Vertex-minors and immersions

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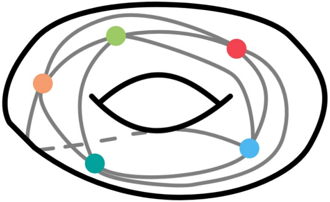
Joint work with Jim Geelen and Paul Wollan

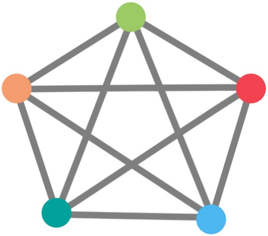
Conjecture (structure)

Every graph with no H-**vertex-minor** *"decomposes" into parts that are "almost"* **circle graphs**.

Like graph minors structure theorem of Robertson & Seymour.



$\longrightarrow$



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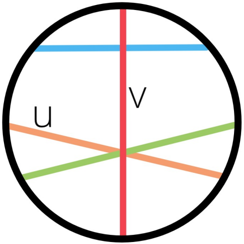
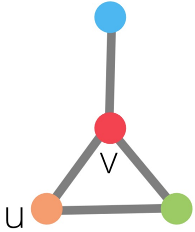
A **circle graph** is the intersection graph of chords on a circle.



chord diagram

$\longrightarrow$



circle graph

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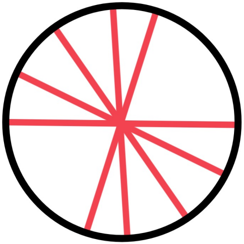

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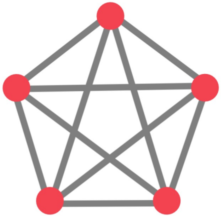

chord diagram | **circle graph**

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Every induced subgraph is a **vertex-minor**,
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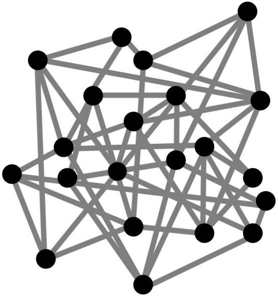

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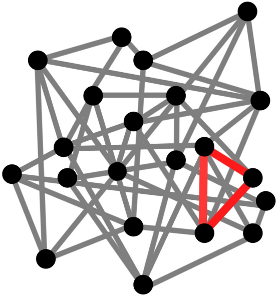

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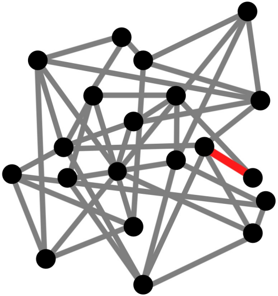

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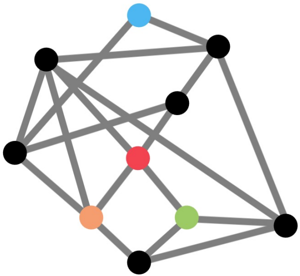
Every induced subdivision is also a **vertex-minor**.



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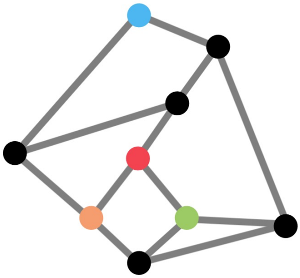

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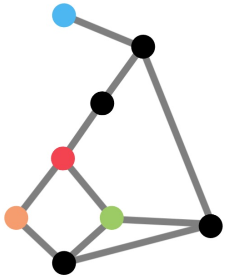

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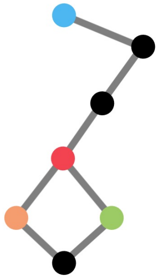

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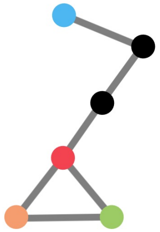

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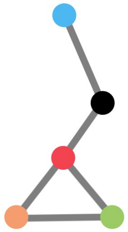

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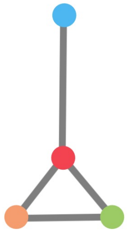
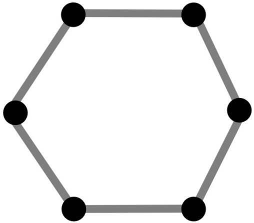


Conjecture (structure)

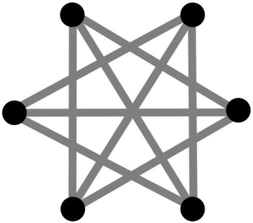
Every graph with no ***H-vertex-minor*** *"decomposes" into parts that are "almost"* ***circle graphs****.*

G has no **H-vertex-minor** $\longrightarrow$ $\overline{G}$ has no **H'-vertex-minor**

(Bouchet)



G



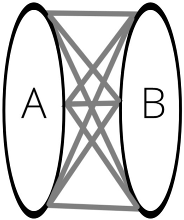
$\overline{G}$

Conjecture (structure)

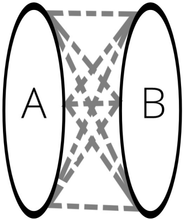
Every graph with no ***H*-vertex-minor** *"decomposes" into parts that are "almost"* **circle graphs**.

Classes with no ***H*-vertex-minor** have strong Erdös-Hajnal property.

(Chudnovsky-Oum via Chudnovsky-Scott-Seymour-Spirkl)



or



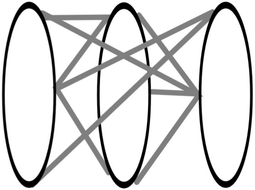
$$|A|, |B| \geq \epsilon_{\mathrm{H}} \cdot |V(G)|$$

Conjecture (structure)

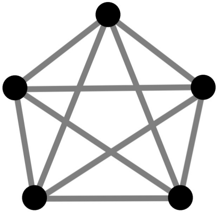
Every graph with no ***H-vertex-minor*** *"decomposes" into parts that are "almost"* ***circle graphs****.*

Classes with no ***H*-vertex-minor** are **χ-bounded**.

(Davies)



chromatic number χ



clique number ω

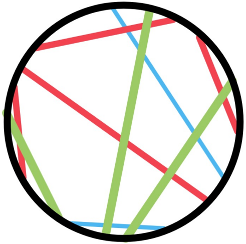
$$\chi \leq f_H(\omega)$$

Conjecture (**structure**)

*Every graph with no H-**vertex-minor** "decomposes" into parts that are "almost" **circle graphs**.*

Circle graphs are **polynomially χ-bounded**.

