THE EXTREMALS OF THE KAHN-SAKS INEQUALITY

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Abstract. A classical result of Kahn and Saks states that given any partially ordered set with two distinguished elements, the number of linear extensions in which the ranks of the distinguished elements differ by k is log-concave as a function of k. The log-concave sequences that can arise in this manner prove to exhibit a much richer structure, however, than is evident from log-concavity alone. The main result of this paper is a complete characterization of the extremals of the Kahn-Saks inequality: we obtain a detailed combinatorial understanding of where and what kind of geometric progressions can appear in these log-concave sequences. This settles a partial conjecture of Chan-Pak-Panova, while the analysis uncovers new extremals that were not previously conjectured. The proof relies on a much more general geometric mechanism—a hard Lefschetz theorem for nef classes that was obtained in the setting of convex polytopes by Shenfeld and Van Handel—which forms a model for the investigation of such structures in other combinatorial problems.

1. Introduction

A sequence $a_1, \ldots, a_n \geq 0$ is called *log-concave* if

$$a_k^2 \geq a_{k-1}a_{k+1}, \qquad k = 2, \ldots, n-1. \tag{1.1}$$

It was observed long ago that many integer sequences that arise in a remarkably broad range of combinatorial problems appear to be log-concave [20]. Whenever the same mathematical phenomenon arises in many different situations, one may wonder whether there is a more fundamental underlying mechanism that explains its appearance. The discovery of such an explanation—that log-concavity arises due to the presence of combinatorial analogues of the Hodge-Riemann relations of algebraic geometry—has led to a striking series of recent breakthroughs in the understanding of log-concavity in combinatorics [1, 12, 3].

While the ubiquity of this mechanism was understood only recently, its first appearance dates back to Stanley's inequalities for matroids and posets [18]. Stanley studied these problems by expressing the combinatorial sequences in question as mixed volumes of convex polytopes, whose log-concavity follows from the Alexandrov-Fenchel inequality [16]. This classical result of convex geometry, which is a far-reaching generalization of the isoperimetric inequality, may also be viewed as a special instance of the Hodge-Riemann relations for toric varieties (see, e.g., [12] and [9, §5.4]). One interpretation of the basic insight behind the recent developments in [1, 12, 3, 5, 6] is that while most combinatorial problems cannot be reformulated in terms of convex polytopes, one can often still prove Alexandrov-Fenchel inequalities directly in the combinatorial context.

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Even within convex geometry, however, the Alexandrov-Fenchel inequality has itself been the subject of an open problem that dates back to Alexandrov's original paper [2, 15]: when does equality hold? This problem is fundamental to the interpretation of the Alexandrov-Fenchel inequality as an isoperimetric theorem, as it characterizes which bodies are extremal for the geometric quantities that appear in the inequality. Recently, this long-standing problem was completely resolved in the setting of convex polytopes by Shenfeld and the first author [17]. This provides a mechanism for obtaining new information on log-concave sequences that arise from the Alexandrov-Fenchel inequality: as equality in (1.1)

$$a_k^2 = a_{k-1}a_{k+i} \qquad \iff \qquad \frac{a_{k+1}}{a_k} = \frac{a_k}{a_{k-1}}$$

corresponds to a geometric progression (when $a_k > 0$), the characterization of the equality cases provides information on where and what kind of geometric progressions can appear in a log-concave sequence. The solution of the extremal problem for the Alexandrov-Fenchel inequality was exploited in [17, §15] and [14] to obtain a detailed combinatorial characterization of the geometric progressions that can appear in Stanley's poset inequalities [18], revealing a much richer structure in these sequences than is evident from log-concavity alone.

The aim of this paper is to develop further insight into such phenomena. We investigate a classical log-concavity inequality for posets due to Kahn and Saks that plays a central role in their work on sorting problems [13]. As the proof of this inequality is a direct modification of that of Stanley's poset inequalities, one may expect that the characterization of its extremals will be similar as well. Surprisingly, however, the Kahn-Saks inequality turns out to be much more delicate, and its extremals exhibit unexpected new features that are not present in Stanley's inequalities. Our results confirm a partial conjecture of Chan-Pak-Panova [8], and uncover further extremals that were not previously conjectured. The key challenge in the proofs is to understand the geometric features that cause the rich structure of the extremals of the Kahn-Saks inequality to appear.

Beside providing a case study of geometric progressions in log-concave sequences, our results and those of [17, 14] may serve as a model for the investigation of such phenomena in a broader context. Like the Alexandrov-Fechel inequality itself, its equality characterization admits an algebraic interpretation as a hard Lefschetz theorem for nef classes [17, §16.2]. The development of analogues of this mechanism outside convex geometry could provide a common explanation for the appearance of geometric progressions in much more general situations (see [11] for recent progress). At the same time, let us note that despite the recent advances in establishing log-concavity in combinatorics, no other method appears as of yet to be able to recover the Kahn-Saks inequality [6, §7.2]. Its rich extremal structure, which perches it at the cusp of log-concavity, may help explain why this is the case.

1.1. **Main results.**

Throughout this paper, we consider an arbitrary partially ordered set (poset) P with n elements. Recall that a *linear extension* of P is a bijection $f : P \to [n]$ such that $f(z) < f(z')$ whenever $z < z'$.

In the sequel, we fix two distinguished elements $x, y \in P$ such that $x \not> y$. For any $k = 1, \ldots, n-1$, we denote by N_k the number of linear extensions f of P so that $f(y) - f(x) = k$. The fundamental result in this setting, due to Kahn and Saks [13, Theorem 2.5], states that the sequence $N_1, \ldots, N_{n-1}$ is log-concave.

Theorem 1.1 (Kahn-Saks inequality). *$N_k^2 \geq N_{k-1}N_{k+1}$ for $k = 2, \ldots, n-2$.*

The aim of our main results is to characterize when equality $N_k^2 = N_{k-1}N_{k+1}$ holds in the Kahn-Saks inequality. Before we proceed to their formulation, let us clarify the notation that will be used throughout the paper.

Notation. Given a clause C, we denote by P_C the subset of elements of the poset P that satisfy this clause. For example,

$$P_{\leq x} := \{\omega \in P : \omega \leq x\},$$
$$P_{x<\cdot<z} := \{\omega \in P : x < \omega < z\},$$
$$P_{>x,\|y} := \{\omega \in P : \omega > x \text{ and } \omega \parallel y\},$$

etc. We use the symbol $z \parallel z'$ to denote that z is incomparable to z', and write $z \lessdot z'$ to indicate that z' covers z (that is, that $z < z'$ and $P_{z<\cdot<z'} = \varnothing$).

We begin by observing that equality in Theorem 1.1 holds trivially when $N_k = 0$. The following lemma characterizes when this happens (cf. [8, Theorem 8.5]).

Lemma 1.2 (Vanishing condition). *For any $k \in [n-1]$, the following are equivalent.*

a. $N_k = 0$.

b. $|P_{<x}| + |P_{>y}| > n - k - 1$ *or* $|P_{x<\cdot<y}| > k - 1$.

The entire difficulty of the problem lies in characterizing the nontrivial equality cases, that is, equality in Theorem 1.1 with $N_k > 0$. To this end, we first characterize what kinds of geometric progressions can appear. In the following results, we take for granted that $2 \leq k \leq n-2$ as in Theorem 1.1.

Theorem 1.3 (Geometric progressions). *If $N_k > 0$, the following are equivalent:*

a. $N_k^2 = N_{k-1}N_{k+1}$.

b. Either $N_{k+1} = N_k = N_{k-1}$, *or* $N_{k+1} = 2N_k = 4N_{k-1}$.

That is, only two types of geometric progressions are possible: flat and doubling progressions. We will characterize each of these situations separately. In order to formulate the equality conditions, we define a number of structural properties of the poset P that will appear in different combinations below.

Definition 1.4. We define the following properties:

(M_k) $|P_{<z}| + |P_{>y}| > n - k$ for all $z \in P_{>x,\not\geq y}$.

(M_k^*) $|P_{>z}| + |P_{<x}| > n - k$ for all $z \in P_{<y,\not\leq x}$.

(E_k) $|P_{z<\cdot<y} \cup \{x\}| > k$ for all $z \in P_{<x}$, and $|P_{<y} \cup \{x\}| > k$.

(E_k^*) $|P_{x<\cdot<z} \cup \{y\}| > k$ for all $z \in P_{>y}$, and $|P_{>x} \cup \{y\}| > k$.

(C_k) $|P_{z<\cdot<y}| + |P_{x<\cdot<z'}| > k - 2$ for all $z \in P_{<y,\|x}$, $z' \in P_{>x,\|y}$ with $z < z'$.

We first characterize the flat progressions, settling a conjecture of Chan-Pak-Panova [8, Conjecture 8.7] (up to minor corrections, see Remark 1.7).

Theorem 1.5 (Flat progressions). *If $N_k > 0$, the following are equivalent:*

a. $N_{k+1} = N_k = N_{k-1}$.

b. There is an element $z \in \{x, y\}$ such that for every linear extension f of P with $f(y) - f(x) = k$, there exist $u, v \in P_{\|z}$ so that $f(u) + 1 = f(z) = f(v) - 1$.

c. Either (M_k) *and* (E_k) *hold, or* (M_k^*) *and* (E_k^*) *hold.*

In contrast, no plausible conjecture has been put forward on the structure of the non-flat equality cases. These are completely settled by the following theorem.

Theorem 1.6 (Doubling progressions). *If $N_k > 0$, the following are equivalent:*

a. $N_{k+1} = 2N_k = 4N_{k-1}$.

b. $P = P_{\le x} \cup P_{<y,\|x} \cup P_{\ge y} \cup P_{>x,\|y}$, *and for every linear extension f of $P_{<x} \cup P_{<y,\|x}$ and every linear extension f' of $P_{>y} \cup P_{>x,\|y}$ the following hold: the k largest elements of f are incomparable to x, the k smallest elements of f' are incomparable to y, and for every $1 \le i \le k-1$ the i largest elements of f are incomparable to the $k-i$ smallest elements of f'.*[1]

c. $P_{\|x,\|y} = P_{x<\cdot<y} = \varnothing$, *and* (E_k), (E^*_k) *and* (C_k) *hold.*

Both Theorems 1.5 and 1.6 provide two distinct combinatorial characterizations of the extremals of the Kahn-Saks inequality. On the one hand, we provide an explicit characterization in terms of the structure of the poset itself. This formulation can be readily used to verify the equality condition in any concrete situation, as the properties in Definition 1.4 can be read off directly from the Hasse diagram of P. On the other hand, we provide a complementary characterization in terms of the structure of the linear extensions of P. While less explicit, this formulation explains the reason that equality arises in each situation (cf. section 2).

Remark 1.7 (On Definition 1.4). Definition 1.4 is formulated for an arbitrary poset P and distinguished elements $x \not\geq y$. In most cases, (E_k) and (E^*_k) can be somewhat simplified. For example, in (E_k), the second condition $|P_{<y} \cup \{x\}| > k$ follows automatically from the first condition as long as $P_{<x} \neq \varnothing$; we included it only to account for the case that $P_{<x} = \varnothing$. Similarly, when $x < y$, we automatically have $x \in P_{z<\cdot<y}$ for any $z \in P_{<x}$ and there is no need to include $\{x\}$ separately in (E_k); the present formulation accounts also for the case that $x \| y$.

On the other hand, other relations among the properties of Definition 1.4 may not be immediately evident. In particular, we will show as part of the proof of our main results that the conditions (M_k) and (E^*_k) are mutually exclusive, and that (M^*_k) and (E_k) are mutually exlusive (Lemma 5.14). This shows, for example, that the two alternatives of Theorem 1.5(c) cannot hold simultaneously.

The first four properties of Definition 1.4 were anticipated (up to minor corrections) by Chan-Pak-Panova [8]. There, the combination of (M_k) and (E_k) was called the *k-midway property*, and the combination of (M^*_k) and (E^*_k) was called the *dual k-midway property*. As these properties turn out to appear in a different combination in Theorem 1.6, we do not adopt this terminology here.

1.2. **Discussion.**

Our results provide a detailed picture of when, where, and why geometric progressions can arise in the log-concave sequences $N_1, \ldots, N_{n-1}$. The aim of this section is to further discuss the significance of these results.

1.2.1. *Shapes of log-concave sequences.* Despite the considerable interest in log-concave sequences of combinatorics [20, 12], log-concavity in itself provides a limited qualitative picture on what such sequences look like. The study of geometric progressions reveals that there can be significant additional structure in the shapes

[1] Here we view a linear extension f of a subset $S \subseteq P$ as defining a linear ordering of S. Thus, e.g., the smallest two elements of f are $f^{-1}(1)$, $f^{-1}(2)$, the largest element of f is $f^{-1}(|S|)$, etc.

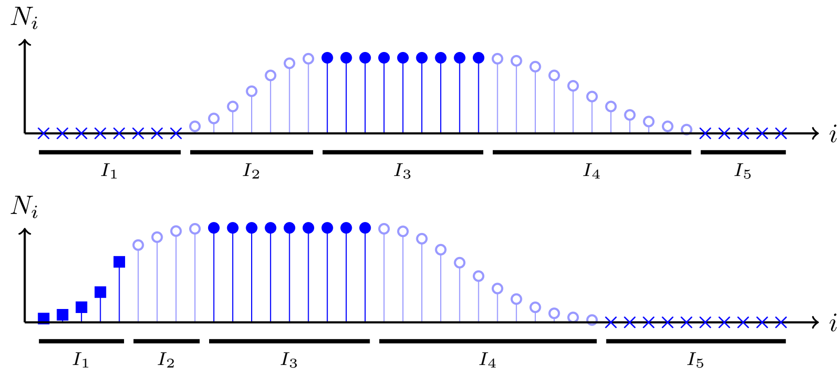

Figure 1.1. Structure of extremal sequences. The symbols denote vanishing (×), flat progressions (●), doubling progressions (■), and strictly log-concave regions (○).

of log-concave sequences that arise in a given combinatorial problem, and provides more detailed qualitative and quantitative information.

In the setting of this paper, the qualitative picture is illustrated in Figure 1.1. Recall that any log-concave sequence is unimodal on its support. Moreover, by Lemma 1.2, the sequences in this paper can vanish only on an initial or final segment. We therefore claim that any extremal sequence (that is, one containing a geometric progression) must look qualitatively like one of the plots in Figure 1.1 (however, some segments I_j may be empty in a given situation):

- The top plot illustrates the situation with no doubling progression. By unimodality, there can be at most one flat segment that is the maximum of the sequence.
- The bottom plot illustrates the situation where there is a doubling progression. Note that the conditions (E_k), (E_k^*), (C_k) automatically imply (E_l), (E_l^*), (C_l) for all $l < k$. Thus Theorem 1.6 and Lemma 1.2 imply that a doubling progression can only arise as the initial segment of the sequence.

