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The Alexandrov-Fenchel Inequality

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Abstract

In the late 1800s, Minkowski discovered the importance of a new kind of geometric object that we now call a convex set. He soon developed a rich theory for understanding such sets, laying the foundations of convex geometry that are widely used to this day. Among the most surprising observations of Minkowski's theory is that the classical isoperimetric theorem—which states that the ball has the smallest surface area among all bodies of a given volume—is just one special case of a much more general phenomenon. When one fixes geometric parameters other than volume, Minkowski discovered that the resulting "bubbles" can be strikingly bizarre. A complete understanding of such objects has been a long-standing problem. In this survey, we briefly discuss the history of and recent progress on these questions, as well as some of their unexpected interactions with other areas.

1 Introduction


H. Minkowski

Hermann Minkowski (1864–1909) had already achieved fame at the young age of 18 when he was awarded the Grand Prix of the Académie des Sciences in Paris for his work on number theory. At that time, he was interested in classical arithmetic questions on positive definite quadratic forms that arose from the work of Lagrange, Gauss, and Hermite. Minkowski realized that such problems can be approached geometrically by reasoning about the volume of the ellipsoid defined by the quadratic form. By replacing the ellipsoid by an arbitrary symmetric convex body, Minkowski obtained a powerful new tool to investigate number-theoretic questions [20].

The unexpected emergence of the notion of convex bodies in the geometry of numbers motivated Minkowski to investigate these objects for their own sake, culminating in a remarkable 1903 paper [25] that recently celebrated its 120$^{\text{th}}$ birthday. This paper not only laid the foundation for the field of convex geometry [31], but also introduced important ideas that had a major impact on other areas of mathematics: e.g., the Minkowski problem that led to seminal work on the theory of nonlinear PDEs [27, 12]; one of the earliest investigations of the stability of geometric inequalities [15, 14]; and a fundamental log-concavity phenomenon that is closely connected to recent breakthroughs in combinatorics [19].

At the same time, some questions that arise from Minkowski's work remain open to this day. This survey reports on recent progress on one of these questions.

2 The Alexandrov-Fenchel inequality

At the heart of Minkowski's theory of convex bodies lies the observation that n-dimensional volume is a *polynomial* of convex bodies in $\mathbb{R}^n$: more precisely, given any convex bodies (nonempty compact convex sets) in $\mathbb{R}^n$, we have

$$\mathrm{Vol}(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \mathsf{V}(K_{i_1}, \ldots, K_{i_n})$$

for all $\lambda_1, \ldots, \lambda_m \geq 0$. The coefficients $\mathsf{V}(C_1, \ldots, C_n)$ of this polynomial, called *mixed volumes*, form a large family of geometric parameters of convex bodies: for example, volume, surface area, mean width, projection volumes, and many other natural quantities are all special cases of mixed volumes.

Given the ubiquity of mixed volumes, it is unsurprising that inequalities between mixed volumes play a central role in understanding the geometry of convex bodies. The most important result of this kind is the following.

Theorem 2.1 (Alexandrov-Fenchel inequality)**.** *We have*

$$\mathsf{V}(K, L, C_1, \ldots, C_{n-2})^2 \geq \mathsf{V}(K, K, C_1, \ldots, C_{n-2})\, \mathsf{V}(L, L, C_1, \ldots, C_{n-2}).$$

for any convex bodies $K, L, C_1, \ldots, C_{n-2}$ in $\mathbb{R}^n$.

In spite of its name, this inequality was first proved by Minkowski [25] in dimension $n = 3$, and was extended by Alexandrov [1, 2] to general n using fundamentally new ideas (Fenchel independently announced the result but did not publish a proof). It lies at the heart of many applications of mixed volumes in convex geometry and in other areas of mathematics. The following basic question already appears in the original papers of Minkowski and Alexandrov:

Question 2.2. *When does the Alexandrov-Fenchel inequality achieve equality?*

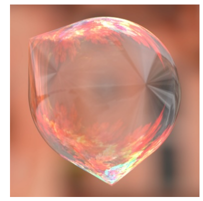
To illustrate the significance of this question, let us discuss a classical example that appears in the work of Minkowski [25] and Bol [3].

