Local structure for vertex-minors

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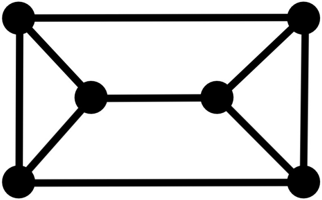
PRINCETON UNIVERSITY

October 19th, 2022

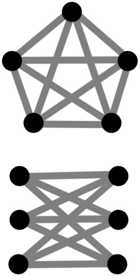
Joint work with Jim Geelen and Paul Wollan.

Kuratowski's Theorem

A graph is planar iff it has no K_5 or $K_{3,3}$ minor.



planar graphs



forbidden minors

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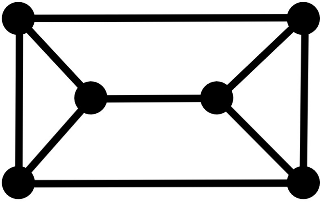

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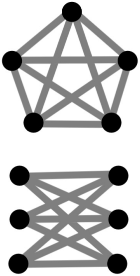


forbidden minors

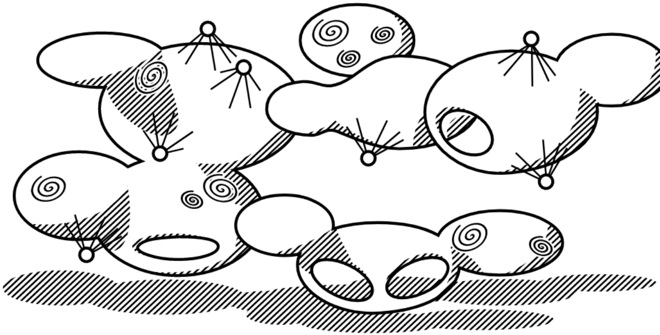
Graph Minors Theorem (Robertson & Seymour 2004)

Every minor-closed class has finitely many forbidden minors.

Theorem (Robertson & Seymour 2003)

The graphs in any proper minor-closed class "decompose" into parts that "almost embed" in a surface of bounded genus.

Figure by Felix Reidl



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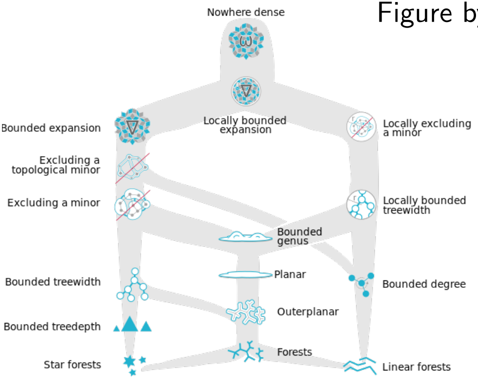

Figure by Felix Reidl

Theory of "**sparsity**" (Nešetřil & Ossona de Mendez)

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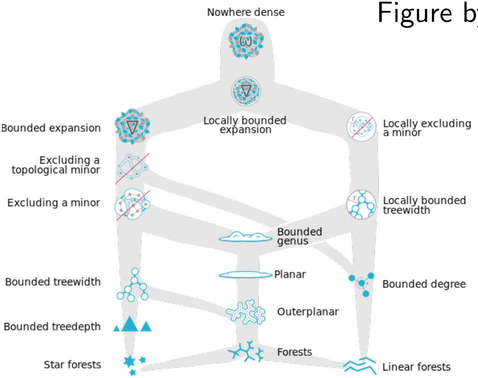

Figure by Felix Reidl

What are the "**dense**" analogs?

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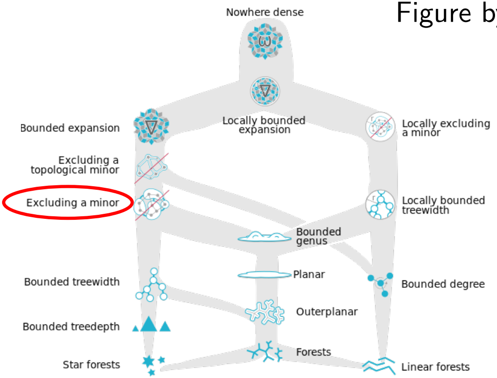

minors ⟶ **vertex-minors**

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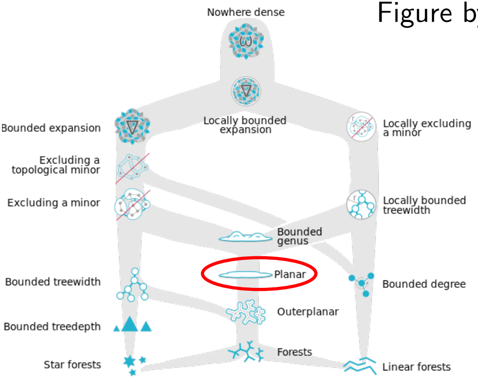
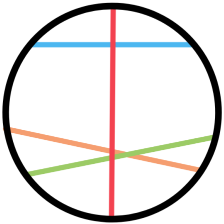