Beyond this qualitative picture, however, our results provide detailed quantitative information: they enable us to compute precisely where in the sequence the geometric progressions appear (that is, we can compute the length of each segment I_j explicitly in terms of the poset structure). It appears rather surprising that such detailed information is accessible for the linear extension numbers of arbitrary posets, which are themselves hard to compute [4].

It is not immediately obvious from the formulation of our results that all regions of the log-concave sequences that are illustrated in Figure 1.1 can in fact appear simultaneously. This is however readily verified by means of simple examples.

Example 1.8. Suppose that x is a globally minimal element of P, that is, $x \leq z$ for all $z \in P$. Then x must appear first in any linear extension of P. Therefore, as noted in [8], this special case reduces to the situation originally studied by Stanley: N_k is the number of linear extensions of $P \backslash \{x\}$ in which y has rank k. In this setting, the explicit construction of [17, Example 15.4] shows that we may engineer the poset P to achieve a log-concave sequence as in the top plot of Figure 1.1 with an essentially arbitrary choice of (positive) lengths of the segments $I_1, \ldots, I_5$.

Example 1.9. We now provide a counterpart of the previous example for the bottom plot of Figure 1.1. Consider the poset P with $|P| = r+s+2$ defined by the relations

$$x \lessdot z_1 \lessdot \cdots \lessdot z_r, \qquad w_1 \lessdot \cdots \lessdot w_s, \qquad w_u \lessdot z_v, \qquad w_u \lessdot y \lessdot z_t$$

for any $v < t \le r$ and $u \le s$. Then $P_{x<\cdot<y} = P_{\|x,\|y} = \varnothing$, and $N_k > 0$ for $k \le t+s$ by Lemma 1.2. A straightforward computation shows that (M_k) holds when $k > t+s$, (M_k^*) holds when $k > u+v$, (E_k) holds when $k \le u$, (E_k^*) holds when $k < t$, and (C_k) holds when $k \le v$. Thus Theorems 1.5 and 1.6 yield

$$\begin{aligned} &N_{k+1} = 2N_k = 4N_{k-1} && \text{if and only if} \quad 2 \le k \le \min(u,v), \\ &N_{k+1} = N_k = N_{k-1} && \text{if and only if} \quad u+v < k < t, \\ &N_k = 0 && \text{if and only if} \quad t+s < k \le |P|. \end{aligned}$$

In particular, we obtain a log-concave sequence as in the bottom plot of Figure 1.1, and the lengths of the segments $I_1, \ldots, I_5$ can be chosen essentially arbitrarily by selecting the parameters r, s, t, u, v appropriately.

1.2.2. *General mechanisms.* One of the main reasons for the interest of the log-concavity phenomenon in combinatorics is that it suggests the presence of a certain universal structure[2] that appears to arise in many combinatorial problems: combinatorial analogues of Alexandrov-Fenchel inequalities [3, 5, 6] or of more general Hodge-Riemann relations [1, 12]. It is remarkable that the same kinds of structures are fundamental to several other areas of mathematics, including convex geometry [16], algebraic geometry [21, 9], and complex geometry [10].

In contrast to log-concavity, which is a robust qualitative property, one might expect that the understanding of geometric progressions in log-concave sequences must be developed on a case-by-case basis. We believe, however, that geometric progressions likely arise in many problems through a common mechanism that is nearly as universal as the structure that gives rise to log-concavity itself. As is explained in [17, §16.2], the equality cases of the Alexandrov-Fenchel inequality that form the basis for the analysis in this paper admit a natural algebraic interpretation (a precise counterpart of the hard Lefschetz theorem of degree 1 for nef classes). In this form, such structures can be meaningfully formulated in much more general situations, and it is natural to conjecture that they do indeed arise in many combinatorial problems. Such questions are largely open to date, though there is recent progress on analogous questions in algebraic geometry [11].

The results of this paper and those of [17, 14] motivate the investigation of such structures, and illustrate the kind of strong qualitative and quantitative information that can be obtained when they are present.

1.2.3. *Interplay between geometry and combinatorics.* The proof of the Kahn-Saks inequality, like that of Stanley's inequality on which it is based, translates the original combinatorial problem to a geometric problem for convex polytopes. As will be explained in section 3, Kahn and Saks construct two $(n-1)$-dimensional convex polytopes K, L so that the linear extension numbers can be represented as a mixed volume $N_k = (n-1)!\,\mathsf{V}(K[n-k], L[k-1])$. The inequality $N_k^2 \ge N_{k-1}N_{k+1}$ then follows immediately from the Alexandrov-Fenchel inequality

$$\mathsf{V}(K, L, C_1, \ldots, C_{n-3})^2 \ge \mathsf{V}(K, K, C_1, \ldots, C_{n-3})\,\mathsf{V}(L, L, C_1, \ldots, C_{n-3}),$$

[2]In fact, the Kahn-Saks inequality is somewhat unusual in that it was developed in [13] as a tool to investigate sorting problems, rather than being motivated by log-concavity in itself.

which is a far-reaching generalization of the isoperimetric inequality in convex geometry. (We refer to [16] for background on convex geometry; the relevant notions for this paper will be briefly reviewed in section 3.) One might view the Kahn-Saks inequality as an isoperimetric inequality for posets, and its equality characterization as the corresponding isoperimetric theorem.

The equality cases of the Alexandrov-Fenchel inequality were completely characterized for convex polytopes in [17]. However, this characterization provides geometric information on the polytopes in question. The main difficulty in the proof of our main results is to understand how to translate this geometric information back to the original combinatorial problem. This translation is far from straightforward, and requires us to develop a detailed understanding of the interplay between the geometric and combinatorial descriptions.

An unexpected consequence of the results of this paper is that they complete the picture of what geometric extremals can arise in combinatorial problems. Let us recall from [17, §2.2] that the equality cases of the Alexandrov-Fenchel inequality arise from a superposition of three distinct mechanisms:

M1. Translation and scaling.

M2. Degeneration of the support of mixed area measures.

M3. "Critical" equality cases caused by dimensional collapse.

As there is enormous freedom in the choice of arbitrary convex polytopes, it may not be too surprising that the Alexandrov-Fenchel inequality has numerous equality cases. What is far more surprising is that *all* the equality mechanisms turn out to arise even in natural combinatorial applications. While only M2 plays a role in the setting of Stanley's inequality [17, §15], M3 can arise in a general form of Stanley's inequality [14] (the latter has striking complexity implications [7]). It will turn out that M1 plays a central role in this paper, completing the picture.

As the role of M1 is a novel feature of the problem investigated in this paper that did not arise in previous works, we briefly discuss it further.

1.2.4. *The role of translation and scaling.* Geometrically, translation and scaling may be viewed as trivial reasons for equality. For example, equality clearly holds in the Alexandrov-Fenchel inequality when $K = L$; therefore, as mixed volumes are homogeneous and translation invariant, we trivially obtain new equality cases $K = aL + v$ by translation and scaling. Nontrivial equality cases arise due to M2 above, which states that these polytopes need not be equal but need only have the same supporting hyperplanes in some directions (cf. section 3.2).

From a combinatorial perspective, however, the appearance of nontrivial translation and scaling is unexpected. The connection between convexity and combinatorics arises when we work with lattice polytopes, in which case mixed volumes are always combinatorial quantities [9, §5.4]. In particular, the polytopes used by Stanley, Kahn and Saks have all their vertices in the set $\{0, 1\}^n$, cf. [19]. One may expect that this property provides sufficient rigidity that two such polytopes cannot have the same supporting hyperplanes if we scale or translate one of them. It is a surprising feature of the Kahn-Saks inequality that nontrivial scaling and translation can nonetheless arise. In particular, the resulting equality cases of the Alexandrov-Fenchel inequality do not respect the lattice structure of the underlying polytopes, as will be illustrated in a simple example in section 5.5.

The consequences of this breakdown of rigidity are fundamental to our main results. Nontrivial scaling is responsible for the presence of the non-flat geometric progressions (cf. section 3.2). At the same time, nontrivial translations arise in the present setting even when we consider flat geometric progressions: the translation determines which of the two cases of Theorem 1.5(c) is in force, as will become clear in the proof (cf. Definition 5.2 and the following discussion).

1.3. **Organization of this paper.** The rest of this paper is organized as follows.

The implications (c)⇒(b)⇒(a) of Theorems 1.5 and 1.6 require only elementary arguments. We first dispense with these implications in section 2. The remainder of the paper is devoted to the core part of our main results—Theorem 1.3 and the implication (a)⇒(c) of Theorems 1.5 and 1.6.

In section 3, we describe the convex geometric construction that underpins the Kahn-Saks inequality. This yields, through the equality cases of the Alexandrov-Fenchel inequality, a geometric description of its extremals. The main difficulty in the proof is now to understand how one can use this geometric information to characterize the combinatorial structure of the poset.

In section 4, we aim to translate the basic geometric data that was obtained in section 3 into combinatorial conditions. This requires us to study the facial structure of the polytopes that appear in the geometric equality characterization. The faces will turn out to be described by combinatorial constraints. As a byproduct, we also obtain a short proof of Lemma 1.2 here.

Now that the geometric data has been converted to combinatorial information, we arrive at the heart of the argument: we must use this combinatorial information to fully characterize the structure of the poset. This is the main part of the proof of our main results, which is contained in section 5.

2. Sufficiency

The main results of this paper provide necessary and sufficient conditions for equality to hold in the Kahn-Saks inequality. One direction of these results is considerably simpler than the other, however: the proof that the combinatorial conditions of Theorems 1.5 and 1.6 are sufficient for equality to hold is entirely elementary. Nonetheless, this direction sheds considerable light on the structure of the problem, as its proof reveals the combinatorial mechanisms that give rise to equality. This direction of our main results will be proved in the present section. The main challenge ahead of us is to show that these are the *only* mechanisms that can give rise to equality, that is, that the sufficient conditions are also necessary. The proof of the latter will occupy the remainder of this paper.

We begin by proving the implications (c)⇒(b)⇒(a) of Theorem 1.5. The argument is essentially contained in [8, §8.3], and is very similar to the argument in the proof of [17, Theorem 15.3]. We include the proof for completeness.

Proof of Theorem 1.5: (c)⇒(b). Suppose that (M_k) and (E_k) hold. Let f be any linear extension of P such that $f(y) - f(x) = k$. Since $|P_{<y} \cup \{x\}| > k$ by (E_k), we must have $f(y) \ge k + 2$ and thus $f(x) = f(y) - k \ge 2$.

Now consider $u \in P$ such that $f(u) = f(x) - 1$, which exists as $f(x) \ge 2$. If it were the case that $u < x$, then (E_k) would imply that

$$k < |P_{u<\cdot<y} \cup \{x\}| \le f(y) - f(u) - 1 = k,$$

which is impossible. But $u > x$ is also impossible as $f(u) < f(x)$. Thus $u \parallel x$.

Similarly, consider $v \in P$ such that $f(v) = f(x) + 1$. Note that $v \not\geq y$ because $f(v) < f(x) + 2 \leq f(y)$ (as $k \geq 2$). If $v > x$, then (M_k) would imply that

$$n - k < |P_{<v}| + |P_{>y}| \leq f(v) - 1 + n - f(y) = n - k,$$

which entails a contradiction. We conclude as above that $v \parallel x$.

We have therefore shown that the validity of (M_k) and (E_k) implies that condition (b) of Theorem 1.5 holds with $z = x$. We omit the completely analogous argument that (M_k^*) and (E_k^*) imply condition (b) of Theorem 1.5 with $z = y$. □

Proof of Theorem 1.5: (b)⇒(a). Suppose that condition (b) of Theorem 1.5 holds with $z = x$. Let f be any linear extension of P such that $f(y) - f(x) = k$ and $f(u) + 1 = f(x) = f(v) - 1$. As $u, v \parallel x$, swapping v and x in the linear order defined by f yields a new linear extension f' of P with $f'(y) - f'(x) = k - 1$, while swapping u and x yields a linear extension f'' of P with $f''(y) - f''(x) = k + 1$.

As the maps $f \mapsto f'$ and $f \mapsto f''$ are clearly injective, it follows that $N_k \leq N_{k-1}$ and $N_k \leq N_{k+1}$. Thus the Kahn-Saks inequality yields

$$N_{k-1}N_k \geq N_k^2 \geq N_{k-1}N_{k+1} \geq N_{k-1}N_k,$$
$$N_kN_{k+1} \geq N_k^2 \geq N_{k-1}N_{k+1} \geq N_kN_{k+1}.$$

Since we assume that $N_k > 0$, it follows that $N_{k-1} = N_k = N_{k+1}$.

We have therefore shown that condition (b) of Theorem 1.5 with $z = x$ implies condition (a). We omit the completely analogous argument in the case $z = y$. □

We now turn to the proof of the implications (c)⇒(b)⇒(a) of Theorem 1.6. The mechanism by which equality arises here is very different than in Theorem 1.5: to prove it, we will establish a bijection between certain sets of linear extensions.

Proof of Theorem 1.6: (c)⇒(b). That $P = P_{\leq x} \cup P_{<y,\parallel x} \cup P_{\geq y} \cup P_{>x,\parallel y}$ is merely a reformulation of the conditions $P_{\parallel x,\parallel y} = P_{x<\cdot<y} = \varnothing$. In the rest of the proof, we fix any linear extension f of $P_{<x} \cup P_{<y,\parallel x}$ and f' of $P_{>y} \cup P_{>x,\parallel y}$.

Let $z \in P_{>y}$. As $P_{x<\cdot<y} = \varnothing$, we have $P_{x<\cdot<z} \backslash \{y\} \subseteq P_{>y} \cup P_{>x,\parallel y}$. Thus

$$k < |P_{x<\cdot<z} \cup \{y\}| = |P_{x<\cdot<z} \backslash \{y\}| + 1 \leq f'(z)$$

by (E_k^*), that is, we have shown that every $z \in P_{>y}$ has rank at least $k + 1$ with respect to f'. On the other hand, if $P_{>y} = \varnothing$ then $|P_{>x,\parallel y}| = |P_{>x} \backslash \{y\}| \geq k$ by (E_k^*). In either case, we conclude that the k smallest elements of f' must lie in $P_{>x,\parallel y}$. The completely analogous argument using (E_k) instead of (E_k^*) shows that the k largest elements of f must lie in $P_{<y,\parallel x}$.

