A combinatorial game for monadic stability

Rose McCarty

UNIVERSITY OF WARSAW

PRINCETON UNIVERSITY

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With Gajarský, Mählmann, Ohlmann, Ossona de Mendez, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, and Toruńczyk.

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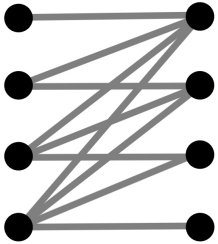

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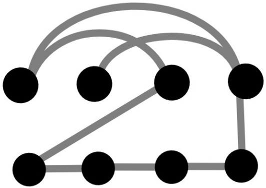

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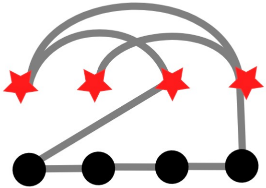

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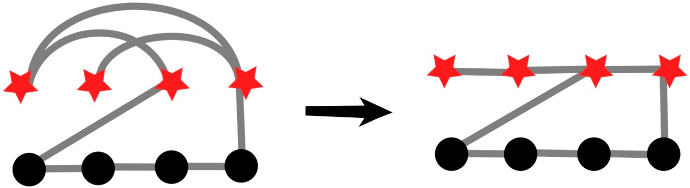

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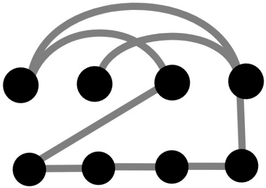

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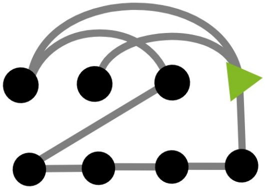

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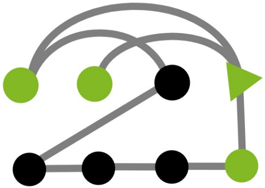

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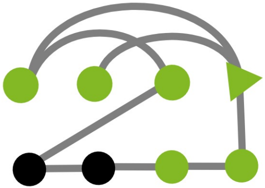

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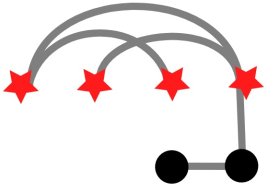

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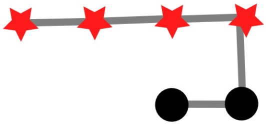

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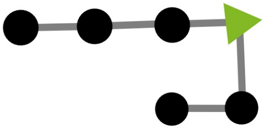

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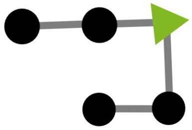

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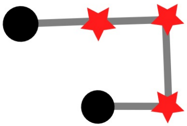

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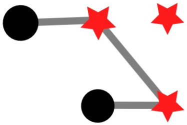

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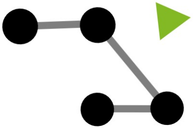

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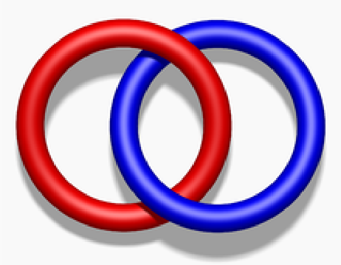
Flipper **wins the game** on a class $\mathcal{C}$ if there exists $t \in \mathbb{N}$ so that Flipper wins in $\leq t$ rounds on each $G \in \mathcal{C}$.

Theorem (Robertson, Seymour, Thomas)

A class of graphs is **linklessly embeddable** *if and only if it contains no minor in the Petersen family.*



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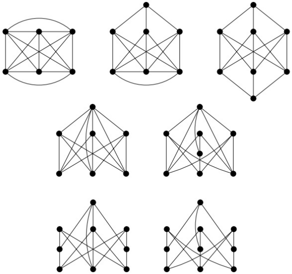

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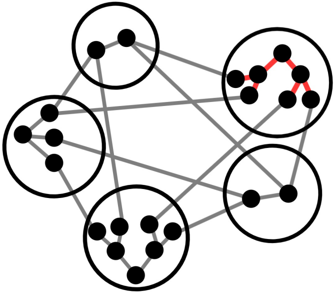
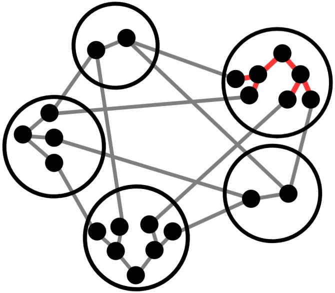
For each $r \in \mathbb{N}$, there exists $t \in \mathbb{N}$ so that no graph with average degree $> t$ is an r-shallow minor of any graph in $\mathcal{C}$.



Theorem (Grohe, Kreutzer, Siebertz)

First-order model-checking is fixed-parameter tractable on any class which is **nowhere dense**.

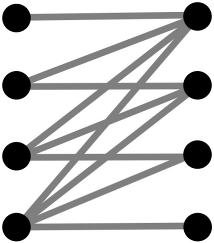
For each $r \in \mathbb{N}$, there exists $t \in \mathbb{N}$ so that K_t is not an r-shallow minor of any graph in $\mathcal{C}$.



