The following statements should have been part of section 1.5 in the lecture notes; unfortunately, I forgot to include them.

Definition 1. A sequence $\{X_n\}$ of random variables in $\mathcal{L}^p$, $p \ge 1$ is called a *Cauchy sequence (in $\mathcal{L}^p$)* if $\sup_{m,n\ge N} \|X_m - X_n\|_p \to 0$ as $N \to \infty$.

Proposition 2 (Completeness of $\mathcal{L}^p$). *Let X_n be a Cauchy sequence in $\mathcal{L}^p$. Then there exists a random variable $X_\infty \in \mathcal{L}^p$ such that $X_n \to X_\infty$ in $\mathcal{L}^p$.*

When is such a result useful? All our previous convergence theorems, such as the dominated convergence theorem etc., assume that we already know that our sequence converges to a particular random variable $X_n \to X$ in some sense; they tell us how to convert between the various modes of convergence. However, we are often just given a sequence X_n, and we still need to establish that X_n converges to *something*. Proving that X_n is a Cauchy sequence is one way to show that the sequence converges, without knowing in advance what it converges to. We will encounter another way to show that a sequence converges in the next chapter (the *martingale convergence theorem*).

Remark 3. As you know from your calculus course, $\mathbb{R}^n$ also has the property that any Cauchy sequence converges: if $\sup_{m,n\ge N} |x_m - x_n| \to 0$ as $N \to \infty$ for some sequence $\{x_n\} \subset \mathbb{R}^n$, then there is some $x_\infty \in \mathbb{R}^n$ such that $x_n \to x_\infty$. In fact, many (but not all) metric spaces have this property, so it is not shocking that it is true also for $\mathcal{L}^p$. A metric space in which every Cauchy sequence converges is called *complete*.

Proof of proposition 2. We need to do two things: first, we need to identify a candidate X_∞. Once we have constructed such an X_∞, we need to show that $X_n \to X_\infty$ in $\mathcal{L}^p$.

Let $M(N) \nearrow \infty$ be a subsequence such that $\sup_{m,n\ge M(N)} \|X_n - X_m\|_p \le 2^{-N}$ for all N. As $\|\cdot\|_1 \le \|\cdot\|_p$ (recall that we assume $p \ge 1$), this implies $\sup_{m,n\ge M(N)} \mathbb{E}(|X_n - X_m|) \le 2^{-N}$, and in particular $\mathbb{E}(|X_{M(N+1)} - X_{M(N)}|) \le 2^{-N}$. Hence

$$\mathbb{E}\left(\sum_{n=1}^{\infty} |X_{M(n+1)} - X_{M(n)}|\right) = \sum_{n=1}^{\infty} \mathbb{E}\left(|X_{M(n+1)} - X_{M(n)}|\right) < \infty,$$

where we have used the monotone convergence theorem to exchange the summation and the expectation. But then the series $X_{M(n)} = X_{M(1)} + \sum_{k=2}^{n} (X_{M(k)} - X_{M(k-1)})$ is a.s. absolutely convergent, so $X_{M(n)}$ converges a.s. to some random variable X_∞. Moreover,

$$\mathbb{E}(|X_{M(k)} - X_\infty|^p) = \mathbb{E}\left(\liminf_{n\to\infty} |X_{M(k)} - X_n|^p\right) \le \liminf_{n\to\infty} \mathbb{E}(|X_{M(k)} - X_n|^p) \le 2^{-kp}$$

using Fatou's lemma, so we conclude that $X_{M(k)} \to X_\infty$ in $\mathcal{L}^p$, and in particular $X_\infty \in \mathcal{L}^p$ itself (the latter follows as evidently $X_{M(k)} - X_\infty \in \mathcal{L}^p$, $X_{M(k)} \in \mathcal{L}^p$ by assumption, and $\mathcal{L}^p$ is linear, so $X_\infty = X_{M(k)} - (X_{M(k)} - X_\infty) \in \mathcal{L}^p$).

It remains to show that $X_n \to X_\infty$ in $\mathcal{L}^p$ (i.e., not necessarily for the subsequence $M(n)$). To this end, note that $\|X_n - X_\infty\|_p \le \|X_n - X_{M(n)}\|_p + \|X_{M(n)} - X_\infty\|_p$; that the second term converges to zero we have already seen, while that the first term converges to zero follows directly from the fact that X_n is a Cauchy sequence. Thus we are done. □