(Davies-McCarty)



coloring

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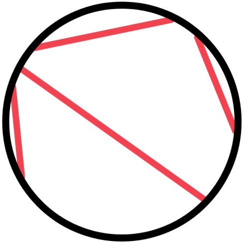

stable set

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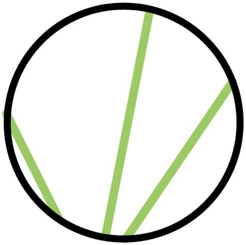

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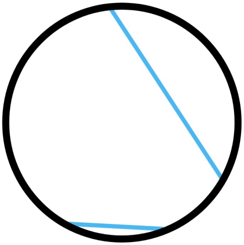

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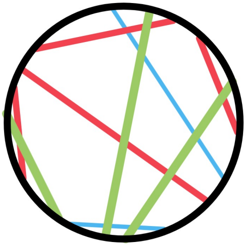

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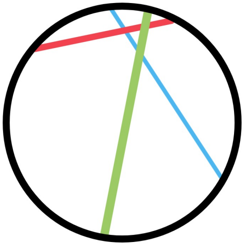

clique

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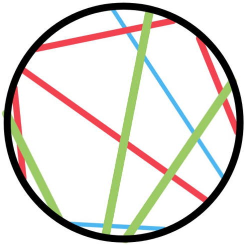

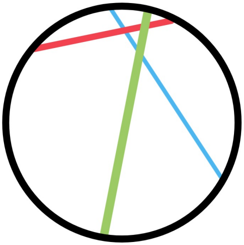

$$\chi \leq 7\omega^2$$

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Conjecture (polynomial χ-boundedness)

*Every graph with no H-**vertex-minor** has $\chi \leq \mathrm{poly}_{\mathrm{H}}(\omega)$.*

Asked by (Kim-Kwon-Oum-Sivaraman).

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Asked by (Kim-Kwon-Oum-Sivaraman).

Follows from (**structure**) since "decomposing" works (Bonamy-Pilipczuk).

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Conjecture (WQO)

*For $H_1, H_2, H_3, \ldots$, some H_i is a **vertex-minor** of H_j, $i < j$.*

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Conjecture (vertex-minor-testing)

*Can test if n-vertex graph has an H-**vertex-minor** in $f(H) \cdot n^c$.*

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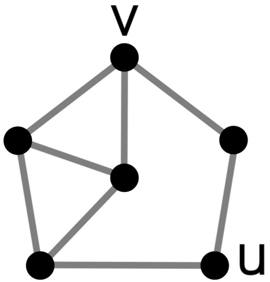
There is a common generalization of minors and **vertex-minors**.

The **vertex-minors** of a graph G are obtained by

1) vertex deletion and
2) **local complementation**.



G

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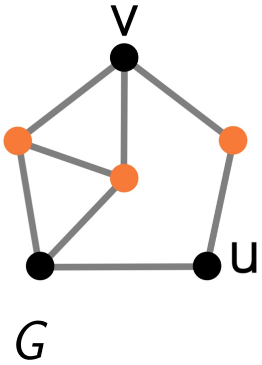

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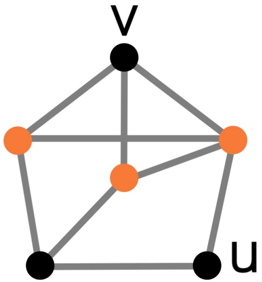

$G * v$

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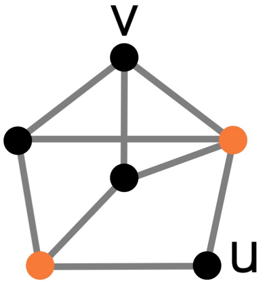

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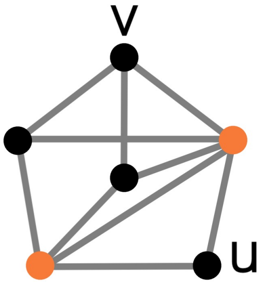

$$G * v * u$$

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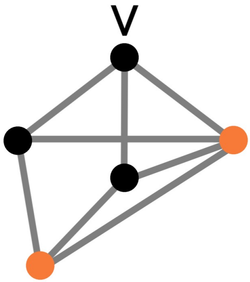

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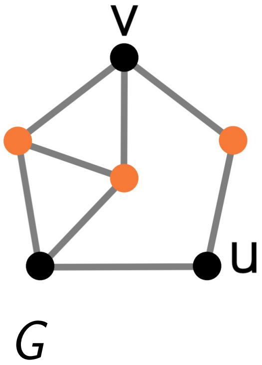

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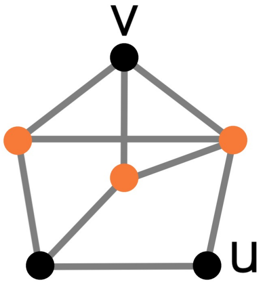

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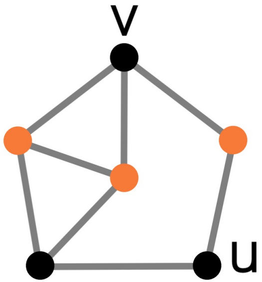

$$G * v * v = G$$

Why **local equivalence** classes?

- graph states in quantum computing

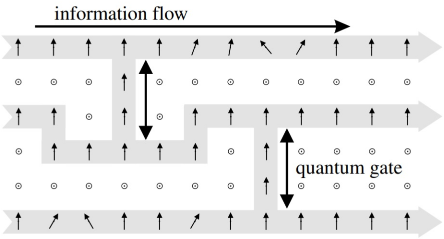

FIG. 1. Quantum computation by measuring two-state parti-

(Raussendorf-Briegel, Van den Nest-Dehaene-De Moor)

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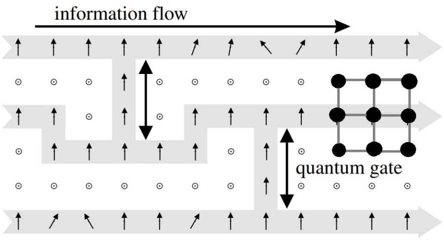

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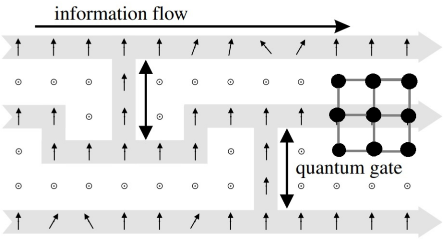


FIG. 1. Quantum computation by measuring two-state parti-

(Raussendorf-Briegel, Van den Nest-Dehaene-De Moor)

Conjecture (Geelen)

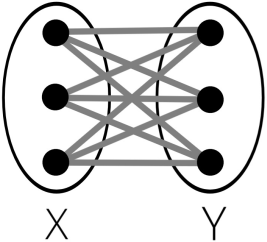
When the graph states that can be prepared have no ***H-vertex-minor***, $BQP_{\mathrm{H}} = BPP$.

