# Structured Random Matrices

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**Abstract** Random matrix theory is a well-developed area of probability theory that has numerous connections with other areas of mathematics and its applications. Much of the literature in this area is concerned with matrices that possess many exact or approximate symmetries, such as matrices with i.i.d. entries, for which precise analytic results and limit theorems are available. Much less well understood are matrices that are endowed with an arbitrary structure, such as sparse Wigner matrices or matrices whose entries possess a given variance pattern. The challenge in investigating such structured random matrices is to understand how the given structure of the matrix is reflected in its spectral properties. This chapter reviews a number of recent results, methods, and open problems in this direction, with a particular emphasis on sharp spectral norm inequalities for Gaussian random matrices.

## 1 Introduction

The study of random matrices has a long history in probability, statistics, and mathematical physics, and continues to be a source of many interesting old and new mathematical problems [2, 25]. Recent years have seen impressive advances in this area, particularly in the understanding of universality phenomena that are exhibited by the spectra of classical random matrix models [8, 26]. At the same time, random matrices have proved to be of major importance in contemporary applied mathematics, see, for example, [28, 32] and the references therein.

Much of classical random matrix theory is concerned with highly symmetric models of random matrices. For example, the simplest random matrix model, the *Wigner matrix*, is a symmetric matrix whose entries above the diagonal are independent and identically distributed. If the entries are chosen to be Gaussian (and the diagonal entries are chosen to have the appropriate variance), this model is additionally invariant under orthogonal transformations. Such strong symmetry properties make it possible to obtain extremely precise analytic results on the asymptotic properties of macroscopic and microscopic spectral statistics of these matrices, and give rise to deep connections with classical analysis, representation theory, combinatorics, and various other areas of mathematics [2, 25].

Much less is understood, however, once we depart from such highly symmetric settings and introduce nontrivial structure into the random matrix model. Such

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models are the topic of this chapter. To illustrate what we mean by "structure," let us describe some typical examples that will be investigated in the sequel.

- A *sparse Wigner matrix* is a matrix with a given (deterministic) sparsity pattern, whose nonzero entries above the diagonal are i.i.d. centered random variables. Such models have interesting applications in combinatorics and computer science (see, for example, [1]), and specific examples such as random band matrices are of significant interest in mathematical physics (cf. [22]). The "structure" of the matrix is determined by its sparsity pattern. We would like to know how the given sparsity pattern is reflected in the spectral properties of the matrix.
- Let $x_1, \ldots, x_s$ be deterministic vectors. Matrices of the form

  $$X = \sum_{k=1}^{s} g_k x_k x_k^*,$$

  where $g_1, \ldots, g_s$ are i.i.d. standard Gaussian random variables, arise in functional analysis (see, for example, [20]). The "structure" of the matrix is determined by the positions of the vectors $x_1, \ldots, x_s$. We would like to know how the given positions are reflected in the spectral properties of the matrix.
- Let $X_1, \ldots, X_n$ be i.i.d. random vectors with covariance matrix $\Sigma$. Consider

  $$Z = \frac{1}{n} \sum_{k=1}^{n} X_k X_k^*,$$

  the *sample covariance matrix* [32, 10]. If we think of $X_1, \ldots, X_n$ are observed data from an underlying distribution, we can think of $Z$ as an unbiased estimator of the covariance matrix $\Sigma = \mathbf{E}Z$. The "structure" of the matrix is determined by the covariance matrix $\Sigma$. We would like to know how the given covariance matrix is reflected in the spectral properties of $Z$ (and particularly in $\|Z - \Sigma\|$).

While these models possess distinct features, we will refer to such models collectively as *structured random matrices*. We emphasize two important features of such models. First, the symmetry properties that characterize classical random matrix models are manifestly absent in the structured setting. Second, it is evident in the above models that it does not make much sense to investigate their asymptotic properties (that is, probabilistic limit theorems): as the structure is defined for the given matrix only, there is no natural way to take the size of these matrices to infinity.

Due to these observations, the study of structured random matrices has a significantly different flavor than most of classical random matrix theory. In the absence of asymptotic theory, our main interest is to obtain nonasymptotic *inequalities* that identify what structural parameters control the macroscopic properties of the underlying random matrix. In this sense, the study of structured random matrices is very much in the spirit of probability in Banach spaces [12], which is heavily reflected in the type of results that have been obtained in this area. In particular, the aspect of structured random matrices that is most well understood is the behavior of matrix norms, and particularly the spectral norm, of such matrices. The investigation of the latter will be the focus of the remainder of this chapter.

In view of the above discussion, it should come as no surprise that some of the earliest general results on structured random matrices appeared in the functional analysis literature [27, 13, 11], but further progress has long remained relatively limited. More recently, the study of structured random matrices has received renewed attention due to the needs of applied mathematics, cf. [28] and the references therein. However, significant new progress was made in the past few years. On the one hand, suprisingly sharp inequalities were recently obtained for certain random matrix models, particularly in the case of independent entries, that yield nearly optimal bounds and go significantly beyond earlier results. On the other hand, very simple new proofs have been discovered for some (previously) deep classical results that shed new light on the underlying mechanisms and that point the way to further progress in this direction. The opportunity therefore seems ripe for an elementary presentation of the results in this area. The present chapter represents the author's attempt at presenting some of these ideas in a cohesive manner.

Due to the limited capacity of space and time, it is certainly impossible to provide an encylopedic presentation of the topic of this chapter, and some choices had to be made. In particular, the following focus is adopted throughout this chapter:

- The emphasis throughout is on spectral norm inequalities for *Gaussian* random matrices. The reason for this is twofold. On the one hand, much of the difficulty of capturing the structure of random matrices arises already in the Gaussian setting, so that this provides a particularly clean and rich playground for investigating such problems. On the other hand, Gaussian results extend readily to much more general distributions, as will be discussed further in section 4.4.
- For simplicity of presentation, no attempt was made to optimize the universal constants that appear in most of our inequalities, even though many of these inequalities can in fact be obtained with surprisingly sharp (even optimal) constants. The original references can be consulted for more precise statements.
- The presentation is by no means exhaustive, and many variations on and extensions of the presented material have been omitted. None of the results in this chapter are original, though I have done my best to streamline the presentation. On the other hand, I have tried to make the chapter as self-contained as possible, and most results are presented with complete proofs.

The remainder of this chapter is organized as follows. The preliminary section 2 sets the stage by discussing the basic methods that will be used throughout this chapter to bound spectral norms of random matrices. Section 3 is devoted to a family of powerful but suboptimal inequalities, the noncommutative Khintchine inequalities, that are applicable to the most general class of structured random matrices that we will encounter. In section 4, we specialize to structured random matrices with independent entries (such as sparse Wigner matrices) and derive nearly optimal bounds. We also discuss a few fundamental open problems in this setting. We conclude this chapter in the short section 5 by investigating sample covariance matrices.

## 2 How to bound matrix norms

As was discussed in the introduction, the investigation of random matrices with arbitrary structure has by its nature a nonasymptotic flavor: we aim to obtain probabilistic inequalities (upper and lower bounds) on spectral properties of the matrices in question that capture faithfully the underlying structure. At present, this program is largely developed in the setting of spectral norms of random matrices, which will be our focus thorughout this chapter. For completeness, we define:

**Definition 2.1.** The *spectral norm* $\|X\|$ is the largest singular value of the matrix $X$.

For convenience, we generally work with symmetric random matrices $X = X^*$. There is no loss of generality in doing so, as will be explained below.

Before we can obtain any meaningful bounds, we must first discuss some basic approaches for bounding the spectral norms of random matrices. The most important methods that are used for this purpose are collected in this section.

### *2.1 The moment method*

Let $X$ be an $n \times n$ symmetric random matrix. The first difficulty one encounters in bounding the spectral norm $\|X\|$ is that the map $X \mapsto \|X\|$ is highly nonlinear. It is therefore not obvious how to efficiently relate the distribution of $\|X\|$ to the distribution of the entries $X_{ij}$. One of the most effective approaches to simplifying this relationship is obtained by applying the following elementary observation.

**Lemma 2.2.** *Let $X$ be an $n \times n$ symmetric matrix. Then*

$$\|X\| \asymp \mathrm{Tr}[X^{2p}]^{1/2p} \quad \textit{for } p \asymp \log n.$$

The beauty of this observation is that unlike $\|X\|$, which is a very complicated function of the entries of $X$, the quantity $\mathrm{Tr}[X^{2p}]$ is a *polynomial* in the matrix entries. This means that $\mathbf{E}[\mathrm{Tr}[X^{2p}]]$, the $2p$-th moment of the matrix $X$, can be evaluated explicitly and subjected to further analysis. As Lemma 2.2 implies that

$$\mathbf{E}[\|X\|^{2p}]^{1/2p} \asymp \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1/2p} \quad \text{for } p \asymp \log n,$$

this provides a direct route to controlling the spectral norm of a random matrix. Various incarnations of this idea are referred to as the *moment method*.

Lemma 2.2 actually has nothing to do with matrices. Given $x \in \mathbb{R}^n$, everyone knows that $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$, so that $\|x\|_p \approx \|x\|_\infty$ when $p$ is large. How large should $p$ be for this to be the case? The following lemma provides the answer.

**Lemma 2.3.** *If $p \asymp \log n$, then $\|x\|_p \asymp \|x\|_\infty$ for all $x \in \mathbb{R}^n$.*

*Proof.* It is trivial that

$$\max_{i\le n}|x_i|^p \le \sum_{i\le n}|x_i|^p \le n\max_{i\le n}|x_i|^p.$$

Thus $\|x\|_\infty \le \|x\|_p \le n^{1/p}\|x\|_\infty$, and $n^{1/p} = e^{(\log n)/p} \asymp 1$ when $\log n \asymp p$. □

The proof of Lemma 2.2 follows readily by applying Lemma 2.3 to the spectrum.

*Proof (Proof of Lemma 2.2).* Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ be the eigenvalues of $X$. Then $\|X\| = \|\lambda\|_\infty$ and $\mathrm{Tr}[X^{2p}]^{1/2p} = \|\lambda\|_{2p}$. The result follows from Lemma 2.3. □

The moment method will be used frequently throughout this chapter as the first step in bounding the spectral norm of random matrices. However, the moment method is just as useful in the vector setting. As a warmup exercise, let us use this approach to bound the maximum of i.i.d. Gaussian random variables (which can be viewed as a vector analogue of bounding the maximum eigenvalue of a random matrix). If $g \sim N(0,I)$ is the standard Gaussian vector in $\mathbb{R}^n$, Lemma 2.3 implies

$$[\mathbf{E}\|g\|_\infty^p]^{1/p} \asymp [\mathbf{E}\|g\|_p^p]^{1/p} \asymp [\mathbf{E}g_1^p]^{1/p} \quad \text{for } p \asymp \log n.$$

Thus the problem of bounding the maximum of $n$ i.i.d. Gaussian random variables is reduced by the moment method to computing the $\log n$-th moment of a single Gaussian random variable. We will bound the latter in section 3.1 in preparation for proving the analogous bound for random matrices. For our present purposes, let us simply note the outcome of this computation $[\mathbf{E}g_1^p]^{1/p} \lesssim \sqrt{p}$ (Lemma 3.1), so that

$$\mathbf{E}\|g\|_\infty \le [\mathbf{E}\|g\|_\infty^{\log n}]^{1/\log n} \lesssim \sqrt{\log n}.$$

This bound is in fact sharp (up to the universal constant).

*Remark 2.4.* Lemma 2.2 implies immediately that

$$\mathbf{E}\|X\| \asymp \mathbf{E}[\mathrm{Tr}[X^{2p}]^{1/2p}] \quad \text{for } p \asymp \log n.$$

Unfortunately, while this bound is sharp by construction, it is essentially useless: the expectation of $\mathrm{Tr}[X^{2p}]^{1/2p}$ is in principle just as difficult to compute as that of $\|X\|$ itself. The utility of the moment method stems from the fact that we can explicitly compute the expectation of $\mathrm{Tr}[X^{2p}]$, a polynomial in the matrix entries. This suggests that the moment method is well-suited in principle only for obtaining sharp bounds on the $p$th moment of the spectral norm

$$\mathbf{E}[\|X\|^{2p}]^{1/2p} \asymp \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1/2p} \quad \text{for } p \asymp \log n,$$

and not on the first moment $\mathbf{E}\|X\|$ of the spectral norm. Of course, as $\mathbf{E}\|X\| \le [\mathbf{E}\|X\|^{2p}]^{1/2p}$ by Jensen's inequality, this yields an *upper* bound on the first moment of the spectral norm. We will see in the sequel that this upper bound is often, but not always, sharp. We can expect the moment method to yield a sharp bound on $\mathbf{E}\|X\|$

when the fluctuations of $\|X\|$ are of a smaller order than its mean; this was the case, for example, in the computation of $\mathbf{E}\|g\|_\infty$ above. On the other hand, the moment method is inherently dimension-dependent (as one must choose $p \sim \log n$), so that it is generally not well suited for obtaining dimension-free bounds.

We have formulated Lemma 2.2 for symmetric matrices. A completely analogous approach can be applied to non-symmetric matrices. In this case, we use that

$$\|X\|^2 = \|X^*X\| \asymp \mathrm{Tr}[(X^*X)^p]^{1/p} \quad \text{for } p \asymp \log n,$$

which follows directly from Lemma 2.2. However, this non-symmetric form is often somewhat inconvenient in the proofs of random matrix bounds, or at least requires additional bookkeeping. Instead, we recall a classical trick that allows us to directly obtain results for non-symmetric matrices from the analogous symmetric results. If $X$ is any $n \times m$ rectangular matrix, then it is readily verified that $\|\tilde{X}\| = \|X\|$, where $\tilde{X}$ denotes the $(n+m) \times (n+m)$ symmetric matrix defined by

$$\tilde{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

Therefore, to obtain a bound on the norm $\|X\|$ of a non-symmetric random matrix, it suffices to apply the corresponding result for symmetric random matrices to the doubled matrix $\tilde{X}$. For this reason, it is not really necessary to treat non-symmetric matrices separately, and we will conveniently restrict our attention to symmetric matrices throughout this chapter without any loss of generality.

*Remark 2.5.* A variant on the moment method is to use the bounds

$$e^{t\lambda_{\max}(X)} \le \mathrm{Tr}[e^{tX}] \le n e^{t\lambda_{\max}(X)},$$

which gives rise to so-called "matrix concentration" inequalities. This approach has become popular in recent years (particularly in the applied mathematics literature) as it provides easy proofs of a number of useful inequalities. Matrix concentration bounds are often stated in terms of tail probabilities $\mathbf{P}[\lambda_{\max}(X) > t]$, and therefore appear at first sight to provide more information than expected norm bounds. This is not the case, however: the resulting tail bounds are highly suboptimal, and much sharper inequalities can be obtained by combining expected norm bounds with concentration inequalities [5] or chaining tail bounds [7]. As in the case of classical concentration inequalities, the moment method essentially subsumes the matrix concentration approach and is often more powerful. We therefore do not discuss this approach further, but refer to [28] for a systematic development.

## 2.2 The random process method

While the moment method introduced in the previous section is very powerful, it has a number of drawbacks. First, while the matrix moments $\mathbf{E}[\mathrm{Tr}[X^{2p}]]$ can typically be computed explicitly, extracting useful information from the resulting expressions is a nontrivial matter that can result in difficult combinatorial problems. Moreover, as discussed in Remark 2.4, in certain cases the moment method *cannot* yield sharp bounds on the expected spectral norm $\mathbf{E}\|X\|$. Finally, the moment method can only yield information on the spectral norm of the matrix; if other operator norms are of interest, this approach is powerless. In this section, we develop an entirely different method that provides a fruitful approach for addressing these issues.

The present method is based on the following trivial fact.

**Lemma 2.6.** *Let $X$ be an $n \times n$ symmetric matrix. Then*

$$\|X\| = \sup_{v \in B} |\langle v, Xv \rangle|,$$

*where $B$ denotes the Euclidean unit ball in $\mathbb{R}^n$.*

When $X$ is a symmetric random matrix, we can view $v \mapsto \langle v, Xv \rangle$ as a *random process* that is indexed by the Euclidean unit ball. Thus controlling the expected spectral norm of $X$ is none other than a special instance of the general probabilistic problem of controlling the expected supremum of a random process. There exist a powerful methods for this purpose (see, e.g., [24]) that could potentially be applied in the present setting to generate insight on the structure of random matrices.