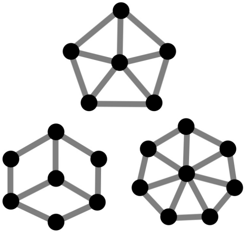
planar graphs ⟶ **circle graphs**

Bouchet's Theorem

A graph is a **circle graph** *iff it has no* W_5, $\hat{W}_6$, *or* W_7 **vertex-minor**.



circle graphs



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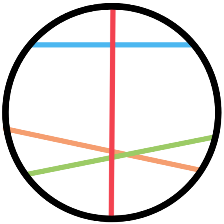

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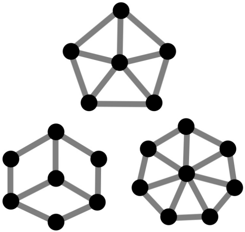


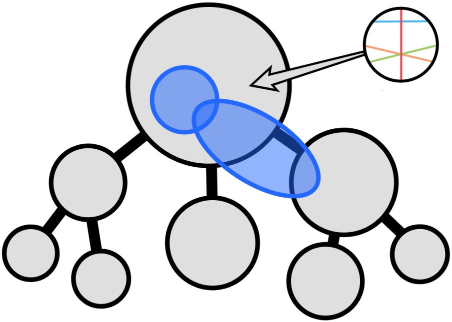
forbidden vertex-minors

Conjecture (Oum 2017)

Every **vertex-minor***-closed class has finitely many forbidden vertex-minors.*

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*



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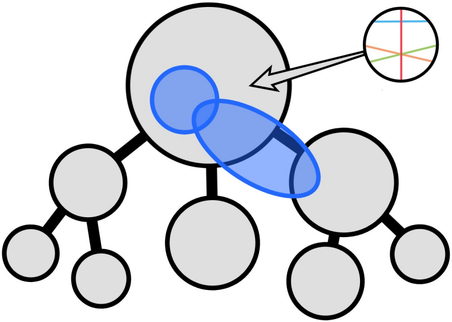
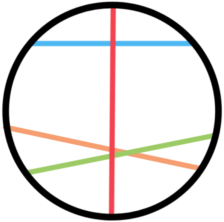


Ongoing project with Jim Geelen & Paul Wollan aiming to prove the conjecture.

Geelen and Oum's Theorem

A graph is a **circle graph** *iff it has no* $W_5, W_6, \ldots$ **pivot-minor**.



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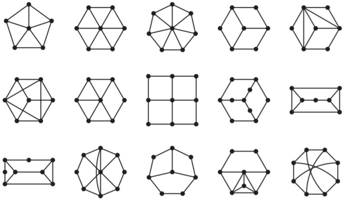

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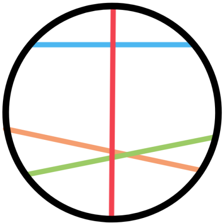

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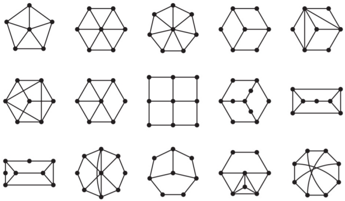

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Common generalization!

(Bouchet 1988; de Fraysseix 1981)

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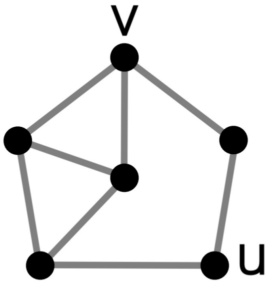
The **vertex-minors** of a graph G are the graphs that can be obtained from G by

1) vertex deletion and
2) **local complementation**



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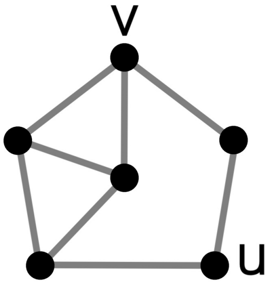

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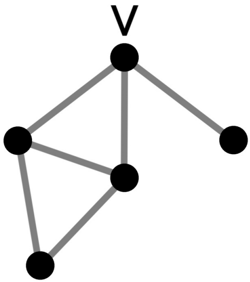

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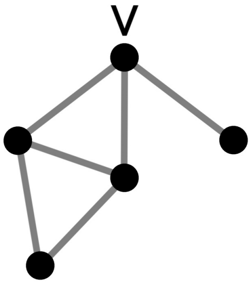