Why **local equivalence** classes?

- graph states in quantum computing
- **rank-connectivity**

$$
\begin{array}{c} \\ X \\ \\ \\ Y \\ \\ \end{array}
\overset{\qquad X \qquad\qquad\qquad Y}{\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}}
$$

adjacency matrix



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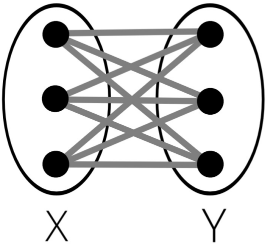

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*Every graph with no H-**vertex-minor** "decomposes" into parts that are "almost" **circle graphs**.*

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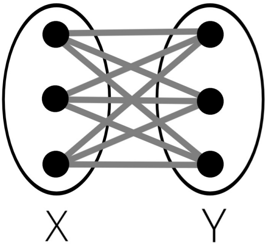

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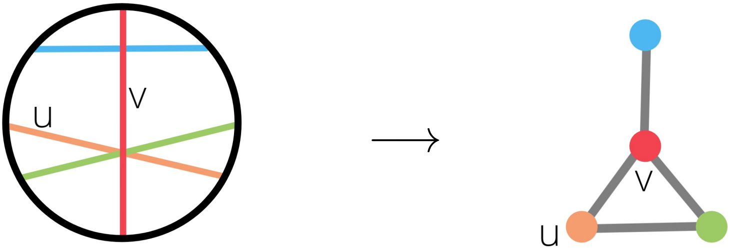
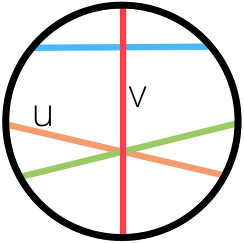
Why **local equivalence** classes?

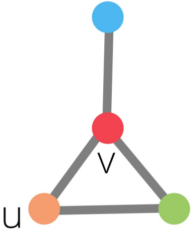
- graph states in quantum computing
- **rank-connectivity**
- has a nice interpretation for **circle graphs**...



chord diagram → **circle graph**

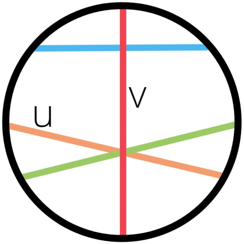


chord diagram

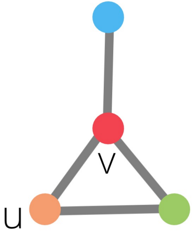


circle graph

tour graph



chord diagram



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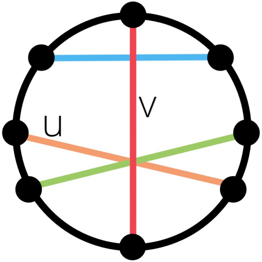
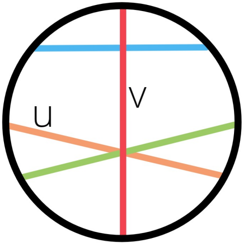
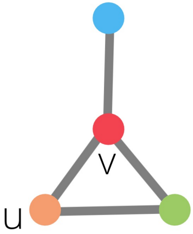


tour graph

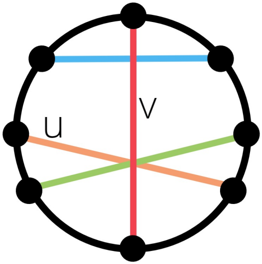
View the **chord diagram** as a 3-regular graph...



chord diagram

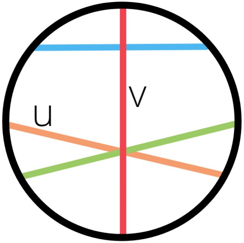


circle graph

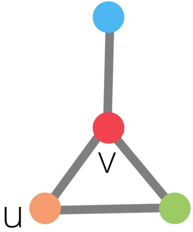


tour graph

View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram



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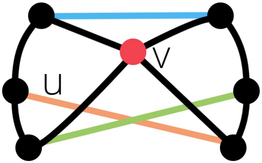

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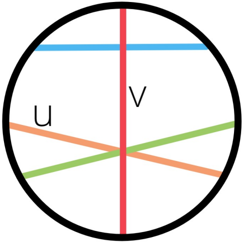

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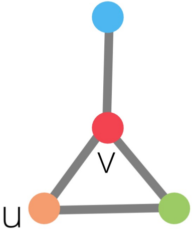
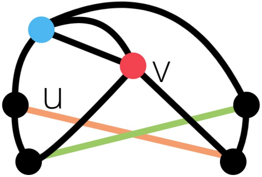


circle graph



tour graph

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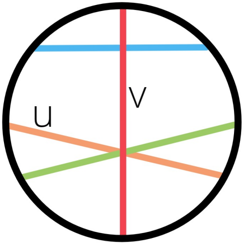
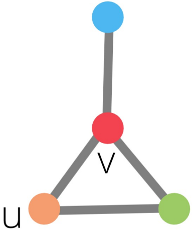
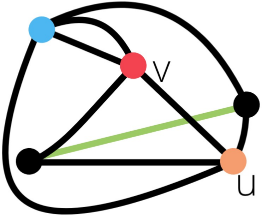


chord diagram



circle graph



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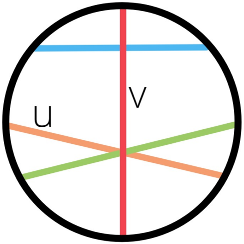
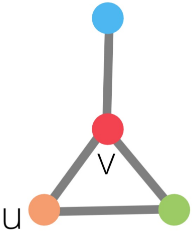
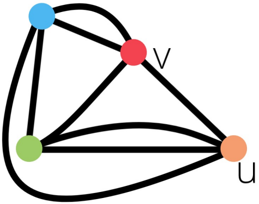


chord diagram



circle graph



tour graph

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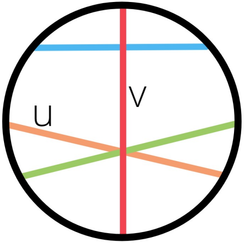
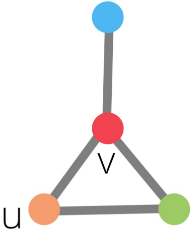
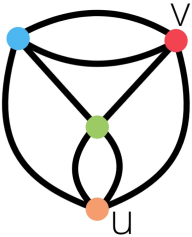


