K \in \mathcal{K}_s^2$ be a zonoid of class $C_+^2$. Applying the Cauchy-Schwarz inequality as in the proof of Proposition 3.3 yields

$$\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle \geqslant \frac{1}{4}\int \frac{\mathsf{V}([-u^\dagger,u^\dagger],f)^2}{\mathsf{V}([-u^\dagger,u^\dagger],K)}\, S_K(du),$$

where we used Lemma 2.3 to compute the generating measure of a planar zonoid. However, as was observed in the proof of Lemma 2.3, we have

$$\mathsf{V}([-u^\dagger,u^\dagger],K) = 2h_K(u), \qquad \mathsf{V}([-u^\dagger,u^\dagger],f) = 2f(u)$$

(the latter follows as $f = a(h_L - h_K)$ for some $L \in \mathcal{K}_s^2$ by Lemma 2.4). Thus

$$\langle \mathscr{A}_K f, \mathscr{A}_K f\rangle \geqslant \frac{1}{2}\int \frac{f^2}{h_K}\, dS_K = \langle f, f\rangle,$$

concluding the proof. □

## 4. Proof of Theorem 1.5

As was already noted in Remark 3.4, we may expect in principle that one may deduce the equality cases of (1.9) by a careful analysis of the Bochner method. The immediate problem with this approach is that the most basic object that appears in the Bochner method—the Hilbert operator $\mathscr{A}_K$—is not even well defined unless $K$ is of class $C_+^2$, and no nontrivial equality cases can arise in that setting. We will nonetheless pursue this strategy in the present section to settle the equality cases. This is possible, in essence, because it suffices for the purposes of characterizing equality to replace $\mathscr{A}_K f$ by $-\frac{1}{n-1}f$ in the Bochner identity, in which case the relevant formulas make sense also in nonsmooth situations.

We begin by making the latter idea precise in section 4.1. We subsequently show in section 4.2 what information on the equality cases may be extracted from the Bochner method. The proof of Theorem 1.5 will be completed in section 4.3.

4.1. **The equality condition.** Before we proceed to the analysis of the equality cases, we state a slight generalization of (1.9) that will be needed in the sequel.

**Lemma 4.1.** *Let $K \in \mathcal{K}_s^n$ be a zonoid. Then*

$$\frac{\mathsf{V}(f,K,\ldots,K)^2}{\mathrm{Vol}(K)} \geqslant \frac{n-1}{n}\mathsf{V}(f,f,K,\ldots,K) + \frac{1}{n^2}\int \frac{f^2}{h_K}\, dS_{K,\ldots,K}$$

*holds whenever $f = h_L - h_M$ for some $L, M \in \mathcal{K}_s^n$.*

*Proof.* This follows from Theorem 1.4 as in the proof of Lemma 3.2. □

We can now obtain a basic reformulation of the equality condition in (1.9). The method is due to Alexandrov [1, pp. 80–81].

**Lemma 4.2.** *For any $L \in \mathcal{K}_s^n$ and any zonoid $K \in \mathcal{K}_s^n$, the following are equivalent:*

1. *Equality holds in* (1.9)*, that is,*

$$\frac{\mathsf{V}(L,K,\ldots,K)^2}{\mathrm{Vol}(K)} = \frac{n-1}{n}\mathsf{V}(L,L,K,\ldots,K) + \frac{1}{n^2}\int \frac{h_L^2}{h_K}\,dS_{K,\ldots,K}.$$

2. *There exists $a > 0$ so that $f = h_L - ah_K$ satisfies*

$$h_K\,dS_{f,K,\ldots,K} = -\frac{1}{n-1}\,f\,dS_{K,\ldots,K}.$$

*Proof.* We first prove that $2 \Rightarrow 1$. Integrating condition 2 yields $\mathsf{V}(f,K,\ldots,K) = 0$, while multiplying condition 2 by $\frac{f}{h_K}$ and integrating yields

$$\frac{n-1}{n}\mathsf{V}(f,f,K,\ldots,K) = -\frac{1}{n^2}\int \frac{f^2}{h_K}\,dS_{K,\ldots,K}.$$

We therefore obtain

$$\frac{\mathsf{V}(f,K,\ldots,K)^2}{\mathrm{Vol}(K)} = \frac{n-1}{n}\mathsf{V}(f,f,K,\ldots,K) + \frac{1}{n^2}\int \frac{f^2}{h_K}\,dS_{K,\ldots,K},$$

and condition 1 follows using $f = h_L - ah_K$ and expanding the squares.