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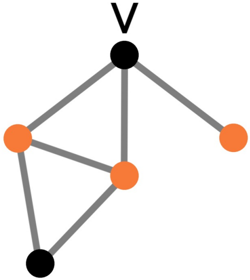

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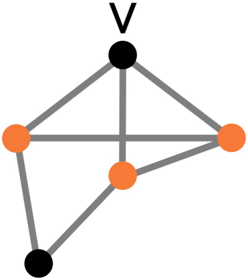

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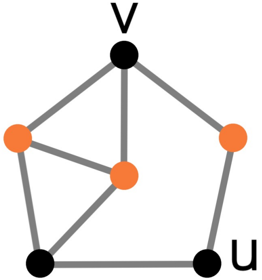

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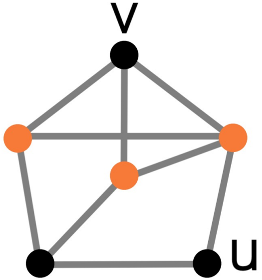

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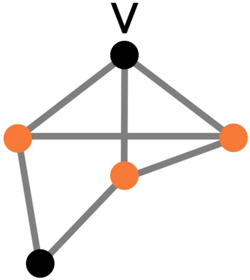
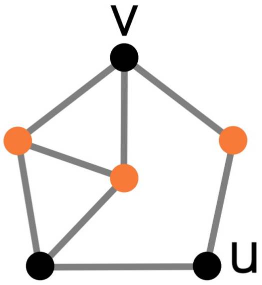


$$G * v - u$$

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G (that is, can be obtained from G by a sequence of local complementations).



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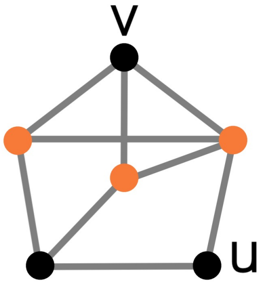

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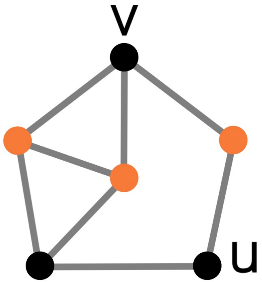

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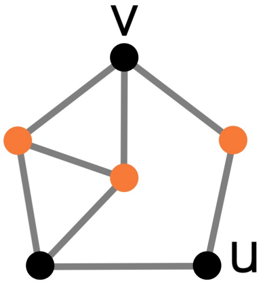

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Why **local equivalence** classes?

- nice interpretation for graph states in quantum computing

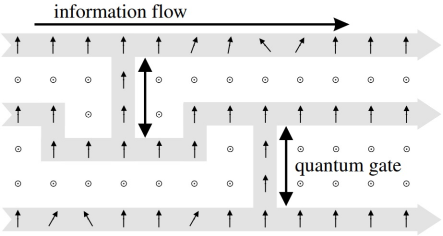

FIG. 1. Quantum computation by measuring two-state parti-

(Raussendorf-Briegel, Van den Nest-Dehaene-De Moor)

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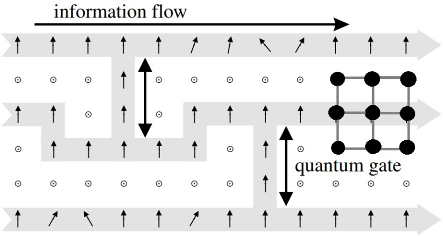

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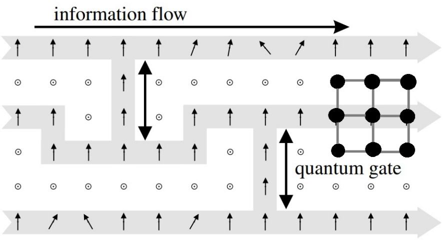

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Conjecture (Geelen)

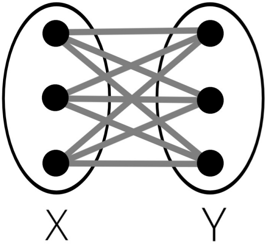
If the graph states that can be prepared come from a proper **vertex-minor-closed** *class $\mathcal{F}$, then $BQP_{\mathcal{F}} = BPP$.*

Why **local equivalence** classes?