Example 2.3. Let K be any convex body in $\mathbb{R}^3$, and denote by B the Euclidean unit ball. Then a special instance of the Alexandrov-Fenchel inequality reads

$$\underbrace{\mathsf{V}(B, K, K)}_{\text{surface area}}{}^2 \geq \underbrace{\mathsf{V}(B, B, K)}_{\text{mean width}}\, \underbrace{\mathsf{V}(K, K, K)}_{\text{volume}}$$

provides a lower bound on the surface area among all convex bodies K in $\mathbb{R}^3$ of a given volume and mean width. In particular, equality is attained precisely when the surface area K is minimized among all convex bodies of a given volume and mean width. The characterization of such bodies may therefore be viewed as a "very remarkable" [25, p. 447] generalization of the classical isoperimetric theorem.

In the present very special case, Question 2.2 was settled by Minkowski [25] and Bol [3]: the extremal bodies K are *cap bodies* obtained by taking the convex hull of a Euclidean ball with a finite or countable number of points outside it, so that the resulting cones do not overlap. In stark contrast to the situation in classical isoperimetric problems, we note that the extremal bodies are non-unique, non-smooth, non-symmetric, and can have highly irregular surface structure.


A cap body

The unusual features of Example 2.3 already hint at the fact that the extremal bodies associated to more general cases of the Alexandrov-Fenchel inequality are likely to possess a rich structure. Unfortunately, the methods used by Minkowski and Bol exploit special features of this specific example, and shed little light on the general situation. While it was frequently mentioned in the literature, further progress on this problem remained elusive for decades, and there was some belief by experts that no meaningful geometric characterization is possible. The situation was summed up, e.g., in the classic monograph of Burago and Zalgaller [6, §20.5]: *"a conclusive study of all these situations when the equality sign holds has not been carried out, probably because they are too numerous."*

This would have likely remained the case if it were not for the crucial insight of R. Schneider, who formulated a detailed conjectural picture for the equality cases of the Alexandrov-Fenchel inequality in 1985 [30]. Schneider's conjectures made it possible, for the first time, to imagine what a geometric solution might look like. This spurred renewed interest in the problem, leading to the characterization of equality in some special cases (most notably for zonoids); we refer to [31, §7.6] for a survey of results obtained prior to the new results to be discussed below. However, the most important cases of the problem remained open.

3 New progress on the equality cases

3.1 Where lies the difficulty?

It is often the case that a characterization of equality cases can be read off in a natural manner from the proof of an inequality: one simply concatenates the conditions for equality arising from each step of the proof. Before we explain recent progress on Question 2.2, let us explain why such an approach fails here.

The classical approach to the Alexandrov-Fenchel inequality uses a continuity argument due to Hilbert [17, Chapter XIX]. The basic scheme of the proof (presented here in an oversimplified manner) is as follows.

1. First, prove that whenever $C_1, \ldots, C_{n-2}$ are *smooth* bodies, there can be *no nontrivial equality cases* of the Alexandrov-Fenchel inequality.

2. Next, prove that the *inequality* holds in a special case that is amenable to explicit computation (e.g., when $C_1 = \cdots = C_{n-2} = B$ are Euclidean balls.)

3. Finally, continuously deform the special case $C_1 = \cdots = C_{n-2} = B$ to general smooth bodies $C_1, \ldots, C_{n-2}$. As equality can never hold, the direction of the inequality must be preserved by continuity.

Once the inequality has been proved for smooth bodies, it extends to arbitrary convex bodies by a routine approximation argument.

It may seem surprising that the Alexandrov-Fenchel inequality possesses a rich and intricate family of equality cases, when the *absence* of nontrivial equality cases plays a fundamental role in its proof. However, the proof works with smooth bodies which indeed cannot achieve equality—e.g., the cap bodies of Example 2.3 are evidently non-smooth. The nontrivial equality cases only emerge in the limiting case of non-smooth bodies, about which the proof provides no information: in the limit all the objects that appear in the proof become singular, and the structure of the equality cases is encoded in the structure of the singularities. The core difficulty of the problem lies in the development of tools that make it possible to make sense of and understand these singularities.

The singular behavior of the Alexandrov-Fenchel inequality for non-smooth bodies in fact creates two distinct sets of challenges.

- **Analytic singularities:** the surfaces of convex bodies can be highly irregular; consider, for example, a cap body as in Example 2.3 with a countable number of caps that form a fractal structure.[1] This requires one to work with highly irregular analytic objects, such as singular elliptic operators.
- **Combinatorial singularities:** perhaps the most important case for applications (cf. section 4) is when $C_1, \ldots, C_{n-2}$ are convex polytopes. In this case the problem is essentially discrete in nature. Nonetheless, the interactions between the combinatorial structures of the polytopes $C_1, \ldots, C_{n-2}$ give rise to a complicated set of singularities that encode the equality cases.