We now prove the converse implication $1 \Rightarrow 2$. Let $g \in C^2(S^{n-1})_{\mathrm{even}}$ and define

$$\beta(t) := \frac{\mathsf{V}(g_t,K,\ldots,K)^2}{\mathrm{Vol}(K)} - \frac{n-1}{n}\mathsf{V}(g_t,g_t,K,\ldots,K) - \frac{1}{n^2}\int \frac{g_t^2}{h_K}\,dS_{K,\ldots,K}$$

where $g_t := h_L + tg$. Condition 1 implies $\beta(0) = 0$, while Lemma 4.1 implies $\beta(t) \geqslant 0$ for all $t$. Thus $\beta$ is minimized at zero, so that $\beta'(0) = 0$ yields

$$\int g\,dS_{f,K,\ldots,K} = -\frac{1}{n-1}\int g\,\frac{f}{h_K}\,dS_{K,\ldots,K}$$

with $f = h_L - \frac{\mathsf{V}(L,K,\ldots,K)}{\mathrm{Vol}(K)}h_K$. As $g \in C^2(S^{n-1})_{\mathrm{even}}$ is arbitrary and as $dS_{f,K,\ldots,K}$ and $\frac{f}{h_K}\,dS_{K,\ldots,K}$ are even measures, condition 2 follows. ◻

It follows from the definition of the Hilbert operator $\mathscr{A}_K$ that when $K, L$ are of class $C^2_+$, Lemma 4.2 states precisely that equality holds in (1.9) if and only if $\mathscr{A}_K f = -\frac{1}{n-1}f$ for $f = h_L - ah_K$. The point of Lemma 4.2 is that the same characterization can be formulated for nonsmooth bodies in the sense of measures. The latter will suffice to apply the Bochner method to study the equality cases.

4.2. **The Bochner method revisited.** Using Lemma 4.2, we can now essentially repeat the proof of Proposition 3.3 in the present setting to extract a necessary condition for equality in (1.9) from the Bochner method.

**Lemma 4.3.** *Let $K \in \mathcal{K}_s^n$ be a zonoid with generating measure $\eta$, and let $L \in \mathcal{K}_s^n$ be such that equality holds in* (1.9)*. Then for every $u \in \operatorname{supp}\eta$, there exists $c(u) \geqslant 0$ such that $h_L(x) = c(u)h_K(x)$ for all $x \in \operatorname{supp} S_{K,\ldots,K}$ with $|\langle u,x\rangle| > 0$.*

*Proof.* Let $a > 0$ be such that $f = h_L - ah_K$ satisfies the second condition of Lemma 4.2, that is, $f\, dS_{K,\ldots,K} = -(n-1)\, h_K\, dS_{f,K,\ldots,K}$. Then we have

$$\begin{aligned}
\int \frac{f^2}{h_K}\, dS_{K,\ldots,K} &= \int\int \frac{f^2}{h_K^2}\, h_{[-u,u]}\, dS_{K,\ldots,K}\, \eta(du) \\
&\geqslant \int \frac{\left(\int \frac{f}{h_K}\, h_{[-u,u]}\, dS_{K,\ldots,K}\right)^2}{\int h_{[-u,u]}\, dS_{K,\ldots,K}}\, \eta(du) \\
&= n(n-1)^2 \int \frac{\mathsf{V}([-u,u], f, K, \ldots, K)^2}{\mathsf{V}([-u,u], K, \ldots, K)}\, \eta(du) \\
&\geqslant n(n-1)(n-2)\mathsf{V}(f, f, K, \ldots, K) + (n-1) \int \frac{f^2}{h_K}\, dS_{K,\ldots,K} \\
&= \int \frac{f^2}{h_K}\, dS_{K,\ldots,K}.
\end{aligned}$$

Here we used $h_K = \int h_{[-u,u]}\, \eta(du)$ in the first line; the Cauchy-Schwarz inequality in the second line; the condition of Lemma 4.2 in the third line; Lemma 3.2 in the fourth line (or by the proof of Lemma 3.6 for $n = 2$); and the fifth line follows as

$$\mathsf{V}(f, f, K, \ldots, K) = \frac{1}{n} \int f\, dS_{f,K,\ldots,K} = -\frac{1}{n(n-1)} \int \frac{f^2}{h_K}\, dS_{K,\ldots,K}$$

by the condition of Lemma 4.2.