Now fix $1 \leq i \leq k - 1$, let $z \in P_{<y,\parallel x}$ be among the i largest elements of f, and let $z' \in P_{>x,\parallel y}$ be among the $k - i$ smallest elements of f'. Note that we must have $P_{z<\cdot<y} \subseteq P_{<y,\parallel x}$ and $P_{x<\cdot<z'} \subseteq P_{>x,\parallel y}$ as $P_{x<\cdot<y} = \varnothing$, $z \parallel x$, and $z' \parallel y$, so that

$$|P_{z<\cdot<y}| < i, \qquad |P_{x<\cdot<z'}| < k - i.$$

Thus (C_k) implies that $z \not< z'$. On the other hand, $z \not\geq z'$ as $P_{x<\cdot<y} = \varnothing$, so we must have $z \parallel z'$. We have therefore shown that the i largest elements of f are incomparable to the $k - i$ smallest elements of f', concluding the proof. □

Proof of Theorem 1.6: (b)⇒(a). Let $P_- := P_{<x} \cup P_{<y,\parallel x}$ and $P_+ := P_{>y} \cup P_{>x,\parallel y}$. Note that $P_-, P_+, \{x, y\}$ are disjoint as $x \not\geq y$, and that $P = P_- \cup P_+ \cup \{x, y\}$.

Let $1 \le l \le k+1$. Denote by $\mathcal{E}_-, \mathcal{E}_+$ the sets of all linear extensions of P_-, P_+, and by $\mathcal{E}_l$ the set of all linear extensions $\bar{f}$ of P such that $\bar{f}(y) - \bar{f}(x) = l$. We aim to construct a bijection $\iota : \mathcal{E}_l \to \mathcal{E}_- \times \mathcal{E}_+ \times \{0,1\}^{l-1}$.

The construction is as follows. Given any $\bar{f} \in \mathcal{E}_l$, let $f \in \mathcal{E}_-$ and $f' \in \mathcal{E}_+$ be defined by restricting the linear order of $\bar{f}$ to P_- and P_+, respectively. Moreover, define $\omega_i = 0$ if the ith smallest element of $\bar{f}$ between x and y is in P_-, and $\omega_i = 1$ if the ith smallest element of $\bar{f}$ between x and y is in P_+. This defines a map $\iota : \bar{f} \mapsto (f, f', \omega)$. We must show this map is injective and surjective.

To show ι is injective, note that by the definitions of $P_-, P_+, \mathcal{E}_l$, every element of P_- must be smaller than y and every element of P_+ must be larger than x in the linear ordering defined by $\bar{f}$. Thus $\bar{f}$ can be uniquely reconstructed from f, f', ω by choosing the elements between x and y in order from the largest elements of f and the smallest elements of f' as defined by ω, and placing the remaining elements of f and f' below x and above y, respectively. This proves injectivity of ι.

Note that the above reconstruction procedure enables us to define a linear ordering $\bar{f}$ of P with $\bar{f}(y) - \bar{f}(x) = l$ starting from *any* $(f, f', \omega) \in \mathcal{E}_- \times \mathcal{E}_+ \times \{0,1\}^{l-1}$. To prove ι is surjective, we must show that any linear ordering $\bar{f}$ thus defined is a linear extension of P, that is, that it is compatible with the partial order of P. In other words, we must show that $\bar{f}(u) < \bar{f}(v)$ implies $u \not> v$.

- For $u, v \in P_- \cup \{y\}$ or $u, v \in P_+ \cup \{x\}$, this follows by construction as f, f' are linear extensions of $P_- \subseteq P_{\not> y}$ and $P_+ \subseteq P_{\not< x}$, respectively.
- For $u \in P_- \cup \{y\}$ and $v \in P_+ \cup \{x\}$, we have $u \not> v$ by the definitions of P_-, P_+ and as $x \not> y$, $P_{x < \cdot < y} = \varnothing$ (except $u = y, v = x$ for which $\bar{f}(u) \not< \bar{f}(v)$).
- For $u \in P_+ \cup \{x\}$ and $v \in P_- \cup \{y\}$, the construction of $\bar{f}$ ensures that if $u \in P_+$ it must be among the j smallest elements of f', and if $v \in P_-$ it must be among the $l - 1 - j$ largest elements of f, where $j = \sum_{i=1}^{l-1} \omega_i$. Thus condition (b) of Theorem 1.6 yields $u \parallel v$, unless $u = x, v = y$ in which case $u \not> v$ by assumption.

Thus $\bar{f}$ is a linear extension of P, proving surjectivity of ι.

As we have now shown that ι is a bijection, it follows that

$$N_l = |\mathcal{E}_l| = 2^{l-1} |\mathcal{E}_-| |\mathcal{E}_+|$$

for every $1 \le l \le k+1$. In particular, $N_{k+1} = 2N_k = 4N_{k-1}$. □

3. Geometric description of the extremals

The aim of this section is to recall the geometric construction that gives rise to the Kahn-Saks inequality [13], and to deduce a geometric characterization of its equality cases from the extremals of the Alexandrov-Fenchel inequality [17]. This geometric description of the equality cases forms the basis for the remainder of the paper: the main challenge in the following sections will be to understand how to use this geometric information to extract combinatorial structure.

Before we proceed, let us recall some basic notions of convex geometry that will be used without comment in the sequel. Given two convex bodies C, C',

$$C + C' := \{x + y : x \in C, y \in C'\}$$

denotes their Minkowski sum. The support function h_C of C is defined by

$$h_C(u) := \sup_{x \in C} \langle u, x \rangle.$$

If $\|u\| = 1$, then $h_C(u)$ may be interpreted as the signed distance from the origin to the supporting hyperplane of C with outer normal u. The set

$$F(C,u) := \{x \in C : \langle u, x\rangle = h_C(u)\}$$

is the (exposed) face of C with outer normal u. Finally, we recall that $h_{C+C'}(u) = h_C(u) + h_{C'}(u)$ and $F(C+C',u) = F(C,u) + F(C',u)$ [16, Theorem 1.7.5].

3.1. **The geometric construction.** The following setting will be used throughout the rest of this paper. Let $\mathbb{R}^P$ be the n-dimensional real vector space that is spanned by the coordinate basis $\{e_z\}_{z\in P}$. We always equip $\mathbb{R}^P$ with the standard inner product that makes the coordinate basis orthonormal. For any $t \in \mathbb{R}^P$, we denote its coordinates as $t_z := \langle e_z, t\rangle$. Finally, we define

$$V := \{e_y - e_x\}^\perp = \{t \in \mathbb{R}^P : t_y - t_x = 0\},$$

where $x \not\geq y$ are the distinguished elements of P.

A fundamental role in the following will be played by the order polytope of P, and in particular by two special slices of the order polytope.

Definition 3.1. The order polytope of P is defined as

$$O_P := \{t \in [0,1]^P : t_z \le t_{z'} \text{ whenever } z < z'\}.$$

Moreover, the polytopes

$$K := \{t \in O_P : t_y - t_x = 0\},$$
$$L := \{t \in O_P : t_y - t_x = 1\}$$

will be fixed throughout the paper.

As $K \subset V$ and $L \subset V + e_y$, every Minkowski combination $(1-\lambda)K + \lambda L$ lies in a translate of the $(n-1)$-dimensional space V. By a classical fact of Minkowski [16, §5.1], the $(n-1)$-dimensional volume then satisfies

$$\mathrm{V}_{n-1}((1-\lambda)K + \lambda L) = \sum_{k=1}^{n} \binom{n-1}{k-1} (1-\lambda)^{n-k} \lambda^{k-1}\, \mathsf{V}(K[n-k], L[k-1]),$$

where $\mathsf{V}(K[n-k], L[k-1])$ denotes the $(n-1)$-dimensional *mixed volume* in which K appears $n-k$ times and L appears $k-1$ times. These mixed volumes turn out to compute the linear extension numbers N_k [13, eq. (2.14)].

Lemma 3.2 (Kahn-Saks)**.** $N_k = (n-1)!\,\mathsf{V}(K[n-k], L[k-1])$.

With Lemma 3.2 in hand, the Kahn-Saks inequality $N_k^2 \ge N_{k-1}N_{k+1}$ follows immediately from the *Alexandrov-Fenchel inequality* [16, Theorem 7.3.1]

$$\mathsf{V}(K, L, K[n-k-1], L[k-2])^2 \ge \\ \mathsf{V}(K, K, K[n-k-1], L[k-2])\,\mathsf{V}(L, L, K[n-k-1], L[k-2]).$$

Equality in the Kahn-Saks inequality is therefore also a special case of equality in the Alexandrov-Fenchel inequality, which is characterized in [17]. The latter will provide an explicit description of the equality conditions in terms of the geometry of the polytopes K and L, which we turn to next.

Remark 3.3. In this paper, mixed volumes will only be used in order to apply the results of [17] and do not appear outside this section. For this reason, we have omitted most background material on mixed volumes, referring the interested reader to the excellent monograph [16] for a detailed treatment.

Let us note that results on mixed volumes are usually formulated for convex bodies lying in a given n-dimensional space, whereas the polytopes K and L lie in different translates of the $(n-1)$-dimensional space V. We can readily reduce the latter setting to the former (with $n \leftarrow n-1$) by replacing L by $L' = L - \frac{e_y - e_x}{2} \subset V$. By translation-invariance of mixed volumes and as $h_L(u) = h_{L'}(u)$ for all $u \in V$, any resulting statements for $K, L' \subset V$ hold verbatim for K, L.

3.2. **Equality cases.** At the heart of the extremal characterization of the Kahn-Saks inequality lies the notion of a k-extreme vector.

Definition 3.4. A vector $u \in V$ is said to be *k-extreme* if the following hold:

$$\dim F(K,u) \geq n-k-1,$$
$$\dim F(L,u) \geq k-2,$$
$$\dim F(K+L,u) \geq n-3.$$

The following is the main result of this section.

Proposition 3.5 (Kahn-Saks extremals: geometric characterization)**.** *Let $a > 0$, and assume that $N_k > 0$. Then the following are equivalent.*

a. $a^2 N_{k+1} = a N_k = N_{k-1}$.

b. There exists $v \in V$ so that $h_K(u) = h_{aL+v}(u)$ for all k-extreme $u \in V$.

Before we prove Proposition 3.5, let us also formulate a geometric characterization of the vanishing condition.

Lemma 3.6 (Vanishing: geometric characterization)**.** *The following are equivalent.*

a. $N_k = 0$.

b. $\dim K < n-k$ *or* $\dim L < k-1$ *or* $\dim(K+L) < n-1$.

Proof. This is immediate from Lemma 3.2 and [16, Theorem 5.1.8]. □

We now complete the proof of Proposition 3.5.

Proof of Proposition 3.5. We begin by observing that a k-extreme vector u is called $(B, K[n-k-1], L[k-2])$-extreme in the terminology of [17, Lemma 2.3]. Thus condition (b) implies, using [17, Lemma 2.8] and translation invariance, that

$$S_{K,K[n-k-1],L[k-2]} = a\, S_{L,K[n-k-1],L[k-2]},$$

where $S_{C_1,\ldots,C_{n-2}}$ denotes the $(n-1)$-dimensional mixed area measure. Integrating this identity against h_K and h_L yields condition (a) by [17, eq. (2.1)] and Lemma 3.2. Thus we have proved the implication (b)⇒(a).

Now suppose condition (a) holds. Then $N_k > 0$ implies $N_{k-1} > 0$ and $N_{k+1} > 0$ as well. Lemma 3.6 yields $\dim K \geq n-k+1$, $\dim L \geq k$, and $\dim(K+L) = n-1$, so that the bodies $(K[n-k-1], L[k-2])$ are supercritical in the terminology of [17, Definition 2.14]. Therefore, by [17, Corollary 2.16] and Lemma 3.2, condition (a) implies that there exist $a' > 0$ and $v \in V$ so that $h_K(u) = h_{a'L+v}(u)$ for all k-extreme vectors $u \in V$. It remains to note that we must have $a' = a$ by the implication (b)⇒(a). Thus we have proved the implication (a)⇒(b). □

Proposition 3.5 provides a complete characterization of the equality cases of the Kahn-Saks inequality in terms of the geometry of the polytopes K and L. It is far from clear, however, how the combinatorial structure of P emerges from this geometric information. Understanding the latter is the main challenge in the proof of our main results, which will be addressed in the following sections.

4. Facial structure

To exploit Proposition 3.5, we must first develop the connection between k-extreme vectors and the combinatorial structure of the poset. Understanding which vectors are k-extreme requires us to compute the dimensions of faces of the polytopes K, L, and $K+L$. In this section we will introduce a basic recipe that enables us to compute face dimensions in terms of the poset structure. With this recipe in hand, we will systematically catalogue the k-extreme directions that will be relevant in the subsequent analysis. It will turn out that some of the combinatorial conditions for vectors to be k-extreme are connected to the properties in Definition 1.4, which explains how these arise in the analysis. Another simple application of the basic recipe will yield a proof of Lemma 1.2 using Lemma 3.6.

Throughout this section, we denote by $\mathrm{aff}' K$ the centered affine hull of a set K, that is, $\mathrm{aff}' K := \mathrm{span}(K-K)$. In particular, note that $\mathrm{aff}'(K+L) = \mathrm{aff}' K + \mathrm{aff}' L$ and that $\dim K = \dim(\mathrm{aff}' K)$ for any convex bodies K, L.

4.1. **The basic recipe.** By definition, the polytopes K, L and their faces are defined as intersections of the order polytope O_P with certain affine hyperplanes. It is straightforward to deduce an upper bound on their dimensions from this information and the poset structure. For example, in $K = O_P \cap \{t \in \mathbb{R}^P : t_y = t_x\}$, the constraint $t_y = t_x$ forces $t_x = t_z = t_y$ for $x < z < y$ by the definition of O_P. Thus

$$K \subset H = \{t \in \mathbb{R}^P : t_x = t_z = t_y \text{ for all } z \in P_{x<\cdot<y}\},$$

which yields $\dim K \leq \dim H = n - 1 - |P_{x<\cdot<y}|$.

To prove a matching lower bound, however, we must show that there are no other constraints than those accounted for in the upper bound. To make this reasoning formal, we introduce the following definition.

Definition 4.1. For any linear extension $f : P \to [n]$, define the simplex

$$\Delta_f := \{t \in [0,1]^P : t_{f^{-1}(1)} \leq t_{f^{-1}(2)} \leq \cdots \leq t_{f^{-1}(n)}\}.$$

Thus $O_P = \bigcup_f \Delta_f$, where the union is over all linear extensions of P.

To illustrate how this will be used, suppose there exists a linear extension f of P so that $P_{x<\cdot<y}$ are the *only* elements of P that lie between x, y in the linear order defined by f (the existence of such a linear extension will be shown in Lemma 4.2). Then it is obvious from the definition of Δ_f that

$$\mathrm{aff}'(\Delta_f \cap \{t \in \mathbb{R}^P : t_y = t_x\}) = H.$$

As $\Delta_f \cap \{t \in \mathbb{R}^P : t_y = t_x\} \subseteq K \subset H$, it follows immediately that $\mathrm{aff}' K = H$, and we can therefore compute $\dim K = n - 1 - |P_{x<\cdot<y}|$.