Already the simplest possible approach to bounding the suprema of random processes, the $\varepsilon$-net method, has proved to be very useful in the study of basic random matrix models. The idea behind this approach is to approximate the supremum over the unit ball $B$ by the maximum over a finite discretization $B_\varepsilon$ of the unit ball, which reduces the problem to computing the maximum of a finite number of random variables (as we did, for example, in the previous section when we computed $\|g\|_\infty$). Let us briefly sketch how this approach works in the following basic example. Let $X$ be the $n \times n$ symmetric random matrix with i.i.d. standard Gaussian entries above the diagonal. Such a matrix is called a *Wigner matrix*. Then for every vector $v \in B$, the random variable $\langle v, Xv \rangle$ is Gaussian with variance at most 2. Now let $B_\varepsilon$ be a finite subset of the unit ball $B$ in $\mathbb{R}^n$ such that every point in $B$ is within distance at most $\varepsilon$ from a point in $B_\varepsilon$. Such a set is called an *$\varepsilon$-net*, and should be viewed as a uniform discretization of the unit ball $B$ at the scale $\varepsilon$. Then we can bound, for small $\varepsilon$,[1]

$$\mathbf{E}\|X\| = \mathbf{E}\sup_{v \in B} |\langle v, Xv \rangle| \lesssim \mathbf{E}\sup_{v \in B_\varepsilon} |\langle v, Xv \rangle| \lesssim \sqrt{\log |B_\varepsilon|},$$

where we used that the expected maximum of $k$ Gaussian random variables with variance $\lesssim 1$ is bounded by $\lesssim \sqrt{\log k}$ (we proved this in the previous section using

[1] The first inequality follows by noting that for every $v \in B$, choosing $\tilde{v} \in B_\varepsilon$ such that $\|v - \tilde{v}\| \le \varepsilon$, we have $|\langle v, Xv \rangle| = |\langle \tilde{v}, X\tilde{v} \rangle + \langle v - \tilde{v}, X(v + \tilde{v}) \rangle| \le |\langle \tilde{v}, X\tilde{v} \rangle| + 2\varepsilon\|X\|$.

the moment method: note that independence was not needed for the upper bound.) A classical argument shows that the smallest $\varepsilon$-net in $B$ has cardinality of order $\varepsilon^{-n}$, so the above argument yields a bound of order $\mathbf{E}\|X\| \lesssim \sqrt{n}$ for Wigner matrices. It turns out that this bound is in fact sharp in the present setting: Wigner matrices satisfy $\mathbf{E}\|X\| \asymp \sqrt{n}$ (we will prove this more carefully in section 3.2 below).

Variants of the above argument have proved to be very useful in random matrix theory, and we refer to [32] for a systematic development. However, $\varepsilon$-net arguments are usually applied to highly symmetric situations, such as is the case for Wigner matrices (all entries are identically distributed). The problem with the $\varepsilon$-net method is that it is sharp essentially only in this situation: this method cannot incorporate nontrivial structure. To illustrate this, consider the following typical structured example. Fix a certain sparsity pattern of the matrix $X$ at the outset (that is, choose a subset of the entries that will be forced to zero), and choose the remaining entries to be independent standard Gaussians. In this case, a "good" discretization of the problem cannot simply distribute points uniformly over the unit ball $B$, but rather must take into account the geometry of the given sparsity pattern. Unfortunately, it is entirely unclear how this is to be accomplished in general. For this reason, $\varepsilon$-net methods have proved to be of limited use for *structured* random matrices, and they will play essentially no role in the remainder of this chapter.

*Remark 2.7.* Deep results from the theory of Gaussian processes [24] guarantee that the expected supremum of any Gaussian process and of many other random processes can be captured sharply by a sophisticated multiscale counterpart of the $\varepsilon$-net method called the generic chaining. Therefore, in principle, it should be possible to capture precisely the norm of structured random matrices if one is able to construct a near-optimal multiscale net. Unfortunately, the general theory only guarantees the existence of such a net, and provides essentially no mechanism to construct one in any given situation. From this perspective, structured random matrices provide a particularly interesting case study of inhomogeneous random processes whose investigation could shed new light on these more general mechanisms (this perspective provided strong motivation for this author's interest in random matrices). At present, however, progress along these lines remains in a very primitive state. Note that even the most trivial of examples from the random matrix perspective, such as the case where $X$ is a diagonal matrix with i.i.d. Gaussian entries on the diagonal, require already a delicate multiscale net to obtain sharp results; see, e.g., [30].

As direct control of the random processes that arise from structured random matrices is largely intractable, a different approach is needed. To this end, the key idea that we will exploit is the use of *comparison theorems* to bound the expected supremum of one random process by that of another random process. The basic idea is to design a suitable comparison process that dominates the random process of Lemma 2.6 but that is easier to control. For this approach to be successful, the comparison process must capture the structure of the original process while at the same time being amenable to some form of explicit computation. In principle there is no reason to expect that this is ever possible. Nonetheless, we will repeatedly apply different variations on this approach to obtain the best known bounds on structured random matrices. Comparison methods are a recurring theme throughout this chapter, and we postpone further discussion to the following sections.

Let us note that the random process method is easily extended also to non-symmetric matrices: if $X$ is an $n \times m$ rectangular matrix, we have

$$\|X\| = \sup_{v,w \in B} \langle v, Xw \rangle.$$

Alternatively, we can use the same symmetrization trick as was illustrated in the previous section to reduce to the symmetric case. For this reason, we will restrict attention to symmetric matrices in the sequel. Let us also note, however, that unlike the moment method, the present approach extends readily to other operator norms by replacing the Euclidean unit ball $B$ by the unit ball for other norms. In this sense, the random process method is substantially more general than the moment method, which is restricted to the spectral norm. However, the spectral norm is often the most interesting norm in practice in applications of random matrix theory.

## 2.3 Roots and poles

The moment method and random process method discussed in the previous sections have proved to be by far the most useful approaches to bounding the spectral norms of random matrices, and all results in this chapter will be based on one or both of these methods. We want to briefly mention a third approach, however, that has recently proved to be useful. It is well-known from linear algebra that the eigenvalues of a symmetric matrix $X$ are the roots of the characteristic polynomial

$$\chi(t) = \det(tI - X),$$

or, equivalently, the poles of the Stieltjes transform

$$s(t) := \mathrm{Tr}[(tI - X)^{-1}] = \frac{d}{dt} \log \chi(t).$$

One could therefore attempt to bound the extreme eigenvalues of $X$ (and therefore the spectral norm $\|X\|$) by controlling the location of the largest root (pole) of the characteristic polynomial (Stieltjes tranform) of $X$, with high probability.

The Stieltjes transform method plays a major role in random matrix theory [2], as it provides perhaps the simplest route to proving limit theorems for the spectral distributions of random matrices. It is possible along these lines to prove asymptotic results on the extreme eigenvalues, see [3] for example. However, as the Stieltjes transform is highly nonlinear, it seems to be very difficult to use this approach to address nonasymptotic questions for structured random matrices where explicit limit information is meaningless. The characteristic polynomial appears at first sight to be more promising, as this is a polynomial in the matrix entries: one can therefore hope to compute $\mathbf{E}\chi$ exactly. This simplicity is deceptive, however, as there is no reason to expect that maxroot($\mathbf{E}\chi$) has any relation to the quantity $\mathbf{E}\,$maxroot($\chi$) that we are interested in. It was therefore long believed that such an approach does not provide any useful tool in random matrix theory. Nonetheless, a determinisitic version of this idea plays the crucial role in the recent breakthrough resolution of the Kadison-Singer conjecture [15], so that it is conceivable that such an approach could prove to be fruitful in problems of random matrix theory (cf. [23] where related ideas were applied to Stieltjes transforms in a random matrix problem). To date, however, these methods have not been successefully applied to the problems investigated in this chapter, and they will make no further appearance in the sequel.

## 3 Khintchine-type inequalities

The main aim of this section is to introduce a very general method for bounding the spectral norm of structured random matrices. The basic idea, due to Lust-Piquard [13], is to prove an analog of the classical Khintchine inequality for scalar random variables in the noncommutative setting. This *noncommutative Khintchine inequality* allows us to bound the moments of structured random matrices, which immediately results in a bound on the spectral norm by Lemma 2.2.

The advantage of the noncommutative Khintchine inequality is that it can be applied in a remarkably general setting: it does not even require independence of the matrix entries. The downside of this inequality is that it almost always gives rise to bounds on the spectral norm that are suboptimal by a multiplicative factor that is logarithmic in the dimension (cf. section 4.2). We will discuss the origin of this suboptimality and some potential methods for reducing it in the general setting of this section. Much sharper bounds will be obtained in section 4 under the additional restriction that the matrix entries are independent.

For simplicity, we will restrict our attention to matrices with Gaussian entries, though extensions to other distributions are easily obtained (for example, see [14]).

### *3.1 The noncommutative Khintchine inequality*

In this section, we will consider the following very general setting. Let $X$ be an $n\times n$ symmetric random matrix with zero mean. The only assumption we make on the distribution is that the entries on and above the diagonal (that is, those entries that are not fixed by symmetry) are centered and jointly Gaussian. In particular, these entries can possess an arbitrary covariance matrix, and are assumed to be neither identically distributed nor independent. Our aim is to bound the spectral norm $\|X\|$ in terms of the given covariance structure of the matrix.

It proves to be convenient to reformulate our random matrix model somewhat. Let $A_1,\ldots,A_s$ be *nonrandom* $n\times n$ symmetric matrices, and let $g_1,\ldots,g_s$ be independent standard Gaussian variables. Then we define the matrix $X$ as

$$X = \sum_{k=1}^{s} g_k A_k.$$

Clearly $X$ is a symmetric matrix with jointly Gaussian entries. Conversely, the reader will convince herself after a moment's reflection that any symmetric matrix with centered and jointly Gaussian entries can be written in the above form for some choice of $s \le n(n+1)/2$ and $A_1, \ldots, A_s$. There is therefore no loss of generality in considering the present formulation (we will reformulate our ultimate bounds in a way that does not depend on the choice of the coefficient matrices $A_k$).

Our intention is to apply the moment method. To this end, we must obtain bounds on the moments $\mathbf{E}[\mathrm{Tr}[X^{2p}]]$ of the matrix $X$. It is instructive to begin by considering the simplest possible case where the dimension $n = 1$. In this case, $X$ is simply a scalar Gaussian random variable with zero mean and variance $\sum_k A_k^2$, and the problem in this case reduces to bounding the moments of a scalar Gaussian variable.

**Lemma 3.1.** *Let $g \sim N(0,1)$. Then $\mathbf{E}[g^{2p}]^{1/2p} \le \sqrt{2p-1}$.*

*Proof.* We use the following fundamental *gaussian integration by parts* property:

$$\mathbf{E}[g f(g)] = \mathbf{E}[f'(g)].$$

To prove it, simply note that integration by parts yields

$$\int_{-\infty}^{\infty} x f(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = \int_{-\infty}^{\infty} \frac{df(x)}{dx} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

for smooth functions $f$ with compact support, and the conclusion is readily extended by approximation to any $C^1$ function for which the formula makes sense.

We now apply the integration by parts formula to $f(x) = x^{2p-1}$ as follows:

$$\mathbf{E}[g^{2p}] = \mathbf{E}[g \cdot g^{2p-1}] = (2p-1)\mathbf{E}[g^{2p-2}] \le (2p-1)\mathbf{E}[g^{2p}]^{1-1/p},$$

where the last inequality is by Jensen. Rearranging yields the conclusion. □

Applying Lemma 3.1 yields immediately that

$$\mathbf{E}[X^{2p}]^{1/2p} \le \sqrt{2p-1} \left[ \sum_{k=1}^{s} A_k^2 \right]^{1/2} \quad \text{when } n = 1.$$

It was realized by Lust-Piquard [13] that the analogous inequality holds in any dimension $n$ (the correct dependence of the bound on $p$ was obtained later, cf. [17]).

**Theorem 3.2 (Noncommutative Khintchine inequality).** *In the present setting*

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1/2p} \le \sqrt{2p-1}\, \mathrm{Tr}\left[ \left( \sum_{k=1}^{s} A_k^2 \right)^{p} \right]^{1/2p}.$$

By combining this bound with Lemma 2.2, we immediately obtain the following conclusion regarding the spectral norm of the matrix $X$.

**Corollary 3.3.** *In the setting of this section,*

$$\mathbf{E}\|X\| \lesssim \sqrt{\log n}\,\Bigg\|\sum_{k=1}^{s} A_k^2\Bigg\|^{1/2}.$$

This bound is expressed directly in terms of the coefficient matrices $A_k$ that determine the structure of $X$, and has proved to be extremely useful in applications of random matrix theory in functional analysis and applied mathematics. To what extent this bound is sharp will be discussed in the next section.

*Remark 3.4.* Recall that our bounds apply to any symmetric matrix $X$ with centered and jointly Gaussian entries. Our bounds should therefore not depend on the choice of representation in terms of the coefficient matrices $A_k$, which is not unique. It is easily verified that this is the case. Indeed, it suffices to note that

$$\mathbf{E}X^2 = \sum_{k=1}^{s} A_k^2,$$

so that we can express the conclusion of Theorem 3.2 and Corollary 3.3 as

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1/2p} \lesssim \sqrt{p}\,\mathrm{Tr}[(\mathbf{E}X^2)^p]^{1/2p}, \qquad \mathbf{E}\|X\| \lesssim \sqrt{\log n}\,\|\mathbf{E}X^2\|^{1/2}$$

without reference to the coefficient matrices $A_k$. We note that the quantity $\|\mathbf{E}X^2\|$ has a natural interpretation: it measures the size of the matrix $X$ "on average" (as the expectation in this quantity is *inside* the spectral norm).

We now turn to the proof of Theorem 3.2. We begin by noting that the proof follows immediately from Lemma 3.1 not just when $n = 1$, but also in any dimension $n$ under the additional assumption that the matrices $A_1, \ldots, A_s$ commute. Indeed, in this case we can work without loss of generality in a basis in which all the matrices $A_k$ are simultaneously diagonal, and the result follows by applying Lemma 3.1 to every diagonal entry of $X$. The crucial idea behind the proof of Theorem 3.2 is that *the commutative case is in fact the worst case situation*! This idea will appear very explicitly in the proof: we will simply repeat the proof of Lemma 3.1, and the result will follow by showing that we can permute the order of the matrices $A_k$ at the pivotal point in the proof. (The simple proof given here follows [29].)

*Proof (Proof of Theorem 3.2).* As in the proof of Lemma 3.1, we obtain

$$\begin{aligned}\mathbf{E}[\mathrm{Tr}[X^{2p}]] &= \mathbf{E}[\mathrm{Tr}[X \cdot X^{2p-1}]] \\ &= \sum_{k=1}^{s} \mathbf{E}[g_k \mathrm{Tr}[A_k X^{2p-1}]] \\ &= \sum_{\ell=0}^{2p-2} \sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X^{\ell} A_k X^{2p-2-\ell}]]\end{aligned}$$

using Gaussian integration by parts. The crucial step in the proof is the observation that permuting $A_k$ and $X^\ell$ inside the trace can only increase the bound.

**Lemma 3.5.** $\mathrm{Tr}[A_k X^\ell A_k X^{2p-2-\ell}] \le \mathrm{Tr}[A_k^2 X^{2p-2}]$.

*Proof.* Let us write $X$ in terms of its eigendecomposition $X = \sum_{i=1}^n \lambda_i v_i v_i^*$, where $\lambda_i$ and $v_i$ denote the eigenvalues and eigenvectors of $X$. Then we can write

$$\mathrm{Tr}[A_k X^\ell A_k X^{2p-2-\ell}] = \sum_{i,j=1}^{n} \lambda_i^\ell \lambda_j^{2p-2-\ell} |\langle v_i, A_k v_j\rangle|^2 \le \sum_{i,j=1}^{n} |\lambda_i|^\ell |\lambda_j|^{2p-2-\ell} |\langle v_i, A_k v_j\rangle|^2.$$

But note that the right-hand side is a convex function of $\ell$, so that its maximum in the interval $[0, 2p-2]$ is attained either at $\ell = 0$ or $\ell = 2p-2$. This yields

$$\mathrm{Tr}[A_k X^\ell A_k X^{2p-2-\ell}] \le \sum_{i,j=1}^{n} |\lambda_j|^{2p-2} |\langle v_i, A_k v_j\rangle|^2 = \mathrm{Tr}[A_k^2 X^{2p-2}],$$

and the proof is complete. □

We now complete the proof of the noncommutative Khintchine inequality. Substituting Lemma 3.5 into the previous inequality yields

$$\begin{aligned}\mathbf{E}[\mathrm{Tr}[X^{2p}]] &\le (2p-1) \sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k^2 X^{2p-2}]] \\ &\le (2p-1)\, \mathrm{Tr}\Bigg[\bigg(\sum_{k=1}^{s} A_k^2\bigg)^p\Bigg]^{1/p} \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1-1/p},\end{aligned}$$

where we used Hölder's inequality $\mathrm{Tr}[YZ] \le \mathrm{Tr}[|Y|^p]^{1/p} \mathrm{Tr}[|Z|^{p/(p-1)}]^{1-1/p}$ in the last step. Rearranging this expression yields the desired conclusion. □

*Remark 3.6.* The proof of Corollary 3.3 given here, using the moment method, is exceedingly simple. However, by its nature, it can only bound the spectral norm of the matrix, and would be useless if we wanted to bound other operator norms. It is worth noting that an alternative proof of Corollary 3.3 was developed by Rudelson, using deep random process machinery described in Remark 2.7, for the special case where the matrices $A_k$ are all of rank one (see [24, Prop. 16.7.4] for an exposition of this proof). The advantage of this approach is that it extends to some other operator norms, which proves to be useful in Banach space theory. It is remarkable, however, that no random process proof of Corollary 3.3 is known to date in the general setting.