Consequently, both inequalities used above must hold with equality. In particular, we have equality in the Cauchy-Schwarz inequality

$$\int \frac{f^2}{h_K^2}\, h_{[-u,u]}\, dS_{K,\ldots,K} = \frac{\left(\int \frac{f}{h_K}\, h_{[-u,u]}\, dS_{K,\ldots,K}\right)^2}{\int h_{[-u,u]}\, dS_{K,\ldots,K}}$$

for every $u \in \operatorname{supp} \eta$. By the equality condition of the Cauchy-Schwarz inequality, this implies that for every $u \in \operatorname{supp} \eta$, there is a constant $c'(u)$ so that $f(x) = c'(u) h_K(x)$ for every $x \in \operatorname{supp} S_{K,\ldots,K}$ with $h_{[-u,u]}(x) = |\langle u, x\rangle| > 0$. But as $f = h_L - ah_K$, the conclusion follows with $c(u) = a + c'(u)$. (Note that it must be the case that $c(u) \geqslant 0$ as $h_K, h_L$ are positive functions.) $\square$

*Remark* 4.4. The proof of Lemma 4.3 actually provides more information than is expressed in its statement: not only do we get equality in Cauchy-Schwarz, but we also get equality in the application of Lemma 3.2. In particular, this implies that if equality holds in (1.9) for given $K, L$ in dimension $n$, then the projections $\mathsf{P}_{u^\perp} K, \mathsf{P}_{u^\perp} L$ must also yield equality in (1.9) in dimension $n-1$ for every $u \in \operatorname{supp} \eta$. It is a curious feature of the present problem that the latter information will not be needed to characterize the equality cases: the equality condition in Cauchy-Schwarz will already suffice to fully characterize the equality cases of (1.9).

4.3. **Characterization of equality.** We are now ready to proceed to the proof of Theorem 1.5. The main difficulty is to show that the stated conditions are necessary for equality, which will be deduced from Lemma 4.3.

In the proof of the following result, we will encounter graphs that may have an uncountable number of vertices and edges. The standard properties of graphs that will be used in the proof—chiefly that a graph can be partitioned into its connected components—are valid at this level of generality; cf. [16, Chapter 2].

**Proposition 4.5.** *Let $K \in \mathcal{K}_s^n$ be a zonoid, and let $L \in \mathcal{K}_s^n$ be such that equality holds in (1.9). Then there exist $1 \leqslant m \leqslant n$, $a_1, \ldots, a_m \geqslant 0$, and zonoids $C_1, \ldots, C_m$ with $\dim(C_1) + \cdots + \dim(C_m) = n$ so that $K = C_1 + \cdots + C_m$ and*

$$h_L(x) = h_{a_1 C_1 + \cdots + a_m C_m}(x) \text{ for all } x \in \operatorname{supp} S_{K,\ldots,K}.$$

*Proof.* We define a graph $(V, E)$ as follows:

- The vertices are $V = \operatorname{supp} \eta$, where $\eta$ denotes the generating measure of $K$.
- There is an edge $\{u, v\} \in E$ between $u, v \in V$ if and only if there exists $x \in \operatorname{supp} S_{K,\ldots,K}$ such that $|\langle u, x\rangle| > 0$ and $|\langle v, x\rangle| > 0$.

Denote by $V = \bigsqcup_{i \in I} V_i$ the partition of $V$ into its connected components $V_i$.

For any edge $\{u, v\} \in E$, Lemma 4.3 implies that

$$c(u) h_K(x) = h_L(x) = c(v) h_K(x)$$

for some $x \in \operatorname{supp} S_{K,\ldots,K}$. As $h_K(x) > 0$, it follows that $c(u) = c(v)$. In particular, the value of $c(u)$ must be constant on each connected component. In the sequel, we will denote this value as $c(u) = a_i$ for $u \in V_i$.

Next, we make a key observation.

**Claim.** For every $x \in \operatorname{supp} S_{K,\ldots,K}$, there exists $i \in I$ so that $x \perp V_j$ for all $j \neq i$.

*Proof.* We can assume that $x \in \operatorname{supp} S_{K,\ldots,K}$ satisfies $|\langle u, x\rangle| > 0$ for some $i \in I$, $u \in V_i$, as otherwise the conclusion is trivial. But then we must have $|\langle v, x\rangle| = 0$ for all $j \neq i$, $v \in V_j$, as distinct connected components have no edge between them. ☐

We also need the following.

**Claim.** For every $u \in S^{n-1}$, there exists $x \in \operatorname{supp} S_{K,\ldots,K}$ so that $|\langle u, x\rangle| > 0$.