Major progress on both fronts was achieved in recent work of the authors, which provides a complete solution to Question 2.2 in two opposite cases: in the combinatorial setting where no analytic difficulties arise, and in the analytic setting where no combinatorial difficulties arise. These results are described below. However, a full solution to Question 2.2 for arbitrary convex bodies would require us to handle the combinatorial and analytic aspects simultaneously. Considerable challenges remain toward achieving this still unfulfilled aim.

Remark 3.1. Beside the classical proof of the Alexandrov-Fenchel inequality by Hilbert's continuity method, several other proofs are now known; see, e.g., [33] and the references therein. However, none of the known proofs can avoid the singularities described above, which appear to be fundamental to the problem.

3.2 The combinatorial setting

The equality cases of the Alexandrov-Fenchel inequality are fully characterized in [35] in the setting where $C_1, \ldots, C_{n-2}$ are arbitrary convex polytopes. This may be viewed as a complete solution to the combinatoral aspect of the problem. As anticipated, the equality cases are numerous and their detailed technical description is beyond the scope of this survey. Instead, we aim here to give an informal flavor of the result, and refer to [35, §2 and §13] for a complete statement.

[1] The authors were surprised to learn from Serge Cantat at the 2024 ICBS that such objects arise naturally in the study of certain questions of birational geometry [7, §2.2.7].

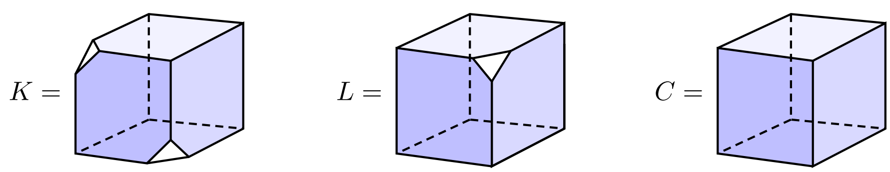
Theorem 3.2 (Informal statement)**.** *When $C_1, \dots, C_{n-2}$ are convex polytopes, the equality cases of Theorem 2.1 arise by a superposition of three distinct mechanisms:*
(1) translation and scaling;
(2) the relative positions of the normal cones of $\mathrm{bd}\, C_1, \dots, \mathrm{bd}\, C_{n-2}$*;*
(3) the relative positions of the affine hulls $\mathrm{aff}\, C_1, \dots, \mathrm{aff}\, C_{n-2}$.

The informal statement of Theorem 3.2 is difficult to interpret in the absence of the relevant definitions. To give a flavor of its meaning, we now illustrate the first two mechanisms, and highlight in particular the role played by the bodies K, L in Theorem 2.1 which is implicit in the informal statement.

Mechanism (1) simply arises from the fact that the Alexandrov-Fenchel inequality is invariant under translation and scaling of any of the bodies involved. We therefore trivially achieve equality when K, L differ by translation and scaling (that is, when $L = aK + v$ for some $a > 0$, $v \in \mathbb{R}^n$).

To give a flavor of mechanism (2), we provide a simple example.

Example 3.3. Let K, L, C be convex polytopes in $\mathbb{R}^3$ defined as follows:



Then equality $\mathsf{V}(K, L, C)^2 = \mathsf{V}(K, K, C)\,\mathsf{V}(L, L, C)$ holds due to mechanism (2). There are three kinds of normal cones of $\mathrm{bd}\, C$: the normal cones of a facet, of an edge, and of a corner of C. Mechanism (2) yields equality when K, L have the same supporting hyperplanes with any normal direction that is *not* normal to a corner of C. Thus we achieve equality when K, L are copies of C with their corners sliced off in an essentially arbitrary manner: slicing a corner only affects the supporting hyperplanes in normal directions to a corner of C. However these are not the only equality cases: for example, we may add additional mass to the sliced-off corners without destroying equality.

The three-dimensional case is especially simple. In higher-dimensional situations, the statement of mechanism (2) is analogous but more complicated: equality arises when K, L have the same supporting hyperplanes with any normal direction that is $(B, C_1, \dots, C_{n-2})$-extreme, where the latter notion is determined by the relative positions of the normal cones of the different bodies $C_1, \dots, C_{n-2}$ [30].