### 3.2 How sharp are Khintchine inequalities?

Corollary 3.3 provides a very convenient bound on the spectral norm $\|X\|$: it is expressed directly in terms of the coefficients $A_k$ that define the structure of the matrix $X$. However, is this structure captured *correctly*? To understand the degree to which Corollary 3.3 is sharp, let us augment it with a lower bound.

**Lemma 3.7.** *Let $X = \sum_{k=1}^{s} g_k A_k$ as in the previous section. Then*

$$\left\|\sum_{k=1}^{s} A_k^2\right\|^{1/2} \lesssim \mathbf{E}\|X\| \lesssim \sqrt{\log n}\left\|\sum_{k=1}^{s} A_k^2\right\|^{1/2}.$$

*That is, the noncommutative Khintchine bound is sharp up to a logarithmic factor.*

*Proof.* The upper bound in Corollary 3.3, and it remains to prove the lower bound. A slightly simpler bound is immediate by Jensen's inequality: we have

$$\mathbf{E}\|X\|^2 \geq \|\mathbf{E}X^2\| = \left\|\sum_{k=1}^{s} A_k^2\right\|.$$

It therefore remains to show that $(\mathbf{E}\|X\|)^2 \gtrsim \mathbf{E}\|X\|^2$, or, equivalently, that $\mathrm{Var}\|X\| \lesssim (\mathbf{E}\|X\|)^2$. To bound the fluctuations of the spectral norm, we recall an important property of Gaussian random variables (see, for example, [16]).

**Lemma 3.8 (Gaussian concentration).** *Let $g$ be a standard Gaussian vector in $\mathbb{R}^n$, let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, and let $p \geq 1$. Then*

$$[\mathbf{E}(f(g) - \mathbf{E}f(g))^p]^{1/p} \lesssim \sqrt{p}\,[\mathbf{E}\|\nabla f(g)\|^p]^{1/p}.$$

*Proof.* Let $g'$ be an independent copy of $g$, and define $g(\varphi) = g\sin\varphi + g'\cos\varphi$. Then

$$f(g) - f(g') = \int_0^{\pi/2} \frac{d}{d\varphi} f(g(\varphi))\,d\varphi = \int_0^{\pi/2} \langle g'(\varphi), \nabla f(g(\varphi))\rangle\,d\varphi,$$

where $g'(\varphi) = \frac{d}{d\varphi} g(\varphi)$. Applying Jensen's inequality twice gives

$$\mathbf{E}(f(g) - \mathbf{E}f(g))^p \leq \mathbf{E}(f(g) - f(g'))^p \leq \frac{2}{\pi}\int_0^{\pi/2} \mathbf{E}(\tfrac{\pi}{2}\langle g'(\varphi), \nabla f(g(\varphi))\rangle)^p\,d\varphi.$$

Now note that $(g(\varphi), g'(\varphi)) \stackrel{d}{=} (g, g')$ for every $\varphi$. We can therefore apply Lemma 3.1 conditionally on $g(\varphi)$ to estimate for every $\varphi$

$$[\mathbf{E}\langle g'(\varphi), \nabla f(g(\varphi))\rangle^p]^{1/p} \lesssim \sqrt{p}\,\mathbf{E}\|\nabla f(g(\varphi))\|^p]^{1/p} = \sqrt{p}\,\mathbf{E}\|\nabla f(g)\|^p]^{1/p},$$

and substituting into the above expression completes the proof. □

We apply Lemma 3.8 to the function $f(x) = \|\sum_{k=1}^{s} x_k A_k\|$. Note that

$$|f(x) - f(x')| \le \left\| \sum_{k=1}^{s} (x_k - x'_k) A_k \right\| = \sup_{v \in B} \left| \sum_{k=1}^{s} (x_k - x'_k) \langle v, A_k v \rangle \right|$$
$$\le \|x - x'\| \sup_{v \in B} \left[ \sum_{k=1}^{s} \langle v, A_k v \rangle^2 \right]^{1/2} =: \sigma_* \|x - x'\|.$$

Thus $f$ is $\sigma_*$-Lipschitz, so $\|\nabla f\| \le \sigma_*$, and Lemma 3.8 yields $\mathrm{Var}\|X\| \lesssim \sigma_*^2$. But as

$$\sigma_* = \sqrt{\frac{\pi}{2}} \sup_{v \in B} \mathbf{E} \left| \sum_{k=1}^{s} g_k \langle v, A_k v \rangle \right| \le \sqrt{\frac{\pi}{2}} \mathbf{E}\|X\|,$$

we have $\mathrm{Var}\|X\| \lesssim (\mathbf{E}\|X\|)^2$, and the proof is complete. □

Lemma 3.7 shows that the structural quantity $\sigma := \| \sum_{k=1}^{s} A_k^2 \|^{1/2} = \|\mathbf{E}X^2\|^{1/2}$ that appears in the noncommutative Khintchine inequality is very natural: the expected spectral norm $\mathbf{E}\|X\|$ is controlled by $\sigma$ up to a logarithmic factor in the dimension. It is not at all clear, *a priori*, whether the upper or lower bound in Lemma 3.7 is sharp. It turns out that either the upper bound or the lower bound may be sharp in different situations. Let us illustrate this in two extreme examples.

*Example 3.9 (Diagonal matrix).* Consider the case where $X$ is a diagonal matrix

$$X = \begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{bmatrix}$$

with i.i.d. standard Gaussian entries on the diagonal. In this case,

$$\mathbf{E}\|X\| = \mathbf{E}\|g\|_\infty \asymp \sqrt{\log n}.$$

On the other hand, we clearly have

$$\sigma = \|\mathbf{E}X^2\|^{1/2} = 1,$$

so the upper bound in Lemma 3.7 is sharp. This shows that the logarithmic factor in the noncommutative Khintchine inequality *cannot* be removed.

*Example 3.10 (Wigner matrix).* Let $X$ be a symmetric matrix

$$X = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{12} & g_{22} & & g_{2n} \\ \vdots & & \ddots & \vdots \\ g_{1n} & g_{2n} & \cdots & g_{nn} \end{bmatrix}$$

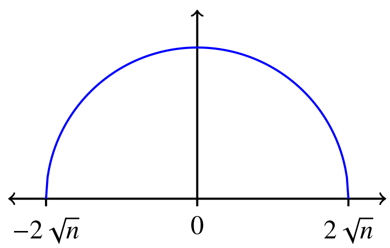
with i.i.d. standard Gaussian entries on and above the diagonal. In this case

$$\sigma = \|\mathbf{E}X^2\|^{1/2} = \sqrt{n}.$$

Thus Lemma 3.7 yields the bounds

$$\sqrt{n} \lesssim \mathbf{E}\|X\| \lesssim \sqrt{n \log n}.$$

Which bound is sharp? A hint can be obtained from what is perhaps the most classical result in random matrix theory: the empirical spectral distribution of the matrix $n^{-1/2}X$ (that is, the random probability measure on $\mathbb{R}$ that places a point mass on every eigenvalue of $n^{-1/2}X$) converges weakly to the Wigner semicircle distribution $\frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx$ [2, 25]. Therefore, when the dimension $n$ is large, the eigenvalues of $X$ are approximately distributed according to the following density:




This picture strongly suggests that the spectrum of $X$ is supported at least approximately in the interval $[-2\sqrt{n}, 2\sqrt{n}]$, which implies that $\|X\| \asymp \sqrt{n}$.

**Lemma 3.11.** *For the Wigner matrix of Example 3.10,* $\mathbf{E}\|X\| \asymp \sqrt{n}$*.*

Thus we see that in the present example it is the *lower* bound in Lemma 3.7 that is sharp, while the upper bound obtained from the noncommutative Khintchine inequality fails to capture correctly the structure of the problem.

We already sketched a proof of Lemma 3.11 using $\varepsilon$-nets in section 2.2. We take the opportunity now to present another proof, due to Chevet [6] and Gordon [9], that provides a first illustration of the *comparison methods* that will play an important role in the rest of this chapter. To this end, we first prove a classical comparison theorem for Gaussian processes due to Slepian and Fernique (see, e.g., [5]).

**Lemma 3.12 (Slepian-Fernique inequality).** *Let* $Y \sim N(0, \Sigma^Y)$ *and* $Z \sim N(0, \Sigma^Z)$ *be centered Gaussian vectors in* $\mathbb{R}^n$*. Suppose that*

$$\mathbf{E}(Y_i - Y_j)^2 \le \mathbf{E}(Z_i - Z_j)^2 \quad \textit{for all } 1 \le i, j \le n.$$

*Then*

$$\mathbf{E}\max_{i\le n} Y_i \le \mathbf{E}\max_{i\le n} Z_i.$$

*Proof.* Let $g, g'$ be independent standard Gaussian vectors. We can assume that $Y = (\Sigma^Y)^{1/2}g$ and $Z = (\Sigma^Z)^{1/2}g'$. Let $Y(t) = \sqrt{t}Z + \sqrt{1-t}Y$ for $t \in [0,1]$. Then

$$
\begin{aligned}
\frac{d}{dt}\mathbf{E}[f(Y(t))] &= \frac{1}{2}\mathbf{E}\Big[\Big\langle \nabla f(Y(t)), \frac{Z}{\sqrt{t}} - \frac{Y}{\sqrt{1-t}}\Big\rangle\Big] \\
&= \frac{1}{2}\mathbf{E}\Big[\frac{1}{\sqrt{t}}\Big\langle (\Sigma^Z)^{1/2}\nabla f(Y(t)), g'\Big\rangle - \frac{1}{\sqrt{1-t}}\Big\langle (\Sigma^Y)^{1/2}\nabla f(Y(t)), g\Big\rangle\Big] \\
&= \frac{1}{2}\sum_{i,j=1}^{n}(\Sigma^Z_{ij} - \Sigma^Y_{ij})\,\mathbf{E}\Big[\frac{\partial^2 f}{\partial x_i \partial x_j}(Y(t))\Big],
\end{aligned}
$$

where we used Gaussian integration by parts in the last step. We would really like to apply this identity with $f(x) = \max_i x_i$: if we can show that $\frac{d}{dt}\mathbf{E}[\max_i Y_i(t)] \geq 0$, that would imply $\mathbf{E}[\max_i Z_i] = \mathbf{E}[\max_i Y_i(1)] \geq \mathbf{E}[\max_i Y_i(0)] = \mathbf{E}[\max_i Y_i]$ as desired. The problem is that the function $x \mapsto \max_i x_i$ is not sufficiently smooth: it does not possess second derivatives. We therefore work with a smooth approximation.

Previously, we used $\|x\|_p$ as a smooth approximation of $\|x\|_\infty$. Unfortunately, it turns out that Slepian-Fernique does *not* hold when $\max_i Y_i$ and $\max_i Z_i$ are replaced by $\|Y\|_\infty$ and $\|Z\|_\infty$, so this cannot work. We must therefore choose instead a *one-sided* approximation. In analogy with Remark 2.5, we choose

$$
f_\beta(x) = \frac{1}{\beta}\log\bigg(\sum_{i=1}^{n} e^{\beta x_i}\bigg).
$$

Clearly $\max_i x_i \leq f_\beta(x) \leq \max_i x_i + \beta^{-1}\log n$, so $f_\beta(x) \to \max_i x_i$ as $\beta \to \infty$. Also

$$
\frac{\partial f_\beta}{\partial x_i}(x) = \frac{e^{\beta x_i}}{\sum_j e^{\beta x_j}} =: p_i(x), \qquad \frac{\partial^2 f_\beta}{\partial x_i \partial x_j}(x) = \beta\{\delta_{ij}p_i(x) - p_i(x)p_j(x)\},
$$

where we note that $p_i(x) \geq 0$ and $\sum_i p_i(x) = 1$. The reader should check that

$$
\frac{d}{dt}\mathbf{E}[f_\beta(Y(t))] = \frac{\beta}{4}\sum_{i\neq j}\{\mathbf{E}(Z_i - Z_j)^2 - \mathbf{E}(Y_i - Y_j)^2\}\,\mathbf{E}[p_i(Y(t))p_j(Y(t))],
$$

which follows by rearranging the terms in the above expressions. The right-hand side is nonnegative by assumption, and thus the proof is easily completed. □

We can now prove Lemma 3.11.

*Proof (Proof of Lemma 3.11).* That $\mathbf{E}\|X\| \gtrsim \sqrt{n}$ follows from Lemma 3.7, so it remains to prove $\mathbf{E}\|X\| \lesssim \sqrt{n}$. To this end, define $X_v := \langle v, Xv\rangle$ and $Y_v = 2\langle v, g\rangle$, where $g$ is a standard Gaussian vector. Then we can estimate

$$
\mathbf{E}(X_v - X_w)^2 \leq 2\sum_{i,j=1}^{n}(v_iv_j - w_iw_j)^2 \leq 4\|v - w\|^2 = \mathbf{E}(Y_v - Y_w)^2
$$

when $\|v\| = \|w\| = 1$, where we used $1 - \langle v, w\rangle^2 \leq 2(1 - \langle v, w\rangle)$ when $|\langle v, w\rangle| \leq 1$. It follows form the Slepian-Fernique lemma that we have

$$\mathbf{E}\lambda_{\max}(X) = \mathbf{E}\sup_{\|v\|=1}\langle v, Xv\rangle \le 2\,\mathbf{E}\sup_{\|v\|=1}\langle v, g\rangle = 2\,\mathbf{E}\|g\| \le 2\sqrt{n}.$$

But as $X$ and $-X$ have the same distribution, so do the random variables $\lambda_{\max}(X)$ and $-\lambda_{\min}(X) = \lambda_{\max}(-X)$. We can therefore estimate

$$\mathbf{E}\|X\| = \mathbf{E}(\lambda_{\max}(X)\vee-\lambda_{\min}(X)) \le \mathbf{E}\lambda_{\max}(X)+2\,\mathbf{E}|\lambda_{\max}(X)-\mathbf{E}\lambda_{\max}(X)| = 2\sqrt{n}+O(1),$$

where we used that $\mathrm{Var}(\lambda_{\max}(X)) = O(1)$ by Lemma 3.8. □

We have seen above two extreme examples: diagonal matrices and Wigner matrices. In the diagonal case, the noncommutative Khintchine inequality is sharp, while the lower bound in Lemma 3.7 is suboptimal. On the other hand, for Wigner matrices, the noncommutative Khintchine inequality is suboptimal, while the lower bound in Lemma 3.7 is sharp. We therefore see that while the structural parameter $\sigma = \|\mathbf{E}X^2\|^{1/2}$ that appears in the noncommutative Khintchine inequality always crudely controls the spectral norm up to a logarithmic factor in the dimension, it fails to capture correctly the structure of the problem and cannot in general yield sharp bounds. The aim of the rest of this chapter is to develop a deeper understanding of the norms of structured random matrices that goes beyond Lemma 3.7.

## 3.3 A second-order Khintchine inequality

Having established that the noncommutative Khintchine inequality falls short of capturing the full structure of our random matrix model, we naturally aim to understand where things went wrong. The culprit is easy to identify. The main idea behind the proof of the noncommutative Khintchine inequality is that the case where the matrices $A_k$ commute is the worst possible, as is made precise by Lemma 3.5. However, when the matrices $A_k$ do not commute, the behavior of the spectral norm can be *strictly better* than is predicted by the noncommutative Khintchine inequality. The crucial shortcoming of the noncommutative Khintchine inequality is that it provides no mechanism to capture the effect of noncommutativity.

*Remark 3.13.* This intuition is clearly visible in the examples of the previous section: the diagonal example corrsponds to choosing coefficient matrices $A_k$ of the form $e_ie_i^*$ for $1 \le i \le n$, while to obtain a Wigner matrix we add additional coefficient matrices $A_k$ of the form $e_ie_j^* + e_je_i^*$ for $1 \le i < j \le n$ (here $e_1, \ldots, e_n$ denotes the standard basis in $\mathbb{R}^n$). Clearly the matrices $A_k$ commute in the diagonal example, in which case noncommutative Khintchine is sharp, but they do not commute for the Wigner matrix, in which case noncommutative Khintchine is suboptimal.

The present insight suggests that a good bound on the spectral norm of random matrices of the form $X = \sum_{k=1}^{s} g_kA_k$ should somehow take into account the algebraic structure of the coefficient matrices $A_k$. Unfortunately, it is not at all clear how this is to be accomplished. In this section we develop an interesting result in this spirit due to Tropp [29]. While this result is still very far from being sharp, the proof contains some interesting ideas, and provides at present the only known approach to improve on the noncommutative Khintchine inequality in the most general setting.