*Proof.* If the conclusion were false, there would exist some $u \in S^{n-1}$ such that $0 = \int |\langle u, x\rangle| \, S_{K,\ldots,K}(dx) = 2 \operatorname{Vol}(\mathsf{P}_{u^\perp} K)$ by Lemma 2.1. The latter is impossible as $K \in \mathcal{K}_s^n$ is assumed to have nonempty interior. ☐

The above two claims imply that distinct $V_i$ must lie in linearly independent subspaces $\mathrm{L}_i = \operatorname{span} V_i$. Indeed, if this is not so, then there exists $z \in S^{n-1}$ so that

$$z = t_1 u_1 + \cdots + t_k u_k = s_1 v_1 + \cdots s_l v_l$$

for some $k, l \geqslant 1$, $i \in I$, $u_1, \ldots, u_k \in V_i$, $v_1, \ldots, v_l \in \bigcup_{j \neq i} V_j$, $t_1, \ldots, t_k, s_1, \ldots, s_l \neq 0$. By the second claim there exists $x \in \operatorname{supp} S_{K,\ldots,K}$ so that $|\langle z, x\rangle| > 0$. But by the first claim we must then have $x \perp v_1, \ldots, u_l$, which entails a contradiction. It follows, in particular, that there can be at most $n$ connected components, so we can write $I = \{1, \ldots, m\}$ for some $1 \leqslant m \leqslant n$.

We now define zonoids $C_1, \ldots, C_m$ as

$$h_{C_i} = \int_{\mathrm{L}_i} h_{[-u,u]} \, \eta(du).$$

As $\mathrm{L}_1, \ldots, \mathrm{L}_m$ are linearly independent and $\operatorname{supp} \eta = V \subseteq S^{n-1} \cap (\mathrm{L}_1 \cup \cdots \cup \mathrm{L}_m)$

$$h_{C_1} + \cdots + h_{C_m} = \int h_{[-u,u]} \, \eta(du) = h_K,$$

that is, $K = C_1 + \cdots + C_m$. Moreover, as $\mathrm{L}_1, \ldots, \mathrm{L}_m$ are linearly independent and $K$ has nonempty interior, we must have $\dim(C_1) + \cdots + \dim(C_m) = n$.

Finally, let $x \in \operatorname{supp} S_{K,\dots,K}$. By the first claim above, there exists $1 \leqslant i \leqslant m$ so that $h_{C_j}(x) = 0$ for all $j \neq i$. As this implies that $h_{C_i}(x) = h_K(x) > 0$, there must exist $u \in V_i$ so that $|\langle u, x\rangle| > 0$. Recalling that $c(u) = a_i$ for $u \in V_i$, we obtain

$$h_L(x) = a_i h_K(x) = a_i h_{C_i}(x) = a_1 h_{C_1}(x) + \cdots + a_m h_{C_m}(x)$$

by Lemma 4.3. As this holds for any $x \in \operatorname{supp} S_{K,\dots,K}$, the proof is complete. □

Before we complete the proof, we must verify the basic case of equality.

**Lemma 4.6.** *Suppose that $K = C_1 + \cdots + C_m$ for some convex bodies $C_1, \dots, C_m$ such that $\dim(C_1) + \cdots + \dim(C_m) = n$, and that $L = a_1 C_1 + \cdots + a_m C_m$ for some $a_1, \dots, a_m \geqslant 0$. Then equality holds in (1.9).*

*Proof.* By [21, Theorem 5.1.8], the condition $\dim(C_1) + \cdots + \dim(C_m) = n$ implies that we have $\mathsf{V}(C_{i_1}, \dots, C_{i_n}) > 0$ if and only if each index $1 \leqslant j \leqslant m$ appears exactly $\dim(C_j)$ times among $(i_1, \dots, i_n)$. Thus for any $b_1, \dots, b_m \geqslant 0$

$$\mathrm{Vol}(b_1 C_1 + \cdots + b_m C_m) = \sum_{i_1,\dots,i_n=1}^{m} b_{i_1} \cdots b_{i_n} \mathsf{V}(C_{i_1}, \dots, C_{i_n}) = \Gamma\, b_1^{\dim(C_1)} \cdots b_m^{\dim(C_m)}$$

for some constant $\Gamma$ that depends only on $C_1, \dots, C_m$. Therefore

$$\begin{aligned} 0 &= -\frac{\mathrm{Vol}(K)}{n^2} \frac{d^2}{dt^2} \log \mathrm{Vol}(e^{ta_1} C_1 + \cdots + e^{ta_m} C_m)\bigg|_{t=0} \\ &= \frac{\mathsf{V}(L, K, \dots, K)^2}{\mathrm{Vol}(K)} - \frac{n-1}{n} \mathsf{V}(L, L, K, \dots, K) - \frac{1}{n^2} \int h_{a_1^2 C_1 + \cdots + a_m^2 C_m} \, dS_{K,\dots,K}. \end{aligned}$$