chord diagram



circle graph



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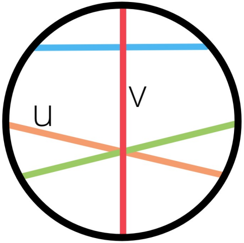
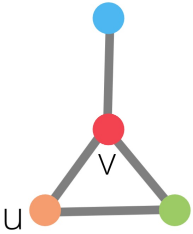
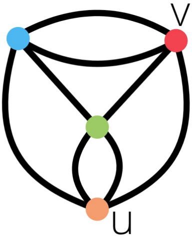


chord diagram

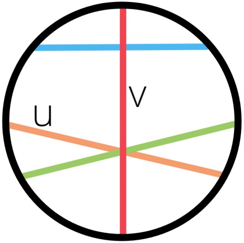


circle graph

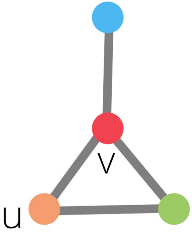


tour graph

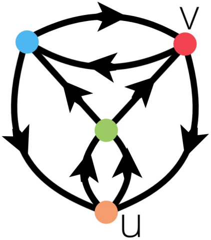
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit.



chord diagram



circle graph



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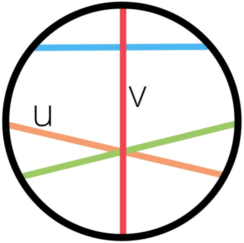

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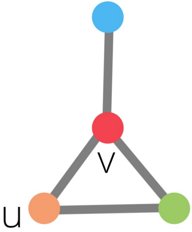
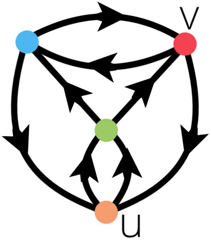
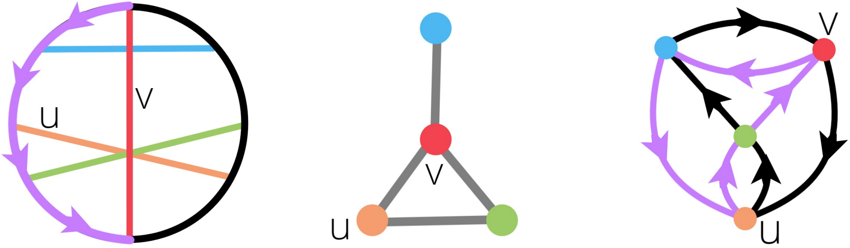


circle graph



tour graph

View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v.



chord diagram **circle graph** **tour graph**

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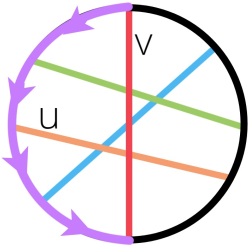
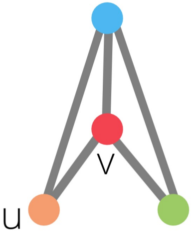
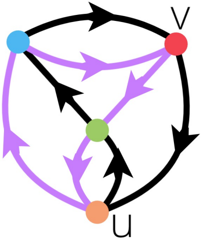


chord diagram



circle graph



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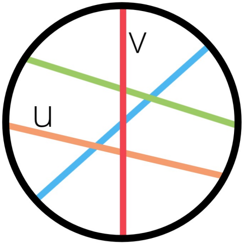
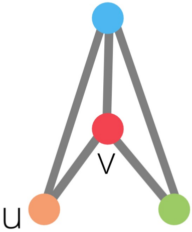


chord diagram



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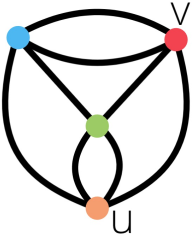

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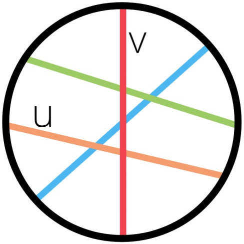
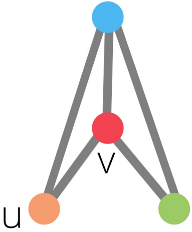
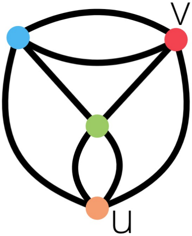


chord diagram



circle graph



tour graph

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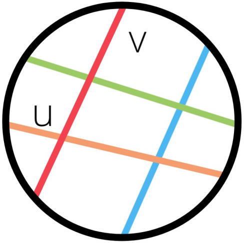
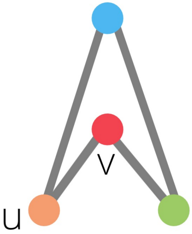


chord diagram



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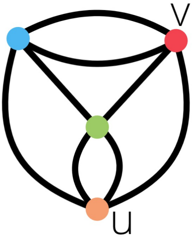

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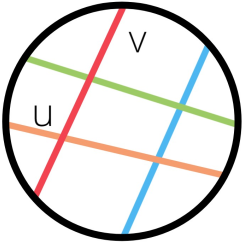

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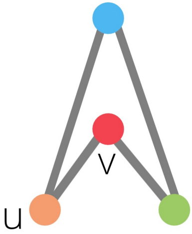

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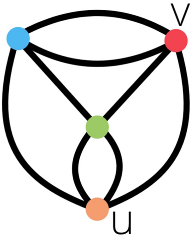
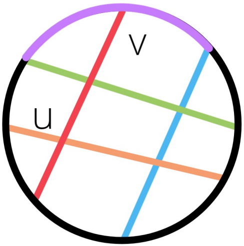


tour graph

View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v...



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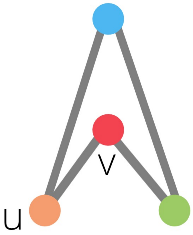

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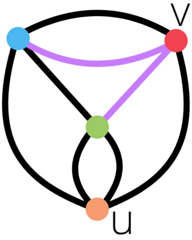

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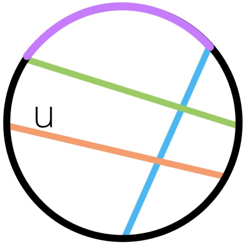
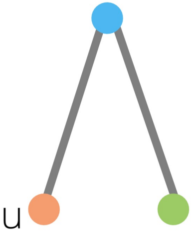


chord diagram



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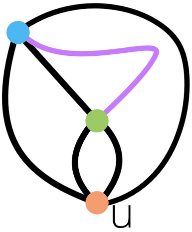