The intuition behind the result of Tropp is that the commutation inequality

$$\mathbf{E}[\mathrm{Tr}[A_k X^\ell A_k X^{2p-2-\ell}]] \le \mathbf{E}[\mathrm{Tr}[A_k^2 X^{2p-2}]]$$

of Lemma 3.5, which captures the idea that the commutative case is the worst case, should incur significant loss when the matrices $A_k$ do not commute. Therefore, rather than apply this inequality directly, we should try to go to second order by integrating again by parts. For example, for the term $\ell = 1$, we could write

$$\begin{aligned}\mathbf{E}[\mathrm{Tr}[A_k X A_k X^{2p-3}]] &= \sum_{l=1}^{s} \mathbf{E}[g_l \mathrm{Tr}[A_k A_l A_k X^{2p-3}]] \\ &= \sum_{l=1}^{s} \sum_{m=0}^{2p-4} \mathbf{E}[\mathrm{Tr}[A_k A_l A_k X^m A_l X^{2p-4-m}]].\end{aligned}$$

If we could again permute the order of $A_l$ and $X^m$ on the right-hand side, we would obtain control of these terms not by the structural parameter

$$\sigma = \left\| \sum_{k=1}^{s} A_k^2 \right\|^{1/2}$$

that appears in the noncommutative Khintchine inequality, but rather by the second-order "noncommutative" structural parameter

$$\left\| \sum_{k,l=1}^{s} A_k A_l A_k A_l \right\|^{1/4}.$$

Of course, when the matrices $A_k$ commute, the latter parameter is equal to $\sigma$ and we recover the noncommutative Khintchine inequality; but when the matrices $A_k$ do not commute, it can be the case that this parameter is much smaller than $\sigma$. This back-of-the-envelope computation suggests that we might indeed hope to capture noncommutativity to some extent through the present approach.

In essence, this is precisely how we will proceed. However, there is a technical issue: the convexity that was exploited in the proof of Lemma 3.5 is no longer present in the second-order terms. We therefore cannot naively exchange $A_l$ and $X^m$ as suggested above, and the parameter $\|\sum_{k,l=1}^{s} A_k A_l A_k A_l\|^{1/4}$ is in fact too small to yield any meaningful bound (as is illustrated by a counterexample in [29]). The key idea in [29] is that a classical complex analysis argument [18, Appendix IX.4] can be exploited to force convexity, at the expense of a larger second-order term.

**Theorem 3.14 (Tropp).** *Let* $X = \sum_{k=1}^{s} g_k A_k$ *as in the previous section. Define*

$$\sigma := \left\| \sum_{k=1}^{s} A_k^2 \right\|^{1/2}, \qquad \tilde{\sigma} := \sup_{U_1,U_2,U_3} \left\| \sum_{k,l=1}^{s} A_k U_1 A_l U_2 A_k U_3 A_l \right\|^{1/4},$$

*where the supremum is taken over all triples $U_1, U_2, U_3$ of commuting unitary matrices.*[2] *Then we have a second-order noncommutative Khintchine inequality*

$$\mathbf{E}\|X\| \lesssim \sigma \, \log^{1/4} n + \tilde{\sigma} \, \log^{1/2} n.$$

Due to the (necessary) presence of the unitaries, the second-order parameter $\tilde{\sigma}$ is not so easy to compute. It is verified in [29] that $\tilde{\sigma} \le \sigma$ (so that Theorem 3.14 is no worse than the noncommutative Khintchine inequality), and that $\tilde{\sigma} = \sigma$ when the matrices $A_k$ commmute. On the other hand, an explicit computation in [29] shows that if $X$ is a Wigner matrix as in Example 3.10, we have $\sigma \asymp \sqrt{n}$ and $\tilde{\sigma} \asymp n^{1/4}$. Thus Theorem 3.14 yields in this case $\mathbf{E}\|X\| \lesssim \sqrt{n}(\log n)^{1/4}$, which is strictly better than the noncommutative Khintchine bound $\mathbf{E}\|X\| \lesssim \sqrt{n}(\log n)^{1/2}$ but falls short of the sharp bound $\mathbf{E}\|X\| \asymp \sqrt{n}$. We therefore see that Theorem 3.14 does indeed improve, albeit ever so slightly, on the noncommutative Khintchine bound. The real interest of Theorem 3.14 is however the very general setting in which it holds, and that it does capture explicitly the noncommutativity of the coefficient matrices $A_k$. In section 4, we will see that much sharper bounds can be obtained if we specialize to random matrices with independent entries. While this is perhaps the most interesting setting in practice, it will require us to depart from the much more general setting provided by the Khintchine-type inequalities that we have seen so far.

The remainder of this section is devoted to the proof of Theorem 3.14. The proof follows essentially along the lines already indicated: we follow the proof of the noncommutative Khintchine inequality and integrate by parts a second time. The new idea in the proof is to understand how to appropriately extend Lemma 3.5.

*Proof (Proof of Theorem 3.14).* We begin as in the proof of Theorem 3.2 by writing

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]] = \sum_{\ell=0}^{2p-2} \sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X^{\ell} A_k X^{2p-2-\ell}]].$$

Let us investigate each of the terms inside the first sum.

**Case $\ell = 0, 2p-2$.** In this case there is little to do: we can estimate

$$\sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k^2 X^{2p-2}]] \le \mathrm{Tr}\left[\left(\sum_{k=1}^{s} A_k^2\right)^p\right]^{1/p} \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1-1/p}$$

precisely as in the proof of Theorem 3.2.

**Case $\ell = 1, 2p-3$.** This is the first point at which something interesting happens. Integrating by parts a second time as was discussed before Theorem 3.14, we obtain

[2] For reasons that will become evident in the proof, it is essential to consider (complex) unitary matrices $U_1, U_2, U_3$, despite that all the matrices $A_k$ and $X$ are assumed to be real.

$$\sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X A_k X^{2p-3}]] = \sum_{m=0}^{2p-4} \sum_{k,l=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k A_l A_k X^m A_l X^{2p-4-m}]].$$

The challenge we now face is to prove the appropriate analogue of Lemma 3.5.

**Lemma 3.15.** *There exist unitary matrices $U_1, U_2$ (dependent on $X$ and $m$) such that*

$$\sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l A_k X^m A_l X^{2p-4-m}] \le \left| \sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l A_k U_1 A_l U_2 X^{2p-4}] \right|.$$

*Remark 3.16.* Let us start the proof as in Lemma 3.5 and see where things go wrong. In terms of the eigendecomposition $X = \sum_{i=1}^{n} \lambda_i v_i v_i^*$, we can write

$$\sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l A_k X^m A_l X^{2p-4-m}] = \sum_{k,l=1}^{s} \sum_{i,j=1}^{n} \lambda_i^m \lambda_j^{2p-4-m} \langle v_j, A_k A_l A_k v_i \rangle \langle v_i, A_l v_j \rangle.$$

Unfortunately, unlike in the analogous expression in the proof of Lemma 3.5, the coefficients $\langle v_j, A_k A_l A_k v_i \rangle \langle v_i, A_l v_j \rangle$ can have arbitrary sign. Therefore, we cannot easily force convexity of the above expression as a function of $m$ as we did in Lemma 3.5: if we replace the terms in the sum by their absolute values, we will no longer be able to interpret the resulting expression as a linear algebraic object (a trace).

However, the above expression is still an an *analytic* function in the complex plane $\mathbb{C}$. The idea that we will exploit is that analytic functions have some hidden convexity built in, as we recall here without proof (cf. [18, p. 33]).

**Lemma 3.17 (Hadamard three line lemma).** *If $\varphi : \mathbb{C} \to \mathbb{C}$ is analytic, the function $t \mapsto \sup_{s \in \mathbb{R}} \log |\varphi(t + is)|$ is convex on the real line (provided it is finite).*

*Proof (Proof of Lemma 3.15).* We can assume that $X$ is nonsingular; otherwise we may replace $X$ by $X + \varepsilon$ and let $\varepsilon \downarrow 0$ at the end of the proof. Write $X = V|X|$ according to its polar decomposition, and note that as $X$ is self-adjoint, $V = \mathrm{sign}(X)$ commutes with $X$ and therefore $X^m = V^m |X|^m$. Define

$$\varphi(z) := \sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l A_k V^m |X|^{(2p-4)z} A_l V^{2p-4-m} |X|^{(2p-4)(1-z)}].$$

As $X$ is nonsingular, $\varphi$ is analytic and $\varphi(t + is)$ is a periodic function of $s$ for every $t$. By the three line lemma, $\sup_{s \in \mathbb{R}} |\varphi(t + is)|$ attains its maximum for $t \in [0, 1]$ at either $t = 0$ or $t = 1$. Moreover, the supremum itself is attained at some $s \in \mathbb{R}$ by periodicity. We have therefore shown that there exists $s \in \mathbb{R}$ such that

$$\left| \sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l A_k X^m A_l X^{2p-4-m}] \right| = \left| \varphi\left( \frac{m}{2p-4} \right) \right| \le |\varphi(is)| \vee |\varphi(1 + is)|.$$

But, for example,

$$|\varphi(is)| = \left|\sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l A_k V^m |X|^{is(2p-4)} A_l V^{2p-4-m} |X|^{-is(2p-4)} X^{2p-4}]\right|,$$

so if this term is the larger we can set $U_1 = V^m|X|^{is(2p-4)}$ and $U_2 = V^{2p-4-m}|X|^{-is(2p-4)}$ to obtain the statement of the lemma (clearly $U_1$ and $U_2$ are unitary). If the term $|\varphi(1+is)|$ is larger, the claim follows in precisely the identical manner. □

Putting together the above bounds, we obtain

$$\begin{aligned}
&\sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X A_k X^{2p-3}]] \\
&\le (2p-3)\,\mathbf{E}\left[\sup_{U_1,U_2}\left|\sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l A_k U_1 A_l U_2 X^{2p-4}]\right|\right] \\
&\le (2p-3)\sup_{U} \mathrm{Tr}\left[\left|\sum_{k,l=1}^{s} A_k A_l A_k U A_l\right|^{p/2}\right]^{2/p} \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1-2/p}.
\end{aligned}$$

This term will evidently yield a term of order $\tilde{\sigma}$ when $p \sim \log n$.

**Case $2 \le \ell \le 2p-4$.** These terms are dealt with much in the same way as in the previous case, except the computation is a bit more tedious. As we have come this far, we might as well complete the argument. We begin by noting that

$$\sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X^{\ell} A_k X^{2p-2-\ell}]] \le \sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X^2 A_k X^{2p-4}]]$$

for every $2 \le \ell \le 2p-4$. This follows by convexity precisely in the same way as in Lemma 3.5, and we omit the (identical) proof. To proceed, we integrate by parts:

$$\begin{aligned}
\sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X^2 A_k X^{2p-4}]] &= \sum_{k,l=1}^{s} \mathbf{E}[g_l \mathrm{Tr}[A_k A_l X A_k X^{2p-4}]] \\
= \sum_{k,l=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k A_l^2 A_k X^{2p-4}]] &+ \sum_{m=0}^{2p-5}\sum_{k,l=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k A_l X A_k X^m A_l X^{2p-5-m}]].
\end{aligned}$$

We deal separately with the two types of terms.

**Lemma 3.18.** *There exist unitary matrices $U_1, U_2, U_3$ such that*

$$\sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l X A_k X^m A_l X^{2p-5-m}] \le \left|\sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l U_1 A_k U_2 A_l U_3 X^{2p-4}]\right|.$$

*Proof.* Let $X = V|X|$ be the polar decomposition of $X$, and define

$$\varphi(y,z) := \sum_{k,l=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k A_l V |X|^{(2p-4)y} A_k V^m |X|^{(2p-4)z} A_l V^{2p-5-m} |X|^{(2p-4)(1-y-z)}]].$$

Now apply the three line lemma to $\varphi$ twice: to $\varphi(\cdot, z)$ with $z$ fixed, then to $\varphi(y, \cdot)$ with $y$ fixed. The omitted details are almost identical to the proof of Lemma 3.15. □

**Lemma 3.19.** *We have for* $p \geq 2$

$$\sum_{k,l=1}^{s} \mathrm{Tr}[A_k A_l^2 A_k X^{2p-4}] \leq \mathrm{Tr}\left[\left(\sum_{k=1}^{s} A_k^2\right)^p\right]^{2/p} \mathrm{Tr}[X^{2p}]^{1-2/p}.$$

*Proof.* We argue essentially as in Lemma 3.5. Define $H = \sum_{l=1}^{s} A_l^2$ and let

$$\varphi(z) := \sum_{k=1}^{s} \mathrm{Tr}[A_k H^{(p-1)z} A_k |X|^{(2p-2)(1-z)}],$$

so that the quantity we would like to bound is $\varphi(1/(p-1))$. By expressing $\varphi(z)$ in terms of the spectral decompositions $X = \sum_{i=1}^{n} \lambda_i v_i v_i^*$ and $H = \sum_{i=1}^{n} \mu_i w_i w_i^*$, we can verify by explicit computation that $z \mapsto \log \varphi(z)$ is convex on $z \in [0,1]$. Therefore

$$\varphi(1/(p-1)) \leq \varphi(1)^{1/(p-1)} \varphi(0)^{(p-2)/(p-1)} = \mathrm{Tr}[H^p]^{1/(p-1)} \mathrm{Tr}[H X^{2p-2}]^{(p-2)/(p-1)}.$$

But $\mathrm{Tr}[H|X|^{2p-2}] \leq \mathrm{Tr}[H^p]^{1/p} \mathrm{Tr}[X^{2p}]^{1-1/p}$ by Hölder's inequality, and the conclusion follows readily by substituting this into the above expression. □

Putting together the above bounds and using Hölder's inequality yields

$$\begin{aligned}\sum_{k=1}^{s} \mathbf{E}[\mathrm{Tr}[A_k X^{\ell} A_k X^{2p-2-\ell}]] &\leq \mathrm{Tr}\left[\left(\sum_{k=1}^{s} A_k^2\right)^p\right]^{2/p} \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1-2/p} \\ &\quad + (2p-4) \sup_{U_1,U_2} \mathrm{Tr}\left[\left|\sum_{k,l=1}^{s} A_k A_l U_1 A_k U_2 A_l\right|^{p/2}\right]^{2/p} \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1-2/p}.\end{aligned}$$

**Conclusion.** Let $p \asymp \log n$. Collecting the above bounds yields

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]] \lesssim \sigma^2 \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1-1/p} + p\,(\sigma^4 + p\,\tilde{\sigma}^4)\,\mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1-2/p},$$

where we used Lemma 2.2 to simplify the constants. Rearranging gives

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]]^{2/p} \lesssim \sigma^2 \mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1/p} + p\,(\sigma^4 + p\,\tilde{\sigma}^4),$$

which is a simple quadratic inequality for $\mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1/p}$. Solve this inequality using the quadratic formula and apply again Lemma 2.2 to conclude the proof. □

## 4 Matrices with independent entries

The Khintchine-type inequalities developed in the previous section have the advantage that they can be applied in a remarkably general setting: they not only allow an arbitrary variance pattern of the entries, but even an arbitrary dependence structure between the entries. This makes such bounds useful in a wide variety of situations. Unfortunately, we have also seen that Khintchine-type inequalities yield suboptimal bounds already in the simplest examples: the mechanism behind the proofs of these inequalities is too crude to fully capture the structure of the underlying random matrices at this level of generality. In order to gain a deeper understanding, we must impose some additional structure on the matrices under consideration.

In this section, we specialize to what is perhaps the most important case of the random matrices investigated in the previous section: we consider symmetric random matrices with *independent* entries. More precisely, in most of this section, we will study the following basic model. Let $g_{ij}$ be independent standard Gaussian random variables and let $b_{ij} \geq 0$ be given scalars for $i \geq j$. We consider the $n \times n$ symmetric random matrix $X$ whose entries are given by $X_{ij} = b_{ij} g_{ij}$, that is,

$$X = \begin{bmatrix} b_{11}g_{11} & b_{12}g_{12} & \cdots & b_{1n}g_{1n} \\ b_{12}g_{12} & b_{22}g_{22} & & b_{2n}g_{2n} \\ \vdots & & \ddots & \vdots \\ b_{1n}g_{1n} & b_{2n}g_{2n} & \cdots & b_{nn}g_{nn} \end{bmatrix}.$$

In other words, $X$ is the symmetric random matrix whose entries above the diagonal are independent Gaussian variables $X_{ij} \sim N(0, b_{ij}^2)$, where the structure of the matrix is controlled by the given variance pattern $\{b_{ij}\}$. As the matrix is symmetric, we will write for simplicity $g_{ji} = g_{ij}$ and $b_{ji} = b_{ij}$ in the sequel.

The present model differs from the model of the previous section only to the extent that we imposed the additional independence assumption on the entries. In particular, the noncommutative Khintchine inequality reduces in this setting to

$$\mathbf{E}\|X\| \lesssim \max_{i\leq n} \sqrt{\sum_{j=1}^n b_{ij}^2}\, \sqrt{\log n},$$

while Theorem 3.14 yields (after some tedious computation)

$$\mathbf{E}\|X\| \lesssim \max_{i\leq n} \sqrt{\sum_{j=1}^n b_{ij}^2}\, (\log n)^{1/4} + \max_{i\leq n} \Bigg(\sum_{j=1}^n b_{ij}^4\Bigg)^{1/4} \sqrt{\log n}.$$

Unfortunately, we have already seen that neither of these results is sharp even for Wigner matrices (where $b_{ij} = 1$ for all $i, j$). The aim of this section is to develop much sharper inequalities for matrices with independent entries that capture *optimally* in many cases the underlying structure. The independence assumption will be crucially exploited to control the structure of these matrices, and it is an interesting open problem to understand to what extent the mechanisms developed in this section persist in the presence of dependence between the entries (cf. section 4.3).

