Conjectures on vertex-minors

Rose McCarty

Department of Combinatorics and Optimization
UNIVERSITY OF WATERLOO

September 29th, 2021

Joint work with Jim Geelen and Paul Wollan.

Figure by Felix Reidl

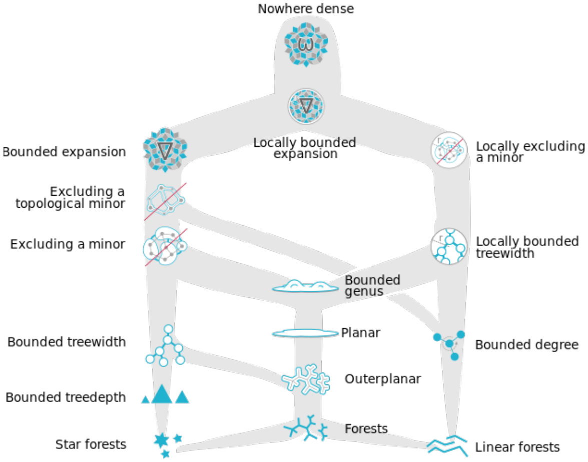
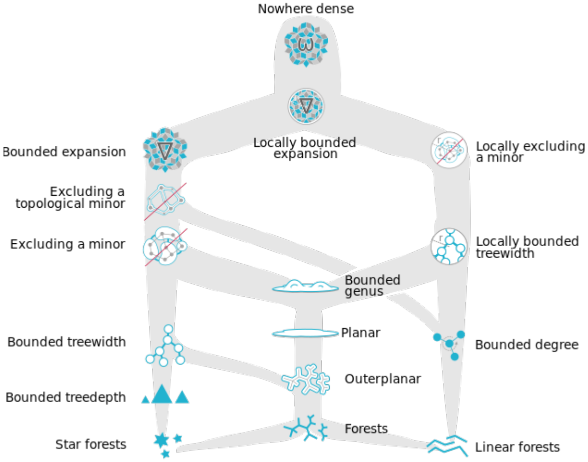
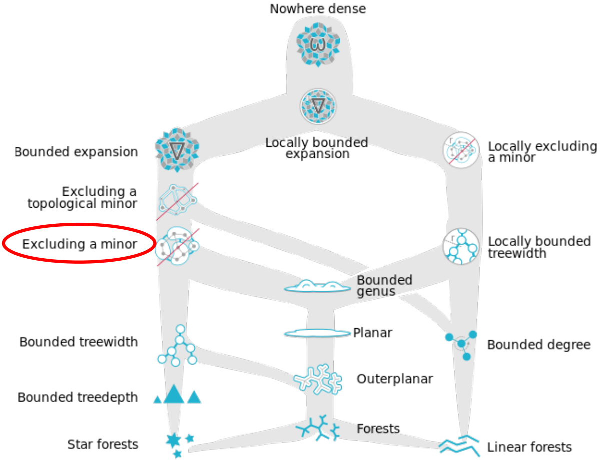


Figure by Felix Reidl



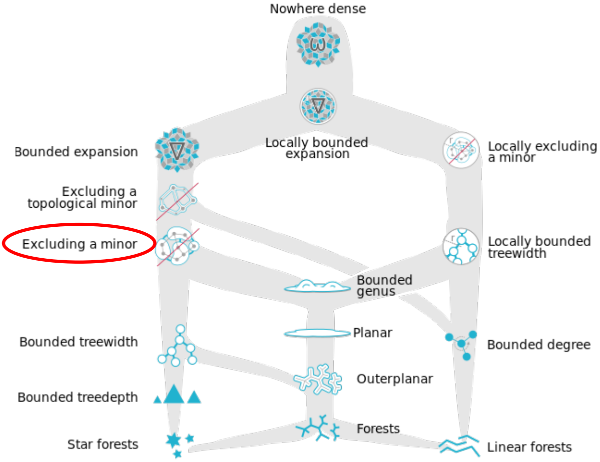
What are the "**dense**" analogs?

Figure by Felix Reidl



What are the “**dense**” analogs?

