8-Connected Graphs are 4-Ordered

Rose McCarty, Yan Wang, Xingxing Yu

School of Mathematics
Georgia Institute of Technology

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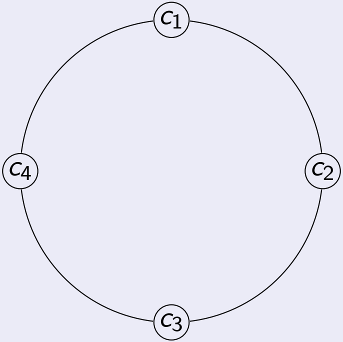
The Problem

Definition

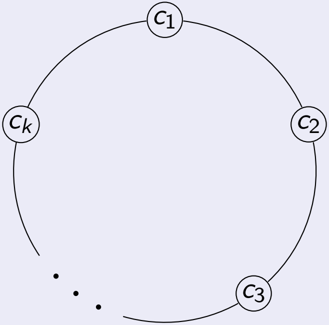
A graph G is **4-ordered** if for every $\{c_1, c_2, c_3, c_4\} \subseteq V(G)$, the graph contains:



The Problem

Definition

A graph G is k-ordered if for every $\{c_1, c_2, \ldots, c_k\} \subseteq V(G)$, the graph contains:



Problem

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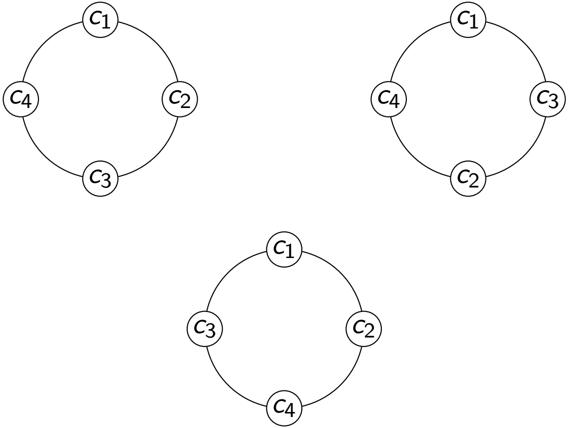
Is a generalization of:

Theorem [Dirac, 60]

For every integer $k \geq 2$, if G is a k-connected graph and S is a set of k vertices in G, then G has a cycle containing every vertex in S. There exist $(k-1)$-connected graphs without this property.

The Problem

For $k \leq 3$ there is only one cyclic ordering of k vertices. For $k = 4$:



Prior Work

Definition

A graph G is ***k*-linked** if for every $\{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\} \subseteq V(G)$, there exist vertex-disjoint paths $P_1, P_2, \ldots, P_k$ so that P_i has ends s_i and t_i.

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[Bollobás and Thomason, 96]

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  [Bollobás and Thomason, 96]
- $12k$-connected graphs are k-linked.
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- $22k$-connected graphs are k-linked.
  [Bollobás and Thomason, 96]
- $12k$-connected graphs are k-linked.
  [Kawarabayashi, Kostochka, and G. Yu, 06]
- $10k$-connected graphs are k-linked.
  [Thomas and Wollan, 05]

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Theorem [R.M., Y. Wang, and X. Yu, 17+] (*In preparation*)

Every 8-connected graph is 4-ordered.

Characterizations

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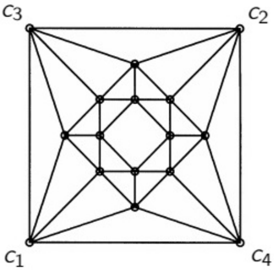
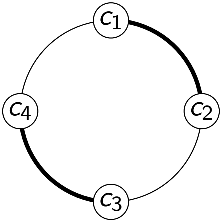
1. A conjecture for a characterization of when a graph with fixed $\{c_1, c_2, c_3, c_4\}$ has no cycle through them in order
2. Main idea of 8-connected implies 4-ordered proof

To show that $f(4) \geq 6$:



[Faudree, 01]