## *4.1 Latała's inequality and beyond*

The earliest nontrivial result on the spectral norm Gaussian random matrices with independent entries is the following inequality due to Latała [11].

**Theorem 4.1 (Latała).** *In the setting of this section, we have*

$$\mathbf{E}\|X\| \lesssim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \left(\sum_{i,j=1}^n b_{ij}^4\right)^{1/4}.$$

Latała's inequality yields a sharp bound $\mathbf{E}\|X\| \lesssim \sqrt{n}$ for Wigner matrices, but is already suboptimal for the diagonal matrix of Example 3.9 where the resulting bound $\mathbf{E}\|X\| \lesssim n^{1/4}$ is very far from the correct answer $\mathbf{E}\|X\| \asymp \sqrt{\log n}$. In this sense, we see that Theorem 4.1 fails to correctly capture the structure of the underlying matrix. Latała's inequality is therefore not too useful for *structured* random matrices; it has however been widely applied together with a simple symmetrization argument [11, Theorem 2] to show that the sharp bound $\mathbf{E}\|X\| \asymp \sqrt{n}$ remains valid for Wigner matrices with general (non-Gaussian) distribution of the entries.

In this section, we develop a nearly sharp improvement of Latała's inequality that can yield optimal results for many structured random matrices.

**Theorem 4.2 ([31]).** *In the setting of this section, we have*

$$\mathbf{E}\|X\| \lesssim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \max_{i\le n}\left(\sum_{j=1}^n b_{ij}^4\right)^{1/4}\sqrt{\log i}.$$

Let us first verify that Latała's inequality does indeed follow.

*Proof (Proof of Theorem 4.1).* As the matrix norm $\|X\|$ is unchanged if we permute the rows and columns of $X$, we may assume without loss of generality that $\sum_{j=1}^n b_{ij}^4$ is decreasing in $i$ (this choice minimizes the upper bound in Theorem 4.2). Now recall the following elementary fact: if $x_1 \ge x_2 \ge \cdots \ge x_n \ge 0$, then $x_k \le \frac{1}{k}\sum_{i=1}^n x_i$ for every $k$. In the present case, this observation and Theorem 4.2 imply

$$\mathbf{E}\|X\| \lesssim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \left(\sum_{i,j=1}^n b_{ij}^4\right)^{1/4} \max_{1\le i<\infty}\frac{\sqrt{\log i}}{i^4},$$

which concludes the proof of Theorem 4.1. □

The inequality of Theorem 4.2 is somewhat reminiscent of the bound obtained in the present setting from Theorem 3.14, with a crucial difference: there is no logarithmic factor in front of the first term. As we already proved in Lemma 3.7 that

$$\mathbf{E}\|X\| \gtrsim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2},$$

we see that Theorem 4.2 provides an *optimal* bound whenever the first term dominates, which is the case for a wide range of structured random matrices. To get a feeling for the sharpness of Theorem 4.2, let us consider an illuminating example.

*Example 4.3 (Block matrices).* Let $1 \le k \le n$ and suppose for simplicity that $n$ is divisible by $k$. We consider the $n \times n$ symmetric block-diagonal matrix $X$ of the form

$$X = \begin{bmatrix} \mathbf{X}_1 & & & \\ & \mathbf{X}_2 & & \\ & & \ddots & \\ & & & \mathbf{X}_{n/k} \end{bmatrix},$$

where $\mathbf{X}_1, \ldots, \mathbf{X}_{n/k}$ are independent $k \times k$ Wigner matrices. This model interpolates between the diagonal matrix of Example 3.9 (the case $k = 1$) and the Wigner matrix of Example 3.10 (the case $k = n$). Note that $\|X\| = \max_i \|\mathbf{X}_i\|$, so we can compute

$$\mathbf{E}\|X\| \lesssim \mathbf{E}[\|\mathbf{X}_1\|^{\log n}]^{1/\log n} \le \mathbf{E}\|\mathbf{X}_1\| + \mathbf{E}[(\|\mathbf{X}_1\| - \mathbf{E}\|\mathbf{X}_1\|)^{\log n}]^{1/\log n} \lesssim \sqrt{k} + \sqrt{\log n}$$

using Lemmas 2.3, 3.11, and 3.8, respectively. On the other hand, Lemma 3.7 implies that $\mathbf{E}\|X\| \gtrsim \sqrt{k}$, while we can trivially estimate $\mathbf{E}\|X\| \ge \mathbf{E}\max_i X_{ii} \asymp \sqrt{\log n}$. Averaging these two lower bounds, we have evidently shown that

$$\mathbf{E}\|X\| \asymp \sqrt{k} + \sqrt{\log n}.$$

This explicit computation provides a simple but very useful benchmark example for testing inequalities for structured random matrices.

In the present case, applying Theorem 4.2 to this example yields

$$\mathbf{E}\|X\| \lesssim \sqrt{k} + k^{1/4}\sqrt{\log n}.$$

Therefore, in the present example, Theorem 4.2 fails to be sharp only when $k$ is in the range $1 \ll k \ll (\log n)^2$. This suboptimal parameter range will be completely eliminated by the sharp bound to be proved in section 4.2 below. But the bound of Theorem 4.2 is already sharp in the vast majority of cases, and is of significant interest in its own right for reasons that will be discussed in detail in section 4.3.

An important feature of the inequalities of this section should be emphasized: unlike all bounds we have encountered so far, the present bounds are *dimension-free*. As was discussed in Remark 2.4, one cannot expect to obtain sharp dimension-free bounds using the moment method, and it therefore comes as no surprise that the bounds of the present section will therefore be obtained by the random process method. The original proof of Latała proceeds by a difficult and very delicate explicit construction of a multiscale net in the spirit of Remark 2.7. We will follow here a much simpler approach that was developed in [31] to prove Theorem 4.2.

The basic idea behind our approach was already encountered in the proof of Lemma 3.11 to bound the norm of a Wigner matrix (where $b_{ij} = 1$ for all $i, j$): we seek a Gaussian process $Y_v$ that dominates the process $X_v := \langle v, Xv\rangle$ whose supremum coincides with the spectral norm. The present setting is significantly more challenging, however. To see the difficulty, let us try to adapt directly the proof of Lemma 3.11 to the present structured setting: we readily compute

$$\mathbf{E}(X_v - X_w)^2 \le 2\sum_{i,j=1}^n b_{ij}^2 (v_i v_j - w_i w_j)^2 \le 4 \max_{i,j\le n} b_{ij}^2 \|v-w\|^2.$$

We can therefore dominate $X_v$ by the Gaussian process $Y_v = 2\max_{i,j} b_{ij}\langle v, g\rangle$. Proceeding as in the proof of Lemma 3.11, this yields the following upper bound:

$$\mathbf{E}\|X\| \lesssim \max_{i,j\le n} b_{ij}\sqrt{n}.$$

This bound is sharp for Wigner matrices (in this case the present proof reduces to that of Lemma 3.11), but is woefully inadequate in any structured example. The problem with the above bound is that it always crudely estimates the behavior of the increments $\mathbf{E}[(X_v - X_w)]^{1/2}$ by a *Euclidean* norm $\|v-w\|$, regardless of the structure of the underlying matrix. However, the geometry defined by $\mathbf{E}[(X_v-X_w)]^{1/2}$ depends strongly on the structure of the matrix, and is typically highly non-Euclidean. For example, in the diagonal matrix of Example 3.9, we have $\mathbf{E}[(X_v-X_w)]^{1/2} = \|v^2-w^2\|$ where $(v^2)_i := v_i^2$. As $v^2$ is in the simplex whenever $v \in B$, we see that the underlying geometry in this case is that of an $\ell_1$-norm and not of an $\ell_2$-norm. In more general examples, however, it is far from clear what is the correct geometry.

The key challenge we face is to design a comparison process that is easy to bound, but whose geometry nonetheless captures faithfully the structure of the underlying matrix. To develop some intuition for how this might be accomplished, let us consider in first instance instead of the increments $\mathbf{E}[(X_v - X_w)^2]^{1/2}$ only the standard deviation $\mathbf{E}[X_v^2]^{1/2}$ of the process $X_v = \langle v, Xv\rangle$. We easily compute

$$\mathbf{E}X_v^2 = 2\sum_{i\ne j} v_i^2 b_{ij}^2 v_j^2 + \sum_{i=1}^n b_{ii}^2 v_i^4 \le 2\sum_{i=1}^n x_i(v)^2,$$

where we defined the nonlinear map $x : \mathbb{R}^n \to \mathbb{R}^n$ as

$$x_i(v) := v_i \sqrt{\sum_{j=1}^n b_{ij}^2 v_j^2}.$$

This computation suggests that we might attempt to dominate the process $X_v$ by the process $Y_v = \langle x(v), g\rangle$, whose increments $\mathbf{E}[(Y_v-Y_w)^2]^{1/2} = \|x(v)-x(w)\|$ capture the non-Euclidean nature of the underlying geometry through the nonlinear map $x$. The reader may readily verify, for example, that the latter process captures automatically the correct geometry of our two extreme examples of Wigner and diagonal matrices.

Unfortunately, the above choice of comparison process $Y_v$ is too optimistic: while we have chosen this process so that $\mathbf{E}X_v^2 \lesssim \mathbf{E}Y_v^2$ by construction, the Slepian-

Fernique inequality requires the stronger bound $\mathbf{E}(X_v - X_w)^2 \lesssim \mathbf{E}(Y_v - Y_w)^2$. It turns out that the latter inequality does not always hold [31]. However, the inequality *nearly* holds, which is the key observation behind the proof of Theorem 4.2.

**Lemma 4.4.** *For every $v, w \in \mathbb{R}^n$*

$$\mathbf{E}(\langle v, Xv\rangle - \langle w, Xw\rangle)^2 \le 4\|x(v) - x(w)\|^2 - \sum_{i,j=1}^n (v_i^2 - w_i^2) b_{ij}^2 (v_j^2 - w_j^2).$$

*Proof.* We simply compute both sides and compare. Define for simplicity the seminorm $\|\cdot\|_i$ as $\|v\|_i^2 := \sum_{j=1}^n b_{ij}^2 v_j^2$, so that $x_i(v) = v_i\|v\|_i$. First, we note that

$$\begin{aligned}\mathbf{E}(\langle v, Xv\rangle - \langle w, Xw\rangle)^2 &= \mathbf{E}\langle v + w, X(v - w)\rangle^2 \\ &= \sum_{i=1}^n (v_i - w_i)^2 \|v + w\|_i^2 + \sum_{i,j=1}^n (v_i^2 - w_i^2) b_{ij}^2 (v_j^2 - w_j^2).\end{aligned}$$

On the other hand, as $2(x_i(v) - x_i(w)) = (v_i + w_i)(\|v\|_i - \|w\|_i) + (v_i - w_i)(\|v\|_i + \|w\|_i)$,

$$\begin{aligned}4\|x(v) - x(w)\|^2 = &\sum_{i=1}^n (v_i + w_i)^2 (\|v\|_i - \|w\|_i)^2 + \sum_{i=1}^n (v_i - w_i)^2 (\|v\|_i + \|w\|_i)^2 \\ &+ 2\sum_{i,j=1}^n (v_i^2 - w_i^2) b_{ij}^2 (v_j^2 - w_j^2).\end{aligned}$$

The result follows readily from the triangle inequality $\|v + w\|_i \le \|v\|_i + \|w\|_i$. □

We can now complete the proof of Theorem 4.2.

*Proof (Proof of Theorem 4.2).* Define the Gaussian processes

$$X_v = \langle v, Xv\rangle, \qquad Y_v = 2\langle x(v), g\rangle + \langle v^2, Y\rangle,$$

where $g \sim N(0, I)$ is a standard Gaussian vector in $\mathbb{R}^n$, $(v^2)_i := v_i^2$, and $Y \sim N(0, B^-)$ is a centered Gaussian vector that is independent of $g$ and whose covariance matrix $B^-$ is the negative part of the matrix of variances $B = (b_{ij}^2)$ (if $B$ has eigendecomposition $B = \sum_i \lambda_i v_i v_i^*$, the negative part $B^-$ is defined as $B^- = \sum_i \max(-\lambda_i, 0) v_i v_i^*$). As $-B \le B^-$ by construction, it is readily seen that Lemma 4.4 implies

$$\mathbf{E}(X_v - X_w)^2 \le 4\|x(v) - x(w)\|^2 + \langle v^2 - w^2, B^-(v^2 - w^2)\rangle = \mathbf{E}(Y_v - Y_w)^2.$$

We can therefore argue by the Slepian-Fernique inequality that

$$\mathbf{E}\|X\| \lesssim \mathbf{E}\sup_{v\in B} Y_v \le 2\,\mathbf{E}\sup_{v\in B}\langle x(v), g\rangle + \mathbf{E}\max_{i\le n} Y_i$$

as in the proof of Lemma 3.11. It remains to bound each term on the right.

Let us begin with the second term. Using the moment method as in section 2.1, one obtains the dimension-dependent bound $\mathbf{E}\max_i Y_i \lesssim \max_i \mathrm{Var}(Y_i)^{1/2}\sqrt{\log n}$.

This bound is sharp when all the variances $\mathrm{Var}(Y_i) = B^-_{ii}$ are of the same order, but can be suboptimal when many of the variances are small. Instead, we will use a sharp *dimension-free* bound on the maximum of Gaussian random variables.

**Lemma 4.5 (Subgaussian maxima).** *Suppose that $g_1, \ldots, g_n$ satisfy $\mathbf{E}[|g_i|^k]^{1/k} \lesssim \sqrt{k}$ for all $k, i$, and let $\sigma_1, \ldots, \sigma_n \geq 0$. Then we have*

$$\mathbf{E} \max_{i \leq n} |\sigma_i g_i| \lesssim \max_{i \leq n} \sigma_i \sqrt{\log(i+1)}.$$

*Proof.* By a union bound and Markov's inequality

$$\mathbf{P}\Big[\max_{i \leq n} |\sigma_i g_i| \geq t\Big] \leq \sum_{i=1}^{n} \mathbf{P}[|\sigma_i g_i| \geq t] \leq \sum_{i=1}^{n} \bigg(\frac{\sigma_i \sqrt{2\log(i+1)}}{t}\bigg)^{2\log(i+1)}.$$

But we can estimate

$$\sum_{i=1}^{\infty} s^{-2\log(i+1)} = \sum_{i=1}^{\infty} (i+1)^{-2}(i+1)^{-2\log s+2} \leq 2^{-2\log s+2} \sum_{i=1}^{\infty} (i+1)^{-2} \lesssim s^{-2\log 2}$$

as long as $\log s > 1$. Setting $s = t / \max_i \sigma_i \sqrt{2\log(i+1)}$, we obtain

$$\begin{aligned}\mathbf{E} \max_{i \leq n} |\sigma_i g_i| &= \max_i \sigma_i \sqrt{2\log(i+1)} \int_0^\infty \mathbf{P}\Big[\max_{i \leq n} |\sigma_i g_i| \geq s \max_i \sigma_i \sqrt{2\log(i+1)}\Big]\, ds \\ &\lesssim \max_i \sigma_i \sqrt{2\log(i+1)} \bigg(e + \int_e^\infty s^{-2\log 2} ds\bigg) \lesssim \max_i \sigma_i \sqrt{\log(i+1)},\end{aligned}$$

which completes the proof. □

*Remark 4.6.* Lemma 4.5 does not require the variables $g_i$ to be either independent or Gaussian. However, if $g_1, \ldots, g_n$ are independent standard Gaussian variables (which satisfy $\mathbf{E}[|g_i|^k]^{1/k} \lesssim \sqrt{k}$ by Lemma 3.1) and if $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ (which is the ordering that optimizes the bound of Lemma 4.5), then

$$\mathbf{E} \max_{i \leq n} |\sigma_i g_i| \asymp \max_{i \leq n} \sigma_i \sqrt{\log(i+1)},$$

cf. [31]. This shows that Lemma 4.5 captures precisely the dimension-free behavior of the maximum of independent centered Gaussian variables.

