Q. 1. Let W_t be a Wiener process.

1. Prove that $\tilde{W}_t = cW_{t/c^2}$ is also a Wiener process for any $c > 0$. Hence the sample paths of the Wiener process are *self-similar* (or *fractal*).
2. Define the stopping time $\tau = \inf\{t > 0 : W_t = x\}$ for some $x > 0$. Calculate the moment generating function $\mathbb{E}(e^{-\lambda\tau})$, $\lambda > 0$ by proceding as follows:

   (a) Prove that $X_t = e^{(2\lambda)^{1/2}W_t - \lambda t}$ is a martingale. Show that $X_t \to 0$ a.s. as $t \to \infty$ (first argue that X_t converges a.s.; it then suffices to show that $X_n \to 0$ a.s. ($n \in \mathbb{N}$), for which you may invoke Q.1 in homework 1.)

   (b) It follows that $Y_t = X_{t\wedge\tau}$ is also a martingale. Argue that Y_t is bounded, i.e., $Y_t < K$ for some $K > 0$ and all t, and that $Y_t \to X_\tau$ a.s. as $t \to \infty$.

   (c) Show that it follows that $\mathbb{E}(X_\tau) = 1$ (this is almost the optional stopping theorem, except that we have not required that $\tau < \infty$!) The rest is easy.

   What is the mean and variance of τ? (You don't have to give a rigorous argument.) In particular, does W_t always hit the level x in finite time?

Q. 2 (Lyapunov functions). In deterministic nonlinear systems and control theory, the notions of (Lyapunov) *stability*, *asymptotic stability*, and *global stability* play an important role. To prove that a system is stable, one generally looks for a suitable *Lyapunov function*, as you might have learned in a nonlinear systems class. Our goal is to find suitable stochastic counterparts of these ideas, albeit in discrete time.

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a sequence $\xi_1, \xi_2, \ldots$ of i.i.d. random variables. We consider the dynamical system defined by the recursion

$$x_n = F(x_{n-1}, \xi_n) \quad (n = 1, 2, \ldots), \qquad x_0 \text{ is non-random},$$

where $F : S \times \mathbb{R} \to S$ is some *continuous* function and S is some compact subset of $\mathbb{R}^d$ (compactness is not essential, but we go with it for simplicity). Let us assume that $F(x^*, \xi) = 0$ for some $x^* \in S$ and all $\xi \in \mathbb{R}$.

The following notions of stability are natural counterparts of the deterministic notions (compare with your favorite nonlinear systems textbook). The equilibrium x^* is

- **stable** if for any $\varepsilon > 0$ and $\alpha \in]0, 1[$, there exists a $\delta < \varepsilon$ such that we have $\mathbb{P}(\sup_{n\geq 0} \|x_n - x^*\| < \varepsilon) > \alpha$ whenever $\|x_0 - x^*\| < \delta$ *("if we start sufficiently close to x^*, then with high probability we will remain close to x^* forever");*
- **asymptotically stable** if it is stable and for every $\alpha \in]0, 1[$, there exists a κ such that $\mathbb{P}(x_n \to x^*) > \alpha$ whenever $\|x_0 - x^*\| < \kappa$ *("if we start sufficiently close to x^*, then we will converge to x^* with high probability");*
- **globally stable** if it is stable and $x_n \to x^*$ a.s. for any x_0.

1. Prove the following theorem:

   Theorem 1. *Suppose that there is a continuous function $V : S \to [0, \infty[$, with $V(x^*) = 0$ and $V(x) > 0$ for $x \neq x^*$, such that*

   $$\mathbb{E}(V(F(x, \xi_n))) - V(x) = k(x) \leq 0 \quad \text{for all} \quad x \in S.$$

   Then x^* is stable. (Note: as ξ_n are i.i.d., the condition does not depend on n.)

   Hint. Show that the process $V(x_n)$ is a supermartingale.

