Q. 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a sequence of i.i.d. Gaussian random variables $\xi_1, \xi_2, \ldots$ with zero mean and unit variance. Consider the following recursion:

$$x_n = e^{a+b\xi_n} x_{n-1}, \qquad x_0 = 1,$$

where a and b are real-valued constants. This is a crude model for some nonnegative quantity that grows or shrinks randomly in every time step; for example, we could model the price of a stock this way, albeit in discrete time.

1. Under which conditions on a and b do we have $x_n \to 0$ in $\mathcal{L}^p$?
2. Show that if $x_n \to 0$ in $\mathcal{L}^p$ for some $p > 0$, then $x_n \to 0$ a.s.

   Hint: prove $x_n \to 0$ in $\mathcal{L}^p \implies x_n \to 0$ in probability $\implies x_n \to 0$ a.s.
3. Show that if there is no $p > 0$ s.t. $x_n \to 0$ in $\mathcal{L}^p$, then $x_n \not\to 0$ in any sense.
4. If we interpret x_n as the price of stock, then x_n is the amount of dollars our stock is worth by time n if we invest one dollar in the stock at time 0. If $x_n \to 0$ a.s., this means we eventually lose our investment with unit probability. However, it is possible for a and b to be such that $x_n \to 0$ a.s., but nonetheless our *expected winnings* $\mathbb{E}(x_n) \to \infty$! Find such a, b. Would you consider investing in such a stock? *[Any answer is acceptable, as long as it is well motivated.]*

Q. 2. We work on the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$, where the probability measure $\mathbb{P}$ is such that the canonical random variable $X : \omega \mapsto \omega$ is a Gaussian random variable with zero mean and unit variance. In addition to $\mathbb{P}$, we consider a probability measure $\mathbb{Q}$ under which $X - a$ is a Gaussian random variable with zero mean and unit variance, where $a \in \mathbb{R}$ is some fixed (non-random) constant.

1. Is it true that $\mathbb{Q} \ll \mathbb{P}$, and if so, what is the Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$? Similarly, is it true that $\mathbb{P} \ll \mathbb{Q}$, and if so, what is $d\mathbb{P}/d\mathbb{Q}$?

We are running a nuclear reactor. That being a potentially dangerous business, we would like to detect the presence of a radiation leak, in which case we should shut down the reactor. Unfortunately, we only have a noisy detector: the detector generates some random value ξ when everything is ok, while in the presence of a radiation leak the noise has a constant offset $a + \xi$. Based on the value returned by the detector, we need to make a decision as to whether to shut down the reactor.

In our setting, the value returned by the detector is modelled by the random variable X. If everything is running ok, then the outcomes of X are distributed according to the measure $\mathbb{P}$. This is called the *null hypothesis* H_0. If there is a radiation leak, however, then X is distributed according to $\mathbb{Q}$. This is called the *alternative hypothesis* H_1. Based on the value X returned by the detector, we decide to shut down the reactor if $f(X) = 1$, with some $f : \mathbb{R} \to \{0, 1\}$. Our goal is to find a suitable function f.

How do we choose the decision function f? What we absolutely cannot tolerate is that a radiation leak occurs, but we do not decide to shut down the reactor—disaster would ensue! For this reason, we fix a tolerance threshold: under the measure corresponding to H_1, the probability that $f(X) = 0$ must be at most some fixed value α (say, 10^{-12}). That is, we insist that any acceptable f must be such that $\mathbb{Q}(f(X) = 0) \leq \alpha$. Given this constraint, we now try to find an acceptable f that minimizes $\mathbb{P}(f(X) = 1)$, the probability of *false alarm* (i.e., there is no radiation leak, but we think there is).

Claim: an f^* that minimizes $\mathbb{P}(f(X) = 1)$ subject to $\mathbb{Q}(f(X) = 0) \leq \alpha$ is given by

$$f^*(x) = \begin{cases} 1 & \text{if } \frac{d\mathbb{Q}}{d\mathbb{P}}(x) > \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta > 0$ is chosen such that $\mathbb{Q}(f^*(X) = 0) = \alpha$. This is called the *Neyman-Pearson test*, and is a very fundamental result in statistics (if you already know it, all the better!). You are going to prove this result.

