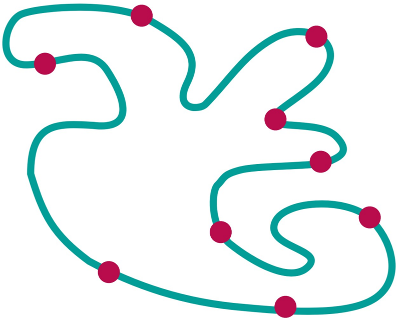
Colouring visibility graphs

Rose McCarty

Joint work with:
James Davies, Tomasz Krawczyk, and Bartosz Walczak

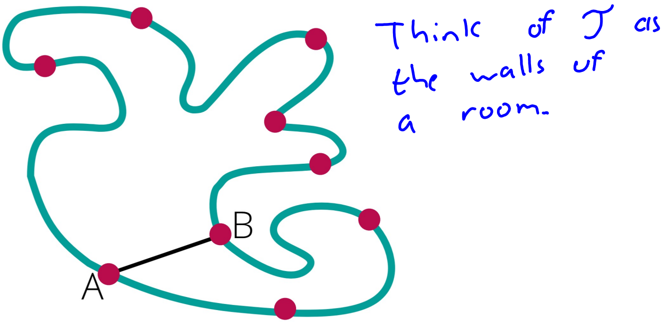
September 2020

Curve visibility graphs



- Consider a (finite) set of points S on a Jordan curve $\mathcal{J}$.
- Points $A, B \in S$ are **mutually visible** if $\overline{AB} \subseteq \text{int}(\mathcal{J})$.
- This defines a **curve visibility graph**.
- It is **ordered** if it comes with a linear ordering of S, ccw.

Curve visibility graphs



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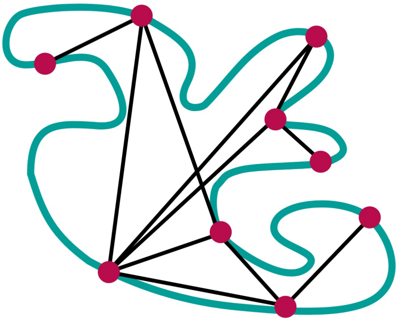

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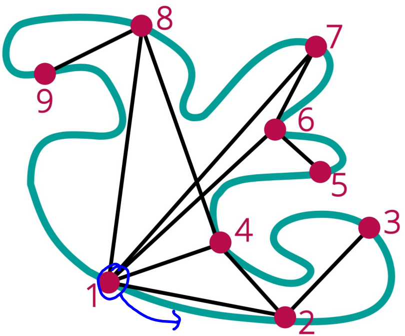

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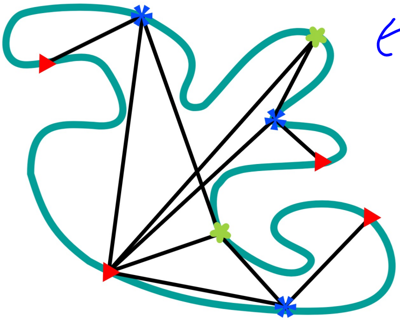
(Start anywhere then go in counterclockwise order)

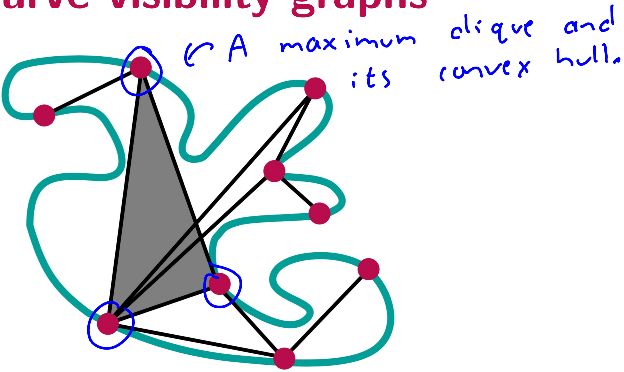
Curve visibility graphs



← 3-colouring

- A **colouring** assigns colours to vertices so that no two vertices of the same colour are mutually visible.
- A **clique** is a set W such that $\mathrm{conv}(W) \subseteq \mathrm{int}(\mathcal{J}) \cup W$.
- **Chromatic number** χ = min # of colours in a colouring
- **Clique number** ω = max # of vertices in a clique

Curve visibility graphs



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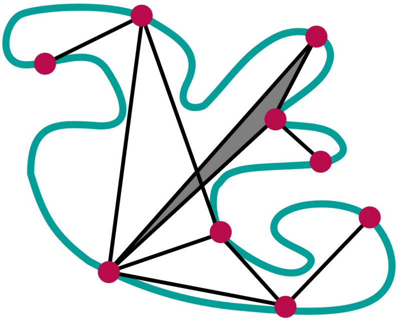

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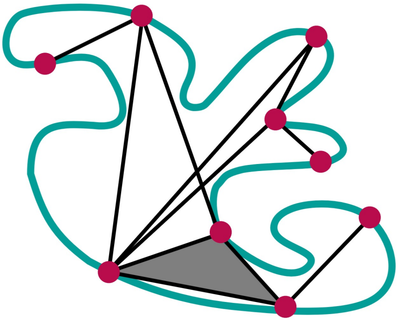

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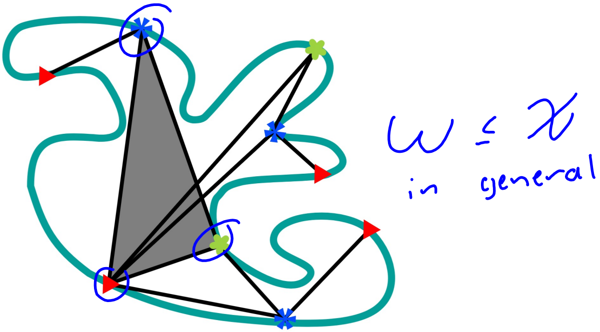