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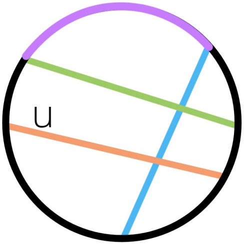
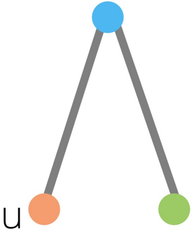
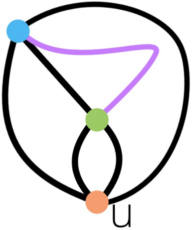


chord diagram

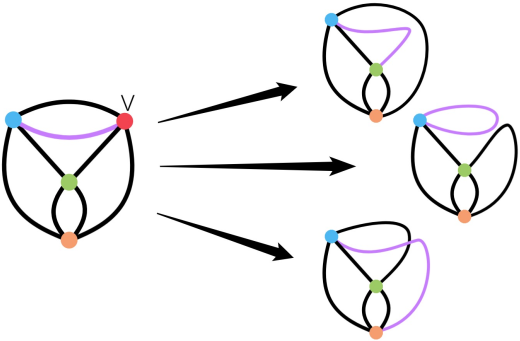


circle graph

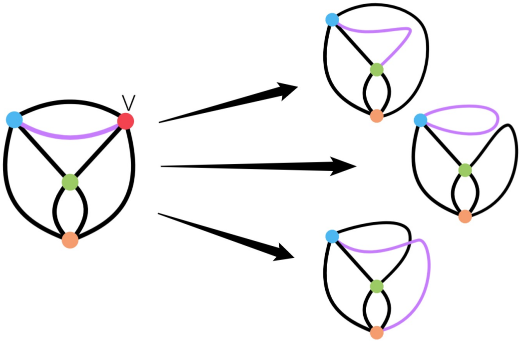


tour graph

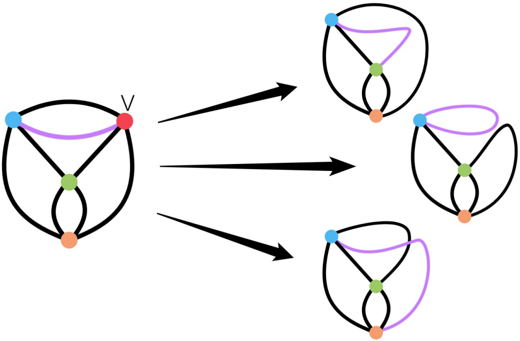
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v, **split it off** in the **tour graph**.



In a 4-regular graph, there are 3 ways to **split off** v.



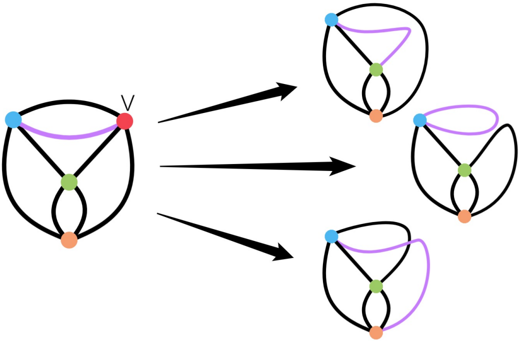
In a 4-regular graph, there are 3 ways to **split off** v. This is how we define **immersions**.



In a 4-regular graph, there are 3 ways to **split off** v. This is how we define **immersions**.

Theorem (Kotzig, Bouchet)

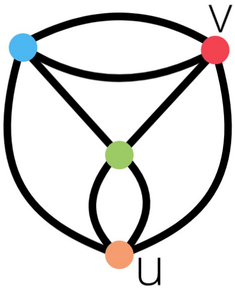
*If H and G are 2-**rank-connected** circle graphs, then H is a vertex-minor of G $\iff$ **tour**(H) **immerses** into **tour**(G).*



Lemma (Bouchet)

If H is a vertex-minor of G and $v \in V(G) \setminus V(H)$, then H is a **vertex-minor** *of either $G - v$, $G * v - v$, or $G * v * u * v - v$ for each neighbour u of v.*

If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

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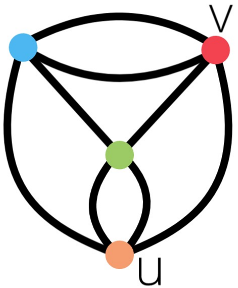
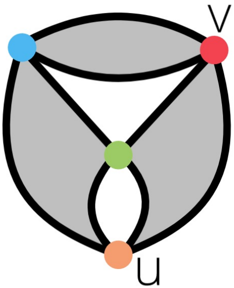


tour graph

Consider a 2-face-coloring.

If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



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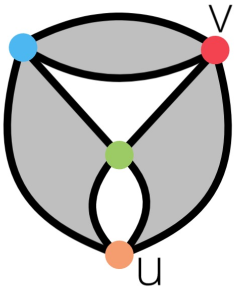
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

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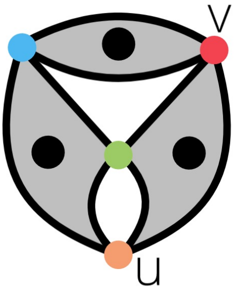

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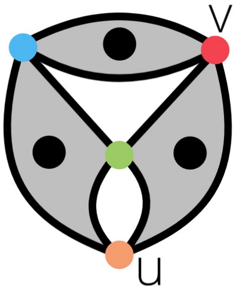

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Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces.

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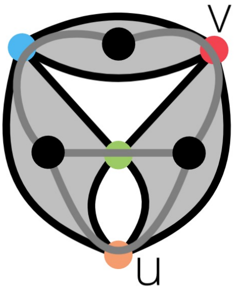

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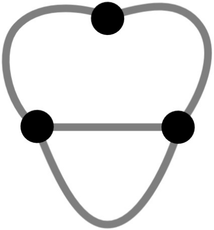

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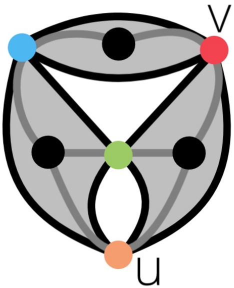

tour graph

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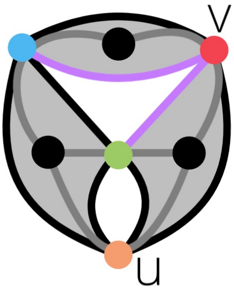

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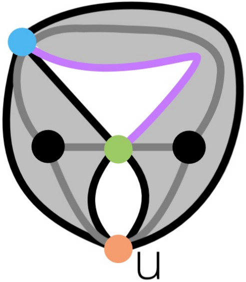