2. Prove the following theorem:

   Theorem 2. *Suppose that there is a continuous function $V : S \to [0, \infty[$ with $V(x^*) = 0$ and $V(x) > 0$ for $x \neq x^*$, such that*

   $$\mathbb{E}(V(F(x, \xi_n))) - V(x) = k(x) < 0 \quad \text{whenever} \quad x \neq x^*.$$

   Then x^* is globally stable.

   Hint. The proof proceeds roughly as follows. Fill in the steps:

   (a) Write $V(x_0) - \mathbb{E}(V(x_n))$ as a telescoping sum. Use this and the condition in the theorem to prove that $k(x_n) \to 0$ in probability "fast enough".

   (b) Prove that if some sequence $s_n \in S$ converges to a point $s \in S$, then $k(s_n) \to k(s)$, i.e., that $k(x)$ is a continuous function.

   (c) As $k(x_n) \to 0$ a.s., k is continuous, and $k(s_n) \to 0$ only if $s_n \to x^*$ (why?), you can now conclude that $x_n \to x^*$ a.s.

3. **(Inverted pendulum in the rain)** A simple discrete time model for a controlled, randomly forced overdamped pendulum is

   $$\theta_{n+1} = \theta_n + (1 + \xi_n)\sin(\theta_n)\Delta + u_{n+1}\Delta \qquad \mod 2\pi,$$

   where θ_n is the angle ($\theta = 0$ is up) of the pendulum at time $n\Delta$, Δ is the time step size (be sure to take it small enough), u_{n+1} an applied control (using a servo motor), and ξ_n are i.i.d. random variables uniformly distributed on $[0, 1]$. The $\sin\theta_n$ term represents the downward gravitational force, while the term $\xi_n \sin\theta_n$ represents randomly applied additional forces in the downward direction—i.e., the force exerted on the pendulum by rain drops falling from above. *(Admittedly, the model is completely contrived! Don't take it too seriously.)*

   Let us represent the circle $\theta \in S^1$ as the unit circle in $\mathbb{R}^2$. Writing $x_n = \sin\theta_n$, $y_n = \cos\theta_n$, and $f(x, \xi, u) = (1 + \xi)x\Delta + u\Delta$, we get

   $$\begin{aligned} x_{n+1} &= x_n \cos(f(x_n, \xi_n, u_{n+1})) + y_n \sin(f(x_n, \xi_n, u_{n+1})), \\ y_{n+1} &= y_n \cos(f(x_n, \xi_n, u_{n+1})) - x_n \sin(f(x_n, \xi_n, u_{n+1})). \end{aligned}$$

   Find some control law $u_{n+1} = g(x_n, y_n)$ that makes the inverted position $\theta = 0$ stable. (Try an intuitive control law and a linear Lyapunov function; you might want to use your favorite computer program to plot $k(\cdot)$.)

4. **Bonus question:** The previous results can be localized to a neighborhood. Prove the following modifications of the previous theorems:

   Theorem 3. *Suppose that there is a continuous function* $V : S \to [0, \infty[$ *with* $V(x^*) = 0$ *and* $V(x) > 0$ *for* $x \neq x^*$*, and a neighborhood* U *of* x^**, such that*

   $$\mathbb{E}(V(F(x, \xi_n))) - V(x) = k(x) \leq 0 \quad \textit{whenever} \quad x \in U.$$

   Then x^* *is stable.*

   Theorem 4. *Suppose that there is a continuous function* $V : S \to [0, \infty[$ *with* $V(x^*) = 0$ *and* $V(x) > 0$ *for* $x \neq x^*$*, and a neighborhood* U *of* x^**, such that*

   $$\mathbb{E}(V(F(x, \xi_n))) - V(x) = k(x) < 0 \quad \textit{whenever} \quad x \in U \backslash \{x^*\}.$$

   Then x^* *is asymptotically stable.*

   Hint. Define a suitable stopping time τ, and apply the previous results to $x_{n \wedge \tau}$.

   You can now show that the controlled pendulum is asymptotically stable.