- nice interpretation for graph states in quantum computing
- locally equivalent graphs have the same **cut-rank** function

$$\begin{array}{c} \\ X \\ \\ \\ Y \\ \\ \end{array} \begin{array}{c} \quad X \qquad\qquad Y \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

adjacency matrix



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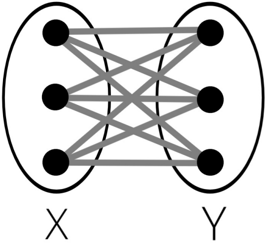

The **cut-rank** of $X \subseteq V(G)$ is the rank of **adj**$[X, \overline{X}]$ over GF_2.

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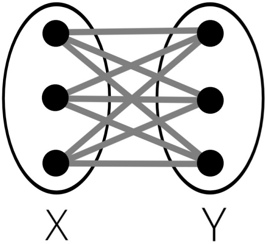

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It is symmetric: **cut-rank**$(X) =$ **cut-rank**$(\overline{X})$.

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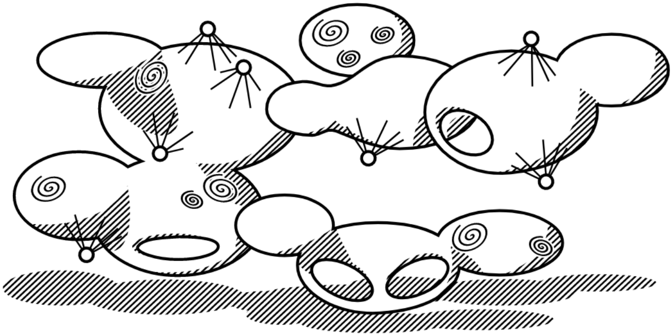
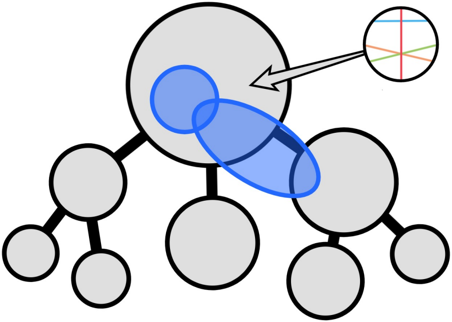
Figure by Felix Reidl



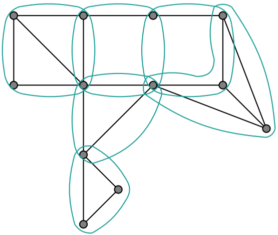
Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*



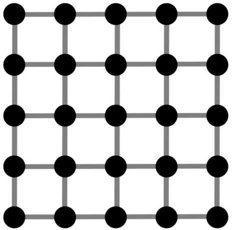
Grid Theorem (Robertson & Seymour 1986)

A class of graphs has bounded tree-width if and only if it does not contain all planar graphs as minors.



$\Longleftrightarrow$

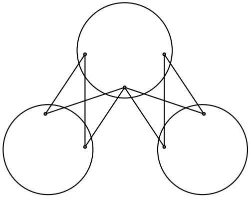
excludes



grid as a minor

Theorem (Geelen, Kwon, McCarty, & Wollan 2020)

A class of graphs has bounded **rank-width** *if and only if it does not contain all* **circle graphs** *as* **vertex-minors***.*



$\iff$

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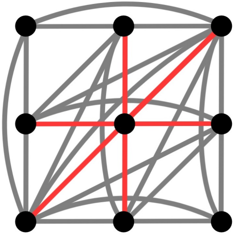
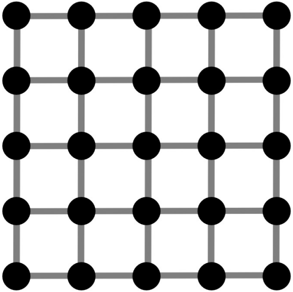


comparability grid
as a **vertex-minor**

Flat Wall Theorem (Robertson & Seymour 1995)

For any proper minor-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a large grid minor, there is a planar subgraph containing a lot of the grid so that the rest of G "almost attaches" onto just the outer face.



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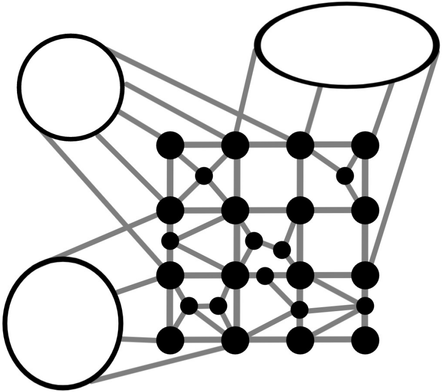
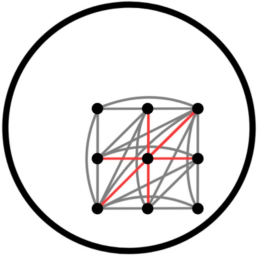


Local Structure Theorem (Geelen, McCarty, & Wollan)