Figure by Felix Reidl

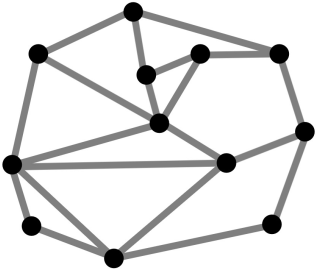


minors ⟶ **vertex-minors/pivot-minors**

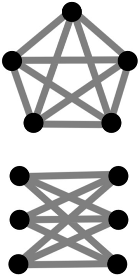
Well-Quasi-Ordering Theorem (Robertson & Seymour 2004)

Every infinite set of graphs contains one that is isomorphic to a minor of another.

Kuratowski's Theorem



planar graphs

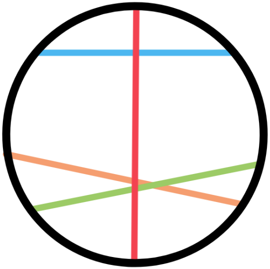


forbidden minors

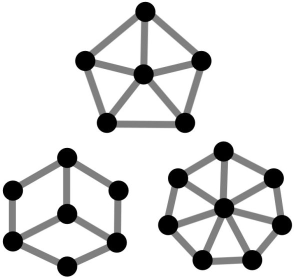
Well-Quasi-Ordering Conjecture (Oum 2017)

Every infinite set of graphs contains one that is isomorphic to a **vertex-minor** *of another.*

Bouchet's Theorem



circle graphs

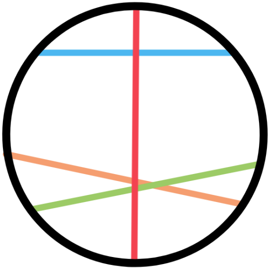


forbidden vertex-minors

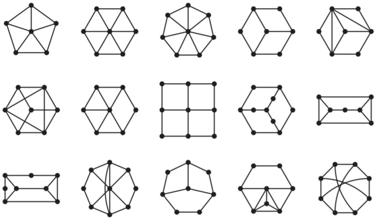
Well-Quasi-Ordering Conjecture (Oum 2017?)

Every infinite set of graphs contains one that is isomorphic to a **pivot-minor** *of another.*

Geelen and Oum's Theorem



circle graphs

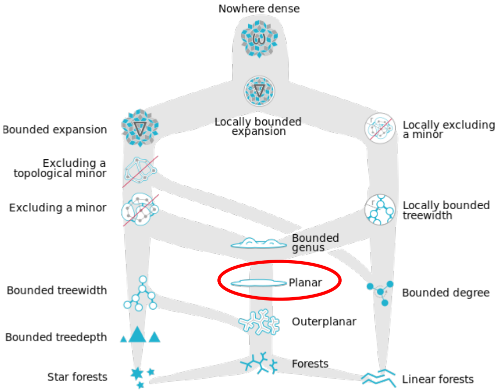


forbidden pivot-minors

Common generalization! (Bouchet 1988; de Fraysseix 1981)

Well-Quasi-Ordering Conjecture (Oum 2017?)

Every infinite set of graphs contains one that is isomorphic to a **pivot-minor** *of another.*



planar graphs $\longrightarrow$ **circle graphs**

Structure Theorem (Robertson & Seymour 2003)

For any proper minor-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that "almost embed" in a surface of bounded genus.

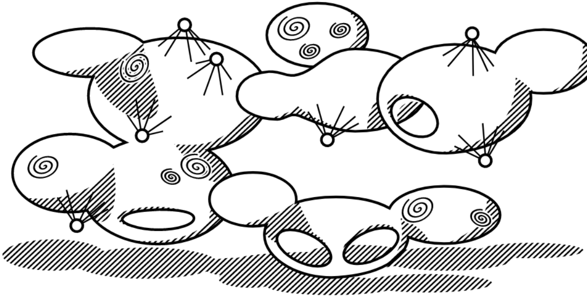
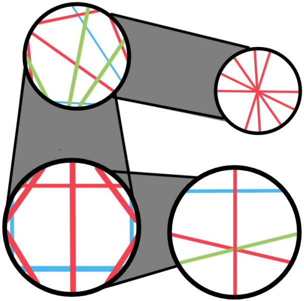