To estimate the second term in our bound on $\mathbf{E}\|X\|$, note that $(B^-)^2 \leq B^2$ implies

$$\mathrm{Var}(Y_i)^2 = (B^-_{ii})^2 \leq (B^-)^2_{ii} \leq (B^2)_{ii} = \sum_{j=1}^{n} (B_{ij})^2 = \sum_{j=1}^{n} b_{ij}^4.$$

Applying Lemma 4.5 with $g_i = Y_i / \mathrm{Var}(Y_i)^{1/2}$ yields the bound

$$\mathbf{E} \max_{i \leq n} Y_i \lesssim \max_{i \leq n} \bigg(\sum_{j=1}^{n} b_{ij}^4\bigg)^{1/4} \sqrt{\log(i+1)}.$$

Now let us estimate the first term in our bound on $\mathbf{E}\|X\|$. Note that

$$\sup_{v\in B}\langle x(v), g\rangle = \sup_{v\in B}\sum_{j=1}^n g_j v_j \sqrt{\sum_{i=1}^n v_i^2 b_{ij}^2} \le \sup_{v\in B}\sqrt{\sum_{i,j=1}^n v_i^2 b_{ij}^2 g_j^2} = \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2 g_j^2},$$

where we used Cauchy-Schwarz and the fact that $v^2$ is in the $\ell_1$-ball whenever $v$ is in the $\ell_2$-ball. We can therefore estimate, using Lemma 3.8 and Lemma 4.5,

$$\begin{aligned}\mathbf{E}\sup_{v\in B}\langle x(v), g\rangle &\le \max_{i\le n}\mathbf{E}\sqrt{\sum_{j=1}^n b_{ij}^2 g_j^2} + \mathbf{E}\max_{i\le n}\left|\sqrt{\sum_{j=1}^n b_{ij}^2 g_j^2} - \mathbf{E}\sqrt{\sum_{j=1}^n b_{ij}^2 g_j^2}\right| \\ &\lesssim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \max_{i,j\le n} b_{ij}\sqrt{\log(i+1)}.\end{aligned}$$

Putting everything together gives

$$\mathbf{E}\|X\| \lesssim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \max_{i,j\le n} b_{ij}\sqrt{\log(i+1)} + \max_{i\le n}\left(\sum_{j=1}^n b_{ij}^4\right)^{1/4}\sqrt{\log(i+1)}.$$

It is not difficult to simplify this (at the expense of a larger universal constant) to obtain the bound in the statement of Theorem 4.2. □

## 4.2 A sharp dimension-dependent bound

The approach developed in the previous section yields optimal results for many structured random matrices with independent entries. The crucial improvement of Theorem 4.2 over the noncommutative Khintchine inequality is that no logarithmic factor appears in the first term. Therefore, when this term dominates, Theorem 4.2 is sharp by Lemma 3.7. However, the second term in Theorem 4.2 is not quite sharp, as is illustrated in Example 4.3. While Theorem 4.2 captures much of the geometry of the underlying model, there remains some residual inefficiency in the proof.

In this section, we will develop an improved version of Theorem 4.2 that is essentially sharp (in a sense that will be made precise below). Unfortunately, it is not known at present how such a bound can be obtained using the random process method, and we revert back to the moment method in the proof. The price we pay for this is that we lose the dimension-free nature of Theorem 4.2.

**Theorem 4.7 ([4]).** *In the setting of this section, we have*

$$\mathbf{E}\|X\| \lesssim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \max_{i,j\le n} b_{ij}\sqrt{\log n}.$$

To understand why this result is sharp, let us recall (Remark 2.4) that the moment method necessarily bounds not the quantity $\mathbf{E}\|X\|$, but rather the larger quantity $\mathbf{E}[\|X\|^{\log n}]^{1/\log n}$. The latter quantity is now however completely understood.

**Corollary 4.8.** *In the setting of this section, we have*

$$\mathbf{E}[\|X\|^{\log n}]^{1/\log n} \asymp \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \max_{i,j\le n} b_{ij}\sqrt{\log n}.$$

*Proof.* The upper bound follows from the proof of Theorem 4.7. The first term on the right is a lower bound by Lemma 3.7. On the other hand, if $b_{kl} = \max_{i,j} b_{ij}$, then $\mathbf{E}[\|X\|^{\log n}]^{1/\log n} \ge \mathbf{E}[|X_{kl}|^{\log n}]^{1/\log n} \gtrsim b_{kl}\sqrt{\log n}$ as $\mathbf{E}[|X_{kl}|^p]^{1/p} \asymp b_{kl}\sqrt{p}$. □

The above result shows that Theorem 4.7 is in fact the *optimal* result that could be obtained by the moment method. This result moreover yields optimal bounds even for $\mathbf{E}\|X\|$ in almost all situations of practical interest, as it is true under mild assumptions that $\mathbf{E}\|X\| \asymp \mathbf{E}[\|X\|^{\log n}]^{1/\log n}$ (as will be discussed in section 4.3). Nonetheless, this is not always the case, and will fail in particular for matrices whose variances are distributed over many different scales; in the latter case, the *dimension-free* bound of Theorem 4.2 can give rise to much sharper results. Both Theorems 4.2 and 4.7 therefore remain of significant independent interest. Taken together, these results strongly support a fundamental conjecture, to be discussed in the next section, that would provide the ultimate understanding of the magnitude of the spectral norm of the random matrix model considered in this chapter.

The proof of Theorem 4.7 is completely different in nature than that of Theorem 4.2. Rather than prove Theorem 4.7 in the general case, we will restrict attention in the rest of this section to the special case of *sparse Wigner matrices*. The proof of Theorem 4.7 in the general case is actually no more difficult than in this special case, but the ideas and intuition behind the proof are particularly transparent when restricted to sparse Wigner matrices (which was how the authors of [4] arrived at the proof). Once this special case has been understood, the reader can extend the proof to the general setting as an exercise, or refer to the general proof given in [4].

*Example 4.9 (Sparse Wigner matrices).* Informally, a sparse Wigner matrix is a symmetric random matrix with a given sparsity pattern, whose nonzero entries are independent standard Gaussian variables. It is convenient to fix the sparsity pattern of the matrix by specifying a given undirected graph $G = ([n], E)$ on $n$ vertices, whose adjacency matrix we denote as $B = (b_{ij})_{1\le i,j\le n}$. The corresponding sparse Wigner matrix $X$ is the symmetric random matrix whose entries are given by $X_{ij} = b_{ij}g_{ij}$, where $g_{ij}$ are independent standard Gaussian variables (up to symmetry $g_{ji} = g_{ij}$). Clearly our previous Examples 3.9, 3.10, and 4.3 are all special cases of this model.

For a sparse Wigner matrix, the first term in Theorem 4.7 is precisely the maximal degree $k = \deg(G)$ of the graph $G$, so that Theorem 4.7 reduces to

$$\mathbf{E}\|X\| \lesssim \sqrt{k} + \sqrt{\log n}.$$

We will see in section 4.3 that this bound is sharp for sparse Wigner matrices.

The remainder of this section is devoted to the proof of Theorem 4.7 in the setting of Example 4.9 (we fix the notation introduced in this example in the sequel). To understand the idea behind the proof, let us start by naively writing out the central quantity that appears in moment method (Lemma 2.2): we evidently have

$$\begin{aligned}\mathbf{E}[\mathrm{Tr}[X^{2p}]] &= \sum_{i_1,\ldots,i_{2p}=1}^{n} \mathbf{E}[X_{i_1i_2}X_{i_2i_3}\cdots X_{i_{2p-1}i_{2p}}X_{i_{2p}i_1}] \\ &= \sum_{i_1,\ldots,i_{2p}=1}^{n} b_{i_1i_2}b_{i_2i_3}\cdots b_{i_{2p}i_1}\,\mathbf{E}[g_{i_1i_2}g_{i_2i_3}\cdots g_{i_{2p}i_1}].\end{aligned}$$

It is useful to think of $\gamma = (i_1,\ldots,i_{2p})$ geometrically as a *cycle* $i_1 \to i_2 \to \cdots \to i_{2p} \to i_1$ of length $2p$. The quantity $b_{i_1i_2}b_{i_2i_3}\cdots b_{i_{2p}i_1}$ is equal to one precisely when $\gamma$ defines a cycle in the graph $G$, and is zero otherwise. We can therefore write

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]] = \sum_{\text{cycle } \gamma \text{ in } G \text{ of length } 2p} c(\gamma),$$

where we defined the constant $c(\gamma) := \mathbf{E}[g_{i_1i_2}g_{i_2i_3}\cdots g_{i_{2p}i_1}]$.

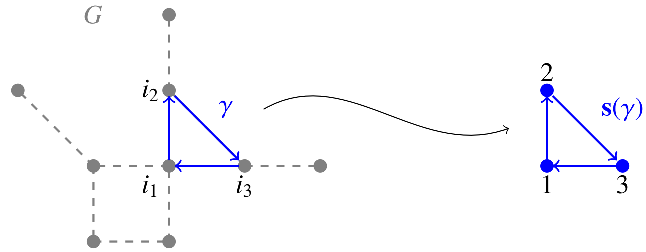
It turns out that $c(\gamma)$ does not really depend on the the position of the cycle $\gamma$ in the graph $G$. While we will not require a precise formula for $c(\gamma)$ in the proof, it is instructive to write down what it looks like. For any cycle $\gamma$ in $G$, denote by $m_\ell(\gamma)$ the number of distinct edges in $G$ that are visited by $\gamma$ precisely $\ell$ times, and denote by $m(\gamma) = \sum_{\ell\ge 1} m_\ell(\gamma)$ the total number of distinct edges visited by $\gamma$. Then

$$c(\gamma) = \prod_{\ell=1}^{\infty} \mathbf{E}[g^\ell]^{m_\ell(\gamma)},$$

where $g \sim N(0,1)$ is a standard Gaussian variable and we have used the independence of the entries. From this formula, we read off two important facts (which are the only ones that will actually be used in the proof):

- If any edge in $G$ is visited by $\gamma$ an odd number of times, then $c(\gamma) = 0$ (as the odd moments of $g$ vanish). Thus the only cycles that matter are *even* cycles, that is, cycles in which every distinct edge is visited an even number of times.
- $c(\gamma)$ depends on $\gamma$ only through the numbers $m_\ell(\gamma)$. Therefore, to compute $c(\gamma)$, we only need to know the *shape* $\mathbf{s}(\gamma)$ of the cycle $\gamma$.

The shape $\mathbf{s}(\gamma)$ is obtained from $\gamma$ by relabeling its vertices in order of appearance; for example, the shape of the cycle $7 \to 3 \to 9 \to 7 \to 3 \to 9 \to 7$ is given by $1 \to 2 \to 3 \to 1 \to 2 \to 3 \to 1$. The shape $\mathbf{s}(\gamma)$ captures the topological properties of $\gamma$ (such as the numbers $m_\ell(\gamma) = m_\ell(\mathbf{s}(\gamma))$) without keeping track of the manner in which $\gamma$ is embedded in $G$. This is illustrated in the following figure:




Putting together the above observations, we obtain the useful formula

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]] = \sum_{\text{shape } \mathbf{s} \text{ of even cycle of length } 2p} c(\mathbf{s}) \times \#\{\text{embeddings of } \mathbf{s} \text{ in } G\}.$$

So far, we have done nothing but bookkeeping. To use the above bound, however, we must get down to work and count the number of shapes of even cycles that can appear in the given graph $G$. The problem we face is that the latter proves to be a difficult combinatorial problem, which is apparently completely intractable when presented with any given graph $G$ that may possess an arbitrary structure (this is already highly nontrivial even in a complete graph when $p$ is large!) To squeeze anything useful out of this bound, it is essential that we find a shortcut.

The solution to our problem proves to be incredibly simple. Recall that $G$ is a given graph of degree $\deg(G) = k$. Of all graphs of degree $k$, which one will admit the most possible shapes? Obviously the graph that admits the most shapes is the one where every potential edge between two vertices is present; therefore, the graph of degree $k$ that possesses the most shapes is the *complete graph on $k$ vertices*. From the random matrix point of view, the latter corrsponds to a Wigner matrix of dimension $k \times k$. This simple idea suggests that rather than directly estimating the quantity $\mathbf{E}[\mathrm{Tr}[X^{2p}]]$ by combinatorial means, we should aim to prove a *comparison principle* between the moments of the $n \times n$ sparse matrix $X$ and the moments of a $k \times k$ Wigner matrix $Y$, which we already know how to bound by Lemma 3.11. Note that such a comparison principle is of a completely different nature than the Slepian-Fernique method used previously: here we are comparing two matrices of *different dimension*. The intuitive idea is that a large sparse matrix can be “compressed” into a much lower dimensional dense matrix without decreasing its norm.

The alert reader will note that there is a problem with the above intuition. While the complete graph on $k$ points admits more shapes than the original graph $G$, there are less potential ways in which each shape can be embedded in the complete graph as the latter possesses less vertices than the original graph. We can compensate for this deficiency by slightly increasing the dimension of the complete graph.

**Lemma 4.10 (Dimension compression).** *Let $X$ be the $n \times n$ sparse Wigner matrix (Example 4.9) defined by a graph $G = ([n], E)$ of maximal degree* $\deg(G) = k$, *and let $Y_r$ be an $r \times r$ Wigner matrix (Example 3.10). Then, for every $p \geq 1$,*

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]] \leq \frac{n}{k+p}\, \mathbf{E}[\mathrm{Tr}[Y_{k+p}^{2p}]].$$

*Proof.* Let $\mathbf{s}$ be the shape of an even cycle of length $2p$, and let $K_r$ be the complete graph on $r > p$ points. Denote by $m(\mathbf{s})$ the number of distinct vertices in $\mathbf{s}$, and note that $m(\mathbf{s}) \le p+1$ as every distinct edge in $\mathbf{s}$ must appear at least twice. Thus

$$\#\{\text{embeddings of } \mathbf{s} \text{ in } K_r\} = r(r-1)\cdots(r-m(\mathbf{s})+1),$$

as any assignment of vertices of $K_r$ to the distinct vertices of $\mathbf{s}$ defines a valid embedding of $\mathbf{s}$ in the complete graph. On the other hand, to count the number of embeddings of $\mathbf{s}$ in $G$, note that we have as many as $n$ choices for the first vertex, while each subsequent vertex can be chosen in at most $k$ ways (as $\deg(G) = k$). Thus

$$\#\{\text{embeddings of } \mathbf{s} \text{ in } G\} \le nk^{m(\mathbf{s})-1}.$$

Therefore, if we choose $r = k + p$, we have $r - m(\mathbf{s}) + 1 \ge r - p \ge k$, so that

$$\#\{\text{embeddings of } \mathbf{s} \text{ in } G\} \le \frac{n}{r}\,\#\{\text{embeddings of } \mathbf{s} \text{ in } K_r\}.$$

The proof now follows from the combinatorial expression for $\mathbf{E}[\mathrm{Tr}[X^{2p}]]$. □

With Lemma 4.10 in hand, it is now straightforward to complete the proof of Theorem 4.7 for the sparse Wigner matrix model of Example 4.9.

*Proof (Proof of Theorem 4.7 in the setting of Example 4.9).* We begin by noting that

$$\mathbf{E}\|X\| \le \mathbf{E}[\|X\|^{2p}]^{1/2p} \le n^{1/2p}\,\mathbf{E}[\|Y_{k+p}\|^{2p}]^{1/2p}$$

by Lemma 4.10, where we used $\|X\|^{2p} \le \mathrm{Tr}[X^{2p}]$ and $\mathrm{Tr}[Y_r^{2p}] \le r\|Y_r\|^{2p}$. Thus

$$\begin{aligned}\mathbf{E}\|X\| &\lesssim \mathbf{E}[\|Y_{k+\lfloor\log n\rfloor}\|^{2\log n}]^{1/2\log n} \\ &\le \mathbf{E}\|Y_{k+\lfloor\log n\rfloor}\| + \mathbf{E}[(\|Y_{k+\lfloor\log n\rfloor}\| - \mathbf{E}\|Y_{k+\lfloor\log n\rfloor}\|)^{2\log n}]^{1/2\log n} \\ &\lesssim \sqrt{k+\log n} + \sqrt{\log n},\end{aligned}$$

where in the last inequality we used Lemma 3.11 to bound the first term and Lemma 3.8 to bound the second term. Thus $\mathbf{E}\|X\| \lesssim \sqrt{k} + \sqrt{\log n}$, completing the proof. □

## *4.3 Three conjectures*

We have obtained in the previous sections two remarkably sharp bounds on the spectral norm of random matrices with independent centered Gaussian entries: the slightly suboptimal dimension-free bound of Theorem 4.2 for $\mathbf{E}\|X\|$, and the sharp dimension-dependent bound of Theorem 4.7 for $\mathbf{E}[\|X\|^{\log n}]^{1/\log n}$. As we will shortly argue, the latter bound is also sharp for $\mathbf{E}\|X\|$ in almost all situations of practical interest. Nonetheless, we cannot claim to have a complete understanding of the mechanisms that control the spectral norm of Gaussian random matrices unless we can obtain a sharp dimension-free bound on $\mathbf{E}\|X\|$. While this problem remains open, the above results strongly suggest what such a sharp bound should look like.

To gain some initial intuition, let us complement the sharp lower bound of Corollary 4.8 for $\mathbf{E}[\|X\|^{\log n}]^{1/\log n}$ by a trivial lower bound for $\mathbf{E}\|X\|$.