For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid**, *the rest of G "almost attaches" in a way that is "mostly compatible".*



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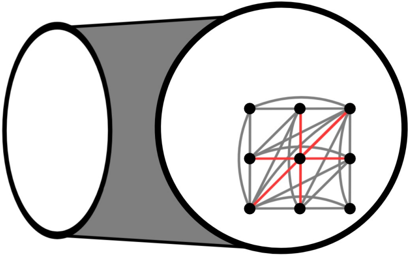
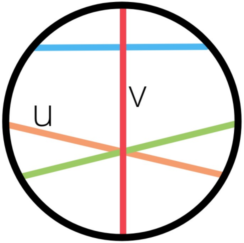
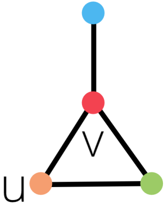


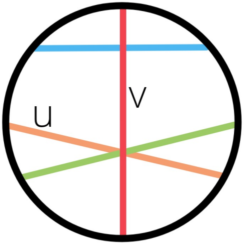


chord diagram

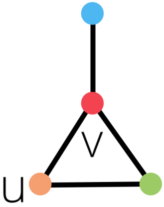


circle graph

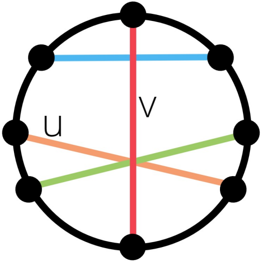
tour graph



chord diagram

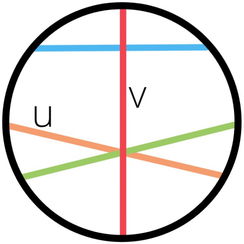


circle graph

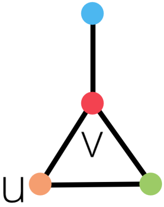


tour graph

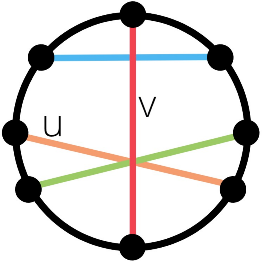
View the **chord diagram** as a 3-regular graph...



chord diagram

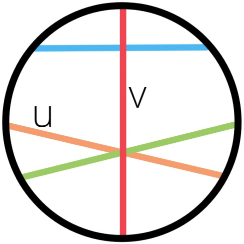


circle graph

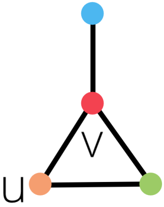


tour graph

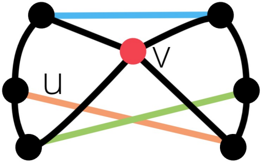
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram



circle graph



tour graph

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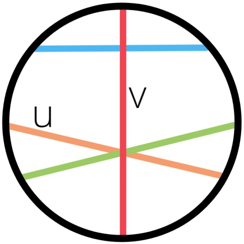
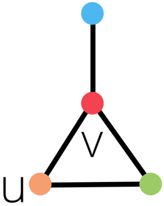
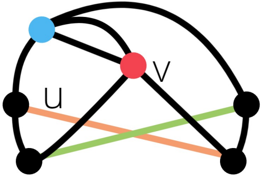


chord diagram

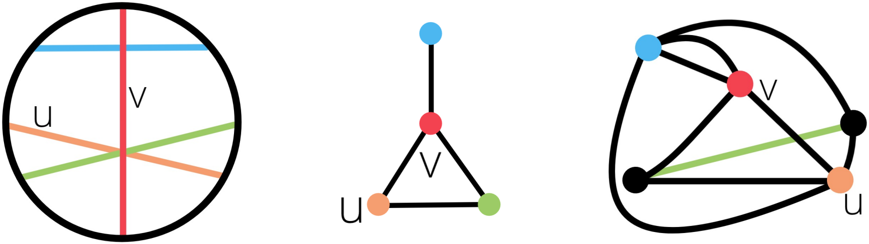


circle graph



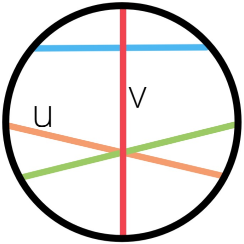
tour graph

View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram **circle graph** **tour graph**

View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



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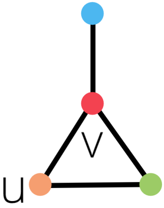
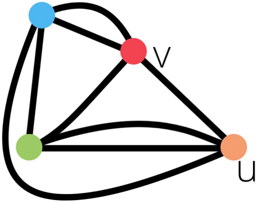


circle graph



tour graph

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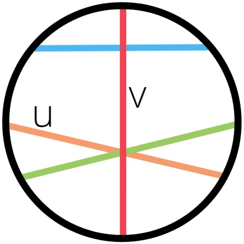