Figure by Felix Reidl

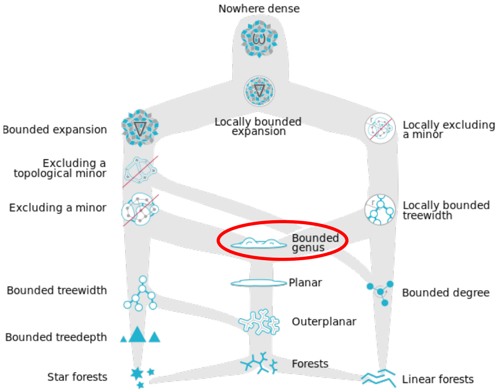
Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*



Structural Conjecture (Geelen)

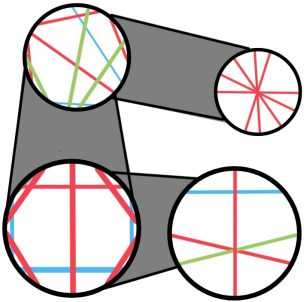
For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*



bounded genus $\rightsquigarrow$ **perturbed circle graphs**

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class* $\mathcal{F}$*, each* $G \in \mathcal{F}$ *"decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

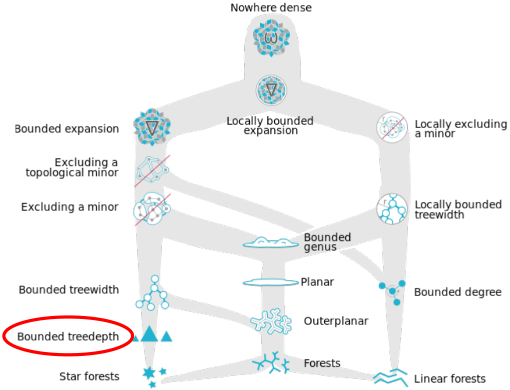


Ongoing project with Jim Geelen & Paul Wollan.
Some also joint with O-joung Kwon & Sang-il Oum.

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class* $\mathcal{F}$*, each* $G \in \mathcal{F}$ *"decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Progress:

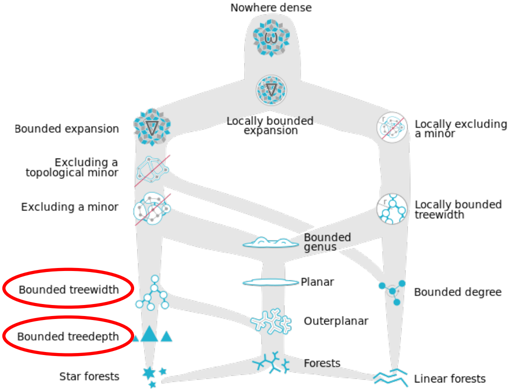


tree-depth $\longrightarrow$ **shrub-depth**/**rank-depth**

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Progress:

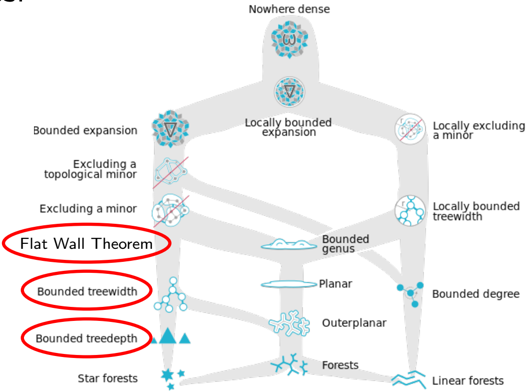


tree-width $\longrightarrow$ **clique-width/rank-width**

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Progress:

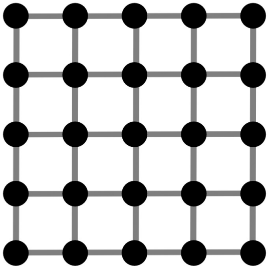


Flat Wall Theorem $\rightsquigarrow$ **Local Structure Theorem**

Grid Theorem (Robertson & Seymour 1986)

For any planar graph H, every graph with tree-width $\geq f(H)$ has a minor isomorphic to H.

grid:

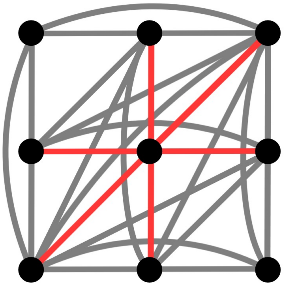


H minor of $G \implies \mathrm{tw}(H) \leq \mathrm{tw}(G)$.