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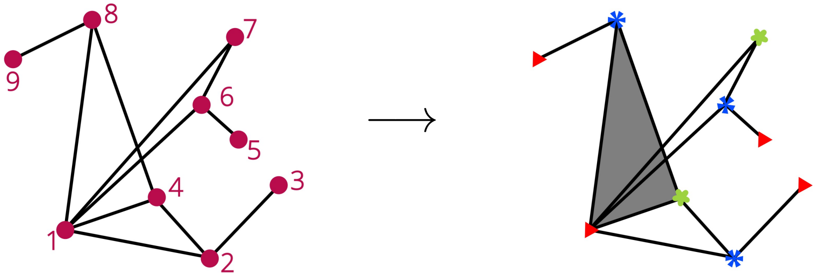
Theorem

There is a polynomial-time algorithm which returns the clique number ω and a $(3 \cdot 4^{\omega-1})$-colouring of an **ordered curve visibility graph**.



$$\omega \leq \chi \leq 3 \cdot 4^{\omega-1}$$

- NP-Complete to test if $\chi \leq 5$ (Çağırıcı, Hliněný, Roy, 19)

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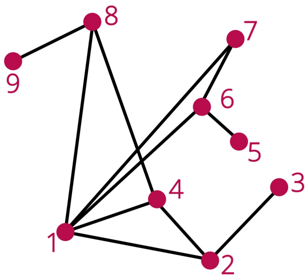
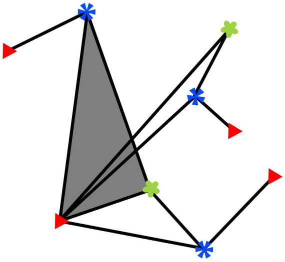


$\longrightarrow$



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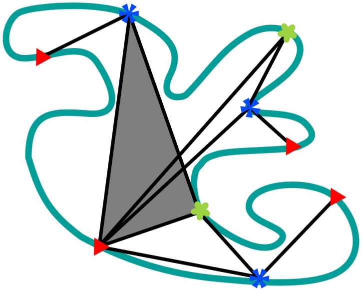
- NP-Complete to test if $\chi \leq 5$ (Çağırıcı, Hliněný, Roy, 19)
(for ordered polygon visibility graphs)

χ-bounded graph classes



- A class is χ-**bounded** if there exists f so that every graph in the class with clique number ω has $\chi \leq f(\omega)$.
- There exist graphs with χ arbitrarily large and $\omega = 2$.
- So the class of **curve visibility graphs** is χ-bounded.
- This was open even for **polygon visibility graphs**. (Kára, Pór, Wood, 05)

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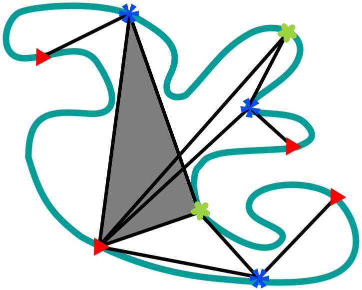

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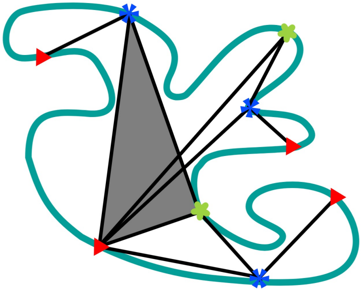

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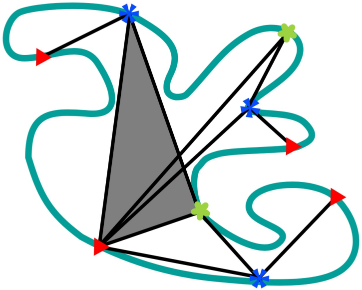

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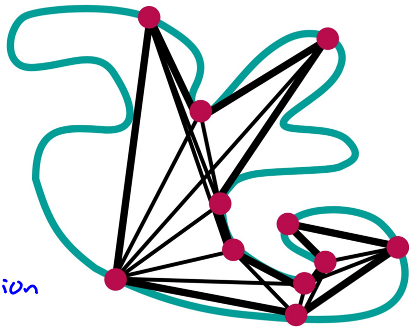
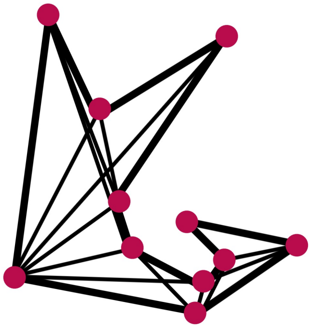
Polygon visibility graphs



general position

- A **GP polygon visibility graph** is a curve visibility graph where S is in GP and consecutive vertices are adjacent.
- There is an $\mathcal{O}(n^2 m)$ algorithm to compute ω. (Ghosh, Shermer, Bhattacharya, Goswami, 07)

Polygon visibility graphs



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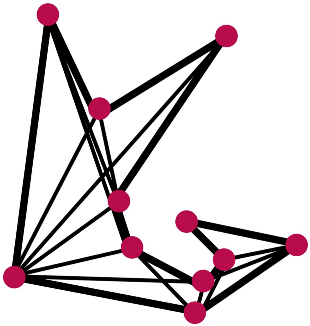

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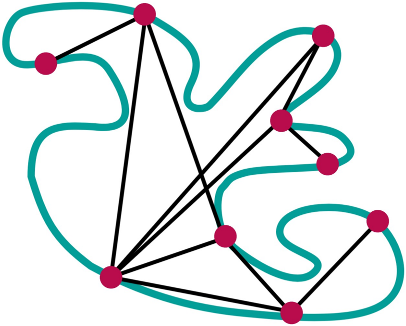
Hereditary graph classes



- A class is **hereditary** if it is closed under deleting vertices.
- The class of **curve visibility graphs** is hereditary.
- The class of **polygon visibility graphs** is NOT.