Mechanisms (1) and (2) were conjectured by Schneider [30]. However, Schneider's conjectures turn out to be incomplete: mechanism (3) is responsible for new equality cases that were not previously conjectured. The latter rather complicated equality cases occur only in certain situations in which some of $C_1, \dots, C_{n-2}$ have empty interior. We omit further discussion of these cases here.

3.3 The analytic setting

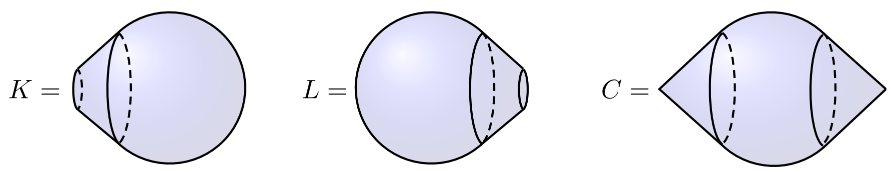
The combinatorial singularities that form the major difficulty in the proof of Theorem 3.2 arise from the interactions between the normal fans of the different bodies $C_1, \dots, C_{n-2}$. From this perspective, the 3-dimensional Example 3.3 that we have chosen for ease of visualization is rather simple, as there is only one body $C_1 = \cdots = C_{n-2} = C$ and thus no combinatorial singularities arise.[2]

However, if we aim to understand the equality cases of the Alexandrov-Fenchel inequality for *arbitrary* convex bodies, we must also resolve the analytic singularities which arise even when there are no combinatorial singularities. In the latter setting, the problem was completely resolved in [34].

Theorem 3.4 (Informal statement)**.** *The characterization of Theorem 3.2 extends to the setting where $C_1 = \cdots = C_{n-2} = C$ is an arbitrary convex body.*

Note, in particular, that Theorem 3.4 fully resolves Question 2.2 in dimension $n = 3$, which was the setting originally considered by Minkowski in [25].

To give a flavor of the statement of Theorem 3.4, we again discuss a simple example. Let K, L, C be convex bodies in $\mathbb{R}^3$ defined as follows:



Even though these bodies are not polytopes, equality arises here for precisely the same reason as in Example 3.3: the bodies K and L have the same supporting hyperplanes with any normal direction that is not normal to a corner of C. It should be clear by now that the same mechanism also explains the cap body theorem of Minkowski and Bol that was described in Example 2.3.

While the works [34, 35] separately resolve analytic and combinatorial singularities, it remains a major problem to handle both phenomena simultaneously. The proof of Theorem 3.4 requires certain quantitative estimates (relying in part on a method due to Kolesnikov and Milman [21]) that are highly specific to the special case $C_1 = \cdots = C_{n-2}$. In contrast, the method of proof of Theorem 3.2 is in principle very general and not specific to polytopes, but the analytic difficulties in extending it to general convex bodies remain to be resolved.

4 Combinatorial applications

A sequence of nonnegative numbers $N_1, \dots, N_n \geq 0$ is said to be *log-concave* if

$$N_i^2 \geq N_{i+1} N_{i-1} \quad \text{for all } i = 2, \dots, n-1.$$

It has long been observed that many integer sequences that arise in combinatorics appear to be log-concave, but a general understanding of this phenomenon was achieved only very recently in a remarkable series of works due to Huh et al., see, e.g., [19, 4]. However, one of the earliest results in this direction, which we presently describe, was obtained by Stanley in 1981 [32, Theorem 3.1].

[2]This special case of Theorem 3.2 was proved by Schneider [31, Theorem 7.6.21] prior to [35].

Example 4.1. Let $P = \{x, y_1, \ldots, y_n\}$ be any partially ordered set, and let N_i be the number of linear extensions of P (i.e., different ways to complete the partial order to a total order) so that $x \in P$ has rank i. Stanley's result, which confirms a conjecture of Rivest and Chung-Fishburn-Graham, is the following.

Theorem 4.2 (Stanley)**.** *The sequence $N_1, \ldots, N_{n+1}$ is log-concave.*

The key insight behind the proof of this theorem is that one can represent

$$N_i = n!\, \mathsf{V}(\underbrace{K, \ldots, K}_{i-1}, \underbrace{L, \ldots, L}_{n+1-i})$$

for suitably constructed convex polytopes K, L (which depend on P). Theorem 4.2 then follows immediately from the Alexandrov-Fenchel inequality.