The above example shows that the key to computing the dimension is to establish the existence of linear extensions that minimally satisfy a given set of constraints. We presently prove a number of lemmas that will be used to construct such minimal linear extensions. The basic recipe illustrated above will be applied systematically in the remainder of this section to compute face dimensions.

In the following, we take for granted the standard fact that every poset has at least one linear extension (such an extension may be found, for example, by iteratively choosing the next element of the linear order to be minimal among the poset elements that have not yet been ordered). We construct three kinds of minimal linear extensions that will be used repeatedly below.

Lemma 4.2. *There exists a linear extension $f : P \to [n]$ so that*

$$\{z \in P : f(x) < f(z) < f(y)\} = P_{x<\cdot<y}.$$

Proof. Choose a linear extension $f : P \to [n]$ that minimizes $f(y)-f(x)$ among all linear extensions of P. By definition, we have $f(x) < f(z) < f(y)$ for $z \in P_{x<\cdot<y}$. On the other hand, if $f(x) < f(z) < f(y)$ for some $z \notin P_{x<\cdot<y}$, then it must be the case that either $z \parallel x$ or $z \parallel y$. We aim to show this cannot occur.

To this end, consider the element $z \parallel x$ so that $f(x) < f(z) < f(y)$ and $f(z)$ is as small as possible. Then for every $f(x) < f(z') < f(z)$, we must have $x < z'$ and thus $z' \parallel z$. We can therefore obtain another linear extension g of P by moving the element z to have rank right below x, while keeping the order of the remaining elements as in f. As $g(y) - g(x) = f(y) - f(x) - 1$, this contradicts minimality of $f(y) - f(x)$. Thus we have ruled out the existence of $f(x) < f(z) < f(y)$ with $z \parallel x$. A completely analogous argument rules out $z \parallel y$, concluding the proof. □

Lemma 4.3. *The following hold.*

a. *Let $S, T \subset P$ satisfy $S \cap T = \varnothing$, S is a lower set ($P_{<z} \subseteq S$ for all $z \in S$), and T is an upper set ($P_{>z} \subseteq T$ for all $z \in T$). Then there is a linear extension $f : P \to [n]$ so that $\{z \in P : f(z) \le |S|\} = S$ and $\{z \in P : f(z) > n-|T|\} = T$.*
b. *There exists a linear extension $f : P \to [n]$ so that*

$$\{z \in P : f(z) < f(x)\} = P_{<x}, \qquad \{z \in P : f(z) > f(y)\} = P_{>y}.$$

Proof. As S is a lower set and T is an upper set, any $z \in S$, $z' \in P\backslash(S \cup T)$, and $z'' \in T$ must satisfy $z' \not\le z$ and $z' \not\ge z''$. Therefore, if we define a linear ordering f of P by choosing its $|S|$ smallest elements to be any linear extension of S, its $|T|$ largest elements to be any linear extension of T, and the remaining elements to be any linear extension of $P\backslash(S \cup T)$, then f is itself a linear extension of P. This proves part (a). Part (b) follows by choosing $S = P_{\le x}$ and $T = P_{\ge y}$. □

Lemma 4.4. *Let $z_1, z_2 \in P$ such that $z_1 \lessdot z_2$, and let $f : P \to [n]$ be any linear extension. Then there is a linear extension $f' : P \to [n]$ so that $f'(z_2) - f'(z_1) = 1$, and so that $f'(z) = f(z)$ whenever $f(z) < f(z_1)$ or $f(z) > f(z_2)$.*

Proof. Choose a linear extension $f' : P \to [n]$ that minimizes $f'(z_2) - f'(z_1)$ among all linear extensions of P such that $f'(z) = f(z)$ when $f(z) < f(z_1)$ or $f(z) > f(z_2)$. If there exists $f'(z_1) < f'(z) < f'(z_2)$, then it must be the case that either $z \parallel z_1$ or $z \parallel z_2$ as $z_1 \lessdot z_2$. This entails a contradiction as in the proof of Lemma 4.2. □

4.2. **The vanishing condition.** As a first illustration of the basic recipe introduced above, we will now give a short proof of Lemma 1.2 using Lemma 3.6. Before we do so, let us define a shorthand notation for the linear spaces that will appear repeatedly throughout the remainder of this section.

Definition 4.5. For any disjoint subsets $S_1, \ldots, S_k \subseteq P$, we define

$$\langle\!\langle S_1, \ldots, S_k \rangle\!\rangle := \operatorname{span}\left\{\textstyle\sum_{z \in S_1} e_z, \ldots, \sum_{z \in S_k} e_z, \mathbb{R}^{P \backslash \{S_1 \cup \cdots \cup S_k\}}\right\}.$$

Note that $\dim \langle\!\langle S_1, \ldots, S_k \rangle\!\rangle = n + k - \sum_{i=1}^{k} |S_i|$.

We now proceed to the proof.

Proof of Lemma 1.2. As $K = O_P \cap \{t \in \mathbb{R}^P : t_y = t_x\}$, every $t \in K$ must satisfy $t_x = t_z = t_y$ for all $z \in P_{x<\cdot<y}$. Thus $\operatorname{aff}' K \subseteq \langle\!\langle P_{x\le\cdot\le y} \rangle\!\rangle$. On the other hand, if f is the linear extension provided by Lemma 4.2, we have

$$\langle\!\langle P_{x\le\cdot\le y} \rangle\!\rangle = \operatorname{aff}'(\Delta_f \cap \{t \in \mathbb{R}^P : t_y = t_x\}) \subseteq \operatorname{aff}' K.$$

We have therefore shown that $\operatorname{aff}' K = \langle\!\langle P_{x\le\cdot\le y} \rangle\!\rangle$.

Similarly, as $L = O_P \cap \{t \in \mathbb{R}^P : t_x = 0, t_y = 1\}$, every $t \in L$ satisfies $t_z = t_x = 0$ for all $z \in P_{<x}$ and $t_z = t_y = 1$ for all $z \in P_{>y}$. Thus $\operatorname{aff}' L \subseteq \mathbb{R}^{P \backslash (P_{\le x} \cup P_{\ge y})}$. On the other hand, if f is the linear extension provided by Lemma 4.3(b), we have

$$\mathbb{R}^{P \backslash (P_{\le x} \cup P_{\ge y})} = \operatorname{aff}'(\Delta_f \cap \{t \in \mathbb{R}^P : t_x = 0, t_y = 1\}) \subseteq \operatorname{aff}' L.$$

We have therefore shown that $\operatorname{aff}' L = \mathbb{R}^{P \backslash (P_{\le x} \cup P_{\ge y})}$.

We can now immediately conclude that

$$\begin{aligned} \dim K &= n + 1 - |P_{x\le\cdot\le y}| = n - 1 - |P_{x<\cdot<y}|, \\ \dim L &= n - |P_{\le x} \cup P_{\ge y}| = n - 2 - |P_{<x}| - |P_{>y}|. \end{aligned}$$

But as $\operatorname{aff}'(K + L) = \operatorname{aff}' K + \operatorname{aff}' L = \langle\!\langle \{x, y\} \rangle\!\rangle$, we also have $\dim(K + L) = n - 1$. The conclusion of Lemma 1.2 now follows from Lemma 3.6. □

The following corollary of Lemma 1.2 will be used frequently.

Corollary 4.6. *If $N_k > 0$ and $N_k^2 = N_{k-1} N_{k+1}$, then*

$$|P_{x<\cdot<y}| + 1 < k < n - 1 - |P_{<x}| - |P_{>y}|.$$

Proof. The assumption implies that $N_{k-1} > 0$ and $N_{k+1} > 0$. The conclusion follows immediately from Lemma 1.2 and Lemma 3.2. □

4.3. **Extreme vectors.** We now turn to the main task of this section, which is to characterize which vectors are k-extreme. More precisely, as we will only use the implication (a)⇒(b) of Proposition 3.5, we do not need to (and will not) characterize *all* k-extreme vectors; it suffices to find vectors that carry significant combinatorial information. To this end, we will consider the following vectors.

Definition 4.7. A vector $u \in V$ is called a

a. *coordinate vector* if $u = \pm e_z$ for $z \in P \backslash \{x, y\}$;
b. *transition vector* if $u = e_{zz'} := e_z - e_{z'}$ for $z, z' \in P \backslash \{x, y\}$;
c. *anchor vector* if $u = e_{zxy} := e_z - \frac{e_x + e_y}{2}$ or $e_{xyz} := \frac{e_x + e_y}{2} - e_z$ for $z \in P \backslash \{x, y\}$.

The motivation for considering these particular vectors is straightforward. By definition, a vector $u \in V$ is k-extreme if the associated faces of K, L, and $K + L$ are sufficiently high-dimensional. As K and L are slices of the order polytope O_P, the most natural candidates for such vectors are the normal directions of the facets (i.e., the highest-dimensional faces) of O_P. It is well known [19, §1] that u is a facet normal of O_P if and only if $u = \pm e_z$ for a maximal (minimal) element z of P, or if $u = e_z - e_{z'}$ for $z \lessdot z'$. This motivates the consideration of coordinate and transition vectors for $z, z' \notin \{x, y\}$. Note, however, that (for example) $e_z - e_x \notin V$ and thus cannot be k-extreme. In these cases we consider instead the projections of such vectors onto V, which are the anchor vectors.

Remark 4.8. The above logic suggests we should consider one additional case $u = \pm\frac{e_x+e_y}{2}$, which corresponds to projecting $\pm e_x$ or $\pm e_y$ onto V. In principle such vectors may indeed be needed in the analysis when x is a minimal element of P or when y is a maximal element of P. However, in the proof of our main results we will be able to assume without loss of generality this is not the case, as inserting a new element in P that is smaller (larger) than all the other elements does not change the numbers N_k. This simple observation does not make any fundamental difference to the proof, but slightly shortens the analysis in a few places.

We now proceed to systematically investigate the combinatorial conditions for coordinate, transition, and anchor vectors to be k-extreme.

4.3.1. *Coordinate vectors.* We begin with a basic observation.

Lemma 4.9. *For $z \in P\backslash\{x, y\}$, the following hold.*

a. If z is a maximal element of P, then $h_K(e_z) = h_L(e_z) = 1$.

b. If z is a minimal element of P, then $h_K(-e_z) = h_L(-e_z) = 0$.

Proof. If z is a maximal element of $P\backslash\{x, y\}$, then $e_z \in K$. Thus

$$1 = \langle e_z, e_z\rangle \leq h_K(e_z) = \sup_{t\in K}\langle e_z, t\rangle \leq 1,$$

where we used that $K \subseteq [0, 1]^P$ in the second inequality. Therefore $h_K(e_z) = 1$. Now let $t' \in \mathbb{R}^P$ be defined by $t'_{z'} = 1_{z'\in P_{\geq y}\cup\{z\}}$. As z is maximal, $t' \in L$ and thus

$$1 = \langle e_z, t'\rangle \leq h_L(e_z) = \sup_{t\in L}\langle e_z, t\rangle \leq 1.$$

We have therefore shown that $h_L(e_z) = 1$, concluding the proof of part (a). The proof of part (b) is completely analogous. □

We can now characterize k-extreme coordinate vectors.

Lemma 4.10. *Let $N_k^2 = N_{k-1}N_{k+1} > 0$. For $z \in P\backslash\{x, y\}$, the following hold.*

a. If z is a maximal element of P, then e_z is k-extreme.

b. If z is a minimal element of P, then $-e_z$ is k-extreme.

Proof. Let z be a maximal element of P. Then Lemma 4.9 yields

$$F(K, e_z) = O_P \cap \{t \in \mathbb{R}^P : t_y = t_x, t_z = 1\},$$
$$F(L, e_z) = O_P \cap \{t \in \mathbb{R}^P : t_x = 0, t_y = t_z = 1\}.$$

Let us compute the affine hulls.

- Clearly $\mathrm{aff}' F(K, e_z) \subseteq \langle\!\langle P_{x\leq\cdot\leq y}\rangle\!\rangle \cap \mathbb{R}^{P\backslash\{z\}}$ (note that $z \notin P_{x\leq\cdot\leq y}$ as z is maximal). Now let f be the linear ordering of P obtained by applying Lemma 4.2 to $P\backslash\{z\}$ and setting $f(z) = n$. As z is maximal, f is a linear extension of P. Thus
$$\langle\!\langle P_{x\leq\cdot\leq y}\rangle\!\rangle \cap \mathbb{R}^{P\backslash\{z\}} = \mathrm{aff}'(\Delta_f \cap \{t \in \mathbb{R}^P : t_y = t_x, t_z = 1\}) \subseteq \mathrm{aff}' F(K, e_z).$$
- Clearly $\mathrm{aff}' F(L, e_z) \subseteq \mathbb{R}^{P\backslash(P_{\leq x}\cup P_{\geq y}\cup\{z\})}$. Now let f be the linear extension of P obtained by applying Lemma 4.3(b) to $P\backslash\{z\}$ and setting $f(z) = n$. Then
$$\mathbb{R}^{P\backslash(P_{\leq x}\cup P_{\geq y}\cup\{z\})} = \mathrm{aff}'(\Delta_f \cap \{t \in \mathbb{R}^P : t_x = 0, t_y = t_z = 1\}) \subseteq \mathrm{aff}' F(L, e_z).$$

Thus $\mathrm{aff}' F(K, e_z) = \langle\!\langle P_{x\le\cdot\le y}\rangle\!\rangle \cap \mathbb{R}^{P\backslash\{z\}}$ and $\mathrm{aff}' F(L, e_z) = \mathbb{R}^{P\backslash(P_{\le x}\cup P_{\ge y}\cup\{z\})}$, which implies $\mathrm{aff}' F(K+L, e_z) = \langle\!\langle \{x,y\}\rangle\!\rangle \cap \mathbb{R}^{P\backslash\{z\}}$. Therefore

$$\begin{aligned}\dim F(K, e_z) &= n-2-|P_{x<\cdot<y}|,\\ \dim F(L, e_z) &= n-2-|P_{<x}|-|P_{>y}\cup\{z\}|,\\ \dim F(K+L, e_z) &= n-2.\end{aligned}$$

It follows readily from Corollary 4.6 that e_z is k-extreme, which concludes the proof of part (a). The proof of part (b) is completely analogous. □

4.3.2. *Transition vectors.* We now turn to the characterization of transition vectors. As above, we must first identify the corresponding supporting hyperplanes.

Lemma 4.11. *Consider $z, z' \in P\backslash\{x, y\}$ such that $z < z'$, and assume that either $z \notin P_{<x}$ or $z' \notin P_{>y}$. Then we have $h_K(e_{zz'}) = h_L(e_{zz'}) = 0$.*

Proof. As $z < z'$, any $t \in O_P$ must satisfy $\langle e_{zz'}, t\rangle = t_z - t_{z'} \le 0$. Thus

$$0 \le h_K(e_{zz'}) = \sup_{t\in K}\langle e_{zz'}, t\rangle \le 0,$$

where we used that $0 \in K$. On the other hand, define $t \in L$ by $t_{z''} = 1_{z''\not\le x}$ if $z \notin P_{<x}$, and by $t_{z''} = 1_{z''\ge y}$ otherwise. Then

$$0 = \langle e_{zz'}, t\rangle \le h_L(e_{zz'}) = \sup_{t'\in L}\langle e_{zz'}, t'\rangle \le 0,$$

concluding the proof. □

We can now characterize k-extreme transition vectors.