Theorem (Geelen, Kwon, McCarty, & Wollan 2020)

For any **circle graph** *H, every graph with* **rank-width** *$\geq f(H)$ has a* **vertex-minor** *isomorphic to H.*

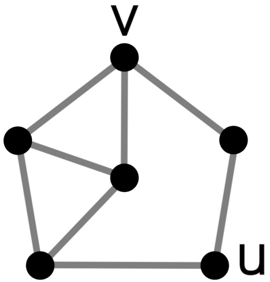
comparability grid:



H vertex-minor of $G \implies \mathrm{rw}(H) \leq \mathrm{rw}(G)$.

The **vertex-minors** of a graph G are obtained by

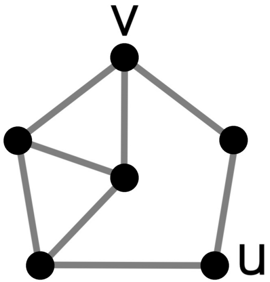
1) vertex deletion and
2) **local complementation**



G

The **vertex-minors** of a graph G are obtained by

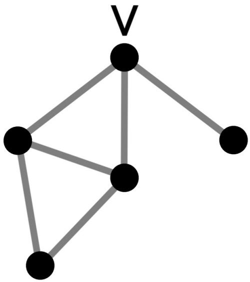
1) vertex deletion and
2) **local complementation**



G

The **vertex-minors** of a graph G are obtained by

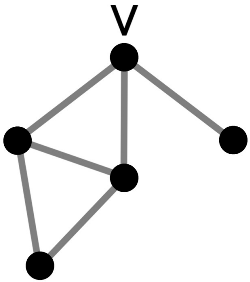
1) vertex deletion and
2) **local complementation**



$$G - u$$

The **vertex-minors** of a graph G are obtained by

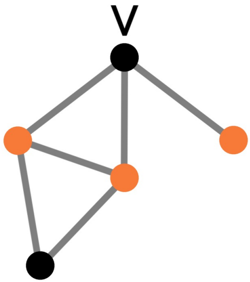
1) vertex deletion and
2) **local complementation**



$G - u$

The **vertex-minors** of a graph G are obtained by

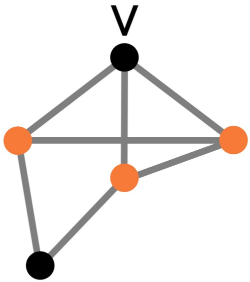
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



$(G - u)$

The **vertex-minors** of a graph G are obtained by

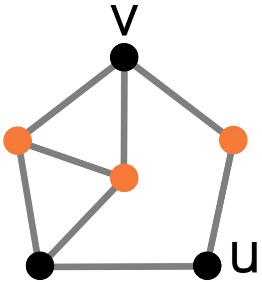
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



$$(G - u) * v$$

The **vertex-minors** of a graph G are obtained by

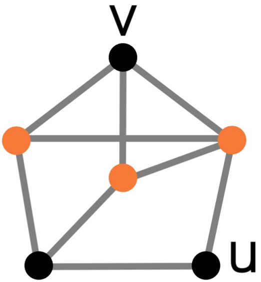
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



G

The **vertex-minors** of a graph G are obtained by

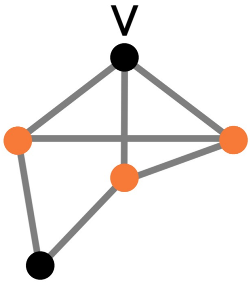
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