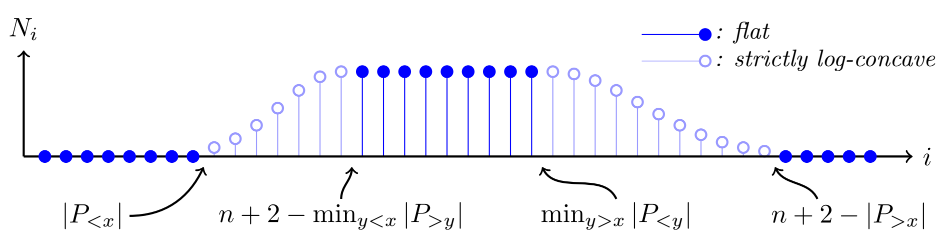
Given a log-concave sequence, the *equality* condition

$$N_i^2 = N_{i+1} N_{i-1} \qquad \Longleftrightarrow \qquad \frac{N_{i+1}}{N_i} = \frac{N_i}{N_{i-1}}$$

corresponds to a *geometric progression* in the sequence. Thus a characterization of the equality cases provides information on where and what kind of geometric progressions can appear in a log-concave sequence.

The equality characterization of the Alexandrov-Fenchel inequality opens the door to answering such questions. In the setting of Theorem 4.2, this has been accomplished in [35, §15] as an application of Theorem 3.2. Let us give an informal statement; here we write $P_{<y} := \{z \in P : z < y\}$ and $P_{>y} := \{z \in P : z > y\}$.

Theorem 4.3 (Informal statement)**.** *In Theorem 4.2, only* flat *geometric progressions $N_{i+1} = N_i = N_{i-1}$ arise at the following locations:*



The phenomenon exhibited by Theorem 4.3 may be viewed as a combinatorial cousin of Minkowski's bizarre "bubbles" of Example 2.3. It is particularly striking that we can achieve not only a qualitative understanding of the kind of geometric progressions that can appear in these log-concave sequences, but we can even compute explicitly where in the sequence the geometric progressions appear. This stands in contrast to the numbers N_i themselves, which are known to be hard to compute in a complexity-theoretic sense [5].

The fact that only flat progressions arise in the setting of Theorem 4.3 is essentially a coincidence: other kinds of progressions can arise in other inequalities.

Theorem 3.2 has been further used in [24, 38] to characterize geometric progressions that arise in other combinatorial inequalities for partially ordered sets that arise from the Alexandrov-Fenchel inequality. One surprising outcome of these results is that all three equality mechanisms in Theorem 3.2 turn out to arise naturally in combinatorial applications: these are not merely esoteric boundary cases the arise only in "weird" geometric examples!

Remark 4.4. Given that Theorem 4.2 is a direct consequence of the Alexandrov-Fenchel inequality, one may expect that Theorem 4.3 is a direct consequence of Theorem 3.2. The difficulty here is that the latter provides geometric information on the polytopes K, L that appear in Example 4.1, but it is not immediately clear how this geometric structure is reflected in the combinatorial structure of the poset P. The main challenge in the proof of Theorem 4.3 and related results is to translate between the geometric and combinatorial aspects of the problem.

The study of log-concavity for the combinatorial problems discussed above is made possible by the fact that the relevant combinatorial quantities can be represented as mixed volumes. However, for other problems where log-concavity has been conjectured, such a representation is not available.

Many classical conjectures of this kind were recently resolved in an impressive series of works reviewed in [19, 4]. One interpretation of the basic insight behind these breakthroughs is that while most combinatorial problems cannot be reformulated in terms of mixed volumes, one can often still prove Alexandrov-Fenchel type inequalities directly in combinatorial applications, so that the log-concavity property arises essentially by the same mechanism (this viewpoint is particularly evident in an approach proposed by Chan and Pak [8]). Whether such a common mechanism can also explain the appearance of geometric progressions in a broad range of combinatorial problems remains to be fully understood.

5 Further questions and developments

5.1 Universality

While the Alexandrov-Fenchel inequality belongs to convex geometry, a surprising series of discoveries (starting from work of Khovanskii and Teissier in the 1970s) have led to the realization that close analogues of the Alexandrov-Fenchel inequality arise in several other areas of mathematics:

- in algebraic geometry [36], [22, §1.6];
- in complex geometry [16, 23];
- in combinatorics [19, 4, 8].

Not only these inequalities themselves, but also the proofs of these inequalities, have many common features, and seem to arise from the same basic mechanism. It therefore appears that there is a certain universal aspect to Alexandrov-Fenchel type inequalities: they are not specific to one mathematical setting, but rather arise in many different settings that share a common algebraic structure.