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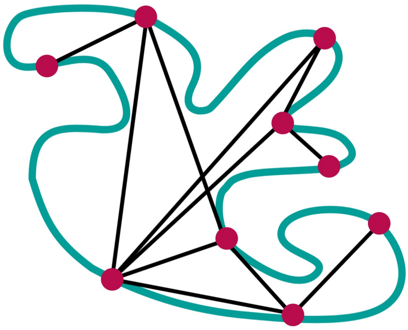

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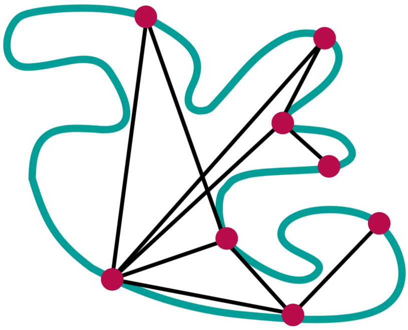

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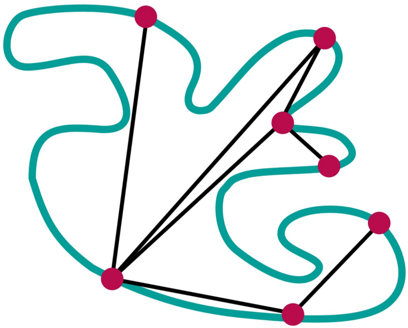

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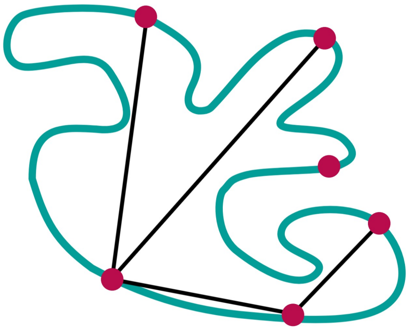

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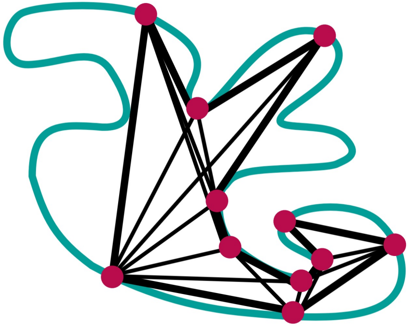

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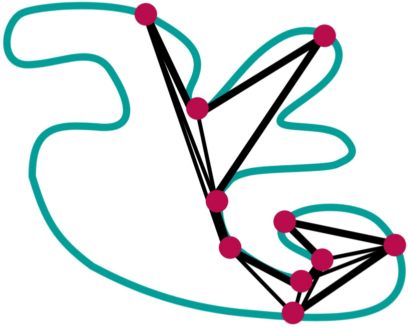

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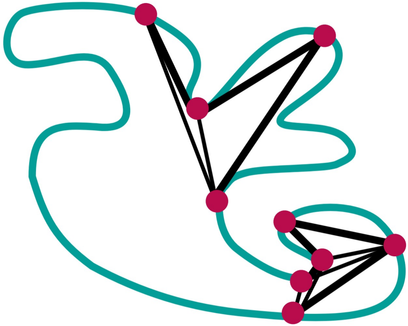

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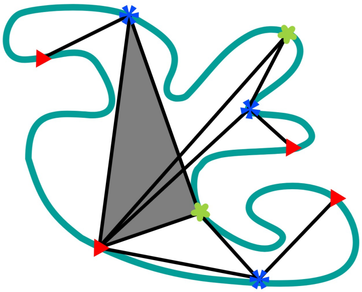

- The class of **curve visibility graphs** is **hereditary** (closed under deleting vertices) and χ**-bounded** ($\chi \leq f(\omega)$).
- The bounds $\omega \leq \chi \leq 3 \cdot 4^{\omega-1}$ can be obtained in polynomial time when the input graph is **ordered**.
- **GP polygon visibility graphs** are a nice, but not hereditary, subclass.

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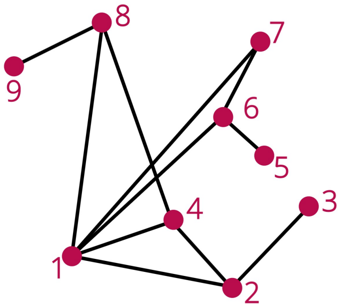

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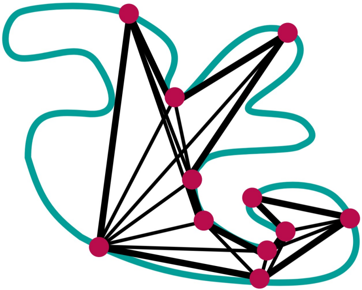

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Hereditary, χ-bounded classes

Conjecture (Esperet, 17)

Every hereditary, χ-bounded class is **polynomially χ-bounded**.

$$\chi \leq 3 \cdot 4^{\omega-1}$$

$$\chi \leq \omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}} \qquad \longrightarrow \qquad \chi \leq \omega^{d}$$

$$\dots$$

- That is, there exists a **polynomial** p so that every graph in the class with clique number ω has $\chi \leq p(\omega)$.
- We believe that **curve visibility graphs** are **polynomially χ-bounded**.