$G * v$

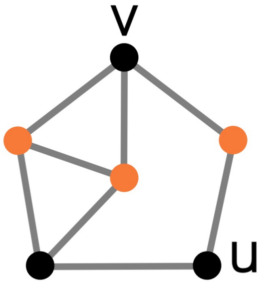
The **vertex-minors** of a graph G are obtained by

1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



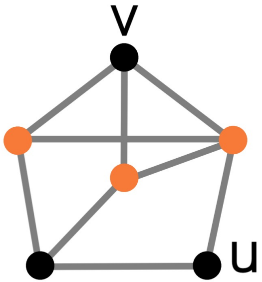
$$G * v - u$$

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.



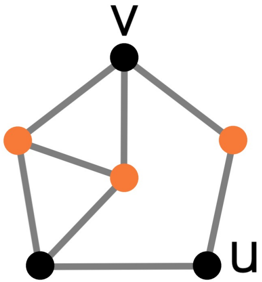
G

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.



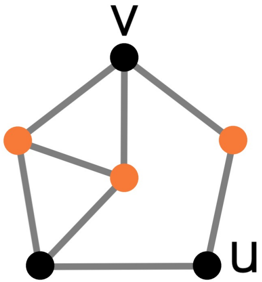
$G * v$

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.



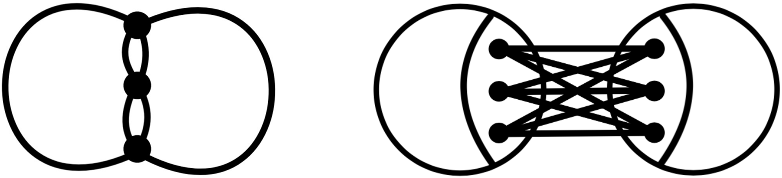
$G * v * v$

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.



$$G * v * v = G$$

Locally equivalent graphs have the same **cut-rank** function.

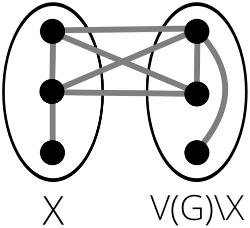


separators $\longrightarrow$ **cut-rank**

Locally equivalent graphs have the same **cut-rank** function.

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$
\begin{array}{c c}
 & \begin{array}{ccc|ccc} & X & & & V(G)\setminus X & \end{array} \\
\begin{array}{c} X \\ \\ V(G)\setminus X \end{array} &
\left[\begin{array}{ccc|ccc}
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$

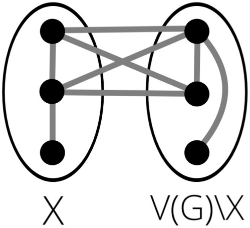


(Oum-Seymour, Bouchet, Cunningham, Oum)

Locally equivalent graphs have the same **cut-rank** function.

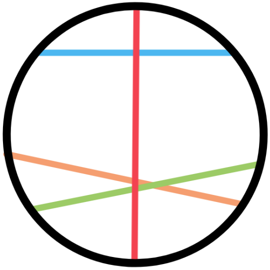
For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

	X			$V(G) \setminus X$		
X	0	1	0	1	1	0
	1	0	1	1	1	0
	0	1	0	0	0	0
$V(G) \setminus X$	1	1	0	0	1	1
	1	1	0	1	0	0
	0	0	0	1	0	0

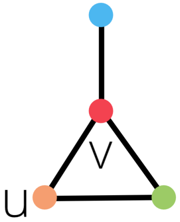


(Oum-Seymour, Bouchet, Cunningham, Oum)

A **circle graph** is the intersection graph of chords on a circle.
Circle graphs are closed under local complementation and vertex-deletion.