At first sight, the characterization of equality cases of the Alexandrov-Fenchel inequality in Theorem 3.2 appears to be overtly convex geometric in nature. This suggests that, in contrast to the inequality itself, these equality conditions may be specific to the convex geometric setting. This is not necessarily the case, however: it is possible to reformulate the three mechanisms of Theorem 3.2 in a purely algebraic fashion, as was sketched in [35, §16.2]. Moreover, many of the core arguments in the proof of Theorem 3.2 are primarily algebraic in nature. These observations lead us to conjecture that the mechanisms for equality are similarly universal, so that analogues of Theorem 3.2 should arise in other areas of mathematics where Alexandrov-Fenchel type inequalities appear.

Significant progress in this direction was recently achieved by Hu and Xiao [18], who proved a partial extension of Theorem 3.2 in algebraic and complex geometry under certain positivity assumptions. Their work provides strong evidence for the universal nature of the equality cases of Alexandrov-Fenchel inequalities. The paper [18] also clarifies the full conjectural picture of the equality cases, which was sketched rather informally in [35], in the context of algebraic geometry.

It should be noted that the first instance of an Alexandrov-Fenchel inequality outside convexity long predates the above developments: Alexandrov already proved a linear algebraic analogue in 1938 [2], whose equality cases were fully settled in a largely overlooked paper by A. A. Panov [29]. The linear algebraic setting is in many ways fundamentally simpler than the convex geometric one; for example, mechanism (2) of Theorem 3.2, which is the main source of nontrivial equality cases in the convex geometric setting, is completely absent in the linear algebraic setting. Nonetheless, Panov's results are entirely consistent with the general conjectural picture (as discussed in [35, 18]) and provides another data point in support of the universal behavior of the equality cases.

5.2 Complexity

Stanley's inequality $N_i^2 \geq N_{i+1}N_{i-1}$ of Theorem 4.2 is a combinatorial inequality, i.e., an inequality between quantities that count the number of certain combinatorial structures (in this case, linear extensions of a partially ordered set). However, its proof is highly non-combinatorial in nature. It is natural to wonder whether it is possible to give a direct combinatorial proof of such inequalities. For example, we may ask whether one can construct an explicit injection $\iota : \mathcal{E}_{i+1} \times \mathcal{E}_{i-1} \to \mathcal{E}_i \times \mathcal{E}_i$ (here $\mathcal{E}_i$ denotes the set of linear extensions of P so that x has rank i), whose existence would immediately imply the desired inequality.

This tantalizing question turns out to be extremely subtle. On the one hand, consider the following apparently innocuous variation on the same problem: rather than count linear extensions, i.e., order-preserving *injective* maps from P to $\{0, \ldots, n\}$, we drop the injectivity assumption and count *all* order-preserving maps. In this case, the analogue of Theorem 4.2 admits a direct combinatorial proof by an explicit injection [13]. On the other hand, no such proof has ever been found for Theorem 4.2, suggesting a combinatorial proof of this inequality may be impossible. It is far from clear, however, how to even meaningfully formalize the latter as a mathematical statement, let alone how to prove it.

Questions of this kind are the subject of a far-reaching research program of Igor Pak [28], which aims to formalize and prove such results through computational complexity theory. A precise explanation of these ideas is beyond our scope, but let us outline one consequence in the present context. If we aim to disprove the existence of an explicit injection, we must formalize what "explicit" means. In combinatorial proofs, explicit injections are typically defined by a relatively simple rule or algorithm. It is therefore natural to formalize the problem as asking whether there exists an injection ι so that both ι and ι^{-1} can be computed in polynomial time. The question whether such an injection can exist in the setting of Theorem 4.2 remains open. However, in a significant achievement, Chan and Pak proved in [10] that a generalization[3] of Theorem 4.2 that appears in the same paper of Stanley [32, Theorem 3.1] cannot be proved in this manner.

Theorem 5.1 (Chan-Pak; informal statement)**.** *Assuming a standard complexity-theoretic hypothesis (a strengthening of* $\mathsf{P} \neq \mathsf{NP}$*), the generalized Stanley inequality cannot be proved by an explicit injection in the above sense.*

Theorem 5.1 is merely a corollary of a much deeper statement that is proved in [10]. This provides the first natural example of a combinatorial inequality that *provably* does not admit a combinatorial proof.