Conjecture (folklore; see Gajarský, Pilipczuk, Toruńczyk)

First-order model-checking is fixed-parameter tractable on any class which is **monadically stable***.*

Recall that we use first-order logic to “exclude”:



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A class is nowhere dense if and only if it is monadically stable and **forbids a $K_{t,t}$-subgraph** *for some $t \in \mathbb{N}$.*

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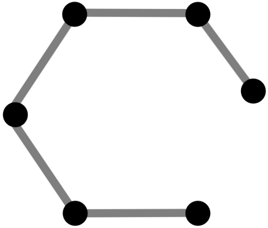

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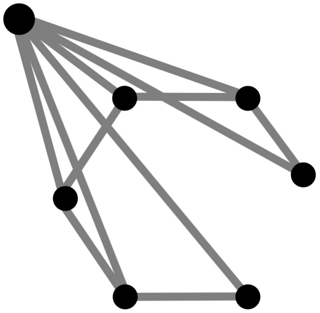

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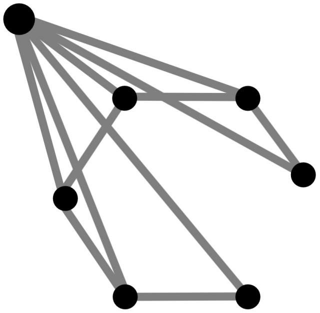

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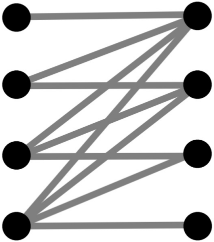
FO model-checking is FPT on $\mathcal{C}$ if there exists $f : \mathbb{N} \to \mathbb{N}$ and $c \in \mathbb{R}$ so that the problem can be solved in time $f(|\phi|)n^c$.

Recall that a class is **monadically stable** if half-graphs are "excluded via" first-order logic.



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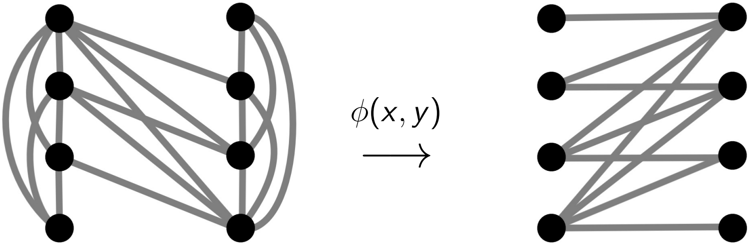

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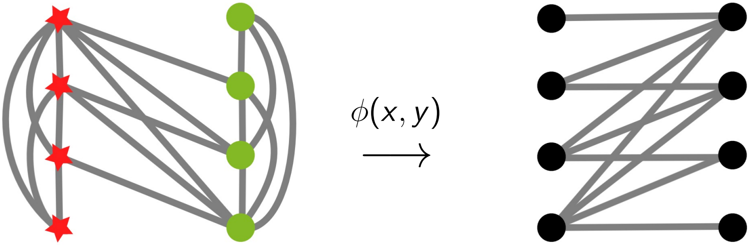

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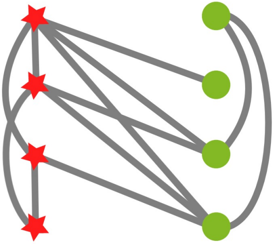
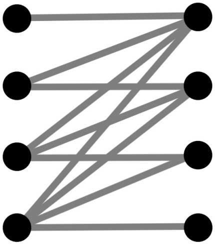


$$\xrightarrow{\phi(x, y)}$$



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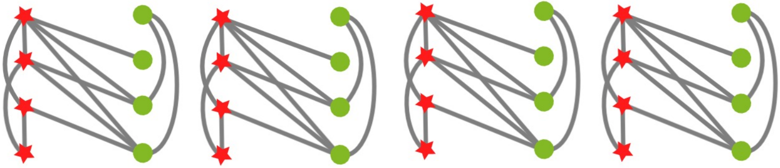

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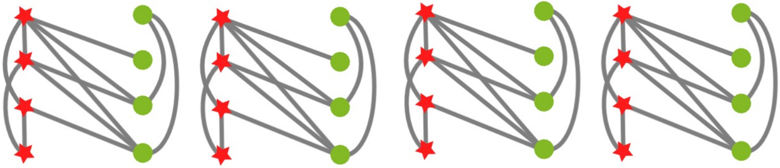

For fixed $\phi(x, y)$, the resulting **transduction** of $\mathcal{C}$ is the class of all induced subgraphs of graphs which can be obtained this way.

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forbidden minor

$\subseteq$

bounded expansion

(shallow minors have bounded average degree)

$\subseteq$

nowhere dense

(have forbidden shallow minors)

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Conjecture (folklore; see Gajarský, Pilipczuk, Toruńczyk)

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Thank you!