Lemma 4.12. *Assume $N_k^2 = N_{k-1}N_{k+1} > 0$. Let $z, z' \in P\backslash\{x, y\}$ such that $z \lessdot z'$. Then the transition vector $e_{zz'}$ is k-extreme in each of the following situations:*

a. $z, z' \in P_{<x}$, *or* $z, z' \in P_{>y}$.
b. $z, z' \in P_{>x,\|y}$, *or* $z, z' \in P_{<y,\|x}$.
c. $z, z' \in P_{x<\cdot<y}$.
d. $z \in P_{>x,\|y}$ *and* $P_{>z} \subseteq P_{>y}$, *or* $z' \in P_{<y,\|x}$ *and* $P_{<z'} \subseteq P_{<x}$.
e. $z \in P_{\|x,\|y}$ *and* $P_{>z} \subseteq P_{>y}$, *or* $z' \in P_{\|x,\|y}$ *and* $P_{<z'} \subseteq P_{<x}$.
f. $z \in P_{\|x,\|y}$ *and* $z' \notin P_{>y}$, *or* $z' \in P_{\|x,\|y}$ *and* $z \notin P_{<x}$.
g. $z \in P_{x<\cdot<y}$, $z' \in P_{>x,\|y}$, *and* $|P_{x<\cdot<z'} \cup P_{x<\cdot<y}| \le k-1$.
h. $z \in P_{<y,\|x}$, $z' \in P_{x<\cdot<y}$, *and* $|P_{z<\cdot<y} \cup P_{x<\cdot<y}| \le k-1$.
i. $z \in P_{<y,\|x}$, $z' \in P_{>x,\|y}$, *and* $|P_{z<\cdot<y} \cup P_{x<\cdot<z'} \cup P_{x<\cdot<y}| \le k-2$.

Proof. We begin by noting that in all cases, we have

$$\begin{aligned}F(K, e_{zz'}) &= O_P \cap \{t \in \mathbb{R}^P : t_y = t_x, t_z = t_{z'}\},\\ F(L, e_{zz'}) &= O_P \cap \{t \in \mathbb{R}^P : t_x = 0, t_y = 1, t_z = t_{z'}\}\end{aligned}$$

by Lemma 4.11. We now consider each case separately.

(a) Assume that $z, z' \in P_{<x}$ (the proof for $z, z' \in P_{>y}$ is completely analogous). Clearly $F(L, e_{zz'}) = L$ and $\mathrm{aff}' F(K, e_{zz'}) \subseteq \langle\!\langle P_{x\le\cdot\le y}, \{z, z'\}\rangle\!\rangle$. On the other hand, by Lemmas 4.2 and 4.4, there is a linear extension f of P so that the only elements between x and y are $P_{x<\cdot<y}$ and such that z, z' are adjacent in the linear order

defined by f. Applying the basic recipe yields $\mathrm{aff}' F(K, e_{zz'}) = \langle\langle P_{x\le\cdot\le y}, \{z, z'\}\rangle\rangle$. As $\mathrm{aff}' L = \mathbb{R}^{P\setminus(P_{\le x}\cup P_{\ge y})}$ by the proof of Lemma 1.2, we obtain

$$\begin{aligned}\dim F(K, e_{zz'}) &= n - 2 - |P_{x<\cdot<y}|,\\ \dim F(L, e_{zz'}) &= n - 2 - |P_{<x}| - |P_{>y}|,\\ \dim F(K + L, e_{zz'}) &= n - 2,\end{aligned}$$

where we use $\mathrm{aff}' F(K+L, e_{zz'}) = \mathrm{aff}' F(K, e_{zz'}) + \mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}, \{x, y\}\rangle\rangle$. It follows readily from Corollary 4.6 that $e_{zz'}$ is k-extreme.

(b) Assume $z, z' \in P_{>x,\|y}$ (the proof for $z, z' \in P_{<y,\|x}$ is completely analogous). Then $\mathrm{aff}' F(K, e_{zz'}) = \langle\langle P_{x\le\cdot\le y}, \{z, z'\}\rangle\rangle$ as in part (a). Moreover, we clearly have $\mathrm{aff}' F(L, e_{zz'}) \subseteq \langle\langle \{z, z'\}\rangle\rangle \cap \mathbb{R}^{P\setminus(P_{\le x}\cup P_{\ge y})}$. By Lemmas 4.3(b) and 4.4, there is a linear extension f of P in which the only elements less than x are $P_{<x}$, the only elements greater than y are $P_{>y}$, and z, z' are adjacent. Applying the basic recipe yields $\mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}\rangle\rangle \cap \mathbb{R}^{P\setminus(P_{\le x}\cup P_{\ge y})}$. We therefore obtain

$$\begin{aligned}\dim F(K, e_{zz'}) &= n - 2 - |P_{x<\cdot<y}|,\\ \dim F(L, e_{zz'}) &= n - 3 - |P_{<x}| - |P_{>y}|,\\ \dim F(K + L, e_{zz'}) &= n - 2,\end{aligned}$$

where we use $\mathrm{aff}' F(K+L, e_{zz'}) = \mathrm{aff}' F(K, e_{zz'}) + \mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}, \{x, y\}\rangle\rangle$. It follows readily from Corollary 4.6 that $e_{zz'}$ is k-extreme.

(c) Clearly $F(K, e_{zz'}) = K$, while $\mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}\rangle\rangle \cap \mathbb{R}^{P\setminus(P_{\le x}\cup P_{\ge y})}$ as in part (b). As $\mathrm{aff}' K = \langle\langle P_{x\le\cdot\le y}\rangle\rangle$ by the proof of Lemma 1.2, we obtain

$$\begin{aligned}\dim F(K, e_{zz'}) &= n - 1 - |P_{x<\cdot<y}|,\\ \dim F(L, e_{zz'}) &= n - 3 - |P_{<x}| - |P_{>y}|,\\ \dim F(K + L, e_{zz'}) &= n - 2,\end{aligned}$$

where we use $\mathrm{aff}' F(K+L, e_{zz'}) = \mathrm{aff}' F(K, e_{zz'}) + \mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}, \{x, y\}\rangle\rangle$. It follows readily from Corollary 4.6 that $e_{zz'}$ is k-extreme.

(d) Assume $z \in P_{>x,\|y}$ and $P_{>z} \subseteq P_{>y}$ (the other case is completely analogous). Then $\mathrm{aff}' F(K, e_{zz'}) = \langle\langle P_{x\le\cdot\le y}, \{z, z'\}\rangle\rangle$ as in part (a). Moreover, we clearly have $\mathrm{aff}' F(L, e_{zz'}) \subseteq \mathbb{R}^{P\setminus(P_{\le x}\cup P_{\ge y}\cup\{z\})}$. Applying Lemma 4.3(a) to $S = P_{\le x}$ and $T = P_{\ge y} \cup \{z\}$ yields $\mathrm{aff}' F(L, e_{zz'}) = \mathbb{R}^{P\setminus(P_{\le x}\cup P_{\ge y}\cup\{z\})}$ using the basic recipe. Thus

$$\begin{aligned}\dim F(K, e_{zz'}) &= n - 2 - |P_{x<\cdot<y}|,\\ \dim F(L, e_{zz'}) &= n - 3 - |P_{<x}| - |P_{>y}|,\\ \dim F(K + L, e_{zz'}) &= n - 2,\end{aligned}$$

where we use $\mathrm{aff}' F(K+L, e_{zz'}) = \mathrm{aff}' F(K, e_{zz'}) + \mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}, \{x, y\}\rangle\rangle$. It follows readily from Corollary 4.6 that $e_{zz'}$ is k-extreme.

(e) Assume $z \in P_{\|x,\|y}$ and $P_{>z} \subseteq P_{>y}$ (the other case is completely analogous). Then $\mathrm{aff}' F(L, e_{zz'}) = \mathbb{R}^{P\setminus(P_{\le x}\cup P_{\ge y}\cup\{z\})}$ as in part (d). Moreover, we clearly have $\mathrm{aff}' F(K, e_{zz'}) \subseteq \langle\langle P_{x\le\cdot\le y}, \{z, z'\}\rangle\rangle$. To prove equality we proceed as in part (a), but we must be more careful in constructing the linear extension f.

By Lemma 4.2, there is a linear extension f of P in which the only elements between x and y are $P_{x<\cdot<y}$. If $f(z) < f(x)$, choose $z_1 \ge z$, $f(z) \le f(z_1) < f(x)$ with maximal $f(z_1)$. We claim that any $f(z_1) < f(z_2) \le f(y)$ must satisfy $z_2 \parallel z_1$:

otherwise $z_2 > z_1 \geq z$, which contradicts maximality of $f(z_1)$ if $f(z_2) < f(x)$ and contradicts $z \in P_{\|x,\|y}$ if $f(x) \leq f(z_2) \leq f(y)$ (as the latter implies $z \in P_{x\leq\cdot\leq y}$). We can therefore obtain a new linear extension of P by moving z_1 right above y in the linear order defined by f, while keeping the remaining ordering fixed. By iterating this process, we can always modify the original linear extension f so that $f(z) > f(y)$. Applying Lemma 4.4 to the latter yields a linear extension in which the only elements between x and y are $P_{x<\cdot<y}$ and z, z' are adjacent. We can now apply the basic recipe to conclude that $\mathrm{aff}' F(K, e_{zz'}) = \langle\langle P_{x\leq\cdot\leq y}, \{z, z'\}\rangle\rangle$.

The rest of the proof of part (e) is identical to that of part (d).

(f) Assume $z \in P_{\|x,\|y}$ and $z' \notin P_{>y}$ (the other case is completely analogous). Then $\mathrm{aff}' F(K, e_{zz'}) = \langle\langle P_{x\leq\cdot\leq y}, \{z, z'\}\rangle\rangle$ as in part (e). Moreover, as $z \in P_{\|x,\|y}$ also implies $z' \notin P_{<x}$, we obtain $\mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}\rangle\rangle \cap \mathbb{R}^{P\setminus(P_{\leq x}\cup P_{\geq y})}$ as in part (b). The rest of the proof of part (f) is identical to that of part (b).

(g) We have $\mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}\rangle\rangle \cap \mathbb{R}^{P\setminus(P_{\leq x}\cup P_{\geq y})}$ as in part (b). Moreover, we clearly have $\mathrm{aff}' F(K, e_{zz'}) \subseteq \langle\langle P_{x\leq\cdot\leq y} \cup P_{x\leq\cdot\leq z'}\rangle\rangle$.

By Lemma 4.2, there is a linear extension f of P in which the only elements between x and y are $P_{x<\cdot<y}$. As $z' \in P_{>x,\|y}$ we must have $f(y) < f(z')$.

Choose $z_1 \notin P_{<z'}$ with $f(y) < f(z_1) < f(z')$, if it exists, so that $f(z_1)$ is maximal. Then any $f(z_1) < f(z_2) \leq f(z')$ must satisfy $z_2 \parallel z_1$: otherwise $z_1 < z_2 \in P_{\leq z'}$ yields a contradiction. We can therefore obtain a new linear extension of P by moving z_1 right above z' while keeping the remaining ordering fixed. Iterating this process, we can ensure that $z'' \in P_{<z'}$ for all $f(y) < f(z'') < f(z')$.

Now choose $z_2 \notin P_{>x}$ with $f(y) < f(z_2) < f(z')$, if it exists, so that $f(z_2)$ is minimal. Then any $f(x) \leq f(z_1) < f(z_2)$ must satisfy $z_1 \parallel z_2$: otherwise $x \leq z_1 < z_2$ or $y = z_1 < z_2$, which contradict $z_2 \notin P_{>x}$ and $z' \in P_{>x,\|y}$ (as we already ensured above that $z_2 \in P_{<z'}$). Thus we can move z_2 right below x while keeping the remaining ordering fixed. Iterating this process, we can always modify the original linear extension f so that $z'' \in P_{x<\cdot<z'}$ for all $f(y) < f(z'') < f(z')$. In particular, the resulting linear extension satisfies

$$\{z'' \in P : f(x) \leq f(z'') \leq f(z')\} = P_{x\leq\cdot\leq y} \cup P_{x\leq\cdot\leq z'}.$$

Applying the basic recipe yields $\mathrm{aff}' F(K, e_{zz'}) = \langle\langle P_{x\leq\cdot\leq y} \cup P_{x\leq\cdot\leq z'}\rangle\rangle$. Thus

$$\begin{aligned} \dim F(K, e_{zz'}) &= n - 2 - |P_{x<\cdot<y} \cup P_{x<\cdot<z'}|, \\ \dim F(L, e_{zz'}) &= n - 3 - |P_{<x}| - |P_{>y}|, \\ \dim F(K + L, e_{zz'}) &= n - 2, \end{aligned}$$

where we use $\mathrm{aff}' F(K+L, e_{zz'}) = \mathrm{aff}' F(K, e_{zz'}) + \mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}, \{x, y\}\rangle\rangle$. It follows from Corollary 4.6 and the assumption that $e_{zz'}$ is k-extreme.

(h) The proof is completely analogous to that of part (g).

(i) We have $\mathrm{aff}' F(L, e_{zz'}) = \langle\langle \{z, z'\}\rangle\rangle \cap \mathbb{R}^{P\setminus(P_{\leq x}\cup P_{\geq y})}$ as in part (b). Next, note that any $t \in F(K, e_{zz'})$ must satisfy $t_z \leq t_y = t_x \leq t_{z'} = t_z$ as $z' > x$ and $z < y$. We therefore clearly have $\mathrm{aff}' F(K, e_{zz'}) \subseteq \langle\langle P_{x\leq\cdot\leq y} \cup P_{x\leq\cdot\leq z'} \cup P_{z\leq\cdot\leq y}\rangle\rangle$.

To prove equality we reason similarly as in part (g). By Lemma 4.2, there is a linear extension f of P in which the only elements between x and y are $P_{x<\cdot<y}$. As $z' > x$ and $z < y$, we must have $f(z) < f(x) < f(y) < f(z')$. By modifying the linear extension as in part (g), we can ensure that all $f(y) < f(z'') < f(z')$ satisfy $z'' \in P_{<z'}$ and all $f(z) < f(z'') < f(x)$ satisfy $z'' \in P_{>z}$.