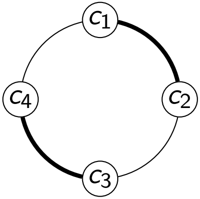
has no



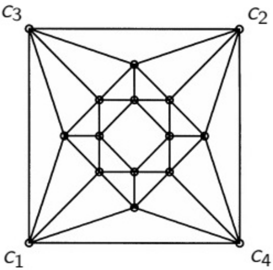
Theorem [Seymour, 80]

If G is a 4-connected graph with $\{c_1, c_2, c_3, c_4\} \subseteq V(G)$ such that G does not contain disjoint paths P_1 and P_3 so that P_i has ends c_i and c_{i+1}, then G can be embedded in the plane with c_1, c_3, c_2, c_4 on the outer face in that order.

If no



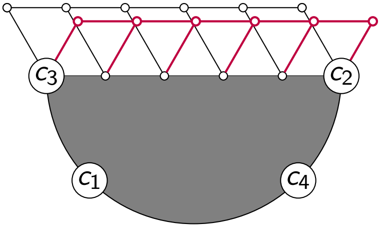
then can be drawn like



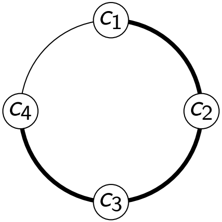
[Faudree, 01]

Characterizations

To show that $f(4) \geq 7$:

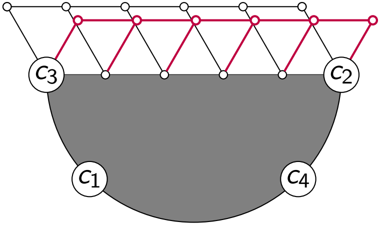


has no

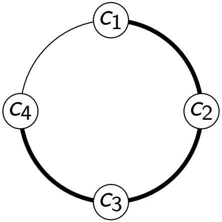


Characterizations

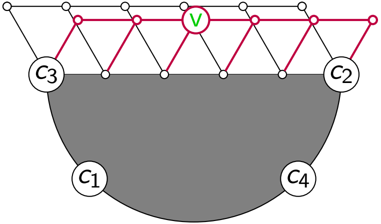
Suppose that



has a path P



Let v be the vertex in P closest to c_1 so that v is not in the planar part of the graph.



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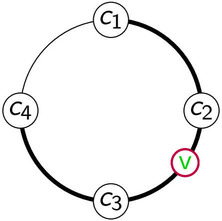
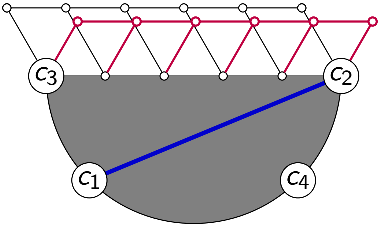
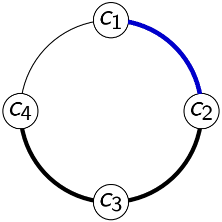


If v does not exist or is not in $P[c_1, c_2]$, then:

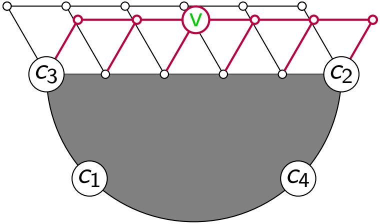


has a path P

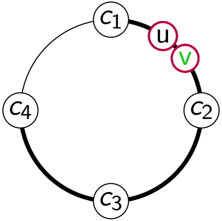


Characterizations

So v is in $P[c_1, c_2]$. Then:



has a path P



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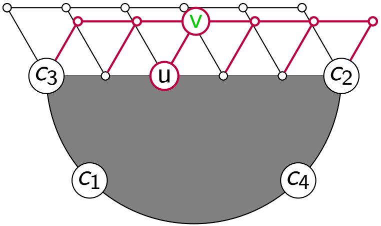

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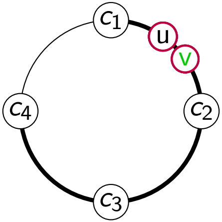