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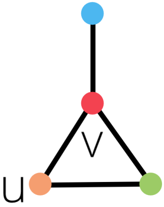
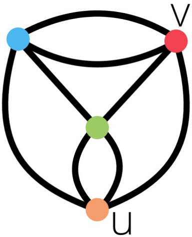


circle graph



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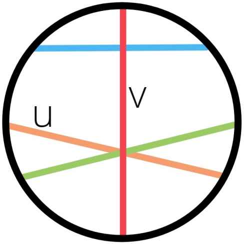
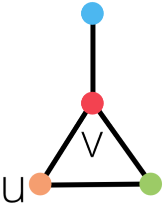
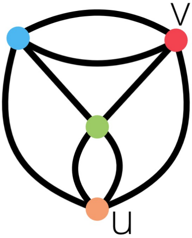


chord diagram

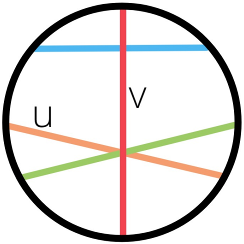


circle graph

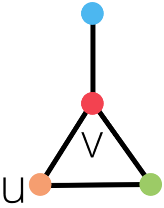


tour graph

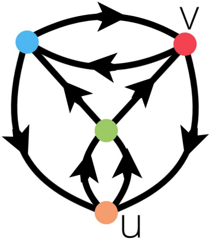
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit.



chord diagram



circle graph



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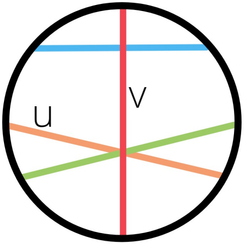
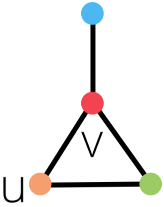
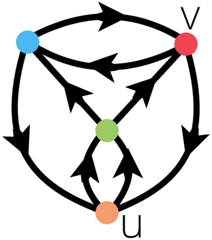


chord diagram

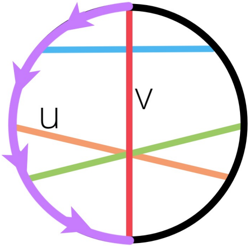


circle graph

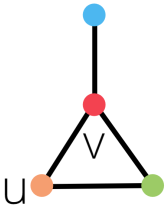


tour graph

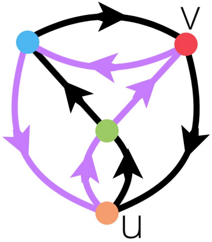
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v.



chord diagram



circle graph



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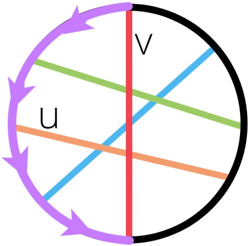
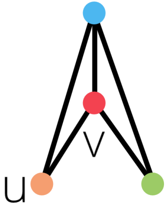
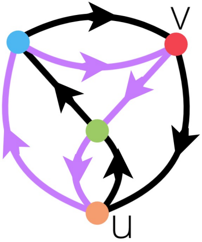


chord diagram



circle graph



tour graph

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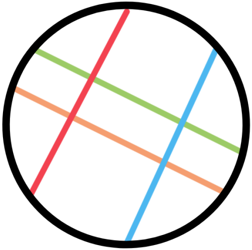
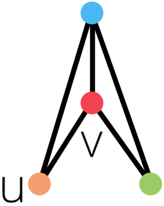
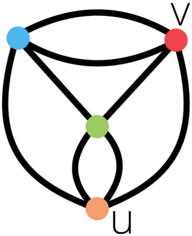


chord diagram



circle graph



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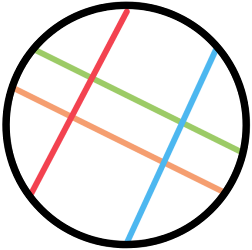
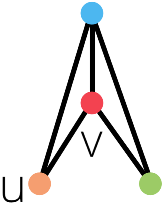
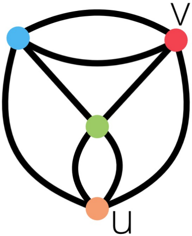


chord diagram

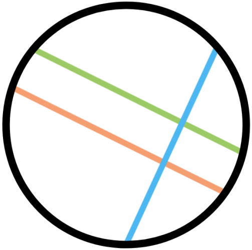


circle graph

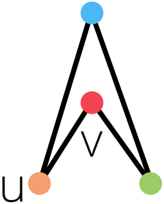


tour graph

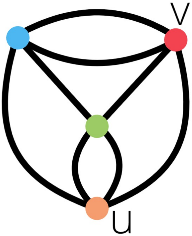
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u.