The proof of Theorem 5.1 is intimately connected to the topic of this survey. If an explicit injection were to exist, there would be a certificate for the inequality to be strict (namely an element that is not in the image of ι) that can be verified in polynomial time by the assumption on ι. It therefore suffices to establish hardness of verifying the equality condition. In particular, as the generalized Stanley inequality is proved using the Alexandrov-Fenchel inequality as in Theorem 4.2, the proof of Theorem 5.1 shows that it can be computationally hard to verify the equality condition of the Alexandrov-Fenchel inequality.

It may seem counterintuitive that there can be an explicit geometric description of the equality cases (Theorem 3.2) that is not computationally effective. There is however no contradiction between these statements.

Example 5.2. Instead of the Alexandrov-Fenchel inequality, consider the following caricature of a geometric inequality for convex bodies K, L:

$$\mathrm{Vol}(K)^2 + \mathrm{Vol}(L)^2 \geq 2\,\mathrm{Vol}(K)\,\mathrm{Vol}(L).$$

This trivial inequality is equivalent to $(\mathrm{Vol}(K) - \mathrm{Vol}(L))^2 \geq 0$. In particular, equality holds if and only if $\mathrm{Vol}(K) = \mathrm{Vol}(L)$. The latter provides a complete geometric understanding of the equality condition: it is clear that the coincidence of the volumes of two convex bodies is merely a matter of scaling, and does not constrain any other geometric features of these bodies.

On the other hand, volumes of convex polytopes are the subject of many hardness results in complexity theory. In particular, it follows from [11, Theorem 1.4(4)] (see also [9]) that is hard to verify the coincidence of the volumes of two convex polytopes. Thus it is hard to verify the equality condition of the above inequality, despite that it can be described explicitly in geometric terms.

[3] The generalized Stanley inequality differs from Theorem 4.2 in that N_i now counts only linear extensions for which in addition some other elements $y_{i_1}, \ldots, y_{i_k}$ have fixed ranks $j_1, \ldots, j_k$.

The Alexandrov-Fenchel inequality bears no resemblance to this trivial inequality. Nonetheless, hidden in the statement of mechanism (3) of Theorem 3.2 is a certain coincidence between mixed volumes that is analogous to the equality condition in the above example. It turns out that mechanism (3) cannot arise in the setting of Theorem 4.2, which explains the simple form of Theorem 4.3. But mechanism (3) can indeed arise in the generalized Stanley inequality [24], which hints at where one should look for hard instances of its equality conditions. The actual implementation of this program in the proof of Theorem 5.1 is independent of Theorem 3.2, and requires many new complexity-theoretic ideas.

5.3 Further open problems

5.3.1 Minkowski's monotonicity problem

One of the most basic properties of mixed volumes is that they are monotone: if $K, L, C_1, \ldots, C_{n-1}$ are convex bodies in $\mathbb{R}^n$, then

$$K \subseteq L \quad \Longrightarrow \quad \mathsf{V}(K, C_1, \ldots, C_{n-1}) \leq \mathsf{V}(L, C_1, \ldots, C_{n-1}).$$

Even though this inequality is far simpler than the Alexandrov-Fenchel inequality, the geometric characterization of its equality cases remains an open problem. Also this question dates back to Minkowski [26, Ch. XXV].

In contrast to the equality cases of the Alexandrov-Fenchel inequality, the solution to Minkowski's monotonicity problem is straightforward for convex polytopes. Motivated by the solution in the case of polytopes, R. Schneider formulated a precise conjecture for general convex bodies, see [30] and [31, Conjecture 7.6.14]. One direction of this conjecture was recently settled for arbitrary convex bodies [37], but only special cases of the converse direction have been verified to date.

5.3.2 Stability of the Alexandrov-Fenchel inequality

Suppose that the left- and right-hand sides of the Alexandrov-Fenchel inequality are nearly equal. Must $K, L, C_1, \ldots, C_{n-2}$ then be close to one of the equality cases described by Theorem 3.2? This is the question of *stability*. Concretely, a stability form of the Alexandrov-Fenchel inequality is an inequality of the form

$$\mathsf{V}(K, L, \mathcal{C})^2 - \mathsf{V}(K, K, \mathcal{C})\, \mathsf{V}(L, L, \mathcal{C}) \geq \Delta(K, L, \mathcal{C}),$$

where $\mathcal{C} = (C_1, \ldots, C_{n-2})$ and $\Delta(K, L, \mathcal{C})$ is a geometric quantity that "explicitly" quantifies the distance of $K, L, \mathcal{C}$ to the equality cases.