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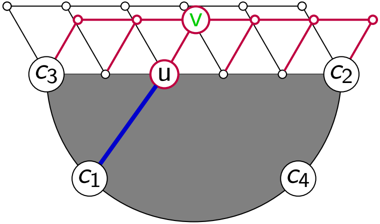
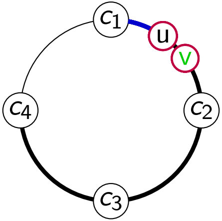
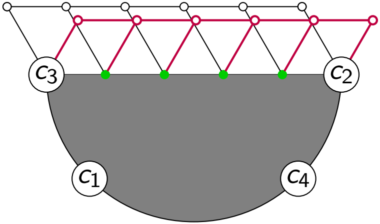


has a path P

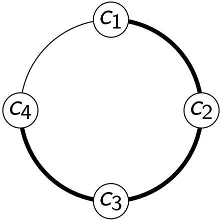


Characterizations

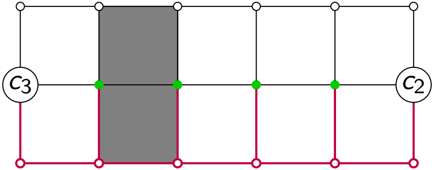
So the graph



has no



There are other possibilities:



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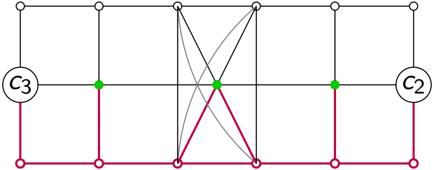
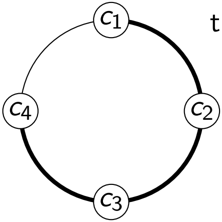


Characterizations

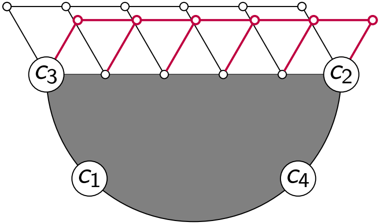
Theorem (informal statement) [X. Yu, 03]

If G is a 4-connected graph with $\{c_1, c_2, c_3, c_4\} \subseteq V(G)$ such that G does not contain a path P with ends c_1 and c_4 that encounters c_1, c_2, c_3, c_4 in order, then G consists of a planar graph attached to a ladder.

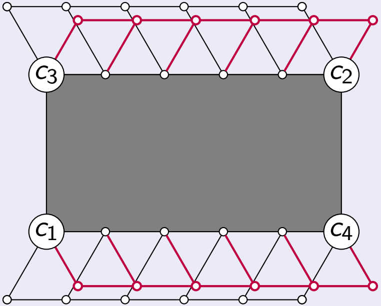
If no



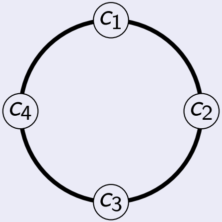
then consists of planar part and a *ladder*



Observation



has no



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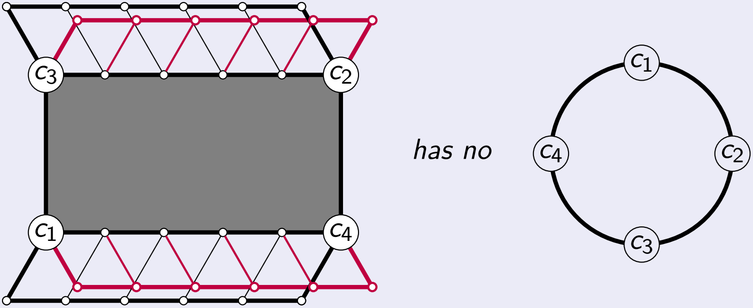