**Lemma 4.11.** *In the setting of this section, we have*

$$\mathbf{E}\|X\| \gtrsim \max_{i\le n} \sqrt{\sum_{j=1}^{n} b_{ij}^2} + \mathbf{E} \max_{i,j\le n} |X_{ij}|.$$

*Proof.* The first term is a lower bound by Lemma 3.7, while the second term is a lower bound by the trivial pointwise inequality $\|X\| \ge \max_{i,j} |X_{ij}|$. □

The simplest possible upper bound on the maximum of centered Gaussian random variables is $\mathbf{E}\max_{i,j} |X_{ij}| \lesssim \max_{i,j} b_{ij}\sqrt{\log n}$, which is sharp for i.i.d. Gaussian variables. Thus the lower bound of Lemma 4.11 matches the upper bound of Theorem 4.7 under a minimal homogeneity assumption: it suffices to assume that the number of entries whose standard deviation $b_{kl}$ is of the same order as $\max_{i,j} b_{ij}$ grows polynomially with dimension (which still allows for a vanishing fraction of entries of the matrix to possess large variance). For example, in the sparse Wigner matrix model of Example 4.9, every row of the matrix that does not correspond to an isolated vertex in $G$ contains at least one entry of variance one. Therefore, if $G$ possesses no isolated vertices, there are at least $n$ entries of $X$ with variance one, and it follows immediately from Lemma 4.11 that the bound of Theorem 4.7 is sharp for sparse Wigner matrices. (There is no loss of generality in assuming that $G$ has no isolated vertices: any isolated vertex yields a row that is identically zero, so we can simply remove such vertices from the graph without changing the norm.)

However, when the variances of the entries of $X$ possess many different scales, the dimension-dependent upper bound $\mathbf{E}\max_{i,j} |X_{ij}| \lesssim \max_{i,j} b_{ij}\sqrt{\log n}$ can fail to be sharp. To obtain a sharp bound on the maximum of Gaussian random variables, we must proceed in a dimension-free fashion as in Lemma 4.5. In particular, combining Remark 4.6 and Lemma 4.11 yields the following explicit lower bound:

$$\mathbf{E}\|X\| \gtrsim \max_{i\le n} \sqrt{\sum_{j=1}^{n} b_{ij}^2} + \max_{i,j\le n} b_{ij}\sqrt{\log i},$$

provided that $\max_j b_{1j} \ge \max_j b_{2j} \ge \cdots \ge \max_j b_{nj} > 0$ (there is no loss of generality in assuming the latter, as we can always permute the rows and columns of $X$ to achieve this ordering without changing the norm of $X$). It will not have escaped the attention of the reader that the latter lower bound is tantalizingly close both to the dimension-dependent upper bound of Theorem 4.7, and to the dimension-free upper bound of Theorem 4.2. This leads us to the following very natural conjecture [31].

**Conjecture 1.** Assume without loss of generality that the rows and columns of $X$ have been permuted such that $\max_j b_{1j} \ge \max_j b_{2j} \ge \cdots \ge \max_j b_{nj} > 0$. Then

$$\mathbf{E}\|X\| \asymp \|\mathbf{E}X^2\|^{1/2} + \mathbf{E}\max_{i,j\le n}|X_{ij}|$$
$$\asymp \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + \max_{i,j\le n} b_{ij}\sqrt{\log i}.$$

Conjecture 1 appears completely naturally from our results, and has a surprising interpretation. There are two simple mechanisms that would certainly force the random matrix $X$ to have large expected norm $\mathbf{E}\|X\|$: the matrix $X$ is can be large "on average" in the sense that $\|\mathbf{E}X^2\|$ is large (note that the expectation here is *inside* the norm), or the matrix $X$ can have an entry that exhibits a large fluctuation in the sense that $\max_{i,j} X_{ij}$ is large. Conjecture 1 suggests that these two mechanisms are, in a sense, the *only* reasons why $\mathbf{E}\|X\|$ can be large.

Given the remarkable similarity between Conjecture 1 and Theorem 4.7, one might hope that a slight sharpening of the proof of Theorem 4.7 would suffice to yield the conjecture. Unfortunately, it seems that the moment method is largely useless for the purpose of obtaining dimension-free bounds: indeed, the Corollary 4.8 shows that the moment method is already exploited optimally in the proof of Theorem 4.7. While it is sometimes possible to derive dimension-free results from dimension-dependent results by a stratification procedure, such methods either fail completely to capture the correct structure of the problem (cf. [19]) or retain a residual dimension-dependence (cf. [31]). It therefore seems likely that random process methods will prove to be essential for progress in this direction.

While Conjecture 1 appears completely natural in the present setting, we should also discuss a competing conjecture that was proposed much earlier by R. Latała. Inspired by certain results of Seginer [21] for matrices with i.i.d. entries, Latała conjectured the following sharp bound in the general setting of this section.

**Conjecture 2.** In the setting of this section, we have

$$\mathbf{E}\|X\| \asymp \mathbf{E}\max_{i\le n}\sqrt{\sum_{j=1}^n X_{ij}^2}.$$

As $\|X\|^2 \ge \max_i \sum_j X_{ij}^2$ holds deterministically, the lower bound in Conjecture 2 is trivial: it states that a matrix that possesses a large row must have large spectral norm. Conjecture 2 suggests that this is the *only* reason why the matrix norm can be large. This is certainly not the case for an arbitrary matrix $X$, and so it is not at all clear *a priori* why this should be true. Nonetheless, no counterexample is known in the setting of the Gaussian random matrices considered in this section.

While Conjectures 1 and 2 appear to arise from different mechanisms, it is observed in [31] that these conjectures are actually equivalent: it is not difficult to show that the right-hand side in both inequalities is equivalent, up to the universal constant, to the explicit expression recorded in Conjecture 1. In fact, let us note that both conjectured mechanisms are essentially already present in the proof of Theorem 4.2: in the comparison process $Y_v$ that arises in the proof, the first term is strongly reminiscent of Conjecture 2, while the second term is reminiscent of the second term in

Conjecture 1. In this sense, the mechanism that is developed in the proof of Theorem 4.2 provides even stronger evidence for the validity of these conjectures. The remaining inefficiency in the proof of Theorem 4.2 is discussed in detail in [31].

We conclude by discussing briefly a much more speculative question. The noncommutative Khintchine inequalities developed in the previous section hold in a very general setting, but are almost always suboptimal. In contrast, the bounds in this section yield nearly optimal results under the additional assumption that the matrix entries are independent. It would be very interesting to understand whether the bounds of the present section can be extended to the much more general setting captured by noncommutative Khintchine inequalities. Unfortunately, independence is used crucially in the proofs of the results in this section, and it is far from clear what mechanism might give rise to analogous results in the dependent setting.

One might nonetheless speculate what such a result might potentially look like. In particular, we note that both parameters that appear in the sharp bound Theorem 4.7 have natural analogues in the general setting: in the setting of this section

$$\|\mathbf{E}X^2\| = \sup_{v\in B}\mathbf{E}\langle v, X^2 v\rangle = \max_i \sum_j b_{ij}^2, \qquad \sup_{v\in B}\mathbf{E}\langle v, Xv\rangle^2 = \max_{i,j} b_{ij}^2.$$

We have already encountered both these quantities also in the previous section: $\sigma = \|\mathbf{E}X^2\|^{1/2}$ is the natural structural parameter that arises in noncommutative Khintchine inequalities, while $\sigma_* := \sup_v \mathbf{E}[\langle v, Xv\rangle^2]^{1/2}$ controls the fluctuations of the spectral norm by Gaussian concentration (see the proof of Lemma 3.7). By analogy with Theorem 4.7, we might therefore speculatively conjecture:

**Conjecture 3.** Let $X = \sum_{k=1}^s g_k A_k$ as in Theorem 3.2. Then

$$\mathbf{E}\|X\| \lesssim \|\mathbf{E}X^2\|^{1/2} + \sup_{v\in B}\mathbf{E}[\langle v, Xv\rangle^2]^{1/2}\sqrt{\log n}.$$

Such a generalization would constitute a far-reaching improvement of the noncommutative Khintchine theory. The problem with Conjecture 3 is that it is completely unclear how such a bound might arise: the only evidence to date for the potential validity of such a bound is the vague analogy with the independent case, and the fact that a counterexample has yet to be found.

## 4.4 Seginer's inequality

Throughout this chapter, we have focused attention on Gaussian random matrices. We depart briefly from this setting in this section to discuss some aspects of structured random matrices that arise under other distributions of the entries.

The main reason that we restricted attention to Gaussian matrices is that most of the difficulty of capturing the structure of the matrix arises in this setting; at the same time, all upper bounds we develop extend without difficulty to more general distributions, so there is no significant loss of generality in focusing on the Gaussian case. For example, let us illustrate the latter statement using the moment method.

**Lemma 4.12.** *Let $X$ and $Y$ be symmetric random matrices with independent entries (modulo symmetry). Assume that $X_{ij}$ are centered and subgaussian, that is,* $\mathbf{E}X_{ij} = 0$ *and* $\mathbf{E}[X_{ij}^{2p}]^{1/2p} \lesssim b_{ij}\sqrt{p}$ *for all* $p \geq 1$*, and let* $Y_{ij} \sim N(0, b_{ij}^2)$*. Then*

$$\mathbf{E}[\mathrm{Tr}[X^{2p}]]^{1/2p} \lesssim \mathbf{E}[\mathrm{Tr}[Y^{2p}]]^{1/2p} \quad \textit{for all } p \geq 1.$$

*Proof.* Let $X'$ be an independent copy of $X$. Then $\mathbf{E}[\mathrm{Tr}[X^{2p}]] = \mathbf{E}[\mathrm{Tr}[(X-\mathbf{E}X')^{2p}]] \leq \mathbf{E}[\mathrm{Tr}[(X-X')^{2p}]]$ by Jensen's inequality. Moreover, $Z = X - X'$ a symmetric random matrix satisfying the same properties as $X$, with the additional property that the entries $Z_{ij}$ have symmetric distribution. Thus $\mathbf{E}[Z_{ij}^p]^{1/p} \lesssim \mathbf{E}[Y_{ij}^p]^{1/p}$ for all $p \geq 1$ (for odd $p$ both sides are zero by symmetry, while for even $p$ this follows from the subgaussian assumption using $\mathbf{E}[Y_{ij}^{2p}]^{1/2p} \asymp b_{ij}\sqrt{p}$). It remains to note that

$$\begin{aligned}\mathbf{E}[\mathrm{Tr}[X^{2p}]] &= \sum_{\text{cycle } \gamma \text{ of length } 2p} \prod_{1\leq i\leq j\leq n} \mathbf{E}[X_{ij}^{\#_{ij}(\gamma)}] \\ &\leq C^{2p} \sum_{\text{cycle } \gamma \text{ of length } 2p} \prod_{1\leq i\leq j\leq n} \mathbf{E}[Y_{ij}^{\#_{ij}(\gamma)}] = C^{2p}\,\mathbf{E}[\mathrm{Tr}[Y^{2p}]]\end{aligned}$$

for a universal constant $C$, where $\#_{ij}(\gamma)$ denotes the number of times the edge $(i, j)$ appears in the cycle $\gamma$. The conclusion follows immediately. □

Lemma 4.12 shows that to upper bound the moments of a subgaussian random matrix with independent entries, it suffices to obtain a bound in the Gaussian case. The reader may readily verify that the completely analogous approach can be applied in the more general setting of the noncommutative Khinthchine inequality. On the other hand, Gaussian bounds using the random process method extend to the subgaussian setting by virtue of a general subgaussian comparison principle [24, Theorem 2.4.12]. Beyond the subgaussian setting, similar methods can be used for entries with heavy-tailed distributions, see for example [4].

The above observations indicate that, in some sense, Gaussian random matrices are the "worst case" among subgaussian matrices. One can go one step further and ask whether there is some form of universality: do all subgaussian random matrices behave like their Gaussian counterparts? The universality phenomenon plays a major role in recent advances in random matrix theory: it turns out that many properties of Wigner matrices do not depend on the distribution of the entries. Unfortunately, we cannot expect universal behavior for structured random matrices: while Gaussian matrices are the "worst case" among subgaussian matrices, matrices with subgaussian entries can sometimes behave much better. The simplest example is the case of diagonal matrices (Example 3.9) with i.i.d. entries on the diagonal: in the Gaussian case $\mathbf{E}\|X\| \asymp \sqrt{\log n}$, but obviously $\mathbf{E}\|X\| \asymp 1$ if the entries are uniformly bounded (despite that uniformly bounded random variables are obviously subgaussian). In view of such examples, there is little hope to obtain a complete understanding of structured random matrices for arbitrary distributions of the entries. This justifies the approach we have taken: we seek sharp bounds for Gaussian matrices, which give rise to powerful upper bounds for general distributions of the entries.

*Remark 4.13.* We emphasize in this context that Conjectures 1 and 2 in the previous section are fundamentally Gaussian in nature, and *cannot* hold as stated for subgaussian matrices. For a counterexample along the lines of Example 4.3, see [21].

Despite these negative observations, it can be of significant interest to go beyond the Gaussian setting to understand whether the bounds we have obtained can be systematically improved under more favorable assumptions on the distributions of the entries. To illustrate how such improvements could arise, we discuss a result of Seginer [21] for random matrices with independent *uniformly bounded* entries.

**Theorem 4.14 (Seginer).** *Let $X$ be an $n\times n$ symmetric random matrix with independent entries (modulo symmetry) and $\mathbf{E}X_{ij} = 0$, $\|X_{ij}\|_\infty \lesssim b_{ij}$ for all $i, j$. Then*

$$\mathbf{E}\|X\| \lesssim \max_{i\le n} \sqrt{\sum_{j=1}^n b_{ij}^2}\,(\log n)^{1/4}.$$

The uniform bound $\|X_{ij}\|_\infty \lesssim b_{ij}$ certainly implies the much weaker subgaussian property $\mathbf{E}[X_{ij}^{2p}]^{1/2p} \lesssim b_{ij}\sqrt{p}$, so that the conclusion of Theorem 4.7 extends immediately to the present setting by Lemma 4.12. In many cases, the latter bound is much sharper than the one provided by Theorem 4.14; indeed, Theorem 4.14 is suboptimal even for Wigner matrices (it could be viewed of a variant of the noncommutative Khintchine inequality in the present setting with a smaller power in the logarithmic factor). However, the interest of Theorem 4.14 is that it *cannot* hold for Gaussian entries: for example, in the diagonal case $b_{ij} = \mathbf{1}_{i=j}$, Theorem 4.14 gives $\mathbf{E}\|X\| \lesssim (\log n)^{1/4}$ while any Gaussian bound must give at least $\mathbf{E}\|X\| \gtrsim \sqrt{\log n}$. In this sense, Theorem 4.14 illustrates that it is possible in some cases to exploit the effect of stronger distributional assumptions in order to obtain improved bounds for non-Gaussian random matrices. The simple proof that we will give (taken from [4]) shows very clearly how this additional distributional information enters the picture.

*Proof (Proof of Theorem 4.14).* The proof works by combining two very different bounds on the matrix norm. On the one hand, due to Lemma 4.12, we can directly apply the Gaussian bound of Theorem 4.7 in the present setting. On the other hand, as the entries of $X$ are uniformly bounded, we can do something that is impossible for Gaussian random variables: we can *uniformly* bound the norm $\|X\|$ as

$$\begin{aligned}\|X\| &= \sup_{v\in B}\left|\sum_{i,j=1}^n v_i X_{ij} v_j\right| \le \sup_{v\in B}\sum_{i,j=1}^n (|v_i|\,|X_{ij}|^{1/2})(|X_{ij}|^{1/2}|v_j|) \\ &\le \sup_{v\in B}\sum_{i,j=1}^n v_i^2 |X_{ij}| = \max_{i\le n}\sum_{j=1}^n |X_{ij}| \le \max_{i\le n}\sum_{j=1}^n b_{ij},\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in going from the first to the second line. The idea behind the proof of Theorem 4.14 is roughly as follows. Many small entries of $X$ can add up to give rise to a large norm; we might expect the cumulative effect of many independent centered random variables to give rise to Gaussian behavior. On the other hand, if a few large entries of $X$ dominate the norm, there is no Gaussian behavior and we expect that the uniform bound provides much better control. To capture this idea, we partition the matrix into two parts $X = X_1 + X_2$, where $X_1$ contains the "small" entries and $X_2$ contains the "large" entries:

$$(X_1)_{ij} = X_{ij}\mathbf{1}_{b_{ij}\le u}, \qquad (X_2)_{ij} = X_{ij}\mathbf{1}_{b_{ij}>u}.$$

Applying the Gaussian bound to $X_1$ and the uniform bound to $X_2$ yields

$$\begin{aligned}\mathbf{E}\|X\| &\le \mathbf{E}\|X_1\| + \mathbf{E}\|X_2\| \\ &\lesssim \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2\mathbf{1}_{b_{ij}\le u}} + u\sqrt{\log n} + \max_{i\le n}\sum_{j=1}^n b_{ij}\mathbf{1}_{b_{ij}>u} \\ &\le \max_{i\le n}\sqrt{\sum_{j=1}^n b_{ij}^2} + u\sqrt{\log n} + \frac{1}{u}\max_{i\le n}\sum_{j=1}^n b_{ij}^2.\end{aligned}$$

The proof is completed by optimizing over $u$. □

The proof of Theorem 4.14 illustrates the improvement that can be achived by trading off between Gaussian and uniform bounds on the norm of a random matrix. Such tradeoffs play a fundamental role in the general theory that governs the suprema of bounded random processes [24, Chapter 5]. Unfortunately, this tradeoff is captured only very crudely by the suboptimal Theorem 4.14.