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Mostly defined by...

- forbidden substructures
- basic classes + operations/decompositions
- geometric representations
    - intersection/disjointness graphs
    - **visibility graphs**?

(Rok, Walczak, 19)

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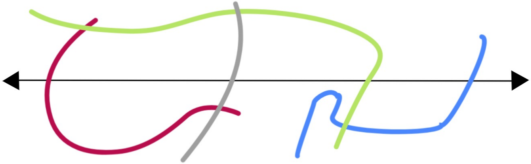

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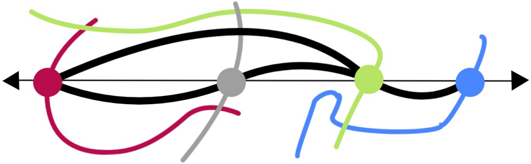

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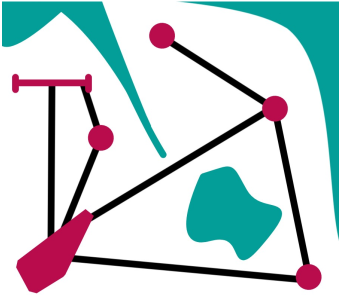
Visibility graphs

- Consider a set S of disjoint shapes in the plane and some (possible empty) obstacle $J \subset \mathbb{R}^2$.
- The **visibility graph** has vertex set S and an edge for each pair of **mutually visible** shapes in S.
- Without extra restrictions, every graph is a visibility graph.

Visibility graphs



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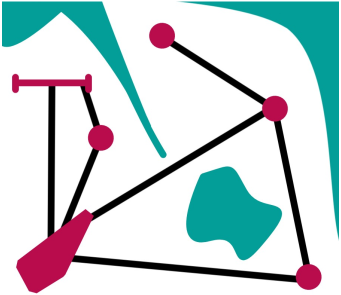
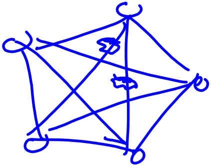
definitions vary

Visibility graphs





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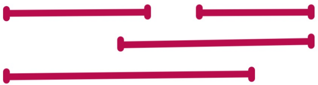
Visibility graphs

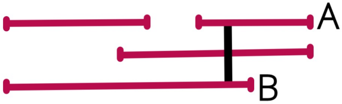
	hereditary?	χ-bounded?
Curve visibility	✓	✓
Bar k-visibility		
Point visibility		
Curve pseudovisibility		

Bar k-visibility graphs (k = ∞ is allowed)



- S is a set of horizontal closed segments
- A and B are **mutually visible** if they can be joined by a vertical segment which intersects $\leq k$ other intervals.

Bar k-visibility graphs



Think of k as the opacity; so if k = ∞ then the segments are clear.

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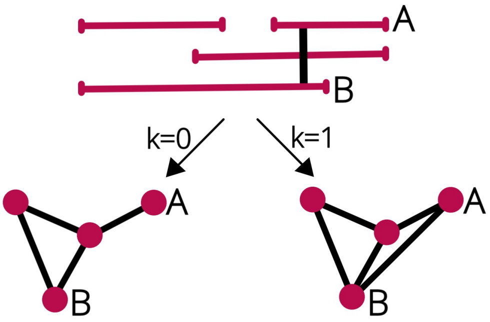

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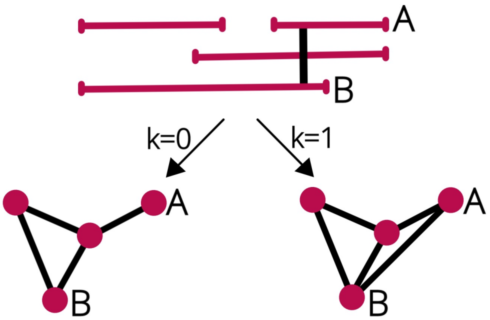

- $k = \infty$: interval graphs
- $k = 0$: can be characterized by a connection to planar triangulations (Luccio, Mazzone, Wong, 87)
- $k < \infty$: bounded average degree (Dean, Evans, Gethner, Laison, Safari, Trotter, 06)

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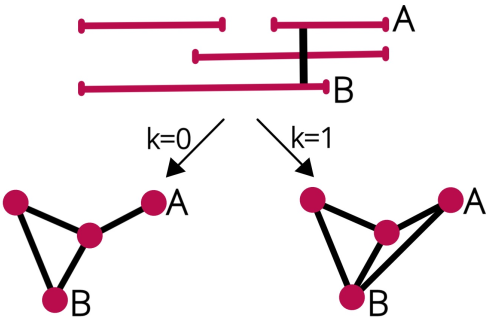

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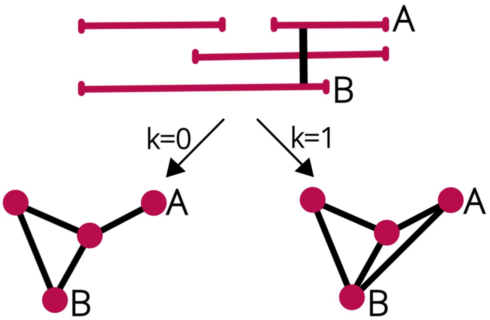

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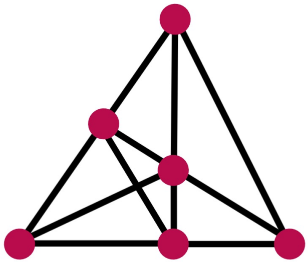
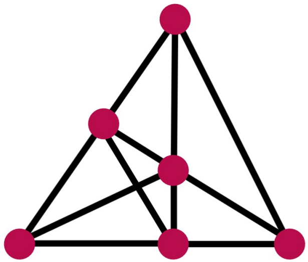
Bar k-visibility	mostly ✗	✓
Point visibility		
Curve pseudovisibility		

Point visibility graphs



- S is a set of points
- A and B are **mutually visible** if $\overline{AB}$ intersects no vertices.
- Point visibility graphs with $\omega \leq 3$ have $\chi \leq 3$ (Kára, Pór, Wood, 05).
- But they are not χ-bounded (Pfender, 08).