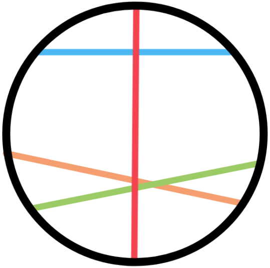


chord diagram

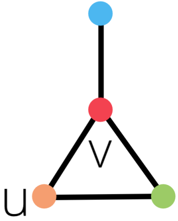


circle graph G

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

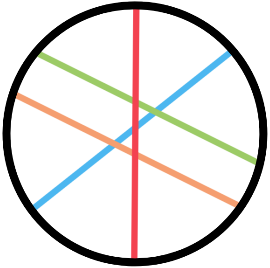


chord diagram

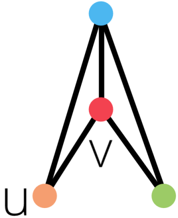


circle graph G

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

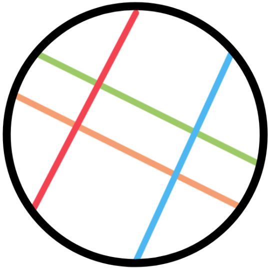


chord diagram

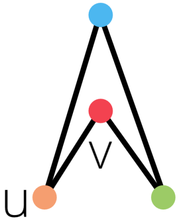


circle graph $G * v$

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

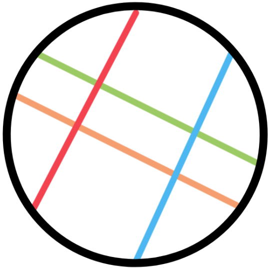


chord diagram

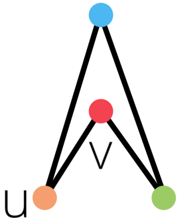


circle graph $G * v * u$

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

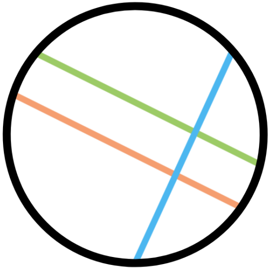


chord diagram

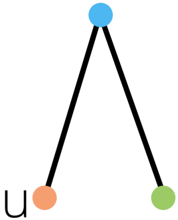


circle graph $G * v * u$

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

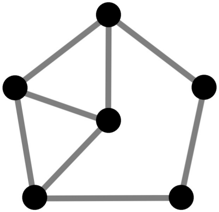


chord diagram

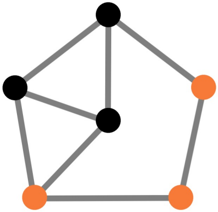


circle graph $G * v * u - v$

A p-**perturbation** of G is any graph that can be obtained from G by **complementing on** p **sets**.



A **p-perturbation** of G is any graph that can be obtained from G by **complementing on p sets**.



A *p*-**perturbation** of G is any graph that can be obtained from G by **complementing on p sets**.

A p-**perturbation** of G is any graph that can be obtained from G by **complementing on p sets**.

G forbids H-vertex-minor

$$\longrightarrow$$

p-**perturbations** of G forbid H'-vertex-minor

(where H' depends on p and H; uses lemma of Bouchet).

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

If a property holds for the following classes, then it is a reasonable conjecture about $\mathcal{F}$.

- classes of bounded **clique-width**/**rank-width**, and
- bounded-order **perturbations** of **circle graphs**.

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

If a property holds for the following classes, then it is a reasonable conjecture about $\mathcal{F}$.

- classes of bounded **clique-width**/**rank-width**, and
- bounded-order **perturbations** of **circle graphs**.

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

If a property holds for the following classes, then it is a reasonable conjecture about $\mathcal{F}$.

- classes of bounded **clique-width**/**rank-width**, and
- bounded-order **perturbations** of **circle graphs**.

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Conjectures: (for any fixed $\mathcal{F}$)

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Conjectures: (for any fixed $\mathcal{F}$)

- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Conjectures: (for any fixed $\mathcal{F}$)

- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number $\omega(G)$ of $G \in \mathcal{F}$ in polynomial time (Geelen).

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Conjectures: (for any fixed $\mathcal{F}$)

- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number $\omega(G)$ of $G \in \mathcal{F}$ in polynomial time (Geelen).
- The chromatic number of $G \in \mathcal{F}$ is $\leq \mathrm{polynomial}(\omega(G))$ (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman).