2. Let $f : \mathbb{R} \to \{0, 1\}$ be an arbitrary measurable function s.t. $\mathbb{Q}(f(X) = 0) \leq \alpha$. Using $\mathbb{Q}(f(X) = 0) \leq \alpha$ and $\mathbb{Q}(f^*(X) = 0) = \alpha$, show that

$$\mathbb{Q}(f^*(X) = 1 \text{ and } f(X) = 0) \leq \mathbb{Q}(f^*(X) = 0 \text{ and } f(X) = 1).$$

3. Using the definition of f^*, show that the previous inequality implies

$$\mathbb{P}(f^*(X) = 1 \text{ and } f(X) = 0) \leq \mathbb{P}(f^*(X) = 0 \text{ and } f(X) = 1).$$

   Finally, complete the proof of optimality of the Neyman-Pearson test by adding a suitable quantity to both sides of this inequality.

A better detector would give a sequence $X_1, \ldots, X_N$ of measurements. Under the measure $\mathbb{P}$ (everything ok), the random variables $X_1, \ldots, X_N$ are independent Gaussian random variables with zero mean and unit variance; under the measure $\mathbb{Q}$ (radiation leak), the random variables $X_1 - a_1, \ldots, X_N - a_N$ are independent Gaussian random variables with zero mean and unit variance, where $a_1, \ldots, a_N$ is a fixed (non-random) alarm signal (for example, a siren $a_n = \sin(n\pi/2)$.)

4. Construct $X_1, \ldots, X_N$, $\mathbb{P}$ and $\mathbb{Q}$ on a suitable product space. What is $d\mathbb{Q}/d\mathbb{P}$? How does the Neyman-Pearson test work in this context?

5. **Bonus question:** Now suppose that we have an entire sequence $X_1, X_2, \ldots$, which are i.i.d. Gaussian random variables with mean zero and unit variance under $\mathbb{P}$, and such that $X_1 - a_1, X_2 - a_2, \ldots$ are i.i.d. Gaussian random variables with mean zero and unit variance under $\mathbb{Q}$. Give a necessary and sufficient condition on the non-random sequence $a_1, a_2, \ldots$ so that $\mathbb{Q} \ll \mathbb{P}$. In the case that $\mathbb{Q} \ll \mathbb{P}$, give the corresponding Radon-Nikodym derivative. If $\mathbb{Q} \not\ll \mathbb{P}$, find an event A so that $\mathbb{P}(A) = 0$ but $\mathbb{Q}(A) \neq 0$. In theory, how would you solve the hypothesis testing problem when $\mathbb{Q} \ll \mathbb{P}$? How about when $\mathbb{Q} \not\ll \mathbb{P}$?

Q. 3. In the lecture notes, some elementary arguments are left for you to work out [marked (why?), or something similar]. Complete the following arguments in ch. 1:

1. Page 20: why is $\bigcap_j \mathcal{F}_j$ a σ-algebra, when $\{\mathcal{F}_j\}$ is a (not necessarily countable) collection of σ-algebras? Show that $\mathcal{F}_1 \cup \mathcal{F}_2$ need not be a σ-algebra (give an example!)

2. Page 24: prove that $\mathbb{P}(\limsup A_k) \geq \limsup \mathbb{P}(A_k)$, where $\{A_k\}$ is a countable collection of measurable sets.

3. Page 27: prove that $X_n \nearrow X$ implies that $\mathbb{E}(X_n)$ has a (not necessarily finite) limit. (Note that you cannot use the monotone convergence theorem here, as that would lead to a circular argument!)

4. Page 28: why is X_n in lemma 1.3.12 measurable?

5. Page 29, in the proof of lemma 1.4.1: prove 2. and 3. Explain the why?s $(2\times)$.

6. Page 35: explain how we obtained the expression in the proof of lemma 1.5.8.

7. Page 36: why is $\mathbb{E}(Z_n) \leq \inf_{k \geq n} \mathbb{E}(X_k)$ in the proof of Fatou's lemma?

8. Page 37: why is μ a probability measure in definition 1.6.1?

9. Page 41: why is $\mathbb{Q}$ a probability measure, and why is $\mathbb{E}_{\mathbb{Q}}(g) = \mathbb{E}_{\mathbb{P}}(gf)$ for any g such that one of the sides makes sense? (Be sure to check also that if one side is well defined, then both sides are).

10. Page 42: if $\mathbb{Q} \ll \mathbb{P}$, why does $\mathbb{P}(A) = 1$ imply $\mathbb{Q}(A) = 1$?

In each case, explain (very briefly) what properties of the measure/expectation or what result you have used to come to the appropriate conclusion.