Now choose $z_2 \notin P_{>x}$ with $f(y) < f(z_2) < f(z')$, if it exists, so that $f(z_2)$ is minimal. Then any $f(z) \le f(z_1) < f(z_2)$ must satisfy $z_1 \parallel z_2$: otherwise $x \le z_1 < z_2$, $y = z_1 < z_2$, or $z \le z_1 < z_2$, which contradict $z_2 \notin P_{>x}$, $z' \in P_{>x,\parallel y}$, and $z \lessdot z'$ (as we already ensured that $z_2 \in P_{<z'}$). Thus we can move z_2 right below z while keeping the remaining ordering fixed. Iterating this process, we can modify the linear extension f so that $z'' \in P_{x<\cdot<z'}$ for all $f(y) < f(z'') < f(z')$. A completely analogous argument ensures also that $z'' \in P_{z<\cdot<y}$ for all $f(z) < f(z'') < f(x)$. In particular, the resulting linear extension satisfies

$$\{z'' \in P : f(z) \le f(z'') \le f(z')\} = P_{x\le\cdot\le y} \cup P_{x\le\cdot\le z'} \cup P_{z\le\cdot\le y},$$

so the basic recipe yields $\mathrm{aff}' F(K, e_{zz'}) = \langle\!\langle P_{x\le\cdot\le y} \cup P_{x\le\cdot\le z'} \cup P_{z\le\cdot\le y}\rangle\!\rangle$. Thus

$$\begin{aligned}
\dim F(K, e_{zz'}) &= n - 3 - |P_{x<\cdot<y} \cup P_{x<\cdot<z'} \cup P_{z<\cdot<y}|,\\
\dim F(L, e_{zz'}) &= n - 3 - |P_{<x}| - |P_{>y}|,\\
\dim F(K + L, e_{zz'}) &= n - 2,
\end{aligned}$$

where we use $\mathrm{aff}' F(K+L, e_{zz'}) = \mathrm{aff}' F(K, e_{zz'}) + \mathrm{aff}' F(L, e_{zz'}) = \langle\!\langle \{z, z'\}, \{x, y\}\rangle\!\rangle$. It follows from Corollary 4.6 and the assumption that $e_{zz'}$ is k-extreme. □

4.3.3. *Anchor vectors.* We conclude with the characterization of anchor vectors. As above, we first identify the corresponding supporting hyperplanes.

Lemma 4.13. *The following hold.*

a. *If $z \in P_{x<\cdot<y} \cup P_{<y,\parallel x}$, then $h_K(e_{zxy}) = 0$ and $h_L(e_{zxy}) = \frac{1}{2}$.*
b. *If $z \in P_{x<\cdot<y} \cup P_{>x,\parallel y}$ then $h_K(e_{xyz}) = 0$ and $h_L(e_{xyz}) = \frac{1}{2}$.*
c. *If $z \in P_{<x}$, then $h_K(e_{zxy}) = 0$ and $h_L(e_{zxy}) = -\frac{1}{2}$.*
d. *If $z \in P_{>y}$, then $h_K(e_{xyz}) = 0$ and $h_L(e_{xyz}) = -\frac{1}{2}$.*

Proof. In all cases, the proof that $h_K(e_{zxy}) = 0$ or $h_K(e_{xyz}) = 0$ is the same as in Lemma 4.11. On the other hand, note that any $t \in L$ satisfies $\langle \frac{e_x+e_y}{2}, t\rangle = \frac{1}{2}$, so

$$h_L(e_{zxy}) = h_L(e_z) - \tfrac{1}{2}, \qquad h_L(e_{xyz}) = h_L(-e_z) + \tfrac{1}{2}.$$

For (a), define $t \in \mathbb{R}^P$ by $t_z = 1_{z \in P_{\ge y} \cup P_{\ge z}}$. As $z \not\le x$, we have $t \in L \subset [0,1]^P$, so

$$1 = \langle e_z, t\rangle \le h_L(e_z) = \sup_{t' \in L} \langle e_z, t'\rangle \le 1.$$

Consequently, we have $h_L(e_{zxy}) = \frac{1}{2}$. The proof of part (b) is completely analogous. For part (c), it suffices to note that $t_z = 0$ for all $t \in L$, so that $h_L(e_z) = 0$ and thus $h_L(e_{zxy}) = -\frac{1}{2}$. The proof of part (d) is completely analogous. □

We can now characterize k-extreme anchor vectors.

Lemma 4.14. *Assume $N_k^2 = N_{k-1}N_{k+1} > 0$.*

a. *If $z \in P_{x<\cdot<y} \cup P_{<y,\parallel x}$ and $z \lessdot y$, then e_{zxy} is k-extreme if $|P_{>z}| + |P_{<x}| \le n - k$.*
b. *If $z \in P_{x<\cdot<y} \cup P_{>x,\parallel y}$ and $x \lessdot z$, then e_{xyz} is k-extreme if $|P_{<z}| + |P_{>y}| \le n - k$.*
c. *If $z \lessdot x$, then e_{zxy} is k-extreme if $|P_{z<\cdot<y} \cup \{x\}| \le k$.*
d. *If $y \lessdot z$, then e_{xyz} is k-extreme if $|P_{x<\cdot<z} \cup \{y\}| \le k$.*

Proof. We first prove (a) (the proof of part (b) is completely analogous). We have

$$F(K, e_{zxy}) = O_P \cap \{t \in \mathbb{R}^P : t_y = t_z = t_x\},$$
$$F(L, e_{zxy}) = O_P \cap \{t_x = 0, t_y = t_z = 1\}$$

by Lemma 4.13. As $z < y$, clearly $\mathrm{aff}' F(L, e_{zxy}) \subseteq \mathbb{R}^{P \setminus (P_{\le x} \cup P_{\ge z})}$. Applying the basic recipe with the linear extension of Lemma 4.3(a) with $S = P_{\le x}$ and $T = P_{\ge z}$ yields $\mathrm{aff}' F(L, e_{zxy}) = \mathbb{R}^{P \setminus (P_{\le x} \cup P_{\ge z})}$. Next, we clearly have $F(K, e_{zxy}) = K$ if $z \in P_{x<\cdot<y}$. If $z \in P_{<y,\|x}$, we obtain $\mathrm{aff}' F(K, e_{zxy}) = \langle\langle P_{x\le\cdot\le y} \cup \{z\}\rangle\rangle$ as in the proof of Lemma 4.12(h) (note that $P_{z<\cdot<y} = \varnothing$ as $z \lessdot y$). In either case

$$\dim F(K, e_{zxy}) \ge n - 2 - |P_{x<\cdot<y}|,$$
$$\dim F(L, e_{zxy}) = n - 2 - |P_{<x}| - |P_{>z}|,$$
$$\dim F(K + L, e_{zxy}) = n - 2,$$

where we use $\mathrm{aff}' F(K + L, e_{zxy}) = \mathrm{aff}' F(K, e_{zxy}) + \mathrm{aff}' F(L, e_{zxy}) = \langle\langle \{x, y, z\}\rangle\rangle$. It follows from Corollary 4.6 and the assumption that e_{zxy} is k-extreme.

We now prove (d) (the proof of part (c) is completely analogous). We have

$$F(K, e_{xyz}) = O_P \cap \{t \in \mathbb{R}^P : t_y = t_z = t_x\},$$
$$F(L, e_{xyz}) = O_P \cap \{t_x = 0, t_y = t_z = 1\} = L$$

by Lemma 4.13. Clearly $\mathrm{aff}' F(K, e_{xyz}) \subseteq \langle\langle P_{x\le\cdot\le y} \cup P_{x\le\cdot\le z}\rangle\rangle$.

To prove equality we reason similarly as for Lemma 4.12(g). By Lemma 4.2, there is a linear extension f of P in which the only elements between x and y are $P_{x<\cdot<y}$. As $y < z$ we must have $f(y) < f(z)$. By modifying f as in the proof of Lemma 4.12(g), we can ensure that all $f(y) < f(z') < f(z)$ satisfy $z' \in P_{<z}$.

Now choose $z_2 \notin P_{>x}$ with $f(y) < f(z_2) < f(z)$, if it exists, so that $f(z_2)$ is minimal. Then any $f(x) \le f(z_1) < f(z_2)$ must satisfy $z_1 \parallel z_2$: otherwise $x \le z_1 < z_2$ or $y = z_1 < z_2$, which contradict $z_2 \notin P_{>x}$ and $y \lessdot z$ (as we already ensured above that $z_2 \in P_{<z}$). We can now repeat the rest of the argument in the proof of Lemma 4.12(g) verbatim to conclude that $\mathrm{aff}' F(K, e_{xyz}) = \langle\langle P_{x\le\cdot\le y} \cup P_{x\le\cdot\le z}\rangle\rangle$. Noting that $P_{x\le\cdot\le y} \cup P_{x\le\cdot\le z} = \{x, y, z\} \cup P_{x<\cdot<z}$, we obtain

$$\dim F(K, e_{xyz}) = n - 1 - |P_{x<\cdot<z} \cup \{y\}|,$$
$$\dim F(L, e_{xyz}) = n - 2 - |P_{<x}| - |P_{>y}|,$$
$$\dim F(K + L, e_{xyz}) = n - 2,$$

where we use $\mathrm{aff}' F(K + L, e_{xyz}) = \mathrm{aff}' F(K, e_{xyz}) + \mathrm{aff}' F(L, e_{xyz}) = \langle\langle \{x, y, z\}\rangle\rangle$. It follows from Corollary 4.6 and the assumption that e_{xyz} is k-extreme. □

5. Proof of the main results

Now that we have characterized k-extreme vectors in combinatorial terms, we aim to combine this information with the equality characterization provided by Proposition 3.5 to prove the main results of this paper. More precisely, our analysis will be based on the following consequences of Proposition 3.5. Here and in the sequel, we will use the notation $v_{xy} := \frac{v_x + v_y}{2}$ for $v \in V$.

Lemma 5.1. *Assume that $N_k > 0$ and $a^2 N_{k+1} = a N_k = N_{k-1}$. Then there exists $v \in V$ so that the following hold for $z, z' \in P \setminus \{x, y\}$.*

a. If z is a maximal element of P, then $v_z = 1 - a$.

b. *If z is a minimal element of P, then $v_z = 0$.*
c. *If $e_{zz'}$ is k-extreme for $z < z'$ with $z \notin P_{<x}$ or $z' \notin P_{>y}$, then $v_z = v_{z'}$.*
d. *If e_{zxy} is k-extreme for $z \in P_{x<\cdot<y} \cup P_{<y,\|x}$, then $v_z = v_{xy} - \frac{a}{2}$.*
e. *If e_{xyz} is k-extreme for $z \in P_{x<\cdot<y} \cup P_{>x,\|y}$, then $v_z = v_{xy} + \frac{a}{2}$.*
f. *If e_{zxy} is k-extreme for $z \in P_{<x}$, then $v_z = v_{xy} + \frac{a}{2}$.*
g. *If e_{xyz} is k-extreme for $z \in P_{>y}$, then $v_z = v_{xy} - \frac{a}{2}$.*

Proof. Proposition 3.5 states that there exists $v \in V$ so that $h_K(u) = ah_L(u)+\langle u, v\rangle$ for every k-extreme vector u. Parts (a) and (b) follow from Lemmas 4.9 and 4.10, (c) follows from Lemma 4.11, and the remaining parts follow from Lemma 4.13. □

Lemma 5.1 shows that each k-extreme direction yields a linear constraint on the vector v. The basic principle behind our main results is that if there are too many k-extreme vectors, the resulting system of linear equations for v has no solution. The latter entails a contradiction, as the existence of v is guaranteed by Proposition 3.5. This will enable us to reason that some vectors must not be k-extreme, which gives rise to the explicit combinatorial conditions in our main results (as in Definition 1.4) using the characterization of k-extreme directions in the previous section. How to reason about the existence of solutions is far from obvious, however, and will require a careful analysis of the structure of the underlying poset.

The fact that many transition vectors, viz. those that are characterized in parts (a)–(f) of Lemma 4.12, are always k-extreme will enable us to fix many entries of v at the outset of the analysis. It will turn out that the nontrivial structure of the extremals is largely (but not entirely) controlled by the value of v_{xy}. In particular, let us define four properties that will play a central role in the analysis; the significance of these properties will become evident in the proofs.

Definition 5.2. We define the following properties:
($\mathscr{M}$) $a \neq \frac{2}{3}(1 - v_{xy})$.
($\mathscr{M}^*$) $a \neq 2v_{xy}$.
($\mathscr{E}$) $a \neq -2v_{xy}$.
($\mathscr{E}^*$) $a \neq 2(1 - v_{xy})$.

We now briefly outline the steps in the proof of our main results. We first show in section 5.1 that ($\mathscr{E}$)⇒(E_k) and ($\mathscr{E}^*$)⇒(E_k^*). In section 5.2, we show that ($\mathscr{M}$)⇒(M_k) and ($\mathscr{M}^*$)⇒(M_k^*). In section 5.3 we argue that if $a \neq \frac{1}{2}$, then either ($\mathscr{M}$) and ($\mathscr{E}$), or ($\mathscr{M}^*$) and ($\mathscr{E}^*$), must hold. This simultaneously proves both Theorems 1.3 and 1.5 (because the implication (c)⇒(a) of Theorem 1.5, which was proved in section 2, then shows that $a \neq \frac{1}{2}$ implies $a = 1$). Finally, we prove Theorem 1.6 in section 5.4 using additional arguments specific to this case.

5.1. **The conditions ($\mathscr{E}$), ($\mathscr{E}^*$).** The aim of this section is to understand the origin of conditions (E_k) and (E_k^*) in Definition 1.4, which are concerned with elements $z \in P_{<x}$ or $P_{>y}$. We begin by computing v_z for these elements.

Lemma 5.3. *Assume that $N_k > 0$ and $a^2N_{k+1} = aN_k = N_{k-1}$, and let $v \in V$ be the vector provided by Lemma 5.1. Then the following hold.*
a. $v_z = 0$ *for every* $z \in P_{<x}$.
b. $v_z = 1 - a$ *for every* $z \in P_{>y}$.

Proof. If $z \in P_{<x}$, let $z \gtrdot z_1 \gtrdot z_2 \gtrdot \cdots \gtrdot z_r$ be a decreasing chain from z to a minimal element z_r of P. Then $z_i \in P_{<x}$ for all i, so that Lemma 4.12(a) and Lemma 5.1(c) yield $v_z = v_{z_1} = v_{z_2} = \cdots = v_{z_r}$. Part (a) follows as $v_{z_r} = 0$ by Lemma 5.1(b). The proof of part (b) follows in a completely analogous fashion using Lemma 5.1(a). □

We now show how the combinatorial conditions of (E_k) and (E_k^*) arise.