Now note that if $\int h_{C_i}\, dS_{C_{i_1},\dots,C_{i_{n-1}}} > 0$, then using [21, Theorem 5.1.8] as above shows that $\int h_{C_j}\, dS_{C_{i_1},\dots,C_{i_{n-1}}} = 0$ for all $j \neq i$. In particular, as we have $S_{K,\dots,K} = \sum_{i_1,\dots,i_{n-1}} S_{C_{i_1},\dots,C_{i_{n-1}}}$, this implies that for every $x \in \operatorname{supp} S_{K,\dots,K}$, there exists an index $i$ so that $h_{C_j}(x) = 0$ for all $j \neq i$. It follows readily that

$$h_{a_1^2 C_1 + \cdots + a_m^2 C_m}(x) = \frac{h_L(x)^2}{h_K(x)} \quad \text{for all } x \in \operatorname{supp} S_{K,\dots,K},$$

and the proof is complete. □

We can now complete the proof of the necessity part of Theorem 1.5. In the proof, we use some nontrivial facts that do not appear elsewhere in this note.

*Proof of Theorem 1.5.* We first prove sufficiency. Suppose that $K = C_1 + \ldots + C_m$ for bodies $C_1, \dots, C_m$ with $\dim(C_1) + \cdots + \dim(C_m) = n$, and that $L$ and $L' := a_1 C_1 + \cdots + a_m C_m$ have the same supporting hyperplanes in all 1-extreme normal directions of $K$. The latter implies by [21, Theorem 4.5.3 and Lemma 7.6.15] that

$$h_L(x) = h_{L'}(x) \quad \text{for all } x \in \operatorname{supp} S_{M,K,\dots,K} \tag{4.1}$$

for any convex body $M$. In particular, every term in (1.9) is unchanged if we replace $L$ by $L'$. Thus equality holds in (1.9) by Lemma 4.6.

We now prove necessity. Suppose equality holds in (1.9). Then Proposition 4.5 provides $C_1, \dots, C_m$ that satisfy all the required properties by construction except the last one: that is, what remains to be shown is that $L$ and $L' := a_1 C_1 + \cdots + a_m C_m$ have the same supporting hyperplanes in all 1-extreme normal directions of $K$.

Let us write $f := h_L - h_{L'}$. By Proposition 4.5, we have $f = 0$ on $\operatorname{supp} S_{K,\dots,K}$. Moreover, as we clearly have $L' + C = (\max_k a_k) K$ for a convex body $C$, it follows that $\operatorname{supp} S_{L',K,\ldots,K} \subseteq \operatorname{supp} S_{K,\ldots,K}$ and thus $f = 0$ on $\operatorname{supp} S_{L',K,\ldots,K}$ as well. Substituting $h_L = h_{L'} + f$ into (1.9) and using that both $L$ and $L'$ yield equality in (1.9) (by assumption and by Lemma 4.6, respectively), we can readily compute

$$\mathsf{V}(f, K, \ldots, K) = 0, \qquad \mathsf{V}(f, f, K, \ldots, K) = 0.$$

Using that $f = h_L - h_{L'}$, this implies that we have equality

$$\mathsf{V}(L, L', K, \ldots, K)^2 = \mathsf{V}(L, L, K, \ldots, K)\,\mathsf{V}(L', L', K, \ldots, K)$$

in Minkowski's quadratic inequality. By the main result of [23], it follows that $L$ and $aL' + v$ have the same supporting hyperplanes in all 1-extreme normal directions of $K$ for some $a \geqslant 0$, $v \in \mathbb{R}^n$. But as $L, L'$ are symmetric we must have $v = 0$, while $\mathsf{V}(f, K, \ldots, K) = 0$ and (4.1) imply $a = 1$. This concludes the proof. □

## 5. Implications

As we recalled in Theorem 1.2, the validity of the local logarithmic Brunn-Minkowski inequality (1.9) for *all* $K \in \mathcal{K}_s^n$ is equivalent to the validity of the logarithmic Brunn-Minkowski and the logarithmic Minkowski inequalities. The proof of these facts is based on several recent deep results on uniqueness in the $L^q$-Minkowski problem for $q < 1$. While this equivalence does not hold for fixed $K \in \mathcal{K}_s^n$, it is explained in [15, §2.4] that the theory behind Theorem 1.2 still yields nontrivial implications when (1.9) is known to hold in a sufficiently rich sub-class of $\mathcal{K}_s^n$. The aim of the final section of this note is to investigate what conclusions may be drawn by combining these results with Theorems 1.4–1.5.

