The Grid Theorem for Rank-Width

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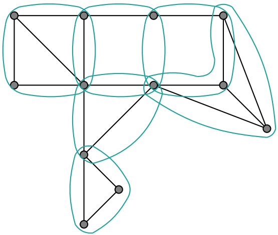
The Grid Theorem [Robertson and Seymour, 86]

A family of graphs has unbounded tree-width iff it contains all planar graphs as minors.



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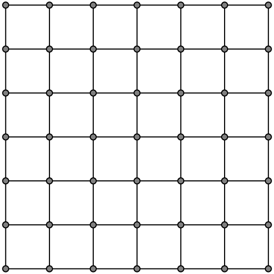
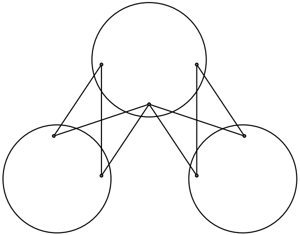


Theorem [Geelen, Kwon, McCarty, Wollan, 18+]

A family of graphs has unbounded **rank-width** *iff it contains all* **circle graphs** *as* **vertex minors***.*



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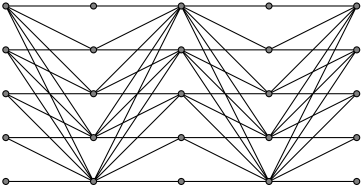


Definition

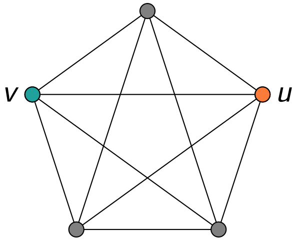
The **vertex minors** of G are graphs obtained by

- deleting vertices
- local complementations

Locally complementing at $v \in V(G)$:

$$E(\mathbf{G*v}) = E(G)\Delta\{\{x, y\} : x \neq y \text{ and } x, y \in N(v)\}$$

G:



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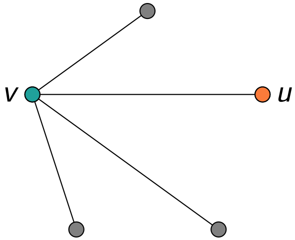

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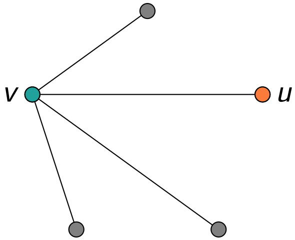

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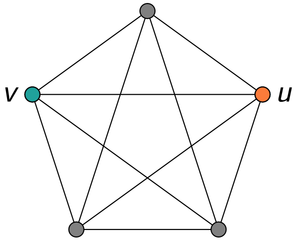
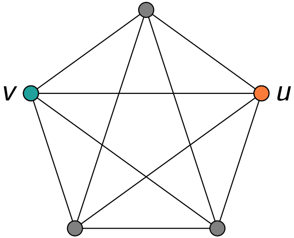


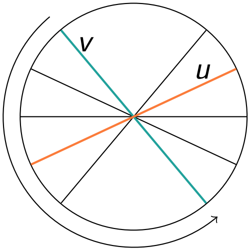
Definition

A **circle graph** is the intersection graph of chords on a circle.

- Circle graphs are vertex minor-closed. [Bouchet, 94]



G



Chord diagram for G

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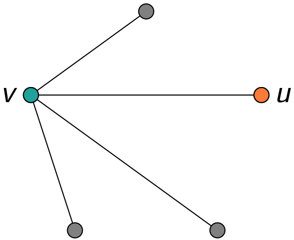
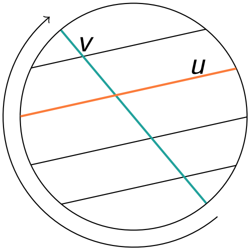