Point visibility graphs



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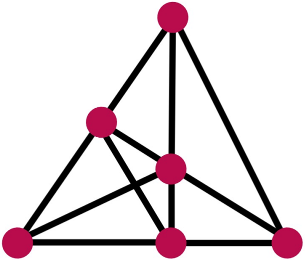

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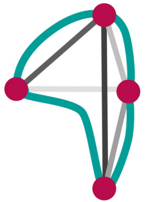
Visibility graphs

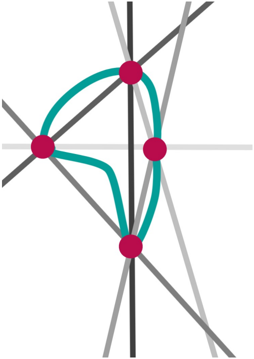
	hereditary?	χ-bounded?
Curve visibility	✓	✓
Bar k-visibility	mostly ✗	✓
Point visibility	✗	✗
Curve pseudovisibility		

Curve pseudovisibility graphs



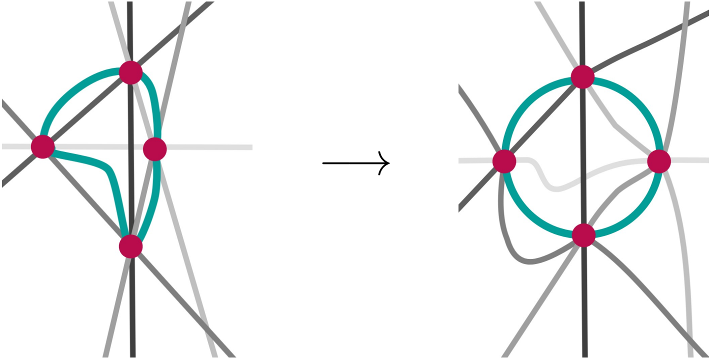
- Consider a **GP** curve visibility graph...

Curve pseudovisibility graphs



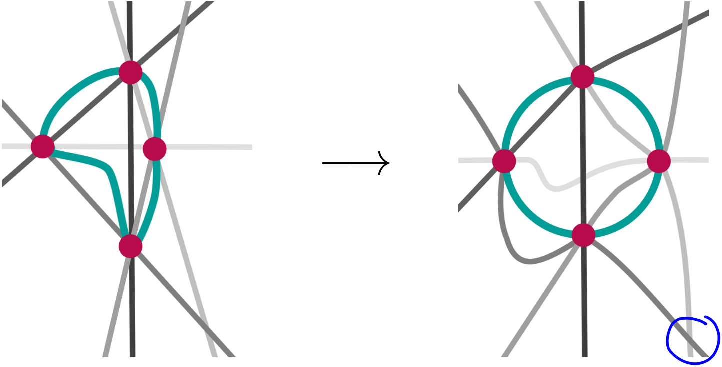
- Consider a **GP** curve visibility graph and line arrangement.

Curve pseudovisibility graphs



- Consider a **GP** curve visibility graph and line arrangement.
- There is a homeomorphism moving $\mathcal{J}$ to the unit circle C.

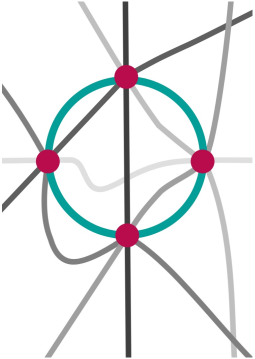
Curve pseudovisibility graphs



- Consider a **GP** curve visibility graph and line arrangement.
- There is a homeomorphism moving $\mathcal{J}$ to the unit circle C.
- The line arrangement yields a **pseudoline arrangement**.

(A set of closed curves which break the plane into two unbounded regions, s.t. every pair intersect exactly once, where they cross.)

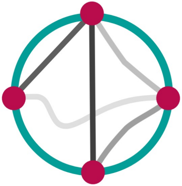
Curve pseudovisibility graphs



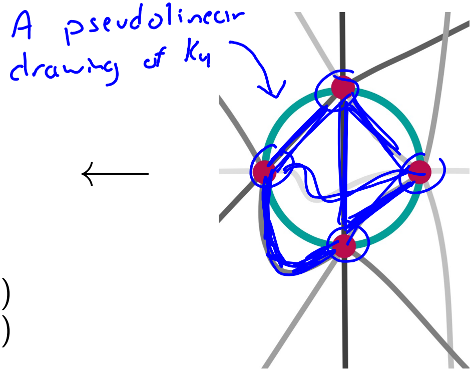
- Consider a **GP** curve visibility graph and line arrangement.
- There is a homeomorphism moving $\mathcal{J}$ to the unit circle C.
- The line arrangement yields a **pseudoline arrangement**.
- If we start with *any* **pseudolinear** drawing of K_n on C...

can be extended to a pseudoline arrangement of $\binom{n}{2}$ pseudolines

Curve pseudovisibility graphs



(O'Rourke, Streinu, 97)
(Abello, Kumar, 02)



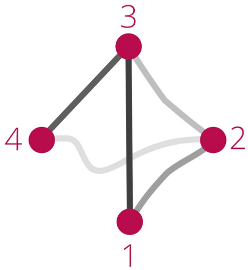
- Consider a **GP** curve visibility graph and line arrangement.
- There is a homeomorphism moving $\mathcal{J}$ to the unit circle C.
- The line arrangement yields a **pseudoline arrangement**.
- If we start with *any* **pseudolinear** drawing of K_n on C, then we obtain a **curve pseudovisibility graph**.