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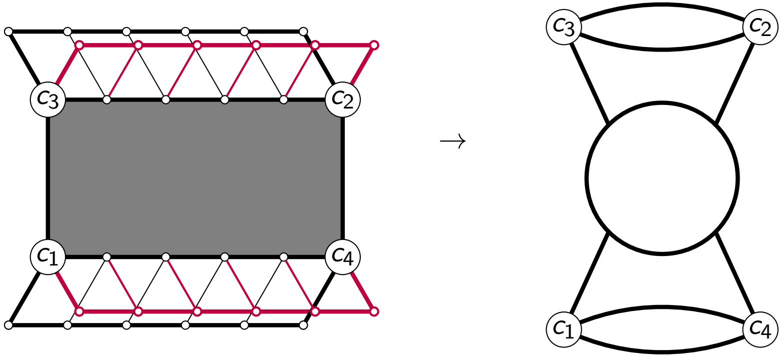
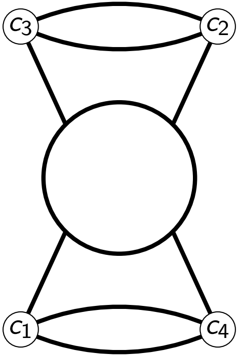


Proof Idea



1. Find illustrated subdivision where c_1, c_2, c_3, c_4 fixed.

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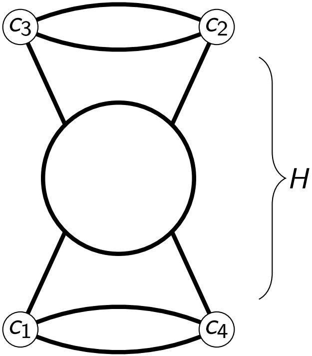

Let H be the induced graph on the set of all vertices of G except those in the $c_3 - c_2$ cycle or $c_1 - c_4$ cycle.

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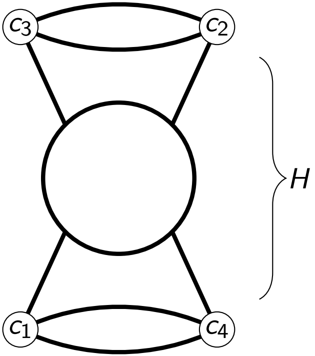

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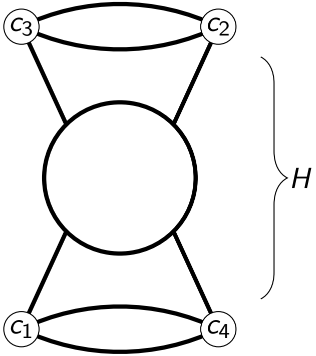
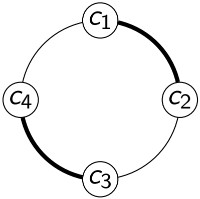
1. Find illustrated subdivision where c_1, c_2, c_3, c_4 fixed.
2. Choose one in a certain way that results in H 2-connected.



Then H has no



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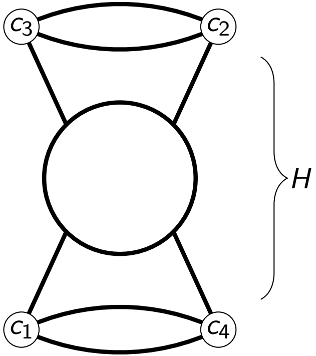

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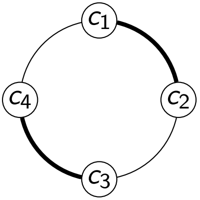

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1. Find illustrated subdivision where c_1, c_2, c_3, c_4 fixed.
2. Choose one in a certain way that results in H 2-connected.
3. Then in fact H is 4-connected. So H is planar.

Future Directions

Do there exist 7 connected graphs that are not 4-ordered?

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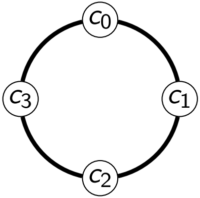
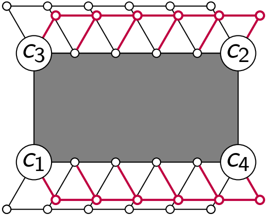


then looks like



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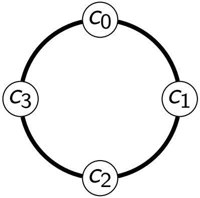
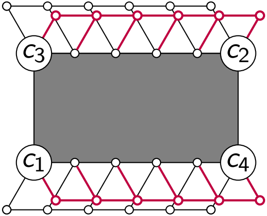
If no 

then looks like 

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Contact: `rmccarty3@gatech.edu`