chord diagram



circle graph



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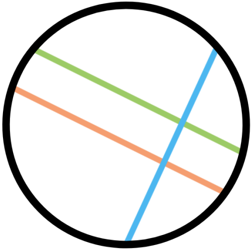
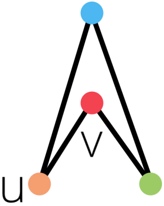
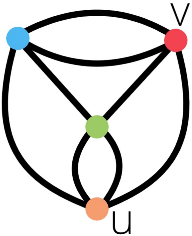


chord diagram

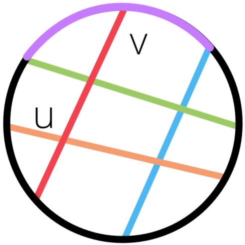


circle graph

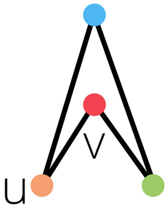


tour graph

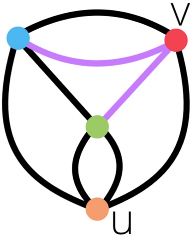
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at *v* then *u*. To delete *v*...



chord diagram

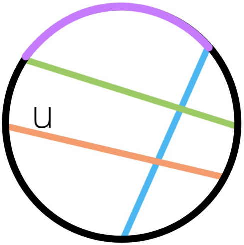


circle graph

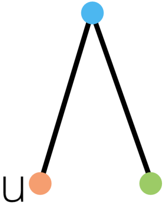


tour graph

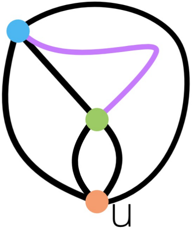
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v...



chord diagram



circle graph



tour graph

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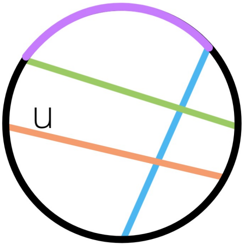
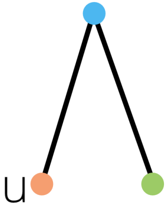
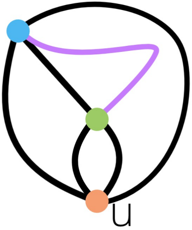


chord diagram

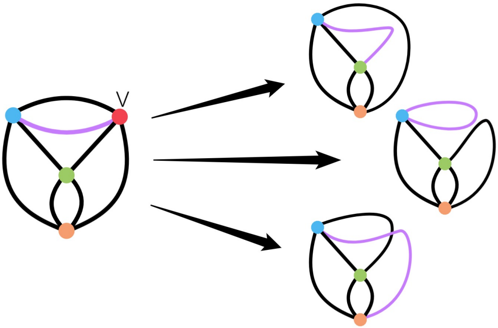


circle graph

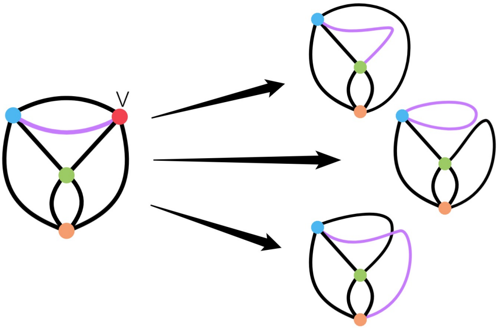


tour graph

View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v, **split it off** in the **tour graph**.



In a 4-regular graph, there are 3 ways to **split off** v.



In a 4-regular graph, there are 3 ways to **split off** v.

Theorem (Kotzig, Bouchet)

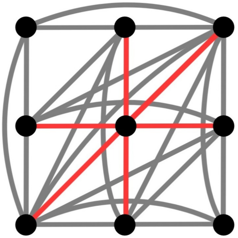
For **prime** *circle graphs H and G, H is a vertex-minor of G iff tour(H) can be obtained from tour(G) by* **splitting off** *vtcs.*

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class “decompose” into parts that are “almost”* **circle graphs***.*

Proof approach:

WMA our favorite circle graph is an induced subgraph.