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Conjectures: (for any fixed $\mathcal{F}$)

- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number $\omega(G)$ of $G \in \mathcal{F}$ in polynomial time (Geelen).
- The chromatic number of $G \in \mathcal{F}$ is $\leq \mathrm{polynomial}(\omega(G))$ (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman).
- Can $n^{1-\epsilon}$-approximate the chromatic number of $G \in \mathcal{F}$.

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class* $\mathcal{F}$*, each* $G \in \mathcal{F}$ *"decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Conjectures: (for any fixed $\mathcal{F}$)

- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number $\omega(G)$ of $G \in \mathcal{F}$ in polynomial time (Geelen).
- The chromatic number of $G \in \mathcal{F}$ is $\leq \mathrm{polynomial}(\omega(G))$ (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman).
- Can $n^{1-\epsilon}$-approximate the chromatic number of $G \in \mathcal{F}$.
- $\mathrm{MBQC}_{\mathcal{F}}$ can be efficiently simulated classically (Geelen).

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs**.

Conjectures: (for any fixed $\mathcal{F}$)

- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number $\omega(G)$ of $G \in \mathcal{F}$ in polynomial time (Geelen).
- The chromatic number of $G \in \mathcal{F}$ is $\leq \mathrm{polynomial}(\omega(G))$ (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman).
- Can $n^{1-\epsilon}$-approximate the chromatic number of $G \in \mathcal{F}$.
- $\mathrm{MBQC}_{\mathcal{F}}$ can be efficiently simulated classically (Geelen).
- For any fixed **permutation graph** H, first-order model-checking is FPT (wrt formula length) on H-free graphs in $\mathcal{F}$.

Structural Conjecture (Geelen)

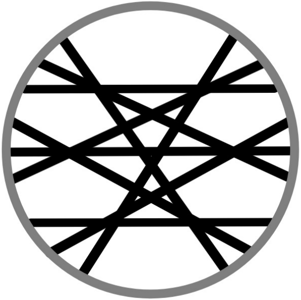
For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

Conjectures: (for any fixed $\mathcal{F}$)

- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number $\omega(G)$ of $G \in \mathcal{F}$ in polynomial time (Geelen).
- The chromatic number of $G \in \mathcal{F}$ is $\leq \mathrm{polynomial}(\omega(G))$ (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman).
- Can $n^{1-\epsilon}$-approximate the chromatic number of $G \in \mathcal{F}$.
- $\mathrm{MBQC}_{\mathcal{F}}$ can be efficiently simulated classically (Geelen).
- For any fixed **permutation graph** H, first-order model-checking is FPT (wrt formula length) on H-free graphs in $\mathcal{F}$. Hliněný & Pokrývka proved for circle graphs via **twin-width**.

Structural Conjecture (Geelen)

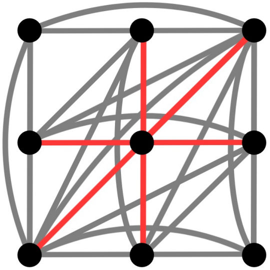
For any proper **vertex-minor***-closed class* $\mathcal{F}$*, each* $G \in \mathcal{F}$ *"decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*



permutation graph

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into bounded-order* **perturbations** *of* **circle graphs***.*

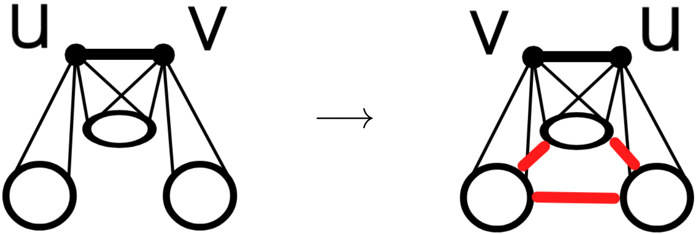


permutation graph

To **pivot** on an edge uv,

1) exchange the labels of u and v, and

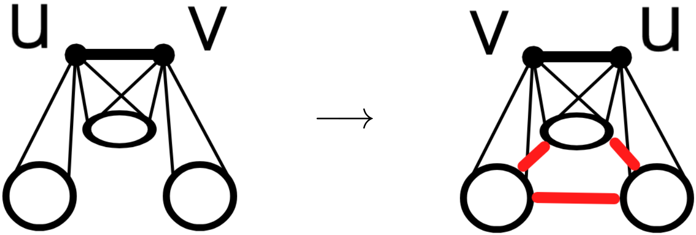
2) complement between the three sets...