Developing a sharp understanding of the behavior of bounded random matrices is a problem of significant interest: the bounded analogue of sparse Wigner matrices (Example 4.9) has interesting connections with graph theory and computer science, cf. [1] for a review of such applications. Unlike in the Gaussian case, however, it is clear that the degree of the graph that defines a sparse Wigner matrix cannot fully explain its spectral norm in the present setting: very different behavior is exhibited in dense vs. locally tree-like graphs of the same degree [4, section 4.2]. To date, a deeper understanding of such matrices beyond the Gaussian case remains limited.

## 5 Sample covariance matrices

We finally turn our attention to a random matrix model that is somewhat different than the matrices we considered so far. The following model will be considered throughout this section. Let $\Sigma$ be a given $d \times d$ positive semidefinite matrix, and let $X_1, X_2, \dots, X_n$ be i.i.d. centered Gaussian random vectors in $\mathbb{R}^d$ with covariance matrix $\Sigma$. We consider in the following the $d \times d$ symmetric random matrix

$$Z = \frac{1}{n}\sum_{k=1}^{n} X_k X_k^* = \frac{XX^*}{n},$$

where we defined the $d \times n$ matrix $X_{ik} = (X_k)_i$. In contrast to the models considered in the previous sections, the random matrix $Z$ is not centered: we have in fact $\mathbf{E}Z = \Sigma$. This gives rise to the classical statistical interpretation of this matrix. We can think of $X_1, \ldots, X_n$ as being i.i.d. data drawn from a centered Gaussian distribution with unknown covariance matrix $\Sigma$. In this setting, the random matrix $Z$, which depends only on the observed data, provides an unbiased estimator of the covariance matrix of the underlying data. For this reason, $Z$ is known as the *sample covariance matrix*. Of primary interest in this setting is not so much the matrix norm $\|Z\| = \|X\|^2/n$ itself, but rather the deviation $\|Z - \Sigma\|$ of $Z$ from its mean.

The model of this section could be viewed as being "semi-structured." On the one hand, the covariance matrix $\Sigma$ is completely arbitrary, and it therefore allows for an arbitrary variance and dependence pattern within each column of the matrix $X$ (as in the most general setting of the noncommutative Khintchine inequality). On the other hand, the columns of $X$ are assumed to be i.i.d., so that no nontrivial structure among the columns is captured by the present model. While the latter assumption is limiting, it allows us to obtain a complete understanding of the structural parameters that control the expected deviation $\mathbf{E}\|Z - \Sigma\|$ in this setting [10].

**Theorem 5.1 (Koltchinskii-Lounici).** *In the setting of this section*

$$\mathbf{E}\|Z - \Sigma\| \asymp \|\Sigma\| \left( \sqrt{\frac{r(\Sigma)}{n}} + \frac{r(\Sigma)}{n} \right),$$

*where* $r(\Sigma) := \mathrm{Tr}[\Sigma]/\|\Sigma\|$ *is the* effective rank *of* $\Sigma$.

The remainder of this section is devoted to the proof of Theorem 5.1.

### Upper bound

The proof of Theorem 5.1 will use the random process method using tools that were already developed in the previous sections. It would be clear how to proceed if we wanted to bound $\|Z\|$: as $\|Z\| = \|X\|^2/n$, it would suffice to bound $\|X\|$ which is the supremum of a Gaussian process. Unfortunately, this idea does not extend directly to the problem of bounding $\|Z - \Sigma\|$: the latter quantity is not the supremum of a centered Gaussian process, but rather of a *squared* Gaussian process

$$\|Z - \Sigma\| = \sup_{v \in B} \left| \frac{1}{n} \sum_{k=1}^{n} \{\langle v, X_k \rangle^2 - \mathbf{E}\langle v, X_k \rangle^2\} \right|.$$

We therefore cannot directly apply a Gaussian comparison method such as the Slepian-Fernique inequality to control the expected deviation $\mathbf{E}\|Z - \Sigma\|$.

To surmount this problem, we will use a simple device that is widely used in the study of squared Gaussian processes (or *Gaussian chaos*), cf. [12, section 3.2].

**Lemma 5.2 (Decoupling).** *Let $\tilde{X}$ be an independent copy of $X$. Then*

$$\mathbf{E}\|Z - \Sigma\| \le \frac{2}{n}\,\mathbf{E}\|X\tilde{X}^*\|.$$

*Proof.* By Jensen's inequality

$$\mathbf{E}\|Z - \Sigma\| = \frac{1}{n}\,\mathbf{E}\|\mathbf{E}[(X+\tilde{X})(X-\tilde{X})^*|X]\| \le \frac{1}{n}\,\mathbf{E}\|(X+\tilde{X})(X-\tilde{X})^*\|.$$

It remains to note that $(X+\tilde{X}, X-\tilde{X})$ has the same distribution as $\sqrt{2}\,(X,\tilde{X})$. □

Roughly speaking, the decoupling device of Lemma 5.2 allows us to replace the square $XX^*$ of a Gaussian matrix by a product of two independent copies $X\tilde{X}^*$. While the latter is still not Gaussian, it becomes Gaussian if we condition on one of the copies (say, $\tilde{X}$). This means that $\|X\tilde{X}^*\|$ is the supremum of a Gaussian process *conditionally* on $\tilde{X}$. This is precisely what we will exploit in the sequel: we use the Slepian-Fernique inequality conditionally on $\tilde{X}$ to obtain the following bound.

**Lemma 5.3.** *In the setting of this section*

$$\mathbf{E}\|Z - \Sigma\| \lesssim \mathbf{E}\|X\|\frac{\sqrt{\mathrm{Tr}[\Sigma]}}{n} + \|\Sigma\|\sqrt{\frac{r(\Sigma)}{n}}.$$

*Proof.* By Lemma 5.2 we have

$$\mathbf{E}\|Z - \Sigma\| \le \frac{2}{n}\,\mathbf{E}\Big[\sup_{v,w\in B} Z_{v,w}\Big], \qquad Z_{v,w} := \sum_{k=1}^{n}\langle v, X_k\rangle\langle w, \tilde{X}_k\rangle.$$

Writing for simplicity $\mathbf{E}_{\tilde{X}}[\cdot] = \mathbf{E}[\cdot|\tilde{X}]$, we can estimate

$$\begin{aligned}\mathbf{E}_{\tilde{X}}(Z_{v,w} - Z_{v',w'})^2 &\le 2\langle v - v', \Sigma(v - v')\rangle \sum_{k=1}^{n}\langle w, \tilde{X}_k\rangle^2 + 2\langle v', \Sigma v'\rangle \sum_{k=1}^{n}\langle w - w', \tilde{X}_k\rangle^2 \\ &\le 2\|\tilde{X}\|^2\|\Sigma^{1/2}(v - v')\|^2 + 2\|\Sigma\|\,\|\tilde{X}^*(w - w')\|^2 \\ &= \mathbf{E}_{\tilde{X}}(Y_{v,w} - Y_{v',w'})^2,\end{aligned}$$

where we defined

$$Y_{v,w} = \sqrt{2}\,\|\tilde{X}\|\,\langle v, \Sigma^{1/2}g\rangle + (2\|\Sigma\|)^{1/2}\,\langle w, \tilde{X}g'\rangle$$

with $g, g'$ independent standard Gaussian vectors in $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively. Thus

$$\mathbf{E}_{\tilde{X}}\bigg[\sup_{v,w\in B} Z_{v,w}\bigg] \le \mathbf{E}_{\tilde{X}}\bigg[\sup_{v,w\in B} Y_{v,w}\bigg] \lesssim \|\tilde{X}\|\,\mathbf{E}\|\Sigma^{1/2}g\| + \|\Sigma\|^{1/2}\,\mathbf{E}_{\tilde{X}}\|\tilde{X}g\|$$
$$\le \|\tilde{X}\|\sqrt{\mathrm{Tr}[\Sigma]} + \|\Sigma\|^{1/2}\,\mathrm{Tr}[\tilde{X}\tilde{X}^*]^{1/2}$$

by the Slepian-Fernique inequality. Taking the expectation with respect to $\tilde{X}$ and using that $\mathbf{E}\|\tilde{X}\| = \mathbf{E}\|X\|$ and $\mathbf{E}[\mathrm{Tr}[\tilde{X}\tilde{X}^*]^{1/2}] \le \sqrt{n\,\mathrm{Tr}[\Sigma]}$ yields the conclusion. □

Lemma 5.3 has reduced the problem of bounding $\mathbf{E}\|Z - \Sigma\|$ to the much more straightforward problem of bounding $\mathbf{E}\|X\|$: as $\|X\|$ is the supremum of a Gaussian process, the latter is amenable to a direct application of the Slepian-Fernique inequality precisely as was done in the proof of Lemma 3.11.

**Lemma 5.4.** *In the setting of this section*

$$\mathbf{E}\|X\| \lesssim \sqrt{\mathrm{Tr}[\Sigma]} + \sqrt{n\|\Sigma\|}.$$

*Proof.* Note that

$$\mathbf{E}(\langle v, Xw\rangle - \langle v', Xw'\rangle)^2 \le 2\,\mathbf{E}(\langle v - v', Xw\rangle)^2 + 2\,\mathbf{E}(\langle v', X(w - w')\rangle)^2$$
$$= 2\|\Sigma^{1/2}(v - v')\|^2\|w\|^2 + 2\|\Sigma^{1/2}v'\|^2\|w - w'\|^2$$
$$\le \mathbf{E}(X'_{v,w} - X'_{v',w'})^2$$

when $\|v\|, \|w\| \le 1$, where we defined

$$X'_{v,w} = \sqrt{2}\,\langle v, \Sigma^{1/2}g\rangle + \sqrt{2}\,\|\Sigma\|^{1/2}\,\langle w, g'\rangle$$

with $g, g'$ independent standard Gaussian vectors in $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively. Thus

$$\mathbf{E}\|X\| = \mathbf{E}\bigg[\sup_{v,w\in B}\langle v, Xw\rangle\bigg] \le \mathbf{E}\bigg[\sup_{v,w\in B} X'_{v,w}\bigg] \lesssim \mathbf{E}\|\Sigma^{1/2}g\| + \|\Sigma\|^{1/2}\mathbf{E}\|g\|$$

by the Slepian-Fernique inequality. The proof is easily completed. □

The proof of the upper bound in Theorem 5.1 is now immediately completed by combining the results of Lemma 5.3 and Lemma 5.4.

*Remark 5.5.* The proof of the upper bound given here reduces the problem of controlling the supremum of a Gaussian chaos process by decoupling to that of controlling the supremum of a Gaussian process. The original proof in [10] uses a different method that exploits a much deeper general result on the suprema of empirical processes of squares, cf. [24, Theorem 9.3.7]. While the route we have taken is much more elementary, the original approach has the advantage that it applies directly to subgaussian matrices. The result of [10] is also stated for norms other than the spectral norm, but proof given here extends readily to this setting.

### Lower bound

It remains to prove the lower bound in Theorem 5.1. The main idea behind the proof is that the decoupling inequality of Lemma 5.2 can be partially reversed.

**Lemma 5.6.** *Let $\tilde{X}$ be an independent copy of $X$. Then for every $v \in \mathbb{R}^d$*

$$\mathbf{E}\|(Z-\Sigma)v\| \geq \frac{1}{n}\,\mathbf{E}\|X\tilde{X}^*v\| - \frac{\|\Sigma v\|}{\sqrt{n}}.$$

*Proof.* The reader may readily verify that the random matrix

$$X' = \left(I - \frac{\Sigma v v^*}{\langle v, \Sigma v\rangle}\right)X$$

is independent of the random vector $X^*v$ (and therefore of $\langle v, Zv\rangle$). Moreover

$$(Z-\Sigma)v = \frac{XX^*v}{n} - \Sigma v = \frac{X'X^*v}{n} + \left(\frac{\langle v, Zv\rangle}{\langle v, \Sigma v\rangle} - 1\right)\Sigma v.$$

As the columns of $X'$ are i.i.d. and independent of $X^*v$, the pair $(X'X^*v, X^*v)$ has the same distribution as $(X_1'\|X^*v\|, X^*v)$ where $X_1'$ denotes the first column of $X'$. Thus

$$\mathbf{E}\|(Z-\Sigma)v\| = \mathbf{E}\left\|\frac{X_1'\|X^*v\|}{n} + \left(\frac{\langle v, Zv\rangle}{\langle v, \Sigma v\rangle} - 1\right)\Sigma v\right\| \geq \frac{1}{n}\,\mathbf{E}\|X^*v\|\,\mathbf{E}\|X_1'\|,$$

where we used Jensen's inequality conditionally on $X'$. Now note that

$$\mathbf{E}\|X_1'\| \geq \mathbf{E}\|X_1\| - \|\Sigma v\|\,\frac{\mathbf{E}|\langle v, X_1\rangle|}{\langle v, \Sigma v\rangle} \geq \mathbf{E}\|X_1\| - \frac{\|\Sigma v\|}{\langle v, \Sigma v\rangle^{1/2}}.$$

We therefore have

$$\mathbf{E}\|(Z-\Sigma)v\| \geq \frac{1}{n}\,\mathbf{E}\|X_1\|\,\mathbf{E}\|\tilde{X}^*v\| - \frac{1}{n}\,\mathbf{E}\|X^*v\|\frac{\|\Sigma v\|}{\langle v, \Sigma v\rangle^{1/2}} \geq \frac{1}{n}\,\mathbf{E}\|X\tilde{X}^*v\| - \frac{\|\Sigma v\|}{\sqrt{n}},$$

as $\mathbf{E}\|X^*v\| \leq \sqrt{n}\,\langle v, \Sigma v\rangle^{1/2}$ and as $X_1\|\tilde{X}^*v\|$ has the same distribution as $X\tilde{X}^*v$. □

As a corollary, we can obtain the first term in the lower bound.

**Corollary 5.7.** *In the setting of this section, we have*

$$\mathbf{E}\|Z-\Sigma\| \gtrsim \|\Sigma\|\sqrt{\frac{r(\Sigma)}{n}}.$$

*Proof.* Taking the supremum over $v \in B$ in Lemma 5.6 yields

$$\mathbf{E}\|Z-\Sigma\| + \frac{\|\Sigma\|}{\sqrt{n}} \geq \sup_{v\in B}\frac{1}{n}\,\mathbf{E}\|X\tilde{X}^*v\| = \frac{1}{n}\,\mathbf{E}\|X_1\|\,\sup_{v\in B}\mathbf{E}\|\tilde{X}^*v\|.$$

Using Gaussian concentration as in the proof of Lemma 3.7, we obtain

$$\mathbf{E}\|X_1\| \gtrsim \mathbf{E}[\|X_1\|^2]^{1/2} = \sqrt{\mathrm{Tr}[\Sigma]}, \qquad \mathbf{E}\|\tilde{X}^* v\| \gtrsim \mathbf{E}[\|\tilde{X}^* v\|^2]^{1/2} = \sqrt{n\,\langle v, \Sigma v\rangle}.$$

This yields

$$\mathbf{E}\|Z - \Sigma\| + \frac{\|\Sigma\|}{\sqrt{n}} \gtrsim \|\Sigma\| \sqrt{\frac{r(\Sigma)}{n}}.$$

On the other hand, we can estimate by the central limit theorem

$$\frac{\|\Sigma\|}{\sqrt{n}} \lesssim \sup_{v \in B} \mathbf{E}|\langle v, (Z - \Sigma)v\rangle| \le \mathbf{E}\|Z - \Sigma\|,$$

as $\langle v, (Z - \Sigma)v\rangle = \langle v, \Sigma v\rangle \frac{1}{n} \sum_{k=1}^n \{Y_k^2 - 1\}$ with $Y_k = \langle v, X_k\rangle / \langle v, \Sigma v\rangle^{1/2} \sim N(0,1)$. □

We can now easily complete the proof of Theorem 5.1.

*Proof (Proof of Theorem 5.1).* The upper bound follows immediately from Lemmas 5.3 and 5.4. For the lower bound, suppose first that $r(\Sigma) \le 2n$. Then $\sqrt{r(\Sigma)/n} \gtrsim r(\Sigma)/n$, and the result follows from Corollary 5.7. On the other hand, if $r(\Sigma) > 2n$,

$$\mathbf{E}\|Z - \Sigma\| \ge \mathbf{E}\|Z\| - \|\Sigma\| \ge \frac{\mathbf{E}\|X_1\|^2}{n} - \|\Sigma\| \frac{r(\Sigma)}{2n} = \|\Sigma\| \frac{r(\Sigma)}{2n},$$

where we used that $Z = \frac{1}{n} \sum_{k=1}^n X_k X_k^* \ge \frac{1}{n} X_1 X_1^*$. □

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