No useful inequality of this kind is known for general convex bodies, and it may be difficult to imagine what such an inequality could even look like given the large number of equality cases. Mechanism (3) of Theorem 3.2 is especially complex, and there is no plausible conjecture for its stability. However, the stability question is already of considerable interest if we restrict attention, e.g., to convex bodies with nonempty interior, in which case mechanism (3) cannot arise. In this setting, a natural conjecture on stability of the Alexandrov-Fenchel inequality is formulated in [34, Remark 7.2]. There is no progress to date toward proving such an inequality (or even a weak inequality as in [34, Theorem 6.1]).

The stability problem is of interest for two reasons. On the one hand, understanding the stability of geometric inequalities is a fundamental question [15, 14]. On the other hand, a sufficiently strong quantitative refinement of the Alexandrov-Fenchel inequality for “nice” bodies (e.g., smooth bodies or polytopes) could provide an approach toward fully resolving Question 2.2 for arbitrary convex bodies by approximation. This approach forms the basis for the proof of Theorem 3.4, but the plausibility of such an approach in the general setting remains unclear.

5.3.3 Local Alexandrov-Fenchel inequalities

We finally formulate an apparently fundamental question that has not previously appeared in print. The following discussion presupposes that the reader is familiar with some basic notions of convex geometry [31].

It is a basic fact that mixed volumes can be represented as

$$\mathsf{V}(K, C_1, \ldots, C_{n-1}) = \frac{1}{n} \int h_K \, dS_{C_1,\ldots,C_{n-1}},$$

where h_K is the support function of K and $S_{C_1,\ldots,C_{n-1}}$ is a measure on S^{n-1} called the mixed area measure. While there is no explicit formula for $S_{C_1,\ldots,C_{n-1}}$ in general, there are two special cases where explicit representations are available:

- When $C_1, \ldots, C_{n-1}$ are convex polytopes, $S_{C_1,\ldots,C_{n-1}}$ is atomic with

$$\frac{dS_{C_1,\ldots,C_{n-1}}}{d\mathcal{H}^0}(u) = \mathsf{V}(F(C_1, u), \ldots, F(C_{n-1}, u)),$$

  where $F(C, u)$ is the exposed face of C with normal direction u.

- When $h_{C_1}, \ldots, h_{C_{n-1}} \in C^\infty$, $S_{C_1,\ldots,C_{n-1}}$ is absolutely continuous with

$$\frac{dS_{C_1,\ldots,C_{n-1}}}{d\mathcal{H}^{n-1}}(u) = \mathsf{D}(D^2 h_{C_1}(u), \ldots, D^2 h_{C_{n-1}}(u)),$$

  where $\mathsf{D}(M_1, \ldots, M_{n-1})$ is the mixed discriminant of matrices $M_1, \ldots, M_{n-1}$.

In both cases, the *density* $\rho_{C_1,\ldots,C_{n-1}}$ of $S_{C_1,\ldots,C_{n-1}}$ with respect to a suitable reference measure on S^{n-1} (the Hausdorff measures $\mathcal{H}^0$ or $\mathcal{H}^{n-1}$, respectively) satisfies a pointwise analogue of the Alexandrov-Fenchel inequality:

$$(\rho_{K,L,C_1,\ldots,C_{n-3}})^2 \geq \rho_{K,K,C_1,\ldots,C_{n-3}} \; \rho_{L,L,C_1,\ldots,C_{n-3}}. \tag{5.1}$$

This follows from the Alexandrov-Fenchel inequality in the case of polytopes, and from the Alexandrov mixed discriminant inequality in the smooth case.

Given that such a property holds in two opposite extremes, one may ask whether this is a general property of mixed area measures. To formulate the question precisely, fix arbitrary convex bodies $K, L, C_1, \ldots, C_{n-3}$ in $\mathbb{R}^n$, and let μ be any measure on S^{n-1} with respect to which all relevant mixed area measures are absolutely continuous (e.g., $\mu = S_{M,\ldots,M}$ where $M = K+L+C_1+\cdots+C_{n-3}$). Now define $\rho_{K,L,C_1,\ldots,C_{n-3}} := \frac{dS_{K,L,C_1,\ldots,C_{n-3}}}{d\mu}$. Then does (5.1) hold μ-a.e.?

This apparently simple question presents the most basic obstruction toward the extension of the method of proof of Theorem 3.2 to general convex bodies. It has so far defied all attempts at its resolution.

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