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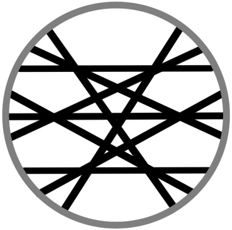

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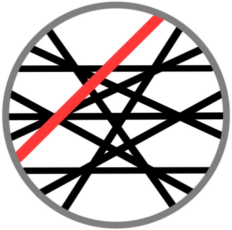

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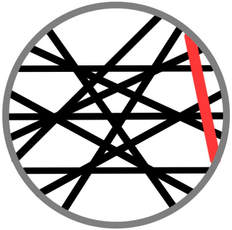

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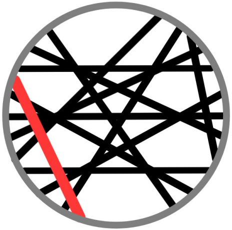

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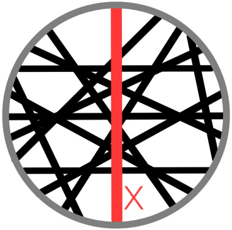

If x can be added as a chord,

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

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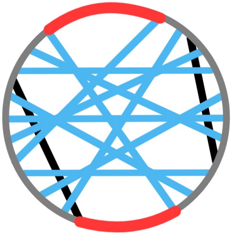

If x can be added as a chord,
then its neighbourhood can be encoded by two arcs.

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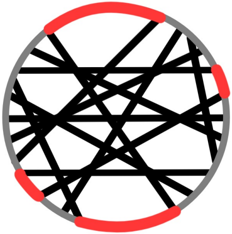

Any neighbourhood can be encoded by even number of arcs.

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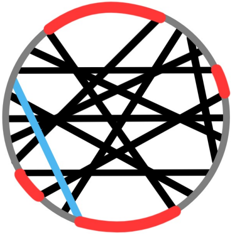

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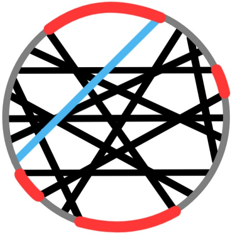

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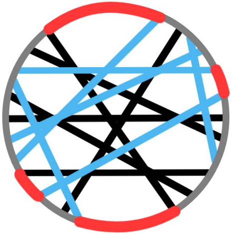

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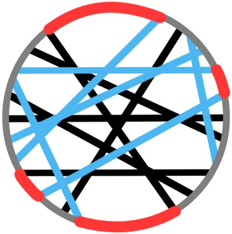
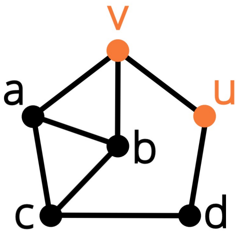
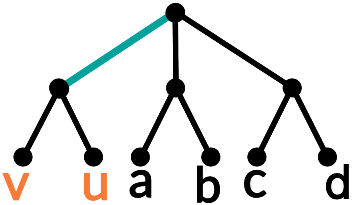


Any neighbourhood can be encoded by even number of arcs.
We can **locally complement** at vertices in the circle graph.

Rank-width(G) is the minimum **width** of a subcubic tree T with leafs $V(G)$.





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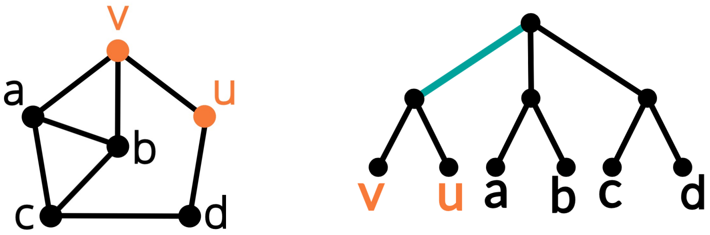
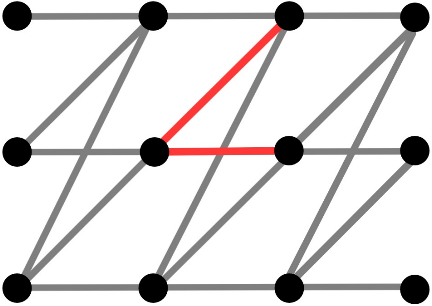


$$\textbf{width}(T) = \max_{e \in E(T)} \text{cut-rank}(X_e)$$

Conjecture (Oum 2009)

A class of graphs has bounded rank-width if and only if it does not contain all **bipartite** *circle graph as* **pivot-minors**.



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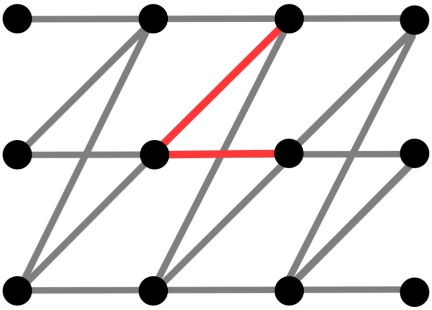

Would be a common generalization!

Thank you!