To **pivot** on an edge uv,

1) exchange the labels of u and v, and

2) complement between the three sets...



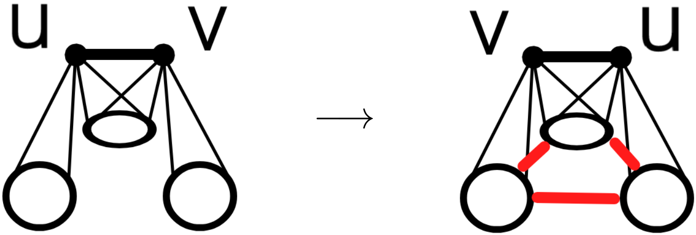
This is the same as $G * v * u * v = G * u * v * u$.

To **pivot** on an edge uv,

1) exchange the labels of u and v, and

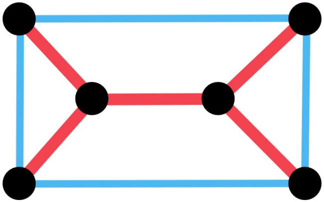
2) complement between the three sets...

Pivot-minors are obtained by deleting vertices and pivoting.



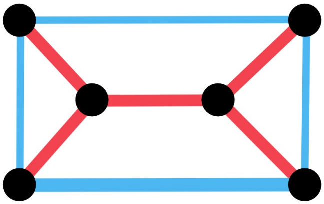
This is the same as $G * v * u * v = G * u * v * u$.

Consider a graph with a spanning tree **T**, and its **fundamental graph** $\mathcal{F}(\mathbf{T})$. **Pivoting** corresponds to changing **T**.

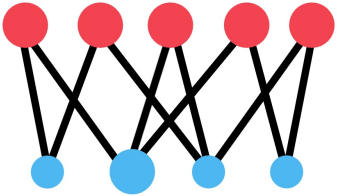


planar graph

Consider a graph with a spanning tree **T**, and its **fundamental graph** $\mathcal{F}(\mathbf{T})$. **Pivoting** corresponds to changing **T**.

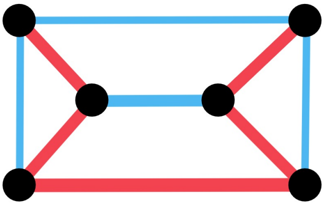


planar graph

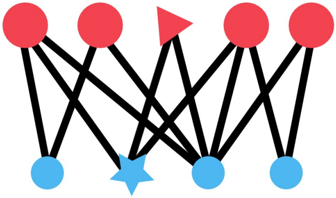


fundamental graph $\mathcal{F}(\mathbf{T})$

Consider a graph with a spanning tree **T**, and its **fundamental graph** $\mathcal{F}(\mathbf{T})$. **Pivoting** corresponds to changing **T**.

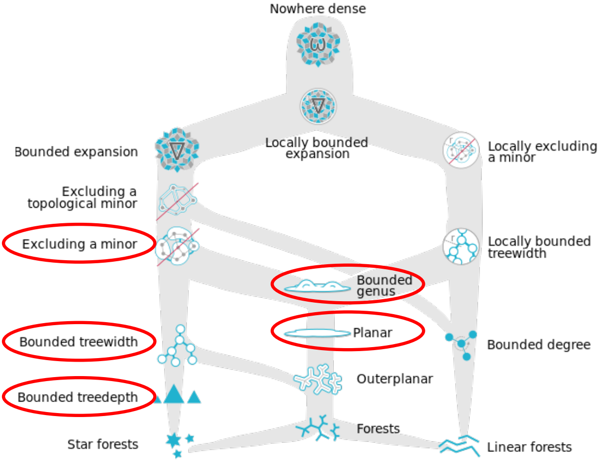


planar graph



fundamental graph $\mathcal{F}(\mathbf{T})$

Figure by Felix Reidl



Pivot-minors?

Thank you!