G*v

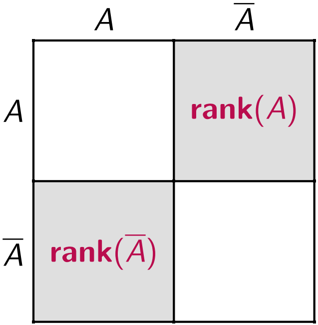


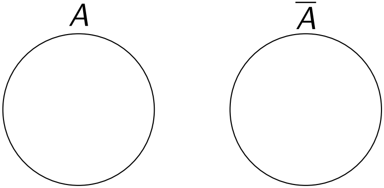
Chord diagram for **G*v**

Definition

For $A \subseteq V(G)$, **rank**(A) is the rank over $\mathbb{GF}_2$ of the submatrix of the adjacency matrix with rows A and columns $\overline{A}$.

- $\text{rank}(A) = \text{rank}(\overline{A})$, and $\text{rank}_G(A) = \text{rank}_{G*v}(A)$!
- $\text{rank}(A)$ large $\Leftrightarrow |\{N(v) \cap \overline{A} : v \in A\}|$ large





$\text{rank}(A) = 0$

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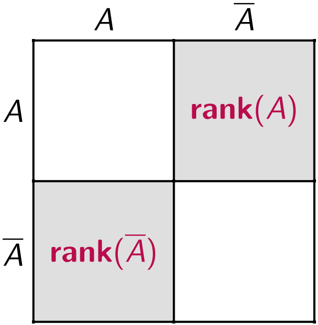
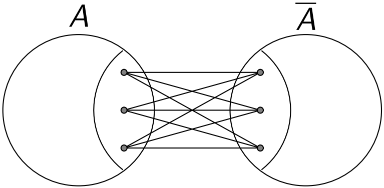




$\text{rank}(A) = 1$

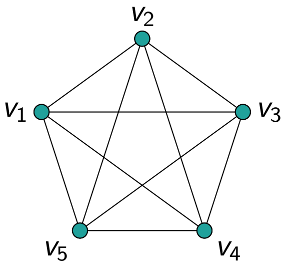
Definition [Oum and Seymour, 06]

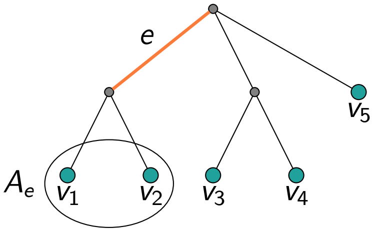
The **rank-width** *of G is*

$$\min_{T} \max_{e \in E(T)} rank(A_e)$$

- *T subcubic tree with leaves$(T) = V(G)$*
- *A_e set of leaves of a component of $T - e$*

- **rank-width**$(G) - 1 \leq$ tree-width(G) [Oum, 07]





Theorem [Geelen, Kwon, McCarty, Wollan, 18+]

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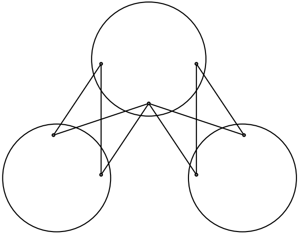

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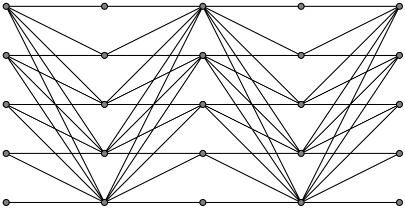
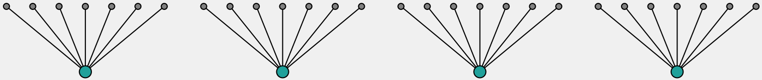


Proof starting points

Corollary

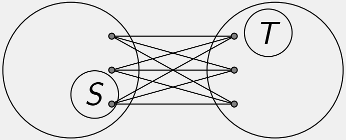
large rank-width ⇒ large



vertex minor

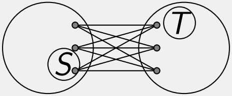
Lemma

G v.m. **minimal** *with rank-width*$(G) \geq k$,
$S, T \subseteq V(G)$ *disjoint and large ⇒ NO small rank separation*