Visibility graphs

	hereditary?	χ-bounded?
Curve visibility	✓	✓
Bar *k*-visibility	mostly ✗	✓
Point visibility	✗	✗
Curve pseudovisibility	✓	?

Theorem

There is a polynomial-time algorithm which returns the clique number ω and a $(3 \cdot 4^{\omega-1})$-colouring of an **ordered curve pseudovisibility graph**.



$\longrightarrow$

the clique number ω

and

a $(3 \cdot 4^{\omega-1})$-colouring

Visibility graphs

	hereditary?	**χ-bounded?**
Curve visibility	✓	✓
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Point visibility	✗	✗
Curve pseudovisibility	✓	✓

Proof Sketch ($\chi \leq 3 \cdot 4^{\omega-1}$).

1) We define an infinite family of ordered graphs $\mathcal{H}$ so that no graph in $\mathcal{H}$ can be obtained by deleting vertices.

i.e. ordered curve pseudovisibility graphs are $\mathcal{H}$-free.

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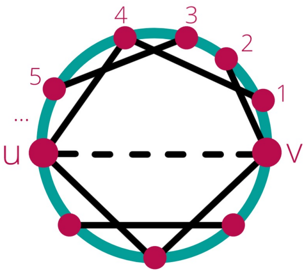
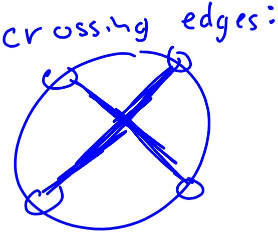


A graph in $\mathcal{H}$



Graphs in $\mathcal{H}$ have non-adjacent vertices u and v which are connected on each side by a "path of crossing edges".

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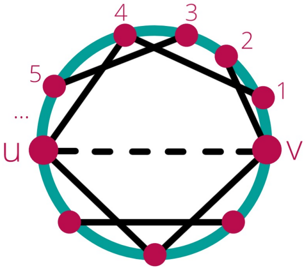

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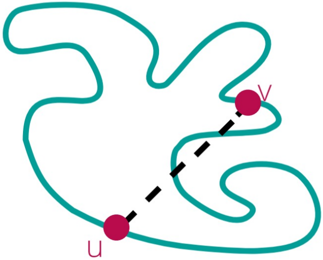

A curve visibility graph

Graphs in $\mathcal{H}$ have non-adjacent vertices u and v which are connected on each side by a "path of crossing edges".

Why are these graphs not curve pseudovisibility graphs?

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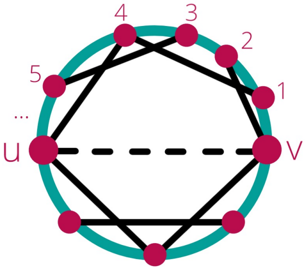

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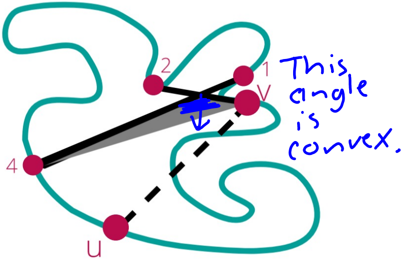

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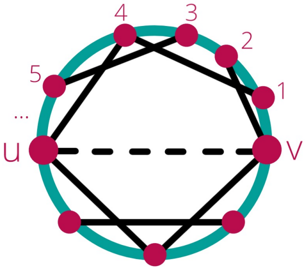

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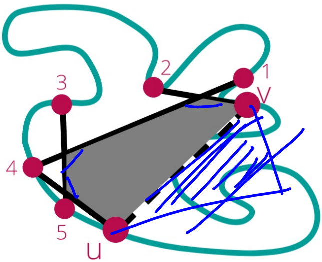

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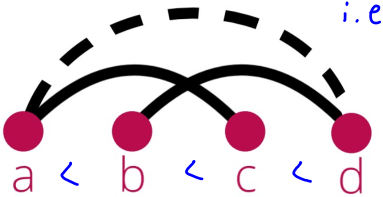

The forbidden configuration.

An ordered graph is **capped** if whenever $a < b < c < d$ and $ac, bd \in E(G)$, then $ad \in E(G)$ as well.

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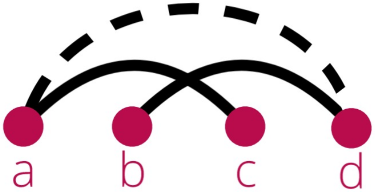

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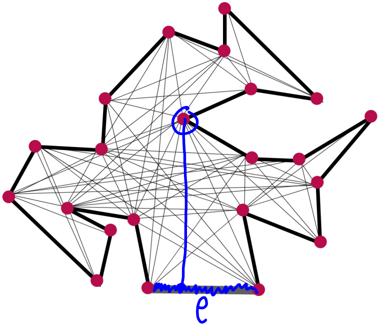

- Fix e; colour all vertices which can see an interior point on e.
- This induces a **capped** subgraph.