We begin with the proof of Corollary 1.6.

*Proof of Corollary 1.6.* By a routine approximation argument as in the proof of Theorem 1.4, it suffices to prove the validity of (1.8) for $K \in \mathcal{K}_s^n$ that are zonoids of class $C_+^\infty$. Let us fix such a zonoid, and let $\mathcal{F} = \{(1-t)K + tB : t \in [0,1]\}$ where $B$ is the Euclidean unit ball. Then every $K' \in \mathcal{F}$ is a zonoid of class $C_+^\infty$. Moreover, it was observed in Remark 3.4 that every even eigenfunction of $\mathscr{A}_{K'}$ that is orthogonal to $h_{K'}$ has eigenvalue $\lambda < -\frac{1}{n-1}$. By the continuity of the eigenvalues of the Hilbert operator (cf. [14, Theorem 5.3]), there exists $\varepsilon > 0$ so that for every $K' \in \mathcal{F}$, every even eigenfunction of $\mathscr{A}_{K'}$ that is orthogonal to $h_{K'}$ has eigenvalue $\lambda \leqslant -\frac{1}{n-1} - \varepsilon$. Thus there exists $p < 0$ so that condition (4) of [15, Theorem 2.1] holds for all $K' \in \mathcal{F}$. The conclusion now follows from the implication (4)⇒(3b) of [15, Theorem 2.1] (as the inequality in (3b) with $q = 0$ is precisely (1.8)). □

The logarithmic Brunn-Minkowski conjecture is intimately connected to the uniqueness problem for cone volume measures; this was in fact the original motivation for the formulation of the conjecture [4]. Let us recall the definition.

**Definition 5.1.** The *cone volume measure* $V_K$ of a convex body $K$ is defined as

$$dV_K := \frac{1}{n}\, h_K\, dS_{K,\ldots,K}.$$

The basic question that arises here is whether the cone volume measure uniquely characterizes the convex body $K$. While this is not always the case, the question is closely connected to the equality cases of the logarithmic Minkowski inequality (1.8) in the case that $K, L \in \mathcal{K}_s^n$ are symmetric. For example, if $K, L \in \mathcal{K}_s^n$ satisfy

$V_K = V_L$ (and thus *a fortiori* $\mathrm{Vol}(K) = \mathrm{Vol}(L)$ as $\mathrm{Vol}(K) = \int dV_K$), the validity of the logarithmic Brunn-Minkowski conjecture would yield

$$0 \leqslant \int h_K \log\left(\frac{h_L}{h_K}\right) dS_{K,\ldots,K} = \int h_L \log\left(\frac{h_L}{h_K}\right) dS_{L,\ldots,L} \leqslant 0$$

using (1.8) in the first inequality, $V_K = V_L$ in the equality, and (1.8) with the roles of $K, L$ reversed in the second inequality. This would imply that $V_K$ is uniquely determined by $K$ whenever (1.8) does not admit nontrivial equality cases.

Unfortunately, even though we obtained a complete characterization of the equality cases of (1.9) when $K$ is a zonoid, this information is lost in Corollary 1.6. The reason is that the proof of Corollary 1.6 required approximation of $K$ by smooth bodies, which destroys the nontrivial equality cases. Nonetheless, for sufficiently smooth zonoids, uniqueness of cone volume measures follows by [15, Theorem 2.1]. Note that while the smoothness assumption on $K$ is restrictive, the following statement requires neither that $L$ is smooth nor that $L$ is a zonoid.

**Corollary 5.2.** *Let $K \in \mathcal{K}^n_s$ be a zonoid of class $C^3_+$. Then for any $L \in \mathcal{K}^n_s$, we have $V_K = V_L$ if and only if $K = L$.*

*Proof.* This follows from the implication (4)⇒(1) of [15, Theorem 2.1] by precisely the same argument as in the proof of Corollary 1.6. ☐

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