Lemma 5.4. *Assume that $N_k > 0$ and $a^2N_{k+1} = aN_k = N_{k-1}$, and let $v \in V$ be the vector provided by Lemma 5.1. Then the following hold.*

a. If ($\mathscr{E}$) holds, then every $z \in P_{<x}$ satisfies $|P_{z<\cdot<y} \cup \{x\}| > k$.

b. If ($\mathscr{E}^*$) holds, then every $z \in P_{>y}$ satisfies $|P_{x<\cdot<z} \cup \{y\}| > k$.

Proof. Suppose there exists $z \in P_{<x}$ such that $|P_{z<\cdot<y} \cup \{x\}| \le k$. Then the latter holds automatically for any $z' \ge z$ (as then $P_{z'<\cdot<y} \subseteq P_{z<\cdot<y}$). In particular, we consider $z' \ge z$ that is a maximal element of $P_{<x}$, so that $z' \lessdot x$. Then Lemma 4.14(c), Lemma 5.1(f), and Lemma 5.3(a) yield $0 = v_{z'} = v_{xy} + \frac{a}{2}$, which is the converse of ($\mathscr{E}$). Thus we proved the contrapositive of part (a).

Similarly, if there exists $z \in P_{>y}$ so that $|P_{x<\cdot<z} \cup \{y\}| \le k$, let $y \lessdot z' \le z$. Then Lemma 4.14(d), Lemma 5.1(g), and Lemma 5.3(b) yield $1 - a = v_{z'} = v_{xy} - \frac{a}{2}$, which is the converse of ($\mathscr{E}^*$). Thus we proved the contrapositive of part (b). □

Lemma 5.4 states that ($\mathscr{E}$) implies the first part of (E_k), and that ($\mathscr{E}^*$) implies the first part of (E_k^*). The second part of (E_k) and of (E_k^*) is already implied by the first part when $P_{<x}$ and $P_{>y}$ are nonempty, so that these additional conditions only need to be established in the special case that x is minimal or that y is maximal. We do not need to consider these cases separately, however, as we can always modify the poset to avoid this situation without changing the linear extension numbers N_k by adding a globally minimal and maximal element to P. We postpone this straightforward argument to the proof of Corollary 5.11.

5.2. **The conditions ($\mathscr{M}$), ($\mathscr{M}^*$).** The aim of this section is to understand the origin of conditions (M_k) and (M_k^*) in Definition 1.4, which are concerned with elements $z \in P_{>x,\not\ge y} = P_{x<\cdot<y} \cup P_{>x,\|y}$ or $P_{<y,\not\le x} = P_{x<\cdot<y} \cup P_{<y,\|x}$. The argument in the case that $z \in P_{>x,\|y}$ or $P_{<y,\|x}$ is similar to section 5.1. The case $z \in P_{x<\cdot<y}$ is more subtle, however, and will require additional insights.

5.2.1. *The case $z \in P_{>x,\|y}$ or $z \in P_{<y,\|x}$.* We begin by computing v_z.

Lemma 5.5. *Assume that $N_k > 0$ and $a^2N_{k+1} = aN_k = N_{k-1}$, and let $v \in V$ be the vector provided by Lemma 5.1. Then the following hold.*

a. $v_z = 0$ for every $z \in P_{<y,\|x}$.

b. $v_z = 1 - a$ for every $z \in P_{>x,\|y}$.

c. $v_z = v_{z'}$ for all $z, z' \in P_{x<\cdot<y}$ with $z < z'$.

Proof. If $z \in P_{<y,\|x}$, let $z \gtrdot z_1 \gtrdot z_2 \gtrdot \cdots \gtrdot z_r$ by a decreasing chain from z to a minimal element z_r of $P_{<y,\|x}$. If z_r is minimal in P, we stop the chain at this point. Otherwise, $z_r \gtrdot z_{r+1}$ for some element $z_{r+1} \in P_{<x}$, and we can continue the chain $z_{r+1} \gtrdot \cdots \gtrdot z_s$ until we reach a minimal element z_s of P.

Now note that Lemma 4.12(a,b) and Lemma 5.1(c) yield $v_z = v_{z_1} = \cdots = v_{z_r}$ and $v_{z_{r+1}} = \cdots = v_{z_s}$. On the other hand, as z_r is a minimal element of $P_{<y,\|x}$, we have $P_{<z_r} \subseteq P_{<x}$, and thus Lemma 4.12(d) and Lemma 5.1(c) yield $z_r = z_{r+1}$.

Part (a) follows as $v_{z_s} = 0$ by Lemma 5.1(b). The proof of part (b) follows in a completely analogous fashion using Lemma 5.1(a).

For part (c), let $z \lessdot z_1 \lessdot \cdots \lessdot z_r \lessdot z'$ be a chain connecting z, z', and note that we must have $z_i \in P_{x<\cdot<y}$ for all i. Thus Lemma 4.12(c) and Lemma 5.1(c) imply that $v_z = v_{z_1} = \cdots = v_{z_r} = v_{z'}$, concluding the proof. □

We now show how the conditions of (M_k) and (M_k^*) arise in the present case.

Lemma 5.6. *Assume that $N_k > 0$ and $a^2 N_{k+1} = aN_k = N_{k-1}$, and let $v \in V$ be the vector provided by Lemma 5.1. Then the following hold.*

a. *If ($\mathscr{M}$) holds, then every $z \in P_{>x,\|y}$ satisfies $|P_{<z}| + |P_{>y}| > n - k$.*
b. *If ($\mathscr{M}^*$) holds, then every $z \in P_{<y,\|x}$ satisfies $|P_{>z}| + |P_{<x}| > n - k$.*

Proof. Suppose there exists $z \in P_{>x,\|y}$ such that $|P_{<z}| + |P_{>y}| \le n - k$. Then the latter condition holds automatically for any $z' \le z$ (as then $P_{<z'} \subseteq P_{<z}$). In particular, we consider $z' \le z$ that is a minimal element of $P_{>x,\|y}$.

Suppose first that $x \lessdot z'$. Lemma 4.14(b), Lemma 5.1(e), and Lemma 5.5(b) then yield $1 - a = v_{z'} = v_{xy} + \frac{a}{2}$, which is the converse of ($\mathscr{M}$). Thus we have proved the contrapositive of part (a) in this case.

On the other hand, if z' does not cover x, then we must have $P_{x<\cdot<z'} \subseteq P_{x<\cdot<y}$ as z' was chosen to be minimal in $P_{>x,\|y}$. Moreover, note that $|P_{x<\cdot<y}| \le k-1$ by Corollary 4.6. Applying Lemma 4.12(g) and Lemma 5.1(c) then shows that $v_{z'} = v_{z''}$ for $x < z'' \lessdot z'$. In particular, Lemma 5.5(b,c) yields $v_{z'''} = 1 - a$ for all $z''' \in P_{x<\cdot\le z''}$. Now choose $x \lessdot z''' \le z'' < z$, and note that $|P_{<z'''}| + |P_{>y}| \le n - k$ still holds. Applying Lemma 4.14(b) and Lemma 5.1(e) yields $1 - a = v_{xy} + \frac{a}{2}$, which is the converse of ($\mathscr{M}$). This completes the proof of part (a).

The proof of part (b) is completely analogous, but now we use Lemma 4.14(a), Lemma 5.1(d), Lemma 5.5(a), and Lemma 4.12(h). □

5.2.2. *The case $z \in P_{x<\cdot<y}$.* Let us begin by explaining the basic difficulty in this case. So far, all our arguments started with the construction of a chain that goes from the element z of interest to a maximal or minimal element of P without passing through x or y. We observed that such chains can be constructed in such a way that all the transition vectors along the chain are k-extreme, so that we can compute the value of v_z using Lemma 5.1 (cf. Lemmas 5.3 and 5.5). However, when $z \in P_{x<\cdot<y}$ it is not even clear that there exists any chain connecting z to a minimal or maximal element of P that does not pass through x or y. In the absence of such a chain we would have no mechanism to obtain information about v_z.

We presently aim to show that when the Kahn-Saks inequality holds with equality, such a chain must always exist. This is not obvious, and arises here in a rather subtle manner from the equality conditions.

We begin by showing that the combinatorial conditions of (M_k) and (M_k^*) cannot simultaneously fail to hold for elements of $P_{x<\cdot<y}$.

Lemma 5.7. *Assume that $N_k > 0$ and $a^2 N_{k+1} = aN_k = N_{k-1}$. Then for any comparable elements $z, z' \in P_{x<\cdot<y}$, the following hold.*

a. *If $|P_{<z}| + |P_{>y}| \le n - k$, then $|P_{>z'}| + |P_{<x}| > n - k$.*
b. *If $|P_{>z}| + |P_{<x}| \le n - k$, then $|P_{<z'}| + |P_{>y}| > n - k$.*

Proof. It suffices to prove part (a), as part (b) is the contrapositive of part (a) with the roles of z, z' reversed. As z, z' are comparable, there is a chain $x \lessdot z_1 \lessdot \cdots \lessdot z_r \lessdot y$

so that $z_i = z$ and $z_j = z'$ for some i, j. Now suppose that part (a) fails, that is, that $|P_{<z}|+|P_{>y}| \leq n-k$ and $|P_{>z'}|+|P_{<x}| \leq n-k$. Then certainly $|P_{<z_1}|+|P_{>y}| \leq n-k$ and $|P_{>z_r}|+|P_{<x}| \leq n-k$ as well. Applying Lemma 4.14(a,b), Lemma 5.1(d,e), and Lemma 5.5(c) yields $v_{xy} + \frac{a}{2} = v_{z_1} = v_{z_r} = v_{xy} - \frac{a}{2}$, which entails a contradiction as $a > 0$. Thus part (a) must hold, concluding the proof. ☐

We now use Lemma 5.7 to reason that if the combinatorial condition of (M_k) or of (M_k^*) fails, then there must exist a chain starting from any point in $P_{x<\cdot<y}$ that leaves this set without passing through x or y.

Lemma 5.8. *Assume that $N_k > 0$ and $a^2N_{k+1} = aN_k = N_{k-1}$, and let $z \in P_{x<\cdot<y}$.*

a. If $|P_{<z}|+|P_{>y}| \leq n-k$, then any $z \leq z_1 \lessdot y$ satisfies $z_1 \lessdot z_2$ for some $z_2 \in P_{>x,\|y}$.

b. If $|P_{>z}|+|P_{<x}| \leq n-k$, then any $z \geq z_1 \gtrdot x$ satisfies $z_1 \gtrdot z_2$ for some $z_2 \in P_{<y,\|x}$.

Proof. Suppose that $|P_{<z}| + |P_{>y}| \leq n - k$, and let $z \leq z_1 \lessdot y$. Suppose that (a) fails, that is, that z_1 is not covered by any element of $P_{>x,\|y}$. Then we must have $P_{>z_1} = P_{\geq y}$. We can therefore estimate using Lemma 5.7(a) and Corollary 4.6

$$n - k < |P_{>z_1}| + |P_{<x}| = |P_{<x}| + |P_{>y}| + 1 < n - k,$$

which entails a contradiction. Thus part (a) is proved, and the proof of part (b) is completely analogous using Lemma 5.7(b). ☐

With this structural information in hand, we can proceed to proving a counterpart of Lemma 5.6 for elements $z \in P_{x<\cdot<y}$.

Lemma 5.9. *Assume that $N_k > 0$ and $a^2N_{k+1} = aN_k = N_{k-1}$, and let $v \in V$ be the vector provided by Lemma 5.1. Then the following hold.*

a. If $(\mathscr{M})$ holds, then every $z \in P_{x<\cdot<y}$ satisfies $|P_{<z}| + |P_{>y}| > n - k$.

b. If $(\mathscr{M}^*)$ holds, then every $z \in P_{x<\cdot<y}$ satisfies $|P_{>z}| + |P_{<x}| > n - k$.

Proof. Let $z \in P_{x<\cdot<y}$ and $x \lessdot z_1 \leq z \leq z_2 \lessdot y$. Suppose that $|P_{<z}|+|P_{>y}| \leq n-k$. Then Lemma 5.8(a) shows that $z_2 \lessdot z_3$ for some $z_3 \in P_{>x,\|y}$, and Lemma 5.7(a) yields $|P_{>z_2}| + |P_{<x}| > n - k$. Moreover, note that the sets $P_{x<\cdot<y} \cup P_{x<\cdot<z_3}$, $P_{<x}$, and $P_{>z_2}$ are disjoint. We can therefore estimate

$$|P_{x<\cdot<y} \cup P_{x<\cdot<z_3}| + n - k + 1 \leq |P_{x<\cdot<y} \cup P_{x<\cdot<z_3}| + |P_{>z_2}| + |P_{<x}| \leq n,$$

which yields $|P_{x<\cdot<y} \cup P_{x<\cdot<z_3}| \leq k - 1$. Thus Lemma 4.12(g), Lemma 5.1(c), and Lemma 5.5(b,c) show that $v_{z_1} = v_{z_2} = v_{z_3} = 1 - a$. On the other hand, as $|P_{<z}| + |P_{>y}| \leq n - k$ clearly implies $|P_{<z_1}| + |P_{>y}| \leq n - k$, Lemma 4.14(b) and Lemma 5.1(e) yield $1 - a = v_{z_1} = v_{xy} + \frac{a}{2}$ which is the converse of $(\mathscr{M})$.

Thus we have proved the contrapositive of part (a). The proof of part (b) is completely analogous, but now we use Lemma 5.8(b), Lemma 5.7(b), Lemma 4.12(h), Lemma 5.5(a,c), Lemma 4.14(a), and Lemma 5.1(d). ☐

Combining Lemma 5.6 and 5.9, we conclude that $(\mathscr{M})\Rightarrow(\mathrm{M}_k)$ and $(\mathscr{M}^*)\Rightarrow(\mathrm{M}_k^*)$.

5.3. **The case $a \neq \frac{1}{2}$.** We are now ready to prove Theorems 1.3 and 1.5. The basic observation is the following simple fact.

Lemma 5.10. *If $a \neq \frac{1}{2}$, then either $(\mathscr{M})$ and $(\mathscr{E})$ hold, or $(\mathscr{M}^*)$ and $(\mathscr{E}^*)$ hold.*

Proof. If the conclusion fails, then it must be the case that either $(\mathscr{M})$ and $(\mathscr{M}^*)$ fail, or $(\mathscr{M})$ and $(\mathscr{E}^*)$ fail, or $(\mathscr{E})$ and $(\mathscr{M}^*)$ fail, or $(\mathscr{E})$ and $(\mathscr{E}^*)$ fail. We now show that each of these possibilities yields a contradiction.

a. If ($\mathscr{M}$) and ($\mathscr{M}^*$) both fail, then $a = \frac{2}{3}(1 - v_{xy}) = 2v_{xy}$ implies that $a = \frac{1}{2}$, which contradicts the assumption.
b. If ($\mathscr{M}$) and ($\mathscr{E}^*$) both fail, then $a = \frac{2}{3}(1 - v_{xy}) = 2(1 - v_{xy})$ implies that $a = 0$, which is impossible as $a > 0$ by assumption.
c. If ($\mathscr{E}$) and ($\mathscr{M}^*$) both fail, then $a = 2v_{xy} = -2v_{xy}$ implies that $a = 0$, which is impossible as $a > 0$ by assumption.
d. If ($\mathscr{E}$) and ($\mathscr{E}^*$) both fail, then $-2v_{xy} = 2(1 - v_{xy})$ is evidently impossible.