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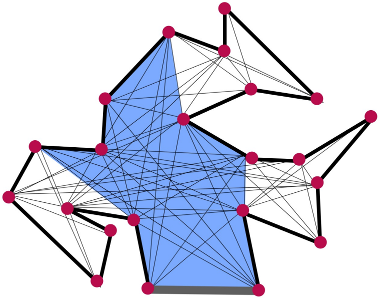

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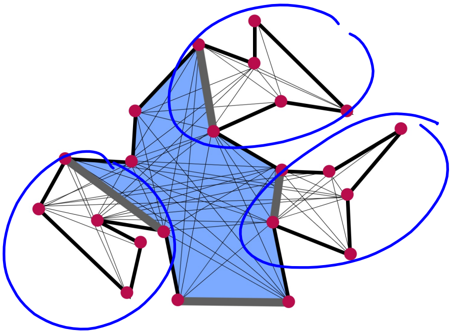

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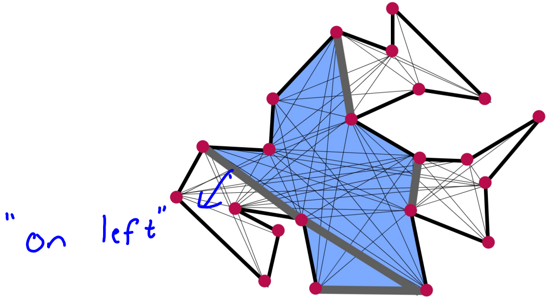

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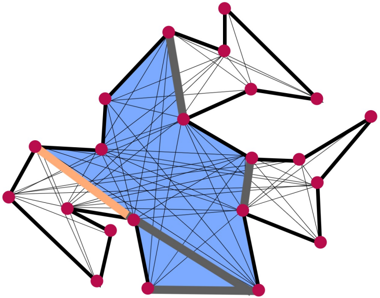

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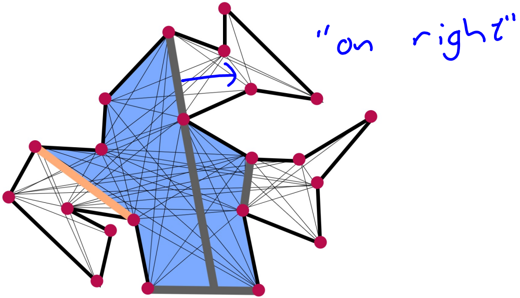

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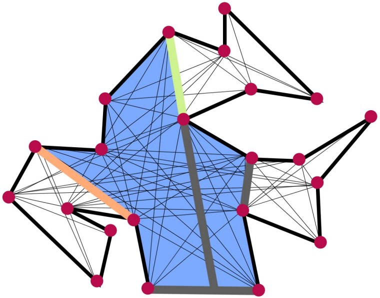

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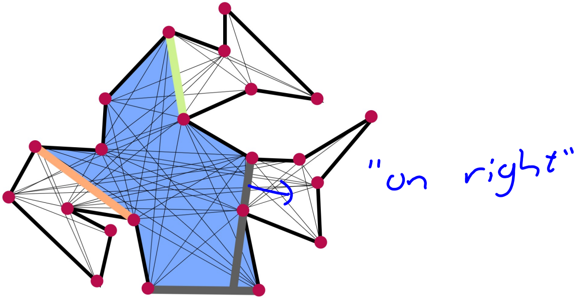

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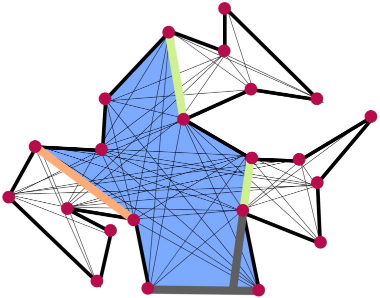

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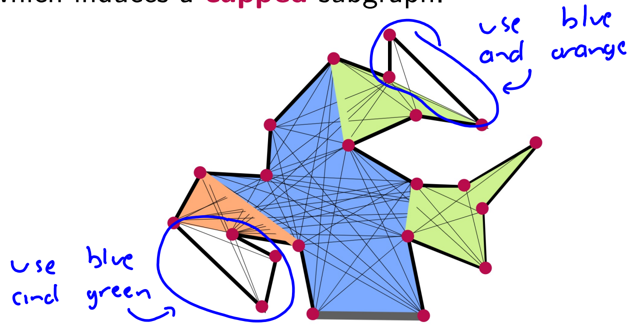

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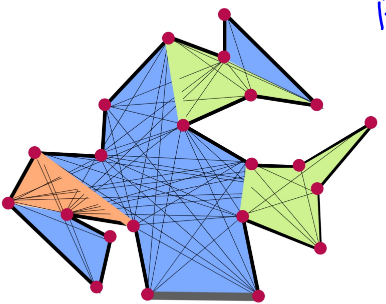

Based on (Suri, 86)

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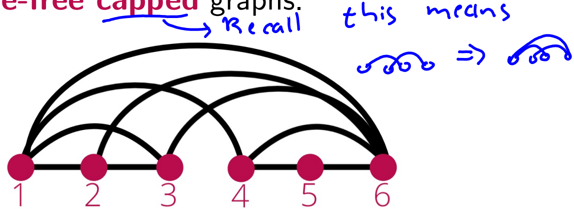

- Consider a capped graph.
- Color all edges "underneath the right side of any triangle" red.
- The black part is **triangle-free** and **capped**.
- The red part has clique number $< \omega$ and is **capped**.
- So continue within red part.