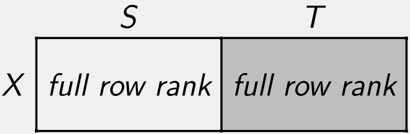
Lemma

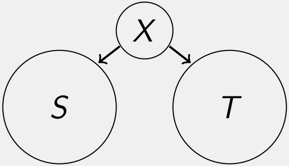
$S, T \subseteq V(G)$ disjoint with NO small rank separation



$\Rightarrow \exists$ *H v.m. of G with*

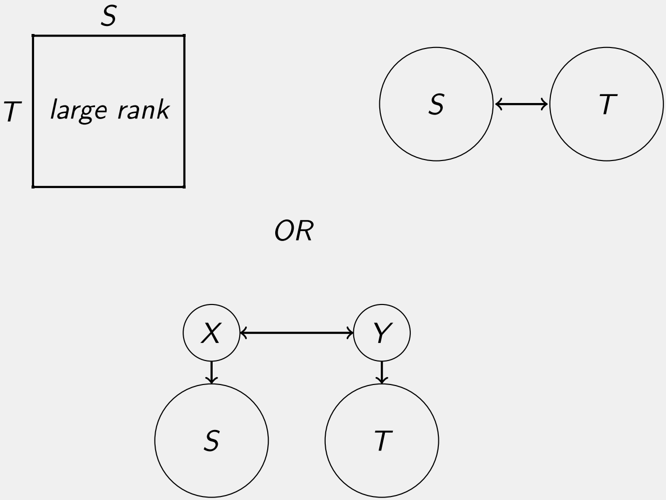
- $H[S \cup T] = G[S \cup T]$ *and*
- $X \subseteq V(G)$ *large s.t.*





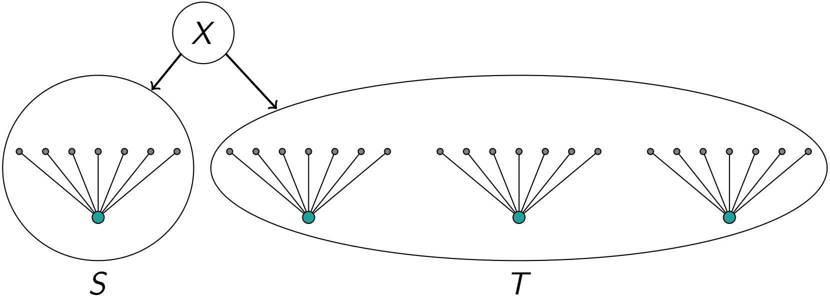
OR ...

Lemma



Applying Lemma

- Use $(G - v) * u = (G * u) - v$
- Use $\text{rank}_G(A) = \text{rank}_{G*v}(A)$



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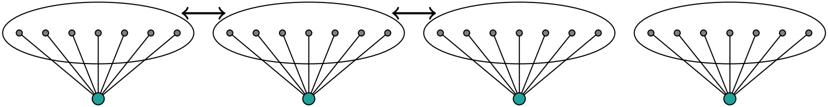

$\mathcal{R}_k := \{G : \text{rank-width}(G) \leq k\}$

Theorem [Dvořák and Kráľ, 11]

$\mathcal{R}_k$ is χ-bounded.

Conjecture [Geelen]: v.m.-free graphs are χ-bounded.

Theorem [Courcelle, Makowsky, and Rotics, 99]

Max-clique is in poly-time on $\mathcal{R}_k$.

Conjecture: Poly-time on v.m-free graphs.

Theorem [Oum, 08]

$\mathcal{R}_k$ has no infinite antichain.

Conjecture: There is no infinite antichain.

Thanks to rosschurchley.com for the Beamer theme.