This concludes the proof. □

We can now conclude the following.

Corollary 5.11. *Assume that $N_k > 0$ and $a^2 N_{k+1} = aN_k = N_{k-1}$ with $a \neq \frac{1}{2}$. Then either* (M_k) *and* (E_k) *hold, or* (M_k^*) *and* (E_k^*) *hold.*

Proof. If $P_{<x} \neq \varnothing$ and $P_{>y} \neq \varnothing$, the second condition of (E_k) and (E_k^*) is subsumed by the first. Then the result follows from Lemmas 5.10, 5.4, 5.6, and 5.9.

Otherwise, we augment the poset P by adding a globally minimal and maximal element, i.e., $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$ with the additional relations $\hat{0} < z < \hat{1}$ for all $z \in P$. As $\hat{0}$ and $\hat{1}$ must appear at the beginning and end of every linear extension, the numbers N_k are unchanged if we replace P by $\hat{P}$. Thus we conclude that either (M_k) and (E_k) hold for $\hat{P}$, or (M_k^*) and (E_k^*) hold for $\hat{P}$. It remains to verify that these conditions for $\hat{P}$ imply the corresponding conditions for P.

To this end, note that (E_k) for $\hat{P}$ states that $|\hat{P}_{z<\cdot<y} \cup \{x\}| > k$ for all $z \in \hat{P}_{<x}$. Applying this condition to $z \in P_{<x}$ yields the first part of (E_k) for P, while applying this condition to $z = \hat{0}$ yields the second part of (E_k) for P. The proof that (E_k^*) for $\hat{P}$ implies (E_k^*) for P is completely analogous. On the other hand, note that (M_k) for $\hat{P}$ states that $|\hat{P}_{<z}| + |\hat{P}_{>y}| > n + 2 - k$ for all $\hat{P}_{>x,\not\geq y}$ (as $|\hat{P}| = n+2$). As $\hat{P}_{<z} = P_{<z} \cup \{\hat{0}\}$ and $\hat{P}_{>y} = P_{>y} \cup \{\hat{1}\}$, the validity of ($\mathrm{M}_k$) for P follows readily. The proof that (M_k^*) for $\hat{P}$ implies (M_k^*) for P is completely analogous. □

We now complete the proofs of Theorems 1.3 and 1.5.

Proof of Theorem 1.5. The implications (c)⇒(b)⇒(a) were proved in section 2. The implication (a)⇒(c) follows by applying Corollary 5.11 with $a = 1$. □

Proof of Theorem 1.3. The implication (b)⇒(a) is trivial. Conversely, suppose that (a) holds. Then we clearly have $a^2 N_{k+1} = aN_k = N_{k-1}$ for some $a > 0$ (as we assumed $N_k > 0$). If $a = \frac{1}{2}$, we have $N_{k+1} = 2N_k = 4N_{k-1}$. If $a \neq \frac{1}{2}$, then Corollary 5.11 and the implication (c)⇒(a) of Theorem 1.5 yield $N_{k+1} = N_k = N_{k-1}$. Thus we have proved (a)⇒(b), concluding the proof. □

5.4. **The case $a = \frac{1}{2}$.** The proof of Theorem 1.6, which is concerned with the equality case $a^2 N_{k+1} = aN_k = N_{k-1}$ for $a = \frac{1}{2}$ (i.e., $N_{k+1} = 2N_k = 4N_{k-1}$), requires us to obtain additional information on the structure of P. Let us begin by explaining why we must have $P_{\|x,\|y} = \varnothing$ in this case.

Lemma 5.12. *Let $N_k > 0$ and $a^2 N_{k+1} = aN_k = N_{k-1}$. If $P_{\|x,\|y} \neq \varnothing$, then $a = 1$.*

Proof. If $P_{\|x,\|y} \neq \varnothing$, there is a chain $z_0 \lessdot z_1 \lessdot \cdots \lessdot z_r$ from a minimal element z_0 to a maximal element z_r of $P_{\|x,\|y}$. Lemma 4.12(f) and Lemma 5.1(c) yield $v_{z_0} = \cdots = v_{z_r}$. To conclude the proof, we show that $v_{z_0} = 0$ and $v_{z_r} = 1 - a$.

If z_r is maximal in P, then $v_{z_r} = 1-a$ by Lemma 5.1(a). Otherwise, if there exists $z_r \lessdot z \in P_{\|y}$, we must have $z \in P_{>x,\|y}$ (as $z \in P_{\|x}$ would contradict maximality of z_r in $P_{\|x,\|y}$, while $z_r \lessdot z \in P_{\le x}$ would contradict $z_r \in P_{\|x}$). Then Lemma 4.12(f), Lemma 5.1(c), and Lemma 5.5(b) yield $v_{z_r} = v_z = 1-a$. Finally, if z_r is not maximal in P and there does not exist $z_r \lessdot z \in P_{\|y}$, we must have $P_{>z_r} \subseteq P_{>y}$ (as $z_r < z \in P_{\le y}$ would contradict $z_r \in P_{\|y}$). Then Lemma 4.12(e), Lemma 5.1(c), and Lemma 5.3(b) yield $v_{z_r} = 1-a$. The proof that $v_{z_0} = 0$ is completely analogous. □

Next, we prove a result that will be used to show that $P_{x<\cdot<y} = \varnothing$ when $a = \frac{1}{2}$. We state a slightly more general form than is needed in the proof of Theorem 1.6, as it will provide some additional information (Lemma 5.14 below): in essence, we show that (M_k) and (E_k^*) (and analogously (M_k^*) and (E_k)) cannot both hold.

Lemma 5.13. *The following hold.*

a. *Let* $x \lessdot z \in P_{>x,\ngeq y}$ *and* $y \lessdot z'$. *If* $|P_{<z}|+|P_{>y}| > n-k$, *then* $|P_{x<\cdot<z'} \cup \{y\}| \le k$.
b. *Let* $y \gtrdot z \in P_{<y,\nleq x}$ *and* $z' \lessdot x$. *If* $|P_{>z}|+|P_{<x}| > n-k$, *then* $|P_{z'<\cdot<y} \cup \{x\}| \le k$.

Proof. Suppose that $x \lessdot z \in P_{>x,\ngeq y}$ satisfies $|P_{<z}| + |P_{>y}| > n-k$, and let $y \lessdot z'$. As the sets $P_{>x,\not>y}$, $P_{<z}$, and $P_{>y}$ are disjoint, and as $P_{x<\cdot<z'} \subseteq P_{>x,\not>y}$, we have

$$|P_{x<\cdot<z'}| + n - k + 1 \le |P_{>x,\not>y}| + |P_{<z}| + |P_{>y}| \le n,$$

so that $|P_{x<\cdot<z'}| \le k-1$. Thus $|P_{x<\cdot<z'} \cup \{y\}| \le k$, which completes the proof of part (a). The proof of part (b) is completely analogous. □

We can now prove Theorem 1.6.

Proof of Theorem 1.6. The implications (c)⇒(b)⇒(a) were proved in section 2, so it remains to prove the implication (a)⇒(c). By the same augmentation argument as in the proof of Corollary 5.11, we can assume without loss of generality in the remainder of the proof that $P_{<x} \neq \varnothing$ and $P_{>y} \neq \varnothing$.

By assumption, we have $N_k > 0$ and $a^2 N_{k+1} = a N_k = N_{k-1}$ with $a = \frac{1}{2}$. Thus the conditions of Definition 5.2 reduce to

$$(\mathscr{M}) \Leftrightarrow (\mathscr{M}^*) \Leftrightarrow v_{xy} \neq \tfrac{1}{4}, \qquad (\mathscr{E}) \Leftrightarrow v_{xy} \neq -\tfrac{1}{4}, \qquad (\mathscr{E}^*) \Leftrightarrow v_{xy} \neq \tfrac{3}{4}.$$

If $v_{xy} \neq \frac{1}{4}$, then ($\mathscr{M}$), ($\mathscr{M}^*$), and either ($\mathscr{E}$) or ($\mathscr{E}^*$) hold. If this were the case, then Lemmas 5.4, 5.6, and 5.9 and the implication (c)⇒(a) yield $N_{k+1} = N_k = N_{k-1}$, which contradicts the assumption. Thus we must have $v_{xy} = \frac{1}{4}$. We therefore conclude that both ($\mathscr{E}$) and ($\mathscr{E}^*$) hold, which implies using Lemma 5.4 (and as we assumed that $P_{<x}, P_{>y}$ are nonempty) that (E_k) and (E_k^*) both hold.

That $P_{\|x,\|y} = \varnothing$ was shown in Lemma 5.12. We claim that also $P_{x<\cdot<y} = \varnothing$. Indeed, if the latter does not hold, then there exist $x \lessdot z_1 \le z_2 \lessdot y$, while there exist $z' \lessdot x$ and $y \lessdot z''$ as we assumed that $P_{<x}, P_{>y}$ are nonempty. By Lemma 5.7, we have either $|P_{<z_1}| + |P_{>y}| > n-k$ or $|P_{>z_2}| + |P_{<x}| > n-k$. Thus Lemma 5.13 shows that either $|P_{z'<\cdot<y} \cup \{x\}| \le k$ or $|P_{x<\cdot<z''} \cup \{y\}| \le k$. But this contradicts the validity of (E_k) and (E_k^*), establishing the claim.

It remains to prove that (C_k) holds. Suppose to the contrary that there exist $z \in P_{<y,\|x}$, $z' \in P_{>x,\|y}$ with $z < z'$ so that $|P_{z<\cdot<y}| + |P_{x<\cdot<z'}| \le k-2$. As $P_{\|x,\|y} = P_{x<\cdot<y} = \varnothing$, we must have $P_{z<\cdot<z'} \subset P_{<y,\|x} \cup P_{>x,\|y}$. Thus there must exist $z \le z_1 \lessdot z_2 \le z'$ so that $z_1 \in P_{<y,\|x}$ and $z_2 \in P_{>x,\|y}$, and

$$|P_{z_1<\cdot<y}| + |P_{x<\cdot<z_2}| \le |P_{z<\cdot<y}| + |P_{x<\cdot<z'}| \le k-2.$$

Consequently $e_{z_1z_2}$ is k-extreme by Lemma 4.12(i). But then Lemma 5.1(c) and Lemma 5.5(a,b) yield $0 = v_{z_1} = v_{z_2} = 1-a$, which contradicts the assumption that $a = \frac{1}{2}$. Thus (C_k) must hold, concluding the proof. □

We conclude this section with an additional fact that is not needed in the proofs of our main results, but helps clarify the conditions of Theorems 1.5 and 1.6.

Lemma 5.14. *The following hold.*

a. If (M_k) *holds, then* (E_k^*) *must fail.*

b. If (M_k^*) *holds, then* (E_k) *must fail.*

Proof. Suppose that (M_k) holds. We consider four cases.

- If $P_{>x,\not\geq y} \neq \varnothing$ and $P_{>y} \neq \varnothing$, then (E_k^*) must fail by Lemma 5.13(a).
- If $P_{>x,\not\geq y} \neq \varnothing$ and $P_{>y} = \varnothing$, then (M_k) implies $|P_{<z}| > n-k$ for $x \lessdot z \in P_{>x,\not\geq y}$. As $P_{<z}$ and $P_{>x}$ are disjoint, $n-k+|P_{>x}| < |P_{<z}|+|P_{>x}| \leq n$ contradicts (E_k^*).
- If $P_{>x,\not\geq y} = \varnothing$ and $P_{>y} \neq \varnothing$, then $|P_{x<\cdot<z} \cup \{y\}| = 1$ for $y \lessdot z$ contradicts (E_k^*).
- Finally, if $P_{>x,\not\geq y} = P_{>y} = \varnothing$, then $|P_{>x} \cup \{y\}| = 1$ which contradicts (E_k^*).

This proves part (a). The proof of part (b) is completely analogous. □

5.5. **An explicit example.** A surprising aspect of the results of this paper is that the equality cases of the Alexandrov-Fenchel inequality that arise here need not respect the lattice structure of the underlying polytopes, as was discussed in section 1.2.4. The following simple example illustrates this phenomenon.

Consider the poset P with $|P| = 6$ defined by the relations

$$z_1 \lessdot z_2 \lessdot y, \qquad x \lessdot z_3 \lessdot z_4.$$

It is readily verified that Theorem 1.6 yields a doubling progression for $k = 2$; in fact, we manually compute $N_1 = 1$, $N_2 = 2$, $N_3 = 4$.

In this example, the polytopes of Kahn and Saks are given by

$$K = \{t \in \mathbb{R}^P : 0 \leq t_{z_1} \leq t_{z_2} \leq t_y = t_x \leq t_{z_3} \leq t_{z_4} \leq 1\},$$
$$L = \{t \in \mathbb{R}^P : 0 \leq t_{z_1} \leq t_{z_2} \leq t_y = 1,\ 0 = t_x \leq t_{z_3} \leq t_{z_4} \leq 1\}.$$

Now note that Proposition 3.5 applies with $a = \frac{1}{2}$, so that $h_K(u) = h_{aL+v}(u)$ for all k-extreme vectors u. We claim that the translation vector v may be chosen as

$$v_{z_1} = v_{z_2} = v_y = 0, \qquad v_{z_3} = v_{z_4} = v_x = \tfrac{1}{2}.$$

This can be read off from the proof of our main results. Indeed, the values of v_{z_i} are given by Lemma 5.5. On the other hand, the proof of Theorem 1.6 shows that $v_{xy} = \frac{1}{4}$, so it is compatible with Proposition 3.5 to choose $v_y = 0$ and $v_x = \frac{1}{2}$ (this choice yields $v \notin V$, but this is irrelevant by Remark 3.3; the present choice was made to make the equality condition easiest to visualize).

From these computations, it is readily seen that

$$\begin{aligned} aL + v &= \{t \in \mathbb{R}^P : 0 \leq t_{z_1} \leq t_{z_2} \leq t_y = \tfrac{1}{2} = t_x \leq t_{z_3} \leq t_{z_4} \leq 1\} \\ &= K \cap \{t \in \mathbb{R}^P : t_x = t_y = \tfrac{1}{2}\}. \end{aligned}$$

Even though the polytopes K and $aL + v$ do not coincide, the proof of our main results shows that they must have the same supporting hyperplanes in all k-extreme directions. On the other hand, note that $aL + v$ is obtained by intersecting K by a non-lattice hyperplane, so that the equality condition of the Alexandrov-Fenchel inequality does not respect the lattice structure of the polytopes K, L.

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