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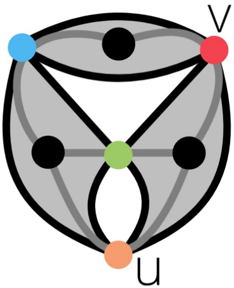

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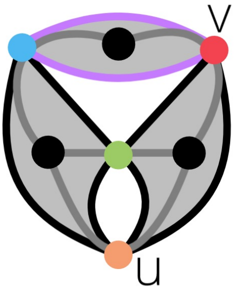

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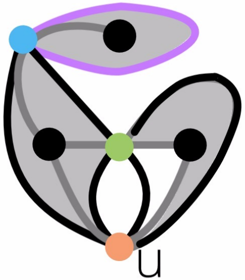

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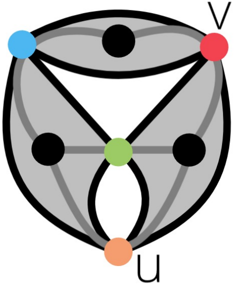
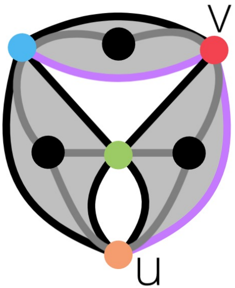


tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v. The split that we do not allow...

If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



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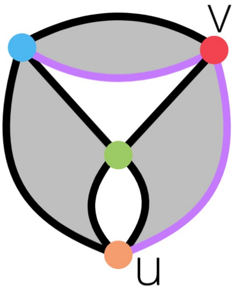

tour graph

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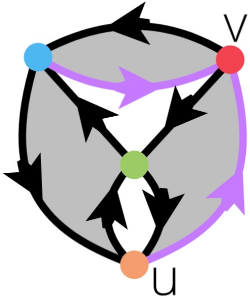
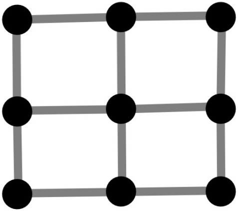


tour graph

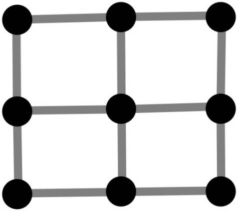
Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v. The split that we do not allow breaks the orientation.

Here is the other direction.



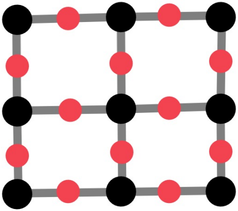
Take a planar graph.

Here is the other direction.



Take a planar graph. Add a vertex to each edge

Here is the other direction.



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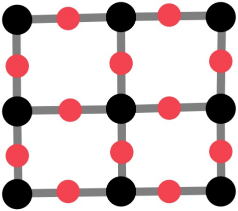
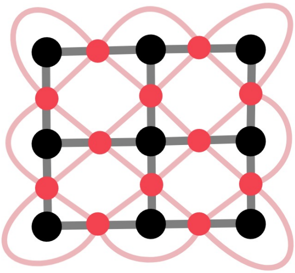


Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face.

Here is the other direction.



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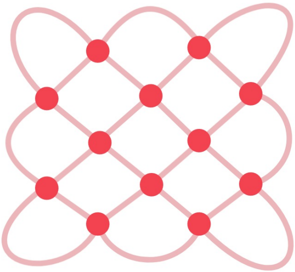
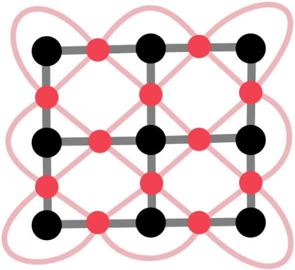


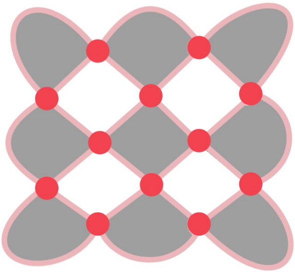
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**.

Here is the other direction.



Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring.

Here is the other direction.



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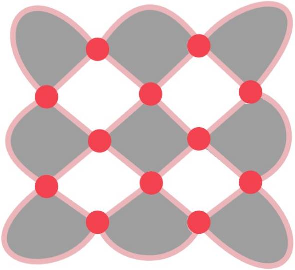
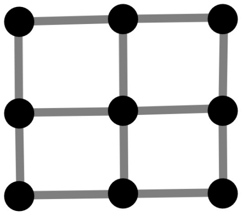


Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.



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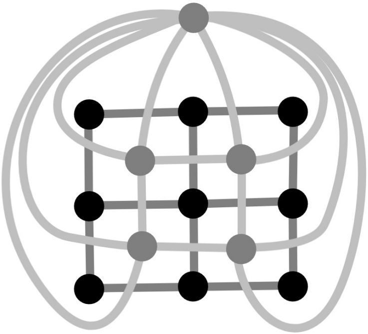

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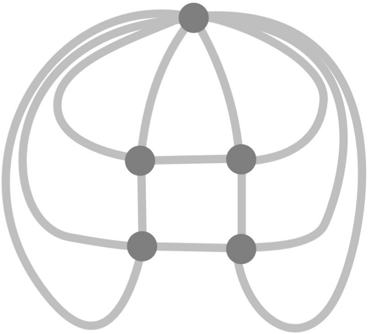

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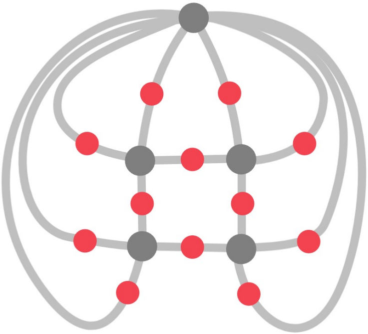

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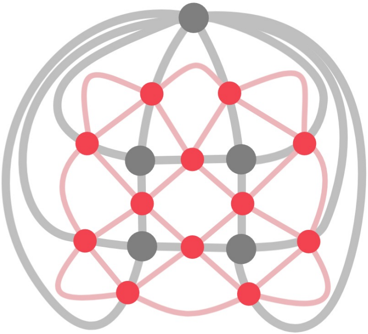

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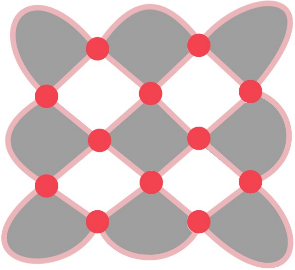