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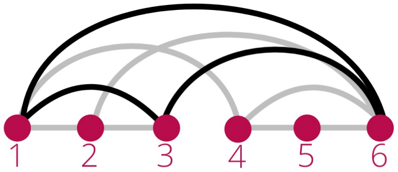

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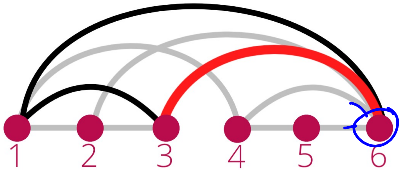

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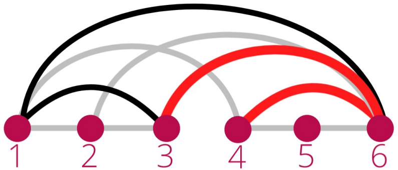

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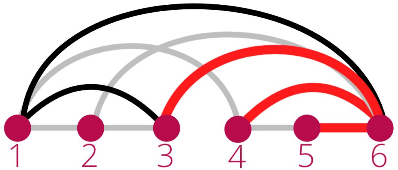

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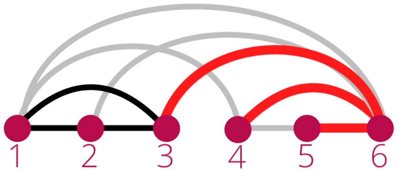

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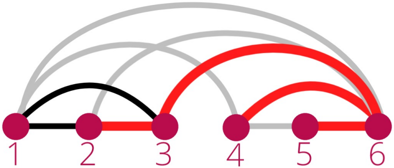

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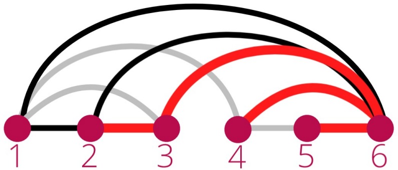

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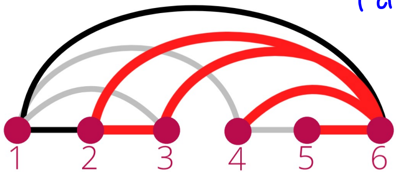

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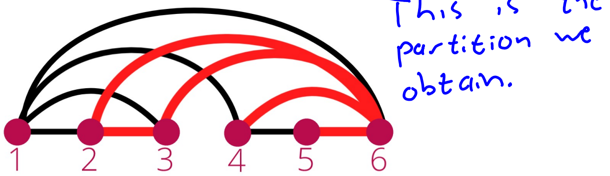

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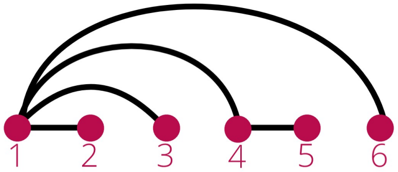

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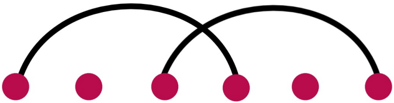

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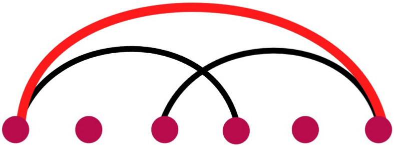

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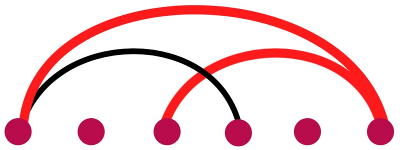

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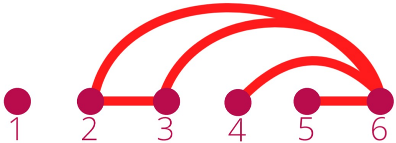

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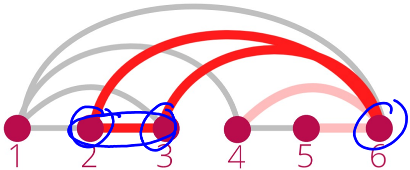

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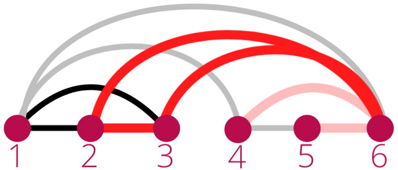

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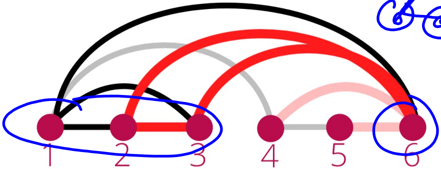

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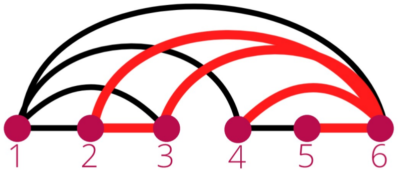

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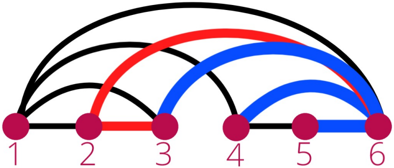

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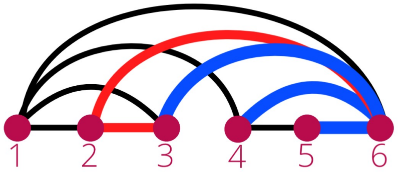

- This is how we compute the clique number ω of a capped graph, which is used as a subroutine.

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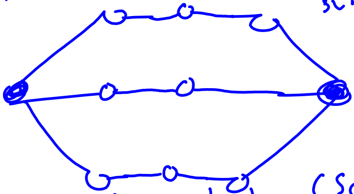
4) Colour **triangle-free capped** graphs.
   - Some extra work is required to get the bound of 4.

Also forbid every induced subdivision of



So χ-bounded by (Scott, Seymour, 20)

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   - vague: Number of holes? How well-structured are they?
2) Can **ordered curve pseudovisibility graphs** be recognized in polynomial time?
   - seems likely: see (Abello, Kumar, 02)
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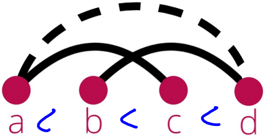

The forbidden configuration for capped graphs.

Thank you!