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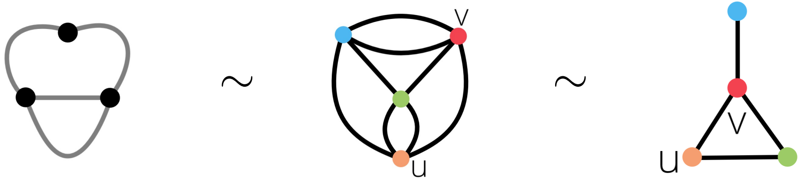
Conjecture (structure)

*Every graph with no H-***vertex-minor** *"decomposes" into parts that are "almost"* **circle graphs***.*

Grid Theorem (Robertson-Seymour)

A class of graphs has unbounded branch-width iff it has all planar graphs as minors.



planar graph | **tour graph** | **circle graph**

Conjecture (**structure**)

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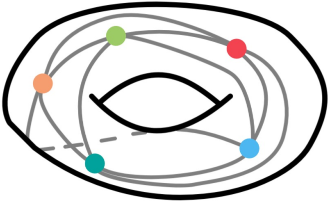

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planar graphs	$\sim$	**circle graphs**
branch-width	$\sim$	**rank-width**

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Grid Theorem (Robertson-Seymour)

A class of graphs has unbounded branch-width iff it has all planar graphs as minors.

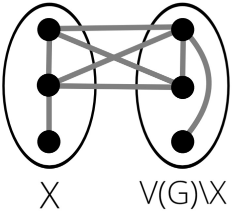
Theorem (Geelen-Kwon-McCarty-Wollan)

*A class of graphs has unbounded **rank-width** iff it has all **circle graphs** as **vertex-minors**.*

Conjectured by Oum.

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

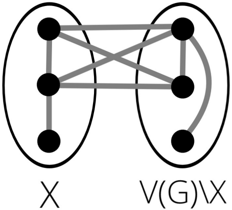
	X			$V(G) \setminus X$		
X	0	1	0	1	1	0
	1	0	1	1	1	0
	0	1	0	0	0	0
$V(G) \setminus X$	1	1	0	0	1	1
	1	1	0	1	0	0
	0	0	0	1	0	0



(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$
\begin{array}{c|c|c|}
 & X & V(G)\setminus X \\
X & \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} & \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \\
\hline
V(G)\setminus X & \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}
\end{array}
$$

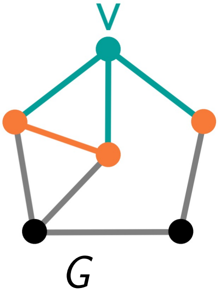


$$\textbf{cut-rank}(X) = \textbf{cut-rank}(V(G) \setminus X)$$

(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$V(G) \begin{array}{c} V(G) \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

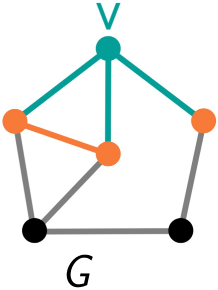


Cut-rank(X) is invariant under local complementation.

(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$
V(G) \overset{\displaystyle V(G)}{\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdot & 1 & 0 & 1 & 0 \\ 1 & 1 & \cdot & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdot & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}}
$$



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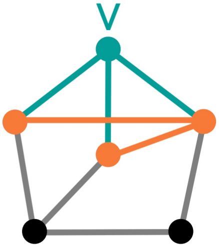
$$
\begin{array}{cc}
 & V(G) \\
V(G) & \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & \cdot & 0 & 1 & 1 & 0 \\
1 & 0 & \cdot & 1 & 1 & 0 \\
1 & 1 & 1 & \cdot & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\begin{array}{l}
 \\ -\mathrm{I} \\ -\mathrm{I} \\ -\mathrm{I} \\ \\ \\
\end{array}
\end{array}
$$



$G * v$

Cut-rank(X) is invariant under local complementation.

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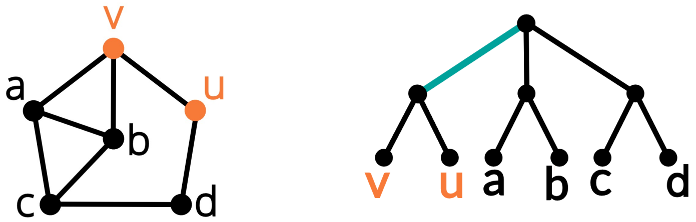
For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

Rank-width(G) is the minimum **width** of a subcubic tree T with leafs $V(G)$.



$$\textbf{width}(T) = \max_{e \in E(T)} \text{cut-rank}(X_e)$$

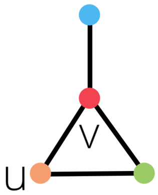
(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

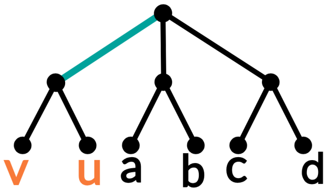
Rank-width(G) is the minimum **width** of a subcubic tree T with leafs $V(G)$.

Theorem (Geelen-Kwon-McCarty-Wollan)

A class of graphs has unbounded **rank-width** *iff it has all* **circle graphs** *as* **vertex-minors***.*



circle graph

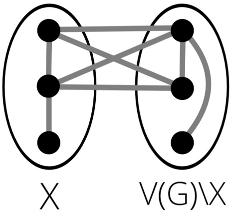


decomposition

Conjecture (structure)

*Every graph with no H-**vertex-minor** "**decomposes**" into parts that are "almost" **circle graphs**.*

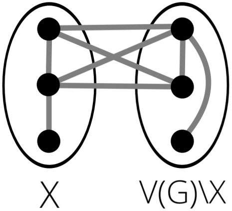
$$
\begin{array}{cc}
 & \begin{array}{ccc|ccc} & X & & & V(G)\setminus X & \end{array} \\
\begin{array}{c} X \\ \\ V(G)\setminus X \end{array} &
\left[\begin{array}{ccc|ccc}
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$



Conjecture (structure)

*Every graph with no H-***vertex-minor** *"decomposes" into parts that are "***almost***"* **circle graphs***.*

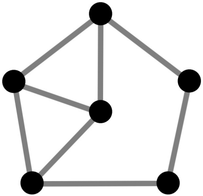
$$
\begin{array}{c}
 \\ X \\ \\ V(G)\setminus X
\end{array}
\begin{array}{c}
\begin{array}{ccc|ccc} & X & & & V(G)\setminus X & \end{array} \\
\left[\begin{array}{ccc|ccc}
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$



Conjecture (**structure**)

*Every graph with no H-**vertex-minor** "decomposes" into parts that are p_H-**perturbations** of **circle graphs**.*

A graph G is a p-**perturbation** of G' if the diagonal of $\mathrm{Adj}(G) + \mathrm{Adj}(G')$ can be filled in to rank $\leq p$.



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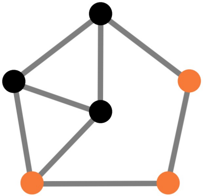

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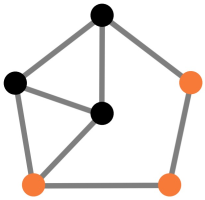

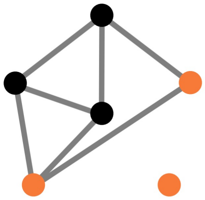

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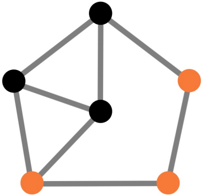

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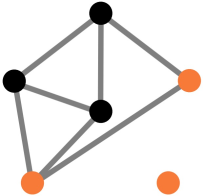

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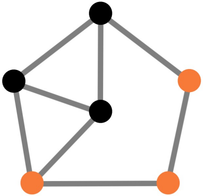

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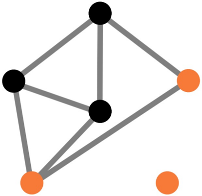

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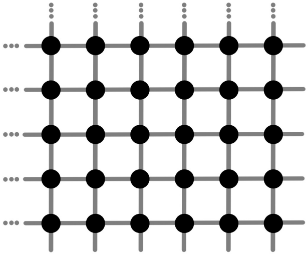

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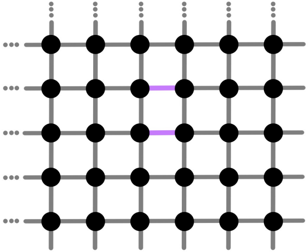

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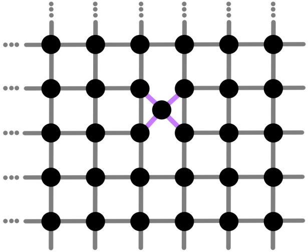

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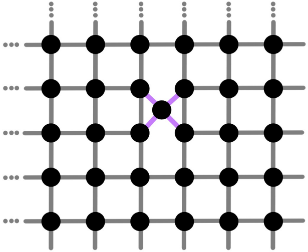

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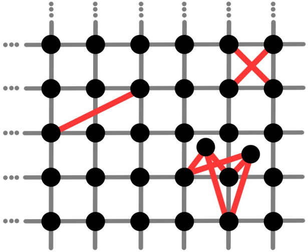

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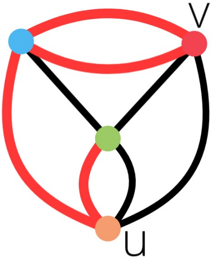

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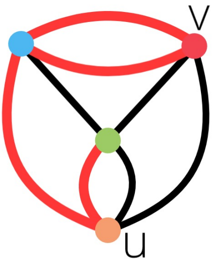

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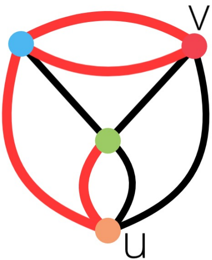

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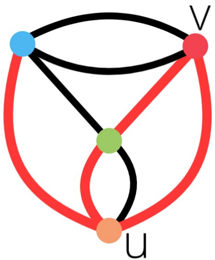

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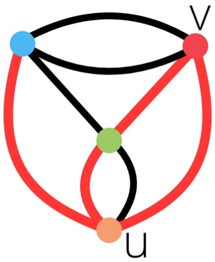

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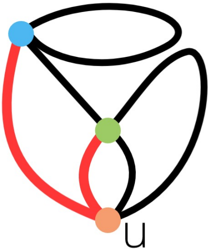

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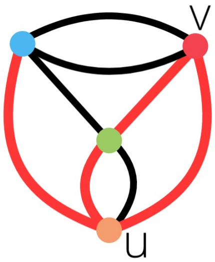

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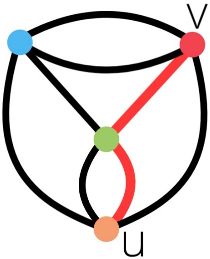

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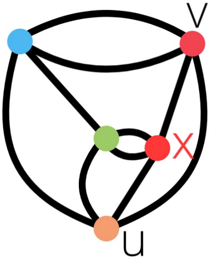

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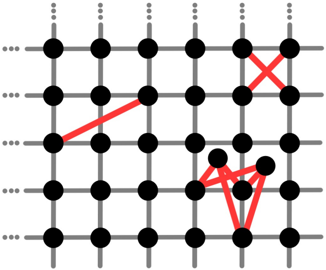

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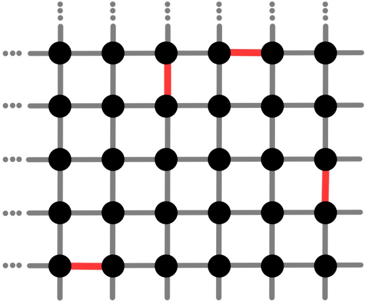

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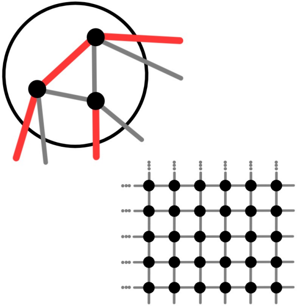

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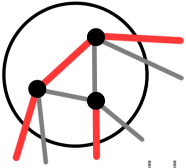

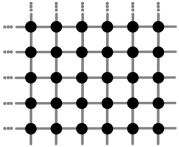

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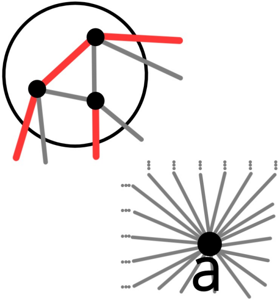

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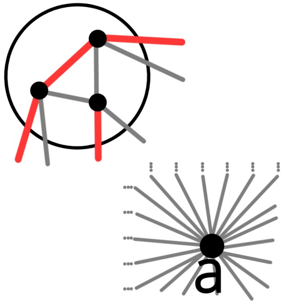

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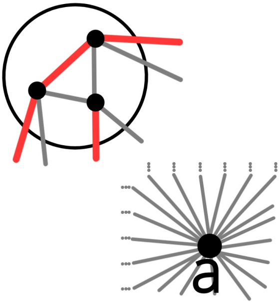

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Signed graphs are $\mathbb{Z}_2$-labelled; for n vertices we have $\mathbb{Z